Chapter 2
Chases and Escapes

2.1 Introduction

The main interest of chases and escapes, or pursuit and evasion, is to obtain the pursuit curve of the chaser(s) and it has a long history [32]. It is said that the original formulation of the problem dates back to Leonardo da Vinci, who considered a cat-chasing-a-mouse problem. Since then, various problems of chases and escapes have attracted mathematicians, and some analytical results were obtained for specific setups. Because the analytical treatments are generally challenging, the setups have been traditionally restricted to one-to-one cases where a chaser pursues a single target.

In this chapter, we introduce the representative problems of chases and escapes. We start by describing a pursuit curve obtained by the French mathematician Pierre Bouguer in 1732 for the cat-chasing-a-mouse problem where the mouse (target) moves on a straight line. Then, a slight extension of the cat-chasing-a-mouse problem is proposed in which the target's path is on an inclined line. Even though this extension appears to be a small change, the closed-form solution in the absolute frame of reference was obtained only recently in 1991.

We subsequently turn our attention to another classic problem of circular pursuit in which the target moves on a circular path. This problem originates from the mid-18th century, and a clear mathematical formulation was done in 1920. It is noted that this problem involves coupled nonlinear differential equations and that a closed-form solution cannot be obtained. Relating to this problem, a new mathematical formulation was developed by Eliezer and Barton, which allows one to consider the problem beyond two-dimensional space. We present this formulation with examples of chases and escapes in three-dimensional space.

Interest in chases and escapes has grown to expand related topics. In particular, connections with game theories have been developed as an interdisciplinary field of mathematics, operations research, and economics. Such research topics are typically called "Discrete Search Games" [86] and "Differential Games" [41]. We do not

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introduce them in detail here, but a brief introduction of discrete search games is given in Appendix A.

2.2 Chases and Escapes with Straight Lines

Let us start here with one of the simplest representative problems, where the target moves on a straight line.

2.2.1 Bouguer's Problem

The first problem is to obtain the pursuit curve of a chaser to a target moving on a straight line at a constant speed (see Fig. 2.1). The chaser, also with a constant speed, is required to have its velocity vector pointing to the position of the target. This problem is called Bouguer's problem after the French mathematician Pierre Bouguer, who proposed and solved it in 1732 [10].

The main problems are (A) to obtain the analytical expression of the path of the chaser in this setup and (B) to obtain the point at which the chaser captures the target. Both of the problems were solved as follows.

(A) The path of the chaser is given by the following equation with the configurations in Fig. 2.1. Even though we just present the expression of the equation below, it will be derived as a special case of the generalized problem in section 2.2.2.

(i) When the speed of the chaser, denoted by v_C, is different from that of the target, denoted by v_T, the pursuit curve is written as

$$y(x) = \frac{n}{1-n^2}x_0 + \frac{1}{2}(x_0 - x)\left\{ \frac{\left(1-\frac{x}{x_0}\right)^n}{1+n} - \frac{\left(1-\frac{x}{x_0}\right)^{-n}}{1-n} \right\}, \tag{2.1}$$

where $n = v_T/v_C$ denotes the ratio of the speeds of the target to the chaser, and x_0 is the initial position of the target in the x-direction.

(ii) When the speeds of the chaser and the target are equal, the pursuit curve is written as

$$y(x) = \frac{1}{2}x_0 \left\{ \frac{1}{2}\left(1 - \frac{x}{x_0}\right)^2 - \ln\left(1 - \frac{x}{x_0}\right) \right\} - \frac{1}{4}x_0. \tag{2.2}$$

(B) The position of the point at which the chaser captures the target is given by

$$(x,y) = \left(x_0, \frac{nx_0}{1-n^2}\right),$$

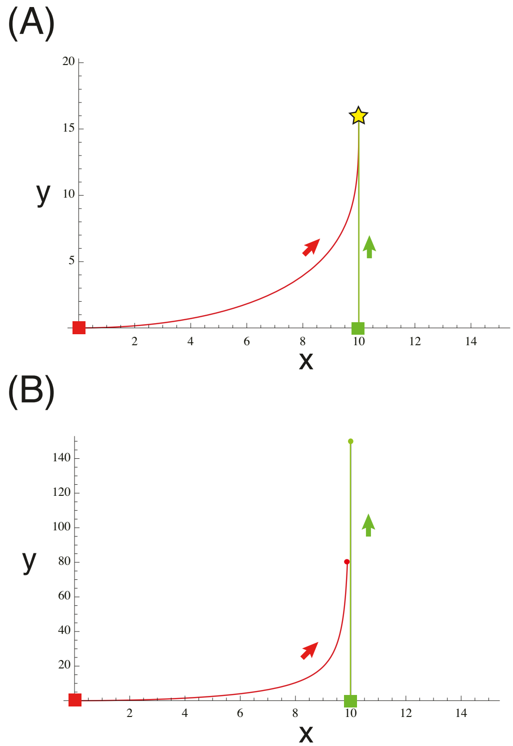

Fig. 2.1 Bouguer's problem: The green line represents the path of the target and the red curve represents the path of the chaser. (A) When the speed of the chaser is larger than that of the target, the target is captured at the position marked by the star. (B) When the speed of the chaser is smaller than that of the target, the chaser cannot capture the target. The initial position of the target, indicated by the green square, is at $(x,y) = (x_0, 0)$ with $x_0 = 10$ and that of the chaser, indicated by the red square, is at $(x,y) = (0,0)$. The green and red arrows, respectively, indicate the directions of the motion of the target and the chaser along the paths.

which can be finite only if $n < 1$, i.e., the speed of the chaser is faster than the target. Otherwise, the capture does not take place as shown in Fig. 2.1(B). Note that this is also the case when the speeds are the same, corresponding to the path of Eq. (2.2).

