Chapter 3
Hypergraphs for Interval-Valued Structures

In this chapter, we present interval-valued fuzzy hypergraphs, $A = [\mu^-, \mu^+]$–tempered interval-valued fuzzy hypergraphs, and some of their properties. Moreover, we discuss the notions of vague hypergraphs, dual vague hypergraphs, and A-tempered vague hypergraphs. Finally, we describe interval-valued intuitionistic fuzzy hypergraphs and interval-valued intuitionistic fuzzy transversals of $\mathscr{H}$. This chapter is due to [4–6, 8, 11, 22, 27].

3.1 Introduction

Zadeh [27] introduced the notion of interval-valued fuzzy sets as an extension of fuzzy set theory [25] for representing vagueness and uncertainty. Interval-valued fuzzy set theory reflects the uncertainty by the length of the interval membership degree $[\mu_1, \mu_2]$. In intuitionistic fuzzy set theory for every membership degree (μ_1, μ_2), the value $\pi = 1 - \mu_1 - \mu_2$ denotes a measure of non-determinacy (or undecidedness). Interval-valued fuzzy sets provide a more adequate description of vagueness than traditional fuzzy sets. It is, therefore, important to use interval-valued fuzzy sets in applications, such as fuzzy control. One of the computationally most intensive parts of fuzzy control is defuzzification [20]. Since interval-valued fuzzy sets are widely studied and used, we describe briefly the work of Gorzalczany [14, 15] on approximate reasoning, Roy and Biswas [23] on medical diagnosis, Turksen [24] on multivalued logic and Mendel [20] on intelligent control. Atanassov and Gargov [6] introduced the notion of interval-valued intuitionistic fuzzy sets which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets.

Graph theory has numerous applications to problems in systems analysis, operations research, economics, and transportation. However, in many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it

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is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic. Hypergraph models are more general types of relations than graphs do and can be used to model networks, social networks, biology networks, process scheduling, data structures, computations, and a variety of other systems where complex relationships between the objects in the system play a dominant role. Fuzzy hypergraphs were proposed by Kaufmann [17] and then generalized and redefined by Lee-kwang and Lee [19]. Goetschel Jr. [12] discussed the concept of hypergraphs by initiating a glimpse of what may be done within a fuzzy setting. Also the idea of transversal of a hypergraph has been extended to fuzzy transversal of a fuzzy hypergraph by Goetschel Jr. et al. [13]. Chen [8] presented the notion of the interval-valued fuzzy hypergraph theory which is based on a combination of the interval-valued fuzzy set and hypergraph models. Akram and Dudek [1] presented some properties of intuitionistic fuzzy hypergraphs and provided its application in clustering problem. Naz et al. [22] proposed the concept of the interval-valued intuitionistic fuzzy hypergraphs by combining the interval-valued intuitionistic fuzzy set and hypergraph models.

Definition 3.1 An *interval number* D is an interval $[a^-, a^+]$ with $0 \le a^- \le a^+ \le 1$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$. Let $D[0, 1]$ be the set of all interval numbers. For interval numbers $D_1 = [a_1^-, b_1^+]$ and $D_2 = [a_2^-, b_2^+]$, we define

- $\min\{D_1, D_2\} = \min\{[a_1^-, b_1^+], [a_2^-, b_2^+]\} = [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$,
- $\max\{D_1, D_2\} = \max\{[a_1^-, b_1^+], [a_2^-, b_2^+]\} = [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$,
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^- \cdot a_2^-, b_1^+ + b_2^+ - b_1^+ \cdot b_2^+]$,
- $D_1 \le D_2 \iff a_1^- \le a_2^-$ and $b_1^+ \le b_2^+$,
- $D_1 = D_2 \iff a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \iff D_1 \le D_2$ and $D_1 \ne D_2$,
- $kD = k[a_1^-, b_1^+] = [ka_1^-, kb_1^+]$, where $0 \le k \le 1$.

Similarly,

$$\sup_{i\in I}\{[a_i^-, b_i^+]\} = [\sup_{i\in I}\{a_i^-\}, \sup_{i\in I}\{b_i^+\}] \quad \text{and} \quad \inf_{i\in I}\{[a_i^-, b_i^+]\} = [\inf_{i\in I}\{a_i^-\}, \inf_{i\in I}\{b_i^+\}].$$

It is known that $(D[0, 1], \le, \vee, \wedge)$ is a complete lattice with $[0, 0]$ as the least element and $[1, 1]$ as the greatest.

Definition 3.2 The *interval-valued fuzzy set* A in X is defined by, $A = \{(x, [\mu_A^-(x), \mu_A^+(x)]) : x \in X\}$, where $\mu_A^-(x)$ and $\mu_A^+(x)$ are fuzzy subsets of X such that $\mu_A^-(x) \le \mu_A^+(x)$, for all $x \in X$.

Let X be a non-empty set, then by an *interval-valued fuzzy relation* B on a set X we mean an interval-valued fuzzy set such that

$$\mu_B^-(xy) \le \min(\mu_A^-(x), \mu_A^-(y)), \mu_B^+(xy) \le \min(\mu_A^+(x), \mu_A^+(y)),$$

for all $xy \in X \times X$. In the clustering, the interval-valued fuzzy set A, is called an *interval-valued fuzzy class*. We define the *support* of A by supp $(A) = \{x \in X \mid [\mu_A^-(x), \mu_A^+(x)] \ne [0, 0]\}$ and say A is nontrivial if supp(A) is non-empty.

Interval-valued fuzzy relations reflect the idea that membership grades are often not precise and the intervals represent such uncertainty.

Definition 3.3 The *height* of an interval-valued fuzzy set $A = [\mu_A^-(x), \mu_A^+(x)]$ is defined as

$$h(A) = \sup_{x \in X}(A)(x) = [\sup_{x \in X} \mu_A^-(x), \sup_{x \in X} \mu_A^+(x)].$$

We shall say that interval-valued fuzzy set A is *normal* if $A = [\mu_A^-(x), \mu_A^+(x)] = [1, 1]$, for all $x \in X$.

Definition 3.4 By an *interval-valued fuzzy graph* on non-empty set X, we mean a pair $G = (A, B)$, where $A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set on X and $B = [\mu_B^-, \mu_B^+]$ is an interval-valued fuzzy relation on X such that

$$\mu_B^-(xy) \leq \min(\mu_A^-(x), \mu_A^-(y)),$$

$$\mu_B^+(xy) \leq \min(\mu_A^+(x), \mu_A^+(y)),$$

for all $x, y \in X$.

For further terminologies and studies on interval-valued fuzzy hypergraphs, readers are referred to [2, 3, 7, 9, 10, 16, 18, 21, 26, 27].

3.2 Interval-Valued Fuzzy Hypergraphs

Definition 3.5 Let X be a finite set and let $E = \{E_1, E_2, \ldots, E_m\}$ be a finite family of nontrivial interval-valued fuzzy subsets of X such that

$$X = \bigcup_j \text{supp}[\mu_j^-, \mu_j^+], \quad j = 1, 2, \ldots, m,$$

where $A = [\mu_j^-, \mu_j^+]$ is an interval-valued fuzzy set defined on $E_j \in E$. Then, the pair $I = (X, E)$ is an *interval-valued fuzzy hypergraph* on X, E is the family of interval-valued fuzzy edges of I and X is the (crisp) vertex set of I. The order of I (number of vertices) is denoted by $|X|$ and the number of edges is denoted by $|E|$.

Definition 3.6 Let $A = [\mu_A^-, \mu_A^+]$ be an interval-valued fuzzy subset of X and let E be a collection of interval-valued fuzzy subsets of X such that for each $B = [\mu_B^-, \mu_B^+] \in E$ and $x \in X$, $\mu_B^-(x) \leq \mu_A^-(x)$, $\mu_B^+(x) \leq \mu_A^+(x)$. Then the pair (A, B) is an *interval-valued fuzzy hypergraph* on the interval-valued fuzzy set A. The interval-valued fuzzy hypergraph (A, B) is also an interval-valued fuzzy hypergraph on X = supp(A), the interval-valued fuzzy set A defines a condition for interval-valued in the edge set E. This condition can be stated separately, so without loss of generality we restrict attention to interval-valued fuzzy hypergraphs on crisp vertex sets.

Example 3.1 Consider an interval-valued fuzzy hypergraph $I = (X, E)$ as shown in Fig. 3.1 such that $X = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3\}$, where

$$E_1 = \left\{\frac{a}{[0.2, 0.3]}, \frac{b}{[0.4, 0.5]}\right\},\ E_2 = \left\{\frac{b}{[0.4, 0.5]}, \frac{c}{[0.2, 0.5]}\right\},\ E_3 = \left\{\frac{a}{[0.2, 0.3]}, \frac{d}{[0.2, 0.4]}\right\}.$$

The corresponding incidence matrix is given in Table 3.1.

Definition 3.7 An interval-valued fuzzy set $A = [\mu_A^-, \mu_A^+] : X \to D[0, 1]$ is an *elementary interval-valued fuzzy set* if A is single valued on supp(A). An *elementary interval-valued fuzzy hypergraph* $I = (X, E)$ is an interval-valued fuzzy hypergraph whose edges are elementary.

We explore the sense in which an interval-valued fuzzy graph is an interval-valued fuzzy hypergraph.

Proposition 3.1 *Interval-valued fuzzy graphs and interval-valued fuzzy digraphs are special cases of the interval-valued fuzzy hypergraphs.*

An interval-valued fuzzy multigraph is a multivalued symmetric mapping $D = [\mu_D^-, \mu_D^+] : X \times X \to D[0, 1]$. An interval-valued fuzzy multigraph can be considered to be the "disjoint union" or "disjoint sum" of a collection of simple interval-valued fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore, these structures can be considered as "disjoint unions" or "disjoint sums" of interval-valued fuzzy hypergraphs.

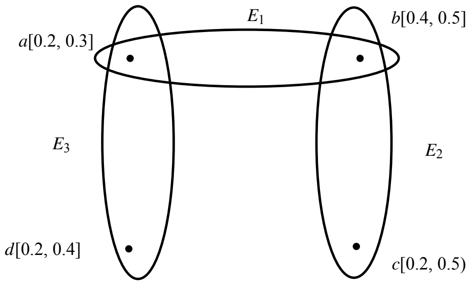


Fig. 3.1 Interval-valued fuzzy hypergraph

Table 3.1 The corresponding incidence matrix

M_I	E_1	E_2	E_3
a	[0.2, 0.3]	[0, 0]	[0.2, 0.3]
b	[0.4, 0.5]	[0.4, 0.5]	[0, 0]
c	[0, 0]	[0.2, 0.5]	[0, 0]
d	[0, 0]	[0, 0]	[0.2, 0.4]

Definition 3.8 An interval-valued fuzzy hypergraph $I = (X, E)$ is *simple* if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+] \in E$ and $\mu_A^- \leq \mu_B^-$, $\mu_A^+ \leq \mu_B^+$ imply that $\mu_A^- = \mu_B^-$, $\mu_A^+ = \mu_B^+$.

An interval-valued fuzzy hypergraph $I = (X, E)$ is *support simple* if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+] \in E$, supp$(A)$ = supp(B), and $\mu_A^- \leq \mu_B^-$, $\mu_A^+ \leq \mu_B^+$ imply that $\mu_A^- = \mu_B^-$, $\mu_A^+ = \mu_B^+$.

An interval-valued fuzzy hypergraph $I = (X, E)$ is *strongly support simple* if $A = [\mu_A^-, \mu_A^+]$, $B = [\mu_B^-, \mu_B^+] \in E$ and supp(A) = supp(B) imply that $A = B$.

Remark 3.1 The Definition 3.8 reduces to familiar definitions in the special case where I is a crisp hypergraph. The interval-valued fuzzy definition of simple is identical to the crisp definition of simple. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges. For interval-valued fuzzy hypergraphs all three concepts imply no multiple edges. Simple interval-valued fuzzy hypergraphs are support simple and strongly support simple interval-valued fuzzy hypergraphs are support simple. Simple and strongly support simple are independent concepts.

Definition 3.9 Let $I = (X, E)$ be an interval-valued fuzzy hypergraph. Suppose that $\alpha, \beta \in [0, 1]$. Let

- $E_{[\alpha,\beta]} = \{A_{[\alpha,\beta]} | \ A \in E\}$, $\quad A_{[\alpha,\beta]} = \{x \mid \mu_A^-(x) \leq \alpha$ or $\mu_A^+(x) \leq \beta\}$, and
- $X_{[\alpha,\beta]} = \bigcup_{A \in E} A_{[\alpha,\beta]}$.

If $E_{[\alpha,\beta]} \neq \emptyset$, then the crisp hypergraph $I_{[\alpha,\beta]} = (X_{[\alpha,\beta]}, E_{[\alpha,\beta]})$ is the $[\alpha, \beta]$–*level hypergraph* of I.

Clearly, it is possible that $A_{[\alpha,\beta]} = B_{[\alpha,\beta]}$ for $A \neq B$, by using distinct markers to identity the various members of E a distinction between $A_{[\alpha,\beta]}$ and $B_{[\alpha,\beta]}$ to represent multiple edges in $I_{[\alpha,\beta]}$. However, we do not take this approach unless otherwise stated, we will always regard $I_{[\alpha,\beta]}$ as having no repeated edges.

