Chapter 5
New Exact Solutions of Fractional-Order Partial Differential Equations

5.1 Introduction

Fractional differential equations (FDEs) have been used nowadays frequently in various applications for modeling anomalous diffusion, heat transfer, seismic wave analysis, signal processing, sound wave propagation, and many other fractional dynamical systems [1–6]. The FDEs are used in modeling many problems in physics and engineering. The fractional derivatives introduced in physical models can describe sound attenuation in complex media. When introduced into the constitutive equations, they build a wave equation in which attenuation obeys a frequency power law characteristic of many media [7].

The last few decades have witnessed rapid development in novel diagnostic and therapeutic applications of ultrasound in biology and medicine. Nonlinear ultrasound modeling has become gradually important for accurate evaluation and simulation of ultrasound in a number of purposes. Ultrasound beams in the therapeutic modalities are finite amplitude in nature. Accurate nonlinear ultrasound models and their competent applications are required for accurate modeling and simulation of those models of ultrasound applications. Additionally, accurate and efficient exact solutions of nonlinear ultrasound models will significantly help us in order to understand the complicated physical phenomena of ultrasound and the associated bioeffects. The main motivation of this work is to develop the exact solutions of fractional-order nonlinear acoustic wave equations.

The study of numerous approximations to the Burgers–Hopf equations in (5.1) has a prominent history concerning the symbiotic interaction of mathematical model and scientific computing to gain insight into the topic.

The propagation of focused and intense ultrasound beams is accompanied by nonlinearity, diffraction, and absorption. For modeling of nonlinear propagation of ultrasound beams in soft tissue, among others, the combined effects of nonlinearity, absorption, and diffraction must be taken into consideration. The description of large amplitude ultrasonic beams requires an accurate representation of

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nonlinearity, absorption, and diffraction. One of the extensively used nonlinear models for the propagation of diffractive ultrasound in dissipative media is the Khokhlov–Zabolotskaya–Kuznetsov (KZK) nonlinear acoustic wave equation [8, 9]. The Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation is a nonlinear beam equation that has been used to model nonlinear wave propagation in therapeutic ultrasound.

Recently, a considerable number of research works have been rendered by the notable researchers to develop the solutions of fractional partial differential equations, fractional ordinary differential equations, and integral equations of physical interest. The fractional differential equations can be described best in discontinuous media, and the fractional order is equivalent to its fractional dimensions. Fractal media which are complex appear in different fields of engineering and physics. In this context, the local fractional calculus theory is very important for modeling problems for fractal mathematics and engineering on Cantorian space in fractal media. Several analytical and numerical methods have been proposed to attain exact and approximate solutions of fractional differential equations [10–22].

With the help of fractional complex transform via the local fractional derivatives, fractional differential equations can be converted into integer-order ordinary differential equations. The fractional complex transform is used to change fractal time-space to continuous time-space. The first integral method [23–27] can be devised to establish the exact solutions for some time fractional differential equations. The present work focuses on the first time the applicability and efficacy of the first integral method on fractional nonlinear acoustic wave equations. To the best information of the author, the exact analytical solutions for the above nonlinear fractional-order acoustic wave equations have been obtained first time ever in this chapter.

In recent years, fractional calculus has played a very important role in various applications for modeling anomalous diffusion, heat transfer, seismic wave analysis, signal processing, control theory, image processing, and many other fractional dynamical systems [1–6]. Fractional differential equations (FDEs) are the generalization of classical differential equations of integer order. The FDEs are inherently multidisciplinary with its application across diverse disciplines of applied science and engineering. Recently, FDEs have attracted great interest due to their applications in various real physical problems. The descriptions of properties of several physical phenomena are found to be best described by fractional differential equations. For this purpose, a reliable and efficient technique is essential for the solution of nonlinear fractional differential equations. In this connection, it is worthwhile to mention the recent notable works on the solutions of fractional differential equations, integral equations, and fractional partial differential equations of physical interest. Several analytical and numerical methods have been employed to develop approximate and exact solutions of fractional differential equations [10, 12–14, 16, 17, 19–22, 28, 29].

The sound propagation in a fluid is determined by nonlinearity, diffraction, absorption, and dispersion. For modeling of nonlinear sound propagation in fluid, the combined effects of nonlinearity, absorption, dispersion, and diffraction should be taken into account. The description of sound propagation in fluid requires an accurate representation of nonlinearity, dispersion, absorption, and diffraction.

The KdV-Khokhlov–Zabolotskaya–Kuznetsov (KdV-KZK) equation describes all the basic physical mechanisms of sound propagation in fluids [30]. The KdV-KZK equation for fluids has profound applications in aerodynamics, acoustics, and also its extension to solids has applications in biomedical engineering and in nonlinear acoustical nondestructive testing.

Nonlinear FDEs can be transformed into integer-order nonlinear ordinary differential equations via fractional complex transform with the help of modified Riemann–Liouville fractional derivative and corresponding useful formulae. The present methods [31–36] under study can be devised to develop the exact analytical solutions for time fractional KdV-KZK equation. The main motivation of this work is to develop the exact solutions of the fractional-order KdV-KZK equation. To the best information of the author, the exact analytical solutions for the fractional KdV-KZK equation have been reported first time ever in this chapter.

In recent decades, FDEs have attracted increasing attention as they are widely used to describe various complex phenomena in many fields [1, 37–41], such as the fluid dynamics, acoustic dissipation, geophysics, relaxation, creep, viscoelasticity, rheology, chaos, control theory, economics, signal and image processing, systems identification, biology, and other areas. Most of the classical mechanic techniques have been used in studies of conservative systems, but most of the processes observed in the physical real world are nonconservative. If the Lagrangian of a conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. In view of the fact that most physical phenomena may be considered as nonconservative, they can be described using fractional-order differential equations. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations rather than integer-order equations [39].

In particular, the fractional derivative is useful in describing the memory and hereditary properties of materials and processes. The fractional differential equations can be described best in discontinuous media, and the fractional order is equivalent to its fractional dimensions. Fractal media which are complex appear in different fields of engineering and physics. In this context, the local fractional calculus theory is very important for modeling problems for fractal mathematics and engineering on Cantorian space in fractal media. Among the investigations for fractional differential equations, finding numerical and exact solutions to fractional differential equations is a prior matter of concern. Many efficient methods have been proposed so far to obtain numerical solutions and exact solutions of fractional differential equations. Most nonlinear physical phenomena that appear in many areas of scientific fields, such as plasma physics, solid state physics, fluid dynamics, optical fibers, mathematical biology, and chemical kinetics, can be best modeled by nonlinear fractional partial differential equations.

With the help of fractional complex transform via the local fractional derivatives, fractional differential equations can be converted into integer-order ordinary differential equations. The fractional complex transform is used to change fractal time-space to continuous time-space.

In this chapter, we present the traveling wave solutions of the fractional (2 + 1)-dimensional Davey–Stewartson equation and doubly periodic solutions of new integrable Davey–Stewartson-type equation. We employ the mixed dn-sn method [42] approach via fractional complex transform in order to obtain exact solutions to the fractional (2 + 1)-dimensional Davey–Stewartson equation and the new integrable Davey–Stewartson-type equation.

5.2 Outline of the Present Study

In this chapter, new exact solutions of fractional nonlinear acoustic wave equations have been devised. The traveling periodic wave solutions of fractional Burgers–Hopf equation and Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation have obtained by the first integral method. Nonlinear ultrasound modeling is found to have an increasing number of applications in both medical and industrial areas where due to high-pressure amplitudes the effects of nonlinear propagation are no longer negligible. Taking nonlinear effects into account, the ultrasound beam analysis makes more accurate in these applications. The Burgers–Hopf equation is one of the extensively studied models in mathematical physics. In addition, the KZK parabolic nonlinear wave equation is one of the most widely employed nonlinear models for the propagation of 3D diffraction sound beams in dissipative media. In the present chapter, these nonlinear equations have solved by the first integral method. As a result, new exact analytical solutions have been obtained first time ever for these fractional-order acoustic wave equations. The obtained results are presented graphically to demonstrate the efficiency of this proposed method.

Also in this chapter, new exact solutions of time fractional KdV-Khokhlov–Zabolotskaya–Kuznetsov (KdV-KZK) equation have been established by classical Kudryashov method and modified Kudryashov method, respectively. In this purpose, modified Riemann–Liouville derivative has been applied to convert nonlinear time fractional KdV-KZK equation into the nonlinear ordinary differential equation. In the present chapter, the classical Kudryashov method and modified Kudryashov method both have been applied successively to compute the analytical solutions of time fractional KdV-KZK equation. As a result, new exact solutions have been obtained first time ever involving symmetric Fibonacci function, hyperbolic function, and exponential function. The methods under consideration are reliable, efficient and can be used as an alternative to establish new exact solutions of different types of fractional differential equations arising in mathematical physics. The obtained results are exhibited graphically in order to demonstrate the efficiency and applicability of these proposed methods for solving nonlinear time fractional KdV-KZK equation.

Moreover, the Jacobi elliptic function method, viz. mixed dn-sn method, has been presented in this chapter for finding the traveling wave solutions of the Davey–Stewartson equations. As a result, some solitary wave solutions and doubly periodic solutions are obtained in terms of Jacobi elliptic functions. Furthermore, solitary wave solutions are obtained as simple limits of doubly periodic functions. These solutions can be useful to explain some physical phenomena, viz. evolution of a three-dimensional wave packet on the water of finite depth. The proposed Jacobi elliptic function method is efficient, powerful and can be used in order to establish more newly exact solutions for other kinds of nonlinear fractional partial differential equations arising in mathematical physics.

5.2.1 Time Fractional Nonlinear Acoustic Wave Equations

Let us consider the time fractional Burgers–Hopf equation [43]

$$\partial_z p = \gamma D_\tau^{2\alpha} p + \beta D_\tau^{\alpha} p^2 \tag{5.1}$$

and the (3 + 1)-dimensional time fractional Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation [44–46]

$$\partial_z D_\tau^{\alpha} p = \frac{c_0}{2}\Delta_\perp p + \gamma D_\tau^{3\alpha} p + \beta D_\tau^{2\alpha} p^2 \tag{5.2}$$

where $0 < \alpha \leq 1$, $\gamma = \frac{D}{2c_0^3}$, and $\beta = \frac{\tilde{\beta}}{2\rho_0 c_0^3}$. Here, p is the acoustic pressure, z is the direction of propagation, $\tau = t - \frac{z}{c_0}$ is the retarded time variable, c_0 is the small signal speed of sound, D is the diffusivity parameter, and ρ_0 is the ambient fluid density.

The first term on the right-hand side of Eq. (5.2) represents diffraction. The second term accounts for thermoviscous attenuation as with Burgers' equation and nonlinearity is described in the third term. The coefficient of nonlinearity $\tilde{\beta}$ is defined by $\tilde{\beta} = 1 + B/2A$, where B/A is the nonlinearity parameter of the medium. The transverse Laplacian can be written in Cartesian coordinates as

$$\Delta_\perp p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \tag{5.3}$$

The Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation is an augmented type of Burgers' equation. In addition to absorption and nonlinearity, it is also involved with diffraction. This last term allows the KZK equation to describe three-dimensional directional nonlinear sound beams; the form generated through the ultrasonic transducer. The nonlinear parabolic KZK wave equation describes the effects of diffraction, absorption, and nonlinearity.

