Chapter 6
Floquet-Bloch Theory for Semiconductor Bragg Structure

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Abstract The problem of Floquet-Bloch waves propagation in a semiconductor magnetophotonic crystal with a transverse magnetic field was solved. The fundamental solutions of the Hill equation in layers that based on the Floquet theory were obtained in an analytical form. The dispersion equation and its roots are found explicitly. The analysis of the dispersion properties of the structures depending on the material parameters of the layers was carried out. The parameters of gyrotropic layers for the full transmission and reflection of a plane wave for different frequencies through a limited magnetophotonic crystal in modes of surface and bulk waves are determined.

6.1 Introduction

In recent years, a lot of theoretical and experimental works [1–12] has been devoted to the problem of the terahertz range electromagnetic waves propagation through the

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periodic thin films (one-dimensional photonic crystals). At the same time, photonic crystals have been already widely used in different applications of modern science and technology of the terahertz, microwave and optical ranges. One of the promising photonic crystals applications is the new terahertz radiation sources developing with frequency tuning possibility [13–17]. Majority of PhC theoretical investigations are based on the characteristic (dispersion) equation solution and on the periodical structure transfer matrix. Due to this approach, the properties of isotropic photonic crystals are well studied both for TE and TM modes [1, 2, 18].

An alternative treatment to the research of PhCs is the Floquet-Bloch theory [19–22], that based on the fundamental solutions of Hill's equations and allows to find in the analytic form not only the dispersion characteristics of PhCs but also expressions for the fields in each PhC layer [18, 23–28]. However, such researches have been carried out only for isotropic photonic crystals based on two-layers periodic dielectric structures. Recently, magnetophotonic crystals (MPhCs) which based on gyrotropic elements with a controlled transverse magnetic field have attracted the particular attention of researchers. In the case of gyrotropy presence in the medium, its material parameters are tensors. The presence of gyrotropic layers in the such structure provides opportunity to change the material parameters values due to the applied magnetic field magnitude and, ultimately, to realize electric-type control for the dispersion properties of the MPhCs and for the wave propagation characteristics. In this case, depending on the direction of the applied magnetic field to the gyromagnetic media, various effects can be observed: the Faraday effect, the magnetic two-beam refraction, the rotation of the polarization plane, the nonreciprocal phenomena for forward and backward waves, the presence of surface gyrotropic waves.

Investigations of MPhCs with gyrotropic elements in general case was carried out mainly by the matrix approach [3, 7, 9, 11]. The Floquet theory approach for gyrotropic MPCs have not been considered. In this paper the problem based on the Floquet theory and new fundamental solutions of the Hill's equation for own waves in MPhC with gyrotropic semiconductor layers was represented.

6.2 Floquet—Bloch Waves Theory

Let us consider the propagation of electromagnetic waves in a stratified two-layers periodic structure in general form with gyrotropic discrete layers (one-dimensional MPhC) (Fig. 6.1). Each of the two layers on the structure period $L = a + b$ is a gyrotropic semiconductor medium, the dielectric constant of which is characterized by a tensor of the standard form with the material parameters $\overset{\leftrightarrow}{\varepsilon}_j$, $(j = 1, 2)$. The width of one layer is a, and the other one is b.

The dielectric constant tensor of the hyrotropic semiconductor medium of each layer $(j = 1, 2)$ has a standard form [29]:

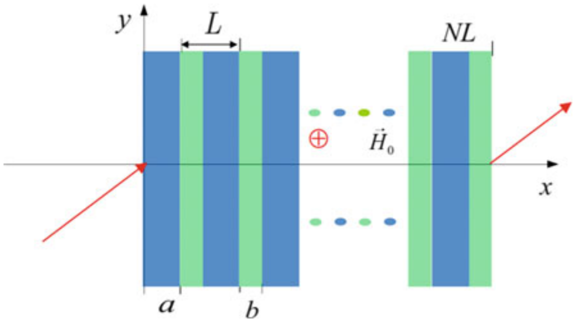

Fig. 6.1 The model of a gyrotropic MphC

$$\overleftrightarrow{\varepsilon}_j = \left\| \begin{matrix} \varepsilon_j & -i\varepsilon_{aj} & 0 \\ i\varepsilon_{aj} & \varepsilon_j & 0 \\ 0 & 0 & \varepsilon_{\parallel j} \end{matrix} \right\|,$$

