Chapter 1
Introduction

The main theme of this book, entitled Group Chase and Escape, is a fusion of two different directions of mathematical and scientific research fields. One field deals with the traditional mathematical problems of chases and escapes, and the other is a more recent emerging field of "self-driven" particles to explain collective motions.

In this introduction, we give an overview of this book by briefly describing backgrounds and contents of the subsequent chapters. We first describe the two fields: chases and escapes (section 1.1) and collective motions (section 1.2). Then we move on to briefly explain our proposal, namely, Group Chase and Escape (section 1.3). It follows potential applications and challenges of our proposal (section 1.4).

1.1 Chases and Escapes

Situations relating to chases and escapes are ubiquitous in our daily lives. We see them in detective stories on TV; kids are playing games of tag; cats are chasing mice, and so on. They have also attracted the interest of mathematicians for a long time [71].

The oldest formulation of the problem is said to have been done by Leonardo da Vinci, who considered a cat-chasing-a-mouse problem [32]. The escaping mouse moves at a constant speed along a straight line, which is perpendicular to the line connecting the initial positions between them. The chasing cat also moves at a constant speed with its direction always pointing to the mouse.

Though the problem statement is simple, it took over 200 years before the pursuit curve of the cat was obtained by Pierre Bouguer in 1732 [10]. After that, this problem was presented in a book on differential equations by George Boole, who was a prominent mathematician known for Boolean algebra and logic [9].

A slight generalization of the problem had been studied where the straight line on which the mouse moves is inclined at an arbitrary angle, but the closed-form analytical solution in the absolute frame of reference was obtained only recently in 1991 [17], about half a millennium after Leonardo da Vinci's time!

A. Kamimura and T. Ohira, *Group Chase and Escape*,
Theoretical Biology, https://doi.org/10.1007/978-981-15-1731-0_1

Another classic problem from this topic is the case where a target moves in circular motions. The first appearance of a circular pursuit problem was reported in the mathematical puzzle section of an English journal called *Ladies' Diary* [71]. In the 1748 issue, the problem was stated as a spider chasing a fly moving on a semicircular pane of glass.

Since then, the circular pursuit problems surfaced occasionally until the end of 19th century, however, progress was not significant probably because the problems involved coupled nonlinear differential equations and it was generally impossible to obtain analytical solutions in closed form. The problem was often associated with A. S. Hathaway, who stated it clearly in 1920 [2].

Apparently, there were no big leaps until we could use computers to draw the pursuit curves as we will show in Chapter 2.

Investigations of pursuit and evasion problems continue. From around 1990, a new mathematical formulation was developed by Eliezer and Barton [26, 27, 5] which allowed one to consider the problems not only in two-dimensional but also in three-dimensional space. This new approach will be described with examples in Chapter 2.

The problems of "chases and escapes" have also developed in different directions. In the mid-twentieth century, connections were found with the field of game theory. The most notable one was developed into theories of "Differential Games", on which R. Issacs published a classic book in 1965 [41]. In these games, the two players, pursuer and pursued, have strategies so that they can, respectively, minimize and maximize specific objective criteria, which are often the time between the beginning and the termination of the game, or the event of "capture". The pursuer wants to catch the pursued as soon as possible, while the pursued wants to evade for as long as possible. The central question is to find the optimal strategy for each of the players to achieve the conflicting objectives.

A representative example in the differential game is called the "homicidal chauffeur" problem. In this game, the chauffeur (pursuer) tries to run over and capture the slower but more maneuverable pedestrian (pursued). The pursuer has a faster speed, but the pursued has a smaller minimum turn-radius. Surprisingly, various types of strategies and situations for the two players have been found associated with the different speeds and the turn-radius ratios.

Other extensions relating to game theories include Search Game Theories [86]. The main theme is that a chaser tries to locate hiders, who can or cannot move among hiding spots. One notable example is the "Princess and Monster" problem [104]. These hiding spots can also take different forms, such as being connected by networks of various kinds.