5.2.2 *Time Fractional KdV-Khokhlov–Zabolotskaya–Kuznetsov Equation*

Let us consider the (3 + 1)-dimensional time fractional KdV-KZK equation

$$\partial_z D_\tau^\alpha p = \frac{c_0}{2}\Delta_\perp p + A_1 D_\tau^{3\alpha} p + A_2 D_\tau^{2\alpha} p^2 - \gamma D_\tau^{4\alpha} p \tag{5.4}$$

where $0 < \alpha \leq 1$, $A_1 = \frac{b}{2c_0^3\rho_0}$, and $A_2 = \frac{\varepsilon}{2\rho_0 c_0^3}$. Here, p is the acoustic pressure, z is the direction of sound propagation, $\tau = t - \frac{z}{c_0}$ is the retarded time variable, c_0 is the small signal speed of sound, ε is the parameter of nonlinearity, b is the diffusivity parameter, ρ_0 is the ambient fluid density, and γ is the adiabatic index defined by $\gamma = c_p/c_v$, where c_p and c_v are the specific heats at constant pressure and constant volume.

The first term on the right-hand side of Eq. (5.4) represents diffraction. The second term accounts for thermoviscous attenuation as with Burgers' equation and nonlinearity is described in the third term. In comparison to KdV–Burgers equation, the KdV-KZK equation has only one extra term. The diffusivity parameter b is defined by $b = \zeta + 4\eta/3$, where ζ and η are the bulk and shear viscosity. The transverse Laplacian can be written in Cartesian coordinates as

$$\Delta_\perp p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \tag{5.5}$$

The KdV-KZK equation is an augmented form of the KdV–Burgers equation. In addition to absorption, dispersion, and nonlinearity, it also accounts for diffraction. The nonlinear parabolic KdV-KZK equation describes the combined effects of diffraction, absorption, dispersion, and nonlinearity.

5.2.3 *Time Fractional (2 + 1)-Dimensional Davey–Stewartson Equations*

Davey–Stewartson (DS) equations have been used for various applications in fluid dynamics. It was proposed initially for the evolution of weakly nonlinear pockets of water waves in the finite depth by Davey and Stewartson [47].

Time Fractional (2 + 1)-Dimensional Davey–Stewartson Equation (Type I)

Let us consider the fractional (2 + 1)-dimensional Davey–Stewartson equation [48]

$$iD_t^{\alpha}q + a(D_x^{2\beta}q + D_y^{2\gamma}q) + b|q|^{2n}q - \lambda qr = 0, \quad (5.6)$$

$$D_x^{2\beta}r + D_y^{2\gamma}r + \delta D_x^{2\beta}\left(|q|^{2n}\right) = 0, \quad (5.7)$$

where $0 < \alpha, \beta, \gamma \leq 1$, $q \equiv q(x, y, t)$, and $r \equiv r(x, y, t)$. Also, a, b, λ, and δ are all constant coefficients. The exponent n is the power law parameter. It is necessary to have $n > 0$. In Eqs. (5.6) and (5.7), $q(x, y, t)$ is a complex-valued function which stands for wave amplitude, while $r(x, y, t)$ is a real-valued function which stands for mean flow. This system of equations is completely integrable and is often used to describe the long-time evolution of a two-dimensional wave packet [49–51].

Time Fractional (2 + 1)-Dimensional New Integrable Davey–Stewartson-Type Equation (Type II)

Let us consider the fractional (2 + 1)-dimensional new integrable Davey–Stewartson-type equation

$$\begin{gathered} iD_\tau^{\alpha}\Psi + L_1\Psi + \Psi\,\Phi + \Psi\chi = 0, \\ L_2\chi = L_3|\Psi|^2, \\ D_\xi^{\beta}\Phi = D_\eta^{\gamma}\chi + \mu D_\eta^{\gamma}\left(|\Psi|^2\right), \quad \mu = \mp 1,\ 0 < \alpha, \beta, \gamma \leq 1 \end{gathered} \quad (5.8)$$

where the linear differential operators are given by

$$\begin{aligned} L_1 &\equiv \left(\tfrac{b^2 - a^2}{4}\right)D_\xi^{2\beta} - aD_\xi^{\beta}D_\eta^{\gamma} - D_\eta^{2\gamma}, \\ L_2 &\equiv \left(\tfrac{b^2 + a^2}{4}\right)D_\xi^{2\beta} + aD_\xi^{\beta}D_\eta^{\gamma} + D_\eta^{2\gamma}, \\ L_3 &\equiv \pm\tfrac{1}{4}\left(b^2 + a^2 + \tfrac{8b^2(a-1)}{(a-2)^2 - b^2}\right)D_\xi^{2\beta} \pm \left(a + \tfrac{2b^2}{(a-2)^2 - b^2}\right)D_\xi^{\beta}D_\eta^{\gamma} \pm D_\eta^{2\gamma}, \end{aligned}$$

where $\Psi \equiv \Psi(\xi, \eta, \tau)$ is complex, while $\Phi \equiv \Phi(\xi, \eta, \tau)$, $\chi \equiv \chi(\xi, \eta, \tau)$ are real and a, b are real parameters. The above equation in integer order was devised firstly by Maccari [52] from the Konopelchenko–Dubrovsky (KD) equation [53].

5.3 Algorithm of the First Integral Method with Fractional Complex Transform

In this section, we deal with the explicit solutions of Eqs. (5.1) and (5.2) by using the first integral method [54]. The main steps of this method are described as follows:

Step 1: Suppose that a nonlinear FPDE, say in four independent variables x, y, z, and t, is given by

$$P(u, u_x, u_{xx}, u_y, u_{yy}, u_z, u_t, D_t^{\alpha}u, D_t^{2\alpha}u, D_t^{3\alpha}u, \partial_z D_t^{\alpha}u, \ldots) = 0,\ 0 < \alpha \le 1 \tag{5.9}$$

where $u = u(x, y, z, t)$ is an unknown function, P is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: By using the fractional complex transform [55–58]:

$$u(x, y, z, t) = \Phi(\xi),\ \xi = lx + my + kz + \frac{\lambda t^{\alpha}}{\Gamma(\alpha + 1)} \tag{5.10}$$

where l, m, k, and λ are constants.

By using the chain rule [55, 58], we have

$$\begin{aligned} D_t^{\alpha}u &= \sigma_t u_{\xi} D_t^{\alpha}\xi, \\ D_x^{\alpha}u &= \sigma_x u_{\xi} D_x^{\alpha}\xi, \\ D_y^{\alpha}u &= \sigma_y u_{\xi} D_y^{\alpha}\xi, \\ D_z^{\alpha}u &= \sigma_z u_{\xi} D_z^{\alpha}\xi, \end{aligned}$$

where σ_t, σ_x, σ_y, and σ_z are the fractal indexes [57, 58], without loss of generality we can take $\sigma_t = \sigma_x = \sigma_y = \sigma_z = \kappa$, where κ is a constant.

Thus, the FPDE (5.9) is transformed to the following ordinary differential equation (ODE) for $u(x, y, z, t) = \Phi(\xi)$:

$$P(\Phi, \lambda\Phi', \lambda^2\Phi'', \lambda^3\Phi''', l\Phi', l^2\Phi'', m\Phi', m^2\Phi'', \ldots, k\lambda\Phi'', \ldots) = 0, \tag{5.11}$$

where prime denotes the derivative with respect to ξ.

Step 3: We suppose that Eq. (5.11) has a solution in the form

$$\Phi(\xi) = X(\xi) \tag{5.12}$$

and introduce a new independent variable $Y(\xi) = \Phi_{\xi}(\xi)$, which leads to a system of ODEs of the form

$$\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi} = Y(\xi), \tag{5.13}$$

$$\frac{\mathrm{d}Y(\xi)}{\mathrm{d}\xi} = F(X(\xi), Y(\xi)).$$

In general, it is very difficult to solve a two-dimensional autonomous planar system of ODEs, such as Eq. (5.13).

Step 4: By using the qualitative theory of differential equations [59], if we can find the integrals to Eq. (5.13) under the same conditions, then the general solutions to Eq. (5.13) can be derived directly. With the aid of the division theorem for two variables in the complex domain **C** which is based on Hilbert's Nullstellensatz theorem [60], one first integral to Eq. (5.13) can be obtained. This first integral can reduce Eq. (5.11) to a first-order integrable ordinary differential equation. Then by solving this equation directly, the exact solution to Eq. (5.9) is obtained.

Now, let us recall the division theorem.

Theorem 5.1 (Division theorem)
Let $Q(x,y)$ and $R(x,y)$ are polynomials in $\boldsymbol{C}[[x,y]]$, and $Q(x,y)$ is irreducible in $\boldsymbol{C}[[x,y]]$. If $R(x,y)$ vanishes at all zero points of $Q(x,y)$, then there exists a polynomial $H(x,y)$ in $\boldsymbol{C}[[x,y]]$ such that

$$R(x,y) = Q(x,y)H(x,y). \tag{5.14}$$

5.4 Algorithm of the Kudryashov Methods Applied with Fractional Complex Transform

In this section, an algorithm has been presented for the analytical solutions of Eq. (5.4) by using both the classical Kudryashov method and modified Kudryashov method [31, 34, 35]. The main steps of this method are described as follows:

Step 1: Suppose that a nonlinear FPDE, say in four independent variables x, y, z, and t, is given by

$$P(u, u_x, u_{xx}, u_y, u_{yy}, u_z, u_t, D_t^{\alpha}u, D_t^{2\alpha}u, D_t^{3\alpha}u, \partial_z D_t^{\alpha}u, \ldots) = 0, \; 0 < \alpha \leq 1 \tag{5.15}$$

where $D_t^{\alpha}u$, $D_t^{2\alpha}u$ and $D_t^{3\alpha}u$ are modified Riemann–Liouville derivatives of u, where $u = u(x,y,z,t)$ is an unknown function, P is a polynomial in u, and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: By using the fractional complex transform [55, 56]:

$$u(x,y,z,t) = U(\xi) \quad \xi = lx + my + kz + \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} \tag{5.16}$$

where l, m, k, and λ are constants.

By using the chain rule [55, 58], we have

$$
\begin{aligned}
D_t^{\alpha}u &= \sigma_t u_{\xi} D_t^{\alpha}\xi,\\
D_x^{\alpha}u &= \sigma_x u_{\xi} D_x^{\alpha}\xi,\\
D_y^{\alpha}u &= \sigma_y u_{\xi} D_y^{\alpha}\xi,\\
D_z^{\alpha}u &= \sigma_z u_{\xi} D_z^{\alpha}\xi,
\end{aligned}
$$

where σ_t, σ_x, σ_y, and σ_z are the fractal indexes [57, 58], without loss of generality we can take $\sigma_t = \sigma_x = \sigma_y = \sigma_z = \kappa$, where κ is a constant.

