Chapter 7
New Techniques on Fractional Reduced Differential Transform Method

7.1 Introduction

The fractional differential equations appear more and more frequently in different research areas and engineering applications. There is a long-standing interest in extending the classical calculus to noninteger orders because fractional differential equations are suitable models for many physical problems. Fractional calculus has been used to model physical and engineering processes which are found to be best described by fractional differential equations. In recent years, considerable interest in fractional differential equations has been stimulated due to their numerous applications in the areas of physics and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, control theory, neutron point kinetics model, anomalous diffusion, vibration and control, continuous time random walk, Lévy statistics, Brownian motion, signal and image processing, relaxation, creep, chaos, fluid dynamics, and material science are well described by differential equations of fractional order [1–8]. The solution of differential equations of fractional order is much involved. Though many exact solutions for linear fractional differential equation had been found, in general, there is a scarcity of analytical method, available in the open literature, which yields an exact solution for nonlinear fractional differential equations.

In the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit and numerical solutions to nonlinear differential equations of integer order. Many methods have been presented [9–19]. Our main interest lies in determining an efficient and accurate method that provides an effective procedure for explicit and numerical solutions of a wide and general class of differential systems representing real physical problems. In this paper, we solve fractional KdV equations by the modified fractional reduced differential transform method (MFRDTM) which is presented with some modification of the reduced differential transformation method [20–22]. In this new approach, the nonlinear term is replaced by its Adomian polynomials. Thus, the nonlinear initial-value

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problem can be easily solved with less computational effort. The main advantage of the method emphasizes the fact that it provides an explicit analytical approximate solution and also numerical solution elegantly. The merits of the new method are as follows: (1) no discretization required and (2) linearization or small perturbation also not required. Thus, it reduces the amount of numerical computation considerably. Application of this attractive new method may be taken into account for further research.

In the past decades, the fractional differential equations have been widely used in various fields of applied science and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, control theory, neutron point kinetics model, anomalous diffusion, vibration and control, continuous time random walk, Levy statistics, Brownian motion, signal and image processing, relaxation, creep, chaos, fluid dynamics, and material science are well described by differential equations of fractional order [1–7, 12, 23–26]. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason, we need a reliable and efficient technique for the solution of fractional differential equations. An immense effort has been expended over the last many years to find robust and efficient numerical and analytical methods for solving such fractional differential equations. In the present analysis, a new approximate numerical technique, coupled fractional reduced differential transform method (CFRDTM), has been proposed which is applicable for coupled fractional differential equations. The proposed method is a very powerful solver for linear and nonlinear coupled fractional differential equations. It is relatively a new approach to provide the solution very efficiently and accurately.

In the field of engineering, physics, and other fields of applied sciences, many phenomena can be obtained very successfully by models using mathematical tools in the form of fractional calculus [1, 4, 12, 23–27]. In the past decades, the fractional differential equations have been widely used in various fields of applied science and engineering. Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, control theory, neutron point kinetics model, anomalous diffusion, vibration and control, continuous time random walk, Lévy statistics, Brownian motion, signal and image processing, relaxation, creep, chaos, fluid dynamics, and material science are well described by differential equations of fractional order. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. For that reason, we need a reliable and efficient technique for the solution of fractional differential equations. An immense effort has been expended over the last many years to find robust and efficient numerical and analytical methods for solving such fractional differential equations. In the present analysis, a new approximate numerical technique, coupled fractional reduced differential transform method (CFRDTM), has been applied which is applicable for coupled fractional differential equations. The new method is a very powerful solver for linear and nonlinear coupled fractional differential equations. It is relatively a new approach to provide the solution very efficiently and accurately.

In the field of engineering, physics, chemistry, and other sciences, many phenomena can be modeled very successfully by using mathematical tools in the form of fractional calculus, e.g., anomalous transport in disordered systems, some percolations in porous media, and the diffusion of biological populations [1, 25–28]. Fractional calculus has been used to model physical and engineering systems that are found to be more accurately described by fractional differential equations. Thus, we need a reliable and competent technique for the solution of fractional differential equations. In this paper, the predator–prey system [29] has been discussed in the form of the fractional coupled reaction-diffusion equation. In the present analysis, a new approximate numerical technique, coupled fractional reduced differential transform method (CFRDTM), has been presented which is appropriate for coupled fractional differential equations. The proposed method is an impressive solver for linear and nonlinear coupled fractional differential equations. It is comparatively a new approach to provide the solution very effectively and competently.

The significant advantage of the proposed method is the fact that it provides its user with an analytical approximation, in many instances an exact solution, in a rapidly convergent sequence with elegantly computed terms. This technique does not involve any linearization, discretization, or small perturbations, and therefore it reduces significantly the numerical computation. This method provides extraordinary accuracy for the approximate solutions when compared to the exact solutions, particularly in large-scale domain. It is not affected by computation round-off errors, and hence one does not face the need for large computer memory and time. The results reveal that the CFRDTM is very effective, convenient, and quite accurate to the system of nonlinear equations.

Several analytical as well as numerical methods have been implemented by various authors to solve fractional differential equations. Wei et al. [30] applied the homotopy method to determine the unknown parameters of solute transport with spatial fractional derivative advection-dispersion equation. Saha Ray and Gupta proposed numerical schemes based on the Haar wavelet method for finding numerical solutions of Burger–Huxley, Huxley, modified Burgers, and mKdV equations [31, 32]. An approximate analytical solution of the time fractional Cauchy reaction diffusion equation by using the fractional-order reduced differential transform method (FRDTM) has been proposed by Shukla et al. [33].

Nonlinear partial differential equations are useful in describing various phenomena. The solutions of the nonlinear evolution equations play an important role in the field of nonlinear wave phenomena. The exact solutions facilitate the verification of numerical methods when they exist. These equations arise in various areas of physics, mathematics, and engineering such as fluid dynamics, nonlinear optics, plasma physics, nuclear physics, mathematical biology, Brusselator model of the chemical reaction–diffusion, and many other areas.

In the past decades, the fractional differential equations have been widely used in various fields of applied science and engineering [1, 4, 23, 25, 27, 28, 34, 35]. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. An immense effort has been expended over the last many years to find robust and efficient numerical and analytical methods for solving nonlinear fractional differential equations [12]. In the present analysis, a new approximate analytic technique, coupled fractional reduced differential transform method (CFRDTM) [34, 35], has been proposed which is applicable for coupled fractional linear and nonlinear differential equations. The proposed method originated from generalized Taylor's formula [36] is a very powerful solver for linear and nonlinear coupled fractional differential equations. It is relatively a new approach to provide the solution very efficiently and accurately.

Nonlinear partial differential equations are useful in describing various phenomena. These equations arise in various areas of physics, mathematics, and engineering such as fluid dynamics, nonlinear optics, plasma physics, nuclear physics, mathematical biology, Brusselator model of the chemical reaction–diffusion, and many other areas. In fluid dynamics, the nonlinear evolution equations show up in the context of shallow water waves. Some of the commonly studied equations are the Korteweg–de Vries (KdV) equation, modified KdV equation, Boussinesq equation, and Whitham–Broer–Kaup equation. In this paper, Whitham–Broer–Kaup equations have been solved by a new novel method revealed by Saha Ray [34, 35] and it is inherited from generalized Taylor's series.

The investigation of the traveling wave solutions to nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena.

In the past decades, the fractional differential equations have been widely used in various fields of applied science and engineering [1, 4, 23, 25, 27, 28, 34, 35]. Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations. An immense effort has been expended over the last many years to find robust and efficient numerical and analytical methods for solving nonlinear fractional differential equations [12]. In the present analysis, a new approximate analytic technique, coupled fractional reduced differential transform method (CFRDTM) [34, 35], has been proposed which is applicable for coupled fractional linear and nonlinear differential equations. The proposed method originated from generalized Taylor's formula [36] is a very powerful solver for linear and nonlinear coupled fractional differential equations. It is relatively a new approach to provide the solution very efficiently and accurately.

7.2 Outline of the Present Study

In this chapter, the modified fractional reduced differential transform method (MFRDTM) has been proposed and it is implemented for solving fractional Korteweg–de Vries (KdV) equations. The fractional derivatives are described in the Caputo sense. The reduced differential transform method is modified to be easily employed to solve wide kinds of nonlinear fractional differential equations. In this new approach, the nonlinear term is replaced by its Adomian polynomials. Thus, the nonlinear initial-value problem can be easily solved with less computational effort. In order to show the power and effectiveness of the present modified method and to illustrate the pertinent features of the solutions, several fractional KdV equations with different types of nonlinearities are considered. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of fractional KdV equations.

A very new technique, coupled fractional reduced differential transform, has been implemented in this chapter to obtain the numerical approximate solution of coupled time fractional KdV equations. The fractional derivatives are described in the Caputo sense. By using the present method, we can solve many linear and nonlinear coupled fractional differential equations. The obtained results are compared with the exact solutions. Numerical solutions are presented graphically to show the reliability and efficiency of the method.

Newly proposed coupled fractional reduced differential transform has been implemented to obtain the soliton solutions of coupled time fractional modified KdV equations. This new method has been revealed by the author. The fractional derivatives are described in the Caputo sense. By using the present method, we can solve many linear and nonlinear coupled fractional differential equations. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of fractional coupled modified KdV equations. Numerical solutions are presented graphically to show the reliability and efficiency of the method. Solutions obtained by this new method have been also compared with Adomian decomposition method (ADM).

A relatively very new technique, viz. coupled fractional reduced differential transform, has been executed to attain the approximate numerical solution of the predator–prey dynamical system. The fractional derivatives are defined in the Caputo sense. Utilizing the present method, we can solve many linear and nonlinear coupled fractional differential equations. The results thus obtained are compared with those of other available methods. Numerical solutions are also presented graphically to show the simplicity and authenticity of the method for solving the fractional predator–prey dynamical system.

Also in this chapter, fractional coupled Schrödinger–Korteweg–de Vries (or Sch–KdV) equation with appropriate initial values has been solved by using a new novel method. The fractional derivatives are described in the Caputo sense. By using the present method, we can solve many linear and nonlinear coupled fractional differential equations. Basically, the present method originated from generalized Taylor's formula [36]. The results reveal that the proposed method is very effective and simple for obtaining approximate solutions of fractional coupled Schrödinger–KdV equation. Numerical solutions are presented graphically to show the reliability and efficiency of the method. The method does not need linearization, weak nonlinearity assumptions, or perturbation theory. The convergence of the method as applied to Sch–KdV is illustrated numerically as well as derived analytically. Moreover, the derived results are compared with those obtained by the Adomian decomposition method (ADM).

The analytical approximate traveling wave solutions of Whitham–Broer–Kaup (WBK) equations, which contain blow-up solutions and periodic solutions, have been obtained by using the coupled fractional reduced differential transform method [34, 35, 37–39]. By using this method, the solutions were calculated in the form of a generalized Taylor's series with easily computable components. The convergence of the method as applied to the Whitham–Broer–Kaup equations is illustrated numerically as well as analytically. By using the present method, we can solve many linear and nonlinear coupled fractional differential equations. The results justify that the proposed method is also very efficient, effective, and simple for obtaining approximate solutions of fractional coupled modified Boussinesq and fractional approximate long wave equations. Numerical solutions are presented graphically to show the reliability and efficiency of the method. Moreover, the results are compared with those obtained by the Adomian decomposition method (ADM) and variational iteration method (VIM) revealing that the present method is superior to others.

7.2.1 Fractional KdV Equation

The aim of this work is to directly apply the MFRDTM to determine the approximate solution of the nonlinear fractional KdV equation with time fractional derivative of the form

$$D_t^{\alpha} u + (u^m)_x + (u^n)_{xxx} = 0,\ m > 0,\ 1 \le n \le 3,\ t > 0,\ 0 < \alpha \le 1 \tag{7.1}$$

which is a generalization of the Korteweg–de Vries equation, denoted by $K(m, n)$ for the different values of m and n, respectively. These $K(m, n)$ equations have the property that for certain values of m and n, their solitary wave solutions have compact support which is known as compactons [40]. Here, the fractional derivative is considered in the Caputo sense [5, 6]. In the case of $\alpha = 1$, fractional Eq. (1.1) reduces to the classical nonlinear KdV equation [14, 16].

7.2.2 Time Fractional Coupled KdV Equations

For solving time fractional coupled KdV equations, two model equations have been considered in the present chapter.

I. Consider the following time fractional coupled KdV equations [41]

$$D_t^{\alpha} u = -\frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} + 3v\frac{\partial v}{\partial x}, \tag{7.2}$$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3u\frac{\partial v}{\partial x}, \tag{7.3}$$

where $t > 0,\ 0 < \alpha, \beta \le 1$.

II. Consider the following time fractional coupled KdV equations [42]

$$D_t^{\alpha} u + 6uu_x - 6vv_x + u_{xxx} = 0, \tag{7.4}$$

$$D_t^{\beta} v + 3uv_x + v_{xxx} = 0, \tag{7.5}$$

where $t > 0,\ 0 < \alpha, \beta \le 1$.

7.2.3 Time Fractional Coupled Modified KdV Equations

In this case, for solving time fractional coupled modified KdV equations, again two model equations have been considered in the present chapter.

I. Consider the following time fractional coupled modified KdV equations [43]

$$D_t^{\alpha} u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial (uv)}{\partial x} - 3\frac{\partial u}{\partial x}, \tag{7.6}$$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial x}, \tag{7.7}$$

where $t > 0,\ 0 < \alpha, \beta \le 1$.

II. Consider the following time fractional coupled modified KdV equations [44]

$$D_t^{\alpha} u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial (uv)}{\partial x} + 3\frac{\partial u}{\partial x} \tag{7.8}$$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial x} \tag{7.9}$$

where $t > 0,\ 0 < \alpha, \beta \le 1$.

7.2.4 Time Fractional Predator–Prey Dynamical System

In the present chapter, a system of two species competitive models with prey population A and predator population B has been also studied. For prey population $A \rightarrow 2A$, at the rate a $(a > 0)$ expresses the natural birthrate. Similarly, for predator population $B \to 2B$, at the rate c $(c > 0)$ represents the natural death rate. The interactive term between predator and prey population is $A + B \to 2B$, at rate b $(b > 0)$ where b denotes the competitive rate. According to the knowledge of fractional calculus and biological population, the time fractional dynamics of a predator–prey system can be described as

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + au - buv, \quad u(x, y, 0) = \varphi(x, y), \tag{7.10}$$

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + buv - cv, \quad v(x, y, 0) = \phi(x, y), \tag{7.11}$$

where $t > 0, x, y \in \mathcal{R}, a, b, c > 0, u(x, y, t)$ denotes the prey population density, and $v(x, y, t)$ represents the predator population density. Here, $\varphi(x, y)$ and $\phi(x, y)$ represent the initial conditions of the population system. The fractional derivatives are considered in Caputo sense. Caputo fractional derivative is used because of its advantage that it permits the initial and boundary conditions included in the formulation of the problem. Here, $u(x, y, t)$ and $v(x, y, t)$ are analytic functions. The physical interpretations of Eqs. (7.10) and (7.11) indicate that the prey–predator population system is analogous to the behavior of fractional-order model of anomalous biological diffusion.

7.2.5 Fractional Coupled Schrödinger–KdV Equation

Nonlinear phenomena play a crucial role in applied mathematics and physics. Calculating exact and numerical solutions, in particular, traveling wave solutions, of nonlinear equations in mathematical physics plays an important role in soliton theory [9, 45]. The investigation of the traveling wave solutions to nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena. Multiple traveling wave solutions of nonlinear evolution equations such as the coupled Schrödinger–KdV equation [46, 47] have been obtained by Fan [48]. The coupled Schrödinger–KdV equation is known to describe various processes in dusty plasma, such as Langmuir, dust-acoustic wave, and electromagnetic waves [48]. The model equation for the coupled fractional Schrödinger–KdV equation can be presented in the following form [48]

$$iD_t^{\alpha} u_t = u_{xx} + uv$$

$$D_t^{\beta} v_t = -6vv_x - v_{xxx} + (|u|^2)_x \tag{7.12}$$

where α, β $(0 < \alpha, \beta \leq 1)$ are the orders of the Caputo fractional time derivatives, respectively, $i = \sqrt{-1}$ and $t \geq 0$.

Recently, Fan [48] applied the unified algebraic method and Kaya et al. [49] applied Adomian's decomposition method for computing solutions to a (classical) integer-order Sch–KdV equation.

7.2.6 Fractional Whitham–Broer–Kaup, Modified Boussinesq, and Approximate Long Wave Equations in Shallow Water

In the present paper, coupled WBK equations introduced by Whitham, Broer, and Kaup [50–52] have been considered. The equations describe the propagation of shallow water waves with different dispersion relations. The fractional-order WBK equations are as follows

$$D_t^{\alpha} u + uu_x + v_x + bu_{xx} = 0, \tag{7.13a}$$

$$D_t^{\beta} v + (uv)_x + au_{xxx} - bv_{xx} = 0, \tag{7.13b}$$

where α, β $(0 < \alpha, \beta \leq 1)$ are the orders of the Caputo fractional time derivatives, respectively, and $t \geq 0$. In WBK equations (7.13a) and (7.13b), the field of horizontal velocity is represented by $u = u(x, t)$, $v = v(x, t)$ which is the height that deviates from the equilibrium position of liquid, and the constants a, b are represented in different diffusion powers [53].

If $a = 1$ and $b = 0$, the following fractional coupled modified Boussinesq equations (7.14a) and (7.14b)

$$D_t^{\alpha} u = -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \tag{7.14a}$$

$$D_t^{\beta} v = -\frac{\partial (uv)}{\partial x} - \frac{\partial^3 u}{\partial x^3} \tag{7.14b}$$

where $t > 0$, $0 < \alpha, \beta \leq 1$, can be obtained as a special case of WBK equations (7.13a) and (7.13b).

If $a = 0$ and $b = 1/2$, the following fractional coupled approximate long wave equations (ALW) equations (7.15a) and (7.15b)

$$D_t^{\alpha} u = -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{7.15a}$$

$$D_t^{\beta} v = -\frac{\partial (uv)}{\partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} \tag{7.15b}$$

where $t > 0$, $0 < \alpha, \beta \leq 1$, can be obtained as a special case of WBK equations (7.13a) and (7.13b).

7.3 Fractional Reduced Differential Transform Methods

In this section, proposed modified fractional reduced differential transform method (MFRDTM) and a newly developed technique, coupled fractional reduced differential transform method (CFRDTM), have been presented.

7.3.1 Modified Fractional Reduced Differential Transform Method

Consider a function of two variables $u(x,t)$ which can be represented as a product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based on the properties of differential transform, the function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k} \tag{7.16}$$

where t-dimensional spectrum function $U_k(x)$ is the transformed function of $u(x,t)$.

The basic definitions and operations of MFRDTM are as follows:

Definition 1 If the function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{\Gamma(\alpha k + 1)} \left[\left(D_t^{\alpha} \right)^k u(x,t) \right]_{t=0}, \tag{7.17}$$

where $(D_t^{\alpha})^k = D_t^{\alpha} \cdot D_t^{\alpha} \cdot D_t^{\alpha} \ldots D_t^{\alpha}$, the k times differentiable Caputo fractional derivative.

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k}. \tag{7.18}$$

Then combining Eqs. (7.17) and (7.18), we can write

$$u(x,t) = \sum_{k=0}^{\infty} \left(\frac{1}{\Gamma(\alpha k + 1)} \left[\left(D_t^{\alpha} \right)^k u(x,t) \right]_{t=0} \right) t^{\alpha k}. \tag{7.19}$$

Some basic properties of the reduced differential transform method are summarized in Table 7.1.

To illustrate the basic concepts for the application of MFRDTM, consider the following general nonlinear partial differential equation:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{7.20}$$

with initial condition

$$u(x,0) = f(x),$$

where $L \equiv D_t^{\alpha}$ is an easily invertible linear operator, R is the remaining part of the linear operator, $Nu(x,t)$ is a nonlinear term, and $g(x,t)$ is an inhomogeneous term.

We can look for the solution $u(x,t)$ of Eq. (7.20) in the form of the fractional power series:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{\alpha k}, \tag{7.21}$$

where t-dimensional spectrum function $U_k(x)$ is the transformed function of $u(x,t)$.

Now, let us write the nonlinear term

$$N(u,t) = \sum_{n=0}^{\infty} A_n(U_0(x), U_1(x), \ldots, U_n(x)) t^{n\alpha}, \tag{7.22}$$

where A_n is the appropriate Adomian's polynomials [13, 17]. In this specific nonlinearity, we use the general form of the formula for A_n Adomian polynomials as

$$A_n(U_0(x), U_1(x), \ldots, U_n(x)) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum_{i=0}^{\infty} \lambda^i U_i(x) \right) \right]_{\lambda=0}. \tag{7.23}$$

Table 7.1 Fundamental operations of MFRDTM

Properties	Function	Transformed function
1	$f(x,t) = au(x,t) \pm bv(x,t)$	$F_k(x) = aU_k(x) \pm bV_k(x)$
2	$f(x,t) = u(x,t)v(x,t)$	$F_k(x) = \sum_{l=0}^{k} U_l(x) V_{k-l}(x)$
3	$f(x,t) = \frac{\partial u(x,t)}{\partial x}$	$F_k(x) = \frac{\partial U_k(x)}{\partial x}$
4	$f(x,t) = D_t^{m\alpha} u(x,t)$, where $\alpha \in \mathcal{R}^+$ and $m \in \mathcal{N}$	$F_k(x) = \frac{\Gamma(\alpha(k+m)+1)}{\Gamma(\alpha k+1)} U_{k+m}(x)$

Now, applying Riemann–Liouville integral J^{α} on both sides of Eq. (7.20), we have

$$u(x,t) = \Phi + J^{\alpha} g(x,t) - J^{\alpha} R u(x,t) - J^{\alpha} N u(x,t), \tag{7.24}$$

where from the initial condition $\Phi = u(x,0) = f(x)$.

Substituting Eqs. (7.21) and (7.22), for $u(x,t)$ and $N(u,t)$, respectively, in Eq. (7.24) yields

$$\sum_{k=0}^{\infty} U_k(x) t^{\alpha k} = f(x) + J^{\alpha}\left(\sum_{k=0}^{\infty} G_k(x) t^{\alpha k}\right) - J^{\alpha}\left(R\left(\sum_{k=0}^{\infty} U_k(x) t^{\alpha k}\right)\right) - J^{\alpha}\left(\sum_{k=0}^{\infty} A_k(x) t^{\alpha k}\right),$$

where $g(x,t) = \left(\sum_{k=0}^{\infty} G_k(x) t^{\alpha k}\right)$, and $G_k(x)$ is the transformed function of $g(x,t)$.

After carrying out Riemann–Liouville integral J^{α}, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} U_k(x) t^{\alpha k} = f(x) &+ \left(\sum_{k=0}^{\infty} G_k(x) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right) \\ &- \left(R\left(\sum_{k=0}^{\infty} U_k(x) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right)\right) \\ &- \left(\sum_{k=0}^{\infty} A_k(x) \frac{t^{\alpha(k+1)}\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right).\end{aligned}$$

Finally, equating coefficients of like powers of t, we derive the following recursive formula

$$U_0(x) = f(x),$$

and

$$\begin{aligned}U_{k+1}(x) = G_k(x) \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)} - R\left(U_k(x) \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right) \\ - A_k(x) \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}, k \geq 0.\end{aligned} \tag{7.25}$$

Using the known $U_0(x)$, all components $U_1(x), U_2(x), \ldots, U_n(x), \ldots,$, etc., are determinable by using Eq. (7.25).

Substituting these $U_0(x), U_1(x), U_2(x), \cdots, U_n(x), \cdots,$, etc., in Eq. (7.21), the approximate solution can be obtained as

$$\tilde{u}_p(x,t) = \sum_{m=0}^{p} U_m(x) t^{m\alpha}, \tag{7.26}$$

where p is the order of approximate solution.

Therefore, the corresponding exact solution is given by

$$u(x,t) = \lim_{p\to\infty} \tilde{u}_p(x,t) \tag{7.27}$$

7.3.2 Coupled Fractional Reduced Differential Transform Method

In order to introduce coupled fractional reduced differential transform, two cases are considered.

For functions with two independent variables

In this case, $U(h, k-h)$ is considered as the coupled fractional reduced differential transform of $u(x,t)$. If the function $u(x,t)$ is analytic and differentiated continuously with respect to time t, then we define the fractional coupled reduced differential transform of $u(x,t)$ as

$$U(h,k-h) = \frac{1}{\Gamma(h\alpha + (k-h)\beta + 1)} \left[D_t^{(h\alpha + (k-h)\beta)} u(x,t) \right]_{t=0}, \tag{7.28}$$

whereas the inverse transform of $U(h, k-h)$ is

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha + (k-h)\beta}, \tag{7.29}$$

which is one of the solutions of coupled fractional differential equations.

Theorem 7.1 *Suppose that $U(h, k-h)$ and $V(h, k-h)$ are coupled fractional reduced differential transform of functions $u(x,t)$ and $v(x,t)$, respectively.*

i. If $u(x,t) = f(x,t) \pm g(x,t)$, then $U(h,k-h) = F(h,k-h) \pm G(h,k-h)$.
ii. If $u(x,t) = af(x,t)$, where $a \in \mathcal{R}$, then $U(h,k-h) = aF(h,k-h)$.
iii. If $f(x,t) = u(x,t)v(x,t)$, then $F(h,k-h) = \sum_{l=0}^{h} \sum_{s=0}^{k-h} U(h-l,s)V(l,k-h-s)$.
iv. If $f(x,t) = D_t^{\alpha} u(x,t)$, then

$$F(h,k-h) = \frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1,k-h).$$

v. If $f(x,t) = D_t^{\beta} v(x,t)$, then

$$F(h,k-h) = \frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h,k-h+1).$$

For functions with three independent variables

In this case, $U(h,k-h)$ is considered as the coupled fractional reduced differential transform of $u(x,y,t)$. If the function $u(x,y,t)$ is analytic and differentiated continuously with respect to time t, then we define the fractional coupled reduced differential transform of $u(x,y,t)$ as

$$U(h,k-h) = \frac{1}{\Gamma(h\alpha + (k-h)\beta + 1)} \left[D_t^{(h\alpha + (k-h)\beta)} u(x,y,t) \right]_{t=0}, \tag{7.30}$$

whereas the inverse transform of $U(h,k-h)$ is

$$u(x,y,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha + (k-h)\beta}, \tag{7.31}$$

which is one of the solutions of coupled fractional differential equations.

Theorem 7.2 *Suppose that $U(h,k-h)$ and $V(h,k-h)$ are coupled fractional reduced differential transform of functions $u(x,y,t)$ and $v(x,y,t)$, respectively.*

i. If $u(x,y,t) = f(x,y,t) \pm g(x,y,t)$, then $U(h,k-h) = F(h,k-h) \pm G(h,k-h)$.
ii. If $u(x,y,t) = af(x,y,t)$, where $a \in \mathcal{R}$, then $U(h,k-h) = aF(h,k-h)$.
iii. If $f(x,y,t) = u(x,y,t)v(x,y,t)$, then $F(h,k-h) = \sum_{l=0}^{h} \sum_{s=0}^{k-h} U(h-l,s) V(l,k-h-s)$.
iv. If $f(x,y,t) = D_t^{\alpha} u(x,y,t)$, then

$$F(h,k-h) = \frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1,k-h).$$

v. If $f(x,y,t) = D_t^{\beta} v(x,y,t)$, then

$$F(h,k-h) = \frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h,k-h+1).$$

7.4 Application of MFRDTM for the Solution of Fractional KdV Equations

We consider the generalized fractional KdV equation of the form

$$D_t^\alpha u + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0,\ 1 \le n \le 3,\ t > 0,\ 0 < \alpha \le 1 \tag{7.32}$$

with initial condition

$$u(x,0) = f(x). \tag{7.33}$$

Applying MFRDTM to Eq. (7.32) and using basic properties of Table 7.1, we can obtain

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{k+1}(x) + \frac{\partial A_k(x)}{\partial x} + \frac{\partial^3 \bar{A}_k(x)}{\partial x^3} = 0, k \ge 0 \tag{7.34}$$

where $U_k(x)$ is the transformed function of $u(x,t)$, and the nonlinear terms u^m and u^n have been considered as Adomian polynomials $\sum_{k=0}^{\infty} A_k(U_0(x), U_1(x), \ldots, U_k(x))$ and $\sum_{k=0}^{\infty} \bar{A}_k(U_0(x), U_1(x), \ldots, U_k(x))$, respectively.

From the initial condition (7.33), we have

$$U_0(x) = f(x). \tag{7.35}$$

Substituting (7.35) into (7.34), we obtain the values of $U_k(x)$ successively. Then, the approximate solution can be obtained as

$$\tilde{u}_p(x,t) = \sum_{m=0}^{p} U_m(x) t^{m\alpha}, \tag{7.36}$$

where p is the order of approximate solution.

7.4.1 Numerical Solutions of Variant Types of Time Fractional KdV Equations

In order to assess the advantages and the accuracy of the modified fractional reduced differential transform method (MFRDTM) for solving nonlinear fractional KdV equation, this method has been applied to solve the following four examples. In the first two examples, we consider quasi-linear time fractional KdV equations, while in the last two examples, we consider a nonlinear time fractional dispersive $K(2,2)$ equation. All the results are calculated by using the symbolic calculus software Mathematica.

Example 7.1

(a) (One-soliton solution)

Consider the following time fractional KdV equation

$$D_t^{\alpha}u + 6uu_x + u_{xxx} = 0, t > 0, 0 < \alpha \le 1 \tag{7.37}$$

with initial condition

$$u(x,0) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right). \tag{7.38}$$

After applying MFRDTM, according to Eq. (7.34), we can obtain the recursive formula

$$U_{k+1}(x) = \left(\frac{-\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right)\left(6\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial U_r(x)}{\partial x} + \frac{\partial^3 U_k(x)}{\partial x^3}\right), k \ge 0 \tag{7.39}$$

where $U_k(x)$ is the transformed function of $u(x,t)$.