Another direction is a fusion of the chase and escape problems with classical mechanics. Analogies and techniques developed in chases and escapes are employed to obtain analytical solutions of curvilinear motion of a particle on an inclined plane [92].

1.2 Collective Motion

The history of investigating multi-particle systems in physics is almost as old as the beginning of classical mechanics. Johannes Kepler, who analyzed astronomical data left by Tycho Brahe, investigated relations between geometry and motions of six planets. As is well known, the Newton's laws were developed based on the work of Kepler.

The work by Daniel Bernoulli in the 18th century is considered as the pioneering work of the kinetic theory of gases. The theory is based on the hypothesis that gases are composed of a huge number of moving particles. In the 19th century, physicists including Rudolf J. E. Clausius, James C. Maxwell and Ludwig E. Boltzmann had further developed the theory leading to statistical physics. Also, the work by Robert Brown led to the theory of Brownian motion by Einstein. Subsequent experiments by Jean B. Perrin confirmed the atomic nature of matter as well as the determination of Avogadro's number.

Collective behaviors and their functions have been of great interest in the field of biology. An example includes investigations of brains composed of neural cells, or neurons [51]. Experimental and anatomical studies begun in the 19th century. J. E. Purkinje, Cammillo Golgi and Ramon Cajal are the names of the pioneering scholars. By the beginning of the 20th century, the notion of the brain and central nervous system as a network of neurons with information processing capability was established.

Theories of the brain as a neural network were sprung in the 20th century. Seminal work was done by the neurophysiologist Warren McCulloch and the mathematician Walter Pitts to connect a network of neurons with computations in 1943 [65]. Following the work, various conceptual and mathematical models and algorithms have been proposed. They include functions of memories, learning, pattern recognition, vision and other sensory systems. Applications to engineering problems and systems, such as constrained optimization problems and robotics, have also been vigorously pursued. In recent years, deep multilayer neural networks and associated algorithms called Deep Learnings have provoked much interest fusing ever more strongly with the field of artificial intelligence [57].

Another line of research has emerged in recent years on collective motions [39, 102] of insects, animals, birds, humans, and automobiles. The phenomena of flocking, grouping and congestions are commonly observed around us. Here, the line of research treats each entity as a "self-propelled" or "self-driven" particle, and a group of them as an aggregate of particles. In contrast to physical matter of atoms and molecules, the movements of the particles are modeled by "coarse-grained" rules and dynamics, and as a result, one tries to elucidate the generality of rich collective behavior.

Investigations of collective motions have been pursued both experimentally and theoretically. For example, careful observation of flying patterns of groups of pigeons has suggested a possible social hierarchy [70]. It is also found that aerodynamic effects are responsible for V-shaped flying patterns of certain birds [82]. Schools of fish such as sardines have been observed and analyzed [33, 34, 81]. The reaction time for collective response to disturbances such as attacks by larger fish is surprisingly short compared to the swimming speed of individuals. Studies of traffic data have shown for natural congestions without bottlenecks on highways that the head of congestion commonly moves backward approximately with the speed of 20km/h [94].

Various mathematical models have been proposed for theoretical investigations, to explain path formation of ants, pedestrian collective motions, flocks of birds and so on. Among them, we will present two representative theoretical frameworks in Chapter 3: Vicsek model [99] for flocking behavior and the Optimal Velocity model [4] for automotive congestions.

1.3 Group Chase and Escape

We have merged these two lines of research to propose a model of "Group Chase and Escape" [44]. On one hand, it is a simple extension of the traditional chase and escape problems to multiple players: one group chases another group. On the other hand, one can also view this proposal as an extension of the self-propelled particles into a mixture of two groups with different motives. In reality, such situations can be observed when one group of animals chases another, such as wolves chasing deer.

