Chapter 6
New Exact Traveling Wave Solutions of the Coupled Schrödinger–Boussinesq Equations and Tzitzéica-Type Evolution Equations

6.1 Introduction

In the recent years, the investigation of finding new exact solutions of nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena such as fluid mechanics, plasma physics, statistical physics, quantum physics, solid state physics, optics, and so on [1, 2]. NLPDEs are widely used to describe complex physical phenomena arising in the various fields of science and engineering. Several methods for finding the exact solutions to nonlinear equations in mathematical physics have been presented, such as the inverse scattering method [3], Bäcklund transformation [4, 5], the truncated Painlevé expansion method [6, 7], Hirota's bilinear method [8], tanh- function method [9, 10], exp-function method [11], (G'/G)-expansion method [12, 13], Jacobi elliptic function method [14–17], the first integral method [18–21], Riccati equation rational expansion method [22], Kudryashov method [23, 24], modified decomposition method [25, 26], and other methods [27–30].

It is commonly known that many problems in applied science and engineering are described by nonlinear partial differential equations (NLPDEs). One of the most significant advances of theoretical physics and nonlinear science has been the development of methods to determine the exact solutions for NLPDEs. When a NLPDE is analyzed, the main objective is the construction of the exact solutions for the equation.

Many powerful methods have been presented, such as the inverse scattering transform method [3] and the Hirota bilinear transform method [8] are known as impressive methods to find solutions of exactly solvable NLPDEs. The truncated Painlevé expansion method [6], Bäcklund transformation method [4], the homogeneous balance method [31], the tanh-function method [32–36], the modified extended tanh-function method [10, 37], the exp-function method [38], the (G'/G)-expansion method [12, 39], the auxiliary equation method [40], the extended auxiliary equation method [41, 42], the Jacobi elliptic function method [14, 43], the

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simplest equation method [44], the extended simplest equation method [45], and the Weierstrass elliptic function method [46] are useful in many applications to find the exact solutions of NLPDEs.

There are many physical phenomena around us that are best described by nonlinear evolution equations. The Tzitzeica-type nonlinear evolution equations, including Tzitzeica, Dodd–Bullough–Mikhailov (DBM), and Tzitzéica–Dodd–Bullough (TDB) equations are a class of such equations which have gained significant attention during the last few decades. The objective of this work is to find the Jacobi elliptic function solutions, including the hyperbolic and trigonometric solutions for the DBM and TDB equations using a new extended auxiliary equation method. These two equations appear in problems varying from fluid flow to quantum field theory. The great deals of efforts have been devoted to solve these equations using a variety of methods that some of them are reviewed here. Abazari [47] used the (G'/G)-expansion method to find more general exact solutions of the Tzitzéica-type nonlinear evolution equations. Manafian and Lakestani [48] utilized the improved $\tan(\Phi(\xi)/2)$-expansion method and gained new and more general exact traveling wave solutions of the Tzitzéica-type nonlinear equations. In [49], Hosseini et al. employed first the Painlevé transformation and Lie symmetry method to convert the DBM and TDB equations into nonlinear ordinary differential equations and then, a modified version of improved $\tan(\Phi(\xi)/2)$-expansion method has been adopted to generate new exact solutions of the reduced equations. Wazwaz [36] exerted the tanh method to generate solitons and periodic solutions of the Tzitzéica-type nonlinear evolution equations, viz. DBM and TDB equations. Hosseini et al. [50] used the modified Kudryashov method and acquired new exact traveling wave solutions of the Tzitzéica-type equations.

6.2 Outline of the Present Study

In this present chapter, an improved algebraic method based on the generalized Jacobi elliptic function method with symbolic computation is used to construct more new exact solutions for coupled Schrödinger–Boussinesq equations. As a result, several families of new generalized Jacobi double periodic elliptic function wave solutions are obtained by using this method, some of them are degenerated to solitary wave solutions in the limiting cases. The present generalized method is efficient, powerful, straightforward, and concise, and it can be used in order to establish more entirely new exact solutions for other kinds of nonlinear partial differential equations arising in mathematical physics.

Also in this chapter, new types of Jacobi elliptic function solutions of Dodd–Bullough–Mikhailov (DBM) and Tzitzeica–Dodd–Bullough (TDB) equations have been obtained using a new extended auxiliary equation method. A new family of explicit traveling wave solutions is derived. The solitary wave solutions and periodic solutions for these equations are formally derived from the Jacobi elliptic function solutions. The proposed method has been efficiently applied to solve the DBM and TDB equations.

6.2.1 Coupled Schrödinger–Boussinesq Equations

The objective in this work is to use a generalized Jacobi elliptic function expansion method to construct the new exact solutions of the coupled Schrödinger–Boussinesq equations (CSBEs)

$$\mathrm{i}u_t + u_{xx} + \alpha u - uv = 0,\ \ x \in R, t > 0, \tag{6.1}$$

$$3v_{tt} - v_{xxxx} + 3\left(v^2\right)_{xx} + \beta v_{xx} = \left(|u|^2\right)_{xx},\ \ x \in R, t > 0, \tag{6.2}$$

where the complex-valued function $u(x, t)$ represents the short-wave amplitude, $v(x, t)$ represents the long-wave amplitude, and α and β are real parameters. Equations (6.1) and (6.2) were considered as a model of the interactions between short and intermediate long waves, and were originated in describing the dynamics of Langmuir soliton formation, the interaction in plasma [51, 52], the diatomic lattice system [53], etc.

6.2.2 Tzitzéica-Type Nonlinear Evolution Equations

A new extended auxiliary equation method is used to produce new exact traveling wave solutions of Dodd–Bullough–Mikhailov and Tzitzeica–Dodd–Bullough equations

The Dodd–Bullough–Mikhailov Equation

Let us consider the Dodd–Bullough–Mikhailov equation as follows

$$u_{xt} + \mathrm{e}^{u} + \mathrm{e}^{-2u} = 0. \tag{6.3}$$

In a traveling wave variable $\xi = kx + \omega t$, Eq. (6.3) reads in the form

$$k\omega f_{\xi\xi} + \mathrm{e}^{f} + \mathrm{e}^{-2f} = 0, \tag{6.4}$$

where $u(x, t) = f(\xi)$.

Using the Painlevé transformation $v = \mathrm{e}^{f}$ or $f = \ln v$, the Dodd–Bullough–Mikhailov Eq. (6.4) can be written as follows

$$k\omega v v_{\xi\xi} - k\omega (v_{\xi})^2 + v^3 + 1 = 0. \tag{6.5}$$

The Tzitzeica–Dodd–Bullough Equation

Now, we consider the Tzitzeica–Dodd–Bullough (TDB) equation as follows

$$u_{xt} = \mathrm{e}^{-u} + \mathrm{e}^{-2u}. \tag{6.6}$$

The traveling wave transformation $\xi = kx + \omega t$ reduces Eq. (6.6) to the following ODE

$$k\omega f_{\xi\xi} - \mathrm{e}^{-f} - \mathrm{e}^{-2f} = 0, \tag{6.7}$$

where $u(x,t) = f(\xi)$.

Using the Painlevé transformation $v = \mathrm{e}^{-f}$ or $f = -\ln v$, the Tzitzeica–Dodd–Bullough (6.7) can be written as follows

$$k\omega v v_{\xi\xi} - k\omega (v_{\xi})^2 + v^3 + v^4 = 0. \tag{6.8}$$

6.3 Algorithms for the Improved Generalized Jacobi Elliptic Function Method and the Extended Auxiliary Equation Method

In this section, algorithms for improved generalized Jacobi elliptic function method and extended auxiliary equation method have been presented.

6.3.1 Algorithm for the Improved Generalized Jacobi Elliptic Function Method

In this present analysis, the determination of exact solutions for coupled Schrödinger–Boussinesq equations have been described using the proposed method. The main steps of this present method are described as follows:

Step 1: Suppose that the coupled nonlinear NLPDEs in the class of coupled Schrödinger–Boussinesq equations, say in two independent variables x, and t are given by

$$F(u, v, u_x, v_x, iu_t, v_t, u_{xx}, v_{xx}, u_{xt}, v_{xt}, \ldots) = 0, \tag{6.9a}$$

$$G(u, v, u_x, v_x, u_t, v_t, u_{xx}, v_{xx}, u_{xt}, v_{xt}, \ldots) = 0, \tag{6.9b}$$

where $u = u(x, t)$ and $v = v(x, t)$ are unknown functions, F and G are polynomials in u, v and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: We introduce the following traveling wave transformations:

$$u(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)},\ v(x,t) = V(\xi), \tag{6.10}$$

$$\xi = x - 2kt + \eta_0 \tag{6.11}$$

where k, and c are real constants to be determined later; and ζ_0, and η_0 are arbitrary constants.