Thus, the FPDE (5.15) is reduced to the following nonlinear ordinary differential equation (ODE) for $u(x, y, z, t) = U(\xi)$:

$$P(U, \lambda U', \lambda^2 U'', \lambda^3 U''', lU', l^2 U'', mU', m^2 U'', \ldots, k\lambda U'', \ldots) = 0. \tag{5.17}$$

Step 3: We assume that the exact solution of Eq. (5.17) can be expressed in the following form

$$U(\xi) = \sum_{i=0}^{N} a_i Q^i(\xi), \tag{5.18}$$

where a_i $(i = 0, 1, 2, \ldots, N)$ are constants to be determined later, such that $a_N \neq 0$, while $Q(\xi)$ has the following form

I. **Classical Kudryashov method**

$$Q(\xi) = \frac{1}{1 + \exp(\xi)}. \tag{5.19}$$

This function $Q(\xi)$ satisfies the first-order differential equation

$$Q_{\xi}(\xi) = Q(\xi)(Q(\xi) - 1). \tag{5.20}$$

II. **Modified Kudryashov method**

$$Q(\xi) = \frac{1}{1 \pm a^{\xi}}. \tag{5.21}$$

This function satisfies the first-order differential equation

$$Q_{\xi}(\xi) = Q(\xi)(Q(\xi) - 1)\ln a. \tag{5.22}$$

Step 4: To determine the dominant term with the highest order of singularity, we substitute

$$U = \xi^{-p}, \tag{5.23}$$

to all terms of Eq. (5.17). Then, the degrees of all terms of Eq. (5.17) are compared, and consequently two or more terms with the lowest degree are chosen. The maximum value of p is the pole of Eq. (5.17), and it is equal to N. This method can be employed when N is integer. If N is noninteger, the equation under study needs to be transformed, and then, the above procedure to be repeated.

Step 5: The necessary number of derivatives of the function $U(\xi)$ with respect to ξ can be calculated using the computer algebra systems of any mathematical software.

Step 6: Substituting the derivatives of function $U(\xi)$ along with Eq. (5.18) in Eq. (5.17) in case of classical Kudryashov method or substituting the derivatives of function $U(\xi)$ along with Eq. (5.18) in Eq. (5.17) in case of modified Kudryashov method, Eq. (5.17) becomes the following form

$$\Phi[Q(\xi)] = 0, \tag{5.24}$$

where $\Phi[Q(\xi)]$ is a polynomial in $Q(\xi)$. Then, after collecting all terms with the same powers of $Q(\xi)$ and equating every coefficient of this polynomial to zero yield a set of algebraic equations for $a_i (i = 0,1,2,\ldots, N)$ and λ.

Step 7: Solving the algebraic equations system thus obtained in step 6 and subsequently substituting these values of the constants $a_i (i = 0, 1, 2,\ldots, N)$ and λ, we can obtain the explicit exact solutions of Eq. (5.4) instantly. The obtained solutions may involve in the symmetric hyperbolic Fibonacci functions [61, 62]. The symmetric Fibonacci sine, cosine, tangent, and cotangent functions are, respectively, defined as follows:

$$sFs(x) = \frac{a^x - a^{-x}}{\sqrt{5}}, \qquad cFs(x) = \frac{a^x + a^{-x}}{\sqrt{5}}$$

$$\tan Fs(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}}, \qquad \cot Fs(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}.$$

5.5 Algorithm of the Mixed Dn-Sn Method with Fractional Complex Transform

In this present analysis, we deal with the determination of explicit solutions of fractional (2 + 1)-dimensional Davey–Stewartson equation by using the mixed dn-sn method. The main steps of this method are described as follows:

Step 1: Suppose that coupled nonlinear FPDEs, say in three independent variables *x*, *y*, and *t*, is given by

$$F(u, v, u_x, v_x, u_y, v_y, u_t, v_t, iD_t^{\alpha}u, D_t^{\alpha}v, D_x^{2\beta}u, D_x^{2\beta}v, D_y^{2\gamma}u, D_y^{2\gamma}v, \ldots) = 0,\ 0 < \alpha, \beta, \gamma \le 1 \tag{5.25a}$$

$$G(u, v, u_x, v_x, u_y, v_y, u_t, v_t, D_t^{\alpha}u, D_t^{\alpha}v, D_x^{2\beta}u, D_x^{2\beta}v, D_y^{2\gamma}u, D_y^{2\gamma}v, \ldots) = 0,\ 0 < \alpha, \beta, \gamma \le 1 \tag{5.25b}$$

where $u = u(x, y, t)$ and $v = v(x, y, t)$ are unknown functions, F and G are polynomials in u, v, and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: We use the fractional complex transform [55–58]:

$$u(x, y, t) = e^{i\theta}u(\xi), \quad v(x, y, t) = v(\xi),$$

$$\theta = \frac{\theta_1 x^{\beta}}{\Gamma(1+\beta)} + \frac{\theta_2 y^{\gamma}}{\Gamma(1+\gamma)} + \frac{\theta_3 t^{\alpha}}{\Gamma(1+\alpha)} \text{ and } \xi = \frac{\xi_1 x^{\beta}}{\Gamma(1+\beta)} + \frac{\xi_2 y^{\gamma}}{\Gamma(1+\gamma)} + \frac{\xi_3 t^{\alpha}}{\Gamma(1+\alpha)}, \tag{5.26}$$

where θ_1, θ_2, θ_3, ξ_1, ξ_2, and ξ_3 are real constants to be determined later.

By using the chain rule [55, 58], we have

$$D_t^{\alpha}u = \sigma_t u_{\xi} D_t^{\alpha}\xi,$$
$$D_x^{\alpha}u = \sigma_x u_{\xi} D_x^{\alpha}\xi,$$
$$D_y^{\alpha}u = \sigma_y u_{\xi} D_y^{\alpha}\xi,$$

where σ_t, σ_x, and σ_y are the fractal indexes [57, 58], without loss of generality we can take $\sigma_t = \sigma_x = \sigma_y = \kappa$, where κ is a constant.

Using fractional complex transform Eq. (5.26), the FPDE (5.25) can be converted to couple nonlinear ordinary differential equations (ODEs) involving $\Phi(\xi) = u(x, y, t)$ and $\Psi(\xi) = v(x, y, t)$. Then eliminating $\Psi(\xi)$ between the resultant coupled ODEs, the following ODE for $\Phi(\xi)$ is obtained

$$F(\Phi, \theta_3\Phi', \theta_3^2\Phi'', \theta_3^3\Phi''', \xi_3\Phi', \xi_3^2\Phi'', \xi_3^3\Phi, \ldots) = 0, \tag{5.27}$$

where prime denotes the derivative with respect to ξ.

Step 3: Let us assume that the exact solution of Eq. (5.27) is to be defined in the polynomial $\phi(\xi)$ of the following form:

$$\Phi(\xi) = \sum_{i=0}^{N} c_i \phi^i(\xi) + \sqrt{k^2 - \phi^2(\xi)} \sum_{i=0}^{N-1} d_i \phi^i(\xi), \quad (5.28)$$

where $\phi(\xi)$ satisfies the following elliptic equation:

$$\phi_\xi = \sqrt{(k^2 - \phi^2)(\phi^2 - k^2(1-m))}. \quad (5.29)$$

The solutions of Eq. (5.29) are given by

$$\phi(\xi) = kdn(k\xi|m),$$

$$\phi(\xi) = k\sqrt{1-m}nd(k\xi|m), \quad (5.30)$$

where $dn(k\xi|m)$ and $nd(k\xi|m) = \frac{1}{dn(k\xi|m)}$ are the Jacobi elliptic functions with modulus m $(0 < m < 1)$.

If $\phi(\xi) = kdn(k\xi|m)$, then Eq. (5.28) becomes

$$\Phi(\xi) = \sum_{i=0}^{N} c_i k^i dn^i(k\xi|m) + k\sqrt{m}sn(k\xi|m) \sum_{i=0}^{N-1} d_i k^i dn^i(k\xi|m),$$

while if $\phi(\xi) = k\sqrt{1-m}nd(k\xi|m)$, then Eq. (5.28) becomes

$$\Phi(\xi) = \sum_{i=0}^{N} c_i k^i (1-m)^{i/2} nd^i(k\xi|m) + k\sqrt{m}cd(k\xi|m) \sum_{i=0}^{N-1} d_i k^i (1-m)^{i/2} nd^i(k\xi|m),$$

where $cd(k\xi|m) = cn((k\xi|m)/dn(k\xi|m)$ and cn is the Jacobi cnoidal function. If $d_i = 0$, $i = 0, 1, 2, \ldots, N-1$, then Eq. (5.28) constitutes the *dn* (or *nd*) expansions.

Step 4: According to the proposed method, we substitute $\Phi(\xi) = \xi^{-p}$ in all terms of Eq. (5.27) for determining the highest order singularity. Then, the degree of all terms of Eq. (5.27) has been taken into the study, and consequently, the two or more terms of lower degree are chosen. The maximum value of p is known as the pole and it is denoted as "*N*." If "*N*" is an integer, then the method only can be implemented, and otherwise if "*N*" is a noninteger, the above Eq. (5.27) may be transferred and the above procedure is to be repeated.

Step 5: Substituting Eq. (5.28) into Eq. (5.27) yields the following algebraic equation

$$P(\phi) + \sqrt{k^2 - \phi^2} Q(\phi) = 0, \quad (5.31)$$

where $P(\phi)$ and $Q(\phi)$ are the polynomials in $\phi(\xi)$. Setting the coefficients of the various powers of ϕ in $P(\phi)$ and $Q(\phi)$ to zero will yield a system of algebraic equations in the unknowns c_i, d_i, k, and m. Solving this system, we can determine the value of these unknowns. Therefore, we can obtain several classes of exact solutions involving the Jacobi elliptic functions *sn*, *dn*, *nd,* and *cd* functions.

The Jacobi elliptic functions $sn(k\xi|m)$, $cn(k\xi|m)$, and $dn(k\xi|m)$ are double periodic and have the following properties:

$$sn^2(k\xi|m) + cn^2(k\xi|m) = 1,$$

$$dn^2(k\xi|m) + msn^2(k\xi|m) = 1.$$

Especially when $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e.,

$$\begin{aligned} sn(k\xi|1) &\to \tanh(k\xi), \\ cn(k\xi|1) &\to \sec h(k\xi), \\ dn(k\xi|1) &\to \sec h(k\xi), \end{aligned}$$

and when $m \to 0$, the Jacobi elliptic functions degenerate to the trigonometric functions, i.e.,

$$\begin{aligned} sn(k\xi|0) &\to \sin(k\xi), \\ cn(k\xi|0) &\to \cos(k\xi), \\ dn(k\xi|0) &\to 1. \end{aligned}$$

Further explanations in detail about the Jacobi elliptic functions can be found in [63].

5.6 Implementation of the First Integral Method for Time Fractional Nonlinear Acoustic Wave Equations

In this section, the new exact analytical solutions of time fractional nonlinear acoustic wave equations have been obtained first time ever using the first integral method.

5.6.1 The Burgers–Hopf Equation

In the present analysis, we introduce the following fractional complex transform in Eq. (5.1):

$$p(z,\tau) = \Phi(\xi),\ \xi = kz + \frac{\lambda\tau^{\alpha}}{\Gamma(\alpha+1)} \tag{5.32}$$

where k and λ are constants.