From the initial condition (7.38), we have

$$U_0(x) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right). \tag{7.40}$$

Substituting (7.40) into (7.39), we obtain the values of $U_k(x)$ for $k = 1, 2, 3, \ldots$ successively.

Then, using Mathematica, the third-order approximate solution can be obtained as

$$u(x,t) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right) + \frac{t^{2\alpha}(-2+\cosh(x))\operatorname{sech}^4\left(\frac{x}{2}\right)}{4\Gamma(1+2\alpha)} + \frac{4t^{\alpha}\operatorname{cosech}^3(x)\sinh^4\left(\frac{x}{2}\right)}{\Gamma(1+\alpha)} + \frac{t^{3\alpha}((39-32\cosh(x)+\cosh(2x))\Gamma(1+\alpha)^2 + 12(-2+\cosh(x))\Gamma(1+2\alpha))\operatorname{sech}^6\left(\frac{x}{2}\right)\tanh\left(\frac{x}{2}\right)}{16\Gamma(1+\alpha)^2\Gamma(1+3\alpha)}. \tag{7.41}$$

If $\alpha = 1$, the solution in Eq. (7.41), which becomes the single soliton solution, is given by

$$u(x,t) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x-t}{2}\right). \tag{7.42}$$

For special case $\alpha = 1$, i.e., for classical integer order, the obtained results for the exact solution (7.42) and the approximate solution in Eq. (7.41) obtained by MFRDTM are shown in Figs. 7.1 and 7.2. It is very much graceful that the approximate solution obtained by the present method and the exact solution are very much identical.

Figures 7.3, 7.4, 7.5, and 7.6 demonstrate the approximate solutions for $\alpha = 0.25$, $\alpha = 0.35$, $\alpha = 0.5$, and $\alpha = 0.75$, respectively.

(b) (Two-soliton solution)

Consider the following time fractional KdV equation

$$D_t^{\alpha} u + 6uu_x + u_{xxx} = 0, t > 0, 0 < \alpha \leq 1 \tag{7.43}$$

with initial condition

$$u(x, 0) = 6\text{sech}^2 x. \tag{7.44}$$

After applying MFRDTM, according to Eq. (7.34), we can obtain the recursive formula

$$U_{k+1}(x) = \left(\frac{-\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1)+1)}\right)\left(3\frac{\partial A_k(x)}{\partial x} + \frac{\partial^3 U_k(x)}{\partial x^3}\right), k \geq 0 \tag{7.45}$$

where $U_k(x)$ is the transformed function of $u(x,t)$, and the nonlinear term u^2 has been considered as Adomian polynomials $\sum_{k=0}^{\infty} A_k(U_0(x), U_1(x), \ldots, U_k(x))$.

From the initial condition (7.44), we have

$$U_0(x) = 6\text{sech}^2 x. \tag{7.46}$$

Substituting Eq. (7.46) into Eq. (7.45), we obtain the values of $U_k(x)$ for $k = 1, 2, 3, \ldots$ successively.

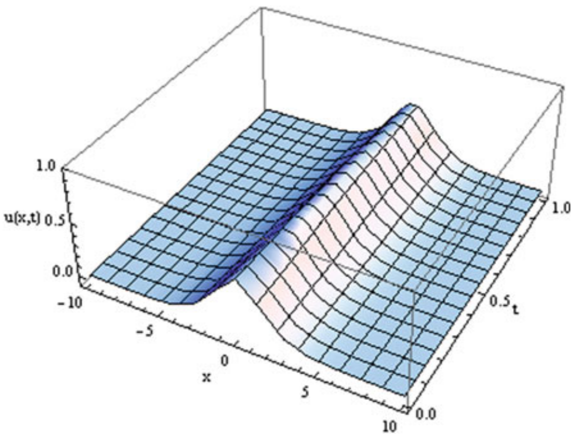


Fig. 7.1 Exact solution $u(x, t)$ for Eq. (7.37)

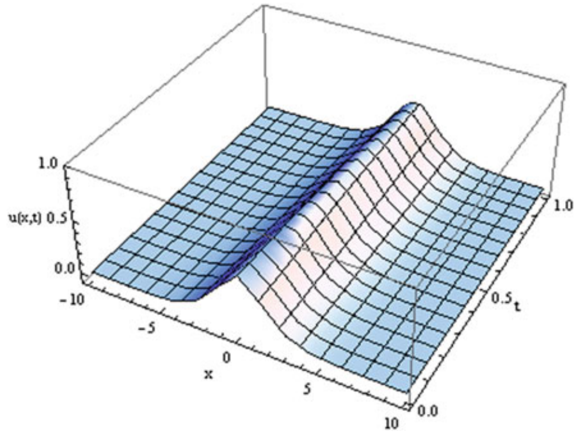


Fig. 7.2 Approximate solution $u(x, t)$ obtained by MFRDTM for Eq. (7.37)

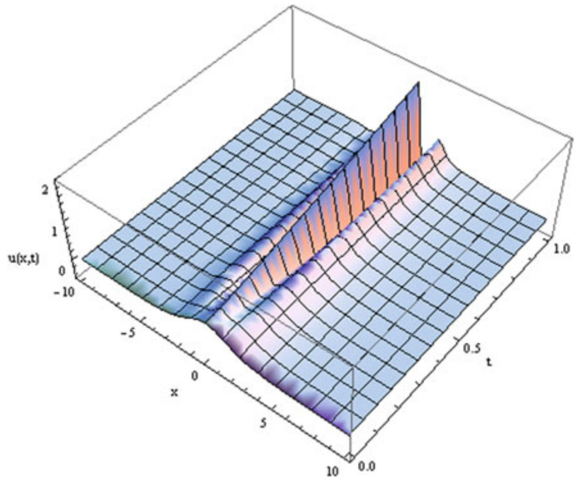


Fig. 7.3 Approximate solution $u(x, t)$ obtained by MFRDTM for Eq. (7.37) when $\alpha = 0.25$

Then, using Mathematica, the second-order approximate solution can be obtained as

$$u(x,t) = 6\mathrm{sech}^2 x + \frac{12t^{2\alpha}(-1064 + 183\cosh(2x) + 240\cosh(4x) + \cosh(6x))\mathrm{sech}^8 x}{\Gamma(1+2\alpha)} + \frac{12t^{\alpha}\mathrm{sech}^5(x)(25\sinh(x) + \sinh(3x))}{\Gamma(1+\alpha)} + O(t^{3\alpha}). \tag{7.47}$$

Fig. 7.4 Approximate solution $u(x, t)$ obtained by MFRDTM for Eq. (7.37) when $\alpha = 0.35$

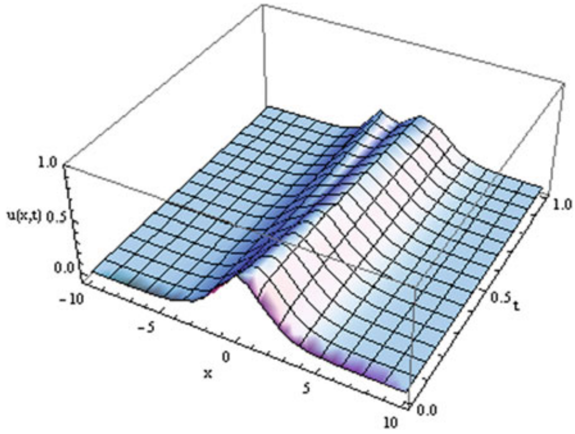


Fig. 7.5 Approximate solution $u(x, t)$ obtained by MFRDTM for Eq. (7.37) when $\alpha = 0.5$

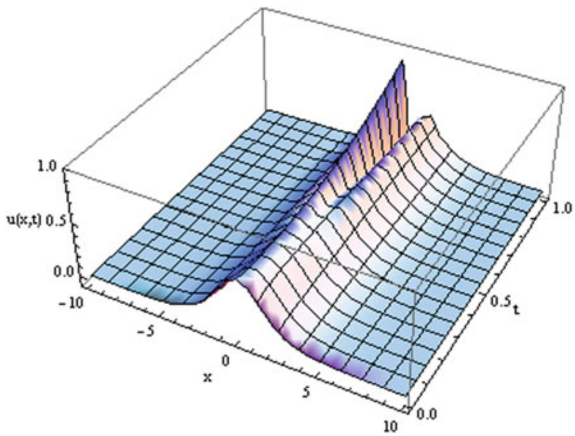
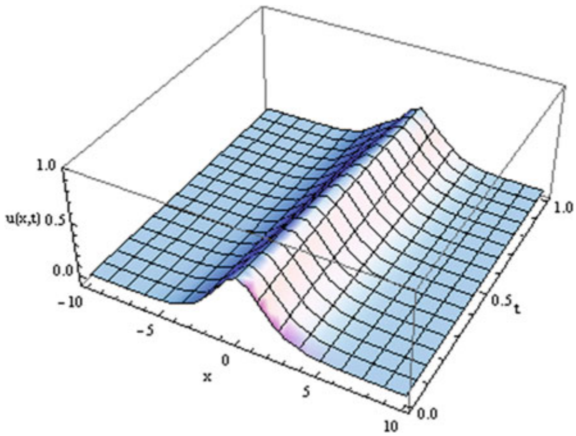


Fig. 7.6 Approximate solution $u(x, t)$ obtained by MFRDTM for Eq. (7.37) when $\alpha = 0.75$



If $\alpha = 1$, the solution in Eq. (7.47), which becomes the two-soliton solution, is given by

$$u(x,t) = \frac{24(4\cosh(x-4t)^2 + \sinh(2x-32t)^2)}{(\cosh(3x-36t) + 3\cosh(x-28t))^2}. \tag{7.48}$$

Figures 7.7, 7.8, and 7.9 exhibit the two-soliton approximate solutions of the KdV equation (7.43) for $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$, respectively.

Example 7.2 Consider the following time fractional KdV equation

$$D_t^{\alpha} u - 3(u^2)_x + u_{xxx} = 0, t > 0, 0 < \alpha \leq 1 \tag{7.49}$$

with initial condition

$$u(x,0) = 6x. \tag{7.50}$$

After applying MFRDTM, according to Eq. (7.34), we can obtain the recursive formula

$$U_{k+1}(x) = \left(\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right)\left(3\frac{\partial A_k(x)}{\partial x} - \frac{\partial^3 U_k(x)}{\partial x^3}\right), k \geq 0 \tag{7.51}$$

where $U_k(x)$ is the transformed function of $u(x,t)$, and the nonlinear term u^2 has been considered as Adomian polynomials $\sum_{k=0}^{\infty} A_k(U_0(x), U_1(x), \ldots, U_k(x))$.

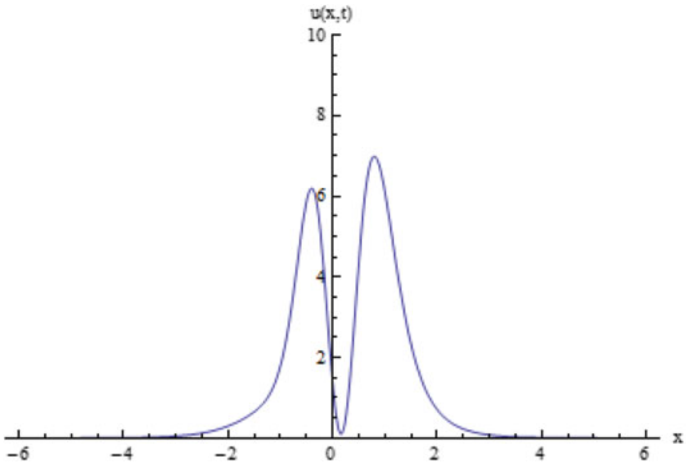


Fig. 7.7 Two-soliton approximate solution $u(x,t)$ of the KdV equation obtained by using Eq. (7.47) for $\alpha = 0.5$, t = 0.0006, and $-6 \leq x \leq 6$

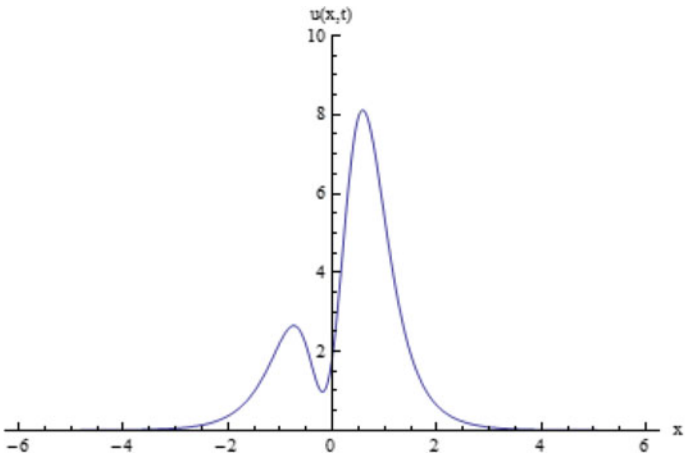


Fig. 7.8 Two-soliton approximate solution $u(x,t)$ of the KdV equation obtained by using Eq. (7.47) for $\alpha = 0.75$, $t = 0.008$, and $-6 \leq x \leq 6$

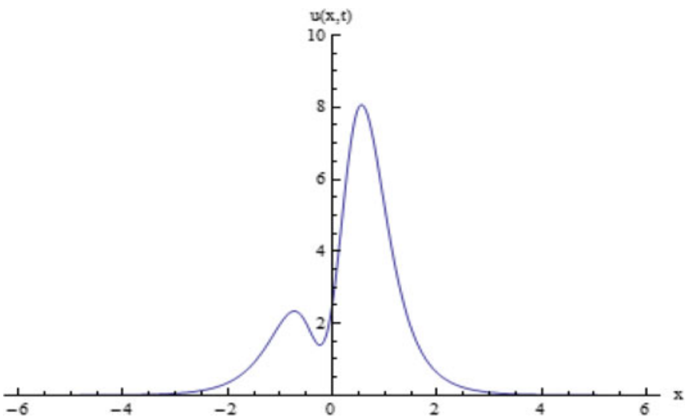


Fig. 7.9 Two-soliton approximate solution $u(x,t)$ of the KdV equation obtained by using Eq. (7.47) for $\alpha = 1$, $t = 0.03$, and $-6 \leq x \leq 6$

From the initial condition Eq. (7.50), we have

$$U_0(x) = 6x. \tag{7.52}$$

Substituting (7.52) into (7.51), we obtain the values of $U_k(x)$ for $k = 1, 2, 3, \ldots$ successively.

Then, using Mathematica, the fourth-order approximate solution can be obtained as

$$u(x,t) = 6x + \frac{216t^{\alpha}x}{\Gamma(1+\alpha)} + \frac{15552t^{2\alpha}x}{\Gamma(1+2\alpha)} + \frac{279936t^{3\alpha}x\left(\frac{1}{\Gamma(1+\alpha)^2} + \frac{4}{\Gamma(1+2\alpha)}\right)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{20155392t^{4\alpha}x(4\Gamma(1+\alpha)^2\Gamma(1+2\alpha) + \Gamma(1+2\alpha)^2 + 2\Gamma(1+\alpha)\Gamma(1+3\alpha))}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+4\alpha)}. \tag{7.53}$$

For the special case $\alpha = 1$, the solution in Eq. (7.53), which becomes the exact solitary wave solution, is given by

$$\begin{aligned} u(x,t) &= 6x + 216tx + 7776t^2x + 279936t^3x + 10077696t^4x + \cdots \\ &= \frac{6x}{1-36t}. \end{aligned} \tag{7.54}$$

Example 7.3 Consider the following time fractional dispersive $K(2,2)$ equation

$$D_t^{\alpha}u + (u^2)_x + (u^2)_{xxx} = 0, t > 0, 0 < \alpha \leq 1, \tag{7.55}$$

with initial condition

$$u(x,0) = x. \tag{7.56}$$

After applying MFRDTM, according to Eq. (7.34), we can obtain the recursive formula

$$U_{k+1}(x) = \left(\frac{-\Gamma(\alpha k+1)}{\Gamma(\alpha(k+1)+1)}\right)\left(\frac{\partial A_k(x)}{\partial x} + \frac{\partial^3 U_k(x)}{\partial x^3}\right), k \geq 0, \tag{7.57}$$

where $U_k(x)$ is the transformed function of $u(x,t)$, and the nonlinear term u^2 has been considered as Adomian polynomials $\sum_{k=0}^{\infty} A_k(U_0(x), U_1(x), \ldots, U_k(x))$.

From the initial condition (7.56), we have

$$U_0(x) = x. \tag{7.58}$$

Substituting (7.58) into (7.57), we obtain the values of $U_k(x)$ for $k = 1, 2, 3, \ldots$ successively.

Then, using Mathematica, the fifth-order approximate solution can be obtained as

$$u(x,t) = x - \frac{2t^{\alpha}x}{\Gamma(1+\alpha)} + \frac{8t^{2\alpha}x}{\Gamma(1+2\alpha)} - \frac{8t^{3\alpha}x\left(\frac{1}{\Gamma(1+\alpha)^2} + \frac{4}{\Gamma(1+2\alpha)}\right)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \frac{32t^{4\alpha}x\left(4\Gamma(1+\alpha)^2\Gamma(1+2\alpha) + \Gamma(1+2\alpha)^2 + 2\Gamma(1+\alpha)\Gamma(1+3\alpha)\right)}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+4\alpha)}$$

$$-\frac{64t^{5\alpha}x\left(2+\frac{\Gamma(1+2\alpha)^2(4\Gamma(1+\alpha)^2+\Gamma(1+2\alpha))}{\Gamma(1+\alpha)^3\Gamma(1+3\alpha)}\right)\Gamma(1+4\alpha)}{\Gamma(1+2\alpha)^2\Gamma(1+5\alpha)}$$
$$+\frac{64t^{5\alpha}x\left(\frac{2\Gamma(1+2\alpha)\left(4\Gamma(1+\alpha)^2\Gamma(1+2\alpha)+\Gamma(1+2\alpha)^2+2\Gamma(1+\alpha)\Gamma(1+3\alpha)\right)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)}\right)\Gamma(1+4\alpha)}{\Gamma(1+2\alpha)^2\Gamma(1+5\alpha)}. \tag{7.59}$$

For the special case $\alpha = 1$, the solution in Eq. (7.59), which becomes the exact solitary wave solution, is given by

$$u(x,t) = x - 2tx + 4t^2x - 8t^3x + 16t^4x - 32t^5x + \cdots = \frac{x}{1+2t}. \tag{7.60}$$

Example 7.4 Consider the following time fractional dispersive $K(2,2)$ equation

$$D_t^{\alpha}u + (u^2)_x + (u^2)_{xxx} = 0,\ t > 0,\ 0 < \alpha \leq 1 \tag{7.61}$$

with initial condition

$$u(x,0) = \frac{4c}{3}\cos^2\left(\frac{x}{4}\right). \tag{7.62}$$

Taking modified fractional reduced differential transform, we can obtain the same recursive formula as in Eq. (7.57).

From the initial condition (5.1.20), here in this case, we have

$$U_0(x) = \frac{4c}{3}\cos^2\left(\frac{x}{4}\right). \tag{7.63}$$

Substituting Eq. (7.63) into Eq. (7.57), we obtain the values of $U_k(x)$ for $k = 1,2,3,\ldots$ successively.

Then, using Mathematica, the third-order approximate solution can be obtained as

$$u(x,t) = \frac{4}{3}c\cos^2\left(\frac{x}{4}\right) + \frac{c^2t^{\alpha}\sin\left(\frac{x}{2}\right)}{3\Gamma(1+\alpha)} - \frac{c^3t^{2\alpha}\cos\left(\frac{x}{2}\right)}{6\Gamma(1+2\alpha)} - \frac{c^4t^{3\alpha}\sin\left(\frac{x}{2}\right)}{12\Gamma(1+3\alpha)}. \tag{7.64}$$

For the special case $\alpha = 1$, the solution in Eq. (7.64), which becomes the exact solitary wave solution, is given by

$$\begin{aligned} u(x,t) = \frac{4}{3}c\cos^2\left(\frac{x}{4}\right) + \frac{1}{3}c^2t\sin\left(\frac{x}{2}\right) - \frac{1}{12}c^3t^2\cos\left(\frac{x}{2}\right) \\ - \frac{1}{72}c^4t^3\sin\left(\frac{x}{2}\right) + \frac{1}{576}c^5t^4\cos\left(\frac{x}{2}\right) + \cdots. \end{aligned} \tag{7.65}$$

Using Taylor series into Eq. (7.65), we can find the closed-form solitary wave solution with compact support, i.e., compacton solution

$$u(x,t) = \begin{cases} \frac{4c}{3}\cos^2\left(\frac{x-ct}{4}\right), & |x-ct| \le 2\pi, \\ 0, & \text{otherwise.} \end{cases}.$$

7.4.2 Convergence Analysis and Error Estimate

Theorem 7.3 *Suppose that,* $D_t^{k\alpha}u(x,t) \in C([0,L]\times[0,T])$ *for* $k = 0,1,2,\ldots,N+1$, *where* $0<\alpha<1$, *then*

$$u(x,t) \cong \sum_{m=0}^{N} U_m(x)t^{m\alpha}.$$

Moreover, there exists a value ξ, *where* $0 \le \xi \le t$ *so that the error term* $E_N(x,t)$ *has the form*

$$|E_N(x,t)| = \underset{t\in[0,T]}{Sup}\left|\frac{D^{(N+1)\alpha}u(x,\xi)t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)}\right|.$$

Proof For $0<\alpha<1$,

$$\begin{aligned} &J^{m\alpha}D^{m\alpha}u(x,t) - J^{(m+1)\alpha}D^{(m+1)\alpha}u(x,t) \\ &\quad = J^{m\alpha}(D^{m\alpha}u(x,t) - J^{\alpha}D^{\alpha}(D^{m\alpha}u(x,t))) \\ &= J^{m\alpha}(D^{m\alpha}u(x,0)) \text{ using Eq. (2.3.2)} \\ &= \frac{D^{m\alpha}u(x,0)t^{m\alpha}}{\Gamma(m\alpha+1)} \\ &= U_m(x)t^{m\alpha}, \text{using Eq. (7.17);} \end{aligned}$$

Now, the Nth order approximation for $u(x,t)$ is

$$\begin{aligned} \sum_{m=0}^{N} U_m(x)t^{m\alpha} &= \sum_{m=0}^{N}\left(J^{m\alpha}D^{m\alpha}u(x,t) - J^{(m+1)\alpha}D^{(m+1)\alpha}u(x,t)\right) \\ &= u(x,t) - J^{(N+1)\alpha}D^{(N+1)\alpha}u(x,t) \\ &= u(x,t) - \frac{1}{\Gamma((N+1)\alpha)}\int_0^t \frac{D^{(N+1)\alpha}u(x,\tau)}{(t-\tau)^{1-(N+1)\alpha}}\mathrm{d}\tau \end{aligned}$$

$$= u(x,t) - \frac{D^{(N+1)\alpha}u(x,\xi)}{\Gamma((N+1)\alpha)} \int_0^t \frac{d\tau}{(t-\tau)^{1-(N+1)\alpha}},$$

applying integral mean value theorem

$$= u(x,t) - \frac{D^{(N+1)\alpha}u(x,\xi)t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \tag{7.66}$$

Therefore,

$$u(x,t) = \sum_{m=0}^{N} U_m(x)t^{m\alpha} + \frac{D^{(N+1)\alpha}u(x,\xi)t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)}. \tag{7.67}$$

Consequently, the error term

$$|E_N(x,t)| = \left| u(x,t) - \sum_{m=0}^{N} U_m(x)t^{m\alpha} \right| = \left| \frac{D^{(N+1)\alpha}u(x,\xi)t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \right|. \tag{7.68}$$

This implies

$$|E_N(x,t)| = \sup_{t\in[0,T]} \left| \frac{D^{(N+1)\alpha}u(x,\xi)t^{(N+1)\alpha}}{\Gamma((N+1)\alpha+1)} \right|. \tag{7.69}$$

As $N \to \infty$, $|E_N| \to 0$.
Hence, $u(x,t)$ can be approximated as

$$u(x,t) = \sum_{m=0}^{\infty} U_m(x)t^{m\alpha} \cong \sum_{m=0}^{N} U_m(x)t^{m\alpha}.$$

with the error term given in Eq. (7.69).

7.5 Application of CFRDTM for the Solutions of Time Fractional Coupled KdV Equations

In the present section, CFRDTM has been applied to determine the approximate solutions for the coupled time fractional KdV equations.

7.5.1 Numerical Solutions of Time Fractional Coupled KdV Equations

In order to examine the efficiency and applicability of the proposed coupled fractional reduced differential transform method (CFRDTM) for solving time fractional coupled KdV equations, this method has been employed to solve the following two examples.

Example 7.5 Consider the following time fractional coupled KdV equations [41]

$$D_t^{\alpha} u = -\frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} + 3v\frac{\partial v}{\partial x}, \tag{7.70a}$$

$$D_t^{\beta} v = -\frac{\partial^3 v}{\partial x^3} - 3u\frac{\partial v}{\partial x}, \tag{7.70b}$$

where $t > 0,\ 0 < \alpha, \beta \leq 1$,

subject to the initial conditions

$$u(x,0) = \frac{4c^2 \exp(cx)}{(1+\exp(cx))^2}, \tag{7.70c}$$

$$v(x,0) = \frac{4c^2 \exp(cx)}{(1+\exp(cx))^2}. \tag{7.70d}$$

The exact solutions of Eqs. (7.70a) and (7.70b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = v(x,t) = \frac{4c^2 \exp(c(x - c^2 t))}{(1+\exp(c(x - c^2 t)))^2}. \tag{7.71}$$

In order to assess the advantages and the accuracy of the CFRDTM, we consider the (2 + 1)-dimensional time fractional coupled Burgers equations. Firstly, we derive the recursive formula from Eqs. (7.70a) and (7.70b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$, $V(0,0) = v(x,0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0,j) = 0,\ \ j = 1,2,3,\ldots \text{ and } V(i,0) = 0,\ \ i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.70a), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = -\frac{\partial^3}{\partial x^3}U(h,k-h) - 6\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)\frac{\partial}{\partial x}U(l,k-h-s)\right) + 3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(h-l,s)\frac{\partial}{\partial x}V(l,k-h-s)\right). \tag{7.72}$$

From the initial condition of Eq. (7.70c), we have

$$U(0,0) = u(x,0). \tag{7.73}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.70b)

$$\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = -\frac{\partial^3}{\partial x^3}V(h,k-h) - 3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right). \tag{7.74}$$

From the initial condition of Eq. (7.70d), we have

$$V(0,0) = v(x,0) \tag{7.75}$$

According to CFRDTM, using recursive Eq. (7.72) with initial condition Eq. (7.73) and also using recursive scheme Eq. (7.74) with initial condition Eq. (7.75) simultaneously, we obtain

$$U(1,0) = \frac{4c^5\exp(cx)(-1+\exp(cx))}{(1+\exp(cx))^3\Gamma(1+\alpha)},$$

$$V(0,1) = \frac{4c^5\exp(cx)(-1+\exp(cx))}{(1+\exp(cx))^3\Gamma(1+\beta)},$$

$$U(1,1) = -\frac{96c^8\exp(2cx)(1-3\exp(cx)+\exp(2cx))}{(1+\exp(cx))^6\Gamma(1+\alpha+\beta)},$$

$$V(0,2) = \frac{4c^8\exp(cx)(1-14\exp(cx)+18\exp(2cx)-14\exp(3cx)+\exp(4cx))}{(1+\exp(cx))^6\Gamma(1+2\beta)},$$

$$U(2,0)=\frac{4c^8\exp(cx)(1+22\exp(cx)-78\exp(2cx)+22\exp(3cx)+\exp(4cx))}{(1+\exp(cx))^6\Gamma(1+2\alpha)},$$

$$V(1,1)=\frac{48c^8\exp(2cx)(-1+\exp(cx))^2}{(1+\exp(cx))^6\Gamma(1+\alpha+\beta)},$$

$$U(2,1)=-\frac{96c^{11}e^{2cx}(-8+81e^{cx}-175e^{2cx}+175e^{3cx}-81e^{4cx}+8e^{5cx})}{(1+e^{cx})^9\Gamma(1+2\alpha+\beta)},$$

$$V(2,1)=\frac{48c^{11}e^{2cx}(-1+e^{cx})(1+22e^{cx}-78e^{2cx}+22e^{3cx}+e^{4cx})}{(1+e^{cx})^9\Gamma(1+2\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned}u(x,t)&=\sum_{k=0}^{\infty}\sum_{h=0}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)}\\&=U(0,0)+\sum_{k=1}^{\infty}\sum_{h=1}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)}\\&=\frac{4c^2e^{cx}}{(1+e^{cx})^2}+\frac{4c^5e^{cx}(-1+e^{cx})t^{\alpha}}{(1+e^{cx})^3\Gamma(1+\alpha)}\\&\quad+\frac{4c^8e^{cx}(1+22e^{cx}-78e^{2cx}+22e^{3cx}+e^{4cx})t^{2\alpha}}{(1+e^{cx})^6\Gamma(1+2\alpha)}\\&\quad-\frac{96c^{11}e^{2cx}(-8+81e^{cx}-175e^{2cx}+175e^{3cx}-81e^{4cx}+8e^{5cx})t^{2\alpha+\beta}}{(1+e^{cx})^9\Gamma(1+2\alpha+\beta)}\cdots\end{aligned}\tag{7.76}$$

$$\begin{aligned}v(x,t)&=\sum_{k=0}^{\infty}\sum_{h=0}^{k}V(h,k-h)t^{(h\alpha+(k-h)\beta)}\\&=V(0,0)+\sum_{k=1}^{\infty}\sum_{h=0}^{k}V(h,k-h)t^{(h\alpha+(k-h)\beta)}\\&=\frac{4c^2e^{cx}}{(1+e^{cx})^2}+\frac{4c^5e^{cx}(-1+e^{cx})t^{\beta}}{(1+e^{cx})^3\Gamma(1+\beta)}\\&\quad+\frac{4c^8e^{cx}(1-14e^{cx}+18e^{2cx}-14e^{3cx}+e^{4cx})t^{2\beta}}{(1+\exp(cx))^6\Gamma(1+2\beta)}\\&\quad+\frac{48c^{11}e^{2cx}(-1+e^{cx})(1+22e^{cx}-78e^{2cx}+22e^{3cx}+e^{4cx})t^{2\alpha+\beta}}{(1+e^{cx})^9\Gamma(1+2\alpha+\beta)}\cdots\end{aligned}\tag{7.77}$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.76) becomes

$$u(x,t) = \frac{4c^2 e^{cx}}{(1+e^{cx})^2} + \frac{4c^5 e^{cx}(-1+e^{cx})t}{(1+e^{cx})^3} + \frac{2c^8 e^{cx}(1-4e^{cx}+e^{2cx})t^2}{(1+e^{cx})^4} + \ldots. \tag{7.78}$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.77) becomes

$$v(x,t) = \frac{4c^2 e^{cx}}{(1+e^{cx})^2} + \frac{4c^5 e^{cx}(-1+e^{cx})t}{(1+e^{cx})^3} + \frac{2c^8 e^{cx}(1-4e^{cx}+e^{2cx})t^2}{(1+e^{cx})^4} + \ldots. \tag{7.79}$$

The solutions in Eqs. (7.78) and (7.79) are exactly the same as the Taylor series expansion of the exact solution

$$u(x,t) = v(x,t) = \frac{4c^2 e^{cx}}{(1+e^{cx})^2} + \frac{4c^5 e^{cx}(-1+e^{cx})t}{(1+e^{cx})^3} + \frac{2c^8 e^{cx}(1-4e^{cx}+e^{2cx})t^2}{(1+e^{cx})^4} + \ldots. \tag{7.80}$$

In order to verify the efficiency and accuracy of the proposed method for the time fractional coupled KdV equations, the graphs have been drawn in Figs. 7.10, 7.11, and 7.12. The numerical solutions for Eqs. (7.76) and (7.77) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 7.10. It can be observed from Fig. 7.10 that the solutions obtained by the proposed method coincide with the exact solution. Figure 7.11 shows the numerical solutions of Eqs. (7.76) and (7.77) when $\alpha = 1/3$ and $\beta = 1/5$. Again, Fig. 7.12 cites the numerical solutions when $\alpha = 0.005$ and $\beta = 0.002$. From Figs. 7.11 and 7.12, it can be observed that the solutions for $u(x,t)$ and $v(x,t)$ bifurcate into waves as the time fractional derivatives α and β decrease.