Using the above traveling wave transformations, the NLPDEs (6.9a and 6.9b) can be transformed to couple nonlinear ordinary differential equations (ODEs) involving $U(\xi)$ and $V(\xi)$. Then, the resultant coupled ODEs are obtained

$$P(U, V, kU, kV, cU, cV, U_\xi, V_\xi, kU_\xi, kV_\xi, cU_\xi, cV_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0, \tag{6.12}$$

$$Q(U, V, kU, kV, cU, cV, U_\xi, V_\xi, kU_\xi, kV_\xi, cU_\xi, cV_\xi, U_{\xi\xi}, V_{\xi\xi}, \ldots) = 0, \tag{6.13}$$

where the suffix denotes the derivative with respect to ξ.

Step 3: Let us assume that the exact solutions of Eqs. (6.12) and (6.13) are to be defined in the polynomial $\varphi(\xi)$ of the following forms:

$$U(\xi) = a_{10} + \sum_{i=1}^{M}\left[a_{1i}\phi^{i}(\xi) + b_{1i}\phi^{-i}(\xi) + c_{1i}\phi^{i-1}(\xi)\phi'(\xi) + d_{1i}\phi^{-i}(\xi)\phi'(\xi)\right], \tag{6.14}$$

$$V(\xi) = a_{20} + \sum_{j=1}^{N}\left[a_{2j}\phi^{j}(\xi) + b_{2j}\phi^{-j}(\xi) + c_{2j}\phi^{j-1}(\xi)\phi'(\xi) + d_{2j}\phi^{-j}(\xi)\phi'(\xi)\right], \tag{6.15}$$

where $\phi(\xi)$ satisfies the following Jacobi elliptic equation:

$$(\phi_\xi(\xi))^2 = p\phi^4(\xi) + q\phi^2(\xi) + r, \tag{6.16}$$

where p, q, r, a_{10}, a_{1i}, b_{1i}, c_{1i}, d_{1i} $(i = 1, 2, \ldots, M)$, a_{20}, a_{2j}, b_{2j}, c_{2j}, d_{2j} $(j = 1, 2, \ldots, N)$ are constants to be determined later.

Step 4: We determine the positive integers M, N in Eqs. (6.14) and (6.15) by balancing the highest order derivatives and the nonlinear terms in Eqs. (6.12) and (6.13), respectively.

Step 5: Substituting Eqs. (6.14) and (6.15) along with Eq.(6.16) into Eqs. (6.12) and (6.13) and collecting all the coefficients of $\varphi^l(\xi)$ $(l = 0, 1, 2, \ldots)$ and $\phi^m(\xi)\phi'(\xi)$ $(m = 0, 1, 2, \ldots)$, then equating these coefficients to zero, yield a set of algebraic equations, which can be solved by using the Mathematica or Maple to find the values of a_{10}, a_{1i}, b_{1i}, c_{1i}, d_{1i} $(i = 1, 2, \ldots, M)$, a_{20}, a_{2j}, b_{2j}, c_{2j}, d_{2j} $(j = 1, 2, \ldots, N)$, k, c.

Step 6: It may be referred to that Eq. (6.16) has families of Jacobi elliptic function solutions as follows [54].

It may be mentioned that there are other Jacobi elliptic function solutions of Eq. (6.16) which are excluded here for simplicity.

Step 7: Substituting the values of a_{10}, a_{1i}, b_{1i}, c_{1i}, d_{1i} $(i = 1, 2, \ldots, M)$, a_{20}, a_{2j}, b_{2j}, c_{2j}, d_{2j} $(j = 1, 2, \ldots, N)$, p, q, r as well as the solutions of Eq. (6.16) provided in Step 6, into Eqs. (6.14) and (6.15), we can obtain several classes of exact solutions for CSBEs involving the Jacobi elliptic functions *sn*, *cn*, *ns*, *nc*, *cs*, and *sc* functions.

In Table 6.1, $sn\xi = sn(\xi, m^2)$, $cn\xi = cn(\xi, m^2)$, $dn\xi = dn(\xi, m^2)$, $ns\xi = ns(\xi, m^2)$, $cs\xi = cs(\xi, m^2)$, $ds\xi = ds(\xi, m^2)$, $sc\xi = sc(\xi, m^2)$, $sd\xi = sd(\xi, m^2)$ are the Jacobi elliptic functions with modulus m, $0 < m < 1$.

The Jacobi elliptic functions $sn\xi$, $cn\xi$, and $dn\xi$ are double periodic and have the following properties:

$$\begin{gathered} sn^2\xi + cn^2\xi = 1, \\ dn^2\xi + m^2 sn^2\xi = 1. \end{gathered}$$

In addition to these, these functions satisfy the followings:

$$\begin{aligned} &(sn\xi)' = cn\xi dn\xi, \quad (cn\xi)' = -sn\xi dn\xi, \quad (dn\xi)' = -m^2 sn\xi cn\xi, \quad (ns\xi)' = -cs\xi ds\xi, \\ &(cs\xi)' = -ns\xi ds\xi, \quad (ds\xi)' = -ns\xi cs\xi, \quad (sc\xi)' = nc\xi dc\xi, \quad (nc\xi)' = sc\xi dc\xi, \\ &(dc\xi)' = (1 - m^2) nc\xi sc\xi, \quad (sd\xi)' = nd\xi cd\xi, \quad (cd\xi)' = (m^2 - 1) sd\xi nd\xi, \\ &\qquad (nd\xi)' = m^2 cd\xi sd\xi. \end{aligned}$$

Further explanations in details about the Jacobi elliptic functions can be found in [55].

Table 6.1 Jacobi elliptic function solutions of Eq. (6.16)

S. no.	p	q	r	$\phi(\xi)$
1.	m^2	$-(1+m^2)$	1	$sn\xi$
2.	1	$-(1+m^2)$	m^2	$ns\xi = (sn\xi)^{-1}$
3.	$-m^2$	$2m^2 - 1$	$1 - m^2$	$cn\xi$
4.	$1 - m^2$	$2m^2 - 1$	$-m^2$	$nc\xi = (cn\xi)^{-1}$
5.	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$ns\xi \pm cs\xi$
6.	$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$nc\xi \pm sc\xi$

6.3.2 Algorithm for the New Extended Auxiliary Equation Method

Let us consider the following nonlinear PDE

$$\Phi(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0, \tag{6.17}$$

where $u = u(x,t)$ is an unknown function, Φ is a polynomial in u and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved. The main steps of the new extended auxiliary equation method [56] can be summarized as follows:

Step 1: The following traveling wave transformation

$$u(x,t) = U(\xi), \xi = kx + \omega t, \tag{6.18}$$

where k and ω are constants, has been considered to reduce Eq. (6.17) to the following nonlinear ordinary differential equation (ODE):

$$H(U, U', U'', \ldots) = 0, \tag{6.19}$$

where H is a polynomial in $U(\xi)$ and its total derivatives $U'(\xi)$, $U''(\xi)$, and so on.

Step 2: Let us assume that Eq. (6.19) has the formal solution

$$U(\xi) = \sum_{i=0}^{2N} a_i F^i(\xi), \tag{6.20}$$

where $F(\xi)$ satisfies the first-order ODE:

$$(F'(\xi))^2 = c_0 + c_2 F^2(\xi) + c_4 F^4(\xi) + c_6 F^6(\xi), \tag{6.21}$$

where $c_j (j = 0, 2, 4, 6)$ and $a_i (i = 0, \ldots, 2N)$ are arbitrary constants to be determined.

Step 3: By balancing the highest order nonlinear terms and the highest order derivatives of $U(\xi)$ in Eq. (6.19), the balance number N of Eq. (6.20) can be determined.

Step 4: Substituting Eq. (6.20) alongwith (6.21) into Eq. (6.19), collecting all the coefficients of $F^j (F')^l$ $(j = 0, 1, 2, \ldots)$ and $(l = 0, 1)$, and set them to zero, leads to a system of algebraic equations for $c_j (j = 0, 2, 4, 6)$, $a_i (i = 0, \ldots, 2N)$, k, and ω.

Step 5: The system of algebraic equations obtained in Step 4 is solved to find $c_j(j = 0, 2, 4, 6)$, $a_i(i = 0, \ldots, 2N)$, k, and ω.

Step 6: It is well familiar that Eq. (6.21) has the following solutions [56, 57]:

$$F(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}(1 \pm \phi_i(\xi))\right]^{1/2}, \tag{6.22}$$

where the function $\phi_i(\xi)$ $(i = 1, 2, \ldots, 12)$ can be expressed through the Jacobi elliptic function $sn(\xi, m)$, $cn(\xi, m)$, $dn(\xi, m)$, and so on, where $0 < m < 1$ is the modulus of the Jacobi elliptic functions. When m approaches to 1 or 0, the Jacobi elliptic functions degenerate to hyperbolic functions and trigonometric functions, respectively. Further explanations in details about the Jacobi elliptic functions can be found in Ref. [55].

