Chapter 2
Hypergraphs in Intuitionistic Fuzzy Environment

In this chapter, we define intuitionistic fuzzy hypergraphs, dual intuitionistic fuzzy hypergraphs, intuitionistic fuzzy line graphs, and 2-section of an intuitionistic fuzzy hypergraph. We describe some applications of intuitionistic fuzzy hypergraphs in planet surface networks, intersecting communities in social network, grouping of incompatible chemical substances, and clustering problem. We design certain algorithms to construct dual intuitionistic fuzzy hypergraphs, intuitionistic fuzzy line graphs and the selection of objects in decision-making problems. Further, we present concept of intuitionistic fuzzy directed hypergraphs and complex intuitionistic fuzzy hypergraphs. This chapter is basically due to [2, 3, 6, 12, 14, 15, 17, 18].

2.1 Introduction

Presently, science and technology is featured with complex processes and phenomena for which complete information are not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large number of these models is based on an extension of the ordinary set theory, namely, fuzzy sets. The notion of fuzzy sets was introduced by Zadeh [25] as a method of representing uncertainty and vagueness. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines, including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, and automata theory. In 1983, Atanassov [5] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Atanassov added in the definition of fuzzy set a new component which determines the degree of nonmembership. Fuzzy sets give the degree of membership of an element in a given set (the nonmembership of degree equals one minus the degree of membership), while intuitionistic fuzzy

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sets give both a degree of membership and a degree of nonmembership, which are more or less independent of each other; the only requirement is that the sum of these two degrees is not greater than 1. Intuitionistic fuzzy sets are higher order fuzzy sets. Application of higher order fuzzy sets makes the solution-procedure more complex, but if the complexity in computation-time, computation-volume or memory-space are not the matter of concern then a better result could be achieved. Fuzzy sets and intuitionistic fuzzy sets cannot handle imprecise, inconsistent, and incomplete information of periodic nature. These theories are applicable to different areas of science, but there is one major deficiency in both sets, that is, a lack of capability to model two-dimensional phenomena. To overcome this difficulty, the concept of complex fuzzy sets was introduced by Ramot et al. [20]. A complex fuzzy set C is characterized by a membership function $\mu(x)$, whose range is not limited to [0, 1] but extends to the unit circle in the complex plane. Hence, $\mu(x)$ is a complex-valued function that assigns a grade of membership of the form $r(x)e^{i\alpha(x)}$, $i = \sqrt{-1}$ to any element x in the universe of discourse. Thus, the membership function $\mu(x)$ of complex fuzzy set consists of two terms, i.e., amplitude term $r(x)$ which lies in the unit interval [0, 1] and phase term (periodic term) $w(x)$ which lies in the interval $[0, 2\pi]$. This phase term distinguishes a complex fuzzy set model from all other models available in the literature. The potential of a complex fuzzy set for representing two-dimensional phenomena makes it superior to handle ambiguous and intuitive information that are prevalent in time-periodic phenomena. To generalize the concepts of intuitionistic fuzzy sets, complex intuitionistic fuzzy sets were introduced by Alkouri and Salleh [4] by adding nonmembership $\nu(x) = s(x)e^{i\beta(x)}$ to the complex fuzzy sets subjected to the constraint $r + s \leq 1$.

Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, and transportation. However, in many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic. Graphs are used to represent the pairwise relationships between objects. However, in many real world phenomena, sometimes relationships are much problematic that they cannot be perceived through simple graphs. By handling such complex relationships by pairwise connections naively, one can face the loss of data which is considered to be worthwhile for learning errands. To overcome these difficulties, we take into account the generalization of simple graphs, named as hypergraphs, to personify the complex relationships. A hypergraph is an extension of a classical graph in this way that a hyperedge can combine two or more than two vertices. Hypergraphs are the generalization of graphs in case of set of multiary relations. It means the expansion of graph models for the modeling complex systems. In case of modeling systems with fuzzy binary and multiary relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realize formal optimization and logical procedures. However, using of the fuzzy graphs and hypergraphs as the models of various systems (social, economic systems, communication networks, and others) leads to difficulties. The graph isomorphic transformations are reduced to redefine the vertices and edges. This redefinition does not change properties the graph determined by an adjacent and an incidence of its vertices and edges. Fuzzy independent sets, domination fuzzy sets, and fuzzy chromatic sets are invariants concerning the isomorphism transformations of the fuzzy graphs and fuzzy hypergraph and allow their structural analysis. Kaufamnn [10] applied the concept of fuzzy sets to hypergraphs. Mordeson and Nair [13] presented fuzzy graphs and fuzzy hypergraphs. Generalization and redefinition of fuzzy hypergraphs were discussed by Lee-Kwang and Lee [11]. The concept of interval-valued fuzzy sets was applied to hypergraphs by Chen [8]. Parvathi et al. [17] established the notion of intuitionistic fuzzy hypergraph, and Myithili and Parvathi [14], Myithili et al. [15] considered intuitionistic fuzzy directed hypergraphs.

Definition 2.1 A mapping $A = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is called an *intuitionistic fuzzy set* on X if $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$, where the mappings $\mu_A : X \to [0, 1]$, and $\nu_A : X \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively.

An intuitionistic fuzzy set A in X can be represented as an object of the form

$$A = (\mu_A, \nu_A) = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\nu_A(x)$) of the element $x \in X$, respectively, and for all $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Obviously, each fuzzy set maybe written as

$$A = \{(x, \mu_A(x), 1 - \mu_A(x)) \mid x \in X\}.$$

The value

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2.1}$$

is called *uncertainty (intuitionistic index)* of the elements $x \in X$ to the intuitionistic fuzzy set A. It represents *hesitancy degree* of x to A.

Clearly, in the case of ordinary fuzzy set, $\pi_A(x) = 0$, for all $x \in X$.

Geometrical Interpretations of an Intuitionistic Fuzzy Set [5]

A geometrical interpretation of an intuitionistic fuzzy set is shown in Fig. 2.1. Atanassov considered a universe X and subset F in the Euclidean plane with the Cartesian coordinates.

This geometrical interpretation can be used as an example when considering a situation at the beginning of negotiations (applications of intuitionistic fuzzy sets for group decision-making, negotiations and other real situations are presented in Fig. 2.2). Each expert i is represented as a point having coordinates $\langle \mu_i, \nu_i, \pi_i \rangle$. Expert

Fig. 2.1 A geometrical interpretation of an intuitionistic fuzzy set

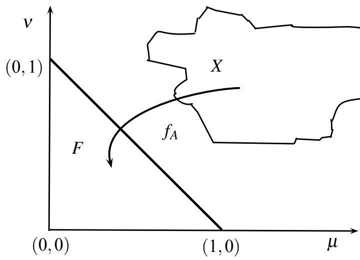
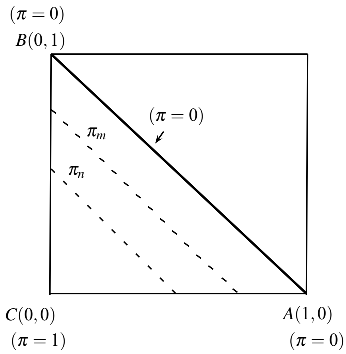


Fig. 2.2 An orthogonal projection (three dimension) representation of an intuitionistic fuzzy set



$A : \langle 1, 0, 0\rangle$—fully accepts a discussed idea. Expert $B : \langle 0, 1, 0\rangle$—fully rejects it. The experts placed on the segment AB fixed their point of view (their hesitation margins equal zero for segment AB, so each expert is convinced to the extent μ_i, is against to the extent ν_i and $\mu_i + \nu_i = 1$; segment AB represents a fuzzy set). Expert $C : \langle 0, 0, 1\rangle$ is absolutely hesitant, i.e., undecided he or she is the most open to the influence of the arguments presented. A line parallel to AB describe a set of experts with the same level of hesitancy. For example, in Fig. 2.2, two sets are presented with intuitionistic indices equal to π_m and π_n, where $\pi_n > \pi_m$. In other words, Fig. 2.2 (the triangle ABC) is an orthogonal projection of the real situation (the triangle ABD) presented in Fig. 2.3.

An element of an intuitionistic fuzzy sets has three coordinates $\langle \mu_i, \nu_i, \pi_i\rangle$, hence the most natural representation of an intuitionistic fuzzy set is to draw a cube (with edge length equal to 1) and because of Eq. (2.1), the triangle ABD (Fig. 2.3) represents an intuitionistic fuzzy set. As before (Fig. 2.2), the triangle ABC is the orthogonal projection of ABD.

Definition 2.2 Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets on a set X. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy relation on a set X, then $A = (\mu_A, \nu_A)$ is called an *intuitionistic fuzzy relation* on $B = (\mu_B, \nu_B)$ if $\mu_A(x, y) \leq \min(\mu_B(x), \mu_B(y))$ and $\nu_A(x, y) \leq \max(\nu_B(x), \nu_B(y))$, for all $x, y \in X$. An intuitionistic fuzzy relation A on X is called *symmetric* if $\mu_A(x, y) = \mu_A(y, x)$ and $\nu_A(x, y) = \nu_A(y, x)$, for all $x, y \in X$.

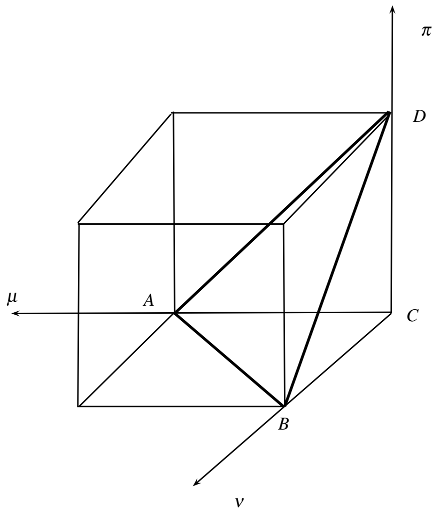


Fig. 2.3 A three-dimension representation of an intuitionistic fuzzy set

Definition 2.3 The *support* of an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$, denoted by supp(A), is defined by

$$\text{supp}(A) = \{x \mid \mu_A(x) \neq 0 \textit{ and } \nu_A(x) \neq 0\}.$$

The support of the intuitionistic fuzzy set is a crisp set.

Definition 2.4 Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on X and let $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$. Then, the set $A_{(\alpha,\beta)} = \{x \mid \mu_A(x) \geq \alpha,\ \nu_A(x) \leq \beta\}$ is called an (α, β)-*level subset* of A. $A_{(\alpha,\beta)}$ is a crisp set.

Definition 2.5 The *height* of an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ is defined as $h(A) = \sup_{x\in X}(A)(x) = (\sup_{x\in X} \mu_A(x), \inf_{x\in X} \nu_A(x))$. We shall say that intuitionistic fuzzy set A is *normal* if there is at least one $x \in X$ such that $\mu_A(x) = 1$.

Definition 2.6 An *intuitionistic fuzzy graph* on X is defined as a pair $\mathscr{G} = (C, D)$, where C is an intuitionistic fuzzy set on X and D is an intuitionistic fuzzy relation in X such that $\lambda_D(yz) \leq \min\{\lambda_C(y), \lambda_C(z)\}$ and $\tau_D(yz) \leq \max\{\tau_C(y), \tau_C(z)\}$, for all $y, z \in X$.

For further terminologies and studies on intuitionistic fuzzy hypergraphs, readers are referred to [1, 7, 9, 16, 19, 21, 22, 24].

2.2 Intuitionistic Fuzzy Hypergraphs

Definition 2.7 An intuitionistic fuzzy hypergraph on a non-empty set X is a pair $\mathscr{H} = (S, R)$ where, $S = \{\eta_1, \eta_2, \ldots, \eta_s\}$ is a family of intuitionistic fuzzy subsets

Table 2.1 Intuitionistic fuzzy subsets on X

$y \in X$	η_1	η_2	η_3	η_4	η_5
a_1	(0.2, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
a_2	(0.3, 0.5)	(0, 1)	(0, 1)	(0.3, 0.5)	(0, 1)
a_3	(0, 1)	(0.4, 0.6)	(0.4, 0.6)	(0, 1)	(0, 1)
a_4	(0.1, 0.3)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
a_5	(0, 1)	(0.3, 0.1)	(0, 1)	(0, 1)	(0, 1)
a_6	(0, 1)	(0, 1)	(0.9, 0.1)	(0, 1)	(0, 1)
a_7	(0, 1)	(0, 1)	(0.5, 0.4)	(0.2, 0.8)	(0, 1)
a_8	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.3)

on X and R is an intuitionistic fuzzy relation on intuitionistic fuzzy subsets η_i's such that

1. $\lambda_R(E_i) = \lambda_R(\{y_1, y_2, \ldots, y_r\}) \leq \min\{\lambda_{\eta_i}(y_1), \lambda_{\eta_i}(y_2), \ldots, \lambda_{\eta_i}(y_r)\}$,
2. $\tau_R(E_i) = \tau_R(\{y_1, y_2, \ldots, y_r\}) \leq \max\{\tau_{\eta_i}(y_1), \tau_{\eta_i}(y_2), \ldots, \tau_{\eta_i}(y_r)\}$,
3. $\lambda_R(E_i) + \tau_R(E_i) \leq 1, \qquad$ for each $E_i \subset X$,
4. $\bigcup_i supp(\eta_i) = X$, for all $\eta_i \in S$.

Example 2.1 Let $S = \{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ be a family of intuitionistic fuzzy subsets on $X = \{a_1, a_2, \ldots, a_8\}$ as shown in Table 2.1.

The intuitionistic fuzzy relation R on each $\eta_i, 1 \leq i \leq 5$, is given as $R(\{a_1, a_2, a_4\}) = (0.1, 0.5)$, $R(\{a_3, a_5\}) = (0.3, 0.6)$, $R(\{a_3, a_6, a_7\}) = (0.4, 0.6)$, $R(\{a_2, a_7\}) = (0.2, 0.8)$, and $R(\{a_8\}) = \eta_5(a_8)$. It is clear From Fig. 2.4 that $\mathscr{H}$ is an intuitionistic fuzzy hypergraph.

Example 2.2 Consider another example of an intuitionistic fuzzy hypergraph consisting of nine vertices $X = \{a_1, a_2, \ldots, a_9\}$ and two hyperedges E_1, E_2. The membership values of vertices are given in (Fig. 2.5) and the membership values of hyperedges are $R(\{a_1, a_2, a_3, a_4, a_9\}) = (0.3, 0.6)$ and $R(\{a_5, a_6, a_7, a_8, a_9\}) = (0.2, 0.5)$. The corresponding intuitionistic fuzzy hypergraph in shown in Fig. 2.6.

Definition 2.8 An intuitionistic fuzzy set $C = (\mu_A, \nu_A) : X \to [0, 1] \times [0, 1]$ is an *elementary intuitionistic fuzzy set* if A is single valued on $supp(A)$. An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is *elementary* if each $\eta_i \in A$ and R are elementary otherwise, it is called *nonelementary*.

Proposition 2.1 *Intuitionistic fuzzy graphs are special cases of the intuitionistic fuzzy hypergraphs.*

An *intuitionistic fuzzy multigraph* is a multivalued symmetric mapping $D = (\mu_D, \nu_D) : V \times V \to [0, 1]$. An intuitionistic fuzzy multigraph can be considered to be the "disjoint union" or "disjoint sum" of a collection of simple intuitionistic fuzzy graphs, as is done with crisp multigraphs. The same holds for multidigraphs. Therefore, these structures can be considered as "disjoint unions" or "disjoint sums" of intuitionistic fuzzy hypergraphs.

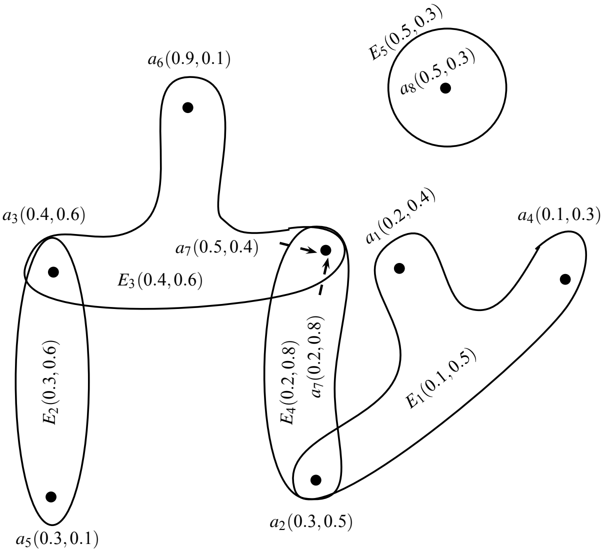


Fig. 2.4 Intuitionistic fuzzy hypergraph

Fig. 2.5 Table of intuitionistic fuzzy subsets on X

$y \in X$	η_1	η_2
a_1	$(0.5, 0.4)$	$(0, 1)$
a_2	$(0.6, 0.3)$	$(0, 1)$
a_3	$(0.4, 0.6)$	$(0, 1)$
a_4	$(0.3, 0.5)$	$(0, 1)$
a_5	$(0, 1)$	$(0.6, 0.3)$
a_6	$(0, 1)$	$(0.4, 0.3)$
a_7	$(0, 1)$	$(0.2, 0.4)$
a_8	$(0, 1)$	$(0.4, 0.3)$
a_9	$(0.5, 0.5)$	$(0.5, 0.5)$

Definition 2.9 An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is called *simple* if every $\eta_i, \eta_j \in S$, $\eta_i \subseteq \eta_j$ implies that $\eta_i = \eta_j$.

An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is called *support simple* if every $\eta_i, \eta_j \in S$, $\eta_i \subseteq \eta_j$, and $supp(\eta_i) = supp(\eta_j)$ imply that $\eta_i = \eta_j$.

An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is called *support simple* if every $\eta_i, \eta_j \in S$, $\eta_i \subseteq \eta_j$, and $supp(\eta_i) = supp(\eta_j)$ imply that $\eta_i = \eta_j$.

$\mathscr{H} = (S, R)$ is called *strongly support simple* if every $\eta_i, \eta_j \in S$, $supp(\eta_i) = supp(\eta_j)$ imply that $\eta_i = \eta_j$.

Remark 2.1 Definition 2.9 reduces to familiar definitions in the special case where H is a crisp hypergraph. The definition of simple intuitionistic fuzzy hypergraph is identical to the definition of simple crisp hypergraph. A crisp hypergraph is support simple and strongly support simple if and only if it has no multiple edges. For intuitionistic fuzzy hypergraphs all three concepts imply no multiple edges. Any simple intuitionistic fuzzy hypergraph is support simple and every strongly support simple intuitionistic fuzzy hypergraph is support simple. Simple and strongly support simple are independent concepts in intuitionistic fuzziness.

Definition 2.10 Let $\mathscr{H} = (S, R)$ be an intuitionistic fuzzy hypergraph on X. For $\alpha, \beta \in [0, 1]$, $0 \le \alpha + \beta \le 1$, the (α, β)-*level hyperedge* of an intuitionistic fuzzy hyperedge η is defined as

$$\eta_{(\alpha,\beta)} = \{u \in X | \lambda_\eta(u) \ge \alpha, \tau_{\eta)}(u) \le \beta\}.$$

$\mathscr{H}_{(\alpha,\beta)} = (S_{(\alpha,\beta)}, R_{(\alpha,\beta)})$ is called an (α, β)-*level hypergraph* of $\mathscr{H}$ where, $S_{(\alpha,\beta)}$ is defined as $S_{(\alpha,\beta)} = \cup_{k=1}^{r} \eta_{k(\alpha,\beta)}$.