Example 7.6 Consider the following time fractional coupled KdV equations [42]

$$D_t^{\alpha} u + 6uu_x - 6vv_x + u_{xxx} = 0, \tag{7.81a}$$

$$D_t^{\beta} v + 3uv_x + v_{xxx} = 0, \tag{7.81b}$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

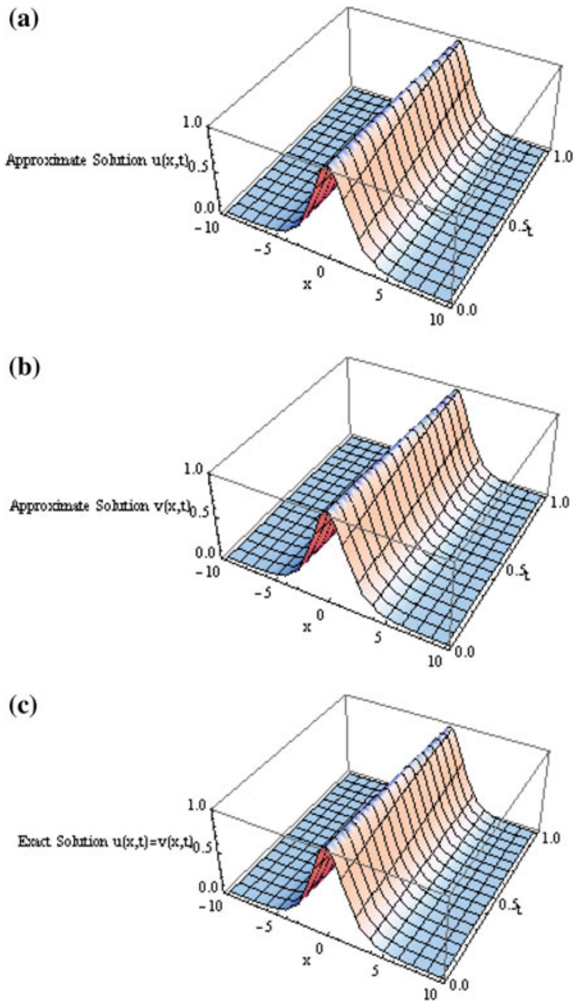


Fig. 7.10 Surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, and **c** the exact solution of $u(x,t) = v(x,t)$ when $\alpha = 1$ and $\beta = 1$

$$u(x,0) = \lambda \operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right), \tag{7.82}$$

$$v(x,0) = \frac{\lambda}{\sqrt{2}} \operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right). \tag{7.83}$$

First, we derive the recursive formula from Eqs. (7.81a) and (7.81b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$,

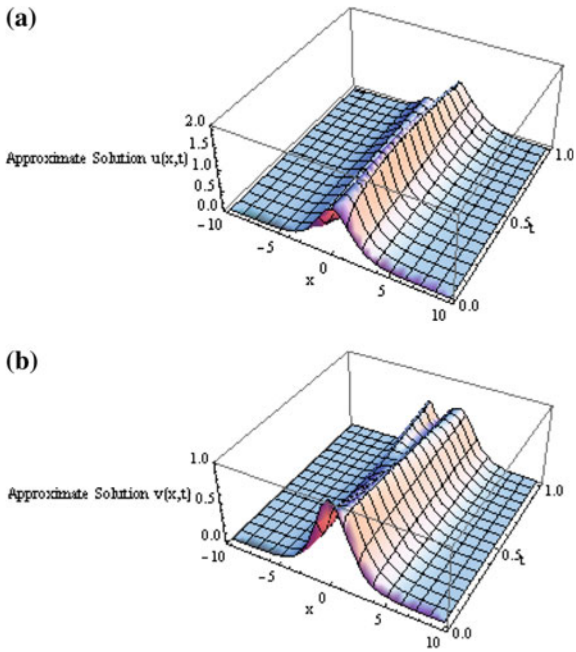


Fig. 7.11 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ when $\alpha = 1/3$ and $\beta = 1/5$

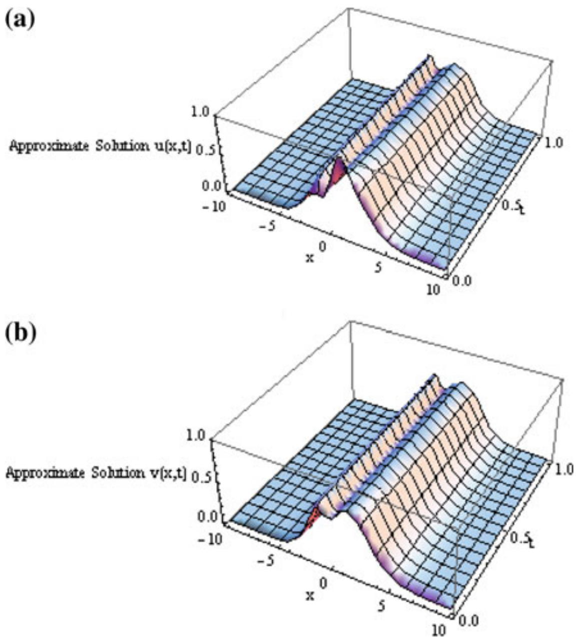


Fig. 7.12 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ when $\alpha = 0.005$ and $\beta = 0.002$

$V(0,0) = v(x,0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0,j) = 0, \quad j = 1,2,3,\ldots \quad \text{and} \quad V(i,0) = 0, \quad i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.81a), we obtain the following recursive formula

$$\begin{aligned}\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = &-\frac{\partial^3}{\partial x^3}U(h,k-h) \\ &-6\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)\frac{\partial}{\partial x}U(l,k-h-s)\right) \\ &+6\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(h-l,s)\frac{\partial}{\partial x}V(l,k-h-s)\right)\end{aligned} \tag{7.84}$$

From the initial condition of Eq. (7.82), we have

$$U(0,0) = u(x,0) \tag{7.85}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.81b)

$$\begin{aligned}\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = &-\frac{\partial^3}{\partial x^3}V(h,k-h) \\ &-3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)\end{aligned} \tag{7.86}$$

From the initial condition of Eq. (7.83), we have

$$V(0,0) = v(x,0) \tag{7.87}$$

According to CFRDTM, using recursive Eq. (7.84) with initial condition Eq. (7.85) and also using recursive scheme Eq. (7.86) with initial condition Eq. (7.87) simultaneously, we obtain successively

$$U(1,0) = \frac{\lambda^{5/2}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)\tanh\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\alpha)},$$

$$V(0,1) = \frac{4\sqrt{2}\lambda^{5/2}\operatorname{cosech}^3\left(x\sqrt{\lambda}\right)\sinh^4\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\beta)},$$

$$U(1,1) = -\frac{3\lambda^4\left(-3+2\cosh\left(x\sqrt{\lambda}\right)\right)\text{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{2\Gamma(1+\alpha+\beta)},$$

$$V(0,2) = \frac{\lambda^4\left(9-14\cosh\left(x\sqrt{\lambda}\right)+\cosh\left(2x\sqrt{\lambda}\right)\right)\text{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{8\sqrt{2}\Gamma(1+2\beta)},$$

$$U(2,0) = \frac{\lambda^4\left(-39+22\cosh\left(x\sqrt{\lambda}\right)+\cosh\left(2x\sqrt{\lambda}\right)\right)\text{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{8\Gamma(1+2\alpha)},$$

$$V(1,1) = \frac{96\sqrt{2}\lambda^4\text{cosech}^6\left(x\sqrt{\lambda}\right)\sinh^8\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= \lambda\text{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right) + \frac{t^{\alpha}\lambda^{5/2}\text{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)\tanh\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\alpha)} \\
&\quad + \frac{t^{2\alpha}\lambda^4\left(-39+22\cosh\left(x\sqrt{\lambda}\right)+\cosh\left(2x\sqrt{\lambda}\right)\right)\text{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{8\Gamma(1+2\alpha)} \\
&\quad - \frac{3t^{\alpha+\beta}\lambda^4\left(-3+2\cosh\left(x\sqrt{\lambda}\right)\right)\text{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{2\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{7.88}$$

$$\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= \frac{\lambda}{\sqrt{2}}\text{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right) + \frac{4\sqrt{2}t^{\beta}\lambda^{5/2}\text{cosech}^3\left(x\sqrt{\lambda}\right)\sinh^4\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\beta)}
\end{aligned}$$

$$+\frac{t^{2\beta}\lambda^4\left(9-14\cosh\left(x\sqrt{\lambda}\right)+\cosh\left(2x\sqrt{\lambda}\right)\right)\operatorname{sech}^6\left(\frac{\sqrt{\lambda}x}{2}\right)}{8\sqrt{2}\Gamma(1+2\beta)}$$
$$+\frac{96\sqrt{2}t^{\alpha+\beta}\lambda^4\operatorname{cosech}^6\left(x\sqrt{\lambda}\right)\sinh^8\left(\frac{\sqrt{\lambda}x}{2}\right)}{\Gamma(1+\alpha+\beta)}+\cdots \tag{7.89}$$

When $\alpha=1$ and $\beta=1$, the solution in Eq. (7.88) becomes

$$\begin{aligned}u(x,t)&=\lambda\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)+\lambda^{5/2}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)\tanh\left(\frac{\sqrt{\lambda}x}{2}\right)t\\&+\frac{\lambda^4}{4}\left(-2+\cosh\left(x\sqrt{\lambda}\right)\right)\operatorname{sech}^4\left(\frac{\sqrt{\lambda}x}{2}\right)t^2\\&+\frac{\lambda^{11/2}}{24}\left(-11\sinh\left(\frac{x\sqrt{\lambda}}{2}\right)+\sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right)\operatorname{sech}^5\left(\frac{\sqrt{\lambda}x}{2}\right)t^3+\cdots\end{aligned} \tag{7.90}$$

When $\alpha=1$ and $\beta=1$, the solution in Eq. (7.89) becomes

$$\begin{aligned}v(x,t)&=\frac{\lambda}{\sqrt{2}}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)+4\sqrt{2}\lambda^{5/2}\operatorname{cosech}^3\left(x\sqrt{\lambda}\right)\sinh^4\left(\frac{\sqrt{\lambda}x}{2}\right)t\\&+\frac{\lambda^4\left(-2+\cosh\left(x\sqrt{\lambda}\right)\right)\operatorname{sech}^4\left(\frac{\sqrt{\lambda}x}{2}\right)t^2}{4\sqrt{2}}\\&+\frac{\lambda^{11/2}\left(-11\sinh\left(\frac{\sqrt{\lambda}x}{2}\right)+\sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right)\operatorname{sech}^5\left(\frac{x\sqrt{\lambda}}{2}\right)t^3}{24\sqrt{2}}+\cdots\end{aligned} \tag{7.91}$$

The solutions in Eqs. (7.90) and (7.91) are exactly the same as the Taylor series expansions of the exact solutions

$$\begin{aligned}u(x,t)&=\lambda\operatorname{sech}^2\left(\frac{\sqrt{\lambda}(x-\lambda t)}{2}\right)\\&=\lambda\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)+\lambda^{5/2}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right)\tanh\left(\frac{\sqrt{\lambda}x}{2}\right)t\\&+\frac{\lambda^4}{4}\left(-2+\cosh\left(x\sqrt{\lambda}\right)\right)\operatorname{sech}^4\left(\frac{\sqrt{\lambda}x}{2}\right)t^2\\&+\frac{\lambda^{11/2}}{24}\left(-11\sinh\left(\frac{x\sqrt{\lambda}}{2}\right)+\sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right)\operatorname{sech}^5\left(\frac{\sqrt{\lambda}x}{2}\right)t^3+\cdots\end{aligned} \tag{7.92}$$

$$
\begin{aligned}
v(x,t) &= \frac{\lambda}{\sqrt{2}}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}(x-\lambda t)}{2}\right) \\
&= \frac{\lambda}{\sqrt{2}}\operatorname{sech}^2\left(\frac{\sqrt{\lambda}x}{2}\right) + 4\sqrt{2}\lambda^{5/2}\operatorname{cosech}^3\left(x\sqrt{\lambda}\right)\sinh^4\left(\frac{\sqrt{\lambda}x}{2}\right)t \\
&\quad + \frac{\lambda^4\left(-2+\cosh\left(x\sqrt{\lambda}\right)\right)\operatorname{sech}^4\left(\frac{\sqrt{\lambda}x}{2}\right)t^2}{4\sqrt{2}} \\
&\quad + \frac{\lambda^{11/2}\left(-11\sinh\left(\frac{\sqrt{\lambda}x}{2}\right)+\sinh\left(\frac{3x\sqrt{\lambda}}{2}\right)\right)\operatorname{sech}^5\left(\frac{x\sqrt{\lambda}}{2}\right)t^3}{24\sqrt{2}} + \cdots
\end{aligned}
\tag{7.93}
$$

Again, in order to verify the efficiency and accuracy of the proposed method for the time fractional coupled KdV equations, the graphs have been drawn in Figs. 7.13, 7.14, and 7.15. The numerical solutions for Eqs. (7.90) and (7.91) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 7.13. It can be observed from Fig. 7.10 that the solutions obtained by the proposed method are exactly identical with the exact solutions. Figure 7.14 shows the numerical solutions of Eqs. (7.88) and (7.89) when $\alpha = 0.4$ and $\beta = 0.25$. Again, Fig. 7.15 cites the numerical solutions when $\alpha = 0.005$ and $\beta = 0.002$. From Figs. 7.14 and 7.15, it can be observed that the solutions for $u(x,t)$ and $v(x,t)$ bifurcate into two waves as the time fractional derivatives α and β decrease.

7.5.2 *Soliton Solutions for Time Fractional Coupled Modified KdV Equations*

In the present section, CFRDTM has been successfully implemented to determine the approximate solutions for the following coupled time fractional modified KdV equations.

Example 7.7 Consider the following time fractional coupled modified KdV equations [43]

$$
D_t^{\alpha}u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial(uv)}{\partial x} - 3\frac{\partial u}{\partial x}, \tag{7.94a}
$$

$$
D_t^{\beta}v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} + 3\frac{\partial v}{\partial x}, \tag{7.94b}
$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

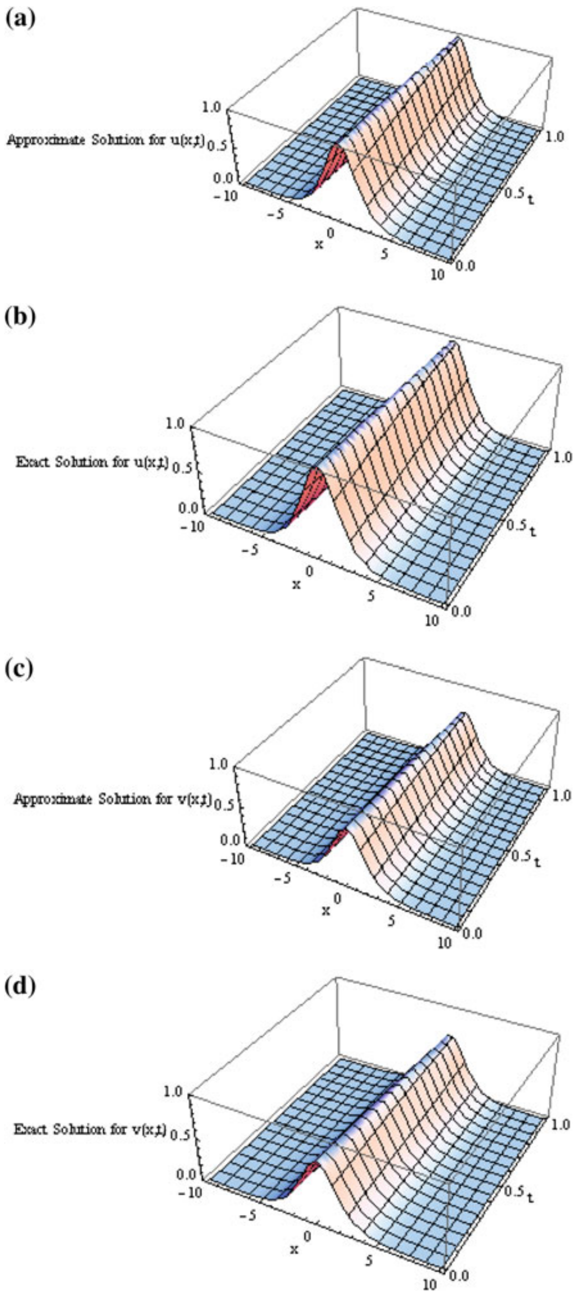


Fig. 7.13 Surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the exact solution of $u(x,t)$, **c** the numerical approximate solution of $v(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

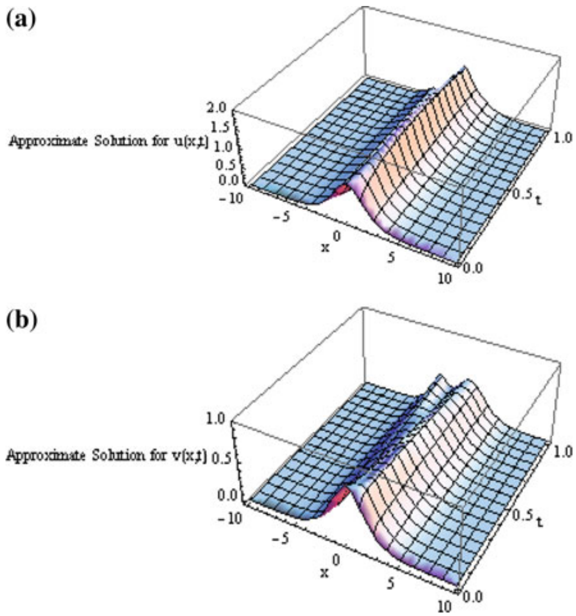


Fig. 7.14 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ when $\alpha = 0.4$ and $\beta = 0.25$

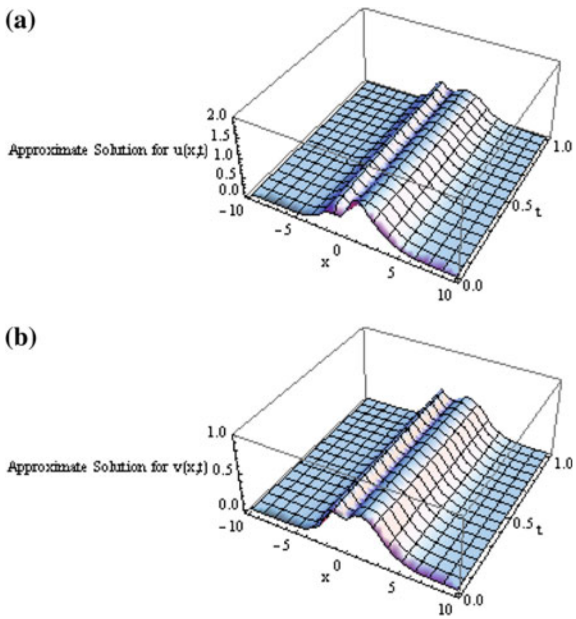


Fig. 7.15 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ when $\alpha = 0.005$ and $\beta = 0.002$

$$u(x,0) = \frac{1}{2} + \tanh(x), \tag{7.94c}$$

$$v(x,0) = 1 + \tanh(x). \tag{7.94d}$$

The exact solutions of Eqs. (7.94a) and (7.94b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \frac{1}{2} + \tanh(x + ct), \tag{7.95a}$$

$$v(x,t) = 1 + \tanh(x + ct). \tag{7.95b}$$

In order to assess the advantages and the accuracy of the CFRDTM for solving time fractional coupled modified KdV equations. Firstly, we derive the recursive formula from Eqs. (7.94a), (7.94b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$, $V(0,0) = v(x,0)$ are the given initial conditions.

Without loss of generality, the following assumptions have been taken

$$U(0,j) = 0,\ \ j = 1,2,3,\ldots \text{ and } V(i,0) = 0,\ \ i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.94a), we obtain the following recursive formula

$$\begin{aligned}\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h) = {} & \frac{1}{2}\frac{\partial^3}{\partial x^3} U(h, k-h) \\ & + \frac{3}{2}\frac{\partial^2}{\partial x^2} V(h, k-h) - 3\frac{\partial}{\partial x} U(h, k-h) \\ & + 3\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l, s) V(l, k-h-s)\right) \\ & - 3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s} U(r, k-h-s-p)\right. \\ & \left.\times U(l,s)\frac{\partial}{\partial x} U(h-r-l, p)\right)\end{aligned} \tag{7.96}$$

From the initial condition of Eq. (7.94c), we have

$$U(0,0) = u(x,0) \tag{7.97}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.94b)

$$\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = -\frac{\partial^3}{\partial x^3}V(h,k-h)+3\frac{\partial}{\partial x}V(h,k-h)$$
$$-3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}\frac{\partial}{\partial x}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)$$
$$-3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right)$$
$$+3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s}U(r,k-h-s-p)\right.$$
$$\left.\times U(l,s)\frac{\partial}{\partial x}U(h-r-l,p)\right) \tag{7.98}$$

From the initial condition of Eq. (7.94d), we have

$$V(0,0) = v(x,0) \tag{7.99}$$

According to CFRDTM, using recursive Eq. (7.96) with initial condition Eq. (7.97) and also using recursive scheme Eq. (7.98) with initial condition Eq. (7.99) simultaneously, we obtain

$$U(1,0) = -\frac{\operatorname{sech}^2(x)}{4\Gamma(1+\alpha)},$$

$$V(0,1) = -\frac{\operatorname{sech}^2(x)}{4\Gamma(1+\beta)},$$

$$U(1,1) = \frac{3\operatorname{sech}^2(x)\tanh(x)}{4\Gamma(1+\alpha+\beta)},$$

$$V(0,2) = \frac{\operatorname{sech}^5(x)(9\cosh(x)-3\cosh(3x)+32\sinh(x)-4\sinh(3x))}{8\Gamma(1+2\beta)},$$

$$U(2,0) = -\frac{7\operatorname{sech}^2(x)\tanh(x)}{8\Gamma(1+2\alpha)},$$

$$V(1,1) = \frac{3\operatorname{sech}^5(x)(-12\cosh(x)+4\cosh(3x)-43\sinh(x)+5\sinh(3x))}{32\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= \frac{1}{2} + \tanh(x) - \frac{t^{\alpha}\operatorname{sech}^2(x)}{4\Gamma(1+\alpha)} - \frac{7t^{2\alpha}\operatorname{sech}^2(x)\tanh(x)}{8\Gamma(1+2\alpha)} \\
&\quad + \frac{3t^{\alpha+\beta}\operatorname{sech}^2(x)\tanh(x)}{4\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{7.100}
$$

$$
\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= 1 + \tanh(x) - \frac{t^{\beta}\operatorname{sech}^2(x)}{4\Gamma(1+\beta)} \\
&\quad + \frac{t^{2\beta}\operatorname{sech}^5(x)(9\cosh(x) - 3\cosh(3x) + 32\sinh(x) - 4\sinh(3x))}{8\Gamma(1+2\beta)} \\
&\quad + \frac{3t^{\alpha+\beta}\operatorname{sech}^5(x)(-12\cosh(x) + 4\cosh(3x) - 43\sinh(x) + 5\sinh(3x))}{32\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned} \tag{7.101}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.100) becomes

$$
\begin{aligned}
u(x,t) = \frac{1}{2} + \tanh(x) - \frac{t\operatorname{sech}^2(x)}{4} - \frac{t^2\operatorname{sech}^2(x)\tanh(x)}{16} \\
- \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots
\end{aligned} \tag{7.102}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.101) becomes

$$
\begin{aligned}
v(x,t) = 1 + \tanh(x) - \frac{t\operatorname{sech}^2(x)}{4} - \frac{t^2\operatorname{sech}^2(x)\tanh(x)}{16} \\
- \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots
\end{aligned} \tag{7.103}
$$

The solutions in Eqs. (7.102) and (7.103) are exactly the same as the Taylor series expansions of the exact solutions

$$\begin{aligned} u(x,t) &= \frac{1}{2} + \tanh\left(x - \frac{t}{4}\right) \\ &= \frac{1}{2} + \tanh(x) - \frac{t\operatorname{sech}^2(x)}{4} - \frac{t^2\operatorname{sech}^2(x)\tanh(x)}{16} \\ &\quad - \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots \end{aligned} \tag{7.104}$$

$$\begin{aligned} v(x,t) &= 1 + \tanh\left(x - \frac{t}{4}\right) \\ &= 1 + \tanh(x) - \frac{t\operatorname{sech}^2(x)}{4} - \frac{t^2\operatorname{sech}^2(x)\tanh(x)}{16} \\ &\quad - \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{192} + \cdots \end{aligned} \tag{7.105}$$

In order to explore the efficiency and accuracy of the proposed method for the time fractional coupled modified KdV equations, the graphs have been drawn in Fig. 7.16a–d. The numerical solutions for Eqs. (7.102) and (7.103) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 7.16a, b. It can be observed from Fig. 7.16a–d that the solutions obtained by the proposed method coincide with the exact solution. In this case, we see that the soliton solutions are kink types for both $u(x,t)$ and $v(x,t)$.

Example 7.8 Consider the following time fractional coupled modified KdV equations [44]

$$D_t^{\alpha}u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u^2\frac{\partial u}{\partial x} + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3\frac{\partial(uv)}{\partial x} + 3\frac{\partial u}{\partial x}, \tag{7.106a}$$

$$D_t^{\beta}v = -\frac{\partial^3 v}{\partial x^3} - 3v\frac{\partial v}{\partial x} - 3\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 3u^2\frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial x}, \tag{7.106b}$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

$$u(x,0) = \tanh(x), \tag{7.106c}$$

$$v(x,0) = 1 - 2\tanh^2(x). \tag{7.106d}$$

The exact solutions of Eqs. (7.106a) and (7.106b) obtained by Adomian decomposition method, for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \tanh(x - t), \tag{7.107a}$$

$$v(x,t) = 1 - 2\tanh^2(x - t). \tag{7.107b}$$

In order to assess the advantages and the accuracy of the CFRDTM for solving time fractional coupled modified KdV equations, firstly we derive the recursive formula

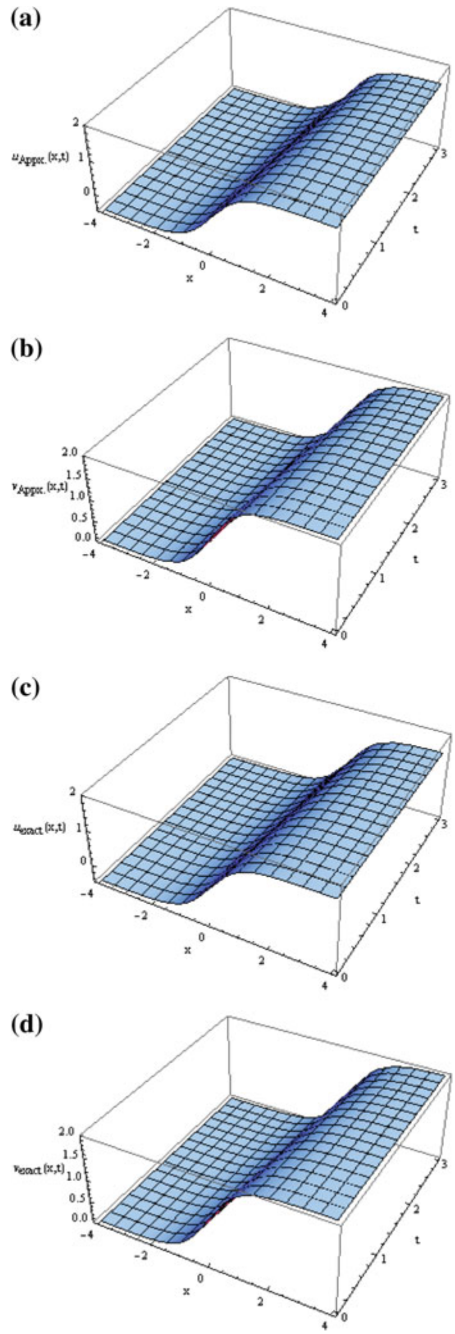


Fig. 7.16 Surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, **c** the exact solution of $u(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

from Eqs. (7.106a), (7.106b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$, $V(0,0) = v(x,0)$ are the given initial conditions.