The function $\phi_i(\xi)$ in Eq. (6.22) has 12 forms as follows [41]:

Type I:

If $c_0 = \frac{c_4^3(m^2-1)}{32c_6^2m^2}$, $c_2 = \frac{c_4^2(5m^2-1)}{16c_6m^2}$, $c_6 > 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_1(\xi) = sn(\kappa\xi),\ \phi_2(\xi) = \frac{1}{msn(\kappa\xi)},\ \kappa = \frac{c_4}{2m}\frac{1}{\sqrt{c_6}}. \tag{6.23}$$

Type II:

If $c_0 = \frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2 = \frac{c_4^2(5-m^2)}{16c_6}$, $c_6 > 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_3(\xi) = msn(\kappa\xi),\ \phi_4(\xi) = \frac{1}{sn(\kappa\xi)},\ \kappa = \frac{c_4}{2}\frac{1}{\sqrt{c_6}}. \tag{6.24}$$

Type III:

If $c_0 = \frac{c_4^3}{32m^2c_6^2}$, $c_2 = \frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6 < 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_5(\xi) = cn(\kappa\xi),\ \phi_6(\xi) = \frac{\sqrt{1-m^2}sn(\kappa\xi)}{dn(\kappa\xi)},\ \kappa = \frac{c_4\sqrt{-c_6}}{2mc_6}. \tag{6.25}$$

Type IV:

If $c_0 = \frac{c_4^3m^2}{32c_6^2(m^2-1)}$, $c_2 = \frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6 < 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_7(\xi) = \frac{\sqrt{1-m^2}dn(\kappa\xi)}{1-m^2},\ \phi_8(\xi) = \frac{1}{dn(\kappa\xi)},\ \kappa = \frac{c_4\sqrt{c_6(m^2-1)}}{2c_6(m^2-1)}. \tag{6.26}$$

Type V:

If $c_0 = \frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2 = \frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6 > 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_9(\xi) = \frac{1}{cn(\kappa\xi)},\ \phi_{10}(\xi) = \frac{\sqrt{1-m^2}dn(\kappa\xi)}{(1-m^2)sn(\kappa\xi)},\ \kappa = \frac{c_4\sqrt{c_6(1-m^2)}}{2c_6(1-m^2)}. \tag{6.27}$$

Type VI:

If $c_0 = \frac{m^2c_4^3}{32c_6^2}$, $c_2 = \frac{c_4^2(m^2+4)}{16c_6}$, $c_6 < 0$, then $\phi_i(\xi)$ in Eq. (6.22) takes the form

$$\phi_{11}(\xi) = dn(\kappa\xi),\ \phi_{12}(\xi) = \frac{\sqrt{1-m^2}}{dn(\kappa\xi)},\ \kappa = \frac{c_4\sqrt{-c_6}}{2c_6}. \tag{6.28}$$

Step 7: Substituting Eq. (6.22) together with Eqs. (6.23–6.28) into Eq. (6.20), some new types of Jacobian elliptic function solutions of Eq. (6.17) can be obtained elegantly.

6.4 New Explicit Exact Solutions of Coupled Schrödinger–Boussinesq Equations

In this present analysis, an investigation has been made in searching the new generalized Jacobi elliptic function solutions for Eqs. (6.1) and (6.2) by using the proposed method discussed in Sect. 6.3.1. According to the technique discussed in the Algorithm of Sect. 6.3.1, we adopt the ansatz solutions of Eqs. (6.1) and (6.2) in the following forms

$$u(x,t) = U(x,t) = U(\xi)\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)}, \tag{6.29}$$

and

$$v(x,t) = V(x,t) = V(\xi), \tag{6.30}$$

respectively. Here, $\xi = x - 2kt + \eta_0$, where k and c are real constants to be evaluated later; and ζ_0 and η_0 are arbitrary constants.

Now, plugging Eqs. (6.29) and (6.30) into Eqs. (6.1) and (6.2) and then, integrating the second Eq. (6.2) of the coupled Schrödinger–Boussinesq equations twice with respect to ξ, we have

$$U_{\xi\xi} - (k^2 + c - \alpha)U - UV = 0, \tag{6.31}$$

$$V_{\xi\xi} - 12k^2V - 3V^2 - \beta V + U^2 = 0, \tag{6.32}$$

Balancing the highest derivative term $U_{\xi\xi}$ with the nonlinear term UV in Eq. (6.31) and the highest derivative term $V_{\xi\xi}$ with the nonlinear term U^2 in Eq. (6.32) leads to $M = N = 2$. Thus, the exact solutions of Eqs. (6.1) and (6.2) have the following forms:

$$U(\xi) = a_{10} + \sum_{i=1}^{2}\left[a_{1i}\phi^{i}(\xi) + b_{1i}\phi^{-i}(\xi) + c_{1i}\phi^{i-1}(\xi)\phi'(\xi) + d_{1i}\phi^{-i}(\xi)\phi'(\xi)\right], \tag{6.33}$$

$$V(\xi) = a_{20} + \sum_{j=1}^{2}\left[a_{2j}\phi^{j}(\xi) + b_{2j}\phi^{-j}(\xi) + c_{2j}\phi^{j-1}(\xi)\phi'(\xi) + d_{2j}\phi^{-j}(\xi)\phi'(\xi)\right]. \tag{6.34}$$

Now, substituting Eqs. (6.33) and (6.34) alongwith Eq. (6.16) into Eqs. (6.31) and (6.32), and then collecting all the coefficients of $\phi^{l}(\xi)$ $(l = 0, 1, 2, \ldots)$ and $\phi^{m}(\xi)\phi'(\xi)$ $(m = 0, 1, 2, \ldots)$, then equating these coefficients to zero, yield a set of over-determined algebraic equations for a_{10}, a_{1i}, b_{1i}, c_{1i}, d_{1i} $(i = 1, 2)$, a_{20}, a_{2j}, b_{2j}, c_{2j}, d_{2j} $(j = 1, 2)$, k, c. Using the *Mathematica* and the *Wu's elimination methods*, the algebraic equations have been solved and thus, the following results have been obtained.

Result 1:

$$a_{10} = 0, a_{11} = 0, a_{12} = 0, b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0, d_{11} = -\frac{4\sqrt{pr}}{\sqrt{q}}, d_{12} = 0;$$
$$a_{20} = 0, a_{21} = 0, a_{22} = 2p, b_{21} = 0, b_{22} = 2r, c_{21} = 0, c_{22} = 0, d_{21} = 0, d_{22} = 0;$$
$$k = -\frac{\sqrt{4q^2 + 8pr - \beta q}}{2\sqrt{3}\sqrt{q}} \text{ and } c = \frac{-4q^2 - 8pr + 12\alpha q + \beta q}{12q}.$$

Result 2:

$$a_{10} = 0, a_{11} = 0, a_{12} = 0, b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0, d_{11} = \frac{4\sqrt{pr}}{\sqrt{q}}, d_{12} = 0;$$
$$a_{20} = 0, a_{21} = 0, a_{22} = 2p, b_{21} = 0, b_{22} = 2r, c_{21} = 0, c_{22} = 0, d_{21} = 0, d_{22} = 0;$$
$$k = \frac{\sqrt{4q^2 + 8pr - \beta q}}{2\sqrt{3}\sqrt{q}} \text{ and } c = \frac{-4q^2 - 8pr + 12\alpha q + \beta q}{12q}.$$

Result 3:

$$a_{10} = 0, a_{11} = 0, a_{12} = 0, b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0, d_{11} = \frac{4\sqrt{pr}}{\sqrt{q}}, d_{12} = 0;$$
$$a_{20} = 0, a_{21} = 0, a_{22} = 2p, b_{21} = 0, b_{22} = 2r, c_{21} = 0, c_{22} = 0, d_{21} = 0, d_{22} = 0;$$
$$k = -\frac{\sqrt{4q^2 + 8pr - \beta q}}{2\sqrt{3}\sqrt{q}} \text{ and } c = \frac{-4q^2 - 8pr + 12\alpha q + \beta q}{12q}.$$

Result 4:

$$a_{10} = 0, a_{11} = 0, a_{12} = 0, b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0, d_{11} = -\frac{4\sqrt{pr}}{\sqrt{q}}, d_{12} = 0;$$
$$a_{20} = 0, a_{21} = 0, a_{22} = 2p, b_{21} = 0, b_{22} = 2r, c_{21} = 0, c_{22} = 0, d_{21} = 0, d_{22} = 0;$$
$$k = \frac{\sqrt{4q^2 + 8pr - \beta q}}{2\sqrt{3}\sqrt{q}} \text{ and } c = \frac{-4q^2 - 8pr + 12\alpha q + \beta q}{12q}.$$

Substituting the results obtained above into Eqs. (6.33) and (6.34) alongwith the Jacobi elliptic function solutions provided in Table 6.1, we can obtain following families of exact solutions to Eqs. (6.1) and (6.2).