Definition 2.11 Let $\mathscr{H} = (S, R)$ be an intuitionistic fuzzy hypergraph. The sequence of order pairs $(\alpha_i, \beta_i) \in [0, 1] \times [0, 1]$, $0 \le \alpha_i + \beta_i \le 1$, $1 \le i \le n$, such that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$, $\beta_1 < \beta_2 < \cdots < \beta_n$ satisfying the properties

1. if $1 \ge \alpha > \alpha_1$ and $0 \le \beta < \beta_1$ then $R_{(\alpha,\beta)} = \emptyset$,
2. if $\alpha_{i+1} < \alpha \le \alpha_i$ and $\beta_i \le \beta < \beta_{i+1}$ then $R_{(\alpha,\beta)} = R_{(\alpha_i,\beta_i)}$,
3. $R_{(\alpha_i,\beta_i)} \sqsubset R_{(\alpha_{i+1},\beta_{i+1})}$,

is called *fundamental sequence* of $\mathscr{H}$, denoted by $f_s(\mathscr{H})$. The corresponding sequence of (α_i, β_i)-level hypergraphs $\mathscr{H}_{(\alpha_1,\beta_1)}, \mathscr{H}_{(\alpha_2,\beta_2)}, \ldots, \mathscr{H}_{(\alpha_n,\beta_n)}$ is called *core set* of $\mathscr{H}$, denoted by $C(\mathscr{H})$. The (α_n, β_n)-level hypergraph, $\mathscr{H}_{(\alpha_n,\beta_n)}$, is called *support level* of $\mathscr{H}$.

Definition 2.12 An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is called a *partial intuitionistic fuzzy hypergraph* of $\mathscr{H}' = (S', R')$ if following conditions are satisfied

1. $supp(S) \subseteq supp(S')$ and $supp(R) \subseteq supp(R')$,
2. if $supp(\eta_i) \in supp(S)$ and $supp(\eta_i^{'}) \in supp(S')$ such that $supp(\eta_i) = supp(\eta_i^{'})$ then $\eta_i = \eta_i^{'}$.

It is denoted by $\mathscr{H} \subseteq \mathscr{H}'$. An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is *ordered* if the core set $C(\mathscr{H}) = \{\mathscr{H}_{(\alpha_1,\beta_1)}, \mathscr{H}_{(\alpha_2,\beta_2)}, \ldots, \mathscr{H}_{(\alpha_n,\beta_n)}\}$ is ordered, that is, $\mathscr{H}_{(\alpha_1,\beta_1)} \subseteq \mathscr{H}_{(\alpha_2,\beta_2)} \subseteq \ldots \subseteq \mathscr{H}_{(\alpha_n,\beta_n)}$. $\mathscr{H}$ is *simply ordered* if $\mathscr{H}$ is ordered and whenever, $R' \subset R_{(\alpha_{i+1},\beta_{i+1})} \setminus R_{(\alpha_i,\beta_i)}$ then $R' \nsubseteq R_{(\alpha_i,\beta_i)}$.

Observation 2.1 *Let $\mathscr{H}$ be an elementary intuitionistic fuzzy hypergraph then $\mathscr{H}$ is ordered. If $\mathscr{H}$ is ordered intuitionistic fuzzy hypergraph and support level $\mathscr{H}_{(\alpha_n,\beta_n)}$ is simple then $\mathscr{H}$ is an elementary intuitionistic fuzzy hypergraph.*

Definition 2.13 Let $\mathscr{H} = (S, R)$ and $\mathscr{H}' = (S', R')$ be any two intuitionistic fuzzy hypergraphs on X and X', respectively, where $S = \{\eta_1, \eta_2, \ldots, \eta_r\}$ and $S' = \{\eta_1', \eta_2', \ldots, \eta_r'\}$. A *homomorphism* between H and H' is a mapping $\psi : X \to X'$ such that

1. $\wedge_{i=1}^{s}\lambda_{\eta_i}(y) \leq \wedge_{i=1}^{s}\lambda_{\eta_i'}(\psi(y))$,
2. $\vee_{i=1}^{s}\tau_{\eta_i}(y) \leq \vee_{i=1}^{s}\tau_{\eta_i'}(\psi(y))$, for all $y \in X$,
3. $\lambda_R(\{y_1, y_2, \ldots, y_s\}) \leq \lambda_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$,
4. $\tau_R(\{y_1, y_2, \ldots, y_s\}) \leq \tau_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$, for all $y_1, y_2, \ldots, y_s \in X$.

Definition 2.14 A *co-weak isomorphism* of two intuitionistic fuzzy hypergraphs $\mathscr{H}$ and $\mathscr{H}'$ is defined as a bijective homomorphism $\psi : X \to X'$ such that

1. $\lambda_R(\{y_1, y_2, \ldots, y_s\}) = \lambda_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$,
2. $\tau_R(\{y_1, y_2, \ldots, y_s\}) = \tau_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$, for all $y_1, y_2, \ldots, y_s \in X$.

Definition 2.15 A *weak isomorphism* of two intuitionistic fuzzy hypergraphs $\mathscr{H}$ and $\mathscr{H}'$ is defined as a bijective homomorphism $\psi : X \to X'$ such that

1. $\wedge_{i=1}^{s}\lambda_{\eta_i}(y) = \wedge_{i=1}^{s}\lambda_{\eta_i'}(\psi(y))$,
2. $\vee_{i=1}^{s}\tau_{\eta_i}(y) = \vee_{i=1}^{s}\tau_{\eta_i'}(\psi(y))$, for all $y \in X$.

Definition 2.16 An *isomorphism* of $\mathscr{H}$ and $\mathscr{H}'$ is a mapping $\psi : X \to X'$ such that

1. $\wedge_{i=1}^{s}\lambda_{\eta_i}(y) = \wedge_{i=1}^{s}\lambda_{\eta_i'}(\psi(y))$,
2. $\vee_{i=1}^{s}\tau_{\eta_i}(y) = \vee_{i=1}^{s}\tau_{\eta_i'}(\psi(y))$, for all $y \in X$,
3. $\lambda_R(\{y_1, y_2, \ldots, y_s\}) = \lambda_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$,
4. $\tau_R(\{y_1, y_2, \ldots, y_s\}) = \tau_{R'}(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$, for all $y_1, y_2, \ldots, y_s \in X$.

Example 2.3 Assume that $S = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ and $S' = \{\eta_1', \eta_2', \eta_3', \eta_4'\}$ are the families of intuitionistic fuzzy subsets on $X = \{a_1, a_2, \ldots, a_6\}$ and $X' = \{a_1', a_2', \ldots, a_6'\}$, respectively, as shown in Tables 2.2 and 2.3.

The intuitionistic fuzzy relations R and R' are defined as $R(\{a_1, a_3, a_4, a_6\}) = (0.1, 0.5)$, $R(\{a_1, a_2, a_3\}) = (0.2, 0.5)$, $D(\{a_3, a_4\}) = (0.5, 0.4)$, $R(\{a_4, a_5, a_6\}) = (0.1, 0.8)$ and $R'(\{a_1', a_2', a_3', a_6'\}) = (0.1, 0.5)$, $D'(\{a_1', a_3', a_4'\}) = (0.2, 0.5)$, $D'(\{a_1', a_2'\}) = (0.5, 0.4)$, $R'(\{a_2', a_5', a_6'\}) = (0.1, 0.8)$. The corresponding intuitionistic fuzzy hypergraphs are given in Figs. 2.7 and 2.8.

Table 2.2 Intuitionistic fuzzy subsets on X

$y \in X$	η_1	η_2	η_3	η_4
a_1	(0.2, 0.5)	(0.2, 0.5)	(0, 1)	(0, 1)
a_2	(0, 1)	(0.5, 0.4, 0.4)	(0, 1)	(0, 1)
a_3	(0.5, 0.4)	(0.5, 0.4)	(0.5, 0.4)	(0, 1)
a_4	(0.8, 0.1)	(0, 1)	(0.8, 0.2)	(0.8, 0.2)
a_5	(0, 0, 0, 0)	(0, 1)	(0, 1)	(0.1, 0.8)
a_6	(0.1, 0.2)	(0, 1)	(0, 1)	(0.1, 0.2)

Table 2.3 Intuitionistic fuzzy subsets on X'

$y' \in X'$	η'_1	η'_2	η'_3	η'_4
a'_1	(0.5, 0.4)	(0.5, 0.4)	(0.5, 0.4)	(0, 1)
a'_2	(0.8, 0.2)	(0, 1)	(0.8, 0.2)	(0.8, 0.2)
a'_3	(0.2, 0.5)	(0.2, 0.5)	(0, 1)	(0, 1)
a'_4	(0, 1)	(0.5, 0.4)	(0, 1)	(0, 1)
a'_5	(0, 1)	(0, 0, 1)	(0, 1)	(0.1, 0.8)
a'_6	(0.1, 0.2)	(0, 0, 1)	(0, 1)	(0.1, 0.2)

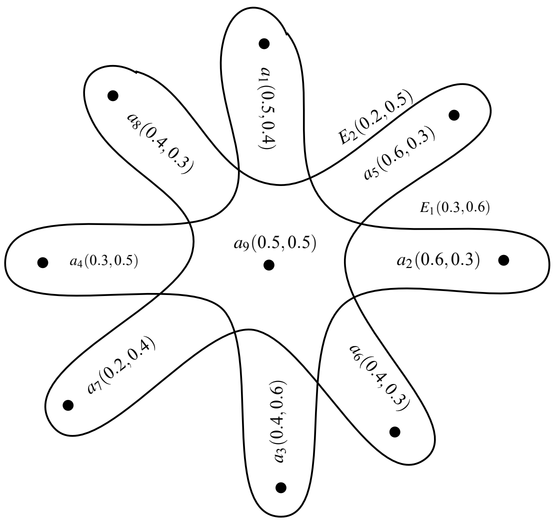


Fig. 2.6 Intuitionistic fuzzy hypergraph

Define a mapping $\psi : X \to X'$ by $\psi(a_1) = a'_3, \psi(a_2) = a'_4, \psi(a_4) = a'_2, \psi(a_3) = a'_1$, $\psi(a_5) = a'_5$, and $\psi(a_6) = a'_6$ then, it can be easily seen that

$$\eta_1(a_1) = (0.2, 0.5) = \eta'_1(a'_3) = \eta'_1(\psi(a_1)), \quad \eta_2(a_1) = (0.2, 0.5) = \eta'_2(a'_3) = \eta'_2(\psi(a_1)),$$
$$\eta_2(a_2) = (0.5, 0.4) = \eta'_2(a'_4) = \eta'_1(\psi(a_2)), \quad \eta_1(a_3) = (0.5, 0.4) = \eta'_1(a'_1) = \eta'_1(\psi(a_3)).$$

Similarly, $\eta_i(y) = \eta'_i(\psi(y))$, and $R(\{y_1, y_2, \ldots, y_s\}) = R'(\{\psi(y_1), \psi(y_2), \ldots, \psi(y_s)\})$, for all $y, y_i \in X$. Therefore, $\mathscr{H}$ and $\mathscr{H}'$ are isomorphic.

Definition 2.17 The *order and size* of an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ can be defined as

$$O(\mathscr{H}) = \sum_{y \in X} (\wedge_j \lambda_{\eta_j}(y), \vee_j \tau_{\eta_j}(y)), \qquad S(\mathscr{H}) = \sum_{E_j \subset X} (\lambda_R(E_j), \tau_R(E_j)).$$

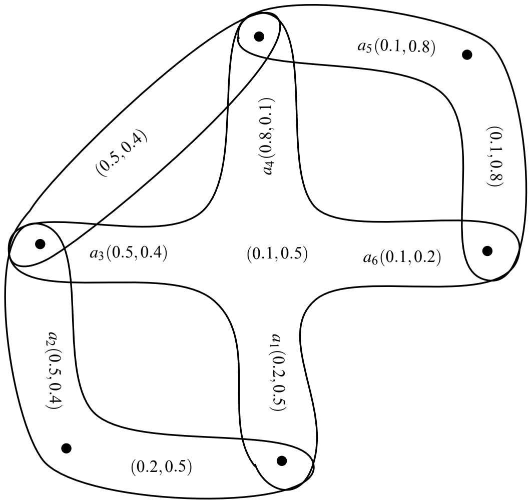


Fig. 2.7 Intuitionistic fuzzy hypergraph $\mathscr{H}$

Theorem 2.2 *For any two isomorphic intuitionistic fuzzy hypergraphs, the order and size are same.*

Proof Let $\mathscr{H}_1 = (S_1, R_1)$ and $\mathscr{H}_2 = (S_2, R_2)$ be any two intuitionistic fuzzy hypergraphs where, $S = \{\eta_1, \eta_2, \ldots, \eta_s\}$ and $S' = \{\eta'_1, \eta_2, \ldots, \eta'_s\}$ are the families of intuitionistic fuzzy subsets on X and X', respectively. If $\psi : X \to X'$ is an isomorphism between $\mathscr{H}$ and $\mathscr{H}'$ then

$$\begin{aligned} O(\mathscr{H}) &= \sum_{y \in X} (\wedge_i \lambda_{\eta_i}(y), \vee_i \tau_{\eta_i}(y)) \\ &= \sum_{y \in X} (\wedge_i \lambda_{\eta'_i}(\psi(y)), \vee_i \tau_{\eta'_i}(\psi(y))) = \sum_{y' \in X'} (\wedge_i \lambda_{\eta'_i}(y'), \vee_i \tau_{\eta'_i}(y')) = O(\mathscr{H}'). \\ S(\mathscr{H}) &= \sum_{E_i \subset X} R(E_i) = \sum_{E_i \subset X} R'(\psi(E_i)) = \sum_{E'_i \subset X'} R'(E'_i) = S(\mathscr{H}'). \end{aligned}$$

It completes the proof.

Remark 2.2 The converse of Theorem 2.2 does not hold, i.e., if the orders and sizes of two intuitionistic fuzzy hypergraphs are same then they may not be isomorphic as given in Example 2.4.

Example 2.4 Consider two intuitionistic fuzzy hypergraphs $\mathscr{H}_1 = (S_1, R_1)$ and $\mathscr{H}_2 = (S_2, R_2)$ given in Figs. 2.9 and 2.10. By Definition 2.17, $O(\mathscr{H}_1) = O(\mathscr{H}_2) = (1.8, 1.3)$ and $S(\mathscr{H}_1) = S(\mathscr{H}_2) = (0.4, 0.9)$. The orders and sizes of intuitionistic fuzzy hypergraphs $\mathscr{H}_1$ and $\mathscr{H}_2$ but $\mathscr{H}_1 \not\approx \mathscr{H}_2$.

Theorem 2.3 *The order of any two weak isomorphic intuitionistic fuzzy hypergraphs is same.*

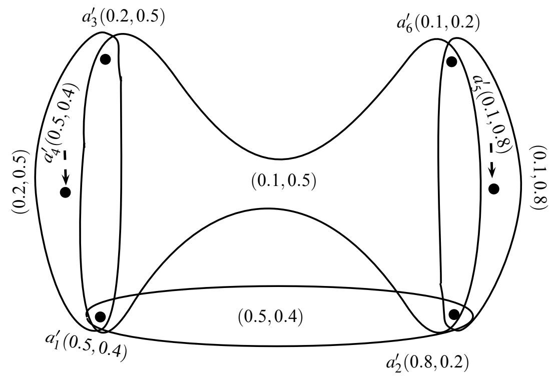


Fig. 2.8 Intuitionistic fuzzy hypergraph $\mathscr{H}'$

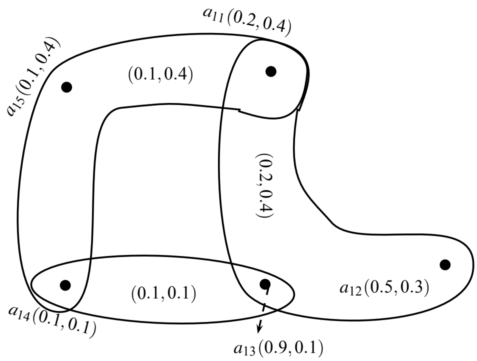


Fig. 2.9 Intuitionistic fuzzy hypergraph $\mathscr{H}_1$

The proof follows from Definition 2.15 and the proof of Theorem 2.2.

Theorem 2.4 *The size of any two co-weak isomorphic intuitionistic fuzzy hypergraphs is same.*

The proof follows from Definition 2.14 and the proof of Theorem 2.2.

Remark 2.3 The intuitionistic fuzzy hypergraphs of same order and size may not be weak isomorphic and co-weak isomorphic, respectively, i.e., the converse of Theorems 2.3 and 2.4 is not true in general as shown in Example 2.5.

Example 2.5 Let $\mathscr{H}_1 = (S_1, R_1)$ and $H_2 = (S_2, R_2)$ be two intuitionistic fuzzy hypergraphs as shown in Figs. 2.11 and 2.12 where $R_1 = \{\eta_{11}, \eta_{12}, \eta_{13}\}$ and $R_2 = \{\eta_{21}, \eta_{22}, \eta_{23}\}$. Clearly, $O(\mathscr{H}_1) = O(\mathscr{H}_2) = (1.6, 1.7)$ and $S(\mathscr{H}_1) = S(\mathscr{H}_2) = (0.5, 1.3)$. Define a mapping $\psi : X_1 \to X_2$ by $\psi(u_1) = u_2$, $\psi(v_1) = v_2$, $\psi(x_1) = x_2$, $\psi(y_1) = y_2$, $\psi(z_1) = z_2$. But $\lambda_{\eta_{12}}(u_1) = 0.5 \nleq 0.2 = \lambda_{\eta_{22}}(u_2)$ so, $\mathscr{H}_1$ and $\mathscr{H}_2$ are not weak isomorphic. Similarly, $\lambda_{R_1}(\{v_1, y_1\}) = 0.2 \nleq \lambda_{R_2}(\{\psi(v_1), \psi(y_1)\}) = 0$. Hence, $\mathscr{H}_1$ and $\mathscr{H}_2$ are not co-weak isomorphic.

Definition 2.18 For any intuitionistic fuzzy hypergraph, the *degree* of a vertex y is defined as, $\deg(y) = \sum_{y \in E_i \subseteq X} R(E_i)$.