Albeit the simplicity of the problem, one can appreciate from the above results that the analytical expressions are rather intricate.

One can consider variations of the problem. For example, the chaser can move in the direction to the target's anticipated position in the future for an interception, instead of pointing to the target's current position.

2.2.2 Chases and Escapes with Inclined Lines

We now generalize Bouguer's problem [17, 26, 27]. Here, the line on which the target moves is an inclined straight line. The schematic view is shown in Fig. 2.2. The target, denoted by E, now moves on an inclined straight line, starting from the point $A = (0, y_0)$. At the same time, the chaser C starts from the origin, $O = (0,0)$. Let us assume that the target and the chaser move with the constant speeds, v_T and v_C, respectively, and that the chaser C always points its velocity vector toward the target E. The position of E is denoted by $(X(t),Y(t))$, and the problem is to find the analytical expression for the path of C whose position is given by $(x(t),y(t))$.

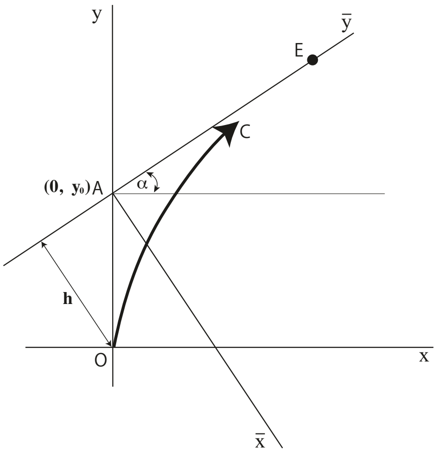

Fig. 2.2 Schematic view of the problem for which the target moves on an inclined line. See the main text for details.

By the condition that the chaser C at $(x(t),y(t))$ always points its velocity vector $(\dot{x}(t),\dot{y}(t))$ to the target E at $(X(t),Y(t))$, there is a relation between their positions,

$$(X(t),Y(t)) = (x(t)+\lambda(t)\dot{x}(t), y(t)+\lambda(t)\dot{y}(t)), \tag{2.3}$$

where $\lambda(t)$ is a function of time and the dot denotes derivative with respect to time t. By denoting the ratio of the speeds v_T and v_C as $n = v_T/v_C$, we also have

$$\dot{X}^2(t)+\dot{Y}^2(t) = n^2(\dot{x}^2(t)+\dot{y}^2(t)). \tag{2.4}$$

The above set of equations (2.3) and (2.4) are the basic equations of the problem: three equations to solve for $x(t), y(t)$, and $\lambda(t)$, given $X(t), Y(t)$, and n. We note that the expression of the equations is generally applicable to chase and escape problems in two-dimensional space.

We now formulate the condition in which the target E moves on a straight line. Here, let us assume, without the loss of generality, that the speed of E is equal to one ($v_T = 1$ and $n = 1/v_C$). By taking t as time with this unit, we have $\dot{X}^2(t) + \dot{Y}^2(t) = 1$. Also the position of the target E is written as $(X(t), Y(t)) = (t\cos\alpha, y_0 + t\sin\alpha)$, where α denotes the angle of the straight line (see Fig. 2.2). Thus, the following set of equations is obtained from Eqs. (2.3) and (2.4) with respect to the position $(x(t), y(t))$ and the velocity $(\dot{x}(t), \dot{y}(t))$ of C,

$$\dot{x}^2 + \dot{y}^2 = \frac{1}{n^2}, \tag{2.5}$$

$$x + \lambda\dot{x} = t\cos\alpha, \tag{2.6}$$

$$y + \lambda\dot{y} = y_0 + t\sin\alpha. \tag{2.7}$$

We have to solve the above set of equations to obtain the analytical expression for the paths of C. In order to make the calculation simpler, we introduce new coordinates $(\bar{x}, \bar{y})$, where the $\bar{y}$ axis is in the direction of the motion of E, while $\bar{x}$ is perpendicular to it with the origin at A (see Fig. 2.2). In the new coordinates, the set of equations (2.5)–(2.7) is given in a simpler form,

$$\dot{\bar{x}}^2 + \dot{\bar{y}}^2 = \frac{1}{n^2}, \tag{2.8}$$

$$\bar{x} + \lambda\dot{\bar{x}} = 0, \tag{2.9}$$

$$\bar{y} + \lambda\dot{\bar{y}} = t. \tag{2.10}$$

We now solve them by eliminating λ from Eqs. (2.9) and (2.10):

$$\bar{y} - \bar{x}\frac{d\bar{y}}{d\bar{x}} = t. \tag{2.11}$$

If we further differentiate both sides by $\bar{x}$, the following equation is obtained:

$$-\bar{x}\frac{d^2\bar{y}}{d\bar{x}^2} = \frac{dt}{d\bar{x}}. \tag{2.12}$$

On the other hand, if we divide Eq. (2.8) by $(\dot{\bar{x}})^2$, we get

$$1 + \left(\frac{d\bar{y}}{d\bar{x}}\right)^2 = \frac{1}{n^2}\left(\frac{dt}{d\bar{x}}\right)^2. \tag{2.13}$$

By eliminating $\frac{dt}{d\bar{x}}$ from Eqs. (2.12) and (2.13), we arrive at

$$1 + \left(\frac{d\bar{y}}{d\bar{x}}\right)^2 = \left(\frac{\bar{x}}{n}\right)^2\left(\frac{d^2\bar{y}}{d\bar{x}^2}\right)^2. \tag{2.14}$$