Set 1:

$$a_{10} = 0, a_{11} = 0, a_{12} = 0, b_{11} = 0, b_{12} = 0, c_{11} = 0, c_{12} = 0, d_{11} = -\frac{4\sqrt{pr}}{\sqrt{q}}, d_{12} = 0;$$
$$a_{20} = 0, a_{21} = 0, a_{22} = 2p, b_{21} = 0, b_{22} = 2r, c_{21} = 0, c_{22} = 0, d_{21} = 0, d_{22} = 0;$$
$$k = -\frac{\sqrt{4q^2 + 8pr - \beta q}}{2\sqrt{3}\sqrt{q}} \text{ and } c = \frac{-4q^2 - 8pr + 12\alpha q + \beta q}{12q}.$$

Case I: If $p = -m^2$, $q = 2m^2 - 1$, $r = 1 - m^2$ and $\phi(\xi) = cn\xi$, then we get the following double periodic solutions in terms of Jacobi elliptic functions

$$u_{11}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = \frac{4\sqrt{m^2(m^2-1)}}{\sqrt{2m^2-1}} \frac{sn\xi dn\xi}{cn\xi} e^{i(kx+ct+\zeta_0)}, 1/2 < m^2 < 1,$$
$$v_{11}(x,t) = V(\xi) = -2m^2 cn^2\xi + 2(1-m^2)nc^2\xi,$$

where

$$\xi = x - 2kt + \eta_0, k = -\frac{\sqrt{4 + 24m^4 + \beta - 2m^2(12 + \beta)}}{2\sqrt{-3 + 6m^2}}, \text{ and}$$
$$c = -\frac{4 + 24m^4 + 12\alpha + \beta - 2m^2(12 + 12\alpha + \beta)}{-12 + 24m^2}.$$

Case II: If $p = 1 - m^2$, $q = 2m^2 - 1$, $r = -m^2$ and $\phi(\xi) = nc\xi$, then we get the following double periodic solutions in terms of Jacobi elliptic functions

$$u_{12}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = -\frac{4\sqrt{m^2(m^2-1)}}{\sqrt{2m^2-1}}\frac{sc\xi dc\xi}{nc\xi}e^{i(kx+ct+\zeta_0)}, 1/2 < m^2 < 1,$$
$$v_{12}(x,t) = V(\xi) = 2\left(1 - m^2\right)nc^2\xi - 2m^2 cn^2\xi,$$

where

$$\xi = x - 2kt + \eta_0, k = -\frac{\sqrt{4 + 24m^4 + \beta - 2m^2(12 + \beta)}}{2\sqrt{-3 + 6m^2}}, \text{ and}$$
$$c = -\frac{4 + 24m^4 + 12\alpha + \beta - 2m^2(12 + 12\alpha + \beta)}{-12 + 24m^2}.$$

Case III: If $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $\phi(\xi) = ns\xi \pm cs\xi$, then we get the following double periodic solutions

$$u_{13}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = \frac{2}{\sqrt{2-4m^2}}\frac{cs\xi ds\xi \pm ns\xi ds\xi}{ns\xi \pm cs\xi}e^{i(kx+ct+\zeta_0)}, m^2 < 1/2,$$
$$v_{13}(x,t) = V(\xi) = \frac{1}{2}(ns\xi \pm cs\xi)^2 + \frac{1}{2}(ns\xi \pm cs\xi)^{-2}$$

where

$$\xi = x - 2kt + \eta_0, k = -\frac{\sqrt{3 + 8m^4 + 2m^2(-4 + \beta) - \beta}}{2\sqrt{3 - 6m^2}}, \text{ and}$$
$$c = \frac{3 + 8m^4 - 12\alpha - \beta + 2m^2(-4 + 12\alpha + \beta)}{-12 + 24m^2}.$$

Case IV: If $p = \frac{1-m^2}{4}$, $q = \frac{1+m^2}{2}$, $r = \frac{1-m^2}{4}$ and $\phi(\xi) = nc\xi \pm sc\xi$, then we get the following Jacobi elliptic function solutions

$$u_{14}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = \mp\frac{\sqrt{2}(m^2-1)}{\sqrt{m^2+1}}dc\xi\, e^{i(kx+ct+\zeta_0)}, \quad 0<m<1,$$
$$v_{14}(x,t) = V(\xi) = \frac{1-m^2}{2}\left(\frac{cn\xi}{1\pm sn\xi}\right)^2 + \frac{1-m^2}{2}\left(\frac{cn\xi}{1\pm sn\xi}\right)^{-2},$$

where

$$\xi = x-2kt+\eta_0, k = -\frac{\sqrt{3+3m^4-m^2(-2+\beta)-\beta}}{2\sqrt{3}\sqrt{1+m^2}}, \text{and}$$
$$c = \frac{-3-3m^4+12\alpha+\beta+m^2(-2+12\alpha+\beta)}{12(1+m^2)}.$$

Case V: If $p=m^2$, $q=-(1+m^2)$, $r=1$ and $\phi(\xi)=sn\xi$, then we get the following Jacobi elliptic function solutions

$$u_{15}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = -\frac{4m}{\sqrt{-m^2-1}}\frac{cn\xi dn\xi}{sn\xi}e^{i(kx+ct+\zeta_0)}$$
$$v_{15}(x,t) = V(\xi) = 2m^2sn^2\xi + 2ns^2\xi,$$

where

$$\xi = x-2kt+\eta_0, k = -\frac{\sqrt{4+24m^4+\beta-2m^2(12+\beta)}}{2\sqrt{-3+6m^2}}, \text{ and}$$
$$c = -\frac{4+24m^4+12\alpha+\beta-2m^2(12+12\alpha+\beta)}{-12+24m^2}.$$

Case VI: If $p=1$, $q=-(1+m^2)$, $r=m^2$ and $\phi(\xi)=sn\xi$, then we get the following Jacobi elliptic function solutions

$$u_{16}(x,t) = U(\xi)e^{i(kx+ct+\zeta_0)} = \frac{4m}{\sqrt{-m^2-1}}\frac{cs\xi ds\xi}{ns\xi}e^{i(kx+ct+\zeta_0)},$$
$$v_{16}(x,t) = V(\xi) = 2ns^2\xi + 2m^2sn^2\xi,$$

where

$$\xi = x-2kt+\eta_0, k = -\frac{\sqrt{4+24m^4+\beta-2m^2(12+\beta)}}{2\sqrt{-3+6m^2}} \text{ and}$$
$$c = -\frac{4+24m^4+12\alpha+\beta-2m^2(12+12\alpha+\beta)}{-12+24m^2}.$$

Set 2:

$$a_{10}=0,a_{11}=0,a_{12}=0,b_{11}=0,b_{12}=0,c_{11}=0,c_{12}=0,d_{11}=\frac{4\sqrt{pr}}{\sqrt{q}},d_{12}=0;$$
$$a_{20}=0,a_{21}=0,a_{22}=2p,b_{21}=0,b_{22}=2r,c_{21}=0,c_{22}=0,d_{21}=0,d_{22}=0;$$
$$k=\frac{\sqrt{4q^2+8pr-\beta q}}{2\sqrt{3}\sqrt{q}}\ \text{and}\ c=\frac{-4q^2-8pr+12\alpha q+\beta q}{12q}.$$

Case I: If $p=-m^2$, $q=2m^2-1$, $r=1-m^2$ and $\phi(\xi)=cn\xi$, then we get the following double periodic solutions in terms of Jacobi elliptic functions

$$u_{21}(x,t)=U(\xi)e^{i(kx+ct+\zeta_0)}=-\frac{4\sqrt{m^2(m^2-1)}}{\sqrt{2m^2-1}}\frac{sn\xi dn\xi}{cn\xi}e^{i(kx+ct+\zeta_0)},1/2<m^2<1,$$
$$v_{21}(x,t)=V(\xi)=-2m^2cn^2\xi+2(1-m^2)nc^2\xi,$$

where

$$\xi=x-2kt+\eta_0,k=\frac{\sqrt{4+24m^4+\beta-2m^2(12+\beta)}}{2\sqrt{-3+6m^2}},\ \text{and}$$
$$c=-\frac{4+24m^4+12\alpha+\beta-2m^2(12+12\alpha+\beta)}{-12+24m^2}.$$