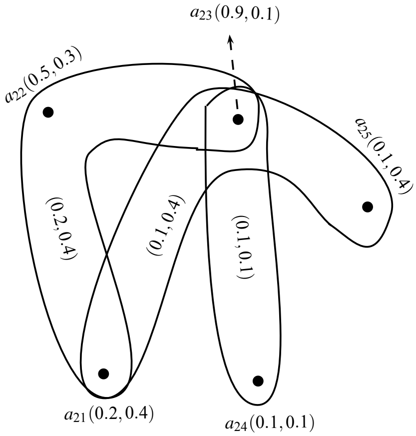


Fig. 2.10 Intuitionistic fuzzy hypergraph $\mathscr{H}_2$

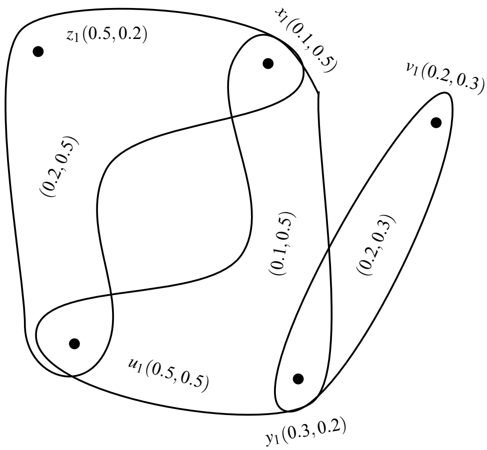


Fig. 2.11 Intuitionistic fuzzy hypergraph $\mathscr{H}_1$

Theorem 2.5 *The degree of vertices of isomorphic intuitionistic fuzzy hypergraphs is preserved.*

Proof Let $\psi : X \to X'$ be an isomorphism of intuitionistic fuzzy hypergraphs $\mathscr{H}$ and $\mathscr{H}'$ where, $X = \{y_1, y_2, \ldots, y_n\}$ and $X' = \{y_1', y_2', \ldots, y_n'\}$. Then Definition 2.18 implies that

$$\deg(y_i) = \sum\nolimits_{y_i \in E_i \subseteq X} R(E_i) = \sum\nolimits_{y_i \in E_i} R'(\phi(E_i)) = \deg(\psi(y_i)).$$

Remark 2.4 If the degrees of vertices of any two intuitionistic fuzzy hypergraphs is preserved then they may not be isomorphic as it is proved in Example 2.6.

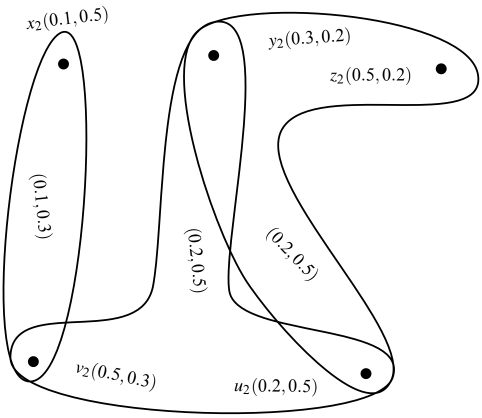


Fig. 2.12 Intuitionistic fuzzy hypergraph $\mathscr{H}_2$

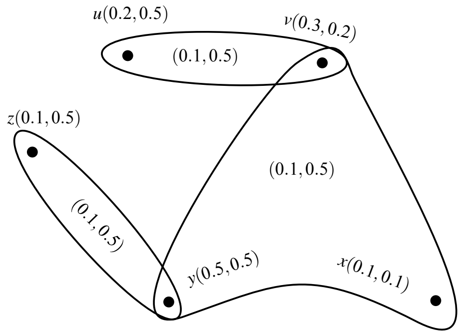


Fig. 2.13 Intuitionistic fuzzy hypergraph $\mathscr{H}$

Example 2.6 Consider two intuitionistic fuzzy hypergraphs $\mathscr{H} = (S, R)$ and $\mathscr{H}' = (S', R')$ as given in Figs. 2.13 and 2.14. Define a mapping $\psi : X \to X'$ by $\psi(u) = u'$, $\psi(v) = x'$, $\psi(x) = v'$, $\psi(y) = y'$, $\psi(z) = z'$. Routine calculations show that

$$\deg(u) = (0.2, 0.5) = \deg(\phi(u)), \quad \deg(v) = (0.3, 1.0) = \deg(\phi(v)).$$

The degree of all other vertices is also preserved but $R(\{x, y, v\}) \neq R'(\{\psi(x), \psi(y), \psi(v)\})$. Hence, $\mathscr{H}$ and $\mathscr{H}'$ are not isomorphic to each other.

Theorem 2.6 *The relation of isomorphism between intuitionistic fuzzy hypergraphs is an equivalence relation.*

Proof Assume that $\mathscr{H}_1 = (S_1, R_1)$, $\mathscr{H}_2 = (S_2, R_2)$ and $\mathscr{H}_3 = (S_3, R_3)$ are intuitionistic fuzzy hypergraphs on X_1, X_2 and X_3, respectively, where $S_1 = \{\eta_{11}, \eta_{12}, \ldots, \eta_{1s}\}$, $S_2 = \{\eta_{21}, \eta_{22}, \ldots, \eta_{2s}\}$ and $S_3 = \{\eta_{31}, \eta_{32}, \ldots, \eta_{3s}\}$.

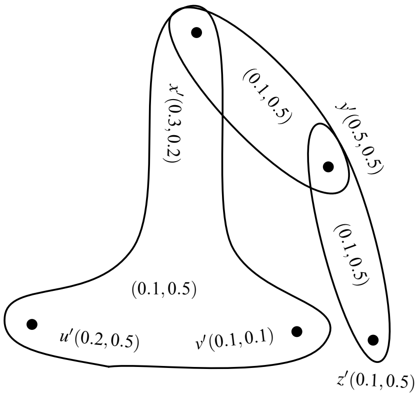


Fig. 2.14 Intuitionistic fuzzy hypergraph $\mathscr{H}'$

1. Reflexivity: Define an identity mapping $I : X_1 \to X_1$ by $I(y_1) = y_1$ for all $y_1 \in X_1$. Clearly I is bijective an
   $(\wedge_j \lambda_{\eta_{1j}}(y_1), \vee_j \tau_{\eta_{1j}}(y_1)) = (\wedge_j \lambda_{\eta_{1j}}(I(y_1)), \vee_j \tau_{\eta_{1j}}(I(y_1)))$ and $R_1(E_{1i}) = R_1(I(E_{1i}))$, for all $y_1 \in X_1$, $E_{1i} \subseteq X_1$.
   So, I is an isomorphism of an intuitionistic fuzzy hypergraph to itself.
2. Symmetry: Let $\psi : X_1 \to X_2$ be an isomorphism defined by $\psi(y_1) = y_2$. Since ψ is bijective therefore, the inverse bijective mapping $\psi^{-1} : X_2 \to X_1$ exists such that $\psi(y_2) = y_1$ for all $y_2 \in X_2$. Then

$$\begin{aligned}(\wedge_j \lambda_{\eta_{2j}}(y_2), \vee_j \tau_{\eta_{2j}}(y_2)) &= (\wedge_j \lambda_{\eta_{2j}}(\psi(y_1)), \vee_j \tau_{\eta_{2j}}(\psi(y_1))),\\ &= (\wedge_j \lambda_{\eta_{1j}}(y_1), \vee_j \tau_{\eta_{1j}}(y_1)),\\ &= (\wedge_j \lambda_{\eta_{1j}}(\psi^{-1}(y_2)), \vee_j \tau_{\eta_{1j}}(\psi^{-1}(y_2))).\end{aligned}$$

$$R_2(E_{2j}) = R_2(\psi(E_{1j})) = R_1(E_{1j}) = R_1(\psi^{-1}(E_{2j})), \quad E_{1j} \subseteq X_1, \ E_{2j} \subseteq X_2.$$

   Thus, ψ^{-1} is an isomorphism.
3. Assume that $\psi_1 : X_1 \to X_2$, $\psi_2 : X_2 \to X_3$ are isomorphisms of $\mathscr{H}_1$ onto $\mathscr{H}_2$ and $\mathscr{H}_2$ onto $\mathscr{H}_3$, respectively, such that $\psi_1(y_1) = y_2$ and $\psi_2(y_2) = y_3$. By Definition 2.16

$$\wedge_j \lambda_{\eta_{1j}}(y_1) = \wedge_j \lambda_{\eta_{2j}}(y_2) = \wedge_j \lambda_{\eta_{3j}}(\psi(y_2)) = \wedge_j \lambda_{\eta_{3j}}(\psi_2(\psi_1(y_1))) = \wedge_j \lambda_{\eta_{3j}}(\psi_2 \circ \psi_1(y_1)),$$

$$\vee_j \tau_{\eta_{1j}}(y_1) = \vee_j \tau_{\eta_{2j}}(y_2) = \vee_j \tau_{\eta_{3j}}(\psi(y_2)) = \vee_j \tau_{\eta_{3j}}(\psi_2(\psi_1(y_1))) = \vee_j \tau_{\eta_{3j}}(\psi_2 \circ \psi_1(y_1)),$$

$$R_1(E_{1j}) = R_2(E_{2j}) = R_3(\psi_2(E_{2j})) = R_3(\psi_2(\psi_1(E_{1j}))) = R_3(\psi_2 \circ \psi_1(E_{1j})).$$

   where, $E_{1j} \subseteq X_1$, $E_{2j} \subseteq X_2$ and $E_{3j} \subseteq X_3$. Clearly, $\psi_2 \circ \psi_1$ is an isomorphism from $\mathscr{H}_1$ onto $\mathscr{H}_3$. Hence, isomorphism of intuitionistic fuzzy hypergraphs is an equivalent relation.

Theorem 2.7 *The relation of weak isomorphism between intuitionistic fuzzy hypergraphs is a partial order relation.*

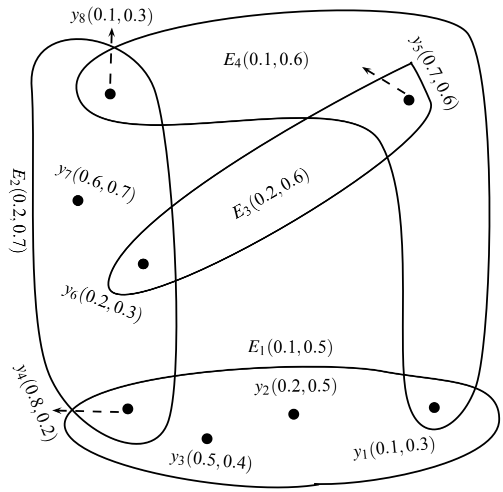


Fig. 2.15 Intuitionistic fuzzy hypergraph $\mathscr{H}$

The proof follows from Definition 2.15 and the proof of Theorem 2.6.

Definition 2.19 An *intuitionistic fuzzy hyperpath* $\mathscr{P}$ of length m in an intuitionistic fuzzy hypergraph is defined as a sequence $y_1, E_1, y_2, E_2, \ldots, y_m, E_m, y_{m+1}$ of distinct vertices y_i's and hyperedges E_i's such that

1. $\lambda_R(E_i) > 0, \quad$ for each $1 \leq i \leq m$,
2. $y_i, y_{i+1} \in E_i, \quad$ for each $1 \leq i \leq m$.

If $y_m = y_{m+1}$ then, $\mathscr{P}$ is called an *intuitionistic fuzzy hypercycle*.

Example 2.7 Let $\mathscr{H} = (S, R)$ be an intuitionistic fuzzy hypergraph as shown in Fig. 2.15. The sequence $y_5, E_3, y_6, E_2, y_4, E_1, y_1, E_4, y_8$ is an intuitionistic fuzzy hyperpath and y_8, E_2, y_4, E_1, y_8 is an intuitionistic fuzzy hypercycle.

Definition 2.20 An intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ on a non-empty set X is *connected* if every two distinct vertices in $\mathscr{H}$ are joined by an intuitionistic fuzzy hyperpath.

Definition 2.21 Let y and z be two distinct vertices of an intuitionistic fuzzy hypergraph $\mathscr{H}$ which are joined by an intuitionistic fuzzy hyperpath $y = y_1, E_1, y_2, E_2, \ldots, y_p, E_p, y_{p+1} = z$ of length p. The *strength* of intuitionistic fuzzy hyperpath $y - z$ is denoted by $S^p(y, z) = (\lambda_{S^p}(y, z), \tau_{S^p}(y, z))$ and defined as,

$$\lambda_{S^p}(y, z) = \lambda_R(E_1) \wedge \lambda_R(E_2) \wedge \ldots \wedge \lambda_R(E_p),$$
$$\tau_{S^p}(y, z) = \tau_R(E_1) \vee \tau_R(E_2) \vee \ldots \vee \tau_R(E_p), u \in E_1, \ v \in E_p.$$

The *strength of connectedness* between y and z is denoted by $S^\infty(y, z) = (\lambda_{S^\infty}(y, z), \tau_{S^\infty}(y, z))$ and defined as

$$\lambda_{S^\infty}(y,z) = \sup_p\{\lambda_{S^p}(y,z)|p = 1,2,\ldots\}, \quad \tau_{S^\infty}(y,z) = \inf_p\{\tau_{S^p}(y,z)|p = 1,2,\ldots\}.$$

Theorem 2.8 *An intuitionistic fuzzy hypergraph $\mathscr{H}$ is connected if and only if $\lambda_{S^\infty}(y,z) > 0$, for all $y, z \in X$.*

Proof Suppose that $\mathscr{H}$ is a connected intuitionistic fuzzy hypergraph then for any two distinct vertices y and z, there exists an intuitionistic fuzzy hyperpath $y - z$ such that

$$\lambda_{S^p}(y,z) > 0 \Rightarrow \sup_p\{\lambda_{S^p}(y,z)|p = 1,2,\ldots\} > 0 \Rightarrow \lambda_{S^\infty}(y,z) > 0.$$

We can prove the converse part on the same lines as above.

Definition 2.22 A *strong intuitionistic fuzzy hypergraph* on a non-empty set X is an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ such that for all $E_i = \{y_1, y_2, \ldots, y_r\} \in E$,

$$R(E_i) = (\min_{j=1}^{r}[\wedge_i \lambda_{\eta_i}(y_j)], \max_{j=1}^{r}[\vee_i \tau_{\eta_i}(y_j)]).$$

Definition 2.23 A *complete intuitionistic fuzzy hypergraph* on a non-empty set X is an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ such that for all $y_1, y_2, \ldots, y_r \in X$,

$$R(E_i) = (\min_{j=1}^{r}[\wedge_i \lambda_{\eta_i}(y_j)], \max_{j=1}^{r}[\vee_i \tau_{\eta_i}(y_j)]).$$

Theorem 2.9 *For any two intuitionistic fuzzy hypergraphs $\mathscr{H}_1$ and $\mathscr{H}_2$, $\mathscr{H}_1$ is connected if and only if $\mathscr{H}_2$ is connected.*

Proof Let $E_1 = \{E_{11}, E_{21}, \ldots, E_{r1}\}$ and $E_2 = \{E_{12}, E_{22}, \ldots, E_{r2}\}$ be the families of hyperedges if intuitionistic fuzzy hypergraphs $\mathscr{H}_1 = (S_1, R_1)$ and $\mathscr{H}_2 = (S_2, R_2)$, respectively. Assume that $\psi : X_1 \to X_2$ is an isomorphism of $\mathscr{H}_1$ onto $\mathscr{H}_2$ and that $\mathscr{H}_1$ is connected then

$$\begin{aligned} 0 < \lambda_{S_1^\infty}(y_1, z_1) &= \sup_p\{\wedge_{k=1}^{p}\lambda_{R_1}(E_{k1}), p = 1,2,\ldots\} \\ &= \sup_p\{\wedge_{k=1}^{p}\lambda_{R_2}(\psi(E_{k1})), p = 1,2,\ldots\} \\ &= \lambda_{S_2^\infty}(\phi(y_1), \phi(z_1)). \end{aligned}$$

It follows that $\mathscr{H}_2$ is connected. The converse part can be proved similarly.

Theorem 2.10 *For any two intuitionistic fuzzy hypergraphs $\mathscr{H}_1$ and $\mathscr{H}_2$, $\mathscr{H}_1$ is strong if and only if $\mathscr{H}_2$ is strong.*

Proof Let $\mathscr{H}_1 = (S_1, R_1)$ and $\mathscr{H}_2 = (S_2, R_2)$ be the intuitionistic fuzzy hypergraphs as defined in Theorem 2.9. Assume that $\mathscr{H}_1$ is strong then

$$R_2(E_{i2}) = R_2(\psi(E_{i1})) = R_1(E_{i1}) = (\min_{j=1}^{r}[\wedge_j \lambda_{\eta_{j1}}(y_{j1})], \max_{j=1}^{r}[\vee_j \tau_{\eta_{j1}}(y_{j1})])$$
$$= (\min_{j=1}^{r}[\wedge_j \lambda_{\eta_{j2}}(\psi(y_{j1})], \max_{j=1}^{r}[\vee_j \tau_{\eta_{j2}}(\psi(y_{j1}))]). \tag{2.2}$$

Equation 2.2 clearly shows that $\mathscr{H}_2$ is strong. Similarly, the converse part.

Definition 2.24 An *intuitionistic fuzzy line graph* of an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is a pair $L(\mathscr{H}) = (S_l, R_l)$ where, $S_l = R$ and two vertices E_i and E_k in $L(\mathscr{H})$ are connected by an edge if $|supp(\eta_i) \cap supp(\eta_k)| \geq 1$ where, $R(E_i) = \eta_i$ and $R(E_k) = \eta_k$. The membership values of sets of vertices and edges are defined as

1. $S_l(E_i) = R(E_i)$,
2. $R_l(E_i E_k) = (\lambda_R(E_i) \wedge \lambda_R(E_k), \tau_R(E_i) \vee \tau_R(E_k))$.

The method for the construction of an intuitionistic fuzzy line graph from an intuitionistic fuzzy hypergraph is explained in Algorithm 2.2.1.

Algorithm 2.2.1 The construction of an intuitionistic fuzzy line graph

1. Input the number of edges n of an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$.
2. Input the degrees of membership of the hyperedges $E_1, E_2, \ldots, E_s$.
3. Construct an intuitionistic fuzzy graph $L(\mathscr{H}) = (S_l, R_l)$ whose vertices are the s hyperedges $E_1, E_2, \ldots, E_s$ such that $S_l(E_i) = R(E_i)$.
4. If $|supp(\eta_i) \cap supp(\eta_k)| \geq 1$ then, draw an edge between E_i and E_k and $R_l(E_i E_k) = (\lambda_R(E_i) \wedge \lambda_R(E_k), \tau_R(E_i) \vee \tau_R(E_k))$.

Example 2.8 An example of an intuitionistic fuzzy hypergraph is shown in Fig. 2.16. The intuitionistic fuzzy line graph is constructed using Algorithm 2.2.1 and represented with dashed lines.