Without loss of generality, the following assumptions have been taken

$$U(0,j) = 0, \;\; j = 1,2,3,\ldots \text{ and } V(i,0) = 0, \;\; i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.106a), we obtain the following recursive formula

$$\begin{aligned}\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = {} & \frac{1}{2}\frac{\partial^3}{\partial x^3}U(h,k-h) \\ & + \frac{3}{2}\frac{\partial^2}{\partial x^2}V(h,k-h) + 3\frac{\partial}{\partial x}U(h,k-h) \\ & + 3\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)V(l,k-h-s)\right) \\ & - 3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s}U(r,k-h-s-p)\right. \\ & \left.\times U(l,s)\frac{\partial}{\partial x}U(h-r-l,p)\right)\end{aligned} \tag{7.108}$$

From the initial condition of Eq. (7.106c), we have

$$U(0,0) = u(x,0) \tag{7.109}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.106b)

$$\begin{aligned}\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = {} & -\frac{\partial^3}{\partial x^3}V(h,k-h) - 3\frac{\partial}{\partial x}V(h,k-h) \\ & - 3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}\frac{\partial}{\partial x}U(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right) \\ & - 3\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}V(l,k-h-s)\frac{\partial}{\partial x}V(h-l,s)\right) \\ & + 3\left(\sum_{r=0}^{h}\sum_{l=0}^{h-r}\sum_{s=0}^{k-h}\sum_{p=0}^{k-h-s}U(r,k-h-s-p)\right. \\ & \left.\times U(l,s)\frac{\partial}{\partial x}U(h-r-l,p)\right)\end{aligned} \tag{7.110}$$

From the initial condition of Eq. (7.106d), we have

$$V(0,0) = v(x,0) \tag{7.111}$$

According to CFRDTM, using recursive Eq. (7.108) with initial condition Eq. (7.109) and also using recursive scheme Eq. (7.110) with initial condition Eq. (7.111) simultaneously, we obtain

$$U(1,0) = -\frac{\operatorname{sech}^2(x)}{\Gamma(1+\alpha)}$$

$$V(0,1) = \frac{4\operatorname{sech}^2(x)\tanh(x)}{\Gamma(1+\beta)}$$

$$U(1,1) = -\frac{24\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+\alpha+\beta)}$$

$$V(0,2) = \frac{\operatorname{sech}^6(x)(21-26\cosh(2x)+\cosh(4x))}{\Gamma(1+2\beta)}$$

$$U(2,0) = -\frac{(-23+\cosh(2x))\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+2\alpha)}$$

$$V(1,1) = \frac{48\operatorname{sech}^4(x)\tanh^2(x)}{\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= \tanh(x) - \frac{t^{\alpha}\operatorname{sech}^2(x)}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}(-23+\cosh(2x))\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+2\alpha)} \\ &\quad - \frac{24t^{\alpha+\beta}\operatorname{sech}^4(x)\tanh(x)}{\Gamma(1+\alpha+\beta)} + \ldots \end{aligned} \tag{7.112}$$

$$
\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= 1 - 2\tanh^2(x) + \frac{4t^{\beta}\operatorname{sech}^2(x)\tanh(x)}{\Gamma(1+\beta)} \\
&\quad + \frac{t^{2\beta}\operatorname{sech}^6(x)(21 - 26\cosh(2x) + \cosh(4x))}{\Gamma(1+2\beta)} \\
&\quad + \frac{48t^{\alpha+\beta}\operatorname{sech}^4(x)\tanh^2(x)}{\Gamma(1+\alpha+\beta)} + \cdots
\end{aligned}
\tag{7.113}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.112) becomes

$$
\begin{aligned}
u(x,t) &= \tanh(x) - t\operatorname{sech}^2(x) - t^2\operatorname{sech}^2(x)\tanh(x) \\
&\quad - \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{3} + \ldots
\end{aligned}
\tag{7.114}
$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.113) becomes

$$
\begin{aligned}
v(x,t) &= 1 - 2\tanh^2(x) + 4t\operatorname{sech}^2(x)\tanh(x) + 2t^2\operatorname{sech}^4(x)(-2+\cosh(2x)) \\
&\quad + \frac{2t^3\operatorname{sech}^5(x)(-11\sinh(x)+\sinh(3x))}{3} + \ldots
\end{aligned}
\tag{7.115}
$$

The solutions in Eqs. (7.114) and (7.115) are exactly the same as the Taylor series expansions of the exact solutions

$$
\begin{aligned}
u(x,t) &= \tanh(x-t) \\
&= \tanh(x) - t\operatorname{sech}^2(x) - t^2\operatorname{sech}^2(x)\tanh(x) \\
&\quad - \frac{t^3\operatorname{sech}^4(x)(-2+\cosh(2x))}{3} + \ldots
\end{aligned}
\tag{7.116}
$$

$$
\begin{aligned}
v(x,t) &= 1 - 2\tanh^2(x-t) \\
&= 1 - 2\tanh^2(x) + 4t\operatorname{sech}^2(x)\tanh(x) \\
&\quad + 2t^2\operatorname{sech}^4(x)(-2+\cosh(2x) \\
&\quad + \frac{2t^3\operatorname{sech}^5(x)(-11\sinh(x)+\sinh(3x))}{3} + \ldots
\end{aligned}
\tag{7.117}
$$

Again, in order to verify the efficiency and reliability of the proposed method for the time fractional coupled modified KdV equations, the graphs have been drawn in Fig. 7.17a–d. The numerical solutions for Eqs. (7.114) and (7.115) for the special case where $\alpha = 1$ and $\beta = 1$ are shown in Fig. 7.17a–d. It can be observed from Fig. 7.17a–d that the soliton solutions obtained by the proposed method are exactly identical with the exact solutions. In this case, we see that the soliton solutions are kink type for $u(x,t)$ and bell type for $v(x,t)$.

Verification of Classical Integer-Order Solutions by ADM

In case of $\alpha = 1$ and $\beta = 1$, to solve Eqs. (7.106a) and (7.106b) by means of Adomian decomposition method (ADM), we rewrite Eqs. (7.106a) and (7.106b) in an operator form

$$L_t u = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3A(u) + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3B(u,v) + 3\frac{\partial u}{\partial x}, \tag{7.118}$$

$$L_t v = -\frac{\partial^3 v}{\partial x^3} - 3C(v) - 3G(u,v) + 3H(u,v) - 3\frac{\partial v}{\partial x}, \tag{7.119}$$

where $L_t \equiv \frac{\partial}{\partial t}$ is the easily invertible linear differential operator with its inverse operator $L_t^{-1}(.) \equiv \int_0^t (.)\mathrm{d}\tau$. Here, the functions $A(u) = u^2\frac{\partial u}{\partial x}$, $B(u,v) = \frac{\partial(uv)}{\partial x}$, $C(v) = v\frac{\partial v}{\partial x}$, $G(u,v) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$, and $H(u,v) = u^2\frac{\partial v}{\partial x}$ are related to the nonlinear terms and they can be expressed in terms of the Adomian polynomials as follows:

$A(u) = \sum_{n=0}^{\infty} A_n$, $B(u,v) = \sum_{n=0}^{\infty} B_n$, $C(v) = \sum_{n=0}^{\infty} C_n$, $G(u,v) = \sum_{n=0}^{\infty} G_n$, and $H(u,v) = \sum_{n=0}^{\infty} H_n$. In particular, for nonlinear operators $A(u)$ and $B(u,v)$, the Adomian polynomials are defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[A\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right]\Bigg|_{\lambda=0}, \quad n \geq 0$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[B\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]\Bigg|_{\lambda=0}, \quad n \geq 0$$

The first few components of $A(u)$, $B(u,v)$, $C(v)$, $G(u,v)$, and $H(u,v)$ are, respectively, given by

$$A_0 = u_0^2 u_{0x},$$

$$A_1 = u_0^2 u_{1x} + 2u_0 u_1 u_{0x},$$

$$A_2 = u_{0x}(2u_0 u_2 + u_1^2) + u_0^2 u_{2x} + 2u_0 u_1 u_{1x}$$

$$\ldots,$$

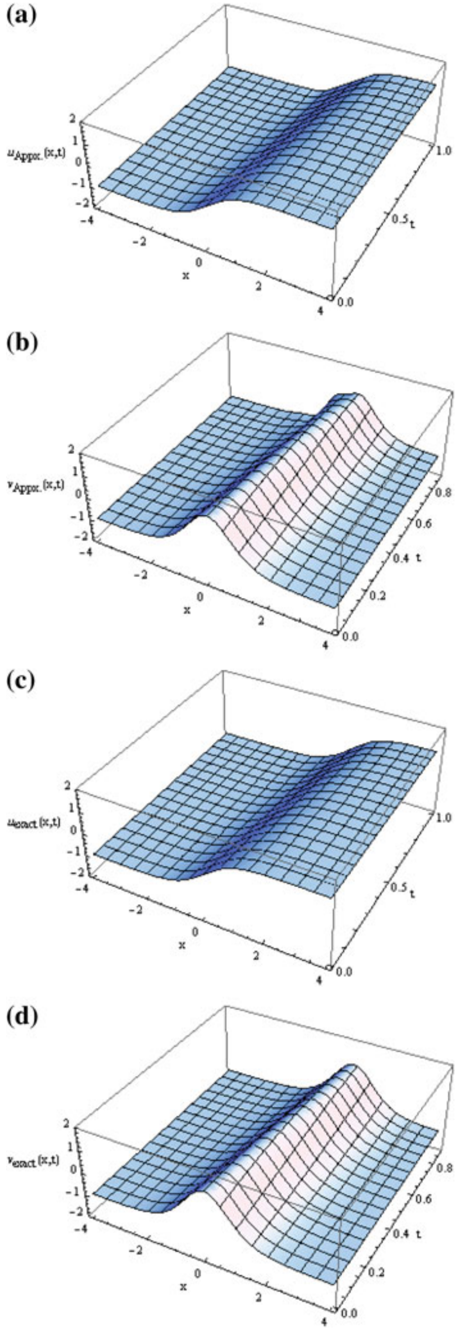


Fig. 7.17 Surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, **c** the exact solution of $u(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

$$B_0 = u_0 v_{0x} + v_0 u_{0x},$$

$$B_1 = u_0 v_{1x} + v_1 u_{0x} + u_1 v_{0x} + v_0 u_{1x},$$

$$B_2 = u_0 v_{2x} + v_2 u_{0x} + u_1 v_{1x} + v_1 u_{1x} + u_2 v_{0x} + v_0 u_{2x},$$

$$\ldots,$$

$$C_0 = v_0 v_{0x},$$

$$C_1 = v_0 v_{1x} + v_1 v_{0x},$$

$$C_2 = v_1 v_{1x} + v_0 v_{2x} + v_2 v_{0x},$$

$$\ldots,$$

$$G_0 = u_{0x} v_{0x},$$

$$G_1 = u_{0x} v_{1x} + v_{0x} u_{1x},$$

$$G_2 = u_{1x} v_{1x} + v_{0x} u_{2x} + u_{0x} v_{2x},$$

$$\ldots,$$

$$H_0 = u_0^2 v_{0x},$$

$$H_1 = u_0^2 v_{1x} + 2u_0 u_1 v_{0x},$$

$$H_2 = v_{0x}(2u_0 u_2 + u_1^2) + u_0^2 v_{2x} + 2u_0 u_1 v_{1x},$$

$$\ldots,$$

and so on, and the rest of the polynomials can be constructed in a similar manner.

Now, operating with L_t^{-1} on the both sides of Eqs. (7.118) and (7.119), yields

$$u(x,t) = u(x,0) + L_t^{-1}\left(\frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3A(u) + \frac{3}{2}\frac{\partial^2 v}{\partial x^2} + 3B(u,v) + 3\frac{\partial u}{\partial x}\right), \quad (7.120)$$

$$v(x,t) = v(x,0) + L_t^{-1}\left(-\frac{\partial^3 v}{\partial x^3} - 3C(v) - 3G(u,v) + 3H(u,v) - 3\frac{\partial v}{\partial x}\right). \quad (7.121)$$

The ADM assumes that the two unknown functions $u(x,t)$ and $v(x,t)$ can be expressed by infinite series in the following forms

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{7.122}$$

$$v(x,t)=\sum_{n=0}^{\infty}v_n(x,t). \tag{7.123}$$

Substituting Eqs. (7.122) and (7.123) into Eqs. (7.120) and (7.121) yields

$$u_0(x,t)=u(x,0),$$

$$u_{n+1}(x,t)=L_t^{-1}\left(\frac{1}{2}\frac{\partial^3 u_n(x,t)}{\partial x^3}-3A_n+\frac{3}{2}\frac{\partial^2 v_n(x,t)}{\partial x^2}+3B_n+3\frac{\partial u_n(x,t)}{\partial x}\right),\quad n\geq 0. \tag{7.124}$$

$$v_0(x,t)=v(x,0),$$

$$v_{n+1}(x,t)=L_t^{-1}\left(-\frac{\partial^3 v_n(x,t)}{\partial x^3}-3C_n-3G_n+3H_n-3\frac{\partial v_n(x,t)}{\partial x}\right),\quad n\geq 0. \tag{7.125}$$

Using known $u_0(x,t)$ and $v_0(x,t)$, all the remaining components $u_n(x,t)$ and $v_n(x,t)$, $n>0$ can be completely determined such that each term is determined by using the previous term. From Eqs. (7.124) and (7.125) with Eqs. (7.106c) and (7.106d), we determine the individual components of the decomposition series as

$$u_0=\tanh(x),$$

$$v_0=1-2\tanh^2(x),$$

$$u_1=-t\,\mathrm{sech}^2(x),$$

$$v_1=4t\,\mathrm{sech}^2(x)\tanh(x),$$

$$u_2=-t^2\,\mathrm{sech}^2(x)\tanh(x),$$

$$v_2=2t^2(-2+\cosh(2x))\mathrm{sech}^4(x),$$

$$u_3=-\frac{1}{3}t^3(-2+\cosh(2x))\mathrm{sech}^4(x),$$

$$v_3=\frac{2}{3}t^3\,\mathrm{sech}^5(x)(-11\sinh(x)+\sinh(3x)),$$

and so on, and the other components of the decomposition series (7.122) and (7.123) can be determined in a similar way.

Substituting these $u_0, u_1, u_2, \ldots$ and $v_0, v_1, v_2, \ldots$ in Eqs. (7.122) and (7.123), respectively, gives the ADM solutions for $u(x,t)$ and $v(x,t)$ in a series form

$$\begin{aligned} u(x,t) = {} & \tanh(x) - t\operatorname{sech}^2(x) - t^2\operatorname{sech}^2(x)\tanh(x) \\ & - \frac{1}{3}t^3(-2 + \cosh(2x))\operatorname{sech}^4(x) + \cdots, \end{aligned} \tag{7.126}$$

$$\begin{aligned} v(x,t) = {} & 1 - 2\tanh^2(x) + 4t\operatorname{sech}^2(x)\tanh(x) \\ & + 2t^2(-2 + \cosh(2x))\operatorname{sech}^4(x) \\ & + \frac{2}{3}t^3\operatorname{sech}^5(x)(-11\sinh(x) + \sinh(3x)) + \cdots. \end{aligned} \tag{7.127}$$

Using Taylor series, we obtain the closed-form solutions

$$u(x,t) = \tanh(x - t), \tag{7.128}$$

$$v(x,t) = 1 - 2\tanh^2(x - t). \tag{7.129}$$

With initial conditions (7.106c) and (7.106d), the solitary wave solutions of Eqs. (7.118) and (7.119) are of kink type for $u(x,t)$ and bell type for $v(x,t)$ which agree to some extent with the results constructed by Fan [44]. According to the learned author Fan [44], the solitary wave solutions of Eqs. (7.118) and (7.119) are kink type for $u(x,t) = \tanh\left(x + \frac{t}{2}\right)$ and bell type for $v(x,t) = \frac{3}{2} - 2\tanh^2\left(x + \frac{t}{2}\right)$, where $k = 1$ and $\lambda = -1$. There is definitely a mistake to be reckoned with and should be taken into account for further study. Since using the same parameters $k = 1$ and $\lambda = -1$, the solitary wave solutions of Eqs. (7.118) and (7.119) have been obtained as in Eqs. (7.128) and (7.129).

In the present analysis, the two methods coupled fractional reduced differential transform and Adomian decomposition method confirm the justification and correctness of the solutions obtained in Eqs. (7.128) and (7.129).

7.5.3 *Approximate Solution for Fractional Predator–Prey Equation*

In order to assess the advantages and the accuracy of the CFRDTM, we consider three cases with different initial conditions of the predator–prey system [54]. Firstly, we derive the recursive formula obtained from predator–prey system of Eqs. (7.10)–(7.11). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,y,t)$ and $v(x,y,t)$, respectively, where $u(x,y,t)$ and $v(x,y,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,y,0)$, $V(0,0) = v(x,y,0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0,j) = 0,\ \ j = 1,2,3,\ldots \text{ and } V(i,0) = 0,\ \ i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.10), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = \frac{\partial^2}{\partial x^2}U(h,k-h) + \frac{\partial^2}{\partial y^2}U(h,k-h) + aU(h,k-h) - b\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)V(l,k-h-s)\right). \tag{7.130}$$

From the initial condition of Eq. (7.10), we have

$$U(0,0) = u(x,y,0). \tag{7.131}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.11)

$$\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = \frac{\partial^2}{\partial x^2}V(h,k-h) + \frac{\partial^2}{\partial y^2}V(h,k-h) + b\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(l,k-h-s)V(h-l,s)\right) - cV(h,k-h). \tag{7.132}$$

From the initial condition of Eq. (7.11), we have

$$V(0,0) = v(x,y,0). \tag{7.133}$$

Applications and Results

Now, let us consider the three cases of the predator–prey system.

Case 1: Here, we consider the fractional predator–prey equation with constant initial condition

$$u(x,y,0) = u_0, \quad v(x,y,0) = v_0 \tag{7.134}$$

According to CFRDTM, using recursive scheme Eq. (7.130) with initial condition Eq. (7.131) and also using recursive scheme Eq. (7.132) with initial condition Eq. (7.133) simultaneously, we obtain

$$U[0,0] = u(x,y,0) = u_0, \quad V[0,0] = v(x,y,0) = v_0,$$

$$U[1,0] = \frac{u_0(a - bv_0)}{\Gamma(1+\alpha)}, \quad V[0,1] = \frac{(bu_0v_0 - cv_0)}{\Gamma(1+\beta)},$$

$$U[2,0] = \frac{u_0(a - bv_0)^2}{\Gamma(1+2\alpha)},$$

$$V[0,2] = \frac{v_0(c - bu_0)^2}{\Gamma(1+2\beta)},$$

$$U[1,1] = -\frac{bu_0(-cv_0 + bu_0v_0)}{\Gamma(1+\alpha+\beta)},$$

$$V[1,1] = \frac{bu_0v_0(a - bv_0)}{\Gamma(1+\alpha+\beta)},$$

$$U[1,2] = -\frac{b(c - bu_0)^2u_0v_0}{\Gamma(1+\alpha+2\beta)},$$

$$V[1,2] = \frac{bu_0(c-bu_0)v_0(-(a-2bv_0)\Gamma(1+\alpha)\Gamma(1+\beta) + (-a+bv_0)\Gamma(1+\alpha+\beta))}{\Gamma(1+\alpha+2\beta)\Gamma(1+\alpha)\Gamma(1+\beta)},$$

$$U[2,1] = \frac{bu_0v_0(a-bv_0)((c-2bu_0)\Gamma(1+\alpha)\Gamma(1+\beta) + (c-bu_0)\Gamma(1+\alpha+\beta))}{\Gamma(1+2\alpha+\beta)\Gamma(1+\alpha)\Gamma(1+\beta)},$$

$$V[2,1] = \frac{bu_0v_0(a - bv_0)^2}{\Gamma(1+2\alpha+\beta)},$$

$$U[3,0] = \frac{u_0(a - bv_0)^3}{\Gamma(1+3\alpha)},$$

$$V[0,3] = -\frac{v_0(c - bu_0)^3}{\Gamma(1+3\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned} u(x,y,t) &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= u_0 + \frac{u_0(a-bv_0)t^{\alpha}}{\Gamma(1+\alpha)} + \frac{u_0(a-bv_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad + \frac{u_0(a-bv_0)^3 t^{3\alpha}}{\Gamma(1+3\alpha)} - \frac{bu_0(-cv_0+bu_0v_0)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots \end{aligned} \tag{7.135}$$

$$\begin{aligned} v(x,y,t) &= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= v_0 + \frac{(bu_0v_0-cv_0)t^{\beta}}{\Gamma(1+\beta)} + \frac{bu_0v_0(a-bv_0)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} \\ &\quad + \frac{bu_0v_0(a-bv_0)^2 t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)}\cdots. \end{aligned} \tag{7.136}$$

Figure 7.18 cites the numerical solutions for Eqs. (7.10)–(7.11) obtained by the proposed CFRDTM method for the constant initial conditions $u_0 = 100$, $v_0 = 10$, $a = 0.05$, $b = 0.03$, and $c = 0.01$. Figure 7.19 shows the time evolution of population of $u(x,y,t)$ and $v(x,y,t)$ obtained from Eqs. (5.2) to (5.3) for different values of α and β. In the present numerical analysis, Table 7.2 shows the comparison of the numerical solutions with the proposed method with homotopy perturbation method and variational iteration method, when $a = 0.05$, $b = 0.03$, and $c = 0.01$. From Table 7.2, it is evidently clear that CFRDTM used in this paper has high accuracy. The numerical results obtained in this proposed method coincide precisely with values obtained in the homotopy perturbation method.

Case 2: In this case, the initial conditions of Eqs. (7.10)–(7.11) are given by

$$u(x,y,0) = e^{x+y}, v(x,y,0) = e^{x+y}. \tag{7.137}$$

By using Eqs. (7.130) to (7.133), we can successively obtain

$$U[0,0] = u(x,y,0) = e^{x+y}, V[0,0] = v(x,y,0) = e^{x+y},$$

$$U[1,0] = \frac{2e^{x+y} + ae^{x+y} - be^{2x+2y}}{\Gamma(1+\alpha)},$$

$$V[0,1] = \frac{2e^{x+y} - ce^{x+y} + be^{2x+2y}}{\Gamma(1+\beta)},$$

$$U[1,1] = \frac{be^{2(x+y)}(2 - c + be^{x+y})}{\Gamma(1+\alpha+\beta)},$$

$$V[1,1] = -\frac{be^{2(x+y)}(-2 - a + be^{x+y})}{\Gamma(1+\alpha+\beta)},$$

$$U[2,0] = \frac{e^{x+y}(4 + a^2 - 10be^{x+y} + b^2e^{2(x+y)} + a(4 - 2be^{x+y}))}{\Gamma(1+2\alpha)},$$

$$V[0,2] = \frac{e^{x+y}(4 + c^2 + 10be^{x+y} + b^2e^{2(x+y)} - 2c(2 + be^{x+y}))}{\Gamma(1+2\beta)},$$

$$U[1,2] = -\frac{be^{2(x+y)}(4 + c^2 + 10be^{x+y} + b^2e^{2(x+y)} - 2c(2 + be^{x+y}))}{\Gamma(1+\alpha+\beta)},$$

$$\begin{aligned} V[1,2] = {} & (be^{2(x+y)}(-(a(-8 + c - be^{x+y}) + 2(-8 + c + 9be^{x+y} \\ & - bce^{x+y} + b^2e^{2(x+y)}))\Gamma(1+\alpha)\Gamma(1+\beta) \\ & + (2 + a - be^{x+y})(2 - c + be^{x+y}) \\ & \times \Gamma(1+\alpha+\beta))/(\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)), \end{aligned}$$

$$U[3,0] = \frac{e^{x+y}(8 + a^3 - 84be^{x+y} + 28b^2e^{2(x+y)} - b^3e^{3(x+y)} + a^2(6 - 3be^{x+y}) + 3a(4 - 10be^{x+y} + b^2e^{2(x+y)}))}{\Gamma(1+3\alpha)},$$

$$V[0,3] = \frac{e^{x+y}(8 - c^3 + 84be^{x+y} + 28b^2e^{2(x+y)} + b^3e^{3(x+y)} + 3c^2(2 + be^{x+y}) - 3c(4 + 10be^{x+y} + b^2e^{2(x+y)}))}{\Gamma(1+3\beta)},$$

and so on.

The explicit approximate solution is

$$\begin{aligned} u(x,y,t) = {} & e^{x+y} + \frac{(2e^{x+y} + ae^{x+y} - be^{2x+2y})t^{\alpha}}{\Gamma(1+\alpha)} \\ & + \frac{e^{x+y}(4 + a^2 - 10be^{x+y} + b^2e^{2(x+y)} + a(4 - 2be^{x+y}))t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots, \end{aligned} \tag{7.138}$$

and

$$\begin{aligned} v(x,y,t) = {} & e^{x+y} + \frac{(2e^{x+y} - ce^{x+y} + be^{2x+2y})t^{\beta}}{\Gamma(1+\beta)} \\ & - \frac{be^{2(x+y)}(-2 - a + be^{x+y})t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots, \end{aligned} \tag{7.139}$$

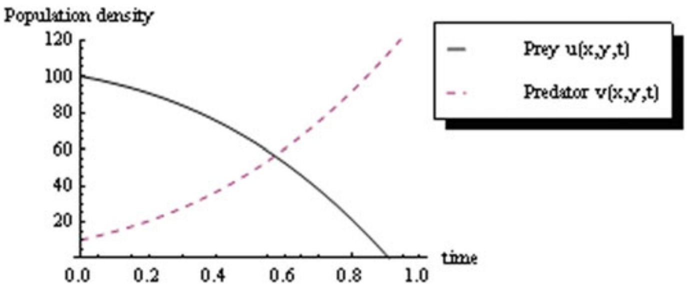


Fig. 7.18 Time evolution of the population for $u(x, y, t)$ and $v(x, y, t)$ obtained from Eqs. (7.135) and (7.136), when $\alpha = 1,\ \beta = 1$

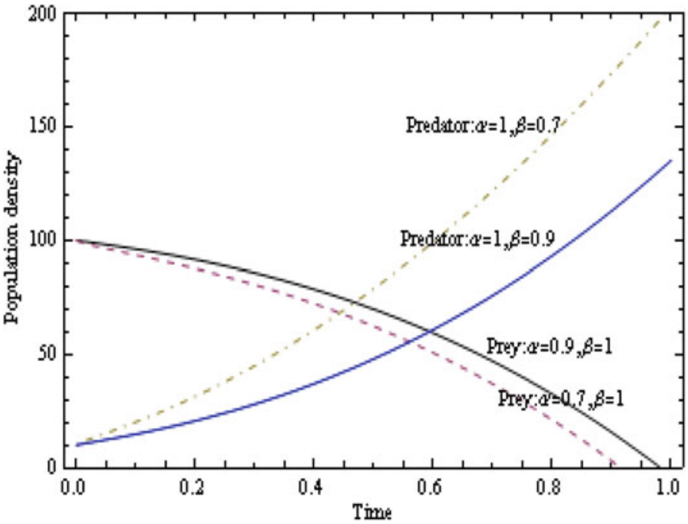


Fig. 7.19 Time evolution of the population for $u(x, y, t)$ and $v(x, y, t)$ obtained from Eqs. (7.135) and (7.136) for different values of α and β

Table 7.2 Comparison of the numerical solutions of the proposed method with homotopy perturbation method and variational iteration method

T	$\alpha = \beta$	Numerical value (u, v) by HPM	Numerical value (u, v) by VIM	Numerical value (u, v) by CFRDTM
0.02	1	(99.4831, 10.6146)	(99. 4834, 10.6323)	(99.4831, 10.6146)
	0.9	(99.1865, 10.9633)	(99.3065, 10. 8375)	(99.1865, 10.9633)
0.2	1	(93.0910, 17.8514)	(93. 3908, 17.7382)	(93.0910, 17.8514)
	0.9	(90.5735, 20.5567)	(92.4584, 18.8198)	(90.5735, 20.5567)
0.3	1	(87.9348, 23.4430)	(88. 9466, 22. 7237)	(87.9348, 23.4430)
	0.9	(83.7993, 27.7785)	(87. 8005, 24.0532)	(83.7993, 27.7785)

Figures 7.20 and 7.21 cite the numerical approximate solutions for the predator–prey system with the appropriate parameter. The obtained results of predator–prey population system indicate that this model exhibits the same behavior observed in the anomalous biological diffusion fractional model.

Figures 7.22 and 7.23 show the numerical solutions for prey population density for different values of parameters a, b, i.e., the natural birthrate of prey population and competitive rate between predator and prey populations. The results depicted in graphs agree with the realistic data.

Case 3: In this case, we consider the initial condition of fractional predator–prey Eqs. (7.10)–(7.11)

$$U[0,0] = u(x,y,0) = \sqrt{xy}, \quad V[0,0] = v(x,y,0) = \mathrm{e}^{x+y}, \tag{7.140}$$

$$U[1,0] = \frac{-\frac{x^2}{4(xy)^{3/2}} - \frac{y^2}{4(xy)^{3/2}} + a\sqrt{xy} - b\mathrm{e}^{x+y}\sqrt{xy}}{\Gamma(1+\alpha)},$$

$$V[0,1] = \frac{2\mathrm{e}^{x+y} - c\mathrm{e}^{x+y} + b\mathrm{e}^{x+y}\sqrt{xy}}{\Gamma(1+\beta)},$$

$$U[1,1] = \frac{b\mathrm{e}^{x+y}\sqrt{xy}(2-c+b\sqrt{xy})}{\Gamma(1+\alpha+\beta)},$$

$$V[1,1] = \frac{-b\mathrm{e}^{x+y}(y^2 + x^2(1-4ay^2+4b\mathrm{e}^{x+y}y^2))}{4(xy)^{3/2}\Gamma(1+\alpha+\beta)},$$

$$\begin{aligned} U[2,0] = {} & \frac{1}{16x^4y^4\Gamma(1+2\alpha)}\sqrt{xy}\big(-15y^4 - 16b\mathrm{e}^{x+y}x^3y^4 + x^2(2y^2 - 8(a-b\mathrm{e}^{x+y})y^4\big) \\ & + x^4(-15 + 16a^2y^4 + 16b^2\mathrm{e}^{2(x+y)}y^4 \\ & - 8b\mathrm{e}^{x+y}y^2(-1+2y+4y^2) - 8ay^2(1+4b\mathrm{e}^{x+y}y^2))), \end{aligned}$$

$$V[0,2] = \frac{\mathrm{e}^{x+y}(4(-2+c)^2(xy)^{3/2} + 4b^2(xy)^{5/2} - b(y^2 - 4xy^2 + x^2(1-4y+8(-2+c)y^2)))}{4(xy)^{3/2}\Gamma(1+2\beta)},$$

and so on.