Case II: If $p=1-m^2$, $q=2m^2-1$, $r=-m^2$ and $\phi(\xi)=nc\xi$, then we get the following double periodic solutions in terms of Jacobi elliptic functions

$$u_{22}(x,t)=U(\xi)e^{i(kx+ct+\zeta_0)}=\frac{4\sqrt{m^2(m^2-1)}}{\sqrt{2m^2-1}}\frac{sc\xi dc\xi}{nc\xi}e^{i(kx+ct+\zeta_0)},1/2<m^2<1,$$
$$v_{22}(x,t)=V(\xi)=2(1-m^2)nc^2\xi-2m^2cn^2\xi,$$

where

$$\xi=x-2kt+\eta_0,k=\frac{\sqrt{4+24m^4+\beta-2m^2(12+\beta)}}{2\sqrt{-3+6m^2}},\ \text{and}$$
$$c=-\frac{4+24m^4+12\alpha+\beta-2m^2(12+12\alpha+\beta)}{-12+24m^2}.$$

Case III: If $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $\phi(\xi) = ns\xi \pm cs\xi$, then we get the following double periodic solutions

$$u_{23}(x,t) = U(\xi)\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)} = -\frac{2}{\sqrt{2-4m^2}}\frac{cs\xi ds\xi \pm ns\xi ds\xi}{ns\xi \pm cs\xi}\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)}, m^2 < 1/2,$$
$$v_{23}(x,t) = V(\xi) = \frac{1}{2}(ns\xi \pm cs\xi)^2 + \frac{1}{2}(ns\xi \pm cs\xi)^{-2},$$

where

$$\xi = x - 2kt + \eta_0, k = \frac{\sqrt{3+8m^4+2m^2(-4+\beta)-\beta}}{2\sqrt{3-6m^2}}, \text{ and}$$
$$c = \frac{3+8m^4-12\alpha-\beta+2m^2(-4+12\alpha+\beta)}{-12+24m^2}.$$

Case IV: If $p = \frac{1-m^2}{4}$, $q = \frac{1+m^2}{2}$, $r = \frac{1-m^2}{4}$ and $\phi(\xi) = nc\xi \pm sc\xi$, then we get the following Jacobi elliptic function solutions

$$u_{24}(x,t) = U(\xi)\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)} = \mp\frac{\sqrt{2}(m^2-1)}{\sqrt{m^2+1}}dc\xi\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)}, 0 < m < 1,$$
$$v_{24}(x,t) = V(\xi) = \frac{1-m^2}{2}\left(\frac{cn\xi}{1 \pm sn\xi}\right)^2 + \frac{1-m^2}{2}\left(\frac{cn\xi}{1 \pm sn\xi}\right)^{-2},$$

where

$$\xi = x - 2kt + \eta_0, k = \frac{\sqrt{3+3m^4-m^2(-2+\beta)-\beta}}{2\sqrt{3}\sqrt{1+m^2}}, \text{ and}$$
$$c = \frac{-3-3m^4+12\alpha+\beta+m^2(-2+12\alpha+\beta)}{12(1+m^2)}.$$

Case V: If $p = m^2$, $q = -(1+m^2)$, $r = 1$ and $\phi(\xi) = sn\xi$, then we get the following double periodic solutions in terms of Jacobi elliptic functions

$$u_{25}(x,t) = U(\xi)\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)} = \frac{4m}{\sqrt{-m^2-1}}\frac{cn\xi dn\xi}{sn\xi}\mathrm{e}^{\mathrm{i}(kx+ct+\zeta_0)},$$
$$v_{25}(x,t) = V(\xi) = 2m^2 sn^2\xi + 2ns^2\xi,$$

where

$$\xi = x - 2kt + \eta_0, k = \frac{\sqrt{4 + 4m^4 + \beta + m^2(16 + \beta)}}{2\sqrt{3}\sqrt{-1 - m^2}}, \text{ and}$$
$$c = \frac{4 + 4m^4 + 12\alpha + \beta + m^2(16 + 12\alpha + \beta)}{12(1 + m^2)}.$$

Case VI: If $p = 1$, $q = -(1 + m^2)$, $r = m^2$ and $\phi(\xi) = ns\xi$, then we get the following double periodic solutions

$$u_{26}(x,t) = U(\xi)\mathrm{e}^{\mathrm{i}(kx + ct + \zeta_0)} = -\frac{4m}{\sqrt{-m^2 - 1}}\frac{cs\xi ds\xi}{ns\xi}\mathrm{e}^{\mathrm{i}(kx + ct + \zeta_0)},$$
$$v_{26}(x,t) = V(\xi) = 2ns^2\xi + 2m^2sn^2\xi,$$

where

$$\xi = x - 2kt + \eta_0, k = \frac{\sqrt{4 + 4m^4 + \beta + m^2(16 + \beta)}}{2\sqrt{3}\sqrt{-1 - m^2}}, \text{ and}$$
$$c = \frac{4 + 4m^4 + 12\alpha + \beta + m^2(16 + 12\alpha + \beta)}{12(1 + m^2)}.$$

Similarly, as the established solutions for **Set 1 and Set 2**, we can construct corresponding exact solutions to Eqs. (6.1) and (6.2) for **Set 3 and Set 4**, which are omitted here.

6.4.1 Numerical Simulations for the Solutions of Coupled Schrödinger–Boussinesq Equations

In the present analysis, the first solutions of **Case IV** of **Set 1** have been used for drawing the solution graphs Figs. 6.1 and 6.2 for coupled Schrödinger–Boussinesq equations.

Again, the solutions of **Case V** of **Set 2** have been used for drawing the solution graphs Figs. 6.3 and 6.4 for coupled Schrödinger–Boussinesq equations.

In the present numerical simulations, the double periodic wave solutions for the first solutions of $u_{14}(x,t)$ and $v_{14}(x,t)$ have been demonstrated in 3D graphs of Figs. 6.1 and 6.2 with elliptic modulus $m = 0.5$. Also, the double periodic wave solutions for $u_{25}(x,t)$ and $v_{25}(x,t)$ have been demonstrated in 3D graphs of Figs. 6.3 and 6.4 with elliptic modulus $m = 0.5$.

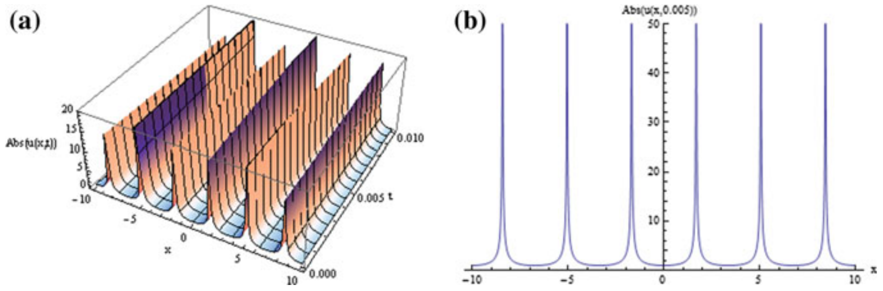

Fig. 6.1 **a** Double periodic wave solutions for the first solution of $u_{14}(x,t)$ when $\alpha = 1$, $\beta = -1$, $\zeta_0 = 0$, $\xi_0 = 0$, and $m = 0.5$, and **b** the corresponding 2D solution graph when $t = 0.005$

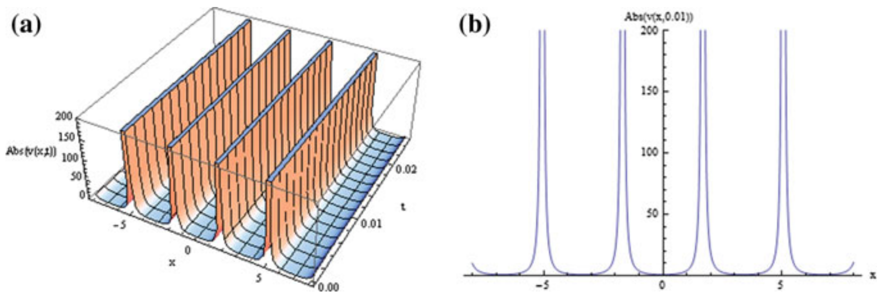

Fig. 6.2 **a** Double periodic wave solutions for the first solution of $v_{14}(x,t)$ when $\alpha = 1$, $\beta = -1$, $\zeta_0 = 0$, $\xi_0 = 0$, and $m = 0.5$, and **b** the corresponding 2D solution graph when $t = 0.01$