The families of crisp sets (hypergraphs) produced by the $[\alpha, \beta]$-cuts of an interval-valued fuzzy hypergraph share an important relationship with each other, as expressed below:

Suppose $\mathbb{X}$ and $\mathbb{Y}$ are two families of sets such that for each set X belonging to $\mathbb{X}$ there is at least one set Y belonging to $\mathbb{Y}$ which contains X. In this case, we say that $\mathbb{Y}$ *absorbs* $\mathbb{X}$ and symbolically write $\mathbb{X} \sqsubseteq \mathbb{Y}$ to express this relationship between $\mathbb{X}$ and $\mathbb{Y}$. Since, it is possible for $\mathbb{X} \sqsubseteq \mathbb{Y}$ while $\mathbb{X} \cap \mathbb{Y} = \emptyset$, we have that $\mathbb{X} \subseteq \mathbb{Y} \Rightarrow \mathbb{X} \sqsubseteq \mathbb{Y}$, whereas the converse is generally false. If $\mathbb{X} \sqsubseteq \mathbb{Y}$ and $\mathbb{X} \neq \mathbb{Y}$, then we write $\mathbb{X} \sqsubset \mathbb{Y}$.

Definition 3.10 Let $I = (X, E)$ be an interval-valued fuzzy hypergraph, and for $[0, 0] < [s, t] \leq h(I)$. Let $I_{[s,t]}$ be the $[s, t]$–level hypergraph of I. The sequence of real numbers $\{[s_1, r_1], [s_2, r_2], \ldots, [s_n, r_n]\}$, $\quad [0, 0] < [s_1, r_1] < [s_2, r_2] < \cdots < [s_n, r_n] = h(I)$, which satisfies the properties,

- if $[s_{i+1}, r_{i+1}] < [u, v] \leq [s_i, r_i]$, then $E_{[u,v]} = E_{[s_i,r_i]}$,
- $E_{[s_i,r_i]} \sqsubset E_{[s_{i+1},r_{i+1}]}$,

is called the *fundamental sequence* of I, and is denoted by $F(I)$ and the set of $[s_i, r_i]$-level hypergraphs $\{I_{[s_1,r_1]}, I_{[s_2,r_2]}, \ldots, I_{[s_n,r_n]}\}$ is called the *set of core hypergraphs* of I or, simply, the *core set* of I, and is denoted by $C(I)$.

Definition 3.11 Suppose $I = (X, E)$ is an interval-valued fuzzy hypergraph with $F(I) = \{[s_1, r_1], [s_2, r_2], \ldots, [s_n, r_n]\}$, and $s_{n+1} = 0$, $r_{n+1} = 0$. Then, I is called *sectionally elementary* if for each edge $A = (\mu_A^-, \mu_A^+) \in E$, each $i = \{1, 2, \ldots, n\}$, and $[s_i, r_i] \in F(I)$, $A_{[s,t]} = A_{[s_i,r_i]}$, for all $[s, t] \in ([s_{i+1}, r_{i+1}], [s_i, r_i]]$.

Clearly I is sectionally elementary if and only if $A(x) = (\mu_A^-(x), \mu_A^+(x)) \in F(I)$ for each $A \in E$ and each $x \in X$.

Definition 3.12 A sequence of crisp hypergraphs $I_i = (X_i, E_i^*), [1, 1] \leq i \leq [n, n]$, is said to be *ordered* if $I_1 \subset I_2 \subset \cdots \subset I_n$. The sequence $\{I_i \mid [1, 1] \leq i \leq [n, n]\}$ is *simply ordered* if it is ordered and if whenever $E^* \in E_{i+[1,1]}^* - E_i^*$, then $E^* \nsubseteq X_i$.

Definition 3.13 An interval-valued fuzzy hypergraph I is *ordered* if the I induced fundamental sequence of hypergraphs is ordered. The interval-valued fuzzy hypergraph I is *simply ordered* if the I induced fundamental sequence of hypergraphs is simply ordered.

Example 3.2 Consider the interval-valued fuzzy hypergraph $I = (X, E)$, where $X = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4, E_5\}$ which is represented by the following incidence matrix Table 3.2.

Clearly, $h(I) = [0.3, 0.9]$.

Now

$$E_{[0.1,0.9]} = \{\{a, b\}, \{b, c\}\},$$

$$E_{[0.2,0.7)} = \{\{a, b\}, \{b, c\}\},$$

$$E_{[0.3,0.4]} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

Thus, for $[0.3, 0.4] < [s, t] \leq [0.1, 0.9]$, $E_{[s,t]} = \{\{a, b\}, \{b, c\}\}$, and for $[0, 0] < [s, t] \leq [0.3, 0.4]$,

$$E_{[s,t]} = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

Table 3.2 Incidence matrix of I

I	E_1	E_2	E_3	E_4	E_5
a	[0.2, 0.7]	[0.0, 0.9]	[0, 0]	[0, 0]	[0.3, 0.4]
b	[0.2, 0.7]	[0.0, 0.9]	[0.0, 0.9]	[0.2, 0.7]	[0, 0]
c	[0, 0]	[0, 0]	[0.0, 0.9]	[0.2, 0.7]	[0.3, 0.4]
d	[0, 0]	[0.3, 0.4]	[0, 0]	[0.3, 0.4]	[0.3, 0.4]

We note that $E_{[0.1,0.9]} \subseteq E_{[0.3,0.4]}$. The fundamental sequence is $F(I) = \{[s_1, r_1] = [0.1, 0.9],\ [s_2, r_2] = [0.3, 0.4]\}$ and the set of core hypergraph is $C(I) = \{I_1 = (X_1, E_1) = I_{[0.1,0.9]},\ I_2 = (X_2, E_2) = I_{[0.3,0.4]}\}$, where

$$X_1 = \{a, b, c\},\ E_1 = \{\{a, b\}, \{b, c\}, \}$$

$$X_2 = \{a, b, c, d\},\ E_2 = \{\{a, b\}, \{a, b, d\}, \{b, c\}, \{b, c, d\}, \{a, c, d\}\}.$$

I is support simple, but not simple. I is not sectionally elementary since $E_{1[s,t]} \neq E_{1[0.1,0.9]}$ for $s = 0.2, t = 0.7$. Clearly, interval-valued fuzzy hypergraph I is simply ordered.

Proposition 3.2 *Let $I = (X, E)$ be an elementary interval-valued fuzzy hypergraph. Then, I is support simple if and only if I is strongly support simple.*

Proof Suppose that I is elementary, support simple and that supp(A) = supp(B). We assume without loss of generality that $h(A) \leq h(B)$. Since, I is elementary, it follows that $\mu_A^- \leq \mu_B^-$, $\mu_A^+ \leq \mu_B^+$ and since I is support simple then $\mu_A^- = \mu_B^-$, $\mu_A^+ = \mu_B^+$. Therefore, I is strongly support simple. The proof of converse part is obvious.

The complexity of an interval-valued fuzzy hypergraph depends in part on how many edges it has. The natural question arises: is there an upper bound on the number of edges of an interval-valued fuzzy hypergraph of order n?

Proposition 3.3 *Let $I = (X, E)$ be a simple interval-valued fuzzy hypergraph of order n. Then, there is no upper bound on $|E|$.*

Proof Let $X = \{x, y\}$, and define $E_N = \{A_i = [\mu_{A_i}^-, \mu_{A_i}^+] \mid i = 1, 2, \ldots, N\}$, where

$$\mu_{A_i}^-(x) = \frac{1}{i+1},\ \ \mu_{A_i}^+(x) = 1 - \frac{1}{i+1},$$

$$\mu_{A_i}^-(y) = \frac{1}{i+1},\ \ \mu_{A_i}^+(y) = \frac{i}{i+1}.$$

Then $I_N = (X, E_N)$ is a simple interval-valued fuzzy hypergraph with N edges. This ends the proof.

Proposition 3.4 *Let $I = (X, E)$ be a support simple interval-valued fuzzy hypergraph of order n. Then, there is no upper bound on $|E|$.*

Proposition 3.5 *Let $I = (X, E)$ be a strongly support simple interval-valued fuzzy hypergraph of order n. Then, there is no upper bound on $|E| \leq 2^n - 1$ if and only if $\{supp(A) \mid A \in E\} = P(X) - \emptyset$.*

Proposition 3.6 *Let $I = (X, E)$ be an elementary simple interval-valued fuzzy hypergraph of order n. Then, there is no upper bound on $|E| \leq 2^n - 1$ if and only if $\{supp(A) \mid A \in E\} = P(X) - \emptyset$.*

Proof Since I is elementary and simple, each nontrivial $W \subseteq X$ can be the support of at most one $A = (\mu_A^-, \mu_A^+) \in E$. Therefore, $|E| \leq 2^n - 1$. To show there exists an elementary, simple I with $|E| = 2^n - 1$, let $E = \{A = (\mu_A^-, \mu_A^+) \mid W \subseteq X\}$ be the set of functions defined by

$$\mu_A^-(x) = \frac{1}{|W|}, \text{ if } x \in W, \quad \mu_A^-(x) = 0, \text{ if } x \notin W,$$

$$\mu_A^+(x) = 1 - \frac{1}{|W|}, \text{ if } x \in W, \quad \mu_A^+(x) = 1, \text{if } x \notin W.$$

Then, each one element has height [0, 1], each two elements have height [0.5, 0.5] and so on. Hence, I is an elementary and simple, and $|E| = 2^n - 1$.

Proposition 3.7 (a) *If $I = (X, E)$ is an elementary interval-valued fuzzy hypergraph, then I is ordered.*
(b) *If I is an ordered interval-valued fuzzy hypergraph with simple support hypergraph, then I is elementary.*

Consider the situation where the vertex of a crisp hypergraph is fuzzified. Suppose that each edge is given a uniform degree of membership consistent with the weakest vertex of the edge. Some constructions describe the following subclass of interval-valued fuzzy hypergraphs.

Definition 3.14 An interval-valued fuzzy hypergraph $I = (X, E)$ is called a $A = [\mu_A^-, \mu_A^+]$-*tempered* interval-valued fuzzy hypergraph of $I = (X, E)$ if there is a crisp hypergraph $I^* = (X, E^*)$ and an interval-valued fuzzy set $A = [\mu_A^-, \mu_A^+] : X \to D(0, 1]$ such that $E = \{B_F = [\mu_{B_F}^-, \mu_{B_F}^+] \mid F \in E^*\}$, where

$$\mu_{B_F}^-(x) = \begin{cases} \min(\mu_A^-(y) \mid y \in F), & \text{if } x \in F, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mu_{B_F}^+(x) = \begin{cases} \min(\mu_A^+(y) \mid y \in F), & \text{if } x \in F, \\ 1, & \text{otherwise.} \end{cases}$$

Let $A \otimes I$ denote the A-tempered interval-valued fuzzy hypergraph of I determined by the crisp hypergraph $I^* = (X, E^*)$ and the interval-valued fuzzy set $A : X \to D(0, 1]$.

Example 3.3 Consider the interval-valued fuzzy hypergraph $I = (X, E)$, where $X = \{a, b, c, d\}$ and $E = \{E_1, E_2, E_3, E_4\}$ which is represented by the following incidence matrix given in Table 3.3.

Table 3.3 Incidence matrix of I

I	E_1	E_2	E_3	E_4
a	[0.2, 0.7]	[0.0, 0.0]	[0, 0]	[0.2, 0.7]
b	[0.2, 0.7]	[0.3, 0.4]	[0.0, 0.9]	[0.0, 0.0]
c	[0, 0]	[0, 0]	[0.0, 0.9]	[0.2, 0.7]
d	[0, 0]	[0.3, 0.4]	[0, 0]	[0.0, 0.0]

Then, $E_{[0.0,0.9]} = \{\{b, c\}\}$, $E_{[0.2,0.7)} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$, and $E_{[0.3,0.4]} = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}\}$. Define $A = [\mu_A^-, \mu_A^+] : X \to D(0, 1]$ by

$$\mu_A^-(a) = 0.2,\ \mu_A^-(b) = \mu_A^-(c) = 0.0,\ \mu_A^-(d) = 0.3,$$

$$\mu_A^+(a) = 0.7,\ \mu_A^+(b) = \mu_A^+(c) = 0.9,\ \mu_A^+(d) = 0.4.$$

Note that

$$\mu^-_{B_{\{a,b\}}}(a) = \min(\mu_A^-(a), \mu_A^-(b)) = 0.0,\ \mu^-_{B_{\{a,b\}}}(b) = \min(\mu_A^-(a), \mu_A^-(b)) = 0.0,$$

$$\mu^-_{B_{\{a,b\}}}(c) = 0.0,\ \mu^-_{B_{\{a,b\}}}(d) = 0.0,$$

$$\mu^+_{B_{\{a,b\}}}(a) = \min(\mu_A^+(a), \mu_A^+(b)) = 0.7,\ \mu^+_{B_{\{a,b\}}}(b) = \min(\mu_A^+(a), \mu_A^+(b)) = 0.7$$

$$\mu^+_{B_{\{a,b\}}}(c) = 1.0,\ \mu^+_{B_{\{a,b\}}}(d) = 1.0.$$

Thus,

$$E_1 = [\mu^-_{B_{\{a,b\}}}, \mu^+_{B_{\{a,b\}}}],\quad E_2 = [\mu^-_{B_{\{b,d\}}}, \mu^+_{B_{\{b,d\}}}],$$

$$E_3 = [\mu^-_{B_{\{b,c\}}}, \mu^+_{B_{\{b,c\}}}],\quad E_4 = [\mu^-_{B_{\{a,c\}}}, \mu^+_{B_{\{a,c\}}}].$$

Hence, I is A-tempered hypergraph.