The solution becomes

$$u(x,y,t) = \sqrt{xy} + \frac{\left(-\frac{x^2}{4(xy)^{3/2}} - \frac{y^2}{4(xy)^{3/2}} + a\sqrt{xy} - b\mathrm{e}^{x+y}\sqrt{xy}\right)t^\alpha}{\Gamma(1+\alpha)} + \cdots, \tag{7.141}$$

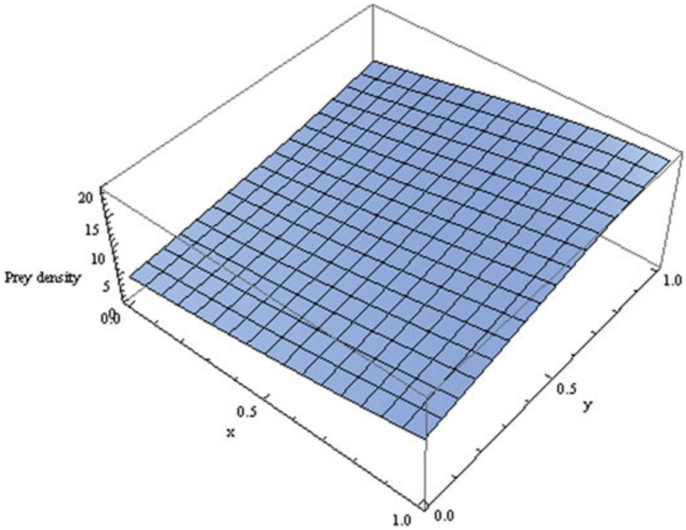


Fig. 7.20 Surface shows the numerical approximate solution of $u(x, y, t)$ when $\alpha = 0.88$, $\beta = 0.54$, a = 0.7, b = 0.03, c = 0.3, and t = 0.53

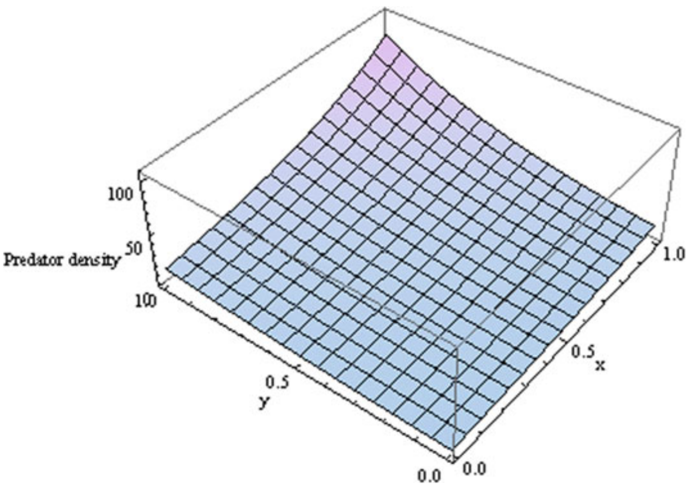


Fig. 7.21 Surface shows the numerical approximate solution of $v(x, y, t)$ when $\alpha = 0.88$, $\beta = 0.54$, a = 0.7, b = 0.03, c = 0.9, and t = 0.6

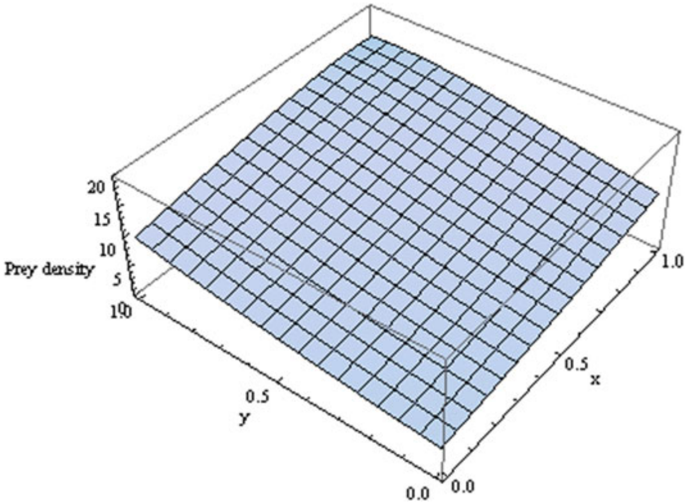


Fig. 7.22 Surface shows the numerical approximate solution of $u(x, y, t)$ when $\alpha = 0.88$, $\beta = 0.54$, $a = 0.5$, $b = 0.03$, $c = 0.3$, and $t = 0.53$

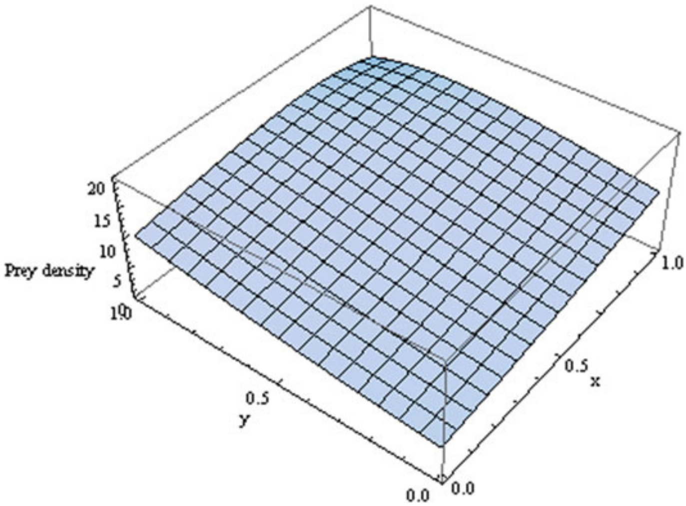


Fig. 7.23 Surface shows the numerical approximate solution of $u(x, y, t)$ when $\alpha = 0.88$, $\beta = 0.54$, $a = 0.7$, $b = 0.04$, $c = 0.3$, and $t = 0.53$

and

$$v(x,y,t) = e^{x+y} + \frac{(2e^{x+y} - ce^{x+y} + be^{x+y}\sqrt{xy})t^{\beta}}{\Gamma(1+\beta)} + \frac{(-be^{x+y}(y^2 + x^2(1 - 4ay^2 + 4be^{x+y}y^2))t^{\alpha+\beta}}{4(xy)^{3/2}\Gamma(1+\alpha+\beta)} + \ldots, \tag{7.142}$$

7.5.4 Solutions for Time Fractional Coupled Schrödinger–KdV Equation

In the present analysis, fractional coupled Schrödinger–KdV equations with appropriate initial conditions have been solved by using the novel method, viz. CFRDTM.

Example 7.9 Consider the following time fractional coupled Schrödinger–KdV equation

$$iD_t^{\alpha}u_t = u_{xx} + uv, \tag{7.143a}$$

$$D_t^{\beta}v_t = -6vv_x - v_{xxx} + (|u|^2)_x, \tag{7.143b}$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

$$u(x,0) = 6\sqrt{2}e^{ipx}k^2\text{sech}^2(kx), \tag{7.143c}$$

$$v(x,0) = \frac{p+16k^2}{3} - 6k^2\tanh^2(kx). \tag{7.143d}$$

The exact solutions of Eqs. (7.143a) and (7.143b), for the special case where $\alpha = \beta = 1$, are given by [55]

$$u(x,t) = 6\sqrt{2}e^{i\theta x}k^2\text{sech}^2(k\xi), \tag{7.144a}$$

$$v(x,t) = \frac{p+16k^2}{3} - 6k^2\tanh^2(k\xi), \tag{7.144b}$$

where

$$\theta = \left(\frac{pt}{3} + p^2t - \frac{10k^2t}{3} + px\right), \xi = x + 2pt,$$

and p, k are arbitrary constants.

In order to assess the advantages and the accuracy of the CFRDTM for solving time fractional coupled Schrödinger–KdV equation, firstly we derive the recursive formula from Eqs. (7.143a), (7.143b). Now, $U(h, k-h)$ and $V(h, k-h)$ are considered as the coupled fractional reduced differential transform of $u(x, t)$ and $v(x, t)$, respectively, where $u(x, t)$ and $v(x, t)$ are the solutions of coupled fractional differential equations. Here, $U(0, 0) = u(x, 0)$, $V(0, 0) = v(x, 0)$ are the given initial conditions. Without loss of generality, the following assumptions have taken

$$U(0,j) = 0, \;\; j = 1, 2, 3, \ldots \text{ and } V(i, 0) = 0, \;\; i = 1, 2, 3, \ldots.$$

Applying CFRDTM to Eq. (7.143a), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1, k-h) = -i\frac{\partial^2}{\partial x^2} U(h, k-h) - i\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l, s)V(l, k-h-s). \tag{7.145}$$

From the initial condition of Eq. (7.143c), we have

$$U(0, 0) = u(x, 0). \tag{7.146}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.143b)

$$\frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h, k-h+1) = \frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(l, k-h-s)\overline{U}(h-l, s)\right) - 6\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} V(l, k-h-s)\frac{\partial}{\partial x} V(h-l, s)\right) - \frac{\partial^3}{\partial x^3} V(h, k-h) \tag{7.147}$$

From the initial condition of Eq. (7.143d), we have

$$V(0, 0) = v(x, 0). \tag{7.148}$$

According to CFRDTM, using recursive equation (7.149) with initial condition Eq. (7.146) and also using recursive scheme Eq. (7.147) with initial condition Eq. (7.148) simultaneously, we obtain

$$U[1,0] = \frac{2\sqrt{2}k^2 \mathrm{sech}^2(kx)(-i\cos(px) + \sin(px))(p - 3p^2 + 10k^2 - 12ipk\tanh(kx))}{\Gamma(1+\alpha)},$$

$$V[0,1] = \frac{24pk^3 \mathrm{sech}^2(kx)\tanh(kx)}{\Gamma(1+\beta)},$$

$$U[1,1] = \frac{72\sqrt{2}pk^5 \mathrm{sech}^6(kx)(-i\cos(px) + \sin(px))\sinh(2kx)}{\Gamma(1+\alpha+\beta)},$$

$$V[0,2] = \frac{12pk^4(-3(p+48k^2) - 2(p-48k^2)\cosh(2kx) + p\cosh(4kx))\mathrm{sech}^6(kx)}{\Gamma(1+2\beta)},$$

$$V[1,1] = \frac{576pk^6(-3+2\cosh(2kx))\mathrm{sech}^6(kx)}{\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned} u(x,t) &= \sum_{k'=0}^{\infty}\sum_{h=0}^{k'} U(h,k'-h)t^{(h\alpha+(k'-h)\beta)} \\ &= U(0,0) + \sum_{k'=1}^{\infty}\sum_{h=1}^{k'} U(h,k'-h)t^{(h\alpha+(k'-h)\beta)} + \cdots \\ &= 6\sqrt{2}k^2 \mathrm{sech}^2(kx)\mathrm{e}^{ipx} \\ &\quad + \frac{72\sqrt{2}pk^5 t^{\alpha+\beta} \mathrm{sech}^6(kx)(-i\cos(px) + \sin(px))\sinh(2kx)}{\Gamma(1+\alpha+\beta)} + \cdots \end{aligned} \tag{7.149}$$

$$\begin{aligned} v(x,t) &= \sum_{k'=0}^{\infty}\sum_{h=0}^{k'} V(h,k'-h)t^{(h\alpha+(k'-h)\beta)} \\ &= V(0,0) + \sum_{k'=1}^{\infty}\sum_{h=0}^{k'} V(h,k'-h)t^{(h\alpha+(k'-h)\beta)} + \ldots \\ &= \frac{p+16k^2}{3} - 6k^2\tanh^2(kx) + \frac{24pk^3 t^{\beta} \mathrm{sech}^2(kx)\tanh(kx)}{\Gamma(1+\beta)} \\ &\quad + \frac{576pk^6 t^{\alpha+\beta}(-3+2\cosh(2kx))\mathrm{sech}^6(kx)}{\Gamma(1+\alpha+\beta)} + \ldots \end{aligned} \tag{7.150}$$

When $\alpha = 1$ and $\beta = 1$, the solutions in Eqs. (7.149) and (7.150) are exactly same as the Taylor series expansions of the exact solutions

$$u(x,t) = 6\sqrt{2}e^{i\theta x}k^2\mathrm{sech}^2(k\xi), \tag{7.151}$$

$$v(x,t) = \frac{p+16k^2}{3} - 6k^2\tanh^2(k\xi). \tag{7.152}$$

In the present numerical experiment, Eqs. (7.149) and (7.150) have been used to draw the graphs as shown in Figs. 7.24, 7.25, 7.26, and 7.27, respectively. The numerical solutions of the coupled Sch–KdV equation (7.143) have been shown in Figs. 7.24, 7.25, 7.26, and 7.27 with the help of third-order approximations for the series solutions of $u(x,t)$ and $v(x,t)$, respectively. In the present numerical computation, we have assumed $p = 0.05$ and $k = 0.05$. Figure 7.28 confirms that exact solution and approximate solutions coincide reasonably well with each other and consequently there is a good agreement of results between these two solutions when $\alpha = 1$ and $\beta = 1$. Figures 7.24, 7.25, 7.26, 7.27, and 7.28 show one-soliton solutions for coupled Sch–KdV equation (7.143). Table 7.3 explores the comparison between CFRDTM and Adomian decomposition method (ADM) results for $\mathrm{Re}(u(x,t))$ and $v(x,t)$ when $\alpha = 1$ and $\beta = 1$. It manifests that CFRDTM solutions are in good agreement with ADM solutions cited in [49].

Figures 7.29, 7.30, and 7.31 exhibit the numerical solutions of the coupled Sch–KdV equations (7.143) when $\alpha = 0.25$ and $\beta = 0.75$.

Example 7.10 Consider the time fractional coupled Schrödinger–KdV equations (7.143a)–(7.143b) with the following initial conditions

$$u(x,0) = \tanh(x)e^{ix}, \tag{7.153a}$$

$$v(x,0) = \frac{11}{12} - 2\tanh^2(x). \tag{7.153b}$$

The exact solutions of Eqs. (7.143a) and (7.143b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \tanh(x+2t)e^{i\left(x+\frac{25}{12}t\right)}, \tag{7.154a}$$

$$v(x,t) = \frac{11}{12} - 2\tanh^2(x+2t) \tag{7.154b}$$

Proceeding in a similar manner, using Eqs. (7.149) and (7.147), we can obtain

$$U[1,0] = \frac{(\cos(x) + i\sin(x))(24\mathrm{sech}^2(x) + 25i\tanh(x))}{12\Gamma(1+\alpha)},$$

$$V[0,1] = -\frac{8\mathrm{sech}^2(x)\tanh(x)}{\Gamma(1+\beta)},$$

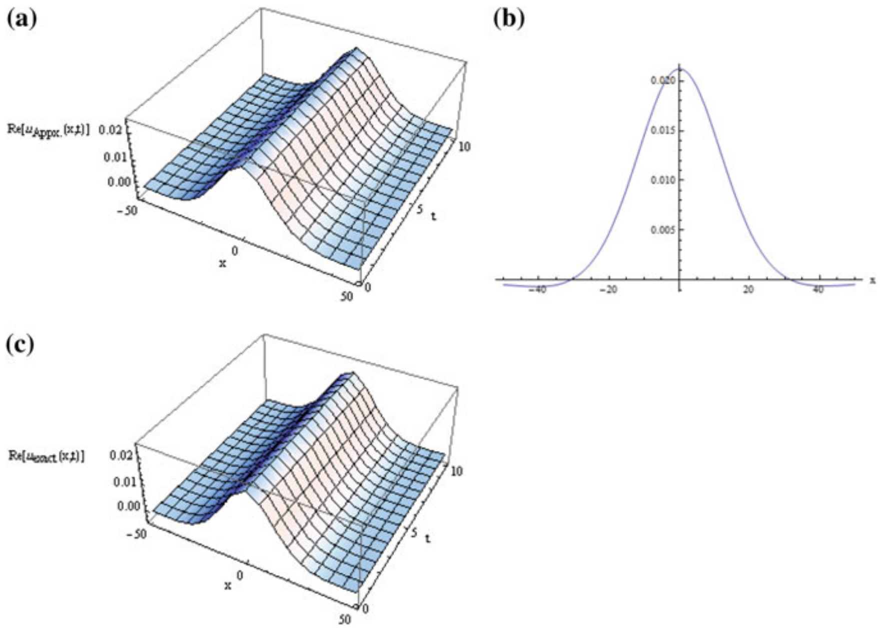


Fig. 7.24 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 1$, and **c** the exact solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

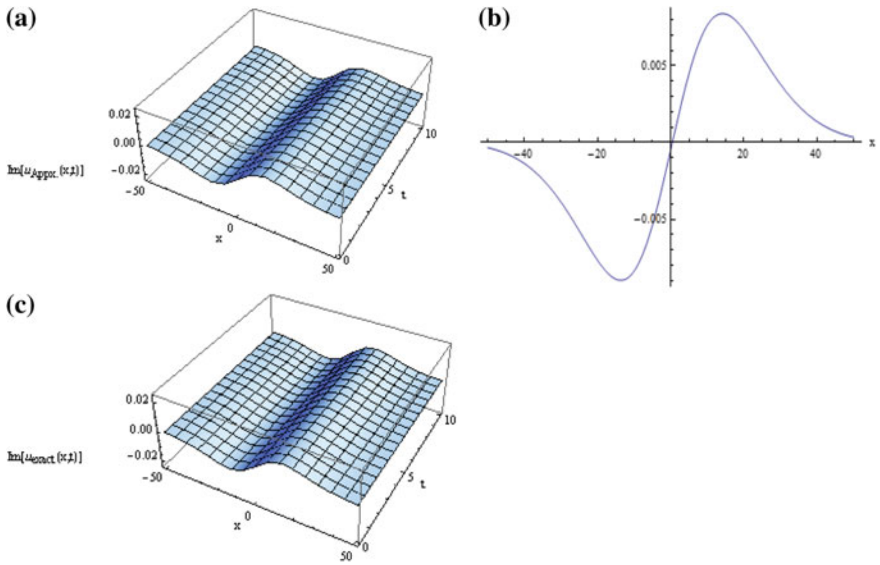


Fig. 7.25 **a** Approximate solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Im}(u(x,t))$ when $t = 1$, and **c** the exact solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

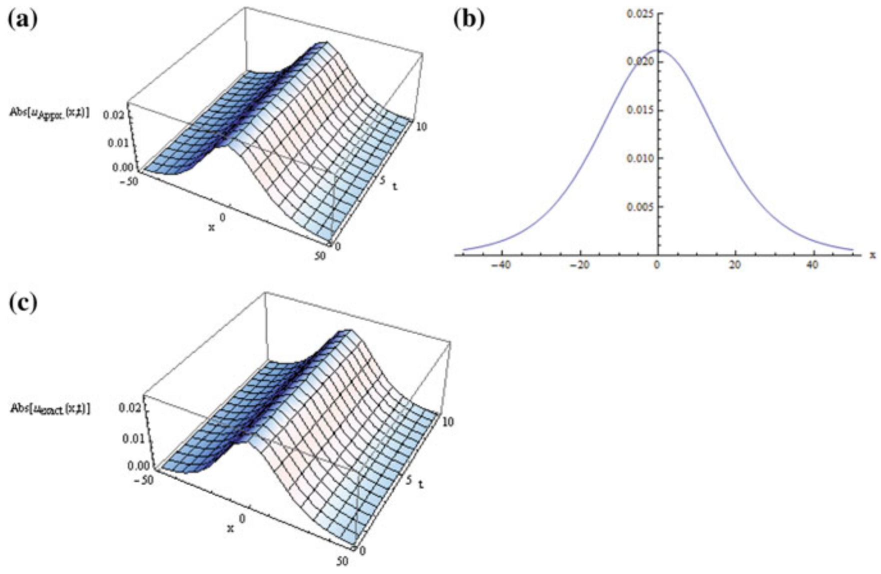


Fig. 7.26 **a** Approximate solution for $\mathrm{Abs}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Abs}(u(x,t))$ when $t = 1$, and **c** the exact solution for $\mathrm{Abs}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

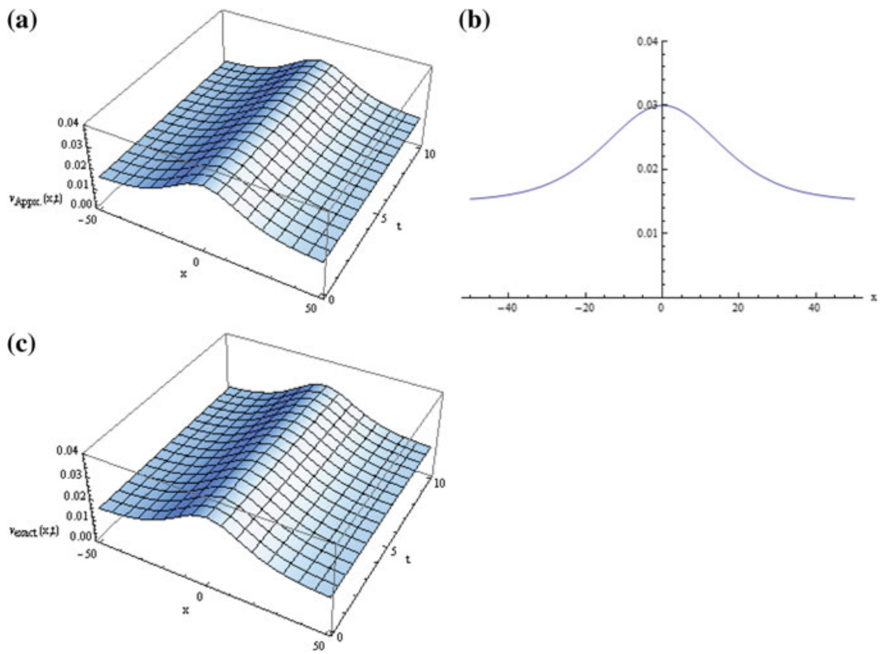


Fig. 7.27 **a** Approximate solution for $v(x,t)$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $v(x,t)$ when $t = 1$, and **c** the exact solution for $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

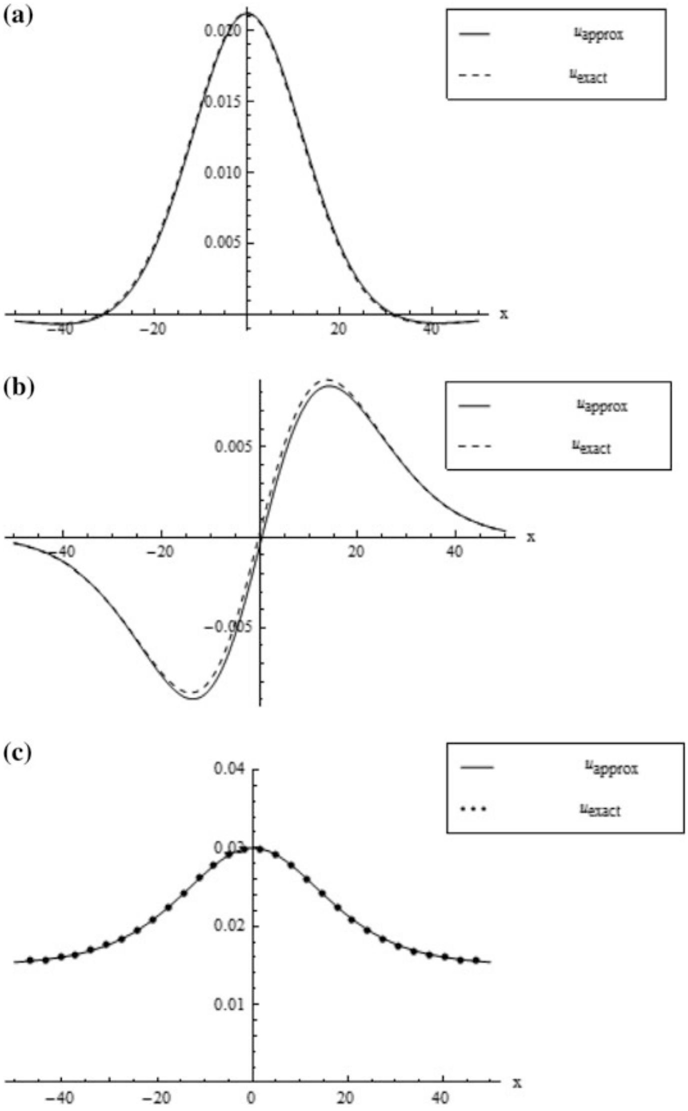


Fig. 7.28 **a** Exact and approximate solutions for $\mathrm{Re}(u(x,t))$, **b** the exact and approximate solutions for $\mathrm{Im}(u(x,t))$, and **c** the exact and approximate solutions for $v(x,t)$ when $t = 1$

Table 7.3 Comparison between CFRDTM and ADM results for $\mathrm{Re}(u(x,t))$ and $v(x,t)$ when $p = 0.05$, $k = 0.05$ for the approximate solution of Eq. (7.143)

(x,t)	$\lvert\mathrm{Re}(u_{\mathrm{Exact}}) - \mathrm{Re}(u_{\mathrm{CRFDTM}})\rvert$	$\lvert\mathrm{Re}(u_{\mathrm{Exact}}) - \mathrm{Re}(u_{\mathrm{ADM}})\rvert$	$\lvert v_{\mathrm{Exact}} - v_{\mathrm{CRFDTM}}\rvert$	$\lvert v_{\mathrm{Exact}} - v_{\mathrm{ADM}}\rvert$
(0.1, 0.1)	3.12296E−7	2.38712E−7	1.45496E−7	7.87471E−8
(0.1, 0.2)	5.42096E−7	5.12923E−7	2.81993E−7	1.64993E−7
(0.1, 0.3)	6.89399E−7	8.22631E−7	4.09489E−7	2.58738 E−7
(0.1, 0.4)	7.54204E−7	1.16783E−6	5.27985E−7	3.59982E−7
(0.1, 0.5)	7.36511E−7	1.54853E−6	6.3748E−7	4.68725E−7
(0.2, 0.1)	6.65778E−7	4.59621E−7	2.95463E−7	1.53728E−7
(0.2, 0.2)	1.24906E−6	9.54729E−7	5.81931E−7	3.14954E−7
(0.2, 0.3)	1.74985E−6	1.48532E−6	8.59402E−7	4.83675E−7
(0.2, 0.4)	2.16815E−6	2.0514E−6	1.12788E−6	6.59893E−7
(0.2, 0.5)	2.50394E−6	2.65296E−6	1.38735E−6	8.43606E−7
(0.3, 0.1)	1.01914E−6	6.80427E−7	4.45372E−7	2.28679E−7
(0.3, 0.2)	1.95579E−6	1.39632E−6	8.81756E−7	4.64851E−7
(0.3, 0.3)	2.80995E−6	2.14769E−6	1.30915E−6	7.08515E−7
(0.3, 0.4)	3.58161E−6	2.93452E−6	1.72755E−6	9.59671E−7
(0.3, 0.5)	4.27078E−6	3.75681E−6	2.13696E−6	1.21832E−6
(0.4, 0.1)	1.37231E−6	9.01083E−7	5.95192E−7	3.03584E−7
(0.4, 0.2)	2.66215E−6	1.83761E−6	1.18141E−6	6.14655E−7
(0.4, 0.3)	3.8695E−6	2.80958E−6	1.75864E−6	9.33213E−7
(0.4, 0.4)	4.99436E−6	3.81699E−6	2.32689E−6	1.25926E−6
(0.4, 0.5)	6.03672E−6	4.85984E−6	2.88616E−6	1.59279E−6
(0.5, 0.1)	1.72524E−6	1.12154E−6	7.44894E−7	3.78428E−7
(0.5, 0.2)	3.36802E−6	2.27849E−6	1.48082E−6	7.64337E−7
(0.5, 0.3)	4.92831E−6	3.47086E−6	2.20779E−6	1.15773E−6
(0.5, 0.4)	6.40613E−6	4.69863E−6	2.92578E−6	1.55859E−6
(0.5, 0.5)	7.80146E−6	5.96181E−6	3.6348E−6	1.96694E−6

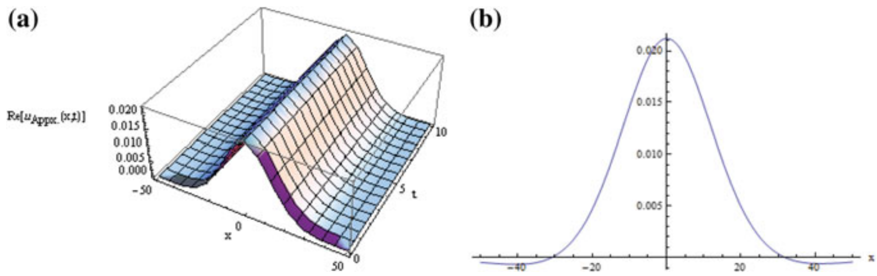


Fig. 7.29 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 0.25$ and $\beta = 0.75$, and **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 1$