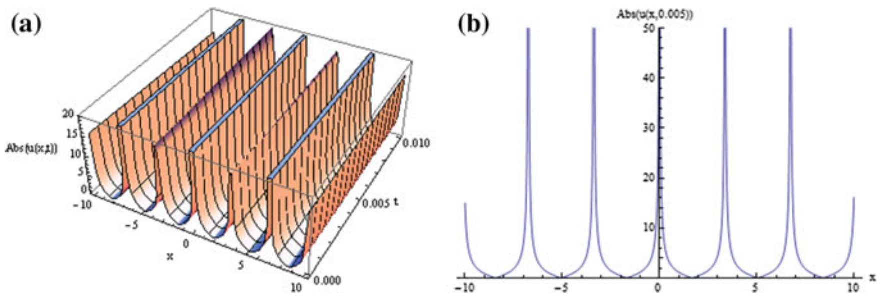

Fig. 6.3 **a** Double periodic wave solutions for $u_{25}(x,t)$ when $\alpha = 1$, $\beta = -1$, $\zeta_0 = 0$, $\xi_0 = 0$, and $m = 0.5$, and **b** the corresponding 2D solution graph when $t = 0.005$

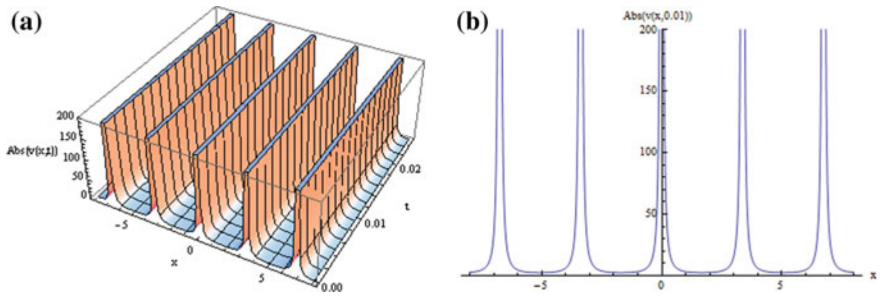

Fig. 6.4 **a** Double periodic wave solutions for $v_{25}(x,t)$ when $\alpha = 1$, $\beta = -1$, $\zeta_0 = 0$, $\xi_0 = 0$, and $m = 0.5$, and **b** the corresponding 2D solution graph when $t = 0.01$

6.5 Implementation of New Extended Auxiliary Equation Method to the Tzitzéica-Type Nonlinear Evolution Equations

In the present section, the Jacobi elliptic function solutions, including the hyperbolic and trigonometric solutions for the DBM and TDB equations have been obtained using a new extended auxiliary equation method.

6.5.1 New Exact Solutions of Dodd–Bullough–Mikhailov (DBM) Equation

In this part, we apply the new extended auxiliary equation method to determine the new exact solutions for Dodd–Bullough–Mikhailov Eq. (6.3).

Suppose the traveling wave solution of Eq. (6.5) can be expressed as

$$U(\xi) = v(\xi) = \sum_{i=0}^{2N} a_i F^i(\xi), \tag{6.35}$$

where $F(\xi)$ satisfies Eq. (6.21).

Balancing the highest order derivative term $vv_{\xi\xi}$ and the nonlinear term v^3 by using homogenous principle the following result could be obtained

$$N + N + 2 = 3N,$$

yielding

$$N = 2.$$

Therefore, the ansatz for the solution of Eq. (6.5) can be written as

$$U(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi) + a_3 F^3(\xi) + a_4 F^4(\xi), \tag{6.36}$$

where $F(\xi)$ satisfies

$$F(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}(1 \pm \phi_i(\xi))\right]^{1/2}, \; i = 1, 2, \ldots, 12. \tag{6.37}$$

By substituting (6.36) and (6.21) into Eq. (6.5), the coefficients of each power of F^i, $i = 0, 1, 2\ldots$ are collected, which are then set to zero. Thus, it leads to a system of algebraic equations.

The derived system of algebraic equations has been solved by using mathematical software, yielding the following results:

$$a_0 = \frac{2^{1/3}3^{1/6} - 2^{1/3}3^{2/3}}{2}, a_2 = -22^{1/6}3^{1/12}\sqrt{a_4},$$
$$c_2 = \frac{2^{5/6}\sqrt{a_4}\left(96 \times 3^{5/12}c_0 + 11 \times 3^{11/12}c_0\right)}{156}, c_4 = -\frac{2}{13}\left(4 \times 2^{2/3}3^{1/3}a_4c_0 + 2^{2/3}3^{5/6}a_4c_0\right),$$
$$c_6 = \frac{1}{26}a_4^{3/2}\left(4\sqrt{2}3^{1/4}c_0 + \sqrt{2}3^{3/4}c_0\right), \omega = \frac{3\sqrt{2}3^{1/4} - 4\sqrt{2}3^{3/4}}{24kl^2},$$

where $l = a_4^{1/4}\sqrt{c_0}$.

Without loss of generality, let us assume $a_4 > 0$ and $c_0 > 0$, and hence $c_6 > 0$. Thus, $\phi(\xi)$ satisfies only the functions (6.23), (6.24), and (6.27).

Set I:

From Eqs. (6.23), (6.36), and (6.37), the Jacobi elliptic function solutions of Eq. (6.5) have been deduced as follows

$$U_{11}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm sn\left(\frac{2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}}{m}\xi\right)\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm sn\left(\frac{2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}}{m}\xi\right)\right)^2, \tag{6.38}$$

$$U_{12}(\xi) = \frac{1}{2^{2/3}}\left(3^{1/6} - 3^{2/3}\right) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{msn\left(\frac{2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}}{m}\xi\right)}\right) + \left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{msn\left(\frac{2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}}{m}\xi\right)}\right)^2, \quad (6.39)$$

where $\xi = kx + \frac{3\sqrt{2}3^{1/4} - 4\sqrt{2}3^{3/4}}{24kl^2}t$ and $l = a_4^{1/4}\sqrt{c_0}$.

If $m \to 1$, then $sn(\xi) \to \tanh(\xi)$, and we have the hyperbolic function solutions of Eq. (6.5)

$$U_{13}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \tanh\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right) + \left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \tanh\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right)^2, \quad (6.40)$$

$$U_{14}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \coth\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right) + \left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \coth\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right)^2, \quad (6.41)$$

Set II:

From Eqs. (6.24), (6.36), and (6.37), the Jacobi elliptic function solutions of Eq. (6.5) have been obtained as follows

$$U_{21}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm msn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right) + \left(2^{1/6}3^{1/12}\right)^2\left(1 \pm msn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}(4+\sqrt{3})}\xi\right)\right)^2, \quad (6.42)$$

$$U_{22}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{sn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)}\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{sn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)}\right)^2. \tag{6.43}$$

If $m \to 0$, then $sn(\xi) \to \sin(\xi)$, and we have the following trigonometric function solutions of Eq. (6.5)

$$U_{23}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - \left(2^{1/6}3^{1/12}\right)^2$$
$$U_{24}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \csc\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \csc\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)^2. \tag{6.44}$$

If $m \to 1$, then we have the same hyperbolic function solutions (6.40) and (6.41).

Set III:

From Eqs. (6.27), (6.36) and (6.37), the Jacobi elliptic function solutions of Eq. (6.5) have been derived as follows

$$U_{31}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{cn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{1}{cn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}\right)^2, \tag{6.45}$$

$$U_{32}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{dn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}{\sqrt{1-m^2}sn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \frac{dn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}{\sqrt{1-m^2}sn\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13(1-m^2)}}(4+\sqrt{3})\xi\right)}\right)^2. \tag{6.46}$$

If $m \to 0$, then $dn(\xi) \to 1$, $sn(\xi) \to \sin(\xi)$, $cn(\xi) \to \cos(\xi)$, and hence, the following trigonometric solutions of Eq. (6.5) have been obtained

$$U_{33}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \sec\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \sec\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)^2, \tag{6.47}$$

$$U_{34}(\xi) = \frac{1}{2^{2/3}}(3^{1/6} - 3^{2/3}) - 2\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \csc\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)$$
$$+\left(2^{1/6}3^{1/12}\right)^2\left(1 \pm \csc\left(2^{\frac{11}{12}}3^{\frac{5}{24}}l\sqrt{\frac{1}{13}}(4+\sqrt{3})\xi\right)\right)^2. \tag{6.48}$$

It may be noted that the solution (6.44) is in agreement with the solution (6.48).

6.5.2 *New Exact Solutions of Tzitzeica–Dodd–Bullough (TDB) Equation*

Suppose the traveling wave solution of Eq. (6.8) can be expressed as

$$\Psi(\xi) = v(\xi) = \sum_{i=0}^{2N} a_i F^i(\xi), \tag{6.49}$$

where $F(\xi)$ satisfies Eq. (6.21).