By applying the fractional complex transform (5.32), Eq. (5.1) can be transformed to the following nonlinear ODE:

$$k\Phi'(\xi) = \gamma\lambda^2\Phi''(\xi) + 2\lambda\beta\Phi(\xi)\Phi'(\xi). \tag{5.33}$$

Using Eqs. (5.12), (5.13), and (5.33) can be written as the following two-dimensional autonomous system

$$\frac{\mathrm{d}X(\xi)}{\mathrm{d}\xi} = Y(\xi), \tag{5.34}$$

$$\frac{\mathrm{d}Y(\xi)}{\mathrm{d}\xi} = \frac{k}{\lambda^2\gamma}Y(\xi) - \frac{2\beta}{\lambda\gamma}X(\xi)Y(\xi).$$

According to the first integral method, we assume that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of Eq. (5.34) and

$$Q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$$

is an irreducible polynomial in the complex domain $\boldsymbol{C}[X,Y]$ such that

$$Q[X(\xi),Y(\xi)] = \sum_{i=0}^{m} a_i(X(\xi))Y(\xi)^i = 0, \tag{5.35}$$

where $a_i(X(\xi)),\ \ i = 0,1,2,\ldots,m$ are polynomials in X and $a_m(X) \neq 0$. Equation (5.35) is called the first integral to Eq. (5.34). Applying the division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $\boldsymbol{C}[X,Y]$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\xi} = \frac{\partial Q}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\partial Q}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i. \tag{5.36}$$

Let us suppose that $m = 1$ in Eq. (5.35), and then by equating the coefficients of Y^i, $i = 0,1$ on both sides of Eq. (5.36), we have

$$Y^0 : \ a_0(X)g(X) = 0 \tag{5.37}$$

$$Y^1 : \dot{a}_0(X) + a_1(X)\left(\frac{k}{\lambda^2\gamma} - \frac{2\beta X}{\lambda\gamma}\right) = a_0(X)h(X) + a_1(X)g(X) \tag{5.38}$$

$$Y^2 : \dot{a}_1(X) = a_1(X)h(X) \tag{5.39}$$

Since, $a_i(X)$, $i = 0, 1$ are polynomials in X, from Eq. (5.39) we infer that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Then balancing the degrees of $g(X)$ and $a_0(X)$ in Eq. (5.38), we conclude that $\deg(g(X)) = 1$ only. Now suppose that

$$g(X) = b_1X + b_0,\ a_0(X) = \frac{A_2}{2}X^2 + A_1X + A_0, \quad (b_1 \neq 0, A_2 \neq 0) \tag{5.40}$$

where b_1, b_0, A_2, A_1, and A_0 are all constants to be determined. Using Eq. (5.38), we find that

$$b_0 = A_1 + \frac{k}{\lambda^2\gamma},$$
$$b_1 = A_2 - \frac{2\beta}{\lambda\gamma}.$$

Next, substituting $a_0(X)$ and $g(X)$ in Eq. (5.37) and consequently equating the coefficients of X^i, $i = 0, 1, 2, 3$ to zero, we obtain the following system of nonlinear algebraic equations:

$$X^0 : A_0\left(A_1 + \frac{k}{\lambda^2\gamma}\right) = 0 \tag{5.41}$$

$$X^1 : A_0\left(A_2 - \frac{2\beta}{\lambda\gamma}\right) + A_1\left(A_1 + \frac{k}{\lambda^2\gamma}\right) = 0, \tag{5.42}$$

$$X^2 : A_1\left(A_2 - \frac{2\beta}{\lambda\gamma}\right) + \frac{A_2}{2}\left(A_1 + \frac{k}{\lambda^2\gamma}\right) = 0, \tag{5.43}$$

$$X^3 : \frac{A_2}{2}\left(A_2 - \frac{2\beta}{\lambda\gamma}\right) = 0. \tag{5.44}$$

Solving the above system of Eqs. (5.41)–(5.44) simultaneously, we get the following nontrivial solution

$$A_0 = 0,\ A_1 = -\frac{k}{\lambda^2\gamma},\ A_2 = \frac{2\beta}{\lambda\gamma}, \tag{5.45}$$

Using Eqs. (5.45) into Eq. (5.35), we obtain

$$Y(\xi) = -\frac{\beta}{\lambda\gamma}X^2 + \frac{k}{\lambda^2\gamma}X. \tag{5.46}$$

Combining Eq. (5.46) with the system given by Eq. (5.34), the exact solution to Eq. (5.33) can be obtained as

$$p(z,\tau) = X(\xi) = \frac{k}{\beta\lambda + \cosh\left(\frac{k\xi}{\lambda^2\gamma} - kC_1\right) - \sinh\left(\frac{k\xi}{\lambda^2\gamma} - kC_1\right)}, \tag{5.47}$$

where C_1 is an arbitrary constant.

5.6.2 *The Khokhlov–Zabolotskaya–Kuznetsov Equation*

First, we introduce the following fractional complex transform in Eq. (5.2):

$$p(x,y,z,\tau) = \Phi(\xi),\ \xi = lx + my + kz + \frac{\lambda\tau^\alpha}{\Gamma(\alpha+1)} \tag{5.48}$$

where l, m, k, and λ are constants.

By applying the fractional complex transform (5.48), Eq. (5.2) can be transferred to the following nonlinear ODE:

$$k\lambda\Phi''(\xi) = \frac{c_0}{2}(l^2+m^2)\Phi''(\xi) + \gamma\lambda^3\Phi'''(\xi) + 2\lambda^2\beta(\Phi(\xi)\Phi''(\xi) + \Phi'(\xi)^2). \tag{5.49}$$

Then integrating Eq. (5.49) once, we obtain

$$\tilde{\xi}_0 + k\lambda\Phi'(\xi) = \frac{c_0}{2}(l^2+m^2)\Phi'(\xi) + \gamma\lambda^3\Phi''(\xi) + \lambda^2\beta(\Phi^2(\xi))', \tag{5.50}$$

where $\tilde{\xi}_0 = \lambda^3\gamma\xi_0$ is an integration constant.

Using Eqs. (5.12), (5.13), and (5.50) can be written as the following two-dimensional autonomous system

$$\frac{dX(\xi)}{d\xi} = Y(\xi), \tag{5.51}$$

$$\frac{dY(\xi)}{d\xi} = \xi_0 + \frac{k}{\lambda^2\gamma}Y(\xi) - \frac{c_0}{2}\frac{(l^2+m^2)}{\lambda^3\gamma}Y(\xi) - \frac{2\beta}{\lambda\gamma}X(\xi)Y(\xi).$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are the nontrivial solutions of Eq. (5.51) and

$$Q(X, Y) = \sum_{i=0}^{m} a_i(X) Y^i$$

is an irreducible polynomial in the complex domain $\boldsymbol{C}[X, Y]$ such that

$$Q[X(\xi), Y(\xi)] = \sum_{i=0}^{m} a_i(X(\xi)) Y(\xi)^i = 0, \tag{5.52}$$

where $a_i(X(\xi))$, $i = 0, 1, 2, \ldots, m$ are polynomials in X and $a_m(X) \neq 0$. Equation (5.52) is called the first integral to Eq. (5.51). Applying the division theorem, there exists a polynomial $g(X) + h(X)Y$ in the complex domain $\boldsymbol{C}[X, Y]$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\xi} = \frac{\partial Q}{\partial X}\frac{\mathrm{d}X}{\mathrm{d}\xi} + \frac{\partial Q}{\partial Y}\frac{\mathrm{d}Y}{\mathrm{d}\xi} = (g(X) + h(X)Y) \sum_{i=0}^{m} a_i(X) Y^i. \tag{5.53}$$

Let us suppose that $m = 1$ in Eq. (5.52), and then by equating the coefficients of Y^i, $i = 0, 1$ on both sides of Eq. (5.53), we have

$$Y^0 : \; a_1(X)\xi_0 = a_0(X) g(X), \tag{5.54}$$

$$Y^1 : \; \dot{a}_0(X) = a_0(X) h(X) - a_1(X)\left(-\frac{c_0}{2}\frac{(l^2 + m^2)}{\lambda^3 \gamma} + \frac{k}{\lambda^2 \gamma} - \frac{2\beta}{\lambda\gamma} X\right) + a_1(X) g(X), \tag{5.55}$$

$$Y^2 : \; \dot{a}_1(X) = a_1(X) h(X), \tag{5.56}$$

Since $a_i(X)$, $i = 0, 1$ are polynomials in X, from Eq. (5.56) we infer that $a_1(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_1(X) = 1$. Then balancing the degrees of $a_0(X)$ and $g(X)$, Eq. (5.55) implies that $\deg(g(X)) \leq \deg(a_0(X))$, and thus from Eq. (5.55), we infer that $\deg(g(X)) = 0$ or 1. If $\deg(g(X)) = 0$, suppose that $g(X) = A$, then from Eq. (5.55), we find

$$\dot{a}_0(X) = A + \frac{c_0}{2}\frac{(l^2 + m^2)}{\lambda^3 \gamma} - \frac{k}{\lambda^2 \gamma} + \frac{2\beta}{\lambda\gamma} X. \tag{5.57}$$

Solving Eq. (5.57), we have

$$a_0(X) = AX + \frac{c_0}{2}\frac{(l^2+m^2)}{\lambda^3\gamma}X - \frac{k}{\lambda^2\gamma}X + \frac{\beta}{\lambda\gamma}X^2 + B, \tag{5.58}$$

where B is an arbitrary constant.

Next, replacing $a_0(X)$, $a_1(X)$, and $g(X)$ in Eq. (5.54) and consequently equating the coefficients of X^i, $i = 0, 1, 2$ to zero, we obtain the following system of nonlinear algebraic equations:

$$X^0 : AB = \xi_0 \tag{5.59}$$

$$X^1 : A^2 + \frac{c_0}{2}\frac{(l^2+m^2)}{\lambda^3\gamma}A - \frac{k}{\lambda^2\gamma}A = 0 \tag{5.60}$$

$$X^2 : \frac{\beta}{\lambda\gamma}A = 0 \tag{5.61}$$

Solving the above system of Eqs. (5.59)–(5.61) simultaneously, we get

$$A = 0. \tag{5.62}$$

Using Eqs. (5.62) into Eq. (5.52), we obtain

$$Y(\xi) = -\frac{c_0}{2}\frac{(l^2+m^2)}{\lambda^3\gamma}X + \frac{k}{\lambda^2\gamma}X - \frac{\beta}{\lambda\gamma}X^2 - B. \tag{5.63}$$

Combining Eq. (5.63) with the system given by Eq. (5.51), the exact solution to Eq. (5.50) can be obtained as

$$p(x, y, z, \tau) = X(\xi) = \frac{-1}{4\lambda^2\beta}\left(c_0(l^2+m^2) - 2k\lambda + \sqrt{\eta}\tan\left(\frac{\sqrt{\eta}}{4\lambda^3\gamma}\left(\xi - 2\lambda^3\gamma C_1\right)\right)\right), \tag{5.64}$$

where $\eta = -c_0^2(l^2+m^2)^2 + 4c_0k\lambda(l^2+m^2) - 4\lambda^2(k^2 - 4B\beta\gamma\lambda^3)$ and C_1 is an arbitrary constant.

The established solutions (5.63) and (5.64) have been checked by putting them into the original Eqs. (5.1) and (5.2). Thus, the new exact solutions (5.63) and (5.64) of fractional Burgers–Hopf and KZK equations, respectively, have been first time obtained in this present work.

5.6.3 Numerical Results and Discussions for Nonlinear Fractional Acoustic Wave Equations

In this present numerical experiment, two exact solutions of Eqs. (5.1) and (5.2) have been used to draw the graphs as shown in Figs. 5.1, 5.2, 5.3, and 5.4 for different fractional-order values of α.