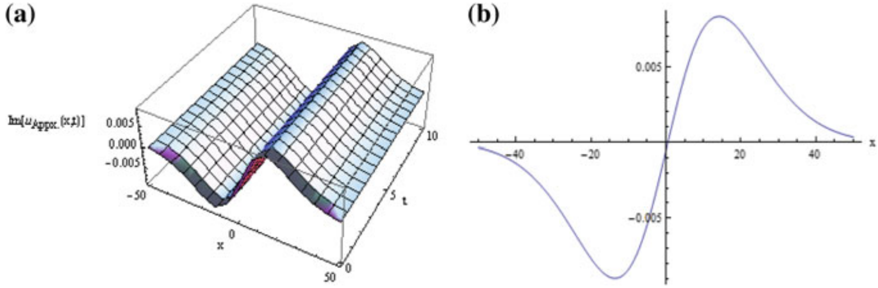


Fig. 7.30 **a** Approximate solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 0.25$ and $\beta = 0.75$, and **b** corresponding solution for $\mathrm{Im}(u(x,t))$ when $t = 1$

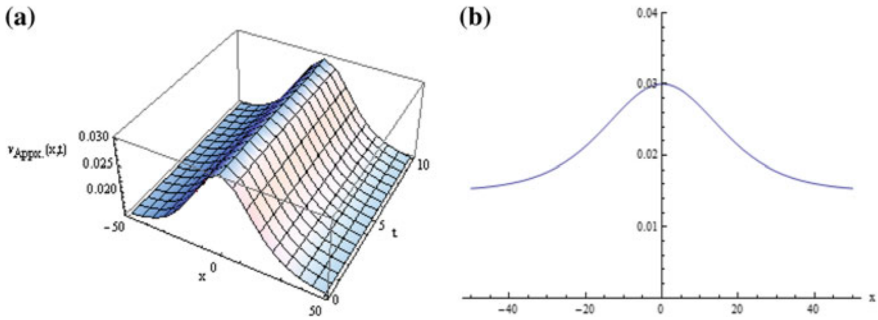


Fig. 7.31 **a** Approximate solution for $v(x,t)$ when $\alpha = 0.25$ and $\beta = 0.75$, and **b** corresponding solution for $v(x,t)$ when $t = 1$

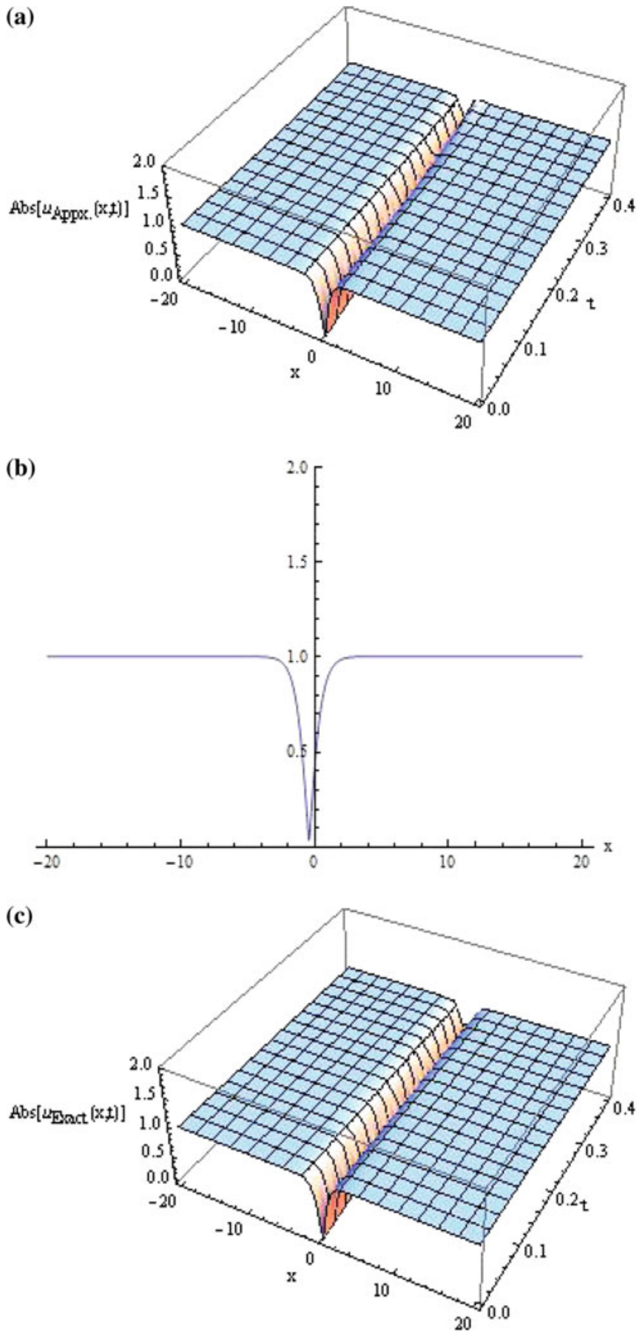


Fig. 7.32 **a** Approximate solution for $\text{Abs}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\text{Abs}(u(x,t))$ when $t = 0.2$, and **c** the exact solution for $Abs(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

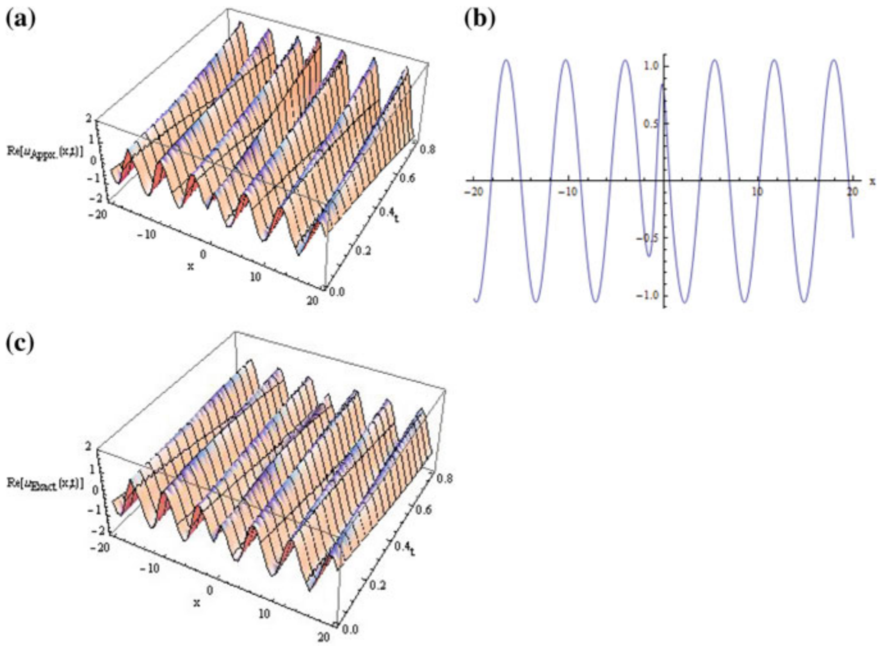


Fig. 7.33 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 0.4$, and **c** the exact solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

$$U[1,1] = \frac{8i\,\mathrm{sech}^2(x)(\cos(x) + i\sin(x))\tanh^2(x)}{\Gamma(1+\alpha+\beta)},$$

$$V[0,2] = \frac{20(-2 + \cosh(2x))\mathrm{sech}^4(x)}{\Gamma(1+2\beta)},$$

$$U[2,0] = \frac{i\,\mathrm{sech}^4(x)\mathrm{e}^{ix}(9408 + 192\cosh(2x) + 5858i\sinh(2x) + 625i\sinh(4x))}{1152\Gamma(1+2\alpha)},$$

$$V[1,1] = -\frac{4(-2 + \cosh(2x))\mathrm{sech}^4(x)}{\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions can be obtained by Eq. (7.29).

Figure 7.35 confirms that exact solution and approximate solutions coincide reasonably well with each other and consequently there is a good agreement of results between these two solutions when $\alpha = 1$ and $\beta = 1$.

Figures 7.32, 7.33, 7.34, 7.35, 7.37, 7.38, 7.39, and 7.40 exhibit the numerical solutions of the coupled Sch–KdV equations (7.143a)–(7.143b) with initial conditions (7.153a)–(7.153b) when $\alpha = 1$, $\beta = 1$ and $\alpha = 0.5$, $\beta = 0.5$, respectively (Fig. 7.36).

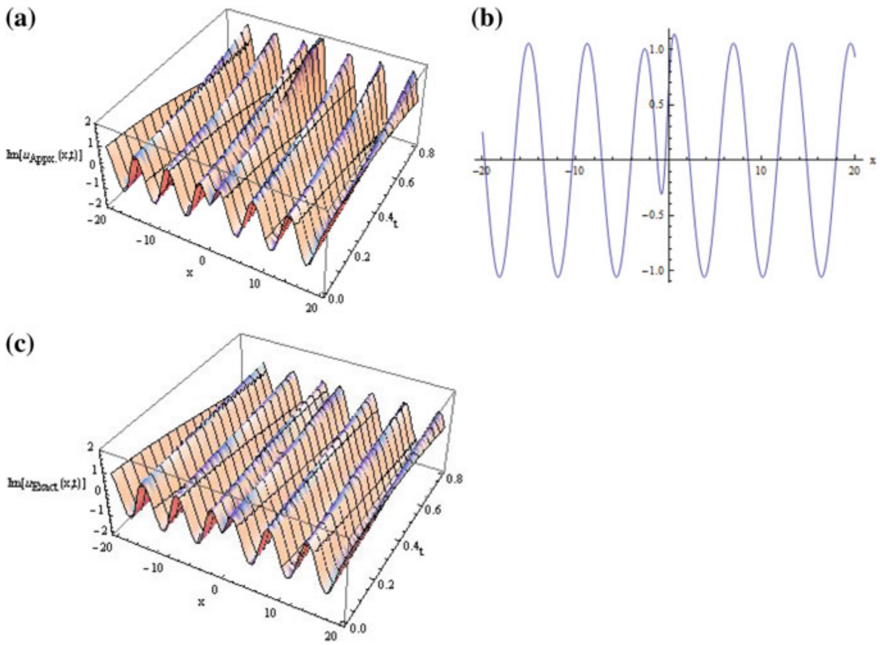


Fig. 7.34 **a** Approximate solution for $\text{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\text{Im}(u(x,t))$ when $t = 0.4$, and **c** the exact solution for $\text{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

Example 7.11 Consider the time fractional coupled Schrödinger–KdV equations (7.143a)–(7.143b) with the following initial conditions

$$u(x,0) = \cos(x) + i\sin(x), \tag{7.155a}$$

$$v(x,0) = \frac{3}{4}. \tag{7.155b}$$

The exact solutions of Eqs. (7.143a) and (7.143b) with initial conditions (7.155), for the special case when $\alpha = \beta = 1$, are given by

$$u(x,t) = \cos\left(x + \frac{t}{4}\right) + i\sin\left(x + \frac{t}{4}\right), \tag{7.156a}$$

$$v(x,t) = \frac{3}{4}. \tag{7.156b}$$

The Jacobi periodic solutions [56] to coupled Sch–KdV equations (7.143a) and (7.143b) are given by

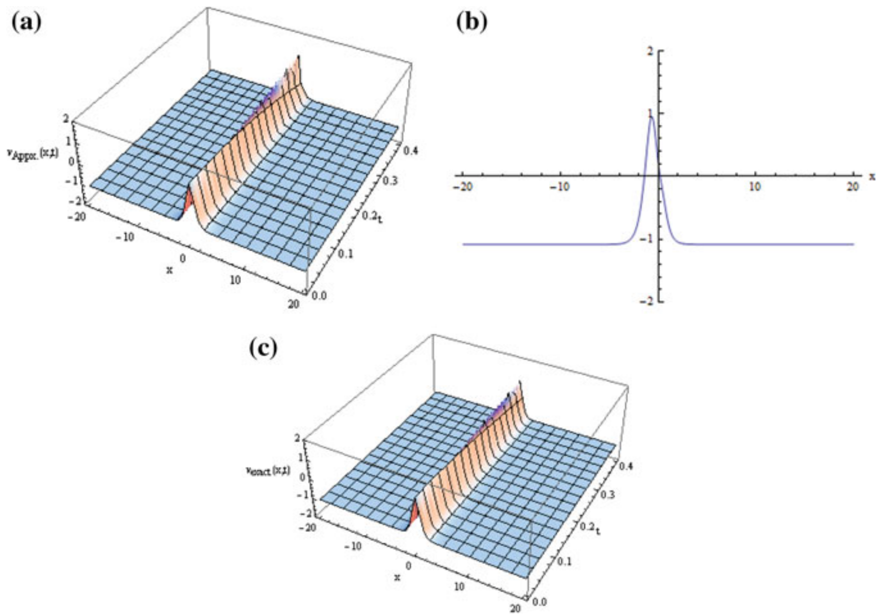


Fig. 7.35 **a** Approximate solution for $v(x,t)$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $v(x,t)$ when $t = 0.3$, and **c** the exact solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

$$u(x,t) = \sqrt{\frac{2}{2-m^2}}\mathrm{e}^{i\theta} dn\left(\frac{1}{\sqrt{2-m^2}}\xi\right), \tag{7.157a}$$

$$v(x,t) = \frac{7}{4} - \frac{2}{2-m^2} dn^2\left(\frac{1}{\sqrt{2-m^2}}\xi\right). \tag{7.157b}$$

where $\theta = \left(x + \frac{t}{4}\right)$ and $\xi = x + 2t$.

For $m = 0$, Eq. (7.157a–b) reduces to Eq. (7.156a–b).

Proceeding in a similar manner, using Eqs. (7.149) and (7.147), we can obtain

$$U[1,0] = \frac{i(\cos(x) + i\sin(x))}{4\Gamma(1+\alpha)},$$

$$V[0,1] = 0,$$

$$U[1,1] = 0,$$

$$V[0,2] = 0,$$

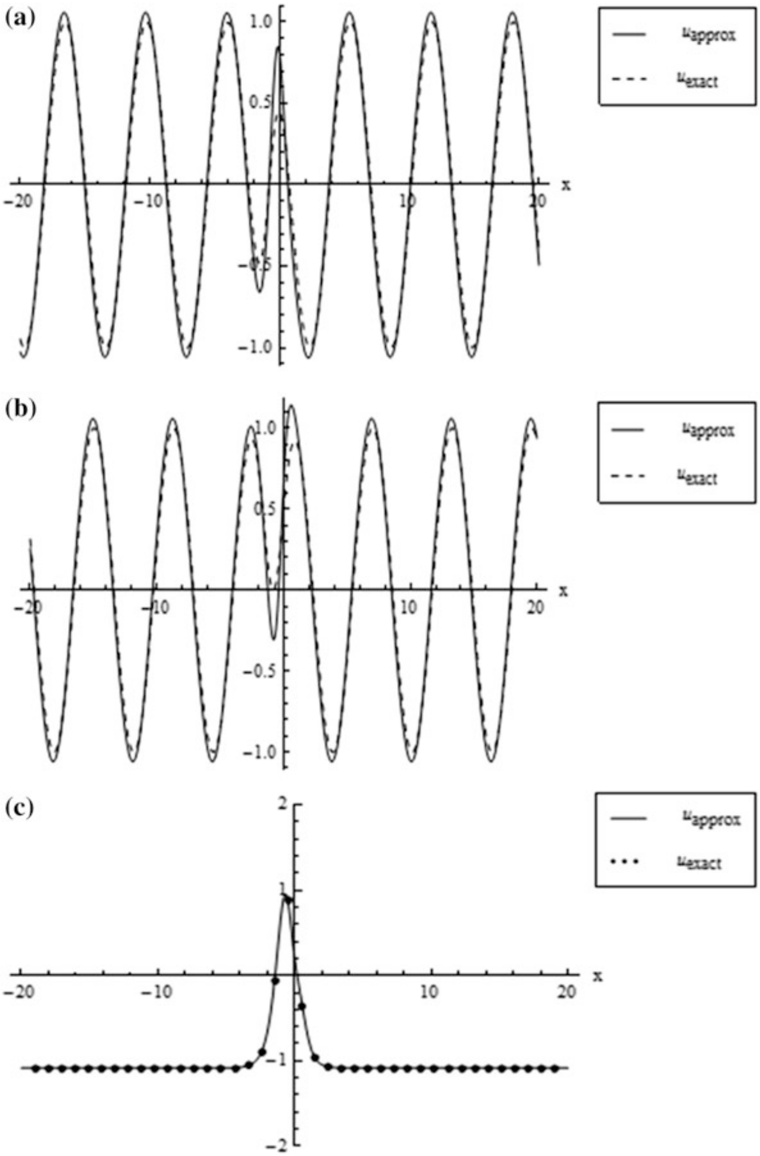


Fig. 7.36 **a** Exact and approximate solutions for $\mathrm{Re}(u(x,t))$ when $t = 0.4$, **b** the exact and approximate solutions for $\mathrm{Im}(u(x,t))$ when $t = 0.4$, and **c** the exact and approximate solutions for $v(x,t)$ when $t = 0.3$

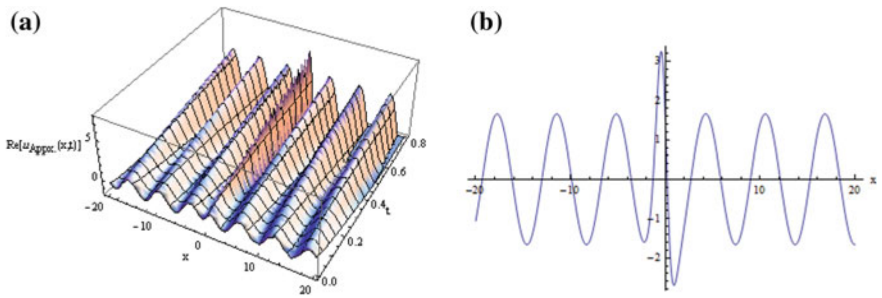


Fig. 7.37 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 0.4$

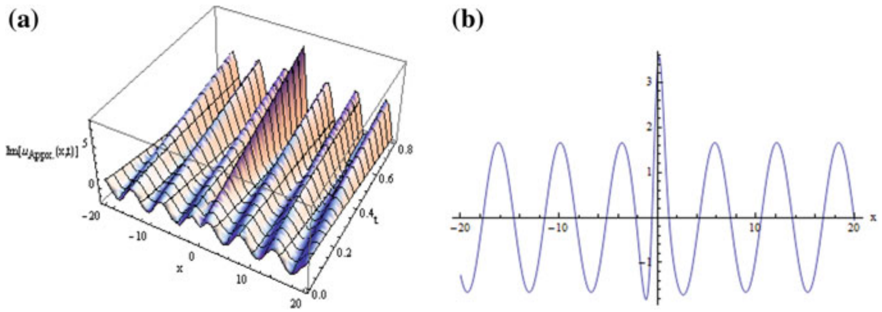


Fig. 7.38 **a** Approximate solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $\mathrm{Im}(u(x,t))$ when $t = 0.4$

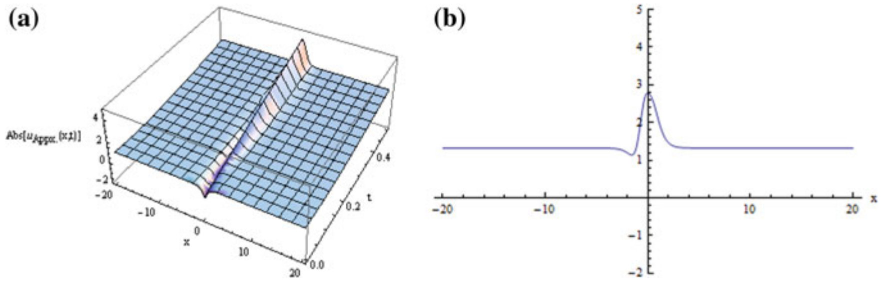


Fig. 7.39 **a** Approximate solution for $\mathrm{Abs}(u(x,t))$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $\mathrm{Abs}(u(x,t))$ when $t = 0.3$

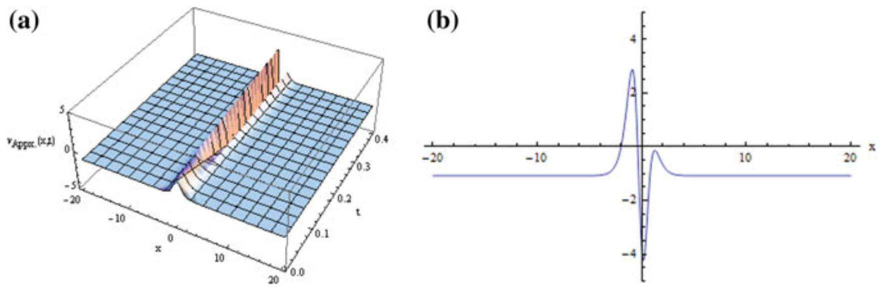


Fig. 7.40 **a** Approximate solution for $v(x,t)$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $v(x,t)$ when $t = 0.3$

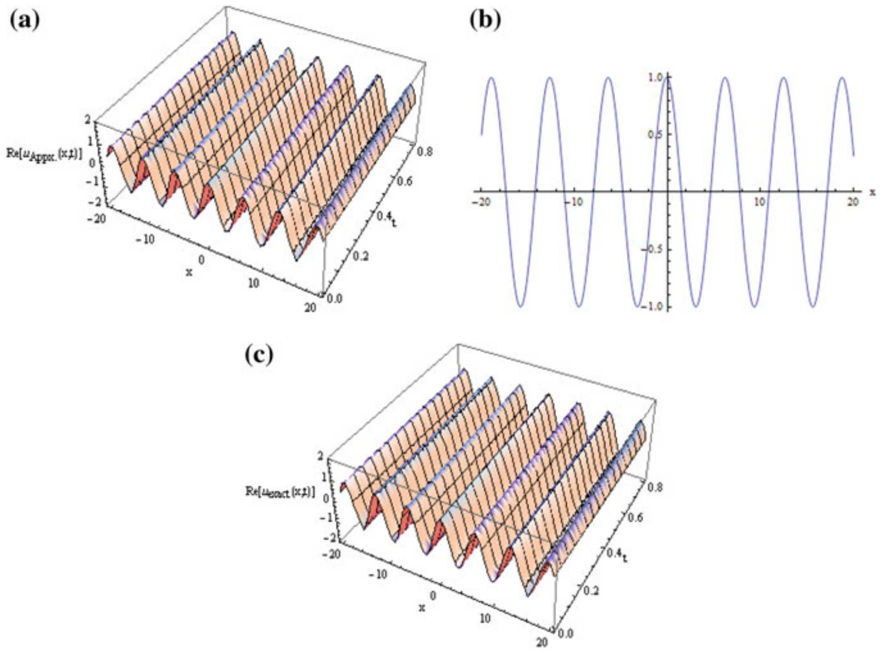


Fig. 7.41 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 0.4$, and **c** the exact solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

$$U[2,0] = -\frac{e^{ix}}{16\Gamma(1+2\alpha)},$$

$$V[1,1] = 0,$$

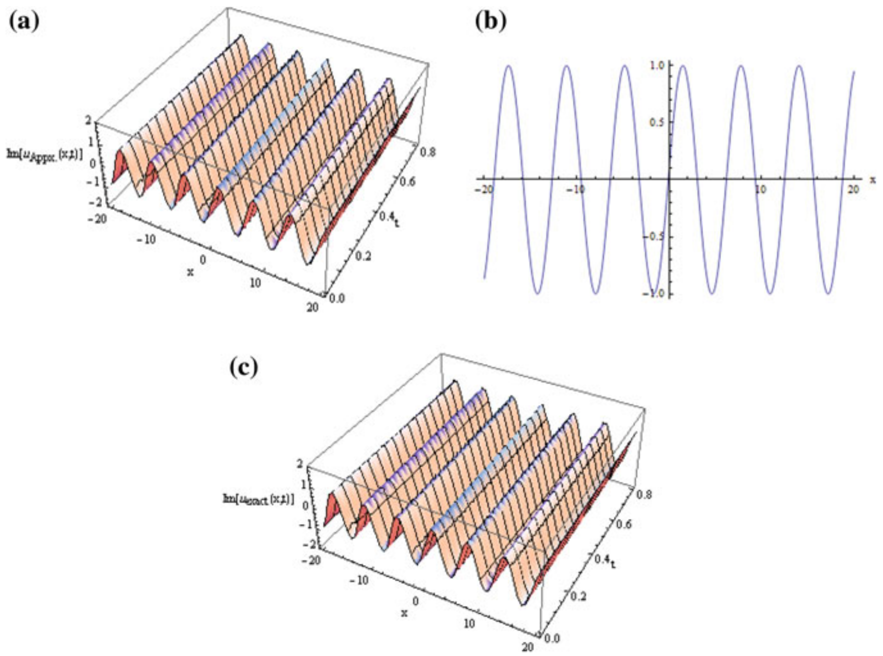


Fig. 7.42 **a** Approximate solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, **b** corresponding solution for $\mathrm{Im}(u(x,t))$ when $t = 0.4$, and **c** the exact solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$

$$U[3,0] = \frac{-i\cos(x) + \sin(x)}{64\Gamma(1+3\alpha)},$$

and so on.

The approximate solutions can be obtained by Eq. (7.29).

Figures 7.41 and 7.42 show the exact and approximate solutions for $\mathrm{Re}(u(x,t))$ and $\mathrm{Im}(u(x,t))$ when $\alpha = 1$ and $\beta = 1$, respectively. Since the obtained approximate solution $v(x,t)$ is exact, it is not drawn.

Figure 7.43 confirms that exact solution and approximate solutions coincide reasonably well with each other and consequently there is a good agreement of results between these two solutions when $\alpha = 1$ and $\beta = 1$.

Figures 7.44 and 7.45 exhibit the numerical solutions of the coupled Sch–KdV equations (7.143a)–(7.143b) with initial conditions (7.155) when $\alpha = 0.5$ and $\beta = 0.5$.

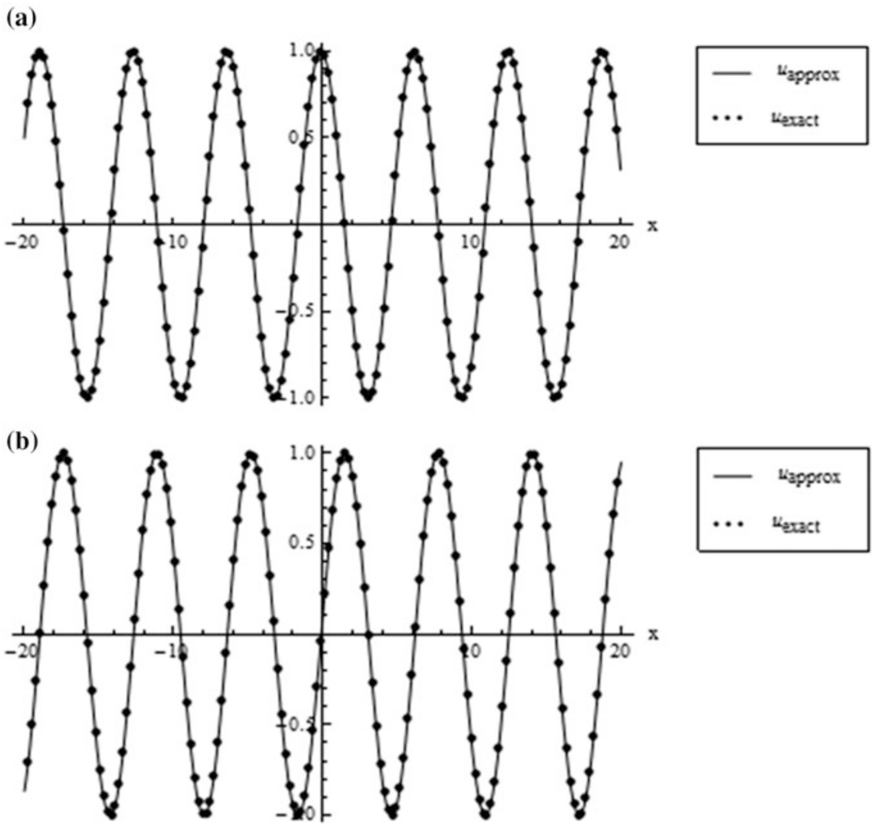


Fig. 7.43 **a** Exact and approximate solutions for $\mathrm{Re}(u(x,t))$ when $t = 0.4$ and **b** the exact and approximate solutions for $\mathrm{Im}(u(x,t))$ when $t = 0.4$

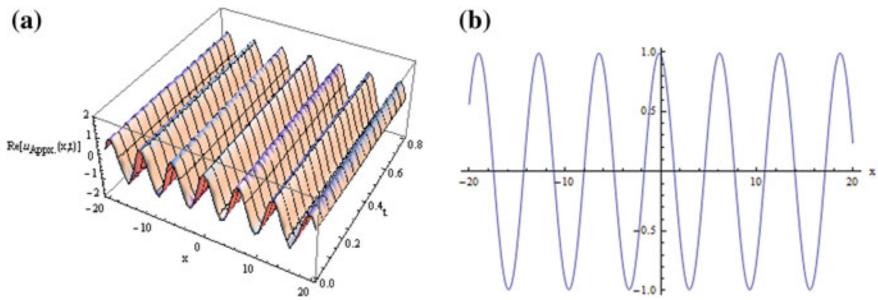


Fig. 7.44 **a** Approximate solution for $\mathrm{Re}(u(x,t))$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $\mathrm{Re}(u(x,t))$ when $t = 0.4$

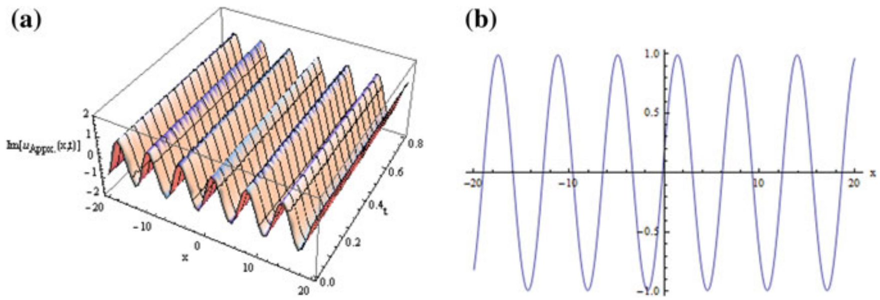


Fig. 7.45 **a** Approximate solution for $\mathrm{Im}(u(x,t))$ when $\alpha = 0.5$ and $\beta = 0.5$, and **b** corresponding solution for $\mathrm{Im}(u(x,t))$ when $t = 0.4$

7.5.5 Traveling Wave Solutions for the Variant of Time Fractional Coupled WBK Equations

In this section, the new proposed CFRDTM [34, 35] is very successfully employed for obtaining approximate traveling wave solutions of fractional coupled Whitham–Broer–Kaup (WBK) equations, fractional coupled modified Boussinesq equations, and fractional approximate long wave equations. By using this proposed method, the solutions were calculated in the form of a generalized Taylor's series with easily computable components. The obtained results justify that the proposed method is also very efficient, effective, and simple for obtaining approximate solutions of fractional coupled evolution equations.