Balancing the highest order derivative term $vv_{\xi\xi}$ and the nonlinear term v^4 by using homogenous principle the following result could be obtained

$$N + N + 2 = 4N,$$

yielding

$$N = 1.$$

Therefore, the ansatz for the solution of Eq. (6.8) can be written as

$$\Psi(\xi) = a_0 + a_1 F(\xi) + a_2 F^2(\xi), \tag{6.50}$$

where $F(\xi)$ satisfies

$$F(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}(1 \pm \phi_i(\xi))\right]^{1/2}, \; i = 1, 2, \ldots, 12. \tag{6.51}$$

Substituting (6.50) and (6.21) into Eq. (6.8) and collecting the coefficients of each power of F^i, $i = 0, 1, 2\ldots$ and set them to zero, we obtain a system of algebraic equations.

Solving this system of algebraic equations by using mathematical software, we obtain the following result:

$$c_2 = \frac{(4+5a_0)a_2c_0}{2a_0(1+a_0)}, c_4 = \frac{(1+2a_0)a_2^2c_0}{a_0^2(1+a_0)}, c_6 = \frac{a_2^3c_0}{2a_0^2(1+a_0)}, \omega = \frac{-a_0^2 - a_0^3}{2ka_2c_0}.$$

Without loss of generality, let us assume $a_0 > 0$, $a_2 > 0$ and $c_0 > 0$, and hence $c_6 > 0$. Thus, $\phi(\xi)$ satisfies only the functions (6.23), (6.24), and (6.27).

Set I:

From Eqs. (6.23), (6.50), and (6.51), the following Jacobi elliptic function solutions of Eq. (6.8) have been derived.

$$\Psi_{11}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm sn\left(\sqrt{\frac{l(1+2a_0)^2}{2m^2a_0^2(1+a_0)}}\xi\right)\right), \tag{6.52}$$

$$\Psi_{12}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \frac{1}{msn\left(\sqrt{\frac{l(1+2a_0)^2}{2m^2a_0^2(1+a_0)}}\xi\right)}\right), \tag{6.53}$$

where $\xi = kx + \frac{-a_0^2 - a_0^3}{2ka_2c_0}t$ and $l = a_2c_0$.

If $m \to 1$, then $sn(\xi) \to \tanh(\xi)$, and we have the hyperbolic function solutions of Eq. (6.8)

$$\Psi_{13}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \tanh\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right), \tag{6.54}$$

$$\Psi_{14}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \coth\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right). \tag{6.55}$$

Set II:

From Eqs. (6.24), (6.50) and (6.51), the following Jacobi elliptic function solutions of Eq. (6.8) have been obtained.

$$\Psi_{21}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm msn\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right), \tag{6.56}$$

$$\Psi_{22}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \frac{1}{sn\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)}\right). \tag{6.57}$$

If $m \to 0$, then $sn(\xi) \to \sin(\xi)$, and we have the following solutions of Eq. (6.8)

$$\Psi_{23}(\xi) = -\frac{1}{2}, \tag{6.58}$$

$$\Psi_{24}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \csc\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right). \tag{6.59}$$

If $m \to 1$, then we can obtain the same hyperbolic function solutions (6.54) and (6.55).

Set III:

From Eqs. (6.27), (6.50), and (6.51), the following Jacobi elliptic function solutions of Eq. (6.8) have been derived.

$$\Psi_{31}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \frac{1}{cn\left(\sqrt{\frac{l(1+2a_0)^2}{2(1-m^2)a_0^2(1+a_0)}}\xi\right)}\right), \tag{6.60}$$

$$\Psi_{32}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \frac{dn\left(\sqrt{\frac{l(1+2a_0)^2}{2(1-m^2)a_0^2(1+a_0)}}\xi\right)}{\sqrt{1-m^2}sn\left(\sqrt{\frac{l(1+2a_0)^2}{2(1-m^2)a_0^2(1+a_0)}}\xi\right)}\right). \quad (6.61)$$

If $m \to 0$, then $dn(\xi) \to 1, sn(\xi) \to \sin(\xi)$, $cn(\xi) \to \cos(\xi)$, and hence, the following trigonometric solutions of Eq. (6.8) have been obtained

$$\Psi_{33}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \sec\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right), \quad (6.62)$$

$$\Psi_{34}(\xi) = a_0 - \frac{(1+2a_0)}{2}\left(1 \pm \csc\left(\sqrt{\frac{l(1+2a_0)^2}{2a_0^2(1+a_0)}}\xi\right)\right). \quad (6.63)$$

It may be noted that the solution (6.59) is in agreement with the solution (6.63).

6.5.3 Physical Interpretations of the Solutions

In the present analysis, three-dimensional and the corresponding two-dimensional graphs of the obtained solutions to the nonlinear evolution equations, viz. Dodd–Bullough–Mikhailov (DBM) and Tzitzeica–Dodd–Bullough (TDB) equations have been presented. To this aim, some special values of the parameters are selected. Here, the physical significance of the obtained solutions of the above equations has been discussed.

In Figs. 6.5, 6.6, 6.7, 6.8, 6.9, 6.10, 6.11 and 6.12, the 3D solution graphs of $U_{11}(\xi)$, $U_{13}(\xi)$, $U_{21}(\xi)$, $U_{34}(\xi)$, $\Psi_{11}(\xi)$, $\Psi_{13}(\xi)$, $\Psi_{21}(\xi)$, $\Psi_{34}(\xi)$, respectively, have been presented with appropriate selection of parameters. The three-dimensional

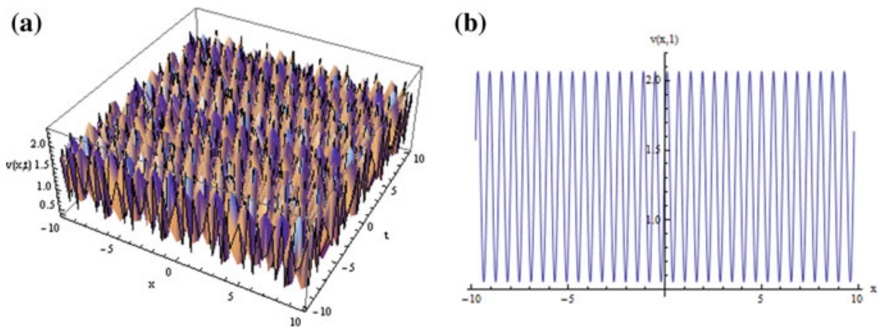

Fig. 6.5 **a** 3D double periodic solution surface for $v(x, t)$ appears in Eq. (6.38) as $U_{11}(\xi)$ in Set 1, when $k = 1$, $l = 1$, $\omega = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

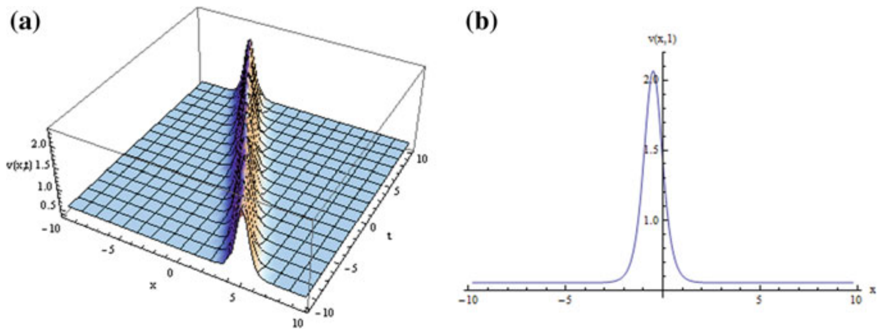

Fig. 6.6 **a** 3D soliton solution surface of $v(x, t)$ appears in Eq. (6.40) as $U_{13}(\xi)$ in Set 1, when $k = 1$, $l = 1$, $\omega = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

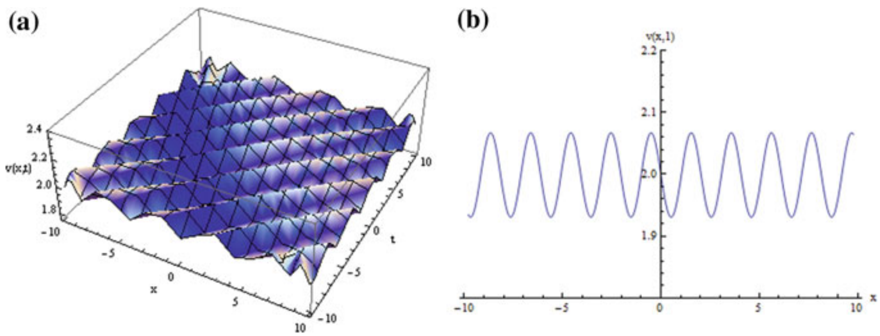

Fig. 6.7 **a** 3D double periodic solution surface of $v(x, t)$ appears in Eq. (6.42) as $U_{21}(\xi)$ in Set 2, when $k = 1$, $l = 1$, $\omega = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