Definition 2.25 An intuitionistic fuzzy hypergraph is known as *linear intuitionistic fuzzy hypergraph* if

$$supp(\eta_i) \subseteq supp(\eta_k) \Rightarrow i = k \quad \text{and} \quad |supp(\eta_i) \cap supp(\eta_k)| \leq 1.$$

Theorem 2.11 *The intuitionistic fuzzy line graph $L(\mathscr{H})$ of an an intuitionistic fuzzy hypergraph $\mathscr{H}$ is connected if and only if $\mathscr{H}$ is connected.*

Proof Let $\mathscr{H} = (S, R)$ be a connected intuitionistic fuzzy hypergraph and $L(\mathscr{H}) = (S_l, R_l)$. Assume that E_i and E_k are two vertices in $L(\mathscr{H})$ such that $y_i \in E_i$, $y_k \in E_k$ and $y_i \neq y_k$. By Definition 2.20, there exists an intuitionistic fuzzy hyperpath $y_i, E_i, y_{i+1}, E_{i+1}, \ldots, E_k, y_k$ between y_i and y_k. Using Definition 2.21, we have

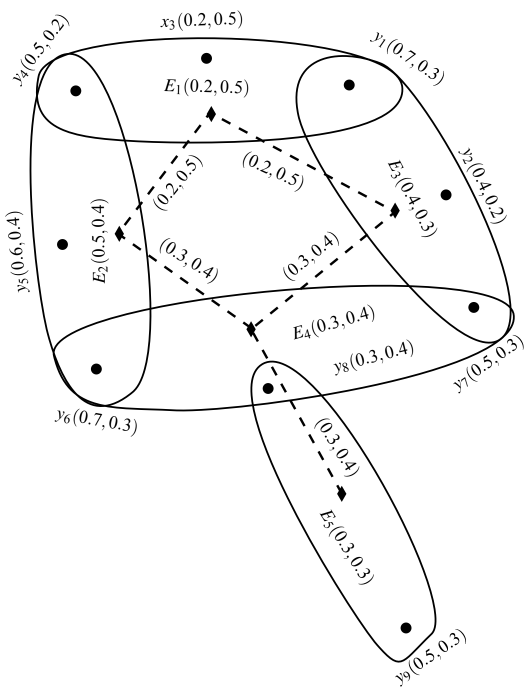


Fig. 2.16 Intuitionistic fuzzy line graph $L(\mathscr{H})$

$$\begin{aligned}
\lambda_{S^\infty}(E_i, E_k) &= \sup\{\lambda_{S^p}(E_i, E_k) | p = 1, 2, \ldots\} \\
&= \sup\{\lambda_{R_l}(E_i E_{i+1}) \wedge \lambda_{R_l}(E_{i+1} E_{i+2}) \wedge \ldots \wedge \lambda_{R_l}(E_{k-1} E_k) | p = 1, 2, \ldots\} \\
&= \sup\{\lambda_R(E_i) \wedge \lambda_R(E_{i+1}) \wedge \ldots \wedge \lambda_R(E_j) | p = 1, 2, \ldots\} \\
&= \sup\{\lambda_{S^p}(y_i, y_k) | p = 1, 2, \ldots\} = \lambda_{S^\infty}(y_i, y_k) > 0.
\end{aligned}$$

Hence, $L(\mathscr{H})$ is connected. Similarly, if $L(\mathscr{H})$ is connected then it can be easily proved that $\mathscr{H}$ is connected.

Definition 2.26 The 2-*section* of an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ is denoted by $[\mathscr{H}]_2 = (S, U)$ and defined as an intuitionistic fuzzy graph whose set of vertices is same as $\mathscr{H}$ and U is an intuitionistic fuzzy set on $\{y_i y_k | y_i, y_k \in E_p, p = 1, 2, \ldots\}$, i.e., any two vertices of the same hyperedge are joined by an edge and

$$U(y_i y_k) = (\min\{\wedge_p \lambda_{\eta_p}(y_i), \wedge_p \lambda_{\eta_p}(y_k)\}, \max\{\vee_p \tau_{\eta_p}(y_i), \vee_p \tau_{\eta_p}(y_k)\}).$$

Example 2.9 An example of a 2-section of an intuitionistic fuzzy hypergraph is shown in Fig. 2.17. The 2-section of $\mathscr{H}$ is represented with dashed lines.

Definition 2.27 Let $\mathscr{H} = (S, R)$ be an intuitionistic fuzzy hypergraph on X then the *dual* of $\mathscr{H}$ is denoted by $\mathscr{H}^D = (S^D, R^D)$ and it is defined as

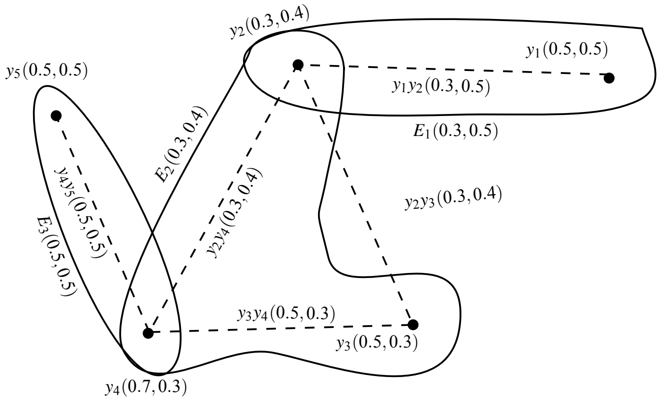


Fig. 2.17 2-section of an intuitionistic fuzzy hypergraph

1. $S^D = R$ is the intuitionistic fuzzy set of vertices of $\mathscr{H}^D$.
2. If $|X| = n$ then, R^D is an intuitionistic fuzzy set on the family of hypeedges $\{X_1, X_2, \ldots, X_n\}$ of $\mathscr{H}^D$ such that $X_k = \{E_i | y_k \in E_i, E_i$ is a hyperedge of $\mathscr{H}\}$. That is, X_k is the collection of those hyperedges which have a common vertex y_k and
$$R^D(X_k) = (\min\{\lambda_R(E_i) | y_k \in E_i\}, \max\{\tau_R(E_i) | y_k \in E_i\}).$$

The method for the construction of dual of an intuitionistic fuzzy hypergraph is presented in Algorithm 2.2.2.

Algorithm 2.2.2 The construction of dual of an intuitionistic fuzzy hypergraph

1. Input $\{y_1, y_2, \ldots, y_n\}$, the set of vertices and $\{E_1, E_2, \ldots, E_r\}$, the set of hyperedges of $\mathscr{H}$.
2. Construct an intuitionistic fuzzy set of vertices of $\mathscr{H}^D$ by defining $S^D = R$.
3. Draw a mapping $g : X \to E$ between sets of vertices and hyperedges. That is, if a vertex y_i belongs to $E_k, E_{k+1}, \ldots, E_r$ then map y_i to $E_k, E_{k+1}, \ldots, E_r$ as drawn in Fig. 2.18.
4. Construct a new family of hyperedges $\{X_1, X_2, \ldots, X_n\}$ of $\mathscr{H}^D$ such that $X_i = \{E_k | g(y_i) = E_k\}$ and $R^D(X_i) = (\min\{\lambda_R(E_k) | g(y_i) = E_k\}, \max\{\tau_R(E_k) | g(y_i) = E_k\})$.

Example 2.10 An intuitionistic fuzzy hypergraph $\mathscr{H}$ on $X = \{y_1, y_2, y_3, y_4, y_5\}$ with a set of hyperedges $E = \{E_1, E_2, E_3, E_4, E_5\}$ is shown in Fig. 2.19. The dual of $\mathscr{H}$ is represented by dashed lines with vertices E_1, E_2, E_3, E_4, E_5 and a family of hyperedges $\{X_1, X_2 = X_3, X_4, X_5\}$.

Theorem 2.12 *For any intuitionistic fuzzy hypergraph $\mathscr{H}$, $[\mathscr{H}^D]_2 = L(\mathscr{H})$.*

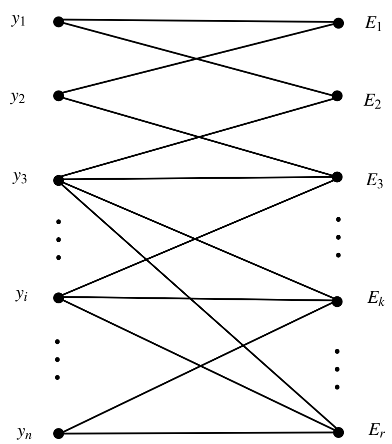


Fig. 2.18 Mapping between sets of vertices and hyperedges

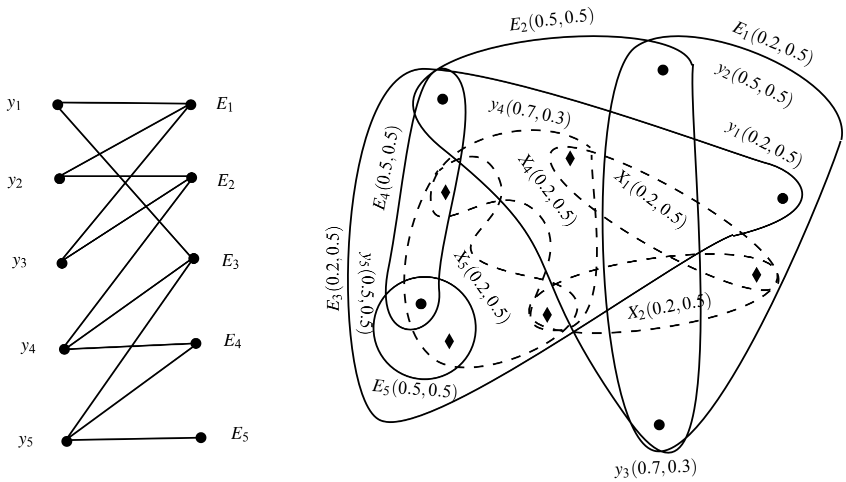


Fig. 2.19 Dual of an intuitionistic fuzzy hypergraph $\mathscr{H}$

Proof Let $\mathscr{H} = (S, R)$ be an intuitionistic fuzzy hypergraph on $X = \{y_1, y_2, \ldots, y_n\}$ with a family of hyperedges $\{E_1, E_2, \ldots, E_s\}$. Assume that $L(\mathscr{H}) = (S_l, R_l)$, $\mathscr{H}^D = (S^D, R^D)$ and $[\mathscr{H}^D]_2 = (S^D, U)$. The 2-section $[\mathscr{H}^D]_2$ has the intuitionistic fuzzy vertex set R which is also an intuitionistic fuzzy vertex set of $L(\mathscr{H})$. Suppose $\{X_1, X_2, \ldots, X_n\}$ is the family of hyperedges of $\mathscr{H}^D$. Clearly $\{E_i E_k | E_i, E_k \in X_i\}$ is the set of edges of $[\mathscr{H}^D]_2$ which the set of edges of $L(\mathscr{H})$. It remains to show that $R_l(E_i E_k) = U(E_i E_k)$.

$$\begin{aligned}R_l(E_iE_k) &= (\lambda_R(E_i)\wedge\lambda_R(E_k), \tau_R(E_i)\vee\tau_R(E_k))\\ &= (\lambda_{S^D}(E_i)\wedge\lambda_{S^D}(E_k), \tau_{S^D}(E_i)\vee\tau_{S^D}(E_k))\\ &= U(E_iE_k).\end{aligned}$$

Theorem 2.13 *For any two isomorphic intuitionistic fuzzy hypergraphs $\mathscr{H}_1$ and $\mathscr{H}_2$, if $\mathscr{H}_1 \simeq \mathscr{H}_2$ then $\mathscr{H}_1^D \simeq \mathscr{H}_2^D$.*

Proof Let $\mathscr{H}_1 = (S_1, R_1)$ and $\mathscr{H}_2 = (S_2, R_2)$ be two isomorphic intuitionistic fuzzy hypergraphs on $X_1 = \{y_{11}, y_{12}, \ldots, y_{1n}\}$ and $X_2 = \{y_{21}, y_{22}, \ldots, y_{2n}\}$, respectively. Take $\psi : X_1 \to X_2$ as an isomorphism of $\mathscr{H}_1$ onto $\mathscr{H}_2$. Let $\{X_{11}, X_{12}, \ldots, X_{1n}\}$ and $\{X_{21}, X_{22}, \ldots, X_{2n}\}$ be the families of hyperedges of $\mathscr{H}_1^D$ and $\mathscr{H}_2^D$. Also, let E_1 and E_2 be the families of hyperedges of $\mathscr{H}_1$ and $\mathscr{H}_2$ then define a mapping $\phi : E_1 \to E_2$. It is to be shown that ψ is an isomorphism. For each $E_{1k} \in E_1$ and $E_{2k} \in E_2$

$$S_1^D(E_{1k}) = R_1(E_{1k}) = R_2(\psi(E_{1k})) = R_2(E_{2k}) = S_2^D(\phi(E_{1k})).$$

$$\begin{aligned}R_1^D(X_{1k}) &= (\lambda_{R_1}(E_{1k})\wedge\lambda_{R_1}(E_{1\overline{k+1}})\wedge\ldots\wedge\lambda_{R_1}(E_{1l}), \tau_{R_1}(E_{1k})\vee\tau_{R_1}(E_{1\overline{k+1}})\vee\ldots\vee\tau_{R_1}(E_{1l})),\\ &= (\lambda_{R_2}(\phi(E_{1k}))\wedge\lambda_{R_2}(\phi(E_{1\overline{k+1}}))\wedge\ldots\wedge\lambda_{R_2}(\phi(E_{1l})),\\ &\qquad \tau_{R_2}(\phi(E_{1k}))\vee\tau_{R_2}(\phi(E_{1\overline{k+1}}))\vee\ldots\vee\tau_{R_2}(\phi(E_{1l}))),\\ &= R_2^D(X_{2k}) = R_2^D(\psi(X_{1k})).\end{aligned}$$

Hence, $\mathscr{H}_1^D \simeq \mathscr{H}_2^D$.

Theorem 2.14 *The dual $\mathscr{H}^D$ of a linear intuitionistic fuzzy hypergraph $\mathscr{H}$ is also linear.*

Proof Let $\mathscr{H} = (S, R)$ and $\mathscr{H}^D = (S^D, R^D)$. On contrary, assume that $\mathscr{H}^D$ is not a linear intuitionistic fuzzy hypergraph then there exist X_i and X_k such that $|supp(\xi_i) \cap supp(\xi_k)| = 2$ where, $R^D(X_i) = \xi_i$ and $R^D(X_k) = \xi_k$. Assume that $supp(\xi_i) \cap supp(\xi_k) = \{E_t, E_s\}$. The definition of duality of $\mathscr{H}^D$ follows that there exist $y_i, y_k \in X$ such that $y_i, y_k \in E_t$ and $y_i, y_k \in E_s$. A contradiction to the given statement that $\mathscr{H}$ is linear. Hence, $\mathscr{H}^D$ is a linear intuitionistic fuzzy hypergraph.

2.3 Applications of Intuitionistic Fuzzy Hypergraphs

Graph theory has proved very useful for solving combinatorial problems of computer science and communication networks. To expand the origin of these applications, graphs were further extended to hypergraphs to model complex systems which arise in operation research, networking, and computer science. In some situations, the given data is fuzzy in nature and contains information about the existence and somewhere non-existence of uncertainty. The intuitionistic fuzzy hypergraphs can be used to formulate these concepts of existence and non-existence of uncertainty in a more generalized form as hypergraphs and intuitionistic fuzzy graphs can do. We now discuss some applications of intuitionistic fuzzy hypergraphs in social networking, chemistry, and planet surface networks.

2.3.1 The Intersecting Communities in Social Network

Nowadays, social networks have become the widely studied areas of research. Social networks are used to represent the lower and higher level interconnections among several communities belonging to social, as well as biological networks. People in society are connected to multiple areas which make them the part of various communities such as companies, universities, colleges, and offices etc. Consider the problem of grouping of authors according to their field of interest. An author can be different from the other regarding his/her critical writing. We present an intuitionistic fuzzy hypergraph $\mathscr{H} = (B, A)$ in which the vertices are authors and membership value of each author represents the degree of good writing and nonmembership value represents that the author's critical writing is not so good. Each hyperedge is the collection of those authors who belong to the same field of interest. The membership value of each hyperedge depicts the common ability of good critical writing of authors and nonmembership value shows the bad writing ability. An example is shown in Fig. 2.20.

The intuitionistic fuzzy hypergraph model can be used for the selection of authors having best writing ability in each field. The method for the selection of authors with best writing is given in Algorithm 2.3.1.

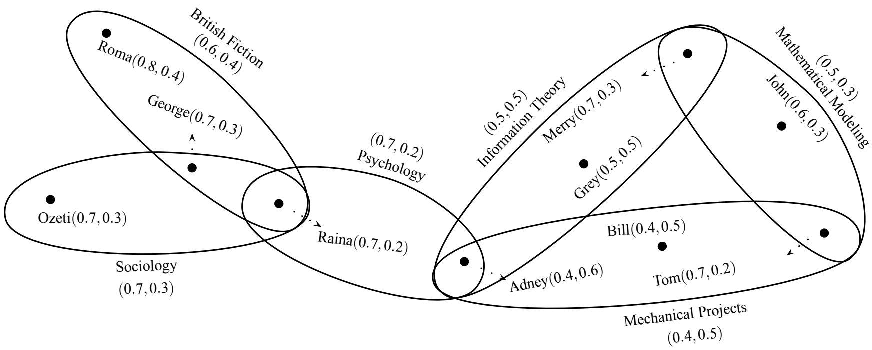


Fig. 2.20 Intuitionistic fuzzy social hypergraph

Algorithm 2.3.1 Selection of authors in an intuitionistic fuzzy social network model

1. Input the set of vertices (authors) $y_1, y_2, \ldots, y_n$.
2. Input the intuitionistic fuzzy set S of vertices such that $S(y_i) = (\lambda_i, \tau_i), 1 \leq i \leq n$.
3. Input the adjacency matrix $\xi = [(\lambda_{ij}, \tau_{ij})]_{n \times n}$ of vertices.
4. **do** i from $1 \to n$
5. $\quad C_i = 0$
6. **do** j from $1 \to n$
7. $\quad \pi_{ij} = 1 - \lambda_{ij} - \tau_{ij}$
8. $\quad S_{ij} = \sqrt{\lambda_{ij}^2 + \pi_{ij}^2 + (1 - \tau_{ij})^2}$
9. $\quad C_i = C_i + S_{ij}$
10. **end do**
11. $\pi_i = 1 - \lambda_i - \tau_i$
12. $C_i = C_i + \sqrt{\lambda_i^2 + \pi_i^2 + (1 - \tau_i)^2}$
13. **end do**
14. Using Algorithm 2.2.1, construct the adjacency matrix ξ_l of intuitionistic fuzzy line graph $L(\mathscr{H})$ of intuitionistic fuzzy hypergraph $\mathscr{H}$ whose adjacency matrix is ξ.
15. Compute the score and choice values of all vertices (fields) in $L(\mathscr{H})$ using steps 4–13.
16. Choose a vertex (field) E in $L(\mathscr{H})$ with maximum choice value.
17. Select a vertex of hyperedge E in $\mathscr{H}$ with maximum choice value which is the best option.

The adjacency matrix of Fig. 2.20 is given in Table 2.4. The score values of intuitionistic fuzzy hypergraph are computed using score function $S_{ij} = \sqrt{\lambda_{ij}^2 + \pi_{ij}^2 + (1 - \tau_{ij})^2}$ and the choice values $C_i = \sum_j S_{ij} + \sqrt{\lambda_i^2 + \pi_i^2 + (1 - \tau_i)^2}$ are given in Table 2.5 where, for any two vertices $y_i, y_j \in E$ (E is a hyperedge), $(\lambda_{ij}, \tau_{ij}) = (\lambda_A(E), \tau_A(E))$, $(\lambda_i, \tau_i) = (\lambda_B(y_i), \tau_B y_i)$.

The intuitionistic fuzzy line graph of Fig. 2.20 is shown in Fig. 2.21. The adjacency matrix of Fig. 2.21 in given in Table 2.6. The score and choice values of Fig. 2.21 are calculated in Table 2.7.

The choice values in Table 2.7 show that the company can gain maximum benefit if it publishes articles and books on Psychology. There are three authors of Psychology, George, Raina, and Adney. The choice values of Table 2.7 show that Adney is the best author of Psychology. The best authors for all the fields are given in Table 2.8 which clearly shows that Raina, Adney, and Merry are the suitable options for all fields.