Proposition 3.8 *An interval-valued fuzzy hypergraph I is an A-tempered interval-valued fuzzy hypergraph of some crisp hypergraph I^* if and only if I is elementary, support simple, and simply ordered.*

Proof Suppose that $I = (X, E)$ is an A-tempered interval-valued fuzzy hypergraph of some crisp hypergraph I^*. Clearly, I is elementary and support simple. We show that I is simply ordered. Let

$$C(I) = \{(I_1^*)^{r_1} = (X_1, E_1^*),\ (I_2^*)^{r_2} = (X_2, E_2^*), \ldots,\ (I_n^*)^{r_n} = (X_n, E_n^*)\}.$$

Since I is elementary, it follows from Proposition 3.7 that I is ordered. To show that I is simply ordered, suppose that there exists $F \in E^*_{i+1} \backslash E^*_i$. Then, there exists $x^* \in F$ such that $\mu^-_A(x^*) = r_{i+1}$, $\mu^+_A(x^*) = \acute{r}_{i+1}$. Since $\mu^-_A(x^*) = r_{i+1} < r_i$ and $\mu^+_A(x^*) = \acute{r}_{i+1} < \acute{r}_i$, it follows that $x^* \notin X_i$ and $F \nsubseteq X_i$, hence I is simply ordered.

Conversely, suppose $I = (X, E)$ is elementary, support simple and simply ordered. Let

$$C(I) = \{(I^*_1)^{r_1} = (X_1, E^*_1),\ (I^*_2)^{r_2} = (X_2, E^*_2), \ldots,\ (I^*_n)^{r_n} = (X_n, E^*_n)\},$$

where $D(I) = \{r_1, r_2, \ldots, r_n\}$ with $0 < r_n < \cdots < r_1$. Since $(I^*)^{r_n} = I^*_n = (X_n, E^*_n)$ and define $A = [\mu^-_A, \mu^+_A] : X_n \rightarrow D(0, 1]$ by

$$\mu^-_A(x) = \begin{cases} r_1, & if\ x \in X_1, \\ r_i, & if\ x \in X_i \backslash X_{i-1}, i = 1, 2, \ldots, n \end{cases} \qquad \mu^+_A(x) = \begin{cases} s_1, & if\ x \in X_1, \\ s_i, & if\ x \in X_i \backslash X_{i-1}, i = 1, 2, \ldots, n \end{cases}$$

We show that $E = \{B_F = [\mu^-_{B_F}, \mu^+_{B_F}] \mid F \in E^*\}$, where

$$\mu^-_{B_F}(x) = \begin{cases} \min(\mu^-_A(y) \mid y \in F), & \text{if } x \in F, \\ 0, & \text{otherwise}, \end{cases} \qquad \mu^+_{B_F}(x) = \begin{cases} \min(\mu^+_A(y) \mid y \in F), & \text{if } x \in F, \\ 1, & \text{otherwise}. \end{cases}$$

Let $F \in E^*_n$. Since I is elementary and support simple, there is a unique interval-valued fuzzy edge $C_F = [\mu^-_{C_F}, \mu^+_{C_F}]$ in E having support E^*. Indeed, distinct edges in E must have distinct supports that lie in E^*_n. Thus, to show that $E = \{B_F = [\mu^-_{B_F}, \mu^+_{B_F}] \mid F \in E^*_n\}$, it suffices to show that for each $F \in E^*_n$, $\mu^-_{C_F} = \mu^-_{B_F}$ and $\mu^+_{C_F} = \mu^+_{B_F}$. As all edges are elementary and different edges have different supports, it follows from the definition of fundamental sequence that $h(C_F)$ is equal to some number r_i of $D(I)$. Consequently, $E^* \subseteq X_i$. Moreover, if $i > 1$, then $F \in E^* \backslash E^*_{i-1}$. Since $F \subseteq X_i$, it follows from the definition of $A = [\mu^-_A, \mu^+_A]$ that for each $x \in F$, $\mu^-_A(x) \geq r_i$ and $\mu^+_A(x) \geq s_i$. We claim that $\mu^-_A(x) = r_i$ and $\mu^+_A(x) = s_i$, for some $x \in F$. If not, then by definition of $A = [\mu^-_A, \mu^+_A]$, $\mu^-_A(x) \geq r_i$ and $\mu^+_A(x) \geq s_i$ for all $x \in F$ which implies that $F \subseteq X_{i-1}$ and so $F \in E^* \backslash E^*_{i-1}$ and since I is simply ordered $F \subsetneq X_{i-1}$, a contradiction. Thus, it follows from the definition of B_F that $B_F = C_F$. This completes the proof.

As a consequence of the above theorem we obtain.

Proposition 3.9 *Suppose that I is a simply ordered interval-valued fuzzy hypergraph and $F(I) = \{r_1, r_2, \ldots, r_n\}$. If I^{r_n} is a simple hypergraph, then there is a partial interval-valued fuzzy hypergraph $\acute{I}$ of I such that the following assertions hold:*

1. *$\acute{I}$ is an A-tempered interval-valued fuzzy hypergraph of I_n.*
2. *$E \sqsubseteq \acute{E}$.*
3. *$F(\acute{I}) = F(I)$ and $C(\acute{I}) = C(I)$.*

3.3 Vague Hypergraphs

Different authors from time to time have made a number of generalizations of Zadeh's [25] fuzzy set theory. The notion of vague set was introduced by Gau and Buehrer [11]. This is because in most cases of judgments, the evaluation is done by human beings and so the certainty is a limitation of knowledge or intellectual functionaries. Naturally, every decision-maker hesitates more or less on every evaluation activity. For example, in order to judge whether a patient has cancer or not, a medical doctor (the decision-maker) will hesitate because of the fact that a fraction of evaluation he thinks in favor of the truthness, another fraction in favor of the falseness and the rest part remains undecided to him. This is the breaking philosophy in the notion of vague set theory introduced by Gau and Buehrer [11].

Definition 3.15 A *vague set* A in the universe of discourse X is a pair (t_A, f_A), where $t_A : X \to [0, 1]$, $f_A : X \to [0, 1]$ are true and false memberships, respectively such that $t_A(x) + f_A(x) \leq 1$, for all $x \in X$.

In the above definition, $t_A(x)$ is considered as the lower bound for degree of membership of x in A (based on evidence), and $f_A(x)$ is the lower bound for negation of membership of x in A (based on evidence against). Therefore, the degree of membership of x in the vague set A is characterized by the interval $[t_A(x), 1 - f_A(x)]$. So, a vague set is a special case of interval-valued sets. The interval $[t_A(x), 1 - f_A(x)]$ is called the *vague value* of x in A, and is denoted by $X_A(x)$. We denote zero vague and unit vague value by $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$, respectively. It is worth to mention here that interval-valued fuzzy sets are not vague sets. In interval-valued fuzzy sets, an interval-valued membership value is assigned to each element of the universe considering the "evidence for x" only, without considering "evidence against x". In vague sets both are independently proposed by the decision-maker. This makes a major difference in the judgment about the grade of membership.

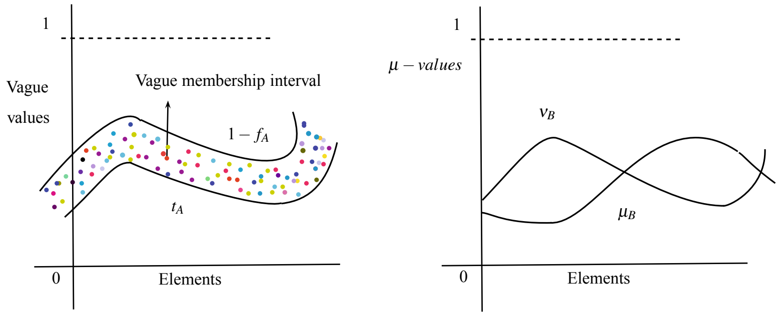
Remark 3.2 The intuitionistic fuzzy sets and vague sets look similar, analytically vague sets are more appropriate when representing vague data. The difference between them is discussed below. The membership interval of element x for vague set A is $[t_A(x), 1 - f_A(x)]$. But, the membership value for element x in an intuitionistic fuzzy set B is $< x, \mu_B(x), \nu_B(x) >$. Here the semantics of t_A is the same as with A and μ_B is the same as with B. However, the boundary is able to indicate the possible existence of a data value. This difference gives rise to a simpler but meaningful graphical view of data sets (see Fig. 3.2).

A vague relation is a generalization of a fuzzy relation.

Definition 3.16 Let X and Y be ordinary finite non-empty sets. We call a *vague relation* to be a vague subset of $X \times Y$, that is, an expression R defined by

$$R = \{< (x, y), t_R(x, y), f_R(x, y) > | x \in X, y \in Y\},$$

where $t_R : X \times Y \to [0, 1]$, $f_R : X \times Y \to [0, 1]$, which satisfies the condition $0 \leq t_R(x, y) + f_R(x, y) \leq 1$, for all $(x, y) \in X \times Y$. A vague relation R on X is called



(a) Vague sets (b) Intuitionistic fuzzy sets

Fig. 3.2 Comparison between vague sets and intuitionistic fuzzy sets

reflexive if $t_R(x, x) = 1$ and $f_R(x, x) = 0$, for all $x \in X$. A vague relation R on X is *symmetric* if $t_R(x, y) = t_R(y, x)$ and $f_R(x, y) = f_R(y, x)$, for all $x, y \in X$.

Definition 3.17 Let $A = (t_A, f_A)$ be a vague set on X and let $\alpha, \beta \in [0, 1]$ be such that $\alpha \leq \beta$. Then, the set $A_{(\alpha,\beta)} = \{x \mid t_A(x) \geq \alpha, \ 1 - f_A(x) \geq \beta\}$ is called a (α, β)-*(weakly) cut set* of A. $A_{(\alpha,\beta)}$ is a crisp set.

Definition 3.18 The *support* of A is defined by $\text{supp}(A) = \{x \in X \mid (t_A(x), f_A(x)) \neq (0, 0)\}$ and we say A is nontrivial if supp(A) is non-empty. The *height* of a vague set A is defined as $h(A) = \sup_{x \in X}(A)(x)$.

Definition 3.19 A vague relation B on a set X is a vague relation from X to X. If A is a vague set on a set X, then a vague relation B on A is a vague relation which satisfies, $t_B(xy) \leq \min(t_A(x), t_A(y))$ and $f_B(xy) \geq \max(f_A(x), f_A(y))$, for all $xy \in E \subseteq X \times X$.

Definition 3.20 Let X be a non-empty set, members of X are called *nodes*. A *vague graph* $G = (A, B)$ with X as the set of nodes, is a pair of functions A, B, where A is a vague set of X and B is a vague relation on X. We note that vague relation B in vague digraph need not to be symmetric.

We now define vague hypergraph,

Definition 3.21 Let X be a finite set and let $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_m\}$ be a finite family of nontrivial vague subsets of X such that $X = \bigcup_j \text{supp}\mathbb{E}_j, \ \ j = 1, 2, \ldots, m$. Then, the pair $\mathbb{H} = (X, \mathbb{E})$ is a *vague hypergraph* on X, $\mathbb{E}$ is the family of vague edges of $\mathbb{H}$ and X is the (crisp) vertex set of $\mathbb{H}$.

Definition 3.22 Let $A = (t_A, f_A)$ be a vague subset of X and let $\mathbb{E}$ be a collection of vague subsets of X such that for each $B = (t_B, f_B) \in \mathbb{E}$ and $x \in X$, $t_A(x) \leq t_B(x)$, $f_B(x) \geq f_A(x)$. Then, the pair (A, B) is a vague hypergraph on the vague set A. The vague hypergraph (A, B) is also a vague hypergraph on $X = \text{supp}(A)$, the vague set A

defines a condition for interval-valued in the edge set $\mathbb{E}$. This condition can be stated separately, so without loss of generality we restrict attention to vague hypergraphs on crisp vertex sets.

Definition 3.23 A vague set A is an elementary vague set if A is single valued on supp(A). An *elementary vague hypergraph* $\mathbb{H} = (X, \mathbb{E})$ is a vague hypergraph whose edges are elementary.

Definition 3.24 A vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ is *simple* if $A = (t_A, f_A)$, $B = (t_B, f_B) \in E$ and $t_A \leq t_B$, $f_A \geq f_B$ imply that $t_A = t_B$, $f_A = f_B$.

A vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ is *support simple* if $A = (t_A, f_A), B = (t_B, f_B) \in E$, supp($A$) = supp($B$), and $t_A \leq t_B$, $f_A \geq f_B$ imply that $t_A = t_B$, $f_A = f_B$.

A vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ is *strongly support simple* if $A = (t_A, f_A)$, $B = (t_B, f_B) \in E$, and supp(A) = supp(B) imply that $A = B$.

Example 3.4 Consider a vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ as shown in Fig. 3.3 such that $X = \{a, b, c, d\}$ and $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\}$, where

$$\mathbb{E}_1 = \left\{\frac{a}{(0.2, 0.3)}, \frac{b}{(0.4, 0.5)}\right\},\ \mathbb{E}_2 = \left\{\frac{b}{(0.4, 0.5)}, \frac{c}{(0.2, 0.5)}\right\},\ \mathbb{E}_3 = \left\{\frac{a}{(0.2, 0.3)}, \frac{d}{(0.2, 0.4)}\right\}.$$

The corresponding incidence matrix is given below in Table 3.4.