Since the chaser's curve is concave downward in the coordinate frame of $(\bar{x}, \bar{y})$, this leads to

$$\frac{d^2\bar{y}}{d\bar{x}^2} > 0, \quad (\forall \bar{x}). \tag{2.15}$$

Thus, by taking the square root of Eq. (2.14), the following is obtained with $p \equiv \frac{d\bar{y}}{d\bar{x}}$:

$$\sqrt{1+p^2} = \left(\frac{\bar{x}}{n}\right)\left(\frac{dp}{d\bar{x}}\right). \tag{2.16}$$

We can integrate this equation by separating the variables $\bar{x}$ and p as

$$\frac{dp}{\sqrt{1+p^2}} = \frac{n}{\bar{x}} d\bar{x}, \tag{2.17}$$

which leads to

$$\ln\left[p+\sqrt{1+p^2}\right] = n\left(\ln\bar{x} - \ln\bar{x}_c\right), \tag{2.18}$$

or

$$p+\sqrt{1+p^2} = \left(\frac{\bar{x}}{\bar{x}_c}\right)^n, \tag{2.19}$$

where $\bar{x}_c$ is a constant of integration. If we take the inverse of both sides, we obtain

$$\frac{1}{p+\sqrt{1+p^2}} = -p+\sqrt{1+p^2} = \left(\frac{\bar{x}}{\bar{x}_c}\right)^{-n}. \tag{2.20}$$

Equations (2.19) and (2.20) give

$$p = \frac{d\bar{y}}{d\bar{x}} = \frac{1}{2}\left[\left(\frac{\bar{x}}{\bar{x}_c}\right)^n - \left(\frac{\bar{x}}{\bar{x}_c}\right)^{-n}\right]. \tag{2.21}$$

We integrate this equation with the following initial conditions at $t = 0$ to obtain the pursuit curve,

$$\bar{x}(t=0) = y_0\cos\alpha, \quad \bar{y}(t=0) = -y_0\sin\alpha, \quad p(t=0) = -\tan\alpha. \tag{2.22}$$

Case 1: The speeds of the chaser and the target are different ($n \neq 1$)

The integration of Eq. (2.21) leads to

$$\bar{y} = \frac{n}{1-n^2} y_0\left(1+n\sin\alpha\right) + \frac{1}{2} y_0\cos\alpha \times \left[\frac{1}{1+n}\left(\frac{1-\sin\alpha}{\cos\alpha}\right)\left(\frac{\bar{x}}{y_0\cos\alpha}\right)^{1+n} - \frac{1}{1-n}\left(\frac{1-\sin\alpha}{\cos\alpha}\right)^{-1}\left(\frac{\bar{x}}{y_0\cos\alpha}\right)^{1-n}\right]. \tag{2.23}$$

This equation represents the analytical expression of the path of the chaser, C, in the coordinates of $(\bar{x}, \bar{y})$. If one wants the corresponding equation in the original coordinates of (x, y), one can make the coordinate transform using

$$\bar{x} = xy_0 \sin\alpha - (y - y_0)\cos\alpha, \quad \bar{y} = x\cos\alpha - (y - y_0)\sin\alpha. \tag{2.24}$$

Examples of the path obtained by numerical simulations are shown in Fig. 2.3.

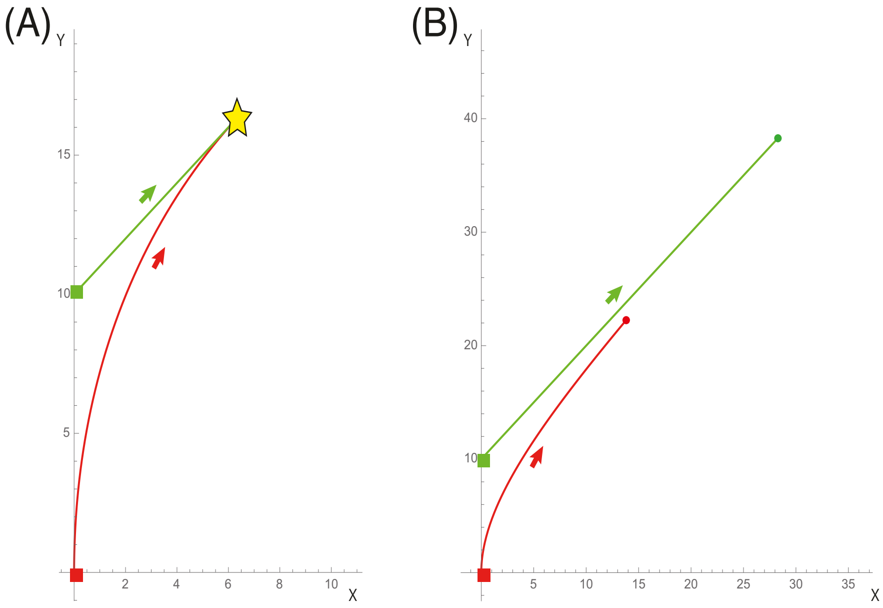

Fig. 2.3 Examples of the pursuit curves with inclined lines. The parameters are (A) $n = 0.5, \alpha = \pi/4, y_0 = 10$ and (B) $n = 1.5, \alpha = \pi/4, y_0 = 10$. The green and red squares indicate the starting points of the target and the chaser, respectively, and the green and red arrows indicate the directions of the motion of the target and the chaser along the paths, respectively.

We also note that by setting

$$\bar{x} = x_0 - x, \quad y_0 = x_0, \quad \alpha = 0, \tag{2.25}$$

the problem reduces to the original Bouguer's problem in section 2.2.1, and Eq. (2.1) is obtained from Eq. (2.23).