Table 2.4 Adjacency matrix

ξ	Roma	George	Ozeti	Raina	Adney	Grey	Merry	John	Bill	Tom
Roma	(0, 1)	(0.6, 0.4)	(0, 1)	(0.6, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
George	(0.6, 0.4)	(0, 1)	(0.7, 0.3)	(0.6, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Ozeti	(0, 1)	(0.7, 0.3)	(0, 1)	(0.7, 0.3)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Raina	(0.6, 0.4)	(0.6, 0.4)	(0.7, 0.3)	(0, 1)	(0.7, 0.2)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Adney	(0, 1)	(0, 1)	(0, 1)	(0.7, 0.2)	(0, 1)	(0.5, 0.5)	(0.5, 0.5)	(0, 1)	(0.4, 0.5)	(0.4, 0.5)
Grey	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.5)	(0, 1)	(0.5, 0.5)	(0, 1)	(0, 1)	(0, 1)
Merry	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.5)	(0.5, 0.5)	(0, 1)	(0.5, 0.3)	(0, 1)	(0.5, 0.3)
John	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.3)	(0, 1)	(0, 1)	(0.5, 0.3)
Bill	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.4, 0.5)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.4, 0.5)
Tom	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.4, 0.5)	(0, 1)	(0.5, 0.3)	(0.5, 0.3)	(0.4, 0.5)	(0, 1)

Table 2.5 Score and choice values

S_{ij}	Roma	George	Ozeti	Raina	Adney	Grey	Merry	John	Bill	Tom	C_i
Roma	0	0.8485	0	0.8485	0	0	0	0	0	0	2.8284
George	0.8485	0	0.9899	0.8485	0	0	0	0	0	0	3.6768
Ozeti	0	0.9899	0	0.9899	0	0	0	0	0	0	2.9697
Raina	0.8485	0.8485	0.9899	0	1.0677	0	0	0	0	0	4.8223
Adney	0	0	0	1.0677	0	0.7071	0.7071	0	0.6481	0.6481	4.3438
Grey	0	0	0	0	0.7071	0	0.7071	0	0	0	2.1213
Merry	0	0	0	0	0.7071	0.7071	0	0.8832	0	0.8832	4.1705
John	0	0	0	0	0	0	0.8832	0	0	0.8832	2.6938
Bill	0	0	0	0	0.6481	0	0	0	0	0.6481	1.9443
Tom	0	0	0	0	0.6481	0	0.8832	0.8832	0.6481	0	4.1303

2.3.2 Planet Surface Networks

There are various types of satellites in space for network communication and exploration of planets. Hypergraphs are a key tool to model such communication links among surface networks and satellites. There exist disturbance and uncertainty in planet surface communication due to climate change and electrical interference of

Table 2.6 Adjacency matrix of intuitionistic fuzzy line graph

ξ_l	British fiction	Sociology	Psychology	Information theory	Mechanical projects	Mathematical modeling
British Fiction	(0, 1)	(0.6,0.3)	(0.6,0.2)	(0, 1)	(0, 1)	(0, 1)
Sociology	(0.6,0.3)	(0, 1)	(0.7,0.2)	(0, 1)	(0, 1)	(0, 1)
Psychology	(0.6,0.2)	(0.7,0.2)	(0, 1)	(0.5, 0.2)	(0.4, 0.2)	(0, 1)
Information Theory	(0, 1)	(0, 1)	(0.5, 0.2)	(0, 1)	(0.4, 0.5)	(0.5,0.3)
Mechanical Projects	(0, 1)	(0, 1)	(0.4, 0.2)	(0.4, 0.5)	(0, 1)	(0.4,0.3)
Mathematical Modeling	(0, 1)	(0, 1)	(0.4, 0.2)	(0.5, 0.5)	(0.4, 0.5)	(0, 1)

Table 2.7 Score and choice values of intuitionistic fuzzy line graph

S_{ij}	British fiction	Sociology	Psychology	Information theory	Mechanical projects	Mathematical modeling	C_j
British Fiction	0	0.9274	1.0198	0	0	0	2.7957
Sociology	0.9274	0	1.0677	0	0	0	2.9850
Psychology	1.0198	1.0677	0	0.9899	0.9798	0	5.1249
Information Theory	0	0	0.9899	0	0.6481	0.8832	3.2283
Mechanical Projects	0	0	0.9798	0.6481	0	0.8602	3.1362
Mathematical Modeling	0	0	0.9798	0.7071	0.6481	0	3.2010

Table 2.8 Authors with best and critical writing

British Fiction	Sociology	Psychology	Information theory	Mechanical projects	Mathematical modeling
Raina	Raina	Raina	Adney	Adney	Merry

devices. This type of uncertainty in planet surface networks can be modeled using intuitionistic fuzzy hypergraphs as given in Fig. 2.22.

The circular dots denote the wireless devices on Earth, square style vertices show the satellites and diamond style vertices show the Earth gateway links. The membership value of each vertex represents the degree of disturbance in signal communication due to climate change and electrical interference. The nonmembership value shows the falsity of disturbance in signal communication. The membership value of each hyperedge represents the disturbance in corresponding access point. This is an application of intuitionistic fuzzy hypergraphs in planet surface networks and this model can be expanded to large-scale networks.

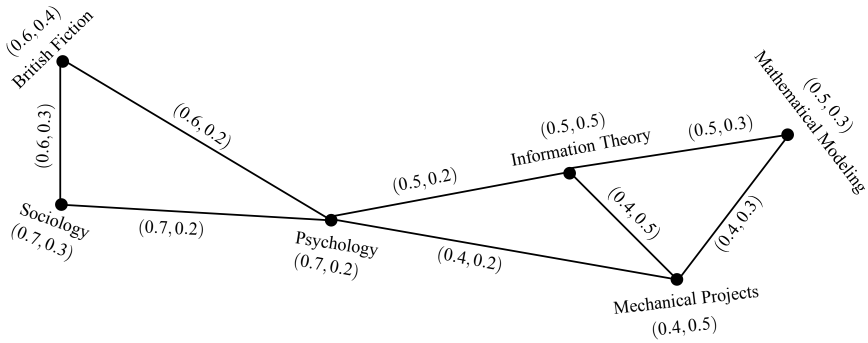


Fig. 2.21 Intuitionistic fuzzy line graph of Fig. 2.20

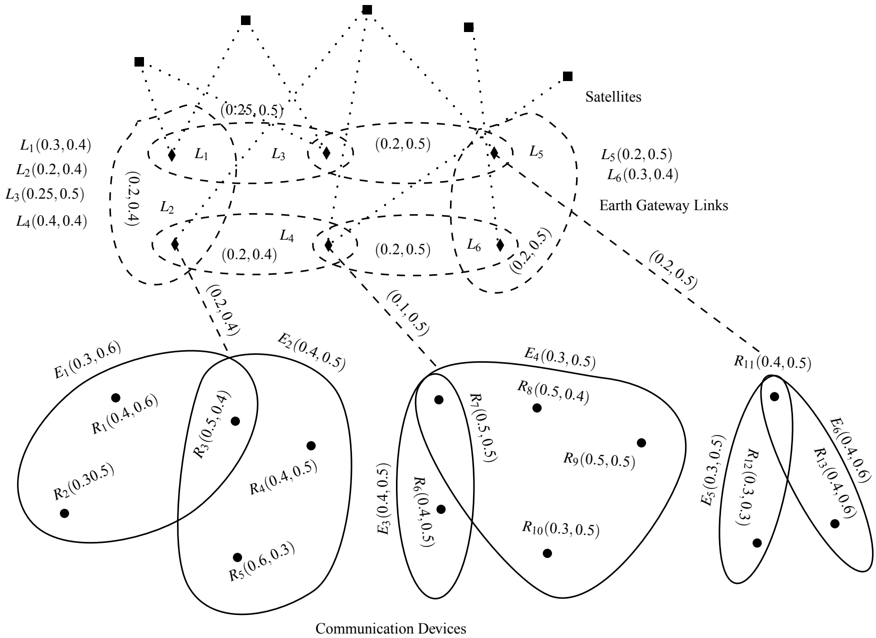


Fig. 2.22 Planet surface communication model

2.3.3 Grouping of Incompatible Chemical Substances

In this modern world, chemical engineers are trying day and night to save the economy by converting raw materials into useful products. Various types of chemicals are used for this purpose. Chemical industries are producing a variety of chemicals to be used by other companies to produce different products. But the major problem is to store the chemicals in order to avoid the accidental mixing to prevent chemical explosions,

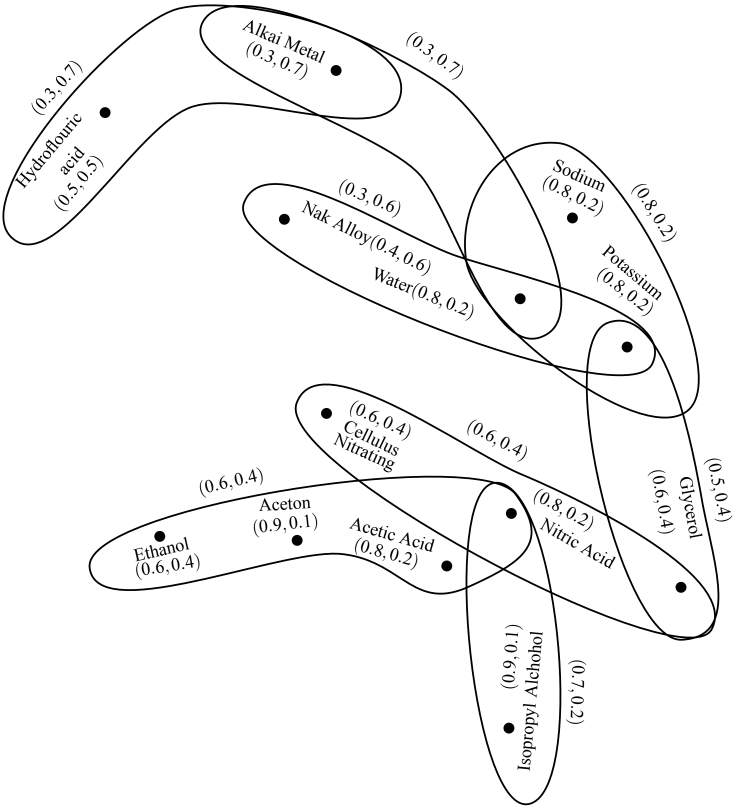


Fig. 2.23 Grouping of incompatible chemicals

oxygen deficiency, and dangerous toxic gases. Intuitionistic fuzzy hypergraphs can be used to model the chemicals in different groups to study the degree of disaster that could happen due to the accidental chemical reactions. An example is shown in Fig. 2.23 in which the vertices represent the chemicals.

Each hyperedge is the collection of those chemicals which can explode when mixed together. The membership value of each chemical show the degree of violent explosion and oxygen deficiency when reacted with various other chemicals. The nonmembership value represents the weakness of disaster. The membership and nonmembership value of each hyperedge represents the strength and weakness of disaster that cause due to chemical reaction. The degree of membership of Sodium is (0.8, 0.2) which shows that sodium is 80% explosive and 20% not explosive when mixed with other chemicals. Intuitionistic fuzzy hypergraphs can also be used for the classification of chemicals which are the most and least destructive in the given group. The method for the computation of such chemicals follows from steps 1–13 of Algorithm 2.3.1. The adjacency matrix of Fig. 2.23 is given in Table 2.9. The score values and choice of chemicals are computed in Table 2.10.

Table 2.9 Adjacency matrix of Fig. 2.23

ξ	Hyd acid	Alkai metal	Sod	Potas	Water	Nak alloy	Glyc	Nitric acid	Cell nitrat	Isop alcho	Acetic acid	Acet	Ethan
Hyd acid	(0, 1)	(0.3, 0.7)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Alkai metal	(0.3, 0.7)	(0, 1)	(0, 1)	(0, 1)	(0.3, 0.7)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Sod	(0, 1)	(0.3, 0.7)	(0, 1)	(0.8, 0.2)	(0.8, 0.2)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Potas	(0, 1)	(0, 1)	(0.8, 0.2)	(0, 1)	(0.3, 0.6)	(0.3, 0.6)	(0.5, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Water	(0, 1)	(0.3, 0.7)	(0.8, 0.2)	(0.8, 0.2)	(0, 1)	(0.3, 0.6)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Nak alloy	(0, 1)	(0, 1)	(0, 1)	(0.3, 0.6)	(0.3, 0.6)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Water	(0, 1)	(0.3, 0.7)	(0.8, 0.2)	(0.8, 0.2)	(0, 1)	(0.3, 0.6)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Glyc	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0.6, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Nitric acid	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0.6, 0.4)	(0.7, 0.2)	(0.6, 0.4)	(0.6, 0.4)	(0.6, 0.4)
Cell nitrat	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0.6, 0.4)	(0.7, 0.2)	(0.6, 0.4)	(0.6, 0.4)	(0.6, 0.4)
Isop alcho	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.7, 0.2)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
Acetic acid	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0.6, 0.4)
Acet	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0.6, 0.4)
Ethan	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.4)	(0, 1)	(0, 1)	(0.6, 0.4)	(0.6, 0.4)	(0, 1)

The score value 0 of some pair of chemicals show that they have no relation in given intuitionistic fuzzy hypergraph. There could be a little hazard due to the mixing of these chemicals. It can be studied on large-scale because it is not in the scope of this article. Table 2.10 shows that Nitric Acid and Cellulus Nitrating are the most explosive chemicals in the given group. These should be stored separately.

2.3.4 Radio Coverage Network

A *hypernetwork* M is a network whose underlying structure is a hypergraph H^*, in which each vertex v_i corresponds to a unique processor p_i of M, and each hyperedge e^*_j corresponds to a connector that connects processors represented by the vertices in e^*_j. A connector is loosely defined as an electronic or a photonic component

Table 2.10 Score and choice values of chemicals

S_{ij}	Hyd acid	Alkai metal	Sod	Potas	Water	Nak alloy	Glyc	Nitric acid	Cell Nitrat	Isop alcho	Acetic acid	Acet	Ethan	C_i
Hyd acid	0	0.4243	0	0	0	0	0	0	0	0	0	0	0	1.1314
Alkai metal	0.4243	0	0	0	0.4243	0	0	0	0	0	0	0	0	1.2729
Sod	0	0.4243	0	1.1314	1.1314	0	0	0	0	0	0	0	0	3.8185
Potas	0	0	1.1314	0	0.5099	0.5099	0.7874	0	0	0	0	0	0	4.07
Water	0	0.4243	1.1314	1.1314	0	0.5099	0	0	0	0	0	0	0	4.3284
Nak alloy	0	0	0	0.5099	0.5099	0	0	0	0	0	0	0	0	1.5855
Glyc	0	0	0	0.7874	0	0	0	0.8485	0.8485	0	0	0	0	3.2718
Nitric acid	0	0	0	0	0	0	0.8485	0	0.8485	1.0677	0.8485	0.8485	0.8485	6.4416
Cell nitrat	0	0	0	0	0	0	0.8485	0.8485	0	1.0677	0.8485	0.8485	0.8485	6.1587
Isop alcho	0	0	0	0	0	0	0	1.0677	0	0	0	0	0	2.4168
Acetic acid	0	0	0	0	0	0	0	0.8485	0	0	0	0.8485	0.8485	3.6769
Acet	0	0	0	0	0	0	0	0.8485	0	0	0.8485	0	0.8485	3.8183
Ethan	0	0	0	0	0	0	0	0.8485	0	0	0.8485	0.8485	0	3.3940

through which messages are transmitted between connected processors, not necessarily simultaneously. We call a connector a *hyperlink*. Unlike a point-to-point network, in which a link is dedicated to a pair of processors, a hyperlink in a hypernetwork is shared by a set of processors. A hyperlink can be implemented by a bus or a crossbar switch. Current optical technologies allow a hyperlink to be implemented by optical waveguides in a foldedbus using time-division multiplexing (TDM). Freespace optical or optoelectronic switching devices such as bulk lens, microlens array, and spatial light modulator (SLM) can also be used to implement hyperlinks. A star coupler, which uses wavelength-division multiplexing (WDM), can be considered either as a generalized bus structure or as a photonic switch, is another implementation of a hyperlink. Similarly, an ATM switch, which uses a variant TDM, is a hyperlink.

In telecommunications, the coverage of a radio station is the geographic area where the station can communicate.

Example 2.11 (**Radio Coverage Network**) Let X be a finite set of radio receivers (vertices); perhaps a set of representative locations at the centroid of a geographic region. For each of m radio transmitters we define the intuitionistic fuzzy set "listening area of station j" where $A_j(x) = (\mu_j(x), \nu_j(x))$ represents the "quality of reception of station j at location x. The membership and nonmembership values near 1 and 0, respectively, could signify "very clear reception on a very poor radio" while membership and nonmembership values near 0 and 1, respectively, could signify "very poor reception on even a very sensitive radio". Also, for a fixed radio the reception will vary between different stations. The stations can be considered as hyperedges. The membership and nonmembership values of the hyperedge indicate the clear and poor communication between stations. This model uses the full definition of an intuitionistic fuzzy hypergraph. The model could be used to determine station programming or marketing strategies or to establish an emergency broadcast network (is there a minimal subset of stations that reaches every radio with at least strength?). Further variables could relate signal strength to changes in time of day, weather and other conditions.

2.3.5 Clustering Problem

A cluster is two or more interconnected computers that create a solution to provide higher availability, higher scalability or both. The advantage of clustering computers for high availability is seen if one of these computers fails, another computer in the cluster can then assume the workload of the failed computer. The users of the system see no interruption of access. The advantages of clustering computers for scalability include increased application performance and the support of a greater number of users.

Definition 2.28 Let X be a reference set. A family of nontrivial intuitionistic fuzzy sets $\{A_1, A_2, A_3, \ldots, A_m\}$, where $A_i = (\mu_i, \nu_i)$, is an *intuitionistic fuzzy partition* if

1. $\bigcup_i \operatorname{supp}(A_i) = X, \quad i = 1, 2, \ldots, m,$

Table 2.11 Intuitionistic fuzzy partition matrix

$\mathscr{H}$	A_t	B_h
x_1	(0.96, 0.04)	(0.04, 0.96)
x_2	(1, 0)	(0, 1)
x_3	(0.61, 0.39)	(0.39, 0.61)
x_4	(0.05, 0.95)	(0.95, 0.05)
x_5	(0.03, 0.97)	(0.97, 0.03)

2. $\sum_{i=1}^{m} \mu_i(x) = 1$, for all $x \in X$,
3. There is at most one i such that $\nu_i(x) = 0$, for all $x \in X$, (there is at most one intuitionistic fuzzy set such that $\mu_i(x) + \nu_i(x) = 1$, for all x).

Note that, this definition generalizes fuzzy partitions because the definition is equivalent to a fuzzy partition when for all x, $\nu_i(x)$ =0. We call a family $\{A_1, A_2, A_3, \ldots, A_m\}$ an *intuitionistic fuzzy covering* of X if it satisfies above conditions $1 - 2$.

The concept of intuitionistic fuzzy partition is essential for cluster analysis. An intuitionistic fuzzy partition can be represented by an intuitionistic fuzzy matrix $[a_{ij}]_{5\times 5}$ where a_{ij} is the membership degree and nonmembership degree of element x_i in class j. We see that this matrix is the same as the incidence matrix in intuitionistic fuzzy hypergraph. Then, we can represent an intuitionistic fuzzy partition by an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ such that

1. X: A set of elements x_i, $i = 1, 2, \ldots, n$.
2. $S = \{\eta_1, \eta_2, \ldots, \eta_m\}$: A set of nontrivial intuitionistic fuzzy classes.
3. $X = \bigcup_j \text{supp}(\eta_j), \quad j = 1, 2, \ldots, m$.
4. An intuitionistic fuzzy relation R on intuitionistic fuzzy classes η_i's satisfying Definition 2.7.
5. $\sum_{i=1}^{m} \mu_i(x) = 1$, for all $x \in X$.
6. There is at most one i such that $\nu_i(x) = 0$, for all $x \in X$,that is, there is at most one intuitionistic fuzzy set such that $\mu_i(x) + \nu_i(x)$ = 1 for all $x \in X$.