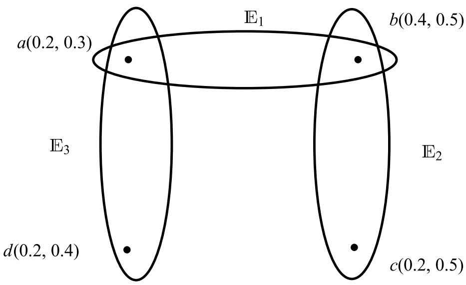


Fig. 3.3 Vague hypergraph

Table 3.4 The incidence matrix of vague hypergraph

$M_{\mathbb{H}}$	$\mathbb{E}_1$	$\mathbb{E}_2$	$\mathbb{E}_3$
a	(0.2, 0.3)	(0, 0)	(0.2, 0.3)
b	(0.4, 0.5)	(0.4, 0.5)	(0, 0)
c	(0, 0)	(0.2, 0.5)	(0,0)
d	(0,0)	(0,0)	(0.2, 0.4)

Definition 3.25 Let $\mathbb{H} = (X, \mathbb{E})$ be a vague hypergraph. Suppose that $\alpha, \beta \in [0, 1]$. Let

- $\mathbb{E}_{(\alpha,\beta)} = \{A_{(\alpha,\beta)} | A \in E\}, \quad A_{(\alpha,\beta)} = \{x \mid t_A(x) \geq \alpha \ , \ 1 - f_A(x) \geq \beta\}$, and
- $X_{(\alpha,\beta)} = \bigcup_{A\in\mathbb{E}} A_{(\alpha,\beta)}$.

If $\mathbb{E}_{(\alpha,\beta)} \neq \emptyset$, then the crisp hypergraph $\mathbb{H}_{(\alpha,\beta)} = (X_{(\alpha,\beta)}, \mathbb{E}_{(\alpha,\beta)})$ is the (α, β)–*level hypergraph* of $\mathbb{H}$.

Clearly, it is possible that $A_{(\alpha,\beta)} = B_{(\alpha,\beta)}$ for $A \neq B$, by using distinct markers to identity the various members of $\mathbb{E}$ a distinction between $A_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)}$ to represent multiple edges in $\mathbb{H}_{(\alpha,\beta)}$. However, we do not take this approach unless otherwise stated, we will always regard $\mathbb{H}_{(\alpha,\beta)}$ as having no repeated edges.

The families of crisp sets (hypergraphs) produced by the (α, β)-cuts of a vague hypergraph share an important relationship with each other, as expressed below:

Suppose $\mathbb{X}$ and $\mathbb{Y}$ are two families of sets such that for each set X belonging to $\mathbb{X}$ there is at least one set Y belonging to $\mathbb{Y}$ which contains X. In this case we say that $\mathbb{Y}$ *absorbs* $\mathbb{X}$ and symbolically write $\mathbb{X} \sqsubseteq \mathbb{Y}$ to express this relationship between $\mathbb{X}$ and $\mathbb{Y}$. Since it is possible for $\mathbb{X} \sqsubseteq \mathbb{Y}$ while $\mathbb{X} \cap \mathbb{Y} = \emptyset$, we have that $\mathbb{X} \subseteq \mathbb{Y} \Rightarrow \mathbb{X} \sqsubseteq \mathbb{Y}$, whereas the converse is generally false. If $\mathbb{X} \sqsubseteq \mathbb{Y}$ and $\mathbb{X} \neq \mathbb{Y}$, then we write $\mathbb{X} \sqsubset \mathbb{Y}$.

Definition 3.26 Let $\mathbb{H} = (X, \mathbb{E})$ be a vague hypergraph, and for $(0, 0) < (s, t) \leq h(\mathbb{H})$. Let $\mathbb{H}_{(s,t)}$ be the (s, t)–level hypergraph of $\mathbb{H}$. The sequence of real numbers,

$$\{(s_1, r_1), (s_2, r_2), \ldots, (s_n, r_n)\}, \quad 0 < s_1 < s_2 < \cdots < s_n \text{ and } 0 > r_1 > r_2 > \cdots > r_n, \\ \text{where } (s_n, r_n) = h(\mathbb{H}),$$

which satisfies the properties

- if $s_i < u \leq s_{i+1}$ and $r_i > v \geq r_{i+1}$, then $\mathbb{E}_{(u,v)} = \mathbb{E}_{(s_{i+1},r_{i+1})}$, and
- $\mathbb{E}_{(s_i,r_i)} \sqsubseteq \mathbb{E}_{(s_{i+1},r_{i+1})}$,

is called the *fundamental sequence* of $\mathbb{H}$, and is denoted by $F(\mathbb{H})$ and the set of (s_i, r_i)-level hypergraphs $\{\mathbb{H}_{((s_1,r_1)}, \mathbb{H}_{(s_2,r_2)}, \ldots, \mathbb{H}_{(s_n,r_n)}\}$ is called the *set of core hypergraphs* of $\mathbb{H}$ or, simply, the *core set* of $\mathbb{H}$, and is denoted by $C(\mathbb{H})$.

Definition 3.27 Suppose $\mathbb{H} = (X, \mathbb{E})$ is vague hypergraph with $F(\mathbb{H}) = \{(s_1, r_1), (s_2, r_2), \ldots, (s_n, r_n)\}$, and $s_{n+1} = 0$, $r_{n+1} = 0$, Then, $\mathbb{H}$ is called *sectionally elementary* if for each edge $A = (t_A, f_A) \in \mathbb{E}$, each $i = \{1, 2, \ldots, n\}$, and $(s_i, r_i) \in F(\mathbb{H})$, $A_{(s,t)} = A_{(s_i,r_i)}$ for all $(s, t) \in ((s_{i+1}, r_{i+1}), (s_i, r_i)]$.

Clearly, $\mathbb{H}$ is sectionally elementary if and only if $A(x) = (t_A(x), f_A(x)) \in F(\mathbb{H})$ for each $A \in \mathbb{E}$ and each $x \in X$.

Definition 3.28 A sequence of crisp hypergraphs $\mathbb{H}_i = (X_i, E_i^*)$, $1 \leq i \leq n$, is said to be *ordered* if $\mathbb{H}_1 \subset \mathbb{H}_2 \subset \ldots \subset \mathbb{H}_n$. The sequence $\{\mathbb{H}_i \mid 1 \leq i \leq n\}$ is *simply ordered* if it is ordered and if whenever $E^* \in E_{i+1}^* - E_i^*$, then $E^* \nsubseteq X_i$.

Definition 3.29 A vague hypergraph $\mathbb{H}$ is *ordered* if the $\mathbb{H}$ induced fundamental sequence of hypergraphs is ordered. The vague hypergraph $\mathbb{H}$ is *simply ordered* if the $\mathbb{H}$ induced fundamental sequence of hypergraphs is simply ordered.

We state the following Propositions without their proofs.

Proposition 3.10 *Let* $\mathbb{H} = (X, \mathbb{E})$ *be an elementary vague hypergraph. Then,* $\mathbb{H}$ *is support simple if and only if* $\mathbb{H}$ *is strongly support simple.*

Proposition 3.11 *Let* $\mathbb{H} = (X, \mathbb{E})$ *be a simple vague hypergraph of order n. Then, there is no upper bound on* $|\mathbb{E}|$.

Proof Let $X = \{x, y\}$, and define $\mathbb{E}_N = \{A_i = (t_{A_i}, f_{A_i}) \mid i = 1, 2, \ldots, N\}$, where

$$t_{A_i}(x) = \frac{1}{i+1}, \quad f_{A_i}(x) = 1 - \frac{1}{i+1},$$

$$t_{A_i}(y) = \frac{1}{i+1}, \quad f_{A_i}(y) = \frac{i}{i+1}.$$

Then $\mathbb{H}_N = (X, \mathbb{E}_N)$ is a simple vague hypergraph with N edges. This ends the proof.

Proposition 3.12 *Let* $\mathbb{H} = (X, \mathbb{E})$ *be a support simple vague hypergraph of order n. Then, there is no upper bound on* $|\mathbb{E}|$.

Proposition 3.13 *Let* $\mathbb{H} = (X, \mathbb{E})$ *be a strongly support simple vague hypergraph of order n. Then, there is no upper bound on* $|\mathbb{E}| \leq 2^n - 1$ *if and only if* $\{supp(A) \mid A \in \mathbb{E}\} = P(X) - \emptyset$.

Proposition 3.14 *Let* $\mathbb{H} = (X, \mathbb{E})$ *be an elementary simple vague hypergraph of order n. Then, there is no upper bound on* $|\mathbb{E}| \leq 2^n - 1$ *if and only if* $\{supp(A) \mid A \in \mathbb{E}\} = P(X) - \emptyset$.

Proof Since $\mathbb{H}$ is elementary and simple, each nontrivial $W \subseteq X$ can be the support of at most one $A = (t_A, f_A) \in \mathbb{E}$. Therefore, $|\mathbb{E}| \leq 2^n - 1$. To show there exists an elementary, simple $\mathbb{H}$ with $|\mathbb{E}| = 2^n - 1$, let $\mathbb{E} = \{A = (t_A, f_A) \mid W \subseteq X\}$ be the set of functions defined by

$$t_A(x) = \frac{1}{|W|}, \text{ if } x \in W, \qquad t_A(x) = 0, \text{ if } x \notin W,$$

$$f_A(x) = 1 - \frac{1}{|W|}, \text{ if } x \in W, \qquad f_A(x) = 1, \text{if } x \notin W.$$

Then, each one element has height $(1, 0)$, each two elements have height $(0.5, 0.5)$ and so on. Hence, $\mathbb{H}$ is an elementary and simple, and $|\mathbb{E}| = 2^n - 1$.

Proposition 3.15 (*a*) *If* $\mathbb{H} = (X, \mathbb{E})$ *is an elementary vague hypergraph, then* $\mathbb{H}$ *is ordered.*
(*b*) *If* $\mathbb{H}$ *is an ordered vague hypergraph with simple support hypergraph, then* $\mathbb{H}$ *is elementary.*

Definition 3.30 The *dual of a vague hypergraph* $\mathbb{H} = (X, \mathbb{E})$ is a vague hypergraph $\mathbb{H}^D = (\mathbb{E}^D, X^D)$ whose vertex set is the edge set of $\mathbb{H}$ and with edges $X^D : \mathbb{E}^D \to [0, 1] \times [0, 1]$ by $X^D(A^D) = (t_A^D(x), f_A^D(x))$. $\mathbb{H}^D$ is a vague hypergraph whose incidence matrix is the transpose of the incidence matrix of $\mathbb{H}$, thus $\mathbb{H}^{DD} = \mathbb{H}$.

Example 3.5 Consider a vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ as shown in Fig. 3.4 such that $X = \{x_1, x_2, x_3, x_4\}$, $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4\}$, where $\mathbb{E}_1 = \left\{\frac{x_1}{(0.5,0.3)}, \frac{x_2}{(0.4,0.2)}\right\}$, $\mathbb{E}_2 = \left\{\frac{x_2}{(0.4,0.2)}, \frac{x_3}{(0.3,0.6)}\right\}$, $\mathbb{E}_3 = \left\{\frac{x_3}{(0.3,0.6)}, \frac{x_4}{(0.5,0.1)}\right\}$, $\mathbb{E}_4 = \left\{\frac{x_4}{(0.5,0.1)}, \frac{x_1}{(0.5,0.3)}\right\}$.

The corresponding incidence matrix of $\mathbb{H}$ is given in Table 3.5.

Consider the dual vague hypergraph $\mathbb{H}^D = (\mathbb{E}^D, X^D)$ of $\mathbb{H}$ such that $\mathbb{E}^D = \{e_1, e_2, e_3, e_4\}$, $X^D = \{A, B, C, D\}$, where $A = \{\frac{e_1}{(0.5,0.3)}, \frac{e_4}{(0.5,0.3)}\}$, $B = \{\frac{e_1}{(0.4,0.2)}, \frac{e_2}{(0.4,0.2)}\}$, $C = \{\frac{e_2}{(0.3,0.6)}, \frac{e_3}{(0.3,0.6)}\}$, $D = \{\frac{e_3}{(0.5,0.1)}, \frac{e_4}{(0.5,0.1)}\}$. The dual vague hypergraph $\mathbb{H}^D = (\mathbb{E}^D, X^D)$ of $\mathbb{H}$ is shown in Fig. 3.5.

The corresponding incidence matrix of $\mathbb{H}^D$ is given in Table 3.6.

Definition 3.31 A vague hypergraph $\mathbb{H} = (X, \mathbb{E})$ is called $A = (t_A, f_A)$-*tempered vague hypergraph* of $\mathbb{H} = (X, \mathbb{E})$ if there is a crisp hypergraph $H^* = (X, E^*)$ and a vague set $A = (t_A, f_A) : X \to (0, 1]$ such that $\mathbb{E} = \{B_F = (t_{B_F}, f_{B_F}) \mid F \in E^*\}$, where

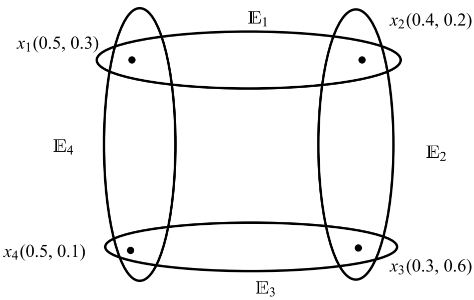


Fig. 3.4 Vague hypergraph

Table 3.5 The corresponding incidence matrix of $\mathbb{H}$

$M_{\mathbb{H}}$	$\mathbb{E}_1$	$\mathbb{E}_2$	$\mathbb{E}_3$	$\mathbb{E}_4$
x_1	(0.5, 0.3)	(0, 0)	(0, 0)	(0.5, 0.3)
x_2	(0.4, 0.2)	(0.4, 0.2)	(0, 0)	(0, 0)
x_3	(0, 0)	(0.3, 0.6)	(0.3, 0.6)	(0, 0)
x_4	(0, 0)	(0, 0)	(0.5, 0.1)	(0.5, 0.1)

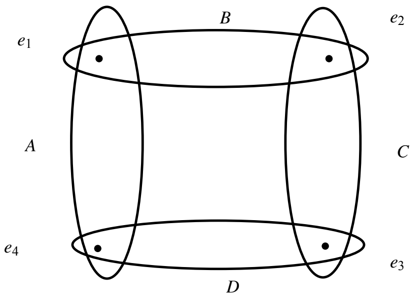


Fig. 3.5 Dual vague hypergraph

Table 3.6 The incidence matrix of $\mathbb{H}^D$

$M_{\mathbb{H}^D}$	A	B	C	D
e_1	(0.5, 0.3)	(0.4, 0.2)	(0, 0)	(0, 0)
e_2	(0, 0)	(0.4, 0.2)	(0.3, 0.6)	(0, 0)
e_3	(0, 0)	(0, 0)	(0.3, 0.6)	(0.5, 0.1)
e_4	(0.5, 0.3)	(0, 0)	(0, 0)	(0.5, 0.1)

$$t_{B_F}(x) = \begin{cases} \min(t_A(y) \mid y \in F), & \text{if } x \in F, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{B_F}(x) = \begin{cases} \max(f_A(y) \mid y \in F), & \text{if } x \in F, \\ 1, & \text{otherwise.} \end{cases}$$

Let $A \otimes \mathbb{H}$ denote the A-tempered vague hypergraph of $\mathbb{H}$ determined by the crisp hypergraph $H^* = (X, E^*)$ and the vague set A.