Example 7.12 Consider the following time fractional coupled WBK equations [57–59]

$$D_t^{\alpha} u = -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - b\frac{\partial^2 u}{\partial x^2}, \tag{7.158a}$$

$$D_t^{\beta} v = -\frac{\partial (uv)}{\partial x} - a\frac{\partial^3 u}{\partial x^3} + b\frac{\partial^2 v}{\partial x^2}, \tag{7.158b}$$

where $t > 0$, $0 < \alpha, \beta \leq 1$, subject to the initial conditions

$$u(x,0) = \lambda - 2Bk\coth(k\xi), \tag{7.158c}$$

$$v(x,0) = -2B(B+b)k^2\mathrm{csch}^2(k\xi), \tag{7.158d}$$

where $B = \sqrt{a+b^2}$, $\xi = x + c$, and c, k, λ are arbitrary constants.

The exact solutions [57, 60] of Eqs. (7.158a) and (7.158b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \lambda - 2Bk\coth(k(\xi - \lambda t)), \tag{7.159a}$$

$$v(x,t) = -2B(B+b)k^2\text{csch}^2(k(\xi - \lambda t)), \tag{7.159b}$$

In order to assess the advantages and the accuracy of the proposed method, CFRDTM has been applied for solving time fractional coupled WBK equations. First, we derive the recursive formula from Eqs. (7.158a) and (7.158b), respectively. Now, $U(h,k-h)$ and $V(h,k-h)$ are considered as the coupled fractional reduced differential transform of $u(x,t)$ and $v(x,t)$, respectively, where $u(x,t)$ and $v(x,t)$ are the solutions of coupled fractional differential equations. Here, $U(0,0) = u(x,0)$, $V(0,0) = v(x,0)$ are the given initial conditions.

Without loss of generality, the following assumptions have been taken

$$U(0,j) = 0, \;\; j = 1,2,3,\ldots \text{ and } V(i,0) = 0, \;\; i = 1,2,3,\ldots.$$

Applying CFRDTM to Eq. (7.158a), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha + (k-h)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} U(h+1,k-h) = -\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(h-l,s)\frac{\partial}{\partial x}U(l,k-h-s)\right) - \frac{\partial}{\partial x}V(h,k-h) - b\frac{\partial^2}{\partial x^2}U(h,k-h). \tag{7.160}$$

From the initial condition of Eq. (7.158c), we have

$$U(0,0) = u(x,0). \tag{7.161}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.158b)

$$\frac{\Gamma(h\alpha + (k-h+1)\beta + 1)}{\Gamma(h\alpha + (k-h)\beta + 1)} V(h,k-h+1) = -\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h} U(l,k-h-s)V(h-l,s)\right) - a\frac{\partial^3}{\partial x^3}U(h,k-h) + b\frac{\partial^2}{\partial x^2}V(h,k-h). \tag{7.162}$$

From the initial condition of Eq. (7.158d), we have

$$V(0,0) = v(x,0). \tag{7.163}$$

According to CFRDTM, using recursive Eq. (7.160) with initial condition Eq. (7.161) and also using recursive scheme Eq. (7.162) with initial condition Eq. (7.163) simultaneously, we obtain

$$U(1,0) = -\frac{2Bk^2\lambda \operatorname{csch}^2(k\xi)}{\Gamma(1+\alpha)},$$

$$V(0,1) = -\frac{4(a+b(b+B))k^3\lambda \coth(k\xi)\operatorname{csch}^2(k\xi)}{\Gamma(1+\beta)},$$

$$U(1,1) = -\frac{4(a+b(b+B))k^4\lambda(2+\cosh(2k\xi))\operatorname{csch}^4(k\xi)}{\Gamma(1+\alpha+\beta)},$$

$$V(1,1) = -\frac{8k^5\lambda(-2b^2(b+B)+a(-2b+3B)+aB\cosh(2k\xi))\coth(k\xi)\operatorname{csch}^4(k\xi)}{\Gamma(1+\alpha+\beta)},$$

and so on.

The approximate solutions, obtained in the series form, are given by

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k} U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= \lambda - 2Bk\coth(k\xi) - \frac{2Bk^2\lambda\operatorname{csch}^2(k\xi)t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad - \frac{4(a+b(b+B))k^4\lambda(2+\cosh(2k\xi))\operatorname{csch}^4(k\xi)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots \end{aligned} \tag{7.164}$$

$$\begin{aligned} v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= -2B(b+B)k^2\operatorname{csch}^2(k\xi) \\ &\quad - \frac{4(a+b(b+B))k^3\lambda\coth(k\xi)\operatorname{csch}^2(k\xi)t^{\beta}}{\Gamma(1+\beta)} - \cdots. \end{aligned} \tag{7.165}$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.164) becomes

$$u(x,t) = \lambda - 2Bk\coth(k\xi) - 2Bk^2\lambda\text{csch}^2(k\xi)t \\ - 2Bk^3\lambda^2\coth(k\xi)\text{csch}^2(k\xi)t^2 + \cdots. \tag{7.166}$$

When $\alpha = 1$ and $\beta = 1$, the solution in Eq. (7.165) becomes

$$\begin{aligned} v(x,t) &= -2B(B+b)k^2\text{csch}^2(k(\xi - \lambda t)) \\ &= -2(B(b+B)k^2\text{csch}^2(k\xi)) - 4(B(b+B)k^3\lambda\coth(k\xi)\text{csch}^2(k\xi))t \\ &\quad - 2(B(b+B)k^4\lambda^2(2+\cosh(2k\xi))\text{csch}^4(k\xi))t^2 - \cdots. \end{aligned} \tag{7.167}$$

The solutions in Eqs. (7.166) and (7.167) are exactly the same as the Taylor series expansions of the exact solutions

$$\begin{aligned} u(x,t) &= \lambda - 2Bk\coth(k(\xi - \lambda t)) \\ &= \lambda - 2Bk\coth(k\xi) - 2Bk^2\lambda\text{csch}^2(k\xi)t \\ &\quad - 2Bk^3\lambda^2\coth(k\xi)\text{csch}^2(k\xi)t^2 + \cdots \end{aligned} \tag{7.168}$$

$$\begin{aligned} v(x,t) &= -2B(B+b)k^2\text{csch}^2(k(\xi - \lambda t)) \\ &= -2(B(b+B)k^2\text{csch}^2(k\xi)) - 4(B(b+B)k^3\lambda\coth(k\xi)\text{csch}^2(k\xi))t \\ &\quad - 2(B(b+B)k^4\lambda^2(2+\cosh(2k\xi))\text{csch}^4(k\xi))t^2 - \cdots \end{aligned} \tag{7.169}$$

Example 7.13 Consider the following time fractional coupled modified Boussinesq (MB) equations [57, 58, 60]

$$D_t^{\alpha}u = -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}, \tag{7.170a}$$

$$D_t^{\beta}v = -\frac{\partial(uv)}{\partial x} - \frac{\partial^3 u}{\partial x^3}, \tag{7.170b}$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

$$u(x,0) = \lambda - 2k\coth(k\xi), \tag{7.170c}$$

$$v(x,0) = -2k^2\text{csch}^2(k\xi). \tag{7.170d}$$

As already mentioned earlier, if $a = 1$ and $b = 0$, the above fractional coupled modified Boussinesq equations (7.170a) and (7.170b) can be obtained as a special case of WBK equations (7.158a) and (7.158b).

The exact solutions [57, 60] of Eqs. (7.170a) and (7.170b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \lambda - 2k\coth(k(\xi - \lambda t)), \tag{7.171a}$$

$$v(x,t) = -2k^2\text{csch}^2(k(\xi - \lambda t)). \tag{7.171b}$$

Proceeding in a similar manner as in Example 7.12, after applying CFRDTM to Eq. (7.170a), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = -\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)\frac{\partial}{\partial x}U(l,k-h-s)\right) - \frac{\partial}{\partial x}V(h,k-h). \tag{7.172}$$

From the initial condition of Eq. (7.170c), we have

$$U(0,0) = u(x,0). \tag{7.173}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.170b)

$$\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = -\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(l,k-h-s)V(h-l,s)\right) - \frac{\partial^3}{\partial x^3}U(h,k-h). \tag{7.174}$$

From the initial condition of Eq. (7.170d), we have

$$V(0,0) = v(x,0). \tag{7.175}$$

According to CFRDTM, using recursive formulae (7.172) and (7.174) along with initial conditions in Eqs. (7.173) and (7.175) simultaneously, we obtain the approximate solutions in the series forms as

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= \lambda - 2k\coth(k\xi) - \frac{2k^2\lambda\text{csch}^2(k\xi)t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad - \frac{4k^4\lambda(2+\cosh(2k\xi))\text{csch}^4(k\xi)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots. \end{aligned} \tag{7.176}$$

$$\begin{aligned} v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= -2k^2\operatorname{csch}^2(k\xi) - \frac{4k^3\lambda\coth(k\xi)\operatorname{csch}^2(k\xi)t^{\beta}}{\Gamma(1+\beta)} - \cdots. \end{aligned} \tag{7.177}$$

When $\alpha = 1$ and $\beta = 1$, the solutions in Eqs. (7.176) and (7.177) are exactly the same as the Taylor series expansions of the exact solutions

$$u(x,t) = \lambda - 2k\coth(k(\xi - \lambda t)), \tag{7.178}$$

$$v(x,t) = -2k^2\operatorname{csch}^2(k(\xi - \lambda t)). \tag{7.179}$$

Example 7.14 Consider the following time fractional coupled approximate long wave (ALW) equations [57, 58, 60]

$$D_t^{\alpha}u = -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \tag{7.180a}$$

$$D_t^{\beta}v = -\frac{\partial(uv)}{\partial x} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}, \tag{7.180b}$$

where $t > 0$, $0 < \alpha, \beta \le 1$, subject to the initial conditions

$$u(x,0) = \lambda - k\coth(k\xi), \tag{7.180c}$$

$$v(x,0) = -k^2\operatorname{csch}^2(k\xi) \tag{7.180d}$$

As already mentioned earlier, if $a = 0$ and $b = 1/2$, the above fractional coupled ALW equations (7.180a) and (7.180b) can be obtained as a special case of WBK equations (7.158a) and (7.158b).

The exact solutions [57, 60] of Eqs. (7.180a) and (7.180b), for the special case where $\alpha = \beta = 1$, are given by

$$u(x,t) = \lambda - k\coth(k(\xi - \lambda t)), \tag{7.181a}$$

$$v(x,t) = -k^2\operatorname{csch}^2(k(\xi - \lambda t)). \tag{7.181b}$$

Proceeding in a similar manner as in Example 7.12, after applying CFRDTM to Eq. (7.180a), we obtain the following recursive formula

$$\frac{\Gamma((h+1)\alpha+(k-h)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}U(h+1,k-h) = -\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(h-l,s)\frac{\partial}{\partial x}U(l,k-h-s)\right) - \frac{\partial}{\partial x}V(h,k-h) - \frac{1}{2}\frac{\partial^2}{\partial x^2}U(h,k-h). \tag{7.182}$$

From the initial condition of Eq. (7.180c), we have

$$U(0,0) = u(x,0). \tag{7.183}$$

In the same manner, we can obtain the following recursive formula from Eq. (7.180b)

$$\frac{\Gamma(h\alpha+(k-h+1)\beta+1)}{\Gamma(h\alpha+(k-h)\beta+1)}V(h,k-h+1) = -\frac{\partial}{\partial x}\left(\sum_{l=0}^{h}\sum_{s=0}^{k-h}U(l,k-h-s)V(h-l,s)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}V(h,k-h). \tag{7.184}$$

From the initial condition of Eq. (7.180d), we have

$$V(0,0) = v(x,0). \tag{7.185}$$

According to CFRDTM, using recursive formulae (7.182) and (7.184) along with initial condition Eqs. (7.183) and (7.185) simultaneously, we obtain the approximate solutions in the series forms as

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= U(0,0) + \sum_{k=1}^{\infty}\sum_{h=1}^{k}U(h,k-h)t^{(h\alpha+(k-h)\beta)} \\ &= \lambda - k\coth(k\xi) - \frac{k^2\lambda\mathrm{csch}^2(k\xi)t^{\alpha}}{\Gamma(1+\alpha)} \\ &\quad - \frac{2k^4\lambda(2+\cosh(2k\xi))\mathrm{csch}^4(k\xi)t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \cdots. \end{aligned} \tag{7.186}$$

$$
\begin{aligned}
v(x,t) &= \sum_{k=0}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= V(0,0) + \sum_{k=1}^{\infty}\sum_{h=0}^{k} V(h,k-h)t^{(h\alpha+(k-h)\beta)} \\
&= -k^2\operatorname{csch}^2(k\xi) - \frac{2k^3\lambda\coth(k\xi)\operatorname{csch}^2(k\xi)t^{\beta}}{\Gamma(1+\beta)} - \cdots.
\end{aligned}
\tag{7.187}
$$

When $\alpha = 1$ and $\beta = 1$, the solutions in Eqs. (7.186) and (7.187) are exactly the same as the Taylor series expansions of the exact solutions

$$u(x,t) = \lambda - k\coth(k(\xi - \lambda t)), \tag{7.188}$$

$$v(x,t) = -k^2\operatorname{csch}^2(k(\xi - \lambda t)). \tag{7.189}$$

Tables 7.4, 7.5, and 7.6 cite the comparison between CFRDTM, Adomian decomposition method (ADM) and variational iteration method (VIM) results for $u(x,t)$ and $v(x,t)$ of WBK equation (7.158), MB equation (7.170), and ALW equation (7.180) when $\alpha = 1$ and $\beta = 1$. It reveals that very good approximations have been obtained.

The comparison results between the proposed method CFRDTM with the other methods ADM and VIM presented in Tables 7.4, 7.5, and 7.6 demonstrate that the proposed method is more accurate and better than ADM and VIM. Therefore, the pertinent feature of the proposed method is that it provides more accurate solution than the existing methods ADM and VIM. Hence, the proposed methodology leads to high accuracy. Moreover, the present approximations show excellent accuracy and sufficiently justify the superiority over other methods.

Figures 7.46, 7.47, and 7.48 explore the numerical approximate solutions obtained by the present method and exact solutions of $u(x,t)$ and $v(x,t)$ for WBK equation (7.158), MB equation (7.170), and ALW equation (7.180) when $\alpha = 1$ and $\beta = 1$.

Figures 7.49, 7.50, and 7.51 exhibit the numerical approximate solutions of $u(x,t)$ and $v(x,t)$ for WBK equation (7.158), MB equation (7.170), and ALW equation (7.180) with regard to different values of α and β.

The comparison of approximate solutions $u(x,t)$ and $v(x,t)$ with regard to exact solutions for WBK equation (7.158), MB equation (7.170), and ALW equation (7.180) has been shown in Figs. 7.52, 7.53, and 7.54 at time instance $t = 5$ for $\alpha = 1$ and $\beta = 1$.

7.5.6 *Convergence and Error Analysis of CFRDTM*

In the present section, the error analysis of CFRDTM has been carried out through the following theorem.

Table 7.4 Comparison between CFRDTM, ADM, and VIM results for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 1.5$, $b = 1.5$, and $c = 10$, for the approximate solutions of WKB equations (7.158a) and (7.158b)

(x, t)	$\lvert u_{\text{Exact}} - u_{\text{ADM}} \rvert$	$\lvert u_{\text{Exact}} - u_{\text{VIM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{ADM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{VIM}} \rvert$	$\lvert u_{\text{Exact}} - u_{\text{CRFDTM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{CRFDTM}} \rvert (0.1,0.1)$
(0.1, 0.1)	1.04892E−04	1.23033E−04	6.41419E−03	1.10430E−04	1.11022E−16	2.77556E−17
(0.1, 0.3)	9.64474E−05	3.69597E−04	5.99783E−03	3.31865E−04	1.11022E−16	3.60822E−16
(0.1, 0.5)	8.88312E−05	6.16873E−04	5.61507E−03	5.54071E−04	1.33227E−15	2.40086E−15
(0.2, 0.1)	4.25408E−04	1.19869E−04	1.33181E−02	1.07016E−04	2.22045E−16	4.16334E−17
(0.2, 0.3)	3.91098E−04	3.60098E−04	1.24441E−02	3.21601E−04	1.66533E−16	3.05311E−16
(0.2, 0.5)	3.60161E−04	6.01006E−04	1.16416E−02	5.36927E−04	1.4988E−15	2.31759E−15
(0.3, 0.1)	9.71922E−04	1.16789E−04	2.07641E−02	1.03737E−04	0	5.55112E−17
(0.3, 0.3)	8.93309E−04	3.50866E−04	1.93852E−02	3.11737E−04	2.77556E−16	2.63678E−16
(0.3, 0.5)	8.22452E−04	5.85610E−04	1.81209E−02	5.20447E−04	1.27676E−15	2.15106E−15
(0.4, 0.1)	1.75596E−03	1.13829E−04	2.88100E−02	1.00579E−04	5.55112E−17	2.77556E−17
(0.4, 0.3)	1.61430E−03	3.41948E−04	2.68724E−02	3.02245E−04	1.66533E−16	2.498E−16
(0.4, 0.5)	1.48578E−03	5.70710E−04	2.50985E−02	5.04593E−04	1.27676E−15	2.04003E−15
(0.5, 0.1)	2.79519E−03	1.10936E−04	3.75193E−02	9.75385E−05	0	0
(0.5, 0.3)	2.56714E−03	3.33274E−04	3.49617E−02	2.93107E−04	2.22045E−16	2.63678E−16
(0.5, 0.5)	2.36184E−03	5.56235E−04	3.26239E−02	4.89335E−04	1.22125E−15	1.90126E−15

Table 7.5 Comparison between CFRDTM, ADM, and VIM results for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $b = 0$, and $c = 10$, for the approximate solutions of MB equations (7.170a) and (7.170b)

(*x*, *t*)	$\lvert u_{Exact} - u_{ADM} \rvert$	$\lvert u_{Exact} - u_{VIM} \rvert$	$\lvert v_{Exact} - v_{ADM} \rvert$	$\lvert v_{Exact} - v_{VIM} \rvert$	$\lvert u_{Exact} - u_{CRFDTM} \rvert$	$\lvert v_{Exact} - v_{CRFDTM} \rvert$
(0.1, 0.1)	8.16297E−07	6.35269E−05	5.88676E−05	1.65942E−05	5.55112E−17	3.46945E−18
(0.1, 0.3)	7.64245E−07	1.90854E−04	5.56914E−05	4.98691E−05	5.55112E−17	5.20417E−17
(0.1, 0.5)	7.16083E−07	3.18549E−04	5.27169E−05	8.32598E−05	6.66134E−16	3.60822E−16
(0.2, 0.1)	3.26243E−06	6.18930E−05	1.18213E−04	1.60813E−05	1.11022E−16	6.93889E−18
(0.2, 0.3)	3.05458E−06	1.85945E−04	1.11833E−04	4.83269E−05	1.11022E−16	4.68375E−17
(0.2, 0.5)	2.86226E−06	3.10352E−04	1.05858E−04	8.06837E−05	7.77156E−16	3.46945E−16
(0.3, 0.1)	7.33445E−06	6.03095E−05	1.78041E−04	1.55880E−05	0	6.93889E−18
(0.3, 0.3)	6.86758E−06	1.81187E−04	1.68429E−04	4.68440E−05	1.66533E−16	3.98986E−17
(0.3, 0.5)	6.43557E−06	3.02408E−04	1.59428E−04	7.82068E−05	6.66134E−16	3.22659E−16
(0.4, 0.1)	1.30286E−05	5.87746E−05	2.38356E−04	1.51135E−05	5.55112E−17	5.20417E−18
(0.4, 0.3)	1.22000E−05	1.76574E−04	2.25483E−04	4.54174E−05	5.55112E−17	3.46945E−17
(0.4, 0.5)	1.14333E−05	2.94707E−04	2.13430E−04	7.58243E−05	6.66134E−16	3.05311E−16
(0.5, 0.1)	2.03415E−05	5.72867E−05	2.99162E−04	1.46569E−05	0	1.73472E−18
(0.5, 0.3)	1.90489E−05	1.72102E−04	2.83001E−04	4.40448E−05	1.11022E−16	4.16334E−17
(0.5, 0.5)	1.78528E−05	2.87241E−04	2.67868E−04	7.35317E−05	6.10623E−16	2.87964E−16

Table 7.6 Comparison between CFRDTM, ADM, and VIM results for $u(x,t)$ and $v(x,t)$ when $\alpha = 1$ and $\beta = 1$, $k = 0.1$, $\lambda = 0.005$, $a = 0$, $b = 0.5$, and $c = 10$, for the approximate solutions of ALW equations (7.180a) and (7.180b)

(x, t)	$\lvert u_{\text{Exact}} - u_{\text{ADM}} \rvert$	$\lvert u_{\text{Exact}} - u_{\text{VIM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{ADM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{VIM}} \rvert$	$\lvert u_{\text{Exact}} - u_{\text{CRFDTM}} \rvert$	$\lvert v_{\text{Exact}} - v_{\text{CRFDTM}} \rvert$
(0.1, 0.1)	8.02989E−06	3.17634E−05	4.81902E−04	8.29712E−06	2.77556E−17	1.73472E−18
(0.1, 0.3)	7.38281E−06	9.54273E−05	4.50818E−04	2.49346E−05	2.77556E−17	2.60209E−17
(0.1, 0.5)	6.79923E−06	1.59274E−04	4.22221E−04	4.16299E−05	3.33067E−16	1.80411E−16
(0.2, 0.1)	3.23228E−05	3.09466E−05	9.76644E−04	8.04063E−06	2.77556E−17	3.46945E−18
(0.2, 0.3)	2.97172E−05	9.29725E−05	9.13502E−04	2.41634E−05	4.16334E−17	2.34188E−17
(0.2, 0.5)	2.73673E−05	1.55176E−04	8.55426E−04	4.03419E−05	3.60822E−16	1.73472E−16
(0.3, 0.1)	7.32051E−05	3.01549E−05	1.48482E−03	7.79401E−06	1.38778E−17	3.46945E−18
(0.3, 0.3)	6.73006E−05	9.05935E−05	1.38858E−03	2.34220E−05	5.55112E−17	1.99493E−17
(0.3, 0.5)	6.19760E−05	1.51204E−04	1.30009E−03	3.91034E−05	3.19189E−16	1.61329E−16
(0.4, 0.1)	1.31032E−04	2.93874E−05	2.00705E−03	7.55675E−06	1.38778E−17	2.60209E−18
(0.4, 0.3)	1.20455E−04	8.82871E−05	1.87661E−03	2.27087E−05	2.77556E−17	1.73472E−17
(0.4, 0.5)	1.10919E−04	1.47354E−04	1.75670E−03	3.79121E−05	3.19189E−16	1.52656E−16
(0.5, 0.1)	2.06186E−04	2.86433E−05	2.54396E−03	7.32847E−06	0	8.67362E−19
(0.5, 0.3)	1.89528E−04	8.60509E−05	2.37815E−03	2.20224E−05	5.55112E−17	2.08167E−17
(0.5, 0.5)	1.74510E−04	1.43620E−04	2.22578E−03	3.67658E−05	3.19189E−16	1.43982E−16

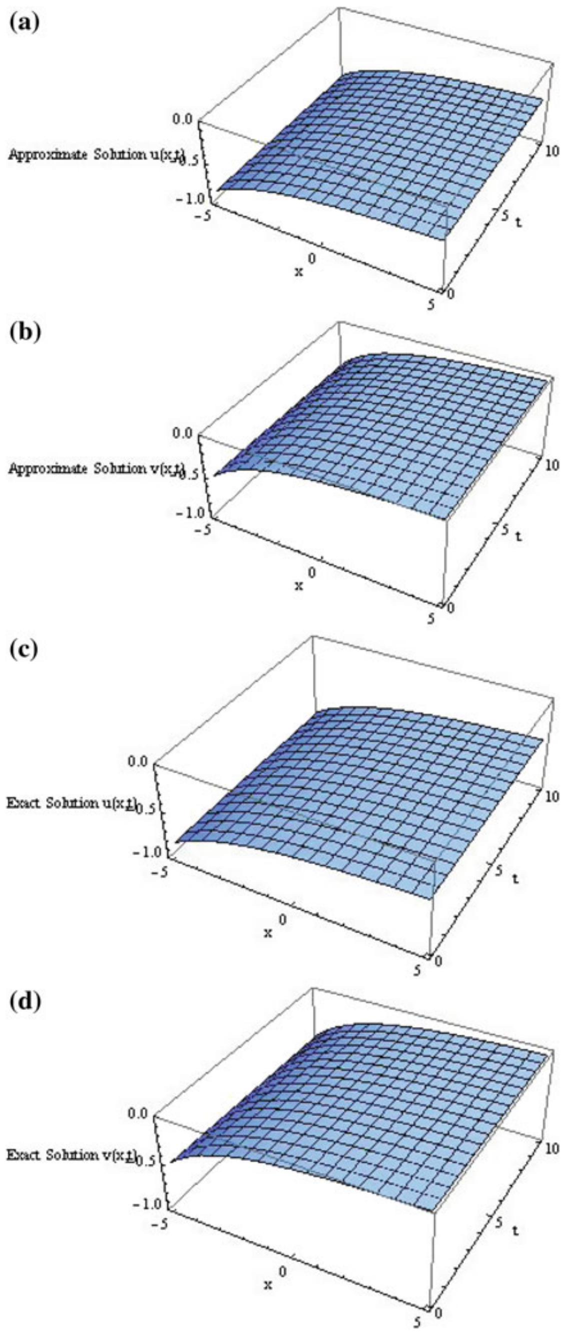


Fig. 7.46 Surfaces show **a** the numerical approximate solution of $u(x,t)$, **b** the numerical approximate solution of $v(x,t)$, **c** the exact solution of $u(x,t)$, and **d** the exact solution of $v(x,t)$ when $\alpha = 1$ and $\beta = 1$

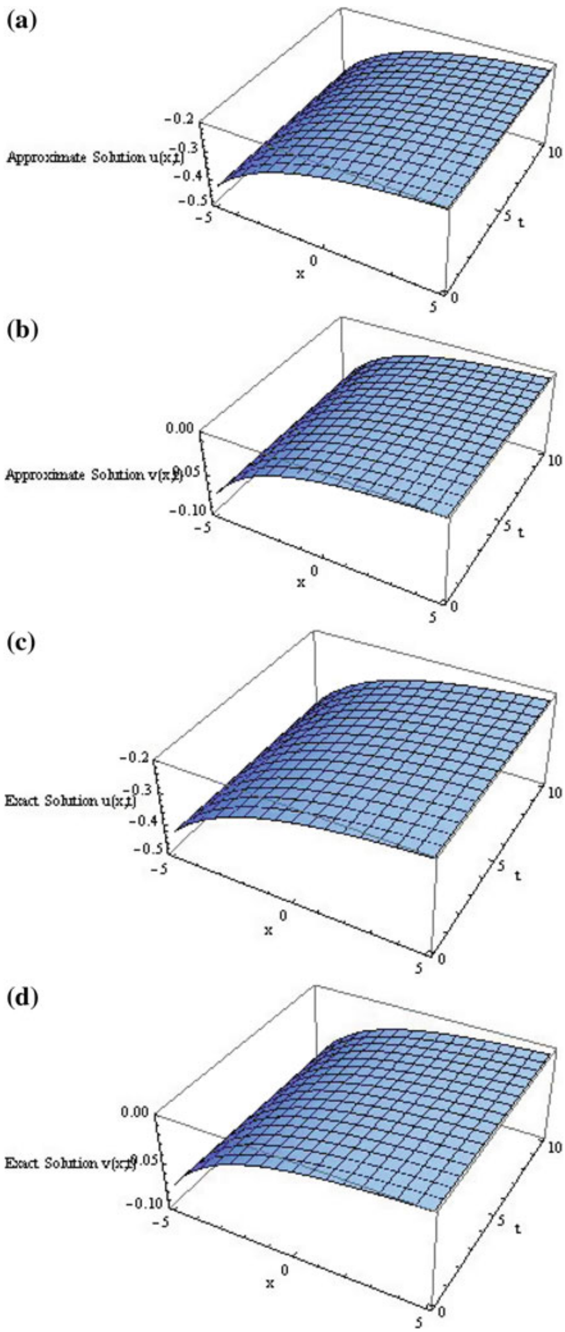


Fig. 7.47 Surfaces show **a** the numerical approximate solution of $u(x, t)$, **b** the numerical approximate solution of $v(x, t)$, **c** the exact solution of $u(x, t)$, and **d** the exact solution of $v(x, t)$ when $\alpha = 1$ and $\beta = 1$