Note that conditions 5–6 are added to intuitionistic fuzzy hypergraph for intuitionistic fuzzy partition. If these conditions are added, the intuitionistic fuzzy hypergraph can represent an intuitionistic fuzzy covering. Naturally, we can apply the (α, β)-cut to the intuitionistic fuzzy partition.

Example 2.12 We consider the clustering problem which is a typical example of an intuitionistic fuzzy partition on the visual image processing. Let us assume that there are five objects classified into two classes: tank and house. To cluster the elements x_1, x_2, x_3, x_4, x_5 into A_t (tank) and B_h (house), an intuitionistic fuzzy partition matrix is given in Table 2.11 in the form of incidence matrix of an intuitionistic fuzzy hypergraph $\mathscr{H} = (S, R)$ such that $S = A_t, B_h$, $R = \{(x_1x_3x_4x_5, 0.03, 0.97), (x_1x_3x_4x_5, 0.04, 0.96)\}$.

We can apply the (α, β)-cut to intuitionistic fuzzy hypergraph and obtain a crisp hypergraph $\mathscr{H}_{(\alpha,\beta)}$. This hypergraph $\mathscr{H}$ represents, generally, the covering because

Table 2.12 Hypergraph $\mathscr{H}_{(0.61,0.04)}$

$\mathscr{H}_{(0.61,0.04)}$	$A_{t(0.61,0.04)}$	$B_{h(0.61,0.04)}$
x_1	1	0
x_2	1	0
x_3	1	0
x_4	0	1
x_5	0	1

Table 2.13 Dual of the above intuitionistic fuzzy hypergraph

$\mathscr{H}^D_{(0.61,0.04)}$	X_1	X_2	X_3	X_4	X_5
A_t	1	1	1	0	0
B_h	0	0	0	1	1

of condition: 5 $\sum_{i=1}^{m} \mu_i(x) = 1$ for all $x \in X$, and 6 for all $x \in X$, there is at most one i such that $\nu_i(x) = 0$, is not always guaranteed. The hypergraph $\mathscr{H}_{(0.61,0.04)}$ is shown in Table 2.12.

We obtain the dual of hypergraph $\mathscr{H}_{(0.61,0.04)}$ as $\mathscr{H}^D_{(0.61,0.04)}$ as given in Table 2.13. The strength (cohesion) of an edge (class) $E_j = \eta_{j(\alpha,\beta)} = \{y_1, y_2, \ldots, y_r\}$ in $\mathscr{H}_{(\alpha,\beta)}$ can be used by taking minimum of membership values and maximum of nonmembership values of vertices y_i's in $\mathscr{H}$. Thus, we can use the strength as a measure of the cohesion or strength of a class in a partition. For example, the strengths of classes $A_{t(0.61,0.04)}$ and $B_{h(0.61,0.04)}$ at $s = 0.61, t = 0.04$ are $\beta(A_{t(0.61,0.04)})$=$(0.61, 0.39)$ and $\beta(B_{h(0.61,0.04)}) = (0.95, 0.05)$, respectively. It can be seen that the class $B_{h(0.61,0.04)}$ is stronger than $A_{t(0.61,0.04)}$ because $\beta(B_{h(0.61,0.04)}) > \beta(A_{t(0.61,0.04)})$. From the above discussion on the hypergraph $\mathscr{H}_{(0.61,0.04)}$ and $\mathscr{H}^D_{(0.61,0.04)}$ we can state that

- The intuitionistic fuzzy hypergraph can represent the fuzzy partition visually. The (α, β)-cut hypergraph also represents the (α, β)-cut partition.
- The dual hypergraph $\mathscr{H}^D_{(0.61,0.04)}$ can represent elements X_i, which can be grouped into a class $\eta_{j(\alpha,\beta)}$. For example, the edges X_1, X_2, X_3 of the dual hypergraph in Table 2.13 represent that the elements x_1, x_2, x_3 that can be grouped into A_t at level $(0.61, 0.04)$.
- In the intuitionistic fuzzy partition, we have $\sum_{i=1}^{m} \mu_i(x) = 1$ for all $x \in X$, and there is at most one i such that $\nu_i(x) = 0$, for all $x \in X$. If we define (α, β)-cut at level ($\alpha > 0.5$ or $\beta < 0.5$), there is no element which is grouped into two classes simultaneously. That is, if $\alpha > 0.5$ or $\beta < 0.5$, every element is contained in only one class in $\mathscr{H}_{(\alpha,\beta)}$. Therefore, the hypergraph $\mathscr{H}_{(\alpha,\beta)}$ represents a partition. (If $s \leq 0.05$ or $t \geq 0.05$ the hypergraph may represent a covering).
- If $(\alpha, \beta) = (0.61, 0.04)$ then the strength of class $B_{h(0.61,0.04)}$ is the highest as $(0.95, 0.05)$, so it is the strongest class. It means that this class can be grouped independently from other parts. Thus, we can eliminate the class B_h from other

classes and continue clustering. Therefore, the discrimination of strong classes from others can allow us to decompose a clustering problem into smaller ones. This strategy allows us to work with the reduced data in a clustering problem.

2.4 Intuitionistic Fuzzy Directed Hypergraphs

In this section, certain types of intuitionistic fuzzy directed hypergraphs including core, simple, elementary, sectionally elementary, and (μ, ν)-tempered intuitionistic fuzzy directed hypergraphs are introduced and some of their properties are discussed. The concept of transversals of intuitionistic fuzzy directed hypergraphs has been studied with the notion of fundamental sequence and locally minimal transversals.

Definition 2.29 If $A_1 = (\lambda_{A_1}, \tau_{A_1})$ and $A_2 = (\lambda_{A_2}, \tau_{A_2})$ are two intuitionistic fuzzy sets on a non-empty set X then the *Cartesian product* of A_1 and A_2 is defined as

$$A_1 \times A_2 = \{\langle (x_1, x_2), \lambda_{A_1}(x_1) \wedge \lambda_{A_2}(x_2), \tau_{A_1}(x_1) \vee \tau_{A_2}(x_2)\rangle | x_1, x_2 \in X\}.$$

The Cartesian product of n intuitionistic fuzzy sets $A_1, A_2, \ldots, A_n$ over the non-empty crisp set X can be defined as

$$A_1 \times A_2 \times \ldots \times A_n = \{\langle (x_1, x_2, \ldots, x_n), \wedge_{i=1}^{n} \lambda_{A_i}(x_i), \vee_{i=1}^{n} \tau_{A_i}(x_i)\rangle | x_1, x_2, \ldots, x_n \in X\}.$$

Definition 2.30 A *directed hyperarc* on a non-empty set of vertices X is defined as a pair $\vec{E} = (t(\vec{E}), h(\vec{E}))$ where, $t(\vec{E})$ and $h(\vec{E})$ are disjoint subsets of X. A vertex x in $\vec{E}$ is said to be a *source vertex* if $x \notin h(\vec{E})$. A vertex d is said to be a *destination vertex* in $\vec{E}$ if $d \notin t(\vec{E})$. An *intuitionistic fuzzy directed hyperedge* or *intuitionistic fuzzy directed hyperarc* is an ordered pair $\vec{\eta} = (t(\vec{\eta}), h(\vec{\eta}))$ of disjoint intuitionistic fuzzy subsets of vertices such that $t(\vec{\eta})$ is the tail of $\vec{\eta}$ while $h(\vec{\eta})$ is its head.

Definition 2.31 An *intuitionistic fuzzy directed hypergraph* on a non-empty set X is a pair $\vec{\mathscr{H}} = (I, R)$, where $I = \{\vec{\zeta}_1, \vec{\zeta}_2, \ldots, \vec{\zeta}_r\}$ is a family of order pairs $\vec{\zeta}_k = (t(\vec{\zeta}_k), h(\vec{\zeta}_k))$, where $t(\vec{\zeta}_k)$ and $h(\vec{\zeta}_k)$ are disjoint intuitionistic fuzzy subsets on X, and R is an intuitionistic fuzzy relation on $\vec{\zeta}_k$'s such that

1. $\lambda_R(\vec{E}_k)=\lambda_R(t(\vec{E}_k), h(\vec{E}_k)) \leq \min\{\wedge_{i=1}^{m} \lambda_{t(\vec{\zeta}_k)}(x_i), \wedge_{i=1}^{n} \lambda_{h(\vec{\zeta}_k)}(y_i)\}$,
2. $\tau_R(\vec{E}_k)=\tau_R(t(\vec{E}_k), h(\vec{E}_k)) \leq \max\{\vee_{i=1}^{m} \tau_{t(\vec{\zeta}_k)}(x_i), \vee_{i=1}^{n} \tau_{h(\vec{\zeta}_k)}(y_i)\}$,
3. $\lambda_R(\vec{E}_k) + \tau_R(\vec{E}_k) \leq 1$, for each $\vec{E}_k$, $1 \leq k \leq r$, where $t(\vec{E}_k) = \{x_1, x_2, \ldots, x_m\} \subset X$ and $h(\vec{E}_k) = \{y_1, y_2, \ldots, y_n\} \subset X$.
4. $\bigcup_k supp(t(\vec{\zeta}_k)) \cup \bigcup_k supp(h(\vec{\zeta}_k)) = X, \quad k = 1, 2, \ldots r.$

Example 2.13 Let $I = \{\vec{\zeta}_1, \vec{\zeta}_2, \vec{\zeta}_3\}$ be a class of intuitionistic fuzzy directed hyperarcs on $X = \{v_1, v_2, v_3, v_4\}$ as given in Table 2.14 and $\vec{E}_1 = supp(\vec{\zeta}_1) = (\{v_2\}, \{v_4\})$, $\vec{E}_2 = supp(\vec{\zeta}_2) = (\{v_3\}, \{v_4\})$, $\vec{E}_3 = supp(\vec{\zeta}_3) = (\{v_1\}, \{v_2, v_4\})$. R is an intuitionistic fuzzy relation on $\vec{\zeta}_k$'s given as, $R(\vec{E}_1) = (0.5, 0.1)$, $R(\vec{E}_2) = (0.4, 0.3)$ and

Table 2.14 Intuitionistic fuzzy directed hyperarcs on X

$x \in X$	$\vec{\zeta}_1$	$\vec{\zeta}_2$	$\vec{\zeta}_3$
v_1	(0, 1)	(0, 1)	(0.3, 0.4)
v_2	(0.5, 0.1)	(0, 1)	(0.5, 0.1)
v_3	(0, 1)	(0.4, 0.3)	(0, 1)
v_4	(0.5, 0.1)	(0.5, 0.1)	(0.2, 0.5)

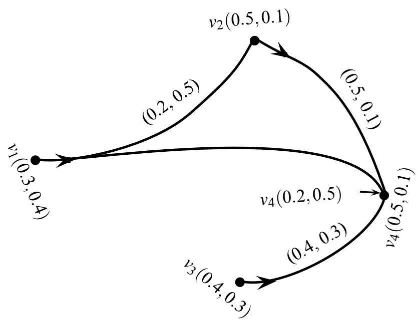


Fig. 2.24 Intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}}$

$R(\vec{E}_3) = (0.2, 0.5)$. The corresponding intuitionistic fuzzy directed hypergraph is shown in Fig. 2.24.

Definition 2.32 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph then *height* of an intuitionistic fuzzy directed hyperarc $\vec{\zeta}$ is denoted by $h(\vec{\zeta})$ and defined as

$$h(\vec{\zeta}) = (\lambda_{h(\vec{\zeta})}, \tau_{h(\vec{\zeta})}) = \big(\max\{\vee_{x\in X}\lambda_{t(\vec{\zeta})}(x), \vee_{x\in X}\lambda_{h(\vec{\zeta})}(x)\},$$
$$\min\{\wedge_{x\in X}\tau_{t(\vec{\zeta})}(x), \wedge_{x\in X}\tau_{h(\vec{\zeta})}(x)\}\big).$$

Definition 2.33 An intuitionistic fuzzy directed hypergraph is called *simple* if for each $\vec{\zeta}_i, \vec{\zeta}_j \in I$, $supp(t(\vec{\zeta}_i)) \subseteq supp(t(\vec{\zeta}_i))$, and $supp(h(\vec{\zeta}_i)) \subseteq supp(h(\vec{\zeta}_i))$ then $i = j$.

Definition 2.34 An intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}} = (I, R)$ is *support simple* if whenever $\vec{\zeta}_i, \vec{\zeta}_j \in I$, $t(\vec{\zeta}_i) \subseteq t(\vec{\zeta}_i)$, $h(\vec{\zeta}_i) \subseteq h(\vec{\zeta}_i)$, and $supp(t(\vec{\zeta}_i)) = supp(t(\vec{\zeta}_i))$, $supp(h(\vec{\zeta}_i)) = supp(h(\vec{\zeta}_i))$ then $\vec{\zeta}_i = \vec{\zeta}_j$, for all i, j.

Example 2.14 Let $I = \{\vec{\zeta}_1, \vec{\zeta}_2, \vec{\zeta}_3, \vec{\zeta}_4\}$ be a family of intuitionistic fuzzy directed hyperarcs on $X = \{v_1, v_2, v_3, v_4\}$ as shown in Table 2.15. Take $\vec{E}_1 = supp(\vec{\zeta}_1) = (\{v_1\}, \{v_2\})$, $\vec{E}_2 = supp(\vec{\zeta}_2) = (\{v_1\}, \{v_2, v_4\})$, $\vec{E}_3 = supp(\vec{\zeta}_3) = (\{v_2\}, \{v_3\})$ and $\vec{E}_4 = supp(\vec{\zeta}_4) = (\{v_2\}, \{v_3, v_4\})$. R is an intuitionistic fuzzy relation on $\vec{\zeta}_k$'s given as, $R(\vec{E}_1) = (0.5, 0.1)$, $R(\vec{E}_2) = (0.4, 0.3)$, $R(\vec{E}_3) = (0.5, 0.2)$, and $R(\vec{E}_4) = (0.4, 0.3)$.

Table 2.15 Intuitionistic fuzzy directed hyperarcs on X

$x \in X$	$\vec{\zeta}_1$	$\vec{\zeta}_2$	$\vec{\zeta}_3$	$\vec{\zeta}_4$
v_1	(0.7, 0.1)	(0.5, 0.2)	(0, 1)	(0, 1)
v_2	(0.7, 0.1)	(0.5, 0.2)	(0.5, 0.2)	(0.5, 0.2)
v_3	(0, 1)	(0, 1)	(0.5, 0.2)	(0.5, 0.2)
v_4	(0, 1)	(0.4, 0.3)	(0, 1)	(0.4, 0.3)

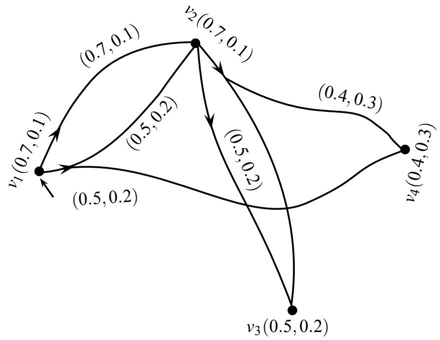


Fig. 2.25 Support simple intuitionistic fuzzy directed hypergraph

The corresponding support simple intuitionistic fuzzy directed hypergraph is shown in Fig. 2.25.

Definition 2.35 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph on X. For $\alpha, \beta \in [0, 1]$, the (α, β)-*level hyperarc* of an intuitionistic fuzzy directed hyperarc $\vec{\zeta}$ is defined as

$$\begin{aligned}\vec{\zeta}_{(\alpha,\beta)} &= (t(\vec{\zeta}_{(\alpha,\beta)}), h(\vec{\zeta}_{(\alpha,\beta)}))\\ &= \left(\{u \in X|\lambda_{t(\vec{\zeta})}(u) \geq \alpha, \tau_{t(\vec{\zeta})}(u) \leq \beta\}, \{v \in X|\lambda_{h(\vec{\zeta})}(v) \geq \alpha, \tau_{h(\vec{\zeta})}(v) \leq \beta\}\right).\end{aligned}$$

$\vec{\mathscr{H}}_{(\alpha,\beta)} = (I_{(\alpha,\beta)}, R_{(\alpha,\beta)})$ is called a (α, β)-*level directed hypergraph* of $\vec{\mathscr{H}}$ where, $I_{(\alpha,\beta)}$ is defined as $I_{(\alpha,\beta)} = \{\cup_{k=1}^{r} h(\vec{\zeta}_{k(\alpha,\beta)}) \bigcup \cup_{k=1}^{r} t(\vec{\zeta}_{k(\alpha,\beta)}), 1 \leq k \leq r\}$.

Definition 2.36 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph. The sequence of order pairs $(\alpha_i, \beta_i) \in [0, 1] \times [0, 1]$, $0 \leq \alpha_i + \beta_i \leq 1$, $1 \leq i \leq n$, such that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$, $\beta_1 < \beta_2 < \cdots < \beta_n$ satisfying the properties

1. if $1 \geq \alpha > \alpha_1$ and $0 \leq \beta < \beta_1$ then $R_{(\alpha,\beta)} = \emptyset$,
2. if $\alpha_{i+1} < \alpha \leq \alpha_i$ and $\beta_i \leq \beta < \beta_{i+1}$ then $R_{(\alpha,\beta)} = R_{(\alpha_i,\beta_i)}$,
3. $R_{(\alpha_i,\beta_i)} \sqsubset R_{(\alpha_{i+1},\beta_{i+1})}$,

is called *fundamental sequence* of $\vec{\mathscr{H}}$, denoted by $f_s(\vec{\mathscr{H}})$. The corresponding sequence of (α_i, β_i)-level directed hypergraphs $\vec{\mathscr{H}}_{(\alpha_1,\beta_1)}, \vec{\mathscr{H}}_{(\alpha_2,\beta_2)}, \ldots, \vec{\mathscr{H}}_{(\alpha_n,\beta_n)}$ is called *core set* of $\vec{\mathscr{H}}$, denoted by $\mathscr{C}(\vec{\mathscr{H}})$. The (α_n, β_n)-level directed hypergraph, $\vec{\mathscr{H}}_{(\alpha_n,\beta_n)}$, is called *support level* of $\vec{\mathscr{H}}$.

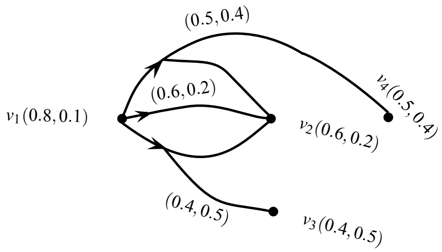


Fig. 2.26 Intuitionistic fuzzy directed hypergraph on four vertices

Fig. 2.27 $\vec{\mathscr{H}}_{(0.8,0.1)}$

Fig. 2.28 $\vec{\mathscr{H}}_{(0.6,0.2)}$

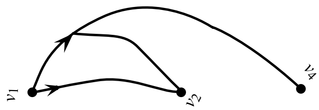


Fig. 2.29 $\vec{\mathscr{H}}_{(0.5,0.4)}$

Example 2.15 Let $\vec{\mathscr{H}}$ be an intuitionistic fuzzy directed hypergraph as shown in Fig. 2.26. Take $(\alpha_1, \beta_1) = (0.8, 0.1)$, $(\alpha_2, \beta_2) = (0.6, 0.2)$, $(\alpha_3, \beta_3) = (0.5, 0.4)$ and $(\alpha_4, \beta_4) = (0.4, 0.5)$. Clearly, the set $\{(\alpha_1, \beta_1),(\alpha_2, \beta_2), (\alpha_3, \beta_3), (\alpha_4, \beta_4)\}$ satisfies all conditions of Definition 2.36 and hence is a fundamental sequence of $\vec{\mathscr{H}}$. The corresponding (α_i, β_i)-level directed hypergraphs are shown in Fig. 2.27, 2.28 and 2.29 whereas, $\vec{\mathscr{H}}_{(0.4,0.5)} = supp(\vec{\mathscr{H}}) = (supp(I), supp(R))$.