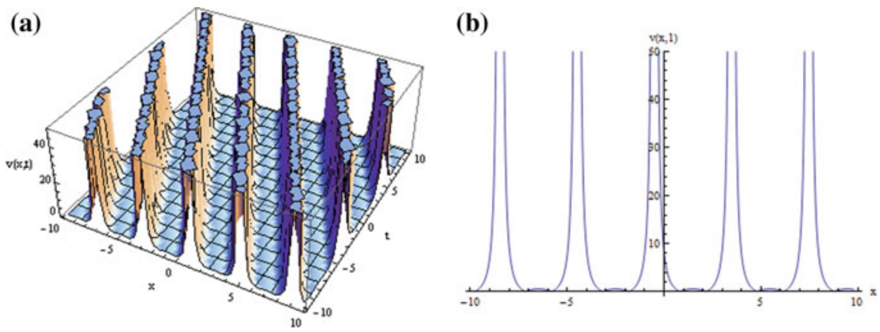

Fig. 6.8 **a** 3D periodic solution surface of $v(x, t)$ appears in Eq. (6.48) as $U_{34}(\xi)$ in Set 3, when $k = 1$, $l = 0.5$, $\omega = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

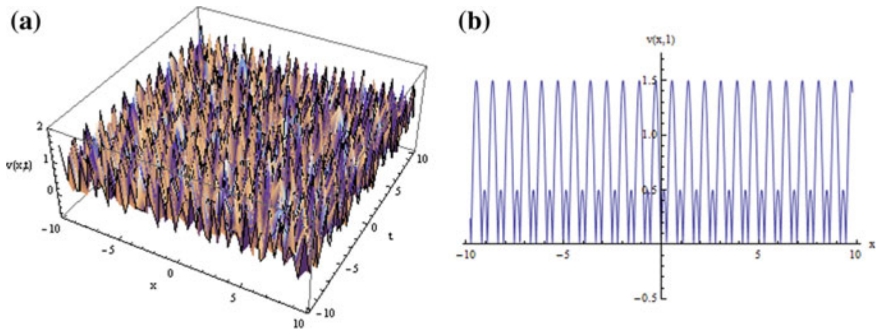

Fig. 6.9 **a** 3D double periodic solution surface of $v(x, t)$ appears in Eq. (6.52) as $\Psi_{11}(\xi)$ in Set I, when $k = 1$, $l = 1$, $\omega = 0.5$, $m = 0.3$, $a_0 = 0.5$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

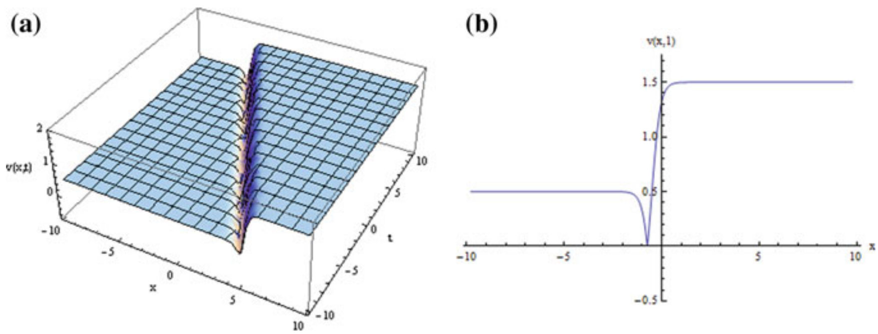

Fig. 6.10 **a** 3D soliton solution surface of $v(x, t)$ appears in Eq. (6.54) as $\Psi_{13}(\xi)$ in Set I, when $k = 1$, $l = 1$, $\omega = 0.5$, $a_0 = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

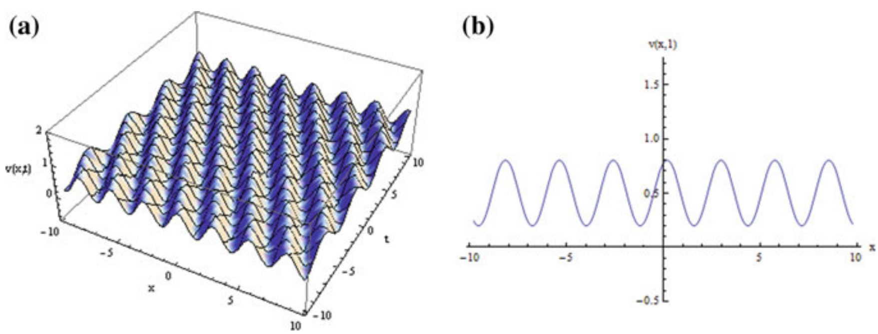

Fig. 6.11 **a** 3D double periodic solution surface of $v(x, t)$ appears in Eq. (6.56) as $\Psi_{21}(\xi)$ in Set II, when $k = 1$, $l = 1$, $\omega = 0.5$, $a_0 = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

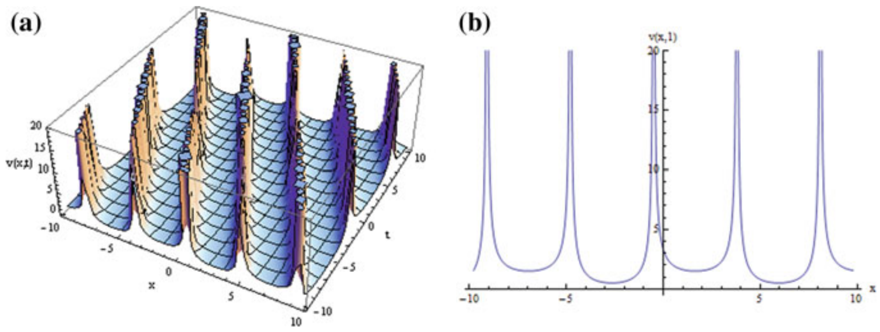

Fig. 6.12 **a** 3D periodic solution surface of $v(x, t)$ appears in Eq. (6.63) as $\Psi_{34}(\xi)$ in Set III, when $k = 1$, $l = 0.1$, $\omega = 0.5$, $a_0 = 0.5$, $m = 0.3$, **b** the corresponding 2D graph for $v(x, t)$, when $t = 1$

graphs of Figs. 6.5, 6.6, 6.7, 6.8, 6.9, 6.10, 6.11 and 6.12 have been depicted when $-10 \leq x \leq 10$, $-10 \leq t \leq 10$. To the best knowledge of information, these solutions have not been reported earlier in the open literature.

In Figs. 6.5 and 6.7, the double periodic solutions for U_{11} and U_{21} of DBM equation, have been displayed. Also, the double periodic solutions for Ψ_{11} and Ψ_{21} of TDB equation have been demonstrated in Figs. 6.9 and 6.11, respectively. Figures 6.6 and 6.10 show the solutions for U_{13} and Ψ_{13} representing the soliton wave solutions of DBM and TDB equations, respectively. Furthermore, the periodic traveling wave solutions for U_{34} and Ψ_{34} of DBM and TDB equations have been illustrated in Figs. 6.8 and 6.12, respectively.

6.6 Conclusion

In this chapter, an improved generalized Jacobi elliptic function method is successfully employed for acquiring new exact solutions of the coupled Schrödinger–Boussinesq equations. By using this present method, some new exact solutions of the coupled Schrödinger–Boussinesq equations are found. More importantly, the present method is more efficient and powerful to determine the new exact solutions to CSBEs. This proposed method can also be utilized for numerous other nonlinear evolution equations or coupled ones. To the best information of the author, these double periodic wave solutions of the CSBEs are new exact solutions which are not reported earlier. Being concise and powerful, this current method can also be extended to solve many other NLPDEs arising in mathematical physics.

Moreover, in the present chapter, a new extended auxiliary equation method is used to construct many new types of Jacobi elliptic function solutions of Dodd–Bullough–Mikhailov and Tzitzeica–Dodd–Bullough equations. Thus, as an achievement, a family of new exact traveling wave solutions of Dodd–Bullough–Mikhailov and Tzitzeica–Dodd–Bullough equations has been formally generated.

It clearly manifests that the employed approach is useful and efficient to find the various kinds of traveling wave solutions. Also, the physical interpretations of the obtained results for Tzitzéica-type nonlinear evolution equations have been surveyed as well. Therefore, the performance of the proposed method is effective and it can be applied to study many other nonlinear evolution equations which frequently arise in nonlinear optics, quantum theory, and other mathematical physics and engineering problems.

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