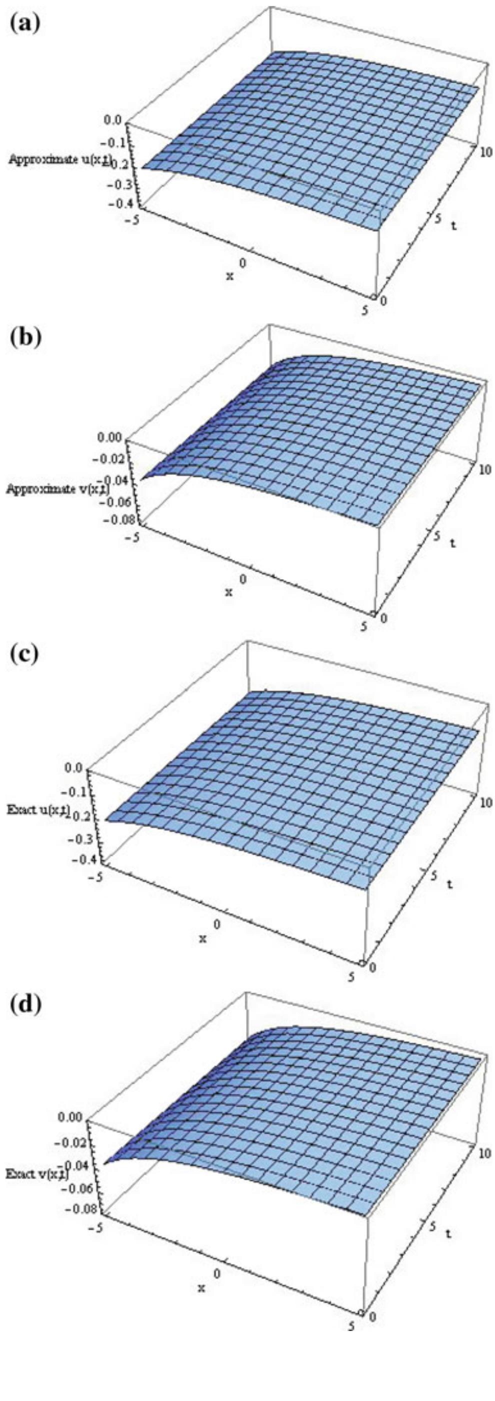
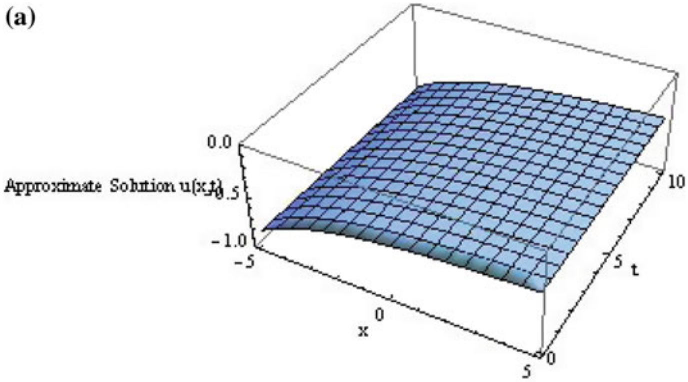


Fig. 7.48 Surfaces show **a** the numerical approximate solution of $u(x, t)$, **b** the numerical approximate solution of $v(x, t)$, **c** the exact solution of $u(x, t)$, and **d** the exact solution of $v(x, t)$ when $\alpha = 1$ and $\beta = 1$

(a)



(b)

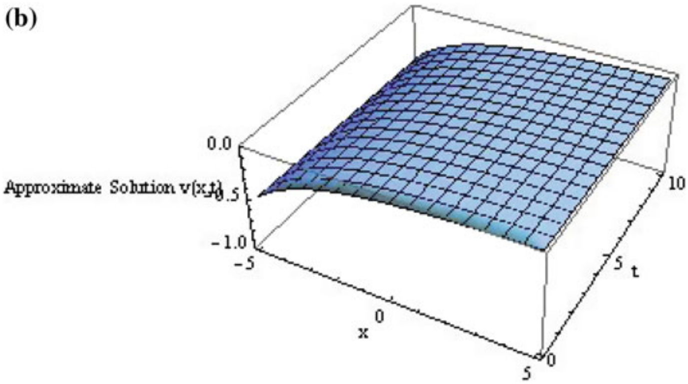


Fig. 7.49 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ for WBK equations (7.158a) and (7.158b) when $\alpha = 1/8$ and $\beta = 1/4$

Theorem 7.4 *Let* $D_t^{\alpha}u = \mathcal{F}(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}, \ldots)$ *and* $D_t^{\beta}v = \mathcal{H}(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}, \ldots)$ *be the general coupled fractional differential equations, and let the Caputo derivatives* $D_t^{k\alpha}u(x,t)$ *and* $D_t^{k\beta}v(x,t)$ *be continuous functions on* $[0, L] \times [0, T]$, i.e.,

$$D_t^{k\alpha}u(x,t) \in C([0,L] \times [0,T]) \text{ and } D_t^{k\beta}v(x,t) \in C([0,L] \times [0,T]),$$

for $k = 0, 1, 2, \ldots, n+1$, where $0 < \alpha, \beta < 1$, then the approximate solutions $\tilde{u}(x,t)$ and $\tilde{v}(x,t)$ of the preceding general coupled fractional differential equations are

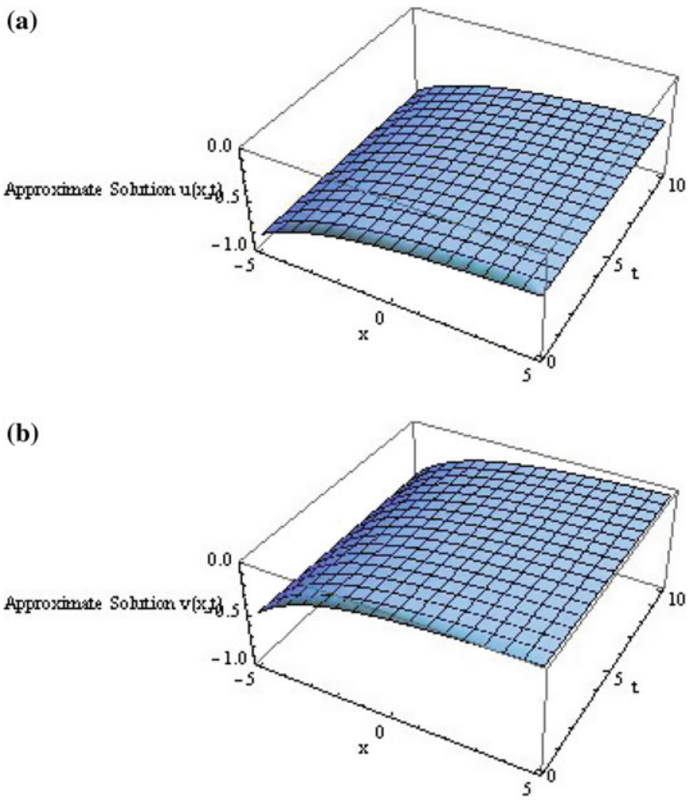


Fig. 7.50 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ for MB equations (7.170a) and (7.170b) when $\alpha = 1/4$ and $\beta = 0.88$

$$\tilde{u}(x,t) \cong \sum_{k=0}^{n} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha + (k-h)\beta},$$

and

$$\tilde{v}(x,t) \cong \sum_{k=0}^{n} \sum_{h=0}^{k} V(h,k-h) t^{h\alpha + (k-h)\beta},$$

where $U(h,k-h)$ and $V(h,k-h)$ are coupled fractional reduced differential transforms of $u(x,t)$ and $v(x,t)$, respectively.

Moreover, there exist values $\xi_1,\ \xi_2$ where $0 \le \xi_1,\ \xi_2 \le t$ so that the error $E_n(x,t)$ for the approximate solution $\tilde{u}(x,t)$ has the form

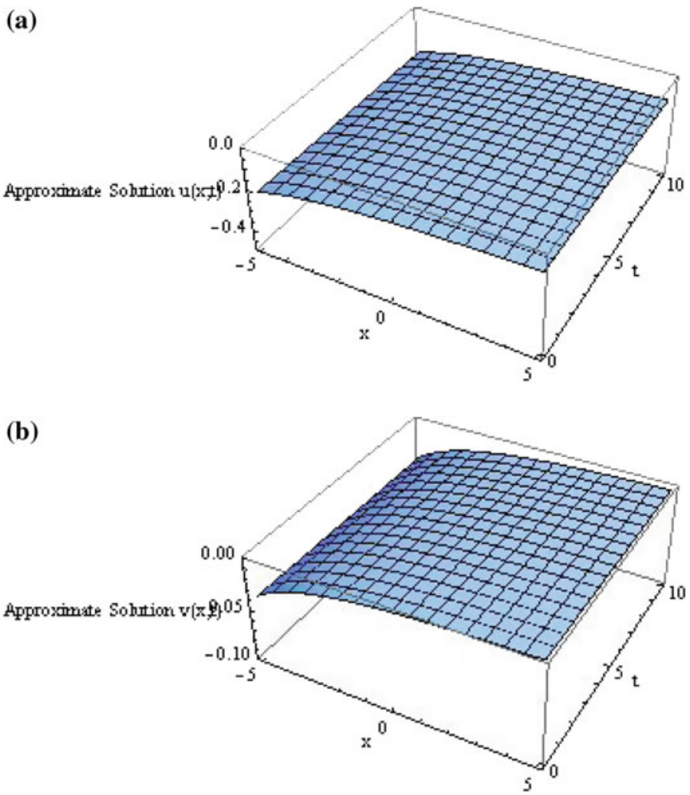


Fig. 7.51 Surfaces show **a** the numerical approximate solution of $u(x,t)$ and **b** the numerical approximate solution of $v(x,t)$ for ALW equations (7.180a) and (7.180b) when $\alpha = 1/2$ and $\beta = 1/2$

$$\|E_n(x,t)\| = \sup_{\substack{0 \le x \le L \\ 0 \le t \le T}} \left| \frac{D^{(n+1)\beta} u(x,0+)}{\Gamma((n+1)\beta+1)} t^{(n+1)\beta} \right|,$$

if $\xi_1,\ \xi_2 \to 0+$.

Proof From Lemma 1 of Chap. 1, we have

$$J^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0+), \quad m-1 < \alpha < m$$

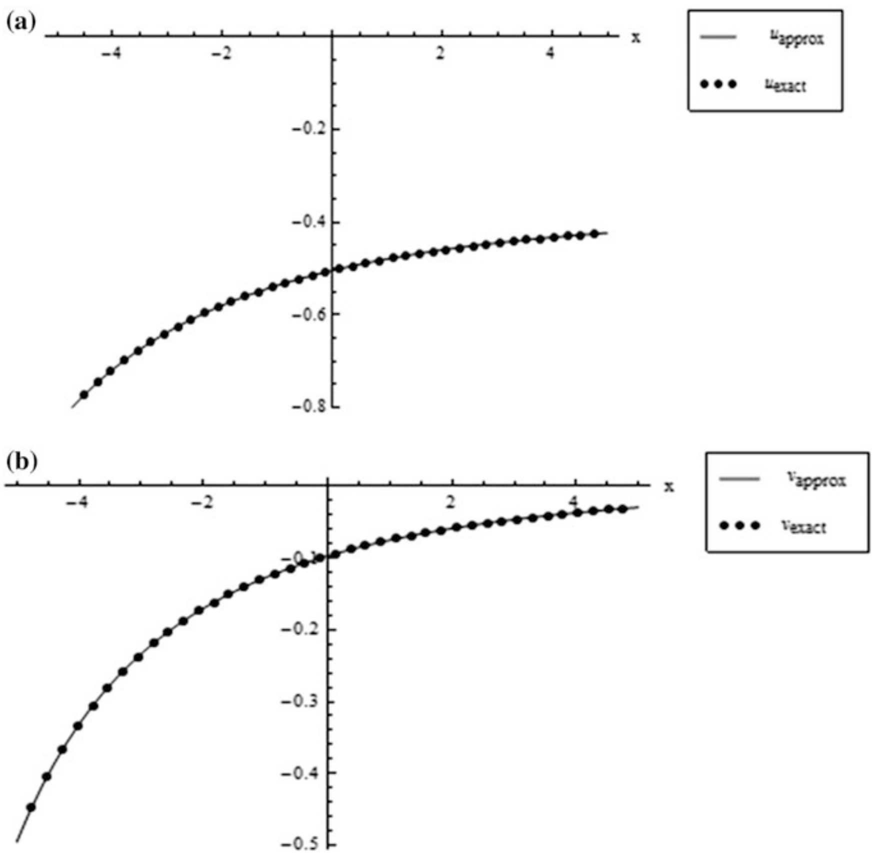


Fig. 7.52 Comparison of approximate solutions **a** $u(x,t)$ and **b** $v(x,t)$ with regard to exact solutions of WBK equation (7.158) at time instance $t = 5$

The error term

$$E_n(x,t) = u(x,t) - \tilde{u}(x,t),$$

where

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} \frac{D^{h\alpha+\beta(k-h)} u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)} t^{h\alpha+\beta(k-h)},$$

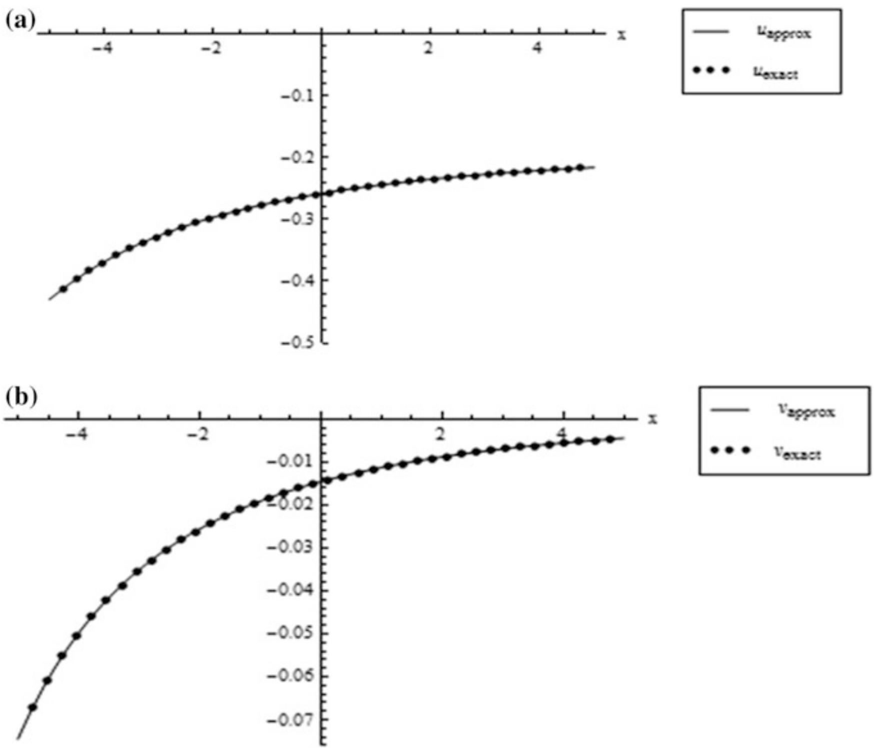


Fig. 7.53 Comparison of approximate solutions **a** $u(x,t)$ and **b** $v(x,t)$ with regard to exact solutions of MB equation (7.170) at time instance $t = 5$

and

$$\tilde{u}(x,t) = \sum_{k=0}^{n} \sum_{h=0}^{k} \frac{D^{h\alpha+\beta(k-h)} u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)} t^{h\alpha+\beta(k-h)}.$$

Now, for $0 < \alpha < 1$,

$$\begin{aligned} &J^{h\alpha+\beta(k-h)} D^{h\alpha+\beta(k-h)} u(x,t) - J^{(h+1)\alpha+\beta(k-h)} D^{(h+1)\alpha+\beta(k-h)} u(x,t) \\ &= J^{h\alpha+\beta(k-h)} \left(D^{h\alpha+\beta(k-h)} u(x,t) - J^{\alpha} D^{\alpha} \left(D^{h\alpha+\beta(k-h)} u(x,t) \right) \right) \\ &= J^{h\alpha+\beta(k-h)} D^{h\alpha+\beta(k-h)} u(x,0), \end{aligned}$$

since $0 < \alpha < 1$, using Eq. (1.14)

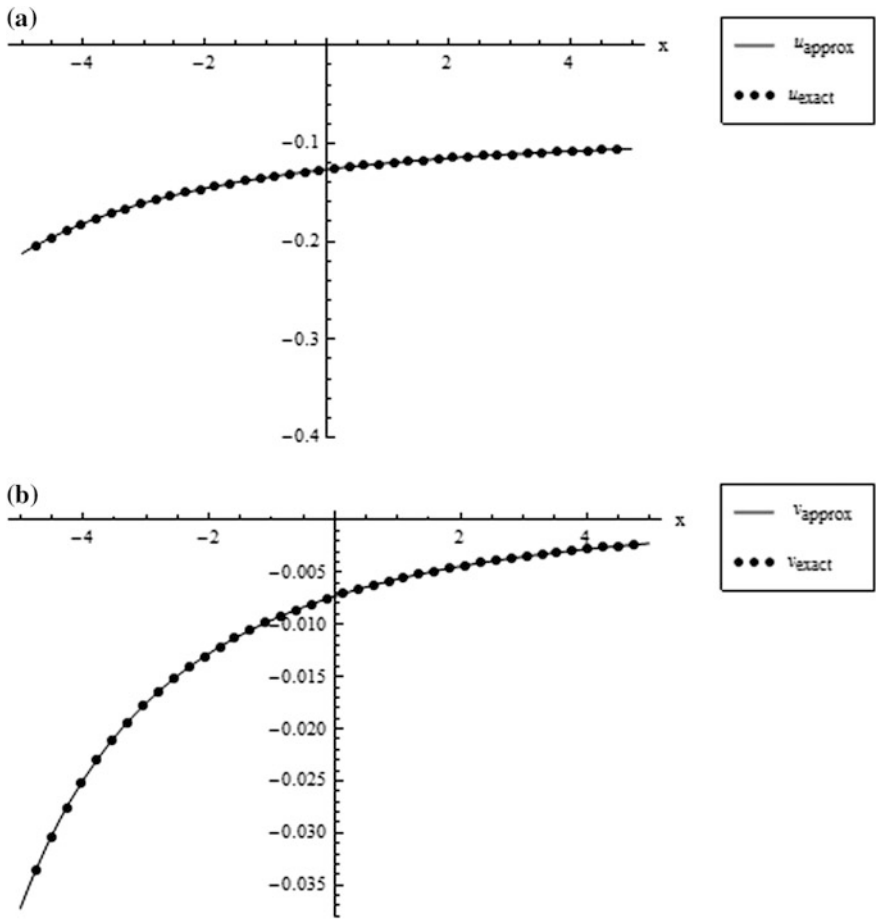


Fig. 7.54 Comparison of approximate solutions **a** $u(x,t)$ and **b** $v(x,t)$ with regard to exact solutions of ALW equation (7.180) at time instance t = 5

$$= \frac{D^{h\alpha+\beta(k-h)}u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}t^{h\alpha+\beta(k-h)} \tag{7.190}$$

The nth order approximation for $u(x,t)$ is

$$\begin{aligned}\tilde{u}(x,t) &= \sum_{k=0}^{n}\sum_{h=0}^{k}\frac{D^{h\alpha+\beta(k-h)}u(x,0)}{\Gamma(h\alpha+\beta(k-h)+1)}t^{h\alpha+\beta(k-h)} \\ &= \sum_{k=0}^{n}\sum_{h=0}^{k}\left(J^{h\alpha+\beta(k-h)}D^{h\alpha+\beta(k-h)}u(x,t) - J^{(h+1)\alpha+\beta(k-h)}D^{(h+1)\alpha+\beta(k-h)}u(x,t)\right),\end{aligned}$$

using Eq. (7.190)

$$
\begin{aligned}
&= \sum_{k=0}^{n} J^{k\beta} D^{k\beta} u(x,t) - \sum_{h=0}^{n} J^{(h+1)\alpha+\beta(n-h)} D^{(h+1)\alpha+\beta(n-h)} u(x,t) \\
&= u(x,t) + \sum_{k=0}^{n-1} J^{(k+1)\beta} D^{(k+1)\beta} u(x,t) - \sum_{h=0}^{n} J^{(h+1)\alpha+\beta(n-h)} D^{(h+1)\alpha+\beta(n-h)} u(x,t)
\end{aligned}
\tag{7.191}
$$

Therefore, from Eq. (7.191), the error term becomes

$$
\begin{aligned}
E_n(x,t) &= u(x,t) - \tilde{u}(x,t) \\
&= \sum_{h=0}^{n} J^{(h+1)\alpha+\beta(n-h)} D^{(h+1)\alpha+\beta(n-h)} u(x,t) - \sum_{k=0}^{n-1} J^{(k+1)\beta} D^{(k+1)\beta} u(x,t) \\
&= \sum_{i=0}^{n} J^{(i+1)\alpha+\beta(n-i)} D^{(i+1)\alpha+\beta(n-i)} u(x,t) - \sum_{i=0}^{n-1} J^{(i+1)\beta} D^{(i+1)\beta} u(x,t) \\
&= \sum_{i=0}^{n} \frac{1}{\Gamma((i+1)\alpha+\beta(n-i))} \int_0^t (t-\tau)^{(i+1)\alpha+\beta(n-i)-1} D^{(i+1)\alpha+\beta(n-i)} u(x,\tau)\mathrm{d}\tau \\
&\quad - \sum_{i=0}^{n-1} \frac{1}{\Gamma((i+1)\beta)} \int_0^t (t-\tau)^{(i+1)\beta-1} D^{(i+1)\beta} u(x,\tau)\mathrm{d}\tau
\end{aligned}
$$

Whence applying integral mean value theorem yielding

$$
\begin{aligned}
E_n(x,t) = \sum_{i=0}^{n} &\frac{D^{(i+1)\alpha+\beta(n-i)} u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)} t^{(i+1)\alpha+\beta(n-i)} \\
&- \sum_{i=0}^{n-1} \frac{D^{(i+1)\beta} u(x,\xi_2)}{\Gamma((i+1)\beta+1)} t^{(i+1)\beta},
\end{aligned}
$$

where $0 \le \xi_1,\ \xi_2 \le t$.

This implies

$$\begin{aligned} E_n(x,t) &= u(x,t) - \tilde{u}(x,t) \\ &= \sum_{i=0}^{n-1} \frac{D^{(i+1)\alpha+\beta(n-i)}u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)} t^{(i+1)\alpha+\beta(n-i)} \\ &\quad + \frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} - \sum_{i=0}^{n-1} \frac{D^{(i+1)\beta}u(x,\xi_2)}{\Gamma((i+1)\beta+1)} t^{(i+1)\beta} \\ &= \sum_{i=0}^{n-1} \left[\frac{D^{(i+1)\alpha+\beta(n-i)}u(x,\xi_1)}{\Gamma((i+1)\alpha+\beta(n-i)+1)} t^{(i+1)\alpha+\beta(n-i)} - \frac{D^{(i+1)\beta}u(x,\xi_2)}{\Gamma((i+1)\beta+1)} t^{(i+1)\beta} \right] \\ &\quad + \frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} \end{aligned} \tag{7.192}$$

Using generalized Taylor's series formula, Eq. (7.192) becomes

$$\begin{aligned} E_n(x,t) = u(x,t) - \frac{D^{(n+1)\alpha}u(x,\zeta_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} - u(x,t) \\ + \frac{D^{(n+1)\beta}u(x,\zeta_2)}{\Gamma((n+1)\beta+1)} t^{(n+1)\beta} + \frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha}, \end{aligned}$$

where $0 \leq \zeta_1,\ \zeta_2 \leq \max\{\xi_1,\ \xi_2\}$ and $\xi_1,\ \xi_2 \to 0+$.

This implies

$$\begin{aligned} \|E_n\| &= \|u(x,t) - \tilde{u}(x,t)\| \\ &= \sup_{\substack{0\leq x\leq L \\ 0\leq t\leq T}} \left| \frac{D^{(n+1)\beta}u(x,\zeta_2)}{\Gamma((n+1)\beta+1)} t^{(n+1)\beta} - \frac{D^{(n+1)\alpha}u(x,\zeta_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} + \frac{D^{(n+1)\alpha}u(x,\xi_1)}{\Gamma((n+1)\alpha+1)} t^{(n+1)\alpha} \right| < \infty \\ &= \sup_{\substack{0\leq x\leq L \\ 0\leq t\leq T}} \left| \frac{D^{(n+1)\beta}u(x,0+)}{\Gamma((n+1)\beta+1)} t^{(n+1)\beta} \right|, \text{ since } \xi_1, \xi_2 \to 0+. \end{aligned} \tag{7.193}$$

As $n \to \infty$, from Eq. (7.193)

$$\|E_n\| \to 0.$$

Hence, $u(x,t)$ can be approximated as

$$u(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha+(k-h)\beta} \cong \sum_{k=0}^{n} \sum_{h=0}^{k} U(h,k-h) t^{h\alpha+(k-h)\beta} = \tilde{u}(x,t),$$

with the error term given in Eq. (7.193).

Following a similar argument, we may also find the error $\left\|\hat{E}_n\right\| = \|v(x,t) - \tilde{v}(x,t)\|$ for the approximate solution $\tilde{v}(x,t)$. ■

7.6 Conclusion

In this chapter, the MFRDTM has been proposed and it is directly applied to obtain explicit and numerical solitary wave solutions of the fractional KdV like $K(m,n)$ equations with initial conditions. In this regard, the reduced differential transform method is modified to be easily employed to solve wide kinds of nonlinear fractional differential equations. In this new approach, the nonlinear term is replaced by its Adomian polynomials. As a result, we obtain the approximate solutions of fractional KdV equation with high accuracy. The obtained results demonstrate the reliability of the proposed algorithm and its wider applicability to fractional nonlinear evolution equations. It also exhibits that the proposed method is a very efficient and powerful technique in finding the solutions of the nonlinear fractional differential equations. The main advantage of the method is the fact that it provides an analytical approximate solution, in many cases an exact solution, in a rapidly convergent series with elegantly computed terms. It requires less amount of computational overhead in comparison with other numerical methods and consequently introduces a significant improvement in solving fractional nonlinear equations over existing methods available in the open literature.

A new approximate numerical technique, coupled fractional reduced differential transform, has been proposed in this chapter for solving nonlinear fractional partial differential equations. The proposed method is only well suited for coupled fractional linear and nonlinear differential equations. In comparison with other analytical methods, the present method is an efficient and simple tool to determine the approximate solution of nonlinear coupled fractional partial differential equations. The obtained results demonstrate the reliability of the proposed algorithm and its applicability to nonlinear coupled fractional evolution equations. It also exhibits that the proposed method is a very efficient and powerful technique in finding the solutions of the nonlinear coupled time fractional differential equations. The main advantage of the proposed method is that it requires less amount of computational overhead in comparison with other numerical and analytical approximate methods and consequently introduces a significant improvement in solving coupled fractional nonlinear equations over existing methods available in the open literature. The application of the proposed method for the solutions of time fractional coupled KdV equations satisfactorily justifies its simplicity and efficiency.

In this chapter, new CFRDTM has been successfully implemented to obtain the soliton solutions of coupled time fractional modified KdV equations. This new method has been revealed by the author. The application of the proposed method for the solutions of time fractional coupled modified KdV equations satisfactorily justifies its simplicity and efficiency. Moreover, in case of integer-order coupled modified KdV equations, the obtained results have been verified by the Adomian decomposition method. This investigation leads to the conclusion that soliton solutions for integer-order coupled modified KdV equations have been wrongly reported by the reverend author Fan [44].

Also in this chapter, the new approximate numerical technique CFRDTM [34, 35] has been proposed for solving nonlinear fractional partial differential equations arising in predator–prey biological population dynamical system. The results thus obtained validate the reliability of the proposed algorithm. It additionally displays that the proposed process is an extraordinarily efficient and strong technique. The main advantage of the proposed method is that it necessitates less amount of computational effort. In a later study, it has been planned to use the proposed process for the solution of the fractional epidemic model, coupled fractional neutron diffusion equations with delayed neutrons, and other physical models with the intention to show the efficiency and wide applicability of the newly proposed method.

In view of the author [61], there is no difference between differential transform method (DTM) and Taylor series method (TSM) both of which (normally) are provided with an analytical continuation via a stepwise procedure, since it is essential to transform the formal series into an approximate solution of the problem (via analytical continuation). The author also wrote in [61] that one may then rightly remember the approach as being "*an extended Taylor series method.*" Thus, the DTM could, eventually, be named as the generalized Taylor series method (GTSM). In the belief of the learned author, "DTM could deserve its name (as a technique) when it extends the Taylor series method to new kinds of expansion (different from a Taylor series expansion)." He, additionally, acknowledges that the DTM has allowed an easy generalization of the Taylor series method to various derivation procedures. "For example, fractional differential equations have been considered using the DTM extended to the fractional derivative procedure via a modified version of the Taylor series." Despite the fact that there is a controversy in the name of DTM, the author of [61] admits that major contribution of the DTM is in the easy generalization of the Taylor series method to problems involving fractional derivatives.

Furthermore, it may be stated that the Taylor series method is used invariably in many mathematical analyses and derivations for the problems of applied science and engineering. Taylor series method of order one is commonly known as the Euler method. However, the Euler method has its independent existence. Like that, DTM is also self-contained for at least in the application of fractional-order calculus and has its own right for its existence.

Also, in this chapter, fractional coupled Schrödinger–Korteweg–de Vries equations with appropriate initial values have been solved by using the novel method, viz. CFRDTM. The applications of the proposed method for the solutions of time fractional coupled Sch–KdV equations reasonably well justify its simplicity, plausibility, and efficiency.

In this chapter, solutions of nonlinear coupled fractional partial differential equations have been proposed by CFRDTM which is only well suited for coupled fractional linear and nonlinear differential equations. The present method is an efficient and simple tool in comparison with other analytical methods. The obtained results quite justify that the proposed method is very well suited and is an efficient and powerful technique in finding the solutions of the nonlinear coupled time fractional differential equations. One of the main advantages of the proposed method is that it requires less amount of computational overhead and consequently introduces a significant achievement in solving coupled fractional nonlinear equations over existing methods available in the open literature. Furthermore, the applications of the proposed method for the solutions of variant types of time fractional coupled WBK equations satisfactorily justify its simplicity and efficiency. The proposed method determines the analytical approximate solutions as well as numerical solutions. This proposed method can be efficiently applied to coupled fractional differential equations more accurately and easily than its comparable methods ADM and VIM. So, this proposed method can be a better substitute than its competitive methods ADM and VIM.

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