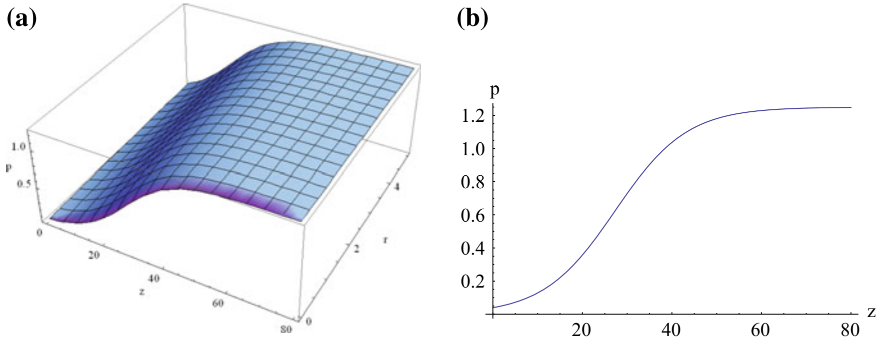

Fig. 5.1 **a** The periodic traveling wave solution for $p(z,\tau)$ appears in Eq. (5.47) of **Case I**, **b** corresponding solution for $p(z,\tau)$, when $\tau = 0$

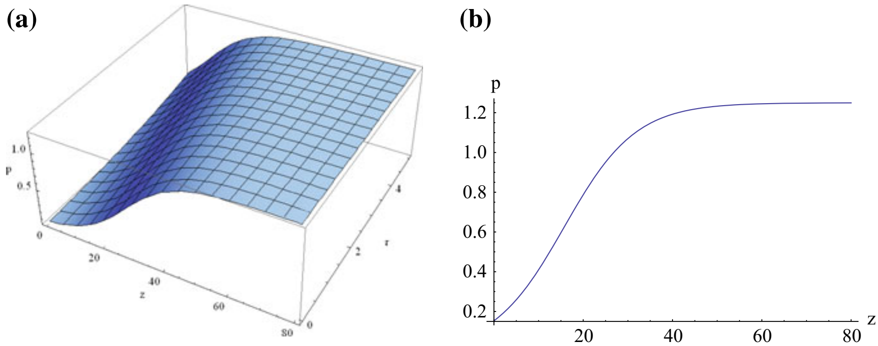

Fig. 5.2 **a** The periodic traveling wave solution for $p(z,\tau)$ appears in Eq. (5.47) of **Case II**, **b** corresponding solution for $p(z,\tau)$, when $\tau = 3$

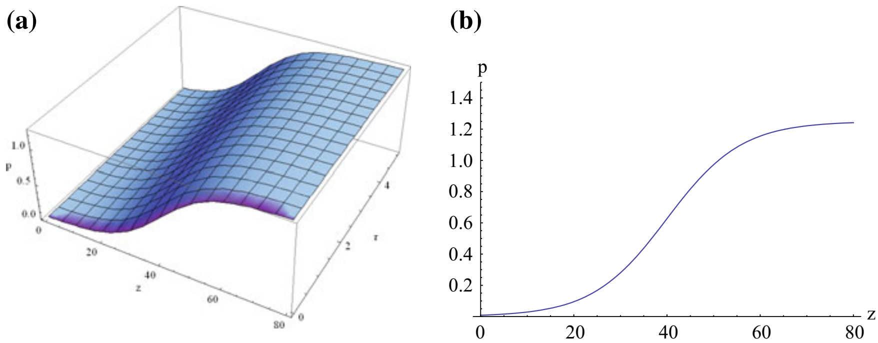

Fig. 5.3 **a** The periodic traveling wave solution for $p(x, y, z, \tau)$ obtained in Eq. (5.64) of **Case III**, **b** corresponding solution for $p(x, y, z, \tau)$, when $\tau = 0$

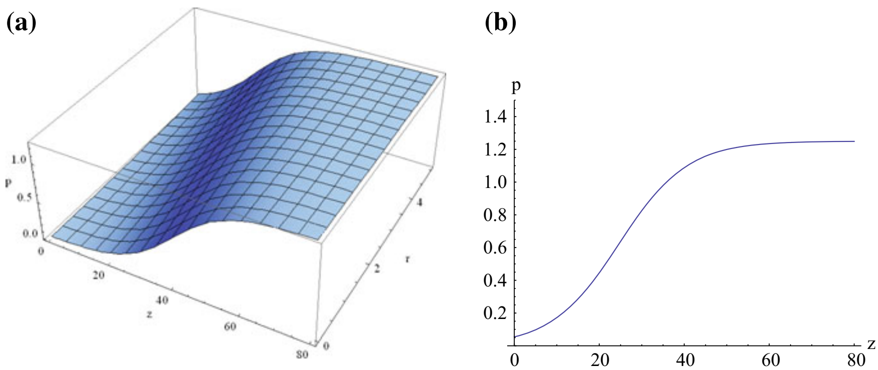

Fig. 5.4 **a** The periodic traveling wave solution for $p(x, y, z, \tau)$ obtained in Eq. (5.64) of **Case IV**, **b** corresponding solution for $p(x, y, z, \tau)$, when $\tau = 4$

Numerical Simulations for Fractional Burgers–Hopf Equation

Case I: For $\alpha = 0.5$ (Fractional order)

Case II: For $\alpha = 0.95$ (Fractional order)

Numerical Simulations for Fractional KZK Equation

Case III: For $\alpha = 0.5$ (Fractional order)

Case IV: For $\alpha = 0.95$ (Fractional order)

In the present numerical simulation, the traveling wave 3-D solutions surfaces and corresponding 2-D solution graphs have been drawn for the obtained exact solutions of Eqs. (5.1) and (5.2) in case of fractional-order time derivative. It can be observed that in all the above cases, the obtained exact solutions represent the kink-type traveling wave solutions with regard to various fractional-order solutions.

5.7 Exact Solutions of Time Fractional KdV-KZK Equation

In the present section, the new exact analytical solutions of time fractional KdV-KZK equation have been obtained first time ever using the Kudryashov method and modified Kudryashov method, respectively.

5.7.1 Kudryashov Method for Time Fractional KdV-KZK Equation

In the present analysis, we introduce the following fractional complex transform in Eq. (5.4):

$$p(x, y, z, \tau) = U(\xi),\ \xi = lx + my + kz + \frac{\lambda\tau^{\alpha}}{\Gamma(\alpha + 1)}, \tag{5.65}$$

where k and λ are constants.

By applying the fractional complex transform (5.65), Eq. (5.4) can be transformed to the following nonlinear ODE:

$$k\lambda U_{\xi\xi} = \frac{c_0}{2}(l^2 + m^2)U_{\xi\xi} + A_1\lambda^3 U_{\xi\xi\xi} + 2A_2\lambda^2[UU_{\xi\xi} + (U_\xi)^2] - \gamma\lambda^4 U_{\xi\xi\xi\xi}. \tag{5.66}$$

Integrating Eq. (5.66) with respect to ξ once, we have

$$C_1 + k\lambda U'(\xi) = \frac{c_0}{2}(l^2 + m^2)U'(\xi) + A_1\lambda^3 U''(\xi) + 2A_2\lambda^2 U(\xi)U'(\xi) - \gamma\lambda^4 U'''(\xi), \tag{5.67}$$

where C_1 is the integration constant.

The dominant terms with highest order of singularity are $\gamma\lambda^4 U'''(\xi)$ and $2A_2\lambda^2 U(\xi)U'(\xi)$. Thus, the pole order of Eq. (5.67) is $N = 2$.

Therefore, we sought for a solution in the form

$$U(\xi) = a_0 + a_1 Q(\xi) + a_2 Q(\xi)^2 \tag{5.68}$$

where a_0, a_1, and a_2 are constants to be determined later.

Substituting the derivatives of function $U(\xi)$ with respect to ξ and taking into account ansatz (5.68) in Eq. (5.67), we obtain a system of algebraic equations in the following form

$$Q^1 : -\frac{1}{2}a_1c_0(l^2+m^2)\ln a + a_1k\lambda\ln a - 2a_0a_1A_2\lambda^2\ln a$$
$$+ a_1A_1\lambda^3(\ln a)^2 + a_1\gamma\lambda^4(\ln a)^3 = 0$$
$$Q^2 : \frac{1}{2}a_1c_0(l^2+m^2)\ln a - a_2c_0(l^2+m^2)\ln a - a_1k\lambda\ln a + 2a_2k\lambda\ln a$$
$$+ 2a_0a_1A_2\lambda^2\ln a - 2a_1^2A_2\lambda^2\ln a - 4a_0a_2A_2\lambda^2\ln a - 3a_1A_1\lambda^3(\ln a)^2$$
$$+ 4a_2A_1\lambda^3(\ln a)^2 - 7a_1\gamma\lambda^4(\ln a)^3 + 8a_2\gamma\lambda^4(\ln a)^3 = 0$$
$$Q^3 : a_2c_0(l^2+m^2)\ln a - 2a_2k\lambda\ln a + 2a_1^2A_2\lambda^2\ln a + 4a_0a_2A_2\lambda^2\ln a$$
$$- 6a_1a_2A_2\lambda^2\ln a + 2a_1A_1\lambda^3(\ln a)^2 - 10A_1a_2\lambda^3(\ln a)^2$$
$$+ 12a_1\gamma\lambda^4(\ln a)^3 - 38a_2\gamma\lambda^4(\ln a)^3 = 0$$
$$Q^4 : 6a_1a_2A_2\lambda^2\ln a - 4a_2^2A_2\lambda^2\ln a + 6a_2A_1\lambda^3(\ln a)^2$$
$$- 6a_1\gamma\lambda^4(\ln a)^3 + 54a_2\gamma\lambda^4(\ln a)^3 = 0$$
$$Q^5 : 4a_2^2A_2\lambda^2\ln a - 24a_2\gamma\lambda^4(\ln a)^3 = 0$$

Solving this system, we obtain the following family of solutions

Case I:

$$a_0 = -\frac{12A_1^4 + 250A_1k\gamma^2 + 625c_0(l^2+m^2)\gamma^3}{100A_1^2A_2\gamma},$$
$$a_1 = 0,$$
$$a_2 = \frac{6A_1^2}{25A_2\gamma},$$
$$\lambda = -\frac{A_1}{5\gamma}.$$

Substituting the above parameter values in the ansatz given by Eq. (5.68), we obtain the following solution of Eq. (5.4)

$$p(x,y,z,\tau) = U(\xi)$$
$$= -\frac{125\gamma^2(2kA_1 + 5c_0(l^2+m^2)\gamma) + 6A_1^4 \sec h^2\left(\frac{\xi}{2}\right)(1+\sinh(\xi))}{100A_1^2A_2\gamma}, \tag{5.69}$$

where $\xi = lx + my + kz + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}$ and $\lambda = -\frac{A_1}{5\gamma}$.

Case II:

$$\begin{aligned} a_0 &= \frac{12A_1^4 + 250A_1k\gamma^2 - 625c_0(l^2+m^2)\gamma^3}{100A_1^2A_2\gamma}, \\ a_1 &= -\frac{12A_1^2}{25A_2\gamma}, \\ a_2 &= \frac{6A_1^2}{25A_2\gamma}, \\ \lambda &= \frac{A_1}{5\gamma}. \end{aligned}$$

Substituting the above parameter values in the ansatz given by Eq. (5.68), we obtain the following solution of Eq. (5.4)

$$p(x,y,z,\tau) = U(\xi) = \frac{125\gamma^2(2kA_1 - 5c_0(l^2+m^2)\gamma) - 6A_1^4\operatorname{sech}^2\left(\frac{\xi}{2}\right)(1-\sinh(\xi))}{100A_1^2A_2\gamma}, \tag{5.70}$$

where $\xi = lx + my + kz + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}$ and $\lambda = \frac{A_1}{5\gamma}$.