Definition 2.37 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph on X then $supp(I) = \{(supp(t(\vec{\zeta_k})), supp(h(\vec{\zeta_k}))) \mid \vec{\zeta_k} \in I\}$. The family of intuitionistic fuzzy directed hyperarcs I is called *elementary* if I is single-valued on $supp(I)$. An intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}}$ is *elementary* if I and R are elementary, otherwise it is *nonelementary*.

Example 2.16 The elementary and nonelementary intuitionistic fuzzy directed hypergraphs are given in Figs. 2.30 and 2.31, respectively.

Definition 2.38 An intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}} = (I, R)$ is called a *partial intuitionistic fuzzy directed hypergraph* of $\vec{\mathscr{H}}' = (I', R')$ if following conditions are satisfied

1. $supp(I) \subseteq supp(I')$ and $supp(R) \subseteq supp(R')$,
2. if $supp(\vec{\zeta_i}) \in supp(I)$ and $supp(\vec{\zeta_i}') \in supp(I')$ such that $supp(\vec{\zeta_i}) = supp(\vec{\zeta_i}')$ then $\vec{\zeta_i} = \vec{\zeta_i}'$.

It is denoted by $\vec{\mathscr{H}} \subseteq \vec{\mathscr{H}}'$. An intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}} = (I, R)$ is *ordered* if the core set $\mathscr{C}(\vec{\mathscr{H}}) = \{\vec{\mathscr{H}}_{(\alpha_1,\beta_1)}, \vec{\mathscr{H}}_{(\alpha_2,\beta_2)}, \ldots \vec{\mathscr{H}}_{(\alpha_n,\beta_n)}\}$ is ordered, that

Fig. 2.30 Elementary intuitionistic fuzzy directed hypergraph

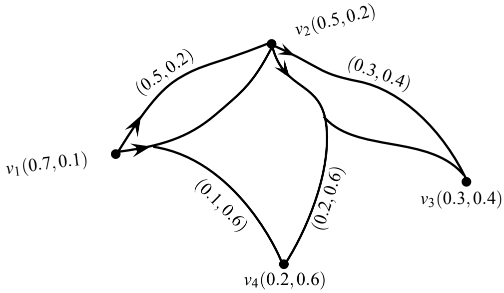
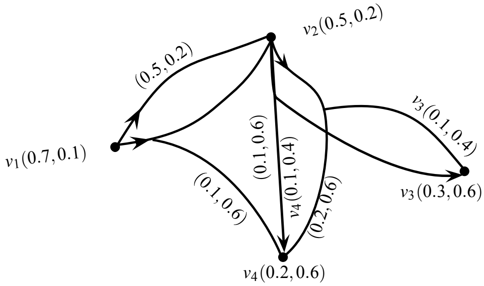


Fig. 2.31 Nonelementary intuitionistic fuzzy directed hypergraph



is $\vec{\mathscr{H}}_{(\alpha_1,\beta_1)} \subseteq \vec{\mathscr{H}}_{(\alpha_2,\beta_2)} \subseteq \ldots \subseteq \vec{\mathscr{H}}_{(\alpha_n,\beta_n)}$. $\vec{\mathscr{H}}$ is *simply ordered* if $\vec{\mathscr{H}}$ is ordered and whenever $R' \subset R_{(\alpha_{i+1},\beta_{i+1})} \setminus R_{(\alpha_i,\beta_i)}$ then $R' \not\subseteq R_{(\alpha_i,\beta_i)}$.

Observation 2.15 *Let $\vec{\mathscr{H}}$ be an elementary intuitionistic fuzzy directed hypergraph then $\vec{\mathscr{H}}$ is ordered. If $\vec{\mathscr{H}}$ is ordered intuitionistic fuzzy directed hypergraph and support level $\vec{\mathscr{H}}_{(\alpha_n,\beta_n)}$ is simple then $\vec{\mathscr{H}}$ is an elementary intuitionistic fuzzy directed hypergraph.*

Note 2.1
1. If $\vec{\mathscr{H}} = (I, R)$ is an intuitionistic fuzzy directed hypergraph with $I = \{\vec{\zeta}_1, \vec{\zeta}_2, \ldots, \vec{\zeta}_r\}$ then $I^* = \{\vec{\zeta}_1^{\,*}, \vec{\zeta}_2^{\,*}, \ldots, \vec{\zeta}_r^{\,*}\}$ is the family of crisp directed hyperarcs corresponding to I.
2. In intuitionistic fuzzy directed hypergraph $\vec{\mathscr{H}} = (I, R)$, if x is a vertex of tail of any intuitionistic fuzzy directed hyperarc $\vec{\zeta}$ then $\vec{\zeta}(x) = (\tau_{t(\vec{\zeta})}, \lambda_{t(\vec{\zeta})})$. If $x \in h(\vec{\zeta})^*$ then $\vec{\zeta}(x) = (\tau_{h(\vec{\zeta})}, \lambda_{t(\vec{\zeta})})$.

Definition 2.39 Let $\vec{\mathscr{H}} = (I, R)$ and $\vec{\mathscr{H}}' = (I', R')$ be any two intuitionistic fuzzy directed hypergraphs on X and X', respectively, where $I = \{\zeta_1, \zeta_2, \ldots, \zeta_r\}$ and $I' = \{\zeta_1', \zeta_2', \ldots, \zeta_r'\}$. A *homomorphism* of intuitionistic fuzzy directed hypergraphs $\vec{\mathscr{H}}$ and $\vec{\mathscr{H}}'$ is a mapping $\phi : X \to X'$ that satisfies

1. $\wedge_{j=1}^{r}\tau_{\vec{\zeta}_j}(x) \leq \wedge_{j=1}^{r}\tau_{\vec{\zeta}_j'}(\phi(x)),\ \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j}(x) \geq \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j'}(\phi(x)),\qquad$ for all $x \in X$.
2. $\tau_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) \leq \tau_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\},$
$\{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\}),$
$\lambda_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) \geq \lambda_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\},$

$\{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\})$,
for all $t_1, t_2, \ldots, t_s, h_1, h_2, \ldots, h_m \in X$.

Definition 2.40 A *weak isomorphism* of intuitionistic fuzzy directed hypergraphs $\vec{\mathscr{H}}$ and $\vec{\mathscr{H}}'$ is a bijective homomorphism $\phi : X \to X'$ that satisfies
$\wedge_{j=1}^{r}\tau_{\vec{\zeta}_j}(x) = \wedge_{j=1}^{r}\tau_{\vec{\zeta}_j'}(\phi(x))$, $\quad \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j}(x) = \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j'}(\phi(x))$, for all $x \in X$.

Definition 2.41 A *co-weak isomorphism* of intuitionistic fuzzy directed hypergraphs $\vec{\mathscr{H}}$ and $\vec{\mathscr{H}}'$ is a bijective homomorphism $\phi : X \to X'$ that satisfies
$\tau_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) = \tau_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\}, \{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\})$,
$\lambda_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) = \lambda_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\}, \{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\})$,
for all $t_1, t_2, \ldots, t_s, h_1, h_2, \ldots, h_m \in X$.

Definition 2.42 An *isomorphism* of intuitionistic fuzzy directed hypergraphs $\vec{\mathscr{H}}$ and $\vec{\mathscr{H}}'$ is a bijective mapping $\phi : X \to X'$ that satisfies

1. $\wedge_{j=1}^{r}\tau_{\vec{\zeta}_j}(x) = \wedge_{j=1}^{r}\tau_{\vec{\zeta}_j'}(\phi(x))$, $\quad \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j}(x) = \vee_{j=1}^{r}\lambda_{\vec{\zeta}_j'}(\phi(x))$, for all $x \in X$.
2. $\tau_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) = \tau_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\}, \{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\})$,
   $\lambda_R(\{t_1, t_2, \ldots, t_s\}, \{h_1, h_2, \ldots, h_m\}) = \lambda_{R'}(\{\phi(t_1), \phi(t_2), \ldots, \phi(t_s)\}, \{\phi(h_1), \phi(h_2), \ldots, \phi(h_m)\})$,
   for all $t_1, t_2, \ldots, t_s, h_1, h_2, \ldots, h_m \in X$.

In this case, $\vec{\mathscr{H}}$ and $\vec{\mathscr{H}}'$ are called *isomorphic* to each other.

Definition 2.43 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph then the *order* $O(\vec{\mathscr{H}})$ and *size* $S(\vec{\mathscr{H}})$ of $\vec{\mathscr{H}}$ are defined as

$$O(\vec{\mathscr{H}}) = \left(\sum_{x \in X} \wedge_j \tau_{\vec{\xi}_j}(x), \sum_{x \in X} \vee_j \lambda_{\vec{\xi}_j}(x)\right), \qquad S(\vec{\mathscr{H}}) = \left(\sum_{\vec{E}_i \in I^*} \tau_R(\vec{E}_i), \sum_{\vec{E}_i \in I^*} \lambda_R(\vec{E}_i)\right).$$

Theorem 2.16 *The order and size of isomorphic intuitionistic fuzzy directed hypergraphs are same.*

Proof Let $\vec{\mathscr{H}_1} = (I_1, R_1)$ and $\vec{\mathscr{H}_2} = (I_2, R_2)$ be any two intuitionistic fuzzy directed hypergraphs on X_1 and X_2, respectively, where $I_1 = \{\zeta_{11}, \zeta_{12}, \ldots, \zeta_{1r}\}$ and $I_2 = \{\zeta_{21}, \zeta_{22}, \ldots, \zeta_{2r}\}$ be the classes of intuitionistic fuzzy directed hyperarcs. Let $\phi : X_1 \to X_2$ be an isomorphism from $\vec{\mathscr{H}_1}$ to $\vec{\mathscr{H}_2}$ then using Definition 2.42,

$$
\begin{aligned}
O(\vec{\mathscr{H}_1}) &= \Big(\sum_{x_1 \in X_1} \wedge_j \tau_{\vec{\zeta}_{1j}}(x_1), \sum_{x_1 \in X_1} \vee_j \lambda_{\vec{\zeta}_{1j}}(x_1)\Big) \\
&= \Big(\sum_{x_1 \in X_1} \wedge_j \tau_{\vec{\zeta}_{2j}}(\phi(x_1)), \sum_{x_1 \in X_1} \vee_j \lambda_{\vec{\zeta}_{2j}}(\phi(x_1))\Big) \\
&= \Big(\sum_{x_2 \in X_2} \wedge_j \tau_{\vec{\zeta}_{2j}}(x_2), \sum_{x_2 \in X_2} \vee_j \lambda_{\vec{\zeta}_{2j}}(x_2)\Big) = O(\vec{\mathscr{H}_2}).
\end{aligned}
$$

$$
\begin{aligned}
S(\vec{\mathscr{H}_1}) &= \Big(\sum_{\vec{E}_{1i} \in I_1^*} \tau_{R_1}(\vec{E}_{1i}), \sum_{\vec{E}_{1i} \in I_1^*} \lambda_{R_1}(\vec{E}_{1i})\Big) \\
&= \Big(\sum_{\vec{E}_{1i} \in I_1^*} \tau_{R_2}(\phi(\vec{E}_{1i})), \sum_{\vec{E}_{1i} \in I_1^*} \lambda_{R_2}(\phi(\vec{E}_{1i}))\Big) \\
&= \Big(\sum_{\vec{E}_{2i} \in I_2^*} \tau_{R_2}(\vec{E}_{2i}), \sum_{\vec{E}_{2i} \in I_2^*} \lambda_{R_2}(\vec{E}_{2i})\Big) = S(\vec{\mathscr{H}_2}).
\end{aligned}
$$

Remark 2.5 1. The order of weak isomorphic intuitionistic fuzzy directed hypergraphs is same.
2. The size of co-weak isomorphic intuitionistic fuzzy directed hypergraphs is same.

Theorem 2.17 *The relation of isomorphism between intuitionistic fuzzy directed hypergraphs is an equivalence relation.*

Proof Let $\vec{\mathscr{H}_1} = (I_1, R_1)$, $\vec{\mathscr{H}_2} = (I_2, R_2)$ and $\vec{\mathscr{H}_3} = (I_3, R_3)$ be intuitionistic fuzzy directed hypergraphs on X_1, X_2 and X_3, respectively, where, $I_1 = \{\zeta_{11}, \zeta_{12}, \ldots, \zeta_{1r}\}$, $I_2 = \{\zeta_{21}, \zeta_{22}, \ldots, \zeta_{2r}\}$ and $I_3 = \{\zeta_{31}, \zeta_{32}, \ldots, \zeta_{3r}\}$.

1. Reflexive: Define $I : X_1 \to X_1$ by $I(x_1) = x_1$, for all $x_1 \in X_1$. Then, I is a bijective homomorphism and
   1. $(\wedge_j \tau_{\vec{\zeta}_{1j}}(x_1), \vee_j \lambda_{\vec{\zeta}_{1j}}(x_1)) = (\wedge_j \tau_{\vec{\zeta}_{1j}}(I(x_1)), \vee_j \lambda_{\vec{\zeta}_{1j}}(I(x_1)))$,
   2. $(\tau_{R_1}(\vec{E}_{1i}), \lambda_{R_1}(\vec{E}_{1i})) = (\tau_{R_1}(I(\vec{E}_{1i})), \lambda_{R_1}(I(\vec{E}_{1i})))$,
   for all $x_1 \in X_1$, $t(\vec{E}_{1i}) \subset X_1$, $h(\vec{E}_{1i}) \subset X_1$.
   I is an isomorphism of an intuitionistic fuzzy directed hypergraph to itself.
2. Symmetric: Let $\phi : X_1 \to X_2$ be an isomorphism defined by $\phi(x_1) = x_2$. Since, ϕ is a bijective mapping therefore, $\phi^{-1} : X_2 \to X_1$ exists and $\phi^{-1}(x_2) = x_1$, for all $x_2 \in X_2$. Then

$$
\begin{aligned}
(\wedge_j \tau_{\vec{\zeta}_{2j}}(x_2), \vee_j \lambda_{\vec{\zeta}_{2j}}(x_2)) &= (\wedge_j \tau_{\vec{\zeta}_{2j}}(\phi(x_1)), \vee_j \lambda_{\vec{\zeta}_{2j}}(\phi(x_1))) \\
&= (\wedge_j \tau_{\vec{\zeta}_{1j}}(x_1), \vee_j \lambda_{\vec{\zeta}_{1j}}(x_1)) \\
&= (\wedge_j \tau_{\vec{\zeta}_{1j}}(\phi^{-1}(x_2)), \vee_j \lambda_{\vec{\zeta}_{1j}}(\phi^{-1}(x_2))).
\end{aligned}
$$

$$
R_2(\vec{E}_{2j}) = R_2(\phi(\vec{E}_{1j})) = R_1(\vec{E}_{1j}) = R_1(\phi^{-1}(\vec{E}_{2j})),\ t(\vec{E}_{2j}) \subseteq X_2,\ h(\vec{E}_{2j}) \subseteq X_2.
$$

Hence, ϕ^{-1} is an isomorphism.

3. Transitive: Let $\phi : X_1 \to X_2$ and $\psi : X_2 \to X_3$ be the isomorphisms of $\vec{\mathscr{H}}_1$ onto $\vec{\mathscr{H}}_2$ and $\vec{\mathscr{H}}_2$ onto $\vec{\mathscr{H}}_3$ defined by $\phi(x_1) = x_2$ and $\psi(x_2) = x_3$, respectively. By Definition 2.42

$$\begin{aligned}(\wedge_j \tau_{\vec{\zeta}_{1j}}(x_1), \vee_j \lambda_{\vec{\zeta}_{1j}}(x_1)) &= (\wedge_j \tau_{\vec{\zeta}_{2j}}(x_2), \vee_j \lambda_{\vec{\zeta}_{2j}}(x_2))\\ &= (\wedge_j \tau_{\vec{\zeta}_{3j}}(\psi(x_2)), \vee_j \lambda_{\vec{\zeta}_{3j}}(\psi(x_2)))\\ &= (\wedge_j \tau_{\vec{\zeta}_{3j}}(\psi(\phi(x_1))), \vee_j \lambda_{\vec{\zeta}_{3j}}(\psi(\phi(x_1))))\\ &= (\wedge_j \tau_{\vec{\zeta}_{3j}}(\psi \circ \phi(x_1)), \vee_j \lambda_{\vec{\zeta}_{3j}}(\psi \circ \phi(x_1))).\end{aligned}$$

$R_1(\vec{E}_{1j}) = R_2(\vec{E}_{2j}) = R_3(\psi(\vec{E}_{2j})) = R_3(\psi(\phi(\vec{E}_{1j}))) = R_3(\psi \circ \phi(\vec{E}_{1j}))$, where $\vec{E}_{ij} = (t(\vec{E}_{ij}), h(\vec{E}_{ij})), t(\vec{E}_{ij}) \subset X_i, h(\vec{E}_{ij}) \subset X_i$. Clearly, $\psi \circ \phi$ is an isomorphism from $\vec{\mathscr{H}}_1$ onto $\vec{\mathscr{H}}_3$. Hence, isomorphism of intuitionistic fuzzy directed hypergraphs is an equivalent relation.

Remark 2.6 The relation of weak isomorphism between intuitionistic fuzzy directed hypergraphs is a partial order relation.

Definition 2.44 Let $\vec{\mathscr{H}} = (I, R)$ be an intuitionistic fuzzy directed hypergraph. A set of intuitionistic fuzzy directed hyperarcs T with the property that $T_{h(\vec{\zeta}_i)} \cap \vec{\zeta}_{ih(\vec{\zeta}_i)} \neq \emptyset$, for each $\vec{\zeta}_i \in I$, is called *intuitionistic fuzzy transversal* of $\vec{\mathscr{H}}$. T is a *minimal intuitionistic fuzzy transversal* of $\vec{\mathscr{H}}$ if whenever $\rho \subset T$, ρ is not an intuitionistic fuzzy transversal of $\vec{\mathscr{H}}$. The family of all minimal intuitionistic fuzzy transversals of $\vec{\mathscr{H}}$ is denoted by $Tr(\vec{\mathscr{H}})$.

Example 2.17 Consider the intuitionistic fuzzy directed hypergraph as shown in Fig. 2.32 where $I = \{\vec{\zeta}_1, \vec{\zeta}_2, \vec{\zeta}_3\}$ is defined in Table 2.16.