Example 3.6 Consider the vague hypergraph $\mathbb{H} = (X, \mathbb{E})$, where $X = \{a, b, c, d\}$ and $\mathbb{E} = \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \mathbb{E}_4\}$ which is represented by the following incidence matrix Table 3.7.

We define a vague set $A = (t_A, f_A)$ by

$$t_A(a) = 0.2,\ t_A(b) = t_A(c) = 0.0,\ t_A(d) = 0.3,\ f_A(a) = 0.7,\ f_A(b) = f_A(c) = 0.9,\ f_A(d) = 0.4.$$

Note that

$$t_{B_{\{a,b\}}}(a) = \min(t_A(a), t_A(b)) = 0.0,\ t_{B_{\{a,b\}}}(b) = \min(t_A(a), t_A(b)) = 0.0, t_{B_{\{a,b\}}}(c) = 0.0,\ t_{B_{\{a,b\}}}(d) = 0.0,$$

$$f_{B_{\{a,b\}}}(a) = \max(f_A(a), f_A(b)) = 0.9,\ f_{B_{\{a,b\}}}(b) = \max(f_A(a), f_A(b)) = 0.9, f_{B_{\{a,b\}}}(c) = 1,\ f_{B_{\{a,b\}}}(d) = 1.$$

Thus,

$$\mathbb{E}_1 = (t_{B_{\{a,b\}}}, f_{B_{\{a,b\}}}),\ \ \mathbb{E}_2 = (t_{B_{\{b,d\}}}, f_{B_{\{b,d\}}}),\ \mathbb{E}_3 = (t_{B_{\{b,c\}}}, f_{B_{\{b,c\}}}),\ \ \mathbb{E}_4 = (t_{B_{\{a,c\}}}, f_{B_{\{a,c\}}}).$$

Hence, $\mathbb{H}$ is A-tempered hypergraph.

Theorem 3.1 *A vague hypergraph* $\mathbb{H}$ *is an* A*-tempered vague hypergraph of some crisp hypergraph* H^* *if and only if* $\mathbb{H}$ *is elementary, support simple, and simply ordered.*

Table 3.7 Incidence matrix of $\mathbb{H}$

$\mathbb{H}$	$\mathbb{E}_1$	$\mathbb{E}_2$	$\mathbb{E}_3$	$\mathbb{E}_4$
a	(0.2, 0.7)	(0, 0)	(0, 0)	(0.2, 0.7)
b	(0.2, 0.7)	(0.3, 0.4)	(0.0, 0.9)	(0, 0)
c	(0, 0)	(0, 0)	(0, 0.9)	(0.2, 0.7)
d	(0, 0)	(0.3, 0.4)	(0, 0)	(0, 0)

Proof Suppose that $\mathbb{H} = (X, \mathbb{E})$ is an A-tempered vague hypergraph of some crisp hypergraph H^*. Clearly, $\mathbb{H}$ is elementary and support simple. We show that $\mathbb{H}$ is simply ordered. Let

$$C(\mathbb{H}) = \{(H_1^*)^{r_1} = (X_1, E_1^*),\ (H_2^*)^{r_2} = (X_2, E_2^*), \ldots,\ (H_n^*)^{r_n} = (X_n, E_n^*)\}.$$

Since, $\mathbb{H}$ is elementary, it follows from Proposition 3.15 that $\mathbb{H}$ is ordered. To show that $\mathbb{H}$ is simply ordered, suppose that there exists $F \in E_{i+1}^* \backslash E_i^*$. Then, there exists $x^* \in F$ such that $t_A(x^*) = r_{i+1}$, $f_A(x^*) = \acute{r}_{i+1}$. Since, $t_A(x^*) = r_{i+1} < r_i$ and $f_A(x^*) = \acute{r}_{i+1} < \acute{r}_i$, it follows that $x^* \notin X_I$ and $F \nsubseteq X_i$, hence $\mathbb{H}$ is simply ordered. Conversely, suppose $\mathbb{H} = (X, \mathbb{E})$ is elementary, support simple, and simply ordered. Let

$$C(\mathbb{H}) = \{(H_1^*)^{r_1} = (X_1, E_1^*),\ (H_2^*)^{r_2} = (X_2, E_2^*), \ldots,\ (H_n^*)^{r_n} = (X_n, E_n^*), \}$$

where $D(\mathbb{H}) = \{r_1, r_2, \ldots, r_n\}$ with $0 < r_n < \cdots < r_1$. Since $(H^*)^{r_n} = H_n^* = (X_n, E_n^*)$ and define $A = (t_A, f_A)$ by,

$$t_A(x) = \begin{cases} r_1, & \text{if } x \in X_1, \\ r_i, & \text{if } x \in X_i \backslash X_{i-1}, i = 1, 2, \ldots, n. \end{cases} \qquad f_A(x) = \begin{cases} s_1, & \text{if } x \in X_1, \\ s_i, & \text{if } x \in X_i \backslash X_{i-1}, i = 1, 2, \ldots, n. \end{cases}$$

We show that $\mathbb{E} = \{B_F = (t_{B_F}, f_{B_F}) \mid F \in E^*\}$, where

$$t_{B_F}(x) = \begin{cases} \min(t_A(y) \mid y \in F), & \text{if } x \in F, \\ 0, & \text{otherwise}, \end{cases} \qquad f_{B_F}(x) = \begin{cases} \max(f_A(y) \mid y \in F), & \text{if } x \in F, \\ 1, & \text{otherwise}. \end{cases}$$

Let $F \in E_n^*$. Since, $\mathbb{H}$ is elementary and support simple, there is a unique vague edge $C_F = (t_{C_F}, f_{C_F})$ in $\mathbb{E}$ having support E^*. Indeed, distinct edges in E must have distinct supports that lie in E_n^*. Thus, to show that $E = \{B_F = (t_{B_F}, f_{B_F}) \mid F \in E_n^*\}$, it suffices to show that for each $F \in E_n^*$, $t_{C_F} = t_{B_F}$ and $f_{C_F} = f_{B_F}$. As all edges are elementary and different edges have different supports, it follows from the definition of fundamental sequence that $h(C_F)$ is equal to some number r_i of $D(\mathbb{H})$. Consequently, $E^* \subseteq X_i$. Moreover, if $i > 1$, then $F \in E^* \backslash E_{i-1}^*$. Since $F \subseteq X_i$, it follows from the definition of $A = (t_A, f_A)$ that for each $x \in F$, $t_A(x) \geq r_i$ and $f_A(x) \leq s_i$. We claim that $t_A(x) = r_i$ and $f_A(x) = s_i$, for some $x \in F$. If not, then by definition of $A = (t_A, f_A)$, $t_A(x) \geq r_i$ and $f_A(x) \leq s_i$ for all $x \in F$ which implies that $F \subseteq X_{i-1}$ and so $F \in E^* \backslash E^*_{i-1}$ and since $\mathbb{H}$ is simply ordered $F \subsetneq X_{i-1}$, a contradiction. Thus it follows from the definition of B_F that $B_F = C_F$. This completes the proof.

As a consequence of the above theorem we obtain.

Proposition 3.16 *Suppose that* $\mathbb{H}$ *is a simply ordered vague hypergraph and* $F(\mathbb{H}) = \{r_1, r_2, \ldots, r_n\}$. *If* $\mathbb{H}^{r_n}$ *is a simple hypergraph, then there is a vague subhypergraph* $\acute{\mathbb{H}}$ *of* $\mathbb{H}$ *such that the following assertions hold,*

(i) $\acute{\mathbb{H}}$ *is an A-tempered vague hypergraph of* $\mathbb{H}_n$.
(ii) $\mathbb{E} \sqsubseteq \acute{\mathbb{E}}$.
(iii) $F(\acute{\mathbb{H}}) = F(\mathbb{H})$ *and* $C(\acute{\mathbb{H}}) = C(\mathbb{H})$.

3.4 Interval-Valued Intuitionistic Fuzzy Hypergraphs

Atanassov and Gargov [6] initiated the concept of interval-valued intuitionistic fuzzy sets as a generalization of intuitionistic fuzzy sets. An interval-valued intuitionistic fuzzy set is characterized by an interval-valued membership degree and an interval-valued nonmembership degree.

Definition 3.32 An *interval-valued intuitionistic fuzzy set V* in X is an object of the form,

$$V = \{\langle x, \mu_V(x), \nu_V(x)\rangle \mid x \in X\},$$

where $\mu_V : X \to \text{Int}([0, 1])$ and $\nu_V : X \to \text{Int}([0, 1])$ such that $\mu^+_V(x) + \nu^+_V(x) \le 1$ for all $x \in X$.

Definition 3.33 The *support* of an interval-valued intuitionistic fuzzy set $V = \{\langle x, \mu_V(x), \nu_V(x)\rangle \mid x \in X\}$ is defined as, $\text{supp}(V) = \{x \mid \mu^-_V(x) \neq 0, \mu^+_V(x) \neq 0, \nu^-_V(x) \neq 1 \text{ and } \nu^+_V(x) \neq 1\}$.

Definition 3.34 The*height* of an interval-valued intuitionistic fuzzy set $V = \{\langle x, \mu_V(x), \nu_V(x)\rangle \mid x \in X\}$ is defined as, $h(V) = \langle[\sup_{x\in X} \mu^-_V(x), \sup_{x\in X} \mu^+_V(x)], [\inf_{x\in X} \nu^-_V(x), \inf_{x\in X} \nu^+_V(x)]\rangle$.

Definition 3.35 For $\alpha, \beta, \gamma, \delta \in [0, 1]$, the $\langle[\alpha, \beta], [\gamma, \delta]\rangle$*-cut of interval-valued intuitionistic fuzzy set V* is

$$V_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = \{x \mid \mu^-_V(x) \ge \alpha, \mu^+_V(x) \ge \beta, \nu^-_V(x) \le \gamma \text{ and } \nu^-_V(x) \le \delta\}.$$

Definition 3.36 Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of vertices and let $\tau = \{\tau_1, \tau_2, ..., \tau_m\}$ be a finite family of nontrivial interval-valued intuitionistic fuzzy sets on X such that

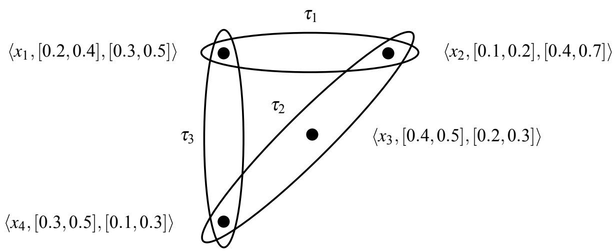


Fig. 3.6 Interval-valued intuitionistic fuzzy hypergraph

$$X = \bigcup_j \text{supp}\langle \mu_j, \nu_j\rangle, \ \ j = 1, 2, ..., m,$$

where μ_j, ν_j are interval-valued membership and interval-valued nonmembership functions defined on $\tau_j \in \tau$. Then, the pair $\mathscr{H} = (X, \tau)$ denotes an interval-valued intuitionistic fuzzy hypergraph on X, τ is the family of interval-valued intuitionistic fuzzy hyperedges of $\mathscr{H}$.

Example 3.7 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ such that $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3\}$ as shown in Fig. 3.6, where
$\tau_1 = \{x_1|\langle[0.2, 0.4], [0.3, 0.5]\rangle, x_2|\langle[0.1, 0.2], [0.4, 0.7]\rangle\}$,
$\tau_2 = \{x_2|\langle[0.1, 0.2], [0.4, 0.7]\rangle, x_3|\langle[0.4, 0.5], [0.2, 0.3]\rangle, x_4|\langle[0.3, 0.5], [0.1, 0.3]\rangle\}$,
$\tau_3 = \{x_1|\langle[0.2, 0.4], [0.3, 0.5]\rangle, x_4|\langle[0.3, 0.5], [0.1, 0.3]\rangle\}$.