Case 2: The speeds of the chaser and the target are equal ($n = 1$)

In the case that the speeds of chaser and target are the same, Eq. (2.21) is written as

$$p = \frac{d\bar{y}}{d\bar{x}} = \frac{1}{2}\left[\left(\frac{\bar{x}}{\bar{x}_c}\right) - \left(\frac{\bar{x}}{\bar{x}_c}\right)^{-1}\right]. \tag{2.26}$$

We integrate this equation with the initial condition

$$\bar{x}_c = \frac{y_0 \cos^2 \alpha}{1 - \sin \alpha}, \tag{2.27}$$

leading to the following results:

$$\bar{y} = -\frac{1}{2}\left(\frac{y_0 \cos^2 \alpha}{1 - \sin \alpha}\right) \times \\ \left[\ln \frac{\bar{x}}{y_0 \cos \alpha} - \frac{1}{2}\left\{\frac{(1 - \sin \alpha)\bar{x}}{y_0 \cos^2 \alpha}\right\}^2 + 2\left(\frac{1 - \sin \alpha}{\cos \alpha}\right) \tan \alpha + \frac{1}{2}\left(\frac{1 - \sin \alpha}{\cos \alpha}\right)^2\right]. \tag{2.28}$$

If we set

$$\bar{x} = x_0 - x, \quad y_0 = x_0, \quad \alpha = 0, \tag{2.29}$$

the expression Eq. (2.2) in the previous section 2.2.1 is obtained from Eq. (2.28).

Catching up or not

Using the formulation, we can predict if the chaser can catch up with the target or not, as a function of the speed ratio $n = v_T/v_C = 1/v_C$ between the two. First, we note that the distance D between the chaser and the target is proportional to λ from Eqs. (2.3) and (2.4),

$$D = \sqrt{(X - x)^2 + (Y - y)^2} = \lambda v_C = \lambda / n. \tag{2.30}$$

From Eqs. (2.9) and (2.10), we can see that $\lambda(t) \geq 0$. Also by Eqs. (2.8)–(2.10) and (2.21), we obtain

$$\begin{aligned} \lambda(t) &= n\bar{x}\left[1 + \left(\frac{d\bar{y}}{d\bar{x}}\right)^2\right]^{\frac{1}{2}} = n\bar{x}\left[1 + \frac{1}{4}\left\{\left(\frac{\bar{x}}{\bar{x}_c}\right)^n - 2 + \left(\frac{\bar{x}}{\bar{x}_c}\right)^{-n}\right\}\right]^{\frac{1}{2}} \\ &= \frac{1}{2} n\bar{x}_c\left[\left(\frac{\bar{x}}{\bar{x}_c}\right)^{1+n} + \left(\frac{\bar{x}}{\bar{x}_c}\right)^{1-n}\right]. \end{aligned} \tag{2.31}$$

For the case $n < 1$, we can see $\lambda \to 0$ as $\bar{x} \to 0$. This means that the positions of the chaser and the target will coincide at $\bar{x} = 0$ [Eq. (2.3)]. Thus, the chaser can catch up. This is intuitively reasonable because, for $n < 1$, the chaser is faster than the target. The point of the catch-up $(\bar{x}^*, \bar{y}^*)$ is given with $\bar{x} = 0$ in Eq. (2.23),

$$(\bar{x}^*, \bar{y}^*) = \left(0, \frac{n y_0 (1 + n \sin \alpha)}{1 - n^2}\right). \tag{2.32}$$

On the other hand, for the case with $n > 1$, the values of λ [from Eq. (2.31)] and $\bar{y}$ [from Eq. (2.23)] approach positive infinity as $\bar{x} \to 0$. This means that the chaser cannot catch up with the target and the distance between the two increases.

For the case of equal speed $n = 1$, we expect that the chaser cannot catch up with the target if they are initially far separated. This can be seen from Eq. (2.31) as

$$\lambda = \frac{\bar{x}^2}{2\bar{x}_c} + \frac{\bar{x}_c}{2}, \tag{2.33}$$

and it leads to

$$\lambda \to \frac{\bar{x}_c}{2} = \frac{y_0 \cos^2\alpha}{1 - \sin\alpha}, \quad (\bar{x} \to 0). \tag{2.34}$$

This means that there exists a constant distance $\bar{x}_c/2$ between the chaser and the target even as the time increases, therefore, the chaser cannot catch up with the target.

2.3 Chases and Escapes with Circular Paths

We now turn our attention to the case of circular motions by the target. While the problem is traditionally investigated in two-dimensional space, we will give a brief review on an extension to three-dimensional space, which was done only recently [5].

2.3.1 The Classic Problem in Two-Dimensional space

Let us first look at the problem in which the target moves in a circle (see Fig. 2.4). The rules for the chaser and the target are basically the same as that of the case in the previous section. The chaser always points its velocity vector to the current position of the target. This problem was first proposed in an English journal called *Ladies' Diary* in 1748 [71].

In contrast to Bouguer's problem, the chaser's path for this problem cannot be solved in an analytically closed form. However, certain characteristics are observed in numerical simulations as illustrated in Fig. 2.4. When the speed of the chaser is slower than the target, the chaser cannot catch up with the target. After a long time, however, it also follows a circular path with a smaller radius [Fig. 2.4(a)(b)]. As we show below, the ratio of radii of the two circles is proved to be the same as the ratio of the speeds of the chaser and the target. When the speed of the chaser is faster than the target, the chaser can catch up with the target [Fig. 2.4(c)(d)].