Traditionally, the analysis of wave propagation in MPhC is carried out on the basis of Maxwell's equations. In the two- a dimensional case $\left(\frac{\partial}{\partial z} = 0\right)$ from the Maxwell equations two independent Helmholtz equations can be obtained, each of which describes TE (transverse electric) or TM (transverse magnetic) waves. For TE waves the electric field component $E_z = 0$ (E_x, E_y, H_z), H_z-polarization (p-polarization). For TM waves the component of the magnetic field $H_z = 0$ (H_x, H_y, E_z), E_z-polarization (s-polarization). The Helmholtz equations with the field time dependence $\exp(-i\omega t)$ for these two types of waves may be given as:

$$\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon_\perp(x)}\frac{\partial H_z}{\partial x}\right) + \frac{1}{\varepsilon_\perp(x)}\frac{\partial^2 H_z}{\partial y^2} + k^2\mu_\parallel H_z = 0, \tag{6.1}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu(x)}\frac{\partial E_z}{\partial x}\right) + \frac{1}{\mu(x)}\frac{\partial^2 E_z}{\partial y^2} + k^2\varepsilon_\parallel E_z = 0. \tag{6.2}$$

Here $\varepsilon_{\perp j}(x) = \varepsilon_j\left(1 - \frac{\varepsilon_{aj}^2}{\varepsilon_j^2}\right)$ is the effective value of dielectric permeability of MPhC layers media; $\mu(x) = \mu_j$ is the magnetic permeability of layers. The relationship of the tangential field components H_y and E_y through the components E_z and H_z (6.2), (6.3), is determined by the equations:

$$E_y = \left(\frac{1}{ik\varepsilon_\perp}\right)\left(\frac{\partial H_z}{\partial x} + i\frac{\varepsilon_a}{\varepsilon}\frac{\partial H_z}{\partial y}\right), \tag{6.3}$$

$$H_y = \left(\frac{1}{-ik\mu_j}\right)\frac{\partial E_z}{\partial x}. \tag{6.4}$$

From the presented (6.1), (6.2) for the both field components E_z, H_z and from the expressions of the tangential fields (6.3) and (6.4) it follows that the principle of permutation duality is fulfilled for the two types of TM and TE waves. Further, when the field component E_z in (6.1) is replaced by H_z and, at the same time, $\overleftrightarrow{\varepsilon} \leftrightarrow -\overleftrightarrow{\mu}$, it passes to (6.2). These facts allow us to simplify the consideration of a general electrodynamics problem and to limit for ourselves only one wave type (TE or TM) implementations for any type of medium. Equations (6.1) and (6.2) can be reduced by using the method of separation of variables to one type of equations of the Hill's equation with periodic coefficients [19–22], namely:

$$\frac{\partial}{\partial x}\left(p(x)\frac{\partial X}{\partial x}\right) + q(x)X_z = 0, \tag{6.5}$$

where $p(x)$ and $q(x)$ is the periodical coefficients which are determined by expressions $p(x) = \frac{1}{\varepsilon_{\perp j}(x)}$, $q(x) = p(x)\big(k^2\mu_{\parallel j}(x)\varepsilon_{\perp j}(x) - \beta^2\big)$ for TE waves and $p(x) = \frac{1}{\mu_{\perp j}(x)}$, $q(x) = p(x)\big(k^2\varepsilon_{\parallel j}(x)\mu_{\perp j}(x) - \beta^2\big)$ for TM waves. β is the wave propagation constant along the axis $Oy(exp(\pm i\beta y))$. Substitution $\frac{\partial^2}{\partial y^2} = -\beta^2$ is used when the variables are separated in (6.1).

The one-dimensional (6.5) is the Hill's equation [19–22] with periodic functions $p(x)$ and $q(x)$ such that $p(x+L) = p(x)$, $q(x+L) = q(x)$. Equation (6.5) with the corresponding boundary conditions is the boundary Sturm-Liouville problem. The boundary conditions for finding solutions to (6.5) are related to the continuity of the tangential components of the magnetic

$$H_z(x, y) = X(x)e^{i\beta y}$$

and electric

$$E_y(x, y) = \left(\frac{1}{ik\varepsilon_\perp}\right)\left(\frac{\partial X(x)}{\partial x} - \beta\frac{\varepsilon_a}{\varepsilon}X(x)\right)$$

fields at the boundaries of the layers and reduced to the following equations:

$$X_1(a) = X_2(a),$$

$$\frac{1}{\varepsilon_{\perp 1}}\left(\frac{\partial X_1(a)}{\partial x} - \beta\frac{\varepsilon_{a1}}{\varepsilon_1}X_1(a)\right) = \frac{1}{\varepsilon_{\perp 2}}\left(\frac{\partial X_2(a)}{\partial x} - \beta\frac{\varepsilon_{a2}}{\varepsilon_2}X_2(a)\right). \tag{6.6}$$