The corresponding incidence matrix $M_{\mathscr{H}}$ is as follows:

$$M_{\mathscr{H}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccc} \tau_1 & \tau_2 & \tau_3 \end{array} \\ \begin{pmatrix} \langle[0.2, 0.4], [0.3, 0.5]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.2, 0.4], [0.3, 0.5]\rangle \\ \langle[0.1, 0.2], [0.4, 0.7]\rangle & \langle[0.1, 0.2], [0.4, 0.7]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0.4, 0.5], [0.2, 0.3]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0.3, 0.5], [0.1, 0.3]\rangle & \langle[0.3, 0.5], [0.1, 0.3]\rangle \end{pmatrix} \end{array}.$$

Definition 3.37 The $\langle[\alpha, \beta], [\gamma, \delta]\rangle$-*cut* of an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}$, denoted by $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ and is defined as $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = (X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}, E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle})$, where

$$\begin{aligned} X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} &= X, \\ E_{j,\langle[\alpha,\beta],[\gamma,\delta]\rangle} &= \{x_i \mid \mu_j^-(x_i) \geq \alpha, \mu_j^+(x_i) \geq \beta, \nu_j^-(x_i) \leq \gamma \text{ and } \nu_j^-(x_i) \leq \delta, j = 1, 2, \ldots, m\}, \\ E_{m+1,\langle[\alpha,\beta],[\gamma,\delta]\rangle} &= \{x_i \mid \mu_j^-(x_i) < \alpha, \mu_j^+(x_i) < \beta, \nu_j^-(x_i) > \gamma \text{ and } \nu_j^-(x_i) > \delta, \forall j\}. \end{aligned}$$

The hyperedge $E_{m+1,\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ is added to group the elements which are not contained in any hyperedge $E_{j,\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ of $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$. The hyperedges in the $\langle[\alpha, \beta], [\gamma, \delta]\rangle$-cut hypergraph are now crisp sets.

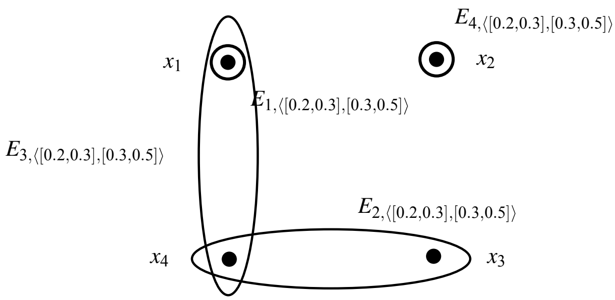


Fig. 3.7 $\langle[0.2, 0.3], [0.3, 0.5]\rangle$-cut hypergraph

Example 3.8 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$, where $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3\}$, given in Example 3.7.

Incidence matrix of $H_{\langle[0.2,0.3],[0.3,0.5]\rangle}$

$$M_{H_{\langle[0.2,0.3],[0.3,0.5]\rangle}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{cccc} E_{1,\langle[0.2,0.3],[0.3,0.5]\rangle} & E_{2,\langle[0.2,0.3],[0.3,0.5]\rangle} & E_{3,\langle[0.2,0.3],[0.3,0.5]\rangle} & E_{4,\langle[0.2,0.3],[0.3,0.5]\rangle} \end{array} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The new hyperedge $E_{4,\langle[0.2,0.3],[0.3,0.5]\rangle}$ is added to group the vertex x_2 as shown in Fig. 3.7.

Definition 3.38 The *dual* interval-valued intuitionistic fuzzy hypergraph of an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ is defined as $\mathscr{H}^* = (X^*, \tau^*)$, where $X^* = \{e_1^{'}, e_2^{'}, ..., e_m^{'}\}$ is the set of vertices corresponding to $\tau_1, \tau_2, ..., \tau_m$, respectively, and $\{X_1, X_2, ..., X_n\}$ is the set of hyperedges corresponding to $x_1, x_2, ..., x_n$, respectively, where $X_i(e_j^{'}) = \tau_j(x_i)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Example 3.9 Consider the dual interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}^* = (X^*, \tau^*)$ (shown in Fig. 3.8) of an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ given in Example 3.7, such that
$X^* = \{e_1^{'}, e_2^{'}, e_3^{'}\}$ and $E^* = \{X_1, X_2, X_3, X_4\}$, where
$X_1 = \{e_1^{'}|\langle[0.2, 0.4], [0.3, 0.5]\rangle, e_3^{'}|\langle[0.2, 0.4], [0.3, 0.5]\rangle\}$,
$X_2 = \{e_1^{'}|\langle[0.1, 0.2], [0.4, 0.7]\rangle, e_2^{'}|\langle[0.1, 0.2], [0.4, 0.7]\rangle\}$,
$X_3 = \{e_2^{'}|\langle[0.4, 0.5], [0.2, 0.3]\rangle\}$,
$X_4 = \{e_2^{'}|\langle[0.3, 0.5], [0.1, 0.3]\rangle, e_3^{'}|\langle[0.3, 0.5], [0.1, 0.3]\rangle\}$.

The corresponding incidence matrix $M_{\mathscr{H}^*}$ is as follows:

$$M_{\mathscr{H}^*} = \begin{array}{c} \\ e_1^{'} \\ e_2^{'} \\ e_3^{'} \end{array} \begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \end{array} \begin{pmatrix} \langle[0.2, 0.4], [0.3, 0.5]\rangle & \langle[0.1, 0.2], [0.4, 0.7]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0.1, 0.2], [0.4, 0.7]\rangle & \langle[0.4, 0.5], [0.2, 0.3]\rangle & \langle[0.3, 0.5], [0.1, 0.3]\rangle \\ \langle[0.2, 0.4], [0.3, 0.5]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.3, 0.5], [0.1, 0.3]\rangle \end{pmatrix}.$$

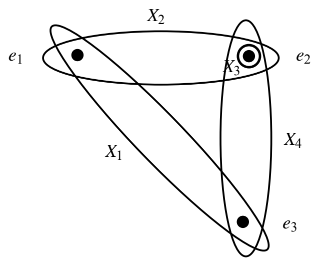


Fig. 3.8 Dual interval-valued intuitionistic fuzzy hypergraph

Definition 3.39 The *strength* ρ of a hyperedge τ_j is defined as

$$\rho(\tau_j) = \{\min(\mu_j^-(x) \mid \mu_j^-(x) > 0), \min(\mu_j^+(x) \mid \mu_j^+(x) > 0),\\ \max(\nu_j^-(x) \mid \nu_j^-(x) > 0), \max(\nu_j^+(x) \mid \nu_j^+(x) > 0)\}.$$

In other words, the minimum membership values $\mu_j^-(x), \mu_j^+(x)$ of vertices and maximum nonmembership values $\nu_j^-(x), \nu_j^+(x)$ of vertices in the hyperedge τ_j. Its interpretation is that the hyperedge τ_j groups elements having participation degree at least $\rho(\tau_j)$ in the hypergraph. The hyperedges with high strength are called the strong hyperedges because the cohesion in them is strong.

Example 3.10 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$, where $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}$ as shown in Fig. 3.9.

Here, $\rho(\tau_1) = \langle[0.5, 0.7], [0.1, 0.2]\rangle$, $\rho(\tau_2) = \langle[0.2, 0.4], [0.2, 0.3]\rangle$, $\rho(\tau_3) = \langle[0.1, 0.3], [0.5, 0.6]\rangle$ and $\rho(\tau_4) = \langle[0.1, 0.3], [0.5, 0.6]\rangle$, respectively. Therefore, the hyperedge τ_1 is stronger than τ_2, τ_3 and τ_4.

Definition 3.40 An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}' = (X', \tau')$ is a *partial* interval-valued intuitionistic fuzzy hypergraph of $\mathscr{H} = (X, \tau)$ if $\tau' \subseteq \tau$ and is written as $\mathscr{H}' \subseteq \mathscr{H}$. If $\mathscr{H}' \subseteq \mathscr{H}$ and $\tau' \subset \tau$, we write $\mathscr{H}' \subset \mathscr{H}$.

Definition 3.41 An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ is *simple* if τ has no repeated interval-valued intuitionistic fuzzy hyperedges and

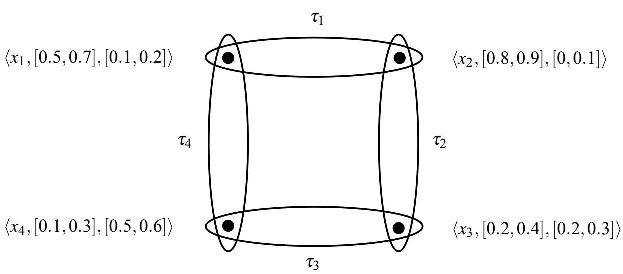


Fig. 3.9 Interval-valued intuitionistic fuzzy hypergraph

whenever $X = \langle \mu_X, \nu_X \rangle$, $Y = \langle \mu_Y, \nu_Y \rangle \in \tau$ and $\mu_X^-(x) \le \mu_Y^-(x)$, $\mu_X^+(x) \le \mu_Y^+(x)$, $\nu_X^-(x) \ge \nu_Y^-(x)$, $\nu_X^+(x) \ge \nu_Y^+(x)$, for all $x \in X$, then $\mu_X^-(x) = \mu_Y^-(x)$, $\mu_X^+(x) = \mu_Y^+(x)$, $\nu_X^-(x) = \nu_Y^-(x)$, $\nu_X^+(x) = \nu_Y^+(x)$.

Definition 3.42 An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ is *support simple* if $X = \langle \mu_X, \nu_X \rangle$, $Y = \langle \mu_Y, \nu_Y \rangle \in \tau$, $\mu_X^-(x) \le \mu_Y^-(x)$, $\mu_X^+(x) \le \mu_Y^+(x)$, $\nu_X^-(x) \ge \nu_Y^-(x)$, $\nu_X^+(x) \ge \nu_Y^+(x)$, for all $x \in X$, and $supp(X) = supp(Y)$, then $\mu_X^-(x) = \mu_Y^-(x)$, $\mu_X^+(x) = \mu_Y^+(x)$, $\nu_X^-(x) = \nu_Y^-(x)$, $\nu_X^+(x) = \nu_Y^+(x)$. An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ is *strongly support simple* if $X = \langle \mu_X, \nu_X \rangle$, $Y = \langle \mu_Y, \nu_Y \rangle \in \tau$ and supp(X)=supp(Y), then $\mu_X^-(x) = \mu_Y^-(x)$, $\mu_X^+(x) = \mu_Y^+(x)$, $\nu_X^-(x) = \nu_Y^-(x)$, $\nu_X^+(x) = \nu_Y^+(x)$.

Definition 3.43 An interval-valued intuitionistic fuzzy set $X = \{\langle x, \mu_X(x), \nu_X(x)\rangle \mid x \in X\}$ is an *elementary interval-valued intuitionistic fuzzy set* if X is single valued on supp(X). An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ whose all interval-valued intuitionistic fuzzy hyperedges are elementary is called an *elementary* interval-valued intuitionistic fuzzy hypergraph.

Example 3.11 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ such that $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, represented by the following incidence matrix:

$$M_{\mathscr{H}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{ccccc} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \end{array} \begin{pmatrix} \langle[0.5,0.7],[0.1,0.2]\rangle & \langle[0.8,0.9],[0,0]\rangle & \langle[0,0],[0,0]\rangle & \langle[0,0],[0,0]\rangle \\ \langle[0.5,0.7],[0.1,0.2]\rangle & \langle[0.8,0.9],[0,0]\rangle & \langle[0.8,0.9],[0,0]\rangle & \langle[0.5,0.7],[0.1,0.2]\rangle \\ \langle[0,0],[0,0]\rangle & \langle[0,0],[0,0]\rangle & \langle[0.8,0.9],[0,0]\rangle & \langle[0.5,0.7],[0.1,0.2]\rangle \\ \langle[0,0],[0,0]\rangle & \langle[0.2,0.4],[0.2,0.3]\rangle & \langle[0,0],[0,0]\rangle & \langle[0.2,0.4],[0.2,0.3]\rangle \end{pmatrix}.$$

Clearly, $\mathscr{H}$ is simple, support simple, and strongly support simple. The partial interval-valued intuitionistic fuzzy hypergraph $\tau' = \{\tau_1, \tau_3\}$ of $\mathscr{H}$ is elementary.

Theorem 3.2 *Let $\mathscr{H} = (X, \tau)$ be an elementary interval-valued intuitionistic fuzzy hypergraph. Then, $\mathscr{H}$ is support simple if and only if $\mathscr{H}$ is strongly support simple.*

Proof Suppose that $\mathscr{H}$ is elementary, support simple, and that supp(X) = supp(Y). Without loss of generality we may assume that $h(X) \le h(Y)$. Since, $\mathscr{H}$ is elementary, it follows that $\mu_X^-(x) \le \mu_Y^-(x)$, $\mu_X^+(x) \le \mu_Y^+(x)$, $\nu_X^-(x) \ge \nu_Y^-(x)$, $\nu_X^+(x) \ge \nu_Y^+(x)$ for all $x \in X$, and since $\mathscr{H}$ is support simple that $\mu_X^-(x) = \mu_Y^-(x)$, $\mu_X^+(x) = \mu_Y^+(x)$, $\nu_X^-(x) = \nu_Y^-(x)$, $\nu_X^+(x) = \nu_Y^+(x)$. Hence, $\mathscr{H}$ is strongly support simple.

Definition 3.44 Let $\mathscr{H} = (X, \tau)$ be an interval-valued intuitionistic fuzzy hypergraph. Let $\alpha, \beta, \gamma, \delta \in [0, 1]$ and

$$E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = \{X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} \neq \emptyset \mid X \in \tau\},\ X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = \bigcup_{X\in\tau} X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}.$$

If $E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} \neq \emptyset$, then the crisp hypergraph $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = (X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}, E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle})$ is the $\langle[\alpha, \beta], [\gamma, \delta]\rangle$-*level hypergraph* of $\mathscr{H}$.