Fig. 2.32 Intuitionistic fuzzy directed hypergraph on four vertices

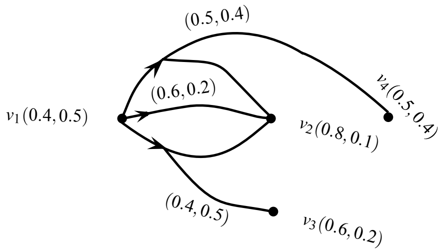


Table 2.16 Intuitionistic fuzzy hyperarcs of $\vec{\mathscr{H}}$ in Fig. 2.32

Intuitionistic fuzzy hyperarc		$h(\vec{\zeta}_i)$	$\vec{\zeta}_{ih(\vec{\zeta}_i)}$
$\vec{\zeta}_1$	$\{(\{(v_1, 0.4, 0.5)\}, \{(v_2, 0.8, 0.1), (v_4, 0.5, 0.4)\})\}$	(0.8, 0.1)	$\{(\{\}, \{v_2\})\}$
$\vec{\zeta}_2$	$\{(\{(v_1, 0.4, 0.5)\}, \{(v_2, 0.8, 0.1)\})\}$	(0.8, 0.1)	$\{(\{\}, \{v_2\})\}$
$\vec{\zeta}_3$	$\{(\{(v_1, 0.4, 0.5)\}, \{(v_2, 0.8, 0.1), (v_3, 0.6, 0.2)\})\}$	(0.8, 0.1)	$\{(\{\}, \{v_2\})\}$

Clearly, for each $1 \le i \le 3$, $\vec{\zeta}_{ih(\vec{\zeta}_i)} \cap T_{ih(\vec{\zeta}_i)} \neq \emptyset$ where, $T = \Big\{\big(\{(v_1, 0.8, 0.1)\}, \{(v_2, 0.6, 0.2)\}\big)\Big\}$. Hence, T is an intuitionistic fuzzy transversal of $\vec{\mathscr{H}}$.

We now provide results and discussions of intuitionistic fuzzy transversals.

Lemma 2.1 *Let $\vec{\mathscr{H}} = (I, \vec{R})$ be an intuitionistic fuzzy directed hypergraph with fundamental sequence $f_s(\vec{\mathscr{H}}) = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$. If T is an intuitionistic fuzzy transversal of $\vec{\mathscr{H}}$ then $\lambda_{h(T)} \ge \lambda_{h(\vec{\zeta}_i)}$ and $\tau_{h(T)} \le \tau_{h(\vec{\zeta}_i)}$. If T is a minimal intuitionistic fuzzy transversals of $\vec{\mathscr{H}}$ then $h(T) = (\max\{\lambda_{h(\vec{\zeta}_i)} | \vec{\zeta}_i \in I\}, \min\{\tau_{h(\vec{\zeta}_i)} | \vec{\zeta}_i \in I\}) = (\alpha_1, \beta_1)$.*

Proposition 2.2 *Let $\vec{\mathscr{H}}$ be an intuitionistic fuzzy directed hypergraph then the following statements are equivalent.*
1. T is an intuitionistic fuzzy transversal of $\vec{\mathscr{H}}$.
2. For each $\vec{\zeta}_i \in I$, $(\alpha, \beta) \in [0, 1] \times [0, 1]$, $0 \le \alpha + \beta \le 1$ with $\alpha < \lambda_{h(\vec{\zeta}_i)}$ and $\beta > \tau_{h(\vec{\zeta}_i)}$ then $T_{(\alpha,\beta)} \cap \vec{\zeta}_{i(\alpha,\beta)} \neq \emptyset$.
3. $T_{(\alpha,\beta)}$ is a transversal of $\vec{\mathscr{H}}_{(\alpha,\beta)}$.

2.5 Complex Intuitionistic Fuzzy Hypergraphs

To generalize the concepts of intuitionistic fuzzy sets, complex intuitionistic fuzzy sets were introduced by Alkouri and Salleh [4]. Complex intuitionistic fuzzy set is a distinctive intuitionistic fuzzy set in which the membership degrees are determined on the unit disc of the complex plane and can more clearly express the imprecision and ambiguity in the data. Yaqoob et al. [23] defined complex intuitionistic fuzzy graphs and discussed an application of these graphs in cellular networks to test the proposed model.

Definition 2.45 A *complex intuitionistic fuzzy set I* on the universal set X is defined as, $I = \{(u, T_I(u)e^{i\phi_I(u)}, F_I(u)e^{i\psi_I(u)}) | u \in X\}$, where $i = \sqrt{-1}$, $T_I(u), F_I(u) \in [0, 1]$, $\phi_I(u), \psi_I(u) \in [0, 2\pi]$, and for every $u \in X$, $0 \le T_I(u) + F_I(u) \le 1$. Here, $T_I(u)$, $F_I(u)$ and $\phi_I(u)$, $\psi_I(u)$ are called the amplitude terms and phase terms for truth membership and falsity membership grades, respectively.

Definition 2.46 A *complex intuitionistic fuzzy graph* on X is an ordered pair $G = (A, B)$, where A is a complex intuitionistic fuzzy set on X and B is complex intuitionistic fuzzy relation on X such that

$$T_B(ab) \le \min\{T_A(a), T_A(b)\},\ F_B(ab) \le \max\{F_A(a), F_A(b)\}, \text{(for amplitude terms)}$$
$$\phi_B(ab) \le \min\{\phi_A(a), \phi_A(b)\},\ \psi_B(ab) \le \max\{\psi_A(a), \psi_A(b)\}, \text{(for phase terms)}$$

$0 \le T_B(ab) + F_B(ab) \le 1$, for all $a, b \in X$.

Definition 2.47 Let X be a nontrivial set of universe. A *complex intuitionistic fuzzy hypergraph* is defined as an ordered pair $H = (\mathscr{C}, \mathscr{D})$, where $\mathscr{C} = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a finite family of complex intuitionistic fuzzy sets on X and $\mathscr{D}$ is a complex intuitionistic fuzzy relation on complex intuitionistic fuzzy sets α_j's such that

(i)

$$
\begin{aligned}
&T_{\mathscr{D}}(\{r_1, r_2, \cdots, r_l\}) \leq \min\{T_{\alpha_j}(r_1), T_{\alpha_j}(r_2), \cdots, T_{\alpha_j}(r_l)\},\\
&F_{\mathscr{D}}(\{r_1, r_2, \cdots, r_l\}) \leq \max\{F_{\alpha_j}(r_1), F_{\alpha_j}(r_2), \cdots, F_{\alpha_j}(r_l)\}, \text{(for amplitude terms)}\\
&\phi_{\mathscr{D}}(\{r_1, r_2, \cdots, r_l\}) \leq \min\{\phi_{\alpha_j}(r_1), \phi_{\alpha_j}(r_2), \cdots, \phi_{\alpha_j}(r_l)\},\\
&\psi_{\mathscr{D}}(\{r_1, r_2, \cdots, r_l\}) \leq \max\{\psi_{\alpha_j}(r_1), \psi_{\alpha_j}(r_2), \cdots, \psi_{\alpha_j}(r_l)\}, \text{(for phase terms)}
\end{aligned}
$$

$0 \leq T_{\mathscr{D}} + F_{\mathscr{D}} \leq 1$, for all $r_1, r_2, \cdots, r_l \in X$.

(ii) $\bigcup_j supp(\alpha_j) = X$, for all $\alpha_j \in \mathscr{C}$.

Note that, $E_k = \{r_1, r_2, \cdots, r_l\}$ is the crisp hyperedge of $H = (\mathscr{C}, \mathscr{D})$.

Example 2.18 Consider a complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ on $X = \{v_1, v_2, v_3, v_4\}$. The complex intuitionistic fuzzy relation is defined as $\mathscr{D}(\{v_1, v_2, v_3, v_4\}) = (0.2e^{i(0.4)\pi}, 0.6e^{i(0.3)\pi})$, $\mathscr{D}(\{v_1, v_2\}) = (0.3e^{i(0.6)\pi}, 0.6e^{i(0.3)\pi})$, and $\mathscr{D}(\{v_3, v_4\}) = (0.2e^{i(0.4)\pi}, 0.5e^{i(0.3)\pi})$. The corresponding complex intuitionistic fuzzy hypergraph is shown in Fig. 2.33.

Definition 2.48 A complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ is *simple* if whenever $\mathscr{D}_j, \mathscr{D}_k \in \mathscr{D}$ and $\mathscr{D}_j \subseteq \mathscr{D}_k$, then $\mathscr{D}_j = \mathscr{D}_k$.

A complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ is *support simple* if whenever $\mathscr{D}_j, \mathscr{D}_k \in \mathscr{D}$, $\mathscr{D}_j \subseteq \mathscr{D}_k$, and $supp(\mathscr{D}_j) = supp(\mathscr{D}_k)$, then $\mathscr{D}_j = \mathscr{D}_k$.

Definition 2.49 Let $H = (\mathscr{C}, \mathscr{D})$ be a complex intuitionistic fuzzy hypergraph. Suppose that $\alpha, \beta \in [0, 1]$ and $\theta, \varphi \in [0, 2\pi]$ such that $0 \leq \alpha + \beta \leq 1$. The $(\alpha e^{i\theta}, \beta e^{i\varphi})$-*level hypergraph* of H is defined as an ordered pair $H^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = (\mathscr{C}^{(\alpha e^{i\theta}, \beta e^{i\varphi})}, \mathscr{D}^{(\alpha e^{i\theta}, \beta e^{i\varphi})})$, where

(i) $\mathscr{D}^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \{D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})} : D_j \in \mathscr{D}\}$ and $D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \{u \in X : T_{D_j}(u) \geq \alpha, \phi_{D_j}(u) \geq \theta,$ and $F_{D_j}(u) \leq \beta, \psi_{D_j}(u) \leq \varphi\}$,

(ii) $\mathscr{C}^{(\alpha e^{i\theta}, \beta e^{i\varphi})} = \bigcup_{D_j \in \mathscr{D}} D_j^{(\alpha e^{i\theta}, \beta e^{i\varphi})}$.

Note that, $(\alpha e^{i\theta}, \beta e^{i\varphi})$-level hypergraph of H is a crisp hypergraph.

Example 2.19 Consider a complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ as shown in Fig. 2.33. Let $\alpha = 0.2$, $\beta = 0.5$, $\theta = 0.5\pi$, and $\varphi = 0.2\pi$. Then, $(\alpha e^{i\theta}, \beta e^{i\varphi})$-level hypergraph of H is shown in Fig. 2.34.

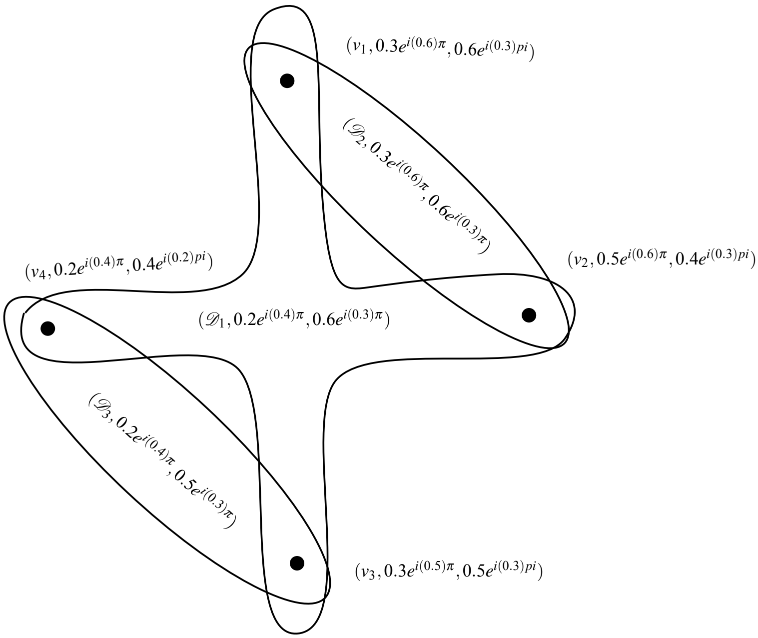


Fig. 2.33 Complex intuitionistic fuzzy hypergraph

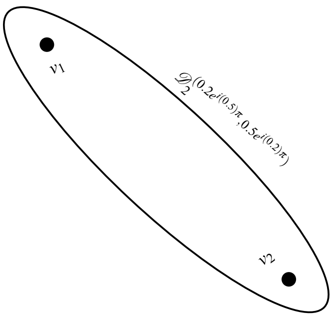


Fig. 2.34 $(0.2e^{i(0.5)\pi}, 0.5e^{i(0.2)\pi})$-level hypergraph of H

Definition 2.50 Let $H = (\mathscr{C}, \mathscr{D})$ be a complex intuitionistic fuzzy hypergraph. The *complex intuitionistic fuzzy line graph* of H is defined as an ordered pair $l(H) = (\mathscr{C}_l, \mathscr{D}_l)$, where $\mathscr{C}_l = \mathscr{D}$ and there exists an edge between two vertices in $l(H)$ if $|supp(D_j) \cap supp(D_k)| \geq 1$. The membership degrees of $l(H)$ are given as

(i) $\mathscr{C}_l(E_k) = \mathscr{D}(E_k)$,
(ii) $\mathscr{D}_l(E_j E_k) = (\min\{T_{\mathscr{D}}(E_j), T_{\mathscr{D}}(E_k)\}e^{i \min\{\phi_{\mathscr{D}}(E_j), \phi_{\mathscr{D}}(E_k)\}}$, $\max\{F_{\mathscr{D}}(E_j), F_{\mathscr{D}}(E_k)\}e^{i \max\{\psi_{\mathscr{D}}(E_j), \psi_{\mathscr{D}}(E_k)\}})$.

Definition 2.51 A complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ is said to be *linear* if for every $D_j, D_k \in \mathscr{D}$,

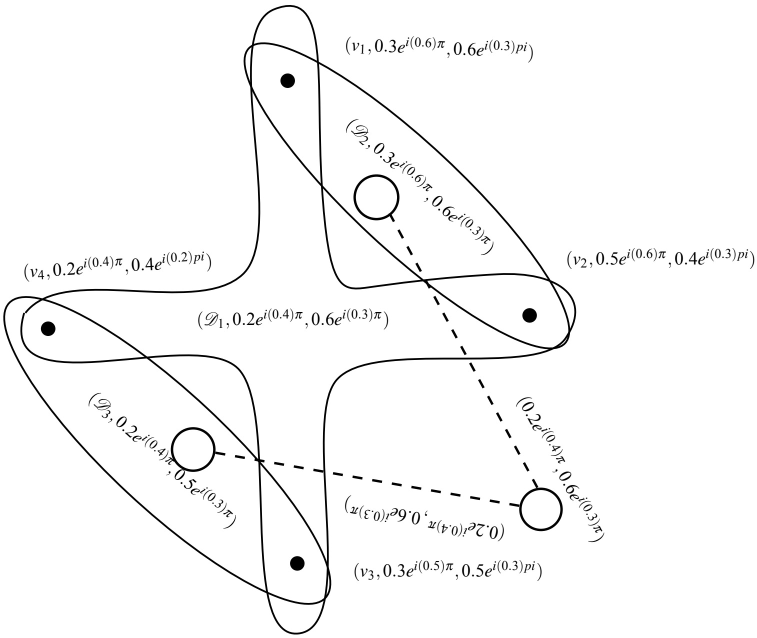


Fig. 2.35 Complex intuitionistic line graph of H

(i) $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$,
(ii) $|supp(D_j) \cap supp(D_k)| \leq 1$.

Example 2.20 Consider a complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ as shown in Fig. 2.33. By direct calculations, we have

$$supp(\mathscr{D}_1) = \{v_1, v_2, v_3, v_4\},\ supp(\mathscr{D}_2) = \{v_1, v_2\},\ supp(\mathscr{D}_3) = \{v_3, v_4\}.$$

Note that, $supp(D_j) \subseteq supp(D_k) \Rightarrow j \neq k$ and $|supp(D_j) \cap supp(D_k)| \nleq 1$. Hence, complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ is not linear. The corresponding complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ and its line graph is shown in Fig. 2.35.

Theorem 2.18 *A simple strong complex intuitionistic fuzzy graph is the complex intuitionistic line graph of a linear complex intuitionistic fuzzy hypergraph.*

Definition 2.52 The 2-*section* $H_2 = (\mathscr{C}_2, \mathscr{D}_2)$ of a complex intuitionistic fuzzy hypergraph $H = (\mathscr{C}, \mathscr{D})$ is a complex intuitionistic fuzzy graph having same set of vertices as that of H, $\mathscr{D}_2$ is a complex intuitionistic fuzzy set on $\{e = u_j u_k | u_j, u_k \in E_l,\quad l = 1, 2, 3, \cdots\}$, and $\mathscr{D}_2(u_j u_k) = (\min\{\min T_{\alpha_l}(u_j), \min T_{\alpha_l}(u_k)\} e^{i\,\min\{\min \phi_{\alpha_l}(u_j), \min \phi_{\alpha_l}(u_k)\}}, \max\{\max F_{\alpha_l}(u_j), \max F_{\alpha_l}(u_k)\} e^{i\,\max\{\max \psi_{\alpha_l}(u_j), \max \psi_{\alpha_l}(u_k)\}})$ such that $0 \leq T_{\mathscr{D}_2}(u_j u_k) + F_{\mathscr{D}_2}(u_j u_k) \leq 1$.

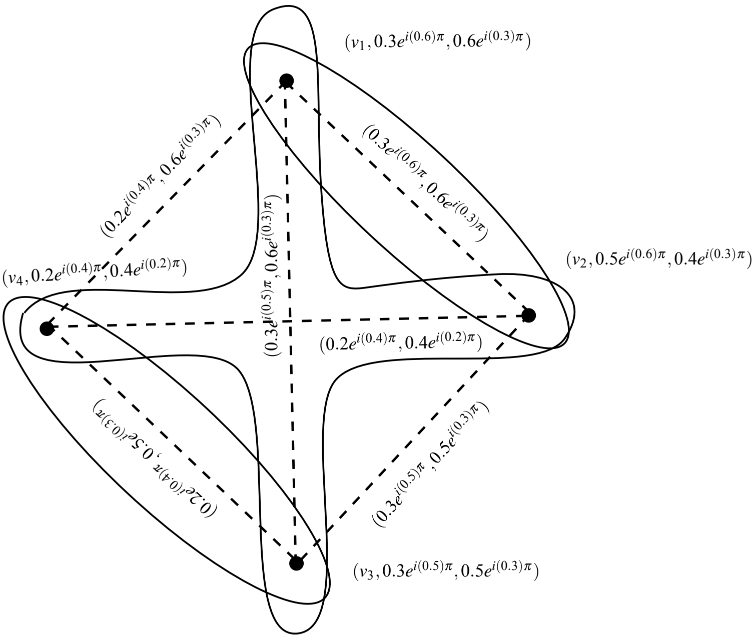


Fig. 2.36 2-section of complex intuitionistic fuzzy hypergraph

Example 2.21 An example of a complex intuitionistic fuzzy hypergraph is given in Fig. 2.36. The 2-section of H is presented with dashed lines.

Definition 2.53 Let $H = (\mathscr{C}, \mathscr{D})$ be a complex intuitionistic fuzzy hypergraph. A *complex intuitionistic fuzzy transversal* τ is a complex intuitionistic fuzzy set of X satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \mathscr{D}$, where $h(\rho)$ is the height of ρ.

A *minimal complex intuitionistic fuzzy transversal* t is the complex intuitionistic fuzzy transversal of H having the property that if $\tau \subset t$, then τ is not a complex intuitionistic fuzzy transversal of H.

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