In addition to these boundary conditions, we use the Floquet theorem to relate solutions on the MPhC period, namely:

$$\rho X_1(0) = X_2(0 + L),$$

$$\rho\frac{1}{\varepsilon_{\perp 1}}\left(\frac{\partial X_1(0)}{\partial x} - \beta\frac{\varepsilon_{a1}}{\varepsilon_1}X_1(0)\right)$$

$$= \frac{1}{\varepsilon_{\perp 2}}\left(\frac{\partial X_2(0+L)}{\partial x} - \beta\frac{\varepsilon_{a2}}{\varepsilon_2}X_2(0+L)\right). \tag{6.7}$$

Using the Hill's equation solution in the form $X(x) = A\psi_1(x) + B\psi_2(x)$ we obtain the following fundamental solutions for the functions $\psi_1(x)$ and $\psi_2(x)$:

$$\psi_1(x) = \begin{cases} \cos\xi_1 x + \beta\frac{\varepsilon_{a1}}{\varepsilon_1}\frac{\sin\xi_1 x}{\xi_1}, & 0 < x < a \\ A\cos\xi_2(x-a) + \\ \quad + B\dfrac{\sin\xi_2(x-a)}{\xi_2}, & a < x < L \end{cases}, \tag{6.8}$$

$$\psi_2(x) = \begin{cases} \varepsilon_{\perp 1}\frac{\sin\xi_1 x}{\xi_1}, & 0 < x < a \\ D\cos\xi_2(x-a) + \\ \quad + C\dfrac{\sin\xi_2(x-a)}{\xi_2}, & a < x < L \end{cases}, \tag{6.9}$$

where

$$A = \cos\xi_1 a + \beta\frac{\varepsilon_{a1}}{\varepsilon_1}\frac{\sin\xi_1 a}{\xi_1}, \quad B = \beta\frac{\varepsilon_{a2}}{\varepsilon_2}\cos\xi_1 a - \frac{\varepsilon_{\perp 2}}{\varepsilon_{\perp 1}}\xi_1\sin\xi_1 a$$
$$+ \beta\beta\frac{\varepsilon_{a1}}{\varepsilon_1}\left(\frac{\varepsilon_{a2}}{\varepsilon_2} - \frac{\varepsilon_{\perp 2}}{\varepsilon_{\perp 1}}\frac{\varepsilon_{a1}}{\varepsilon_1}\right)\frac{\sin\xi_1 a}{\xi_1}, \quad D = \varepsilon_{\perp 1}\frac{\sin\xi_1 a}{\xi_1},$$
$$C = \varepsilon_{\perp 2}\cos\xi_1 a + \beta\varepsilon_{\perp 2}\left(\frac{\varepsilon_{a2}}{\varepsilon_2}\frac{\varepsilon_{\perp 1}}{\varepsilon_{\perp 2}} - \frac{\varepsilon_{a1}}{\varepsilon_1}\right)\frac{\sin\xi_1 a}{\xi_1}.$$

Using the Floquet theorem and (6.7), we obtain in general the characteristic equation for the definition of a constant $\rho = e^{iKL}$, namely:

$$\left(\rho + \rho^*\right) = 2\cos KL = \frac{1}{\varepsilon_{\perp 2}}\left[\psi_2'(L) - \beta\frac{\varepsilon_{a2}}{\varepsilon_2}\psi_2(L)\right] + \psi_1(L),$$

which, taking into account fundamental solutions of the Hill's (6.8), (6.9), takes the following form for determining the Floquet wave number $K = K_{TE}$ in the MPhC TE waves:

$$\cos K_{TE}L = \cos\xi_1 a\cos\xi_2 b$$
$$-\frac{1}{2}\left[\begin{array}{l}\dfrac{\varepsilon_{\perp 2}}{\varepsilon_{\perp 1}}\dfrac{\xi_1}{\xi_2} + \dfrac{\varepsilon_{\perp 1}}{\varepsilon_{\perp 2}}\dfrac{\xi_2}{\xi_1} \\ + \dfrac{\beta^2}{\xi_1\xi_2}\left(\dfrac{\varepsilon_{\perp 2}}{\varepsilon_{\perp 1}}\right)\left(\dfrac{\varepsilon_{a1}}{\varepsilon_1} - \dfrac{\varepsilon_{\perp 1}}{\varepsilon_{\perp 2}}\dfrac{\varepsilon_{a2}}{\varepsilon_2}\right)^2\end{array}\right]\sin\xi_1 a\sin\xi_2 b. \tag{6.10}$$

Note that the dispersion (6.10) exactly coincides with the equation which was obtained by the transfer matrix method [11].