To analyze the movements, we define the problem as shown in Fig. 2.5. The target moves on a circular path of a radius a, and is pursued by a chaser who can move n times faster than the target. Suppose that the initial position of the target at

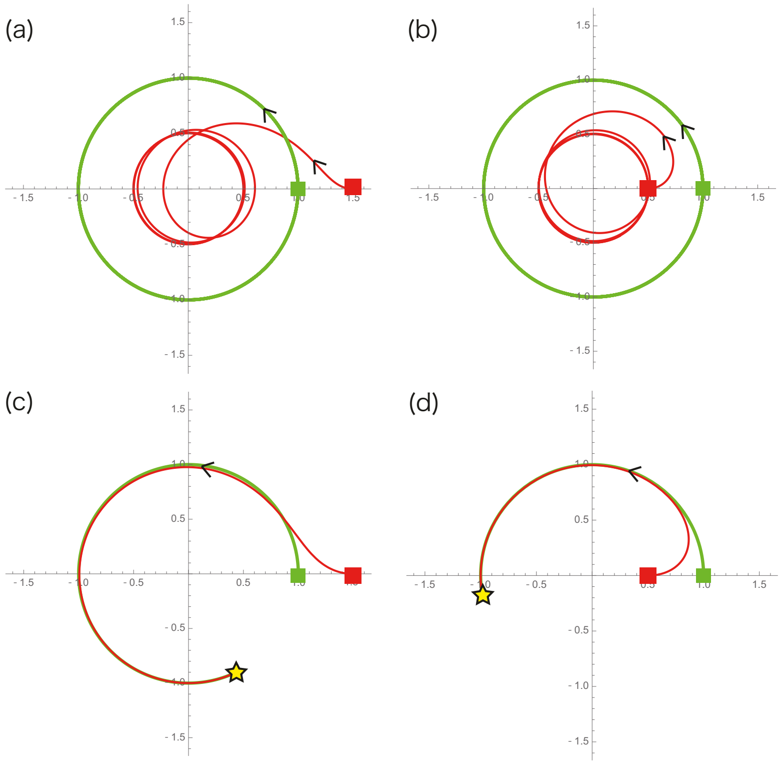

Fig. 2.4 Circular chase and escape: The green circle is the path of the target moving at a constant speed starting from $(x,y) = (1,0)$ indicated by the green square. The red curve represents the pursuit path of the chaser starting from $(x,y) = (1.5,0)$ for (a) and (c), and $(x,y) = (0.5,0)$ for (b) and (d). Both of the starting points are indicated by the red squares. The ratio of the speeds between the chaser and the target are $n = v_T/v_C = 2/1$ for (a) and (b), and $n = 0.95/1$ for (c) and (d). The chaser can catch up with the target for (c) and (d), and never catch up for (a) and (b).

time $t = 0$ is on the x–axis, at $(a,0)$. Because the target moves on the circular path, the position of the target is represented by an angle θ as $(a\cos\theta, a\sin\theta)$. When the target moves from the initial position ($\theta = 0$) to the present position at θ, a distance traveled by the target is $a\theta$. Hence, during the time interval, the chaser moves by a distance of $s = na\theta$ to reach the present position at (x,y). It is assumed that the chaser always points its direction of motion to the target. This makes the tangent at (x,y) to the chaser's pursuit curve pass through the target's instantaneous position.

We denote the angle made by the tangent line and the x–axis by φ, and denote the distance between the chaser and the target by ρ (see Fig. 2.5A). Then, the unit vector in the direction of the tangent line is written as $\mathbf{u} = (\cos\varphi, \sin\varphi)$, and the vector normal to the direction is written as $\mathbf{v} = (\sin\varphi, -\cos\varphi)$. The length of side 1 (see Fig. 2.5B) is obtained by considering the inner product of $\mathbf{p} = (x,y)$ and $\mathbf{v}$ as $\mathbf{p}\cdot\mathbf{v} = x\sin\varphi - y\cos\varphi$. On the other hand, the length is also obtained by considering

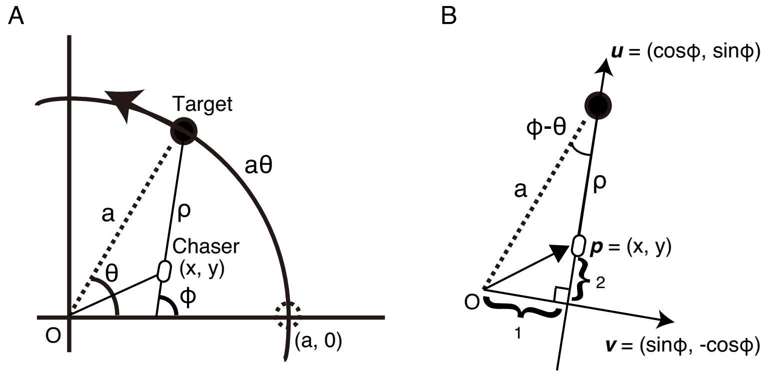

Fig. 2.5 (A) Definition of variables to characterize positions and movements of the chaser and the target. (B) The relative positions of the target and the chaser are extracted from (A).

the triangle of which side 1 is the base and the apex is the position of the target, as $a\sin(\varphi-\theta)$. Thus, the equation of the tangent line to the chaser's pursuit curve is written as

$$x\sin\varphi - y\cos\varphi = a\sin(\varphi-\theta). \tag{2.35}$$

Similarly, the length of side 2 (see Fig. 2.5B) is also obtained in two ways, and the equation of the line normal to the tangent line that passes through (x,y) is

$$x\cos\varphi + y\sin\varphi = a\cos(\varphi-\theta) - \rho. \tag{2.36}$$

By differentiating Eq. (2.35) with respect to θ, one gets

$$\frac{dx}{d\theta}\sin\varphi + x\cos\varphi\frac{d\varphi}{d\theta} - \frac{dy}{d\theta}\cos\varphi + y\sin\varphi\frac{d\varphi}{d\theta} = a\cos(\varphi-\theta)\left(\frac{d\varphi}{d\theta}-1\right).$$

This equation is rewritten as

$$\frac{dx}{d\theta}\sin\varphi - \frac{dy}{d\theta}\cos\varphi + \frac{d\varphi}{d\theta}(x\cos\varphi + y\sin\varphi) = a\cos(\varphi-\theta)\left(\frac{d\varphi}{d\theta}-1\right).$$