5.7.2 *Modified Kudryashov Method for Time Fractional KdV-KZK Equation*

Following the same preceding argument, Eq. (5.67) is to be acquired. Then substituting the derivatives of function $U(\xi)$ with respect to ξ into Eq. (5.67) and the ansatz given by Eq. (5.68) into the resulting Eq. (5.67), we obtain a system of algebraic equations in the following form

$$Q^1 : -\tfrac{1}{2}a_1c_0(l^2+m^2)\ln a + a_1k\lambda\ln a - 2a_0a_1A_2\lambda^2\ln a + a_1A_1\lambda^3(\ln a)^2 + a_1\gamma\lambda^4(\ln a)^3 = 0,$$

$$Q^2 : \tfrac{1}{2}a_1c_0(l^2+m^2)\ln a - a_2c_0(l^2+m^2)\ln a - a_1k\lambda\ln a + 2a_2k\lambda\ln a + 2a_0a_1A_2\lambda^2\ln a - 2a_1^2A_2\lambda^2\ln a$$
$$-4a_0a_2A_2\lambda^2\ln a - 3a_1A_1\lambda^3(\ln a)^2 + 4a_2A_1\lambda^3(\ln a)^2 - 7a_1\gamma\lambda^4(\ln a)^3 + 8a_2\gamma\lambda^4(\ln a)^3 = 0,$$

$$Q^3 : a_2c_0(l^2+m^2)\ln a - 2a_2k\lambda\ln a + 2a_1^2A_2\lambda^2\ln a + 4a_0a_2A_2\lambda^2\ln a - 6a_1a_2A_2\lambda^2\ln a + 2a_1A_1\lambda^3(\ln a)^2$$
$$-10A_1a_2\lambda^3(\ln a)^2 + 12a_1\gamma\lambda^4(\ln a)^3 - 38a_2\gamma\lambda^4(\ln a)^3 = 0,$$

$$Q^4 : 6a_1a_2A_2\lambda^2\ln a - 4a_2^2A_2\lambda^2\ln a + 6a_2A_1\lambda^3(\ln a)^2 - 6a_1\gamma\lambda^4(\ln a)^3 + 54a_2\gamma\lambda^4(\ln a)^3 = 0,$$

$$Q^5 : 4a_2^2A_2\lambda^2\ln a - 24a_2\gamma\lambda^4(\ln a)^3 = 0.$$

Solving this system we obtain the following family of solutions

Case I:

$$
\begin{aligned}
a_0 &= -\frac{12A_1^4 + 250A_1k\gamma^2\ln a + 625c_0(l^2+m^2)\gamma^3(\ln a)^2}{100A_1^2A_2\gamma},\\
a_1 &= 0,\\
a_2 &= \frac{6A_1^2}{25A_2\gamma},\\
\lambda &= -\frac{A_1}{5\gamma\ln a}.
\end{aligned}
$$

Substituting the above parameter values in the ansatz given by Eq. (5.68), we obtain the following solutions of Eq. (5.4)

$$
\begin{aligned}
p_1(x,y,z,\tau) = -\frac{1}{100A_1^2A_2\gamma}\Bigg[12\left(1-\frac{\left(1-\tan Fs\left(\frac{\xi}{2}\right)\right)^2}{2}\right)\\
A_1^4 + 250A_1k\gamma^2\ln a + 625c_0(l^2+m^2)\gamma^3(\ln a)^2\Bigg],
\end{aligned} \tag{5.71}
$$

$$
\begin{aligned}
p_2(x,y,z,\tau) = -\frac{1}{100A_1^2A_2\gamma}\Bigg[12\left(1-\frac{\left(1-\cot Fs\left(\frac{\xi}{2}\right)\right)^2}{2}\right)\\
A_1^4 + 250A_1k\gamma^2\ln a + 625c_0(l^2+m^2)\gamma^3(\ln a)^2\Bigg],
\end{aligned} \tag{5.72}
$$

where $\xi = lx + my + kz + \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}$ and $\lambda = -\frac{A_1}{5\gamma\ln a}$.

Case II:

$$
\begin{aligned}
a_0 &= \frac{12A_1^4 + 250A_1k\gamma^2\ln a - 625c_0(l^2+m^2)\gamma^3(\ln a)^2}{100A_1^2A_2\gamma},\\
a_1 &= -\frac{12A_1^2}{25A_2\gamma},\\
a_2 &= \frac{6A_1^2}{25A_2\gamma},\\
\lambda &= \frac{A_1}{5\gamma\ln a}.
\end{aligned}
$$

Substituting the above parameter values in the ansatz given by Eq. (5.68), we obtain the following solutions of Eq. (5.4)

$$p_1(x,y,z,\tau) = \frac{12(-1-2a^{\xi}+a^{2\xi})A_1^4 + 250(1+a^{\xi})^2 A_1 k\gamma^2 \ln a - 625(1+a^{\xi})^2 c_0(l^2+m^2)\gamma^3(\ln a)^2}{100(1+a^{\xi})^2 A_1^2 A_2 \gamma}, \tag{5.73}$$

$$p_2(x,y,z,\tau) = \frac{12(-1+2a^{\xi}+a^{2\xi})A_1^4 + 250(-1+a^{\xi})^2 A_1 k\gamma^2 \ln a - 625(-1+a^{\xi})^2 c_0(l^2+m^2)\gamma^3(\ln a)^2}{100(-1+a^{\xi})^2 A_1^2 A_2 \gamma}, \tag{5.74}$$

where $\xi = lx + my + kz + \frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)}$ and $\lambda = \frac{A_1}{5\gamma \ln a}$.

5.7.3 Numerical Results and Discussions

In this section, the numerical simulations of time fractional KdV-KZK equation have been presented graphically. Here, the exact solutions (5.69) and (5.70) obtained by classical Kudryashov method and also the exact solutions (5.71)–(5.74) obtained by modified Kudryashov method have been used to draw the 3-D solution graphs.

Numerical Simulations for the Solutions Obtained by Classical Kudryashov Method

In the present analysis, Eqs. (5.69) and (5.70) have been used for drawing the solution graphs for time fractional KdV-KZK equation in case of both fractional and classical orders (Figs. 5.5, 5.6, 5.7, 5.8, 5.9, and 5.10).

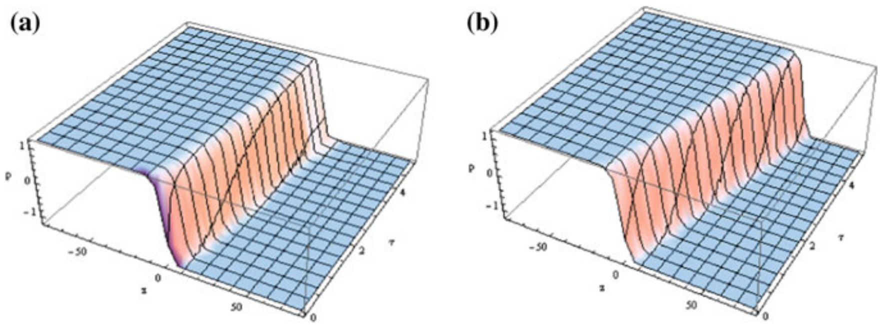

Fig. 5.5 Solitary wave solutions for Eq. (5.69) at $A_1 = 10$, $A_2 = 20$, $\gamma = 0.5$, $k = l = m = 0.5$, $c_0 = 1$, **a** when $\alpha = 0.5$ and **b** when $\alpha = 1$

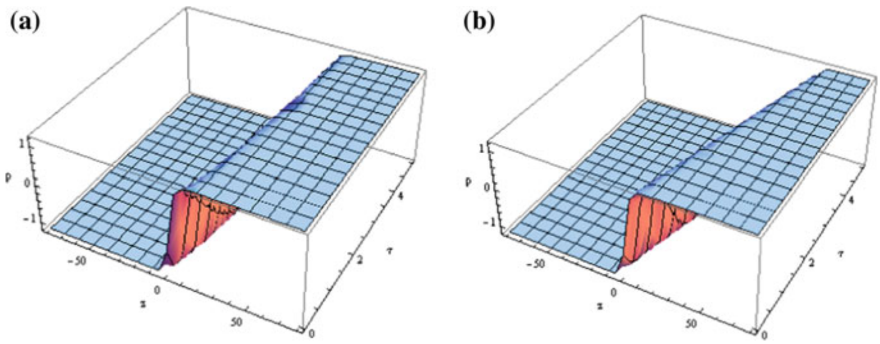

Fig. 5.6 Solitary wave solutions for Eq. (5.70) at $A_1 = 10$, $A_2 = 20$, $\gamma = 0.5$, $k = l = m = 0.5$, $c_0 = 1$, **a** when $\alpha = 0.5$ and **b** when $\alpha = 1$

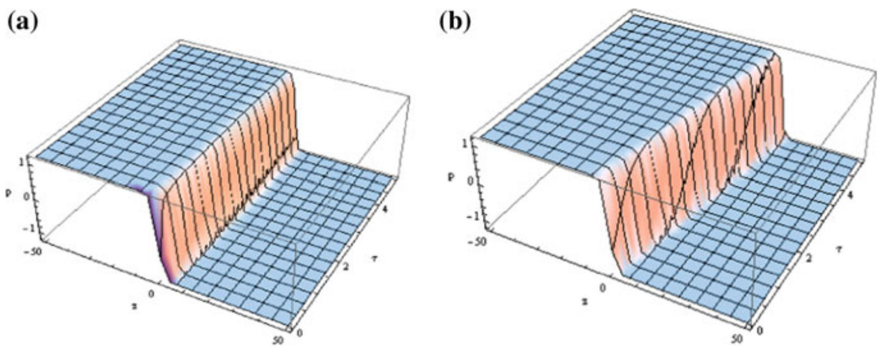

Fig. 5.7 Solitary wave solutions for Eq. (5.71) at $A_1 = 10$, $A_2 = 20$, $\gamma = 0.5$, $k = l = m = 0.5$, $c_0 = 1$, $a = 10$ **a** when $\alpha = 0.25$ and **b** when $\alpha = 1$

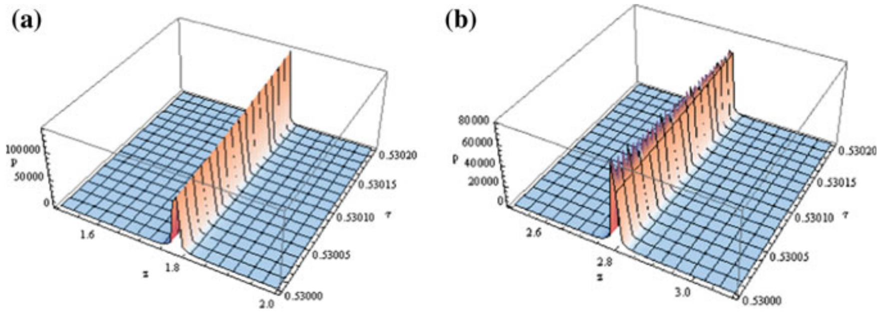

Fig. 5.8 Solitary wave solutions for Eq. (5.72) at $A_1 = 10$, $A_2 = 20$, $\gamma = 0.5$, $k = l = m = 0.5$, $c_0 = 1$, $a = 10$ **a** when $\alpha = 1$ and **b** when $\alpha = 0.5$