The fundamental solutions of the Hill's (6.8), (6.9) make it possible analytically to determine the reflection and transmission coefficients for a plane wave that propagates through a bounded MPhC.

$$
\begin{aligned}
T_N &= 2\left[\left(W_{22}^N + \frac{\varepsilon_{in}}{\xi_{in}}\frac{\xi_{ex}}{\varepsilon_{ex}}W_{11}^N\right) - \left(\frac{\varepsilon_{in}}{\xi_{in}}kW_{21}^N + \frac{\xi_{ex}}{\varepsilon_{ex}}\frac{1}{k}W_{12}^N\right)\right]^{-1}, \\
R_N &= 1 - T_N\left(\frac{\varepsilon_{in}}{\xi_{in}}\frac{\xi_{ex}}{\varepsilon_{ex}}W_{11}^N + i\frac{\varepsilon_{in}}{\xi_{in}}ikW_{21}^N\right),
\end{aligned} \tag{6.11}
$$

where

$$
\boldsymbol{W}^N = \boldsymbol{W}\frac{\sin(NKL)}{\sin Kl} - \vec{\mathbf{1}}\frac{\sin[(N-1)KL]}{\sin Kl},
$$

$$
\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} \psi_1(L) & ik\psi_2(L) \\ \frac{1}{ik\varepsilon_{\perp 2}}\begin{bmatrix} \psi_1'(L) \\ -\beta\frac{\varepsilon_{a2}}{\varepsilon}\psi_1(L) \end{bmatrix} & \frac{1}{\varepsilon_{\perp 2}}\begin{bmatrix} \psi_2'(L) \\ -\beta\frac{\varepsilon_{a2}}{\varepsilon}\psi_2(L) \end{bmatrix} \end{pmatrix}.
$$

The minimum values of the $|T_N|^2_{\min}$ are determined by the expression

$$
|T_N|^2_{\min} = \left[1 + \left(\left(\frac{1}{2\sin KL}\right)^2 \left|\left(\frac{\varepsilon_{in}}{\xi_{in}}kW_{21} + \frac{\xi_{ex}}{\varepsilon_{ex}}\frac{1}{k}W_{12}\right)\right|^2 - 1\right)\right]^{-1}
$$

6.3 Analysis of Results

Let's turn to analysis of the waves propagation in MPhC that contains the semiconductor plasma and the dielectric layers. Two cases are considered for positive and negative values of the effective permittivity of semiconductor plasma for the single MPhC layer.

Figure 6.2 shows the dispersion characteristic as a dependence of the Floquet-Bloch wave number $K_{TE}L$ on the frequency parameter $\frac{\omega L}{2\pi c}$ (dotted blue curves). This dispersion characteristic was calculated with following parameters of the MPhC: $a = 0.2L$; $\varepsilon_1 = 18$; $\varepsilon_2 = 2.5$; $\varepsilon_{a1} = 10$. Forbidden zones are accented by shaded areas. The dependence of the limited MPhC reflection coefficient $|R|^2_N$ on $\frac{\omega L}{2\pi c}$ for $N = 6$ periods is represented on the same figure.

The same dependencies as in Fig. 6.2 only for negative values $\varepsilon_{1\perp} < 0$ and $\varepsilon_{a1} = 20$ are shown in Fig. 6.3. The transition from positive to negative magnitudes is accompanied not only by the transmission and forbidden zones location changes but also by their width.