From Eq. (2.36), the last term on the left-hand side (in the bracket) is $a\cos(\varphi-\theta)-\rho$. Then, we have

$$\frac{dx}{d\theta}\sin\varphi - \frac{dy}{d\theta}\cos\varphi + a\cos(\varphi-\theta)\frac{d\varphi}{d\theta} - \rho\frac{d\varphi}{d\theta} = a\cos(\varphi-\theta)\frac{d\varphi}{d\theta} - a\cos(\varphi-\theta),$$

or

$$\frac{dx}{d\theta}\sin\varphi - \frac{dy}{d\theta}\cos\varphi - \rho\frac{d\varphi}{d\theta} = -a\cos(\varphi-\theta). \tag{2.37}$$

From the definition of φ, one sees

$$\frac{dy}{dx} = \tan\varphi.$$

Also, since the chaser moves by a distance of $s = na\theta$ to reach the present position at (x, y), and

$$\frac{dx}{ds} = \cos\varphi,$$

then

$$ds = \frac{dx}{\cos\varphi} = na\,d\theta,$$

and so

$$\frac{dx}{d\theta} = na\cos\varphi. \tag{2.38}$$

The chain rule of calculus gives

$$\frac{dy}{d\theta} = \frac{dy}{dx}\cdot\frac{dx}{d\theta} = \tan\varphi\, na\cos\varphi,$$

or

$$\frac{dy}{d\theta} = na\sin\varphi. \tag{2.39}$$

Thus, Eq. (2.37) is rewritten by inserting Eqs. (2.38) and (2.39) as

$$na\cos\varphi\sin\varphi - na\sin\varphi\cos\varphi - \rho\frac{d\varphi}{d\theta} = -a\cos(\varphi-\theta),$$

or

$$\rho\frac{d\varphi}{d\theta} = a\cos(\varphi-\theta). \tag{2.40}$$

This equation is one of the differential equations to explain the movements of the chaser.

Another equation is also obtained as follows. By differentiating Eq. (2.36) with respect to θ, one gets

$$\frac{dx}{d\theta}\cos\varphi - x\sin\varphi\frac{d\varphi}{d\theta} + \frac{dy}{d\theta}\sin\varphi + y\cos\varphi\frac{d\varphi}{d\theta} = -a\sin(\varphi-\theta)\left(\frac{d\varphi}{d\theta}-1\right) - \frac{d\rho}{d\theta}.$$

This equation is rewritten as

$$\frac{dx}{d\theta}\cos\varphi + \frac{dy}{d\theta}\sin\varphi - \frac{d\varphi}{d\theta}(x\sin\varphi - y\cos\varphi) = -a\sin(\varphi-\theta)\left(\frac{d\varphi}{d\theta}-1\right) - \frac{d\rho}{d\theta}.$$

From Eq. (2.35), the last term on the left-hand side (in the bracket) is $a\sin(\varphi-\theta)$, and we have

$$\frac{dx}{d\theta}\cos\varphi + \frac{dy}{d\theta}\sin\varphi - a\sin(\varphi-\theta)\frac{d\varphi}{d\theta} = -a\sin(\varphi-\theta)\frac{d\varphi}{d\theta} + a\sin(\varphi-\theta) - \frac{d\rho}{d\theta},$$

or

$$\frac{dx}{d\theta}\cos\varphi + \frac{dy}{d\theta}\sin\varphi = a\sin(\varphi-\theta) - \frac{d\rho}{d\theta}. \tag{2.41}$$

Thus, Eq. (2.41) is rewritten by inserting Eqs. (2.38) and (2.39) as

$$na = a\sin(\varphi-\theta) - \frac{d\rho}{d\theta},$$

or

$$\frac{d\rho}{d\theta} = a\left[\sin(\varphi-\theta) - n\right]. \tag{2.42}$$

To sum up, the following set of differential equations is obtained by differentiating Eqs. (2.35) and (2.36) with respect to θ:

$$\rho\frac{d\varphi}{d\theta} = a\cos(\varphi-\theta), \tag{2.43}$$

$$\frac{d\rho}{d\theta} = a\left[\sin(\varphi-\theta) - n\right]. \tag{2.44}$$

It is not possible to solve these equations analytically. However, we are still able to give some insights into the nature of the chaser's trajectory. By introducing a variable $\omega = \varphi - \theta$, we have $d\omega/dt = d\varphi/dt - 1$. Here, we assume $\theta = t$ because the target moves on the circular path at a constant speed. Then, Eq. (2.43) becomes

$$\rho\frac{d\omega}{dt} + \rho = a\cos\omega. \tag{2.45}$$

Also, Eq. (2.44) is written as

$$\frac{d\rho}{dt} = a\sin\omega - an. \tag{2.46}$$

Differentiating Eq. (2.46) with respect to t, we have

$$\frac{d^2\rho}{dt^2} = a\cos\omega\frac{d\omega}{dt},$$

or

$$\frac{d\omega}{dt} = \frac{d^2\rho/dt^2}{a\cos\omega}.$$

By substituting it into Eq. (2.45), we obtain

$$\rho\frac{d^2\rho}{dt^2} + a\rho\cos\omega = a^2\cos^2\omega. \tag{2.47}$$

This equation suggests that in a steady-state solution which satisfies

$$\frac{d\rho}{dt} = \frac{d^2\rho}{dt^2} = 0,$$

the chaser also moves on a circular path (see Fig. 2.6), so that the chord between the chaser and the target rotates but its length ρ is a constant, where

$$\rho = a \cos \omega.$$

At the steady state, one also gets $n = \sin \omega$ from Eq. (2.46). Then,

$$\rho = a\sqrt{1-n^2}. \tag{2.48}$$

Writing the radius of the circular path of the chaser by R, the Pythagorean theorem says

$$R^2 + \rho^2 = a^2.$$

Then,

$$R = \sqrt{a^2-\rho^2} = \sqrt{a^2-a^2(1-n^2)} = \sqrt{a^2n^2} = na.$$

Hence, when $n \leq 1$, the chaser cannot catch up with the target, and that the chaser's pursuit curve eventually becomes a circle with the radius $R = na$.