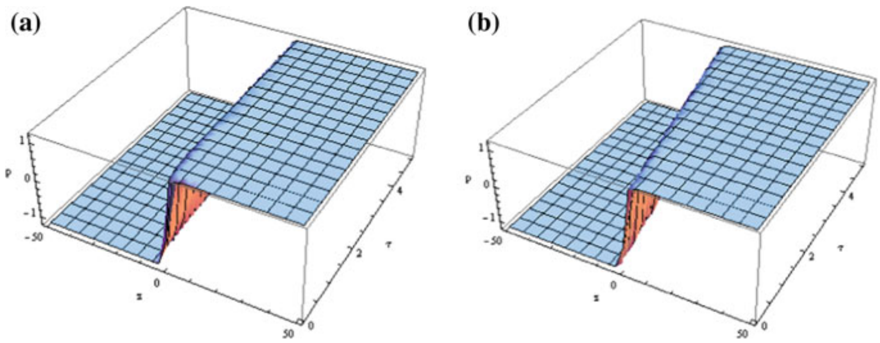

Fig. 5.9 Solitary wave solutions for Eq. (5.73) at $A_1 = 10$, $A_2 = 20$, $\gamma = 0.5$, $k = l = m = 0.5$, $c_0 = 1$, $a = 10$ **a** when $\alpha = 0.25$ and **b** when $\alpha = 1$

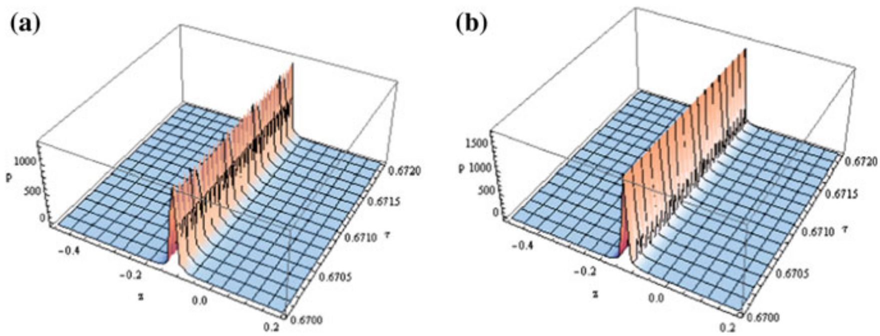

Fig. 5.10 Solitary wave solutions for Eq. (5.74) at $A_1 = A_2 = \gamma = k = l = m = c_0 = 1$, $a = 10$, **a** when $\alpha = 1$ and **b** when $\alpha = 0.75$

Numerical Simulations for the Solutions Obtained by the Modified Kudryashov Method

In the present analysis, Eqs. (5.71)–(5.74) have been used for drawing the solution graphs for time fractional KdV-KZK equation in case of both fractional and classical orders.

In the present numerical simulations, the solitary wave solutions for Eqs. (5.69)–(5.74) have been demonstrated in 3-D graphs. From the above figures, it may be observed that the solution surfaces obtained by classical Kudryashov for Eq. (5.69) are anti-kink solitary waves. On the other hand, the solution surfaces obtained by classical Kudryashov for Eq. (5.70) show the kink solitary waves. Similarly, the solution surfaces obtained by modified Kudryashov for Eqs. (5.71) and (5.73) show the anti-kink and kink solitary waves, respectively. However, in case of the solution surfaces obtained by modified Kudryashov for Eqs. (5.72) and (5.74), single soliton solitary waves of different shapes have been observed.

5.7.4 Physical Significance for the Solution of KdV-KZK Equation

The KdV-KZK equation covers all the four basic physical mechanisms of nonlinear acoustics, viz. diffraction, nonlinearity, dissipation, and dispersion. The solution of the KdV-KZK equation describes a shock wave as a transition between two constant velocity values. This transition can undergo oscillations due to the dispersion.

The obtained results are related to the physical phenomenon in Cantorian time-space. These results enrich the properties of the genuinely nonlinear phenomenon. To the best of the author information, the obtained solutions of this work have not been reported earlier in the open literature. The reported results have a potential application in observing the structure of KdV-KZK equation from micro-physical to macro-physical behavior of substance in the real world.

5.8 Implementation of the Jacobi Elliptic Function Method

In this section, the new exact analytical solutions of fractional (2 + 1)-dimensional Davey–Stewartson equation and new integrable Davey–Stewartson-type equation have been obtained using the mixed dn-sn method.

5.8.1 Exact Solutions of Fractional (2 + 1)-Dimensional Davey–Stewartson Equation

Let us consider the fractional (2 + 1)-dimensional Davey–Stewartson equation [48]

$$iD_t^{\alpha}q + a(D_x^{2\beta}q + D_y^{2\gamma}q) + b|q|^{2n}q - \lambda qr = 0, \tag{5.75}$$

$$D_x^{2\beta}r + D_y^{2\gamma}r + \delta D_x^{2\beta}\left(|q|^{2n}\right) = 0, \tag{5.76}$$

where $0 < \alpha, \beta, \gamma \leq 1$, $q \equiv q(x, y, t)$, and $r \equiv r(x, y, t)$. Also, a, b, λ, and δ are all constant coefficients. The exponent n is the power law parameter. It is necessary to have $n > 0$. In Eqs. (5.75) and (5.76), $q(x, y, t)$ is a complex-valued function which stands for wave amplitude, while $r(x, y, t)$ is a real-valued function which stands for mean flow. This system of equations is completely integrable and is often used to describe the long-time evolution of a two-dimensional wave packet [49–51].

We first transform the fractional (2 + 1)-dimensional Davey–Stewartson Eqs. (5.75) and (5.76) to a system of nonlinear ordinary differential equations in order to derive its exact solutions.

By applying the following fractional complex transform

$$q(x,y,t) = e^{i\theta}u(\xi),\ r(x,y,t) = v(\xi),$$

$$\theta = \frac{\theta_1 x^{\beta}}{\Gamma(1+\beta)} + \frac{\theta_2 y^{\gamma}}{\Gamma(1+\gamma)} + \frac{\theta_3 t^{\alpha}}{\Gamma(1+\alpha)} \text{ and } \xi = \frac{\xi_1 x^{\beta}}{\Gamma(1+\beta)} + \frac{\xi_2 y^{\gamma}}{\Gamma(1+\gamma)} + \frac{\xi_3 t^{\alpha}}{\Gamma(1+\alpha)},$$

Equations (5.75) and (5.76) can be reduced to the following couple nonlinear ODEs:

$$-(\theta_3 + a\theta_1^2 + a\theta_2^2)u + (a\xi_1^2 + a\xi_2^2)u_{\xi\xi} + bu^{2n+1} - \lambda uv = 0, \tag{5.77}$$

$$\xi_1^2 v_{\xi\xi} + \xi_2^2 v_{\xi\xi} + \delta\xi_1^2\left(u^{2n}\right)_{\xi\xi} = 0, \tag{5.78}$$

where ξ_3 has been set to $-2a\xi_1\theta_1 - 2a\xi_2\theta_2$. Equation (5.78) is then integrated term by term twice with respect to ξ where integration constants are considered zero. Thus, we obtain

$$v = -\frac{\delta\xi_1^2 u^{2n}}{\xi_1^2 + \xi_2^2}. \tag{5.79}$$

Substituting Eq. (5.79) into Eq. (5.77) yields

$$-(\theta_3 + a\theta_1^2 + a\theta_2^2)u + (a\xi_1^2 + a\xi_2^2)u_{\xi\xi} + bu^{2n+1} + \lambda\frac{\delta\xi_1^2 u^{2n+1}}{\xi_1^2 + \xi_2^2} = 0. \tag{5.80}$$

Using the transformation

$$u(\xi) = \Phi^{\frac{1}{n}}(\xi),$$

Equation (5.80) further reduces to

$$\begin{aligned} &-(\theta_3 + a\theta_1^2 + a\theta_2^2)n^2\Phi^2 + (a\xi_1^2 + a\xi_2^2)(1-n)\Phi_{\xi}^2 \\ &+ (a\xi_1^2 \quad + a\xi_2^2)n\Phi_{\xi\xi} + bn^2\Phi^4 + \lambda\frac{\delta\xi_1^2 n^2\Phi^4}{\xi_1^2 + \xi_2^2} = 0 \end{aligned} \tag{5.81}$$

By balancing the terms $\Phi\Phi_{\xi\xi}$ and Φ^4 in Eq. (5.81), the value of N can be determined, which is $N = 1$ in this problem.

Therefore, the solution of Eq. (5.81) can be written in the following ansatz as

$$\Phi(\xi) = c_0 + c_1\phi(\xi) + d_0\sqrt{k^2 - \phi^2(\xi)}, \tag{5.82}$$

where c_0, c_1, and d_0 are constants to be determined later and $\phi(\xi)$ satisfies Eq. (5.29).

Now substituting Eq. (5.82) along with Eq. (5.29) into Eq. (5.81) and then equating each coefficient of $\phi^i(\xi)$, i = 0,1,2,… to zero, we can get a set of algebraic equations for c_0, c_1, d_0, θ_3, and m as follows:

$$\begin{aligned}
&-(a\xi_1^2+a\xi_2^2)(-k^4(-1+m)(\xi_1^2+\xi_2^2)c_1^2+k^4(-1+m)n(\xi_1^2+\xi_2^2)(c_1^2+d_0^2)\\
&\quad+n^2(\theta_1^2+\theta_2^2)(c_0^2+k^2d_0^2))\\
&\quad+n^2(-\theta_3(\xi_1^2+\xi_2^2)(c_0^2+k^2d_0^2)+(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))\\
&\quad(c_0^4+6k^2c_0^2d_0^2+k^4d_0^4))=0\\
&-nc_0c_1(a(\xi_1^2+\xi_2^2)(2n(\theta_1^2+\theta_2^2)+k^2(-2+m)(\xi_1^2+\xi_2^2))\\
&\quad-2n(-\theta_3(\xi_1^2+\xi_2^2)+2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(c_0^2+3k^2d_0^2)))=0\\
&-(a\xi_1^2+a\xi_2^2)(2k^2n(\xi_1^2+\xi_2^2)d_0^2+n^2(\theta_1^2+\theta_2^2)(c_1^2+d_0^2)\\
&\quad+k^2(\xi_1^2+\xi_2^2)((-2+m))c_1^2-(-1+m)d_0^2))\\
&\quad-n^2(\theta_3(\xi_1^2+\xi_2^2)(c_1^2-d_0^2)-2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(3c_0^2(c_1^2-d_0^2)\\
&\quad-k^2d_0^2(-3c_1^2+d_0^2)))=0\\
&-2nc_0c_1(a(\xi_1^2+\xi_2^2)^2-2n(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(c_1^2-3d_0^2))=0\\
&-a(1+n)(\xi_1^2+\xi_2^2)^2(c_1^2-d_0^2)+n^2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(c_1^4-6c_1^2d_0^2+d_0^4)=0\\
&nc_0d_0(-a(\xi_1^2+\xi_2^2)(2n(\theta_1^2+\theta_2^2)+k^2(-1+m)(\xi_1^2+\xi_2^2))\\
&\quad+2n(-\theta_3(\xi_1^2+\xi_2^2)+2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(c_0^2+k^2d_0^2)))=0\\
&c_1d_0(a(\xi_1^2+\xi_2^2)(-2n^2(\theta_1^2+\theta_2^2)-2k^2(-1+m)(\xi_1^2+\xi_2^2)\\
&\quad+k^2n(\xi_1^2+\xi_2^2))+2n^2(-\theta_3(\xi_1^2+\xi_2^2)\\
&\quad+2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(3c_0^2+k^2d_0^2)))=0\\
&-2nc_0d_0(a(\xi_1^2+\xi_2^2)^2-2n(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(3c_1^2-d_0^2))=0\\
&-2c_1d_0(a(1+n)(\xi_1^2+\xi_2^2)^2-2n^2(\lambda\delta\xi_1^2+b(\xi_1^2+\xi_2^2))(c_1^2-d_0^2))=0
\end{aligned} \tag{5.83}$$

Solving the above algebraic Eqs. (5.83), we have the set of coefficients for the nontrivial solutions of Eq. (5.81) as given below:

Case 1:

$$\begin{aligned}
&c_0=0, c_1=-\frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)}{\sqrt{-bn^2\xi_1^2-n^2\delta\lambda\xi_1^2-bn^2\xi_2^2}}, d_0=0, m=1, \theta_3\\
&=-\frac{a(n^2\theta_1^2+n^2\theta_2^2-k^2\xi_1^2-k^2\xi_2^2)}{n^2},
\end{aligned} \tag{5.84}$$

where $\xi_3 = -2a\xi_1\theta_1 - 2a\xi_2\theta_2$ and k is the free parameter.