The families of crisp sets (hypergraphs) produced by the $\langle[\alpha, \beta], [\gamma, \delta]\rangle$-cuts of an interval-valued intuitionistic fuzzy hypergraph share an important relationship with each other, as expressed below:

Suppose $\mathbb{A}$ and $\mathbb{B}$ are two families of sets such that for each set $A \in \mathbb{A}$ there is at least one set $B \in \mathbb{B}$ which contains A. In this case we say that $\mathbb{B}$ absorbs $\mathbb{A}$ and symbolically write $\mathbb{A} \sqsubseteq \mathbb{B}$. Since it is possible for $\mathbb{A} \sqsubseteq \mathbb{B}$ while $\mathbb{A} \cap \mathbb{B} = \emptyset$, we have that $\mathbb{A} \subseteq \mathbb{B}$ implies $\mathbb{A} \sqsubseteq \mathbb{B}$, whereas the converse is generally false. If $\mathbb{A} \sqsubseteq \mathbb{B}$ and $\mathbb{A} \neq \mathbb{B}$, then we write $\mathbb{A} \sqsubset \mathbb{B}$.

Definition 3.45 Let $\mathscr{H} = (X, \tau)$ be an interval-valued intuitionistic fuzzy hypergraph, and for $\langle[0, 0], [0, 0]\rangle < \langle[\alpha, \beta], [\gamma, \delta]\rangle \leq h(\mathscr{H})$, let $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = (X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}, E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle})$ be the $\langle[\alpha, \beta], [\gamma, \delta]\rangle$-level hypergraph of $\mathscr{H}$. The sequence of real numbers $\{\langle[r_i, s_i], [t_i, q_i]\rangle \mid 1 \leq i \leq n\}$, $0 < r_n < \ldots < r_1$, $0 < s_n < \ldots < s_1$, $1 > t_n > \ldots > t_1$ and $1 > q_n > \ldots > q_1$, where $h(\mathscr{H}) = \langle[r_1, s_1], [t_1, q_1]\rangle$, which satisfies the properties

(i) if $r_{i+1} < u \leq r_i$, $s_{i+1} < v \leq s_i$, $t_{i+1} > l \geq t_i$, and $q_{i+1} > m \geq q_i$, then $E_{\langle[u,v],[l,m]\rangle} = E_{\langle[r_i,s_i],[t_i,q_i]\rangle}$, $i = 1, 2, \ldots, n$,
(ii) $E_{\langle[r_i,s_i],[t_i,q_i]\rangle} \sqsubset E_{\langle[r_{i+1},s_{i+1}],[t_{i+1},q_{i+1}]\rangle}$, $i = 1, 2, \ldots, n-1$,

is called the *fundamental sequence* of $\mathscr{H}$, denoted by $F(\mathscr{H})$. The set of $\langle[r_i, s_i], [t_i, q_i]\rangle$–level hypergraphs $\{H_{\langle[r_i,s_i],[t_i,q_i]\rangle} \mid 1 \leq i \leq n\}$ is the *set of core hypergraphs* of $\mathscr{H}$ or, the *core set* of $\mathscr{H}$, denoted by $C(\mathscr{H})$.

Definition 3.46 Let $\mathscr{H} = (X, \tau)$ be an interval-valued intuitionistic fuzzy hypergraph and $F(\mathscr{H}) = \{\langle[r_i, s_i], [t_i, q_i]\rangle \mid 1 \leq i \leq n\}$. Then, $\mathscr{H}$ is called *sectionally elementary* if for each X, where X is an interval-valued intuitionistic fuzzy set defined on $\tau_j \in \tau$ and each $\langle[r_i, s_i], [t_i, q_i]\rangle \in F(\mathscr{H})$, $X_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = X_{\langle[r_i,s_i],[t_i,q_i]\rangle}$ for all $\langle[\alpha, \beta], [\gamma, \delta]\rangle \in (\langle[r_{i+1}, s_{i+1}], [t_{i+1}, q_{i+1}]\rangle, \langle[r_i, s_i], [t_i, q_i]\rangle]$. (Take $r_{n+1} = 0, s_{n+1} = 0, t_{n+1} = 0, q_{n+1} = 0$.)

Definition 3.47 An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}$ is *ordered* if $C(\mathscr{H}) = \{H_{\langle[r_i,s_i],[t_i,q_i]\rangle} \mid 1 \leq i \leq n\}$ is ordered, and is *simply ordered* if $C(\mathscr{H})$ is simply ordered.

Example 3.12 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$, represented by incidence matrix, as in Example 3.11. Clearly, $h(\mathscr{H}) = \langle[0.8, 0.9], [0, 0]\rangle$. Now

$$E_{\langle[0.8,0.9],[0,0]\rangle} = E_{\langle[0.5,0.7],[0.1,0.2]\rangle} = \{\{x_1, x_2\}, \{x_2, x_3\}\},$$
$$E_{\langle[0.2,0.4],[0.2,0.3]\rangle} = \{\{x_1, x_2\}, \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}\}.$$

Thus, for $0.2 < \alpha \leq 0.8$, $0.4 < \beta \leq 0.9$, $0.2 > \gamma \geq 0$, $0.3 > \delta \geq 0$,

$$E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = \{\{x_1, x_2\}, \{x_2, x_3\}, \}$$

and for $0 < \alpha \leq 0.2, 0 < \beta \leq 0.4, 1 > \gamma \geq 0.2, 1 > \delta \geq 0.3$,

$$E_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} = \{\{x_1, x_2\}, \{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}\}.$$

It is easy to see that, $E_{\langle[0.8,0.9],[0,0]\rangle} \sqsubset E_{\langle[0.2,0.4],[0.2,0.3]\rangle}$. Therefore, the fundamental sequence is $F(\mathscr{H}) = \{\langle[r_1, s_1],\ [t_1, q_1]\rangle = \langle[0.8, 0.9], [0, 0]\rangle, \langle[r_2, s_2], [t_2, q_2]\rangle = \langle[0.2, 0.4], [0.2, 0.3]\rangle\}$ and the set of core hypergraphs is $C(\mathscr{H}) = \{H_{\langle[0.8,0.9],[0,0]\rangle}, H_{\langle[0.2,0.4],[0.2,0.3]\rangle}\}$. $\mathscr{H}$ is not sectionally elementary, as $\tau_{1,\langle[\alpha,\beta],[\gamma,\delta]\rangle} \neq \tau_{1,\langle[0.8,0.9],[0,0]\rangle}$ for $\langle[\alpha, \beta],\ [\gamma, \delta]\rangle = \langle[0.5, 0.7], [0.1, 0.2]\rangle$. Clearly, $\mathscr{H}$ is simply ordered.

Proposition 3.17 *(i) An elementary interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}(X, \tau)$ is ordered.*
(ii) An ordered interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}(X, \tau)$ with $C(\mathscr{H}) = \{H_{\langle[r_i,s_i],[t_i,q_i]\rangle} \mid 1 \leq i \leq n\}$ and simple $H_{\langle[r_n,s_n],[t_n,q_n]\rangle}$, is elementary.

The complexity of an interval-valued intuitionistic fuzzy hypergraph depends in part on how many hyperedges it has. The natural question arises: is there an upper bound on the number of hyperedges of an interval-valued intuitionistic fuzzy hypergraph of order n?

Proposition 3.18 *Let $\mathscr{H} = (X, \tau)$ be a simple interval-valued intuitionistic fuzzy hypergraph of order n. Then, there is no upper bound on $|\tau|$.*

Proof Let $X = \{x, y\}$, and define $\tau_N = \{X_i = \langle[\mu^-_{X_i}, \mu^+_{X_i}][\nu^-_{X_i}, \nu^+_{X_i}]\rangle \mid i = 1, 2, \ldots, N\}$, where

$$\mu^-_{X_i}(x) = 1/1+i,\ \mu^+_{X_i}(x) = 1/1+i,\ \nu^-_{X_i}(x) = 1/1+i,\ \nu^+_{X_i}(x) = 1/1+i,$$
$$\mu^-_{X_i}(y) = i/1+i,\ \mu^+_{X_i}(y) = i/1+i,\ \nu^-_{X_i}(y) = i/1+i,\ \nu^+_{X_i}(y) = i/1+i.$$

Then, $\mathscr{H}_N = (X, \tau_N)$ is a simple interval-valued intuitionistic fuzzy hypergraph with N hyperedges.

Proposition 3.19 *Let $\mathscr{H} = (X, \tau)$ be a support simple interval-valued intuitionistic fuzzy hypergraph of order n. Then, there is no upper bound on $|\tau|$.*

Proof The proof follows at once from Proposition 3.18, as the class of support simple interval-valued intuitionistic fuzzy hypergraphs contains the class of simple interval-valued intuitionistic fuzzy hypergraphs.

Proposition 3.20 *Let $\mathscr{H} = (X, \tau)$ be a strongly support simple interval-valued intuitionistic fuzzy hypergraph of order n. Then, $|\tau| \leq 2^n - 1$, with equality if and only if $\{\text{supp}(X) \mid X \in \tau\} = P(X) - \emptyset$.*

Proof Each nontrivial $U \subseteq X$ can be the support of at most one $X \in \tau$, therefore $|\tau| \leq 2^n - 1$. The second statement is obvious.

Consider the situation where the node set of a (crisp) hypergraph is fuzzified. Suppose that each hyperedge is given a uniform degree of interval-valued membership and interval-valued nonmembership consistent with the weakest node of the hyperedge. Such constructions describe the following subclass of interval-valued intuitionistic fuzzy hypergraphs.

Definition 3.48 An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ is said to be a $V = \langle[\mu_V^-, \mu_V^+], [\nu_V^-, \nu_V^+]\rangle$-*tempered interval-valued intuitionistic fuzzy hypergraph* of H^*, if there is a crisp hypergraph $H^* = (X, E^*)$ and an interval-valued intuitionistic fuzzy set $X = \langle[\mu_X^-, \mu_X^+], [\nu_X^-, \nu_X^+]\rangle : X \to \text{Int}((0, 1])$ such that $\tau = \{Y_e = \langle[(\mu_Y^-)_e, (\mu_Y^+)_e], [(\nu_Y^-)_e, (\nu_Y^+)_e]\rangle \mid e \in E\}$, where

$$(\mu_Y^-)_e(x) = \begin{cases} \min(\mu_X^-(y) \mid y \in e), & \text{if } x \in e, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\mu_Y^+)_e(x) = \begin{cases} \min(\mu_X^+(y) \mid y \in e), & \text{if } x \in e, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\nu_Y^-)_e(x) = \begin{cases} \max(\nu_X^-(y) \mid y \in e), & \text{if } x \in e, \\ 0, & \text{otherwise,} \end{cases}$$

$$(\nu_Y^+)_e(x) = \begin{cases} \max(\nu_X^+(y) \mid y \in e), & \text{if } x \in e, \\ 0, & \text{otherwise,} \end{cases}$$

The V-tempered interval-valued intuitionistic fuzzy hypergraph of H^* will be denoted by $V \otimes H^*$.

Example 3.13 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ such that $X = \{x_1, x_2, x_3, x_4\}$ and $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, represented by the following incidence matrix:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{cccc} \tau_1 & \tau_2 & \tau_3 & \tau_4 \end{array} \\ \begin{pmatrix} \langle[0.5, 0.7], [0.1, 0.3]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.5, 0.7], [0.1, 0.3]\rangle \\ \langle[0.5, 0.7], [0.1, 0.3]\rangle & \langle[0.2, 0.4], [0.2, 0.3]\rangle & \langle[0.6, 0.8], [0.1, 0.2]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.6, 0.8], [0.1, 0.2]\rangle & \langle[0.5, 0.7], [0.1, 0.3]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0.2, 0.4], [0.2, 0.3]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle \end{pmatrix} \end{array}.$$

Define $V = \langle[\mu_V^-, \mu_V^+], [\nu_V^-, \nu_V^+]\rangle : X \to \text{Int}((0, 1])$ by,

$$\mu_V^-(x_1) = 0.5,\ \mu_X^-(x_2) = \mu_V^-(x_3) = 0.6,\ \mu_V^-(x_4) = 0.2,$$
$$\mu_V^+(x_1) = 0.7,\ \mu_X^+(x_2) = \mu_V^+(x_3) = 0.8,\ \mu_V^+(x_4) = 0.4,$$
$$\nu_V^-(x_1) = 0.1,\ \nu_X^-(x_2) = \nu_V^-(x_3) = 0.1,\ \nu_V^-(x_4) = 0.2,$$
$$\nu_V^+(x_1) = 0.3,\ \nu_X^+(x_2) = \nu_V^+(x_3) = 0.2,\ \nu_V^+(x_4) = 0.3.$$

Now

$$
\begin{aligned}
(\mu_Y^-)_{\{x_1,x_2\}}(x_1) &= (\mu_Y^-)_{\{x_1,x_2\}}(x_2) = \min(\mu_V^-(x_1), \mu_V^-(x_2)) = 0.5,\\
(\mu_Y^-)_{\{x_1,x_2\}}(x_3) &= (\mu_Y^-)_{\{x_1,x_2\}}(x_4) = 0,\\
(\mu_Y^+)_{\{x_1,x_2\}}(x_1) &= (\mu_Y^+)_{\{x_1,x_2\}}(x_2) = \min(\mu_V^+(x_1), \mu_V^+(x_2)) = 0.7,\\
(\mu_Y^+)_{\{x_1,x_2\}}(x_3) &= (\mu_Y^+)_{\{x_1,x_2\}}(x_4) = 0,\\
(\nu_Y^-)_{\{x_1,x_2\}}(x_1) &= (\nu_Y^-)_{\{x_1,x_2\}}(x_2) = \max(\nu_V^-(x_1), \nu_V^-(x_2)) = 0.1,\\
(\nu_Y^-)_{\{x_1,x_2\}}(x_3) &= (\nu_Y^-)_{\{x_1,x_2\}}(x_4) = 0,\\
(\nu_Y^+)_{\{x_1,x_2\}}(x_1) &= (\nu_Y^+)_{\{x_1,x_2\}}(x_2) = \max(\nu_V^+(x_1), \nu_V^+(x_2)) = 0.3,\\
(\nu_Y^+)_{\{x_1,x_2\}}(x_3) &= (\nu_Y^+)_{\{x_1,x_2\}}(x_4) = 0.
\end{aligned}
$$

Therefore, $\tau_1 = \langle[(\mu_Y^-)_{\{x_1,x_2\}}, (\mu_Y^+)_{\{x_1,x_2\}}], [(\nu_Y^-)_{\{x_1,x_2\}}.(\nu_Y^+)_{\{x_1,x_2\}}]\rangle$.