Mathematical analysis of group chase and escape is very challenging. Therefore, we have relied mostly on computational simulations to study the problem and show that a simple model can give rise to rich and complex behavior. In the original proposal of the framework, we first focused on certain quantities such as time for the total catch, average lifetimes of targets, and "cost" of capture by varying the number of chasers and targets. We found two qualitatively different regions for the quantities as a function of the number of chasers.

The most notable behavior of the simple model is the spatial organization of chasers and targets. To connect the two qualitatively-different regions with the spatial organization, we classify chasing processes into several patterns and introduce order parameters to characterize them. It is found that the number of targets decreases intermittently along with the capture of targets. At each event of the capture, the spatial organization of chasers changes drastically. Before the event, the chasers surround the target to be captured, and after that, they move to other remaining targets toward the next capture. Correspondingly, the order parameters change drastically at the events as well.

After introducing the basic model, we will discuss some of the recent developments of group chase and escape by extending the model. The developments will be reviewed by roughly categorizing them into three directions: abilities, reactions and motions.

The first category, "abilities", deals with modifications of abilities of the chasers and the targets for detecting the opponents' positions. Examples include the effects of changing and distributing the detection ranges of the chasers. It also suggests a benefit of division of labor in chasing and escaping.

The second category, “reactions”, deals with modifications of the relationship of pursuit and evasion, and the outcome of the “capture”. One investigated example extends the chasers and the targets to a three-member case [88]. In the example, three different species, A, B, and C, are considered. The species form a triangular relation of chasing and escaping: the first species A chases the second species B, and the species B chases the third species C, which in turn chases the first species A. Here, the captured member is converted to the member of the chasing group, for example, the captured B is converted to another member of A. Another related extension is considered as a vampire problem [75]. In this work, the basic rules for the chasers and the targets are the same as in the basic model. However, with some probabilities, the captured target can be converted into a new chaser rather than simply being removed from the field. It is observed that with this inclusion of conversion, there is a maximum for the time to catch all the targets when we change the initial number of targets.

The third category, “motions”, deals with modifications of the movements of the chaser and the target. Examples include the changes of lattice structures, off-lattice models, and errors in the movement. We also extend the model to facilitate interactions within each group. One investigated example is a model where chasers interact among themselves to repel within a certain distance. This has an effect that the chasers get less in each other’s way, and they tend to spread over the field. It is observed that when the number of chasers is small, this interaction works more effectively to reduce the time for the entire catch.

1.4 Potential Applications and Challenges

The topic of group chases and escapes described above is still at the beginning phase and there are a number of open problems and potentials for further developments and applications.

We will first discuss open problems such as the effects of boundary conditions, characterization of chasing patterns, and development of macroscopic descriptions, that require some theoretical challenges for further developments.

Then, we will present and discuss possible applications of group chases and escapes. The first topic is hunting in nature, in which we will take wolves as one typical example. One promising direction is to model cooperative hunting and compare with ethological studies. The second topic is to apply the framework to problems in engineering such as optimization problems. One attempt with preliminary results is described to employ the idea of chases and escapes for a combinatorial optimization problem. Finally, we make a remark by providing a general perspective of living together.

In the following chapters, we present the main body of the topics as we outlined in this introduction. The structure of the following chapters is as follows.

In Chapter 2, we introduce two classical examples of chases and escapes: a cat-chasing-a-mouse problem (Bouguer’s problem), and chases and escapes with circular paths. Recent developments are also described to extend the problems to three-dimensional space.

In Chapter 3, we provide a brief introduction of statistical mechanical approaches to collective motions. Then, we review two representative models, namely, the Vicsek model and the Optimal Velocity model.

In Chapter 4, we introduce a basic model to demonstrate group chase and escape. We review the behavior of the model obtained by computer simulations and present analysis to understand the dynamics. In addition, recent developments are described to modify and extend the basic model.

In Chapter 5, as described above, we discuss some open questions and challenges. Also, we point out possible applications of group chase and escape to cooperative hunting by wolves and optimization problems as promising directions for further developments.