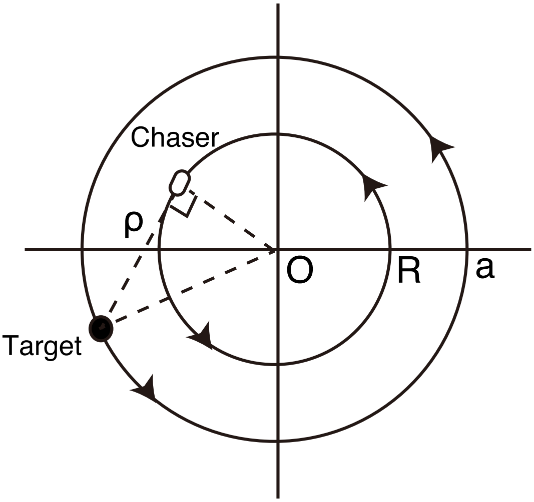

Fig. 2.6 In the steady state $d\rho/dt = 0$ and $d^2\rho/dt^2 = 0$, the chord joining the chaser and the target rotates, but its length (ρ) remains unchanged.

2.3.2 *Extension to Three-Dimensional Space*

In considering general problems of chases and escapes in three-dimensional space, the following approach can be taken by extending the method described for two-dimensional space in the previous section. Here, we denote the position of the target by (X,Y,Z) and that of the chaser by (x,y,z). The ratio of the speeds between the two is given by $n = v_T/v_C$. We derive the set of equations by the condition that the chaser always points its velocity vector to the position of the target.

In the same manner as for Eqs. (2.3) and (2.4), one writes

$$(X,Y,Z) = (x+\lambda\dot{x}, y+\lambda\dot{y}, z+\lambda\dot{z}), \tag{2.49}$$

$$\dot{X}^2+\dot{Y}^2+\dot{Y}^2 = n^2(\dot{x}^2+\dot{y}^2+\dot{z}^2). \tag{2.50}$$

The problem is to obtain the solution for the four functions, $x(t), y(t), z(t), \lambda(t)$, from the four first-order non-linear differential equations, given $X(t), Y(t), Z(t)$, and n. We have assumed that the chasers and the targets are both moving, respectively, with constant speeds, but this condition can be relaxed as long as the ratio n remains constant.

Because the set of the differential equations is nonlinear, it is generally difficult to obtain solutions. Here, we will present only a few special solutions for the setups explained below without showing derivations in detail. The interested readers may refer to the paper by Barton and Eliezer [5] for their derivations and other solutions.

2.3.2.1 Circular Cylindrical Helices

The first example is where the target moves in a circular cylindrical helix of constant pitch, a straightforward extension of the two-dimensional circular chases and escapes in section 2.3.1. The derivation of the path is somehow in the reverse direction because we first specify the path of the chaser. Let us consider the case where the chaser is moving in a circular cylindrical helix of equal pitch, which is represented by the following equation with a constant pitch p:

$$x = \cos t, \quad y = \sin t, \quad z = \left(\frac{p}{2\pi}\right)t. \tag{2.51}$$

Then, from Eq. (2.49), the target is at the location given below:

$$X = \cos t - \lambda\sin t, \quad Y = \sin t + \lambda\cos t, \quad Z = \left(\frac{p}{2\pi}\right)(t+\lambda). \tag{2.52}$$

If we assume that λ is a constant at this point and introduce a new parameter α by

$$\cos\alpha = \frac{1}{\sqrt{1+\lambda^2}}, \tag{2.53}$$

one can show that the position of the target (2.52) is rewritten as

$$X = \sqrt{1+\lambda^2}\cos(t+\alpha), \quad Y = \sqrt{1+\lambda^2}\sin(t+\alpha), \quad Z = \left(\frac{p}{2\pi}\right)(t+\lambda). \quad (2.54)$$

This expression represents that the path of the target is a circular cylindrical helix with the radius of $\sqrt{1+\lambda^2}$ and the pitch p. For these paths, it can be further shown that $n = v_T/v_C > 1$ and that the distance between the chaser and the target is a constant $\lambda\sqrt{1+p^2/4\pi^2}$. This indicates that the distance between the two does not change when the chaser moves slower than the target. Projecting the motions onto the two-dimensional $x-y$ plane corresponds to the long-time limit case of the circular chase and escape in section 2.3.1. One example of paths of the chaser and the target is shown in Fig. 2.7.

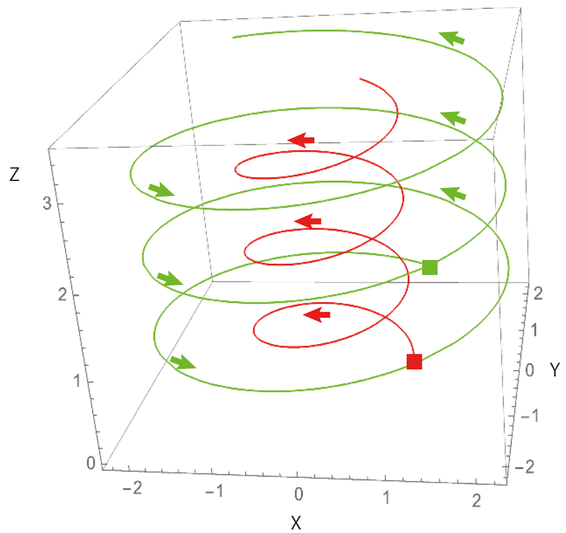

Fig. 2.7 Circular chase and escape in three-dimensional space: The red helix represents the path of the chaser starting from the point $(1,0,0)$, while the green one represents the path of the target starting from $(1,2,1/\pi)$. The parameters are $p = 1.0$ and $\lambda = 2.0$. The green and red arrows, respectively, indicate the directions of the motion of the target and the chaser along the paths.