Also, it is easy to see that
$\tau_2 = \langle[(\mu_Y^-)_{\{x_2,x_4\}}, (\mu_Y^+)_{\{x_2,x_4\}}], [(\nu_Y^-)_{\{x_2,x_4\}}, (\nu_Y^+)_{\{x_2,x_4\}}]\rangle$,
$\tau_3 = \langle[(\mu_Y^-)_{\{x_2,x_3\}}, (\mu_Y^+)_{\{x_2,x_3\}}], [(\nu_Y^-)_{\{x_2,x_3\}}, (\nu_Y^+)_{\{x_2,x_3\}}]\rangle$,
$\tau_4 = \langle[(\mu_Y^-)_{\{x_1,x_3\}}, (\mu_Y^+)_{\{x_1,x_3\}}], [(\nu_Y^-)_{\{x_1,x_3\}}, (\nu_Y^+)_{\{x_1,x_3\}}]\rangle$.
Thus, $\mathscr{H}$ is $X = \langle[\mu_V^-, \mu_V^+], [\nu_V^-, \nu_V^+]\rangle$-tempered interval-valued intuitionistic fuzzy hypergraph.

Theorem 3.3 *An interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}$ is a V-tempered interval-valued intuitionistic fuzzy hypergraph of some crisp hypergraph H^* if and only if $\mathscr{H}$ is elementary, support simple, and simply ordered.*

Proof Suppose that $\mathscr{H} = (X, \tau)$ is a V-tempered interval-valued intuitionistic fuzzy hypergraph of $H^* = (X, E^*)$. Clearly, $\mathscr{H}$ is elementary, support simple and ordered (being elementary). To show that $\mathscr{H}$ is simply ordered, let $C(\mathscr{H}) = \{H_{\langle[r_i,s_i],[t_i,q_i]\rangle}(X_i, E_i) \mid 1 \le i \le n\}$. Suppose there exists $e \in E_{i+1} \backslash E_i$, then there exists $z \in e$ such that $\mu_X^-(z) = r_{i+1}$, $\mu_X^+(z) = s_{i+1}$, $\nu_X^-(z) = t_{i+1}$ and $\nu_X^+(z) = q_{i+1}$. Since $\mu_X^-(z) = r_{i+1} < r_i$, $\mu_X^+(z) = s_{i+1} < s_i$, $\nu_X^-(z) = t_{i+1} > t_i$ and $\nu_X^+(z) = q_{i+1} > q_i$, it follows that $z \notin X_i$ and $e \nsubseteq X_i$, hence $\mathscr{H}$ is simply ordered.

Conversely, suppose that $\mathscr{H} = (X, \tau)$ is elementary, support simple, and simply ordered. Define $V = \langle[\mu_V^-, \mu_V^+], [\nu_V^-, \nu_V^+]\rangle : X_n \to \text{Int}((0, 1])$ by

$$\mu_Y^-(x) = \begin{cases} r_1, & \text{if } x \in X_1,\\ r_i, & \text{if } x \in X_i \backslash X_{i-1},\ i = 2, 3, \ldots, n, \end{cases}$$

$$\mu_Y^+(x) = \begin{cases} s_1, & \text{if } x \in X_1,\\ s_i, & \text{if } x \in X_i \backslash X_{i-1},\ i = 2, 3, \ldots, n, \end{cases}$$

$$\nu_Y^-(x) = \begin{cases} t_1, & \text{if } x \in X_1,\\ t_i, & \text{if } x \in X_i \backslash X_{i-1},\ i = 2, 3, \ldots, n, \end{cases}$$

$$\nu_Y^+(x) = \begin{cases} q_1, & \text{if } x \in X_1, \\ q_i, & \text{if } x \in X_i \backslash X_{i-1}, \ i = 2, 3, \ldots, n. \end{cases}$$

We show that $\tau = \{Y_e = \langle[(\mu_Y^-)_e, (\mu_Y^+)_e], [(\nu_Y^-)_e, (\nu_Y^+)_e]\rangle \mid e \in E_n\}$. Since, $\mathscr{H}$ is elementary and support simple, there is a unique interval-valued intuitionistic fuzzy hyperedge Z_e in τ having support e. Since distinct hyperedges in τ must have distinct supports that lie in E_n. Thus, to show that $\tau = \{Y_e = \langle[(\mu_Y^-)_e, (\mu_Y^+)_e], [(\nu_Y^-)_e, (\nu_Y^+)_e]\rangle \mid e \in E_n\}$, it suffices to show that $Y_e = Z_e$, for each $e \in E_n$.

Since, all hyperedges are elementary and different hyperedges have different supports, it follows from Definition 3.45 that $h(Z_e) = \langle[r_i, s_i], [t_i, q_i]\rangle \in F(\mathscr{H})$. Consequently, $e \subseteq X_i$. Moreover, $e \in E_i \backslash E_{i-1}, i = 2, 3, \ldots, n$. As $e \subseteq X_i$, it follows from the definition of $V = \langle[\mu_V^-, \mu_V^+], [\nu_V^-, \nu_V^+]\rangle$ that $\mu_V^-(x) \geq r_i$, $\mu_V^+(x) \geq s_i$, $\nu_V^-(x) \leq t_i$ and $\nu_V^+(x) \leq q_i$ for each $x \in e$. We claim that $\mu_V^-(x) = r_i$, $\mu_V^+(x) = s_i$, $\nu_V^-(x) = t_i$ and $\nu_V^+(x) = q_i$, for some $x \in e$. For if not, then, by definition of V, $\mu_V^-(x) \geq r_{i-1}$, $\mu_V^+(x) \geq s_{i-1}$, $\nu_V^-(x) \leq t_{i-1}$ and $\nu_V^+(x) \leq q_{i-1}$ for all $x \in e$ which implies that $e \subseteq X_{i-1}$ and so $e \in E_i \backslash E_{i-1}$ and since $\mathscr{H}$ is simply ordered $e \nsubseteq X_{i-1}$, a contradiction. Hence, $Y_e = Z_e$, by definition of Y_e.

Corollary 3.1 *Suppose that $\mathscr{H} = (X, \tau)$ is a simply ordered interval-valued intuitionistic fuzzy hypergraph with $F(\mathscr{H}) = \{\langle[r_i, s_i], [t_i, q_i]\rangle \mid 1 \leq i \leq n\}$. If $H_{\langle[r_n,s_n],[t_n,q_n]\rangle}$ is a simple hypergraph, then there is a partial interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}' = (X, \tau')$ of $\mathscr{H}$ such that the following assertions hold.*

(i) $\mathscr{H}' = (X, \tau')$ *is a V-tempered interval-valued intuitionistic fuzzy hypergraph of H_n.*
(ii) $\tau \sqsubseteq \tau'$.
(iii) $F(\mathscr{H}') = F(\mathscr{H})$ *and* $C(\mathscr{H}') = C(\mathscr{H})$.

Definition 3.49 Let $\mathscr{H} = (X, \tau)$ be an interval-valued intuitionistic fuzzy hypergraph. An *interval-valued intuitionistic fuzzy transversal* $\mathscr{T}$ of $\mathscr{H} = (X, \tau)$ is an interval-valued intuitionistic fuzzy set defined on X such that $\mathscr{T}_{h(\tau_j)} \cap (\tau_j)_{h(\tau_j)} \neq \emptyset$, for each $\tau_j \in \tau, \ j = 1, 2, \ldots, m$.

Definition 3.50 A *minimal interval-valued intuitionistic fuzzy transversal* $\mathscr{T}$ of $\mathscr{H}$ is a transversal of $\mathscr{H}$ such that if $\mathscr{T}' \subset \mathscr{T}$, then $\mathscr{T}'$ is not an interval-valued intuitionistic fuzzy transversal of $\mathscr{H}$. The class of all minimal interval-valued intuitionistic fuzzy transversals of $\mathscr{H}$ will be denoted by $Tr(\mathscr{H})$.

Example 3.14 Consider an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$ such that $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $\tau = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, represented by the following incidence matrix:

$$M_{\mathscr{H}} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} \tau_1 & \tau_2 & \tau_3 & \tau_4 \end{array} \\ \begin{pmatrix} \langle[0.3, 0.5], [0.2, 0.4]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0.3, 0.5], [0.2, 0.4]\rangle & \langle[0.6, 0.8], [0.1, 0.2]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0.2, 0.4], [0.3, 0.5]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.4, 0.6], [0.2, 0.3]\rangle & \langle[0.5, 0.7], [0.1, 0.2]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0.2, 0.4], [0.3, 0.5]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0, 0], [0, 0]\rangle & \langle[0.5, 0.7], [0.1, 0.2]\rangle \end{pmatrix} \end{array}.$$

$$Tr(\mathscr{H}) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \mathscr{T} \\ \begin{pmatrix} \langle[0, 0], [0, 0]\rangle \\ \langle[0.6, 0.8], [0.1, 0.2]\rangle \\ \langle[0.5, 0.7], [0.1, 0.2]\rangle \\ \langle[0, 0], [0, 0]\rangle \\ \langle[0, 0], [0, 0]\rangle \end{pmatrix} \end{array}.$$

Theorem 3.4 *If $\mathscr{T}$ is an interval-valued intuitionistic fuzzy transversal of an interval-valued intuitionistic fuzzy hypergraph $\mathscr{H} = (X, \tau)$, then $h(\mathscr{T}) \geq h(\tau_j)$ for each $\tau_j \in \tau$. Moreover, if $\mathscr{T}$ is a minimal interval-valued intuitionistic fuzzy transversal of $\mathscr{H}$, then $h(\mathscr{T}) = h(\mathscr{H})$.*

Proof The proof follows at once from above definitions.

Theorem 3.5 *Let $\mathscr{H} = (X, \tau)$ be an interval-valued intuitionistic fuzzy hypergraph. Then the following statements are equivalent:*

(i) $\mathscr{T}$ is an interval-valued intuitionistic fuzzy transversal of $\mathscr{H}$,
(ii) for each $\tau_j \in \tau$ and each $\langle[\alpha, \beta], [\gamma, \delta]\rangle$, $\langle[0, 0], [0, 0]\rangle < \langle[\alpha, \beta], [\gamma, \delta]\rangle \leq h(\tau_j)$, $\mathscr{T}_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} \cap (\tau_j)_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} \neq \emptyset$,
(iii) for each $\langle[\alpha, \beta], [\gamma, \delta]\rangle$, $\langle[0, 0], [0, 0]\rangle < \langle[\alpha, \beta], [\gamma, \delta]\rangle \leq h(\mathscr{H})$, $\mathscr{T}_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ is a transversal of $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$.

If $\mathscr{T}$ is a minimal interval-valued intuitionistic fuzzy transversal of $\mathscr{H}$, then $\mathscr{T}_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ need not be a minimal transversal of $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ for each $\langle[\alpha, \beta], [\gamma, \delta]\rangle$, $\langle[0, 0], [0, 0]\rangle < \langle[\alpha, \beta], [\gamma, \delta]\rangle \leq h(H)$. However, interval-valued intuitionistic fuzzy transversals satisfying this condition are of interest.

Definition 3.51 An interval-valued intuitionistic fuzzy set $\mathscr{T}$ with the property that $\mathscr{T}_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$ is a minimal transversal of $H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle}$, for each $\langle[\alpha, \beta], [\gamma, \delta]\rangle$, $\langle[0, 0], [0, 0]\rangle < \langle[\alpha, \beta], [\gamma, \delta]\rangle \leq h(H)$ is called a *locally minimal interval-valued intuitionistic fuzzy transversal* of $\mathscr{H}$. The class of all locally minimal interval-valued intuitionistic fuzzy transversals of $\mathscr{H}$ will be denoted by $Tr^*(\mathscr{H})$. That is

$$Tr^*(\mathscr{H}) = \{\mathscr{T} \mid h(\mathscr{T}) = h(\mathscr{H}) \ \&\ \mathscr{T}_{\langle[\alpha,\beta],[\gamma,\delta]\rangle} \in Tr(H_{\langle[\alpha,\beta],[\gamma,\delta]\rangle})\}.$$

Remark 3.3 For any interval-valued intuitionistic fuzzy hypergraph $\mathscr{H}$, $Tr^*(\mathscr{H}) \subseteq Tr(\mathscr{H})$.

Theorem 3.6 *Suppose $\mathscr{H} = (X, \tau)$ is an ordered interval-valued intuitionistic fuzzy hypergraph with $F(\mathscr{H}) = \{\langle[r_i, s_i], [t_i, q_i]\rangle \mid 1 \leq i \leq n\}$ and $C(\mathscr{H}) = \{\mathscr{H}_{\langle[r_i,s_i],[t_i,q_i]\rangle} \mid 1 \leq i \leq n\}$. Then, $Tr^*(\mathscr{H}) \neq \emptyset$.*

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