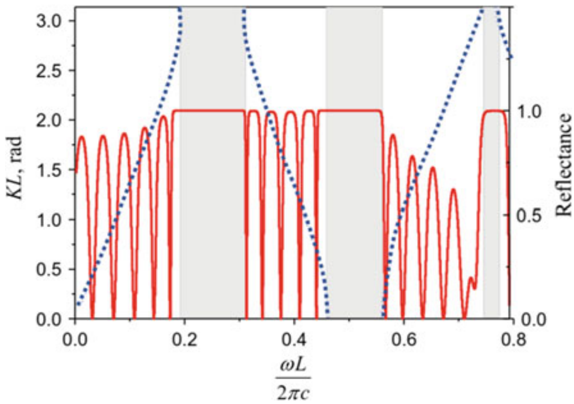

Fig. 6.2 Dispersion characteristics of the magnetophotonic crystal (dotted curves) and reflectance for positive effective value of permittivity (solid curve)

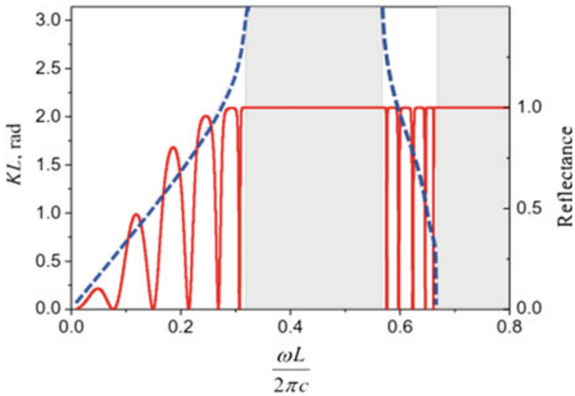

Fig. 6.3 Dispersion characteristics of the magnetophotonic crystal (dotted curves) and reflectance for negative effective value of permittivity (solid curve)

The number of the reflection (transmission) coefficient maximums in the bandwidth is equal $(N-1)$, and their value is determined by the expression $|R_N|^2_{\max} = 1 - |T_N|^2_{\min}$.

Figures 6.4 and 6.5 show spatial distributions of fields $\mathrm{Re}\, E_y$ in layers for two cases: full passing (Fig. 6.4, $\frac{\omega L}{2\pi c} = 0.6231$) in the transmission zone and full reflection in the forbidden zone (Fig. 6.5, $\frac{\omega L}{2\pi c} = 0.7$).

6.4 Conclusions

An analytical theory of gyrotropic plasma MPhC for determining eigenfunctions, dispersion characteristics, reflection and transmission coefficients of TM modes with arbitrary plasma layers material parameters have been developed. The obtained characteristics allow to construct the theory of controlled waveguide structures, whose guided surfaces can be Bragg reflecting surfaces from the considered MPhCs.

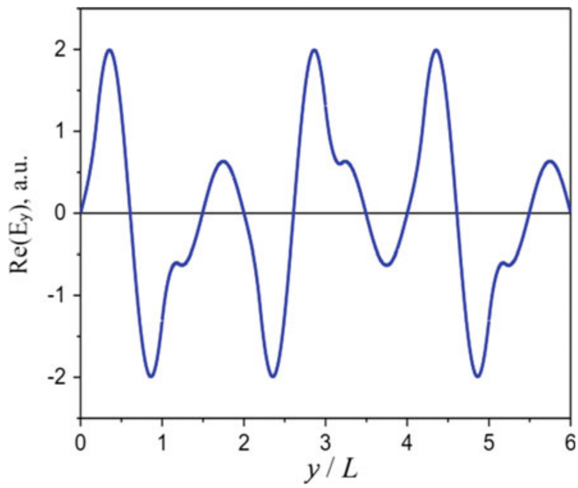

Fig. 6.4 Spatial distribution of the tangential component of electric field in magnetophotonic crystal (transmission zone)

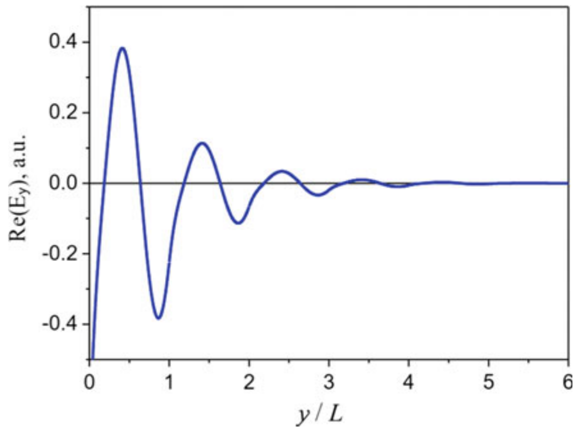

Fig. 6.5 Spatial distribution of the tangential component of electric field in magnetophotonic crystal (forbidden zone)

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