2.3.2.2 Equiangular Spiral Helix

The second example is a spiral. The positions of the chaser and the target are, respectively, given by the following equations:

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad z = Bt, \quad (2.55)$$

and

$$X = e^{-t}\cos(t+\pi/2), \quad Y = e^{-t}\sin(t+\pi/2), \quad Z = B(t+1), \tag{2.56}$$

where B is a parameter. One example is shown in Fig. 2.8. At $t = 0$, the chaser starts at $(1,0,0)$, while the target starts at $(0,1,B)$ (both starting points are located outside the space in Fig. 2.8).

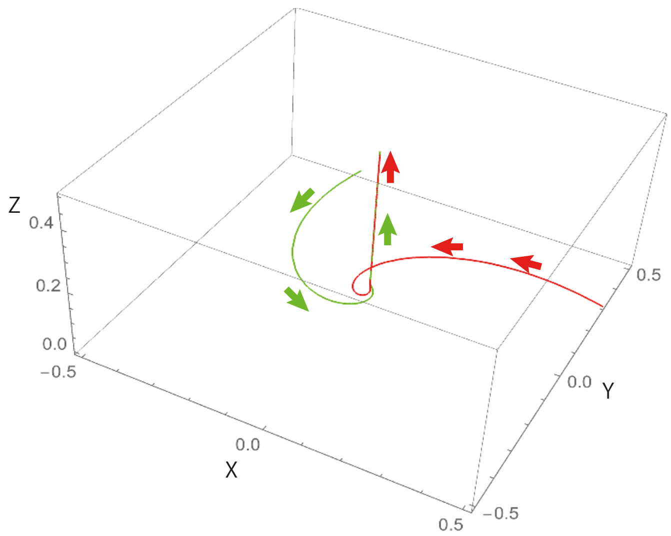

Fig. 2.8 Circular chase and escape with equiangular spiral helix in three-dimensional space: The red helix represents the path of the chaser starting from $(1,0,0)$, while the green one represents the path of the target starting from $(0,1,B)$. The parameter B is set as $B = 0.2$.

Here, we assume that the chaser and the target move at an equal constant speed ($n = 1$). As time progresses, the paths of both the chaser and the target appear as if they climb up and wind around the z axis. The distance D between the chaser and the target is greater than B, and approaches B in the limit of infinitely long time, i.e., the chaser cannot catch up with the target.

If we set $B = 0$, the motions of the chaser and the target are restricted to the two-dimensional plane as shown in Fig. 2.9. In this case, both of them fall into the origin in the long-time limit.

We can also consider another kind of spiral. For example, the following equations represent the spirals for both moving toward the origin:

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad z = Be^{-2t}, \tag{2.57}$$

$$X = e^{-t}\cos(t+\pi/2), \quad Y = e^{-t}\sin(t+\pi/2), \quad Z = -Be^{-2t}. \tag{2.58}$$

In this example, the chaser starts at $(1,0,B)$ to move downwards to the origin, and the target starts at $(0,1,-B)$ to move upwards to the origin. Typical paths are

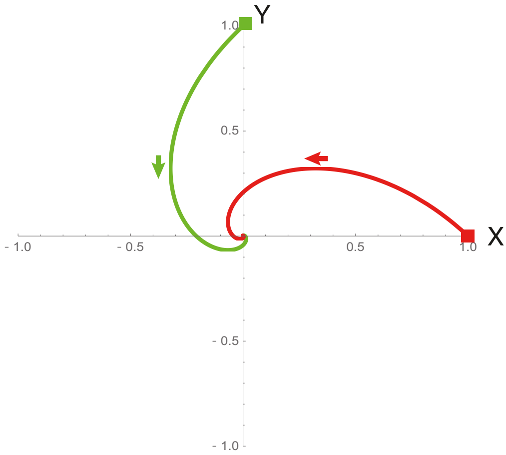

Fig. 2.9 Circular chase and escape with equiangular spiral helix to be projected onto the two-dimensional plane by fixing $B = 0$: The red helix represents the path of the chaser, while the green one represents the path of the target. The green and red arrows, respectively, indicate the directions of the motion of the target and the chaser along the paths.

shown in Fig. 2.10. We note that, by fixing $B = 0$, this example also reduces to the two-dimensional motions as described in Fig. 2.9.

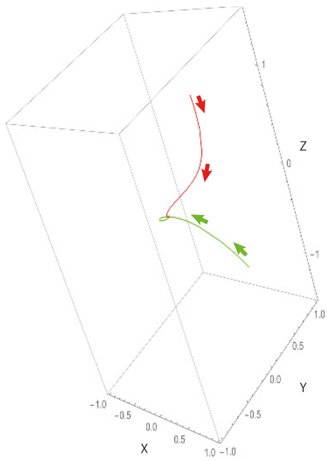

Fig. 2.10 Circular chase and escape with equiangular spiral-helix in three-dimensional space: The red helix represents the path for the chaser starting from $(1,0,B)$, while the green one represents the path for the target starting from $(0,1,-B)$. The parameter B is fixed as $B = 1.3$. The green and red arrows, respectively, indicate the directions of the motion of the target and the chaser along the paths.