Substituting Eqs. (5.84) into Eq. (5.28) and using special solutions (5.30) of Eq. (5.29), we obtain

$$\Phi(\xi) = -\frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)k \sec h(k\xi)}{\sqrt{-bn^2\xi_1^2 - n^2\delta\lambda\xi_1^2 - bn^2\xi_2^2}}$$

which yields the following solitary wave solutions of Eqs. (5.75) and (5.76):

$$u(x,y,t) = \Phi(\xi)^{\frac{1}{n}} = \left(-\frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)k \sec h(k\xi)}{\sqrt{-bn^2\xi_1^2 - n^2\delta\lambda\xi_1^2 - bn^2\xi_2^2}}\right)^{\frac{1}{n}}, \quad (5.85a)$$

$$v(x,y,t) = -\frac{a(1+n)\delta\xi_1^2(\xi_1^2+\xi_2^2)k \sec h^2(k\xi)}{(bn^2\xi_1^2 + n^2\delta\lambda\xi_1^2 + bn^2\xi_2^2)}. \quad (5.85b)$$

Case 2:

$$c_0 = 0, c_1 = \frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)}{\sqrt{-bn^2\xi_1^2 - n^2\delta\lambda\xi_1^2 - bn^2\xi_2^2}}, d_0 = 0, m = 1, \theta_3 = -\frac{a(n^2\theta_1^2 + n^2\theta_2^2 - k^2\xi_1^2 - k^2\xi_2^2)}{n^2}, \quad (5.86)$$

where $\xi_3 = -2a\xi_1\theta_1 - 2a\xi_2\theta_2$ and k is the free parameter.

Substituting Eqs. (5.86) into Eq. (5.28) and using special solutions (5.30) of Eq. (5.29), we obtain

$$\Phi(\xi) = \frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)k \sec h(k\xi)}{\sqrt{-bn^2\xi_1^2 - n^2\delta\lambda\xi_1^2 - bn^2\xi_2^2}}$$

which yields the following solitary wave solutions of Eqs. (5.75) and (5.76):

$$u(x,y,t) = \Phi(\xi)^{\frac{1}{n}} = \left(\frac{i\sqrt{a}\sqrt{1+n}(\xi_1^2+\xi_2^2)k \sec h(k\xi)}{\sqrt{-bn^2\xi_1^2 - n^2\delta\lambda\xi_1^2 - bn^2\xi_2^2}}\right)^{\frac{1}{n}}, \quad (5.87a)$$

$$v(x,y,t) = -\frac{a(1+n)\delta\xi_1^2(\xi_1^2+\xi_2^2)k \sec h^2(k\xi)}{(bn^2\xi_1^2 + n^2\delta\lambda\xi_1^2 + bn^2\xi_2^2)}. \quad (5.87b)$$

5.8.2 Exact Solutions of the Fractional (2 + 1)-Dimensional New Integrable Davey–Stewartson-Type Equation

Let us consider the fractional (2 + 1)-dimensional new integrable Davey–Stewartson-type equation

$$\begin{aligned} &iD_\tau^\alpha \Psi + L_1\Psi + \Psi\,\Phi + \Psi\chi = 0, \\ &L_2\chi = L_3|\Psi|^2, \\ &D_\xi^\beta \Phi = D_\eta^\gamma \chi + \mu D_\eta^\gamma\left(|\Psi|^2\right), \quad \mu = \mp 1,\ 0 < \alpha, \beta, \gamma \le 1 \end{aligned} \tag{5.88}$$

where the linear differential operators are given by

$$\begin{aligned} L_1 &\equiv \left(\frac{b^2 - a^2}{4}\right) D_\xi^{2\beta} - a D_\xi^\beta D_\eta^\gamma - D_\eta^{2\gamma}, \\ L_2 &\equiv \left(\frac{b^2 + a^2}{4}\right) D_\xi^{2\beta} + a D_\xi^\beta D_\eta^\gamma + D_\eta^{2\gamma}, \\ L_3 &\equiv \pm\frac{1}{4}\left(b^2 + a^2 + \frac{8b^2(a-1)}{(a-2)^2 - b^2}\right) D_\xi^{2\beta} \pm \left(a + \frac{2b^2}{(a-2)^2 - b^2}\right) D_\xi^\beta D_\eta^\gamma \pm D_\eta^{2\gamma}, \end{aligned}$$

where $\Psi \equiv \Psi(\xi, \eta, \tau)$ is complex while $\Phi \equiv \Phi(\xi, \eta, \tau)$, $\chi \equiv \chi(\xi, \eta, \tau)$ are real and a, b are real parameters. The above equation in integer order was devised firstly by Maccari [52] from the Konopelchenko–Dubrovsky (KD) equation [53].

In the present analysis, the Jacobi elliptic function method has been used to investigate for new types of doubly periodic exact solutions in terms of Jacobi elliptic functions.

According to the algorithm discussed in Sect. 5.5, let us consider the following fractional complex transform

$$\begin{aligned} &\Psi(\xi, \eta, \tau) = \Psi(X)e^{i\theta},\ \Phi(\xi, \eta, \tau) = \Phi(X),\ \chi(\xi, \eta, \tau) = \chi(X), \\ &X = k\left(\frac{\xi^\beta}{\Gamma(1+\beta)} + l\frac{\eta^\gamma}{\Gamma(1+\gamma)} + \lambda\frac{\tau^\alpha}{\Gamma(1+\alpha)}\right),\ \theta \\ &\quad = \frac{\theta_1\xi^\beta}{\Gamma(1+\beta)} + \frac{\theta_2\eta^\gamma}{\Gamma(1+\gamma)} + \frac{\theta_3\tau^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{5.89}$$

where k, l, λ, θ_1, θ_2, and θ_3 are constants.

By applying the fractional complex transform (5.89), Eq. (5.88) can be reduced to the following couple nonlinear ODEs:

$$k^2 M_1 \frac{d^2\Psi(X)}{dX^2} + M_0\Psi(X) + \Psi(X)\Phi(X) + \Psi(X)\chi(X) = 0, \tag{5.90}$$

$$k^2 M_2 \frac{d^2\chi(X)}{dX^2} = k^2 M_3 \frac{d^2\Psi^2(X)}{dX^2}, \tag{5.91}$$

$$k\frac{d\Phi(X)}{dX} = kl\frac{d\chi(X)}{dX} + \mu kl\frac{d\Psi^2(X)}{dX}, \tag{5.92}$$

where λ has been set to $a(l\theta_1 + \theta_2) + 2l\theta_2 - \frac{\theta_1(b^2-a^2)}{2}$.

Here,

$$M_0 = -\theta_3 - \frac{(b^2-a^2)}{4}\theta_1^2 + a\theta_1\theta_2 + \theta_2^2,$$
$$M_1 = -al - l^2 + \frac{(b^2-a^2)}{4},$$
$$M_2 = al + l^2 + \frac{(b^2+a^2)}{4},$$
$$M_3 = \pm l^2 \pm \left(a + \frac{2b^2}{(a-2)^2 - b^2}\right)l \pm \frac{1}{4}\left(b^2 + a^2 + \frac{8b^2(a-1)}{(a-2)^2 - b^2}\right).$$

Now, Eqs. (5.92) and (5.91) are integrated once and twice term by term with respect to X where integration constants are considered zero. Thus, we obtain

$$\chi(X) = \frac{M_3}{M_2}\Psi^2(X),$$

$$\Phi(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi^2(X). \tag{5.93}$$

Eliminating $\chi(X)$, $\Phi(X)$ from Eqs. (5.90) and (5.93), we arrive at

$$k^2 M_1 \frac{d^2\Psi(X)}{dX^2} + M_0\Psi(X) + \left(\frac{lM_3}{M_2} + \mu l + \frac{M_3}{M_2}\right)\Psi^3(X) = 0 \tag{5.94}$$

By balancing the nonlinear term $\Psi^3(X)$ and highest order derivative term $\frac{d^2\Psi(X)}{dX^2}$ in Eq. (5.94), the value of N can be determined, which is $N = 1$ in this problem.

Therefore, the solution of Eq. (5.94) can be written in the following ansatz as

$$\Psi(X) = c_0 + c_1\phi(X) + d_0\sqrt{p^2 - \phi^2(X)}, \tag{5.95}$$

where c_0, c_1, and d_0 are constants to be determined later, and $\phi(X)$ satisfies the elliptic equation:

$$\frac{\mathrm{d}\phi(X)}{\mathrm{d}X} = \sqrt{(p^2 - \phi^2(X))(\phi^2(X) - p^2(1-m))}, \tag{5.96}$$

whose solutions are given by

$$\begin{aligned} \phi(X) &= pdn(pX|m), \\ \phi(X) &= p\sqrt{1-m}nd(pX|m), \end{aligned} \tag{5.97}$$

Now substituting Eq. (5.95) along with Eq. (5.96) into Eq. (5.94) and then equating each coefficient of $\phi^i(X)$, i = 0,1,2,… to zero, we can get a set of algebraic equations for c_0, c_1, d_0, p, and m as follows:

$$\begin{gathered} c_0(M_0M_2 + (M_3 + lM_3 + lM_2\mu)(c_0^2 + 3p^2d_0^2)) = 0, \\ c_1(M_0M_2 - k^2(-2+m)M_1M_2p^2 + 3(M_3 + lM_3 + lM_2\mu)(c_0^2 + p^2d_0^2)) = 0, \\ 3(M_3 + lM_3 + lM_2\mu)c_0(c_1^2 - d_0^2)) = 0, \\ c_1(-2k^2M_1M_2 + (M_3 + lM_3 + lM_2\mu)(c_1^2 - 3d_0^2)) = 0, \\ d_0(M_0M_2 + k^2M_1M_2p^2 - k^2mM_1M_2p^2 + 3M_3c_0^2 + 3lM_3c_0^2 + 3lM_2\mu c_0^2 \\ + M_3p^2d_0^2 + lM_3p^2d_0^2 + lM_2\mu p^2d_0^2) = 0, \\ 6(M_3 + lM_3 + lM_2\mu)c_0c_1d_0 = 0, \\ d_0(-2k^2M_1M_2 + (M_3 + lM_3 + lM_2\mu)(3c_1^2 - d_0^2)) = 0. \end{gathered} \tag{5.98}$$

Solving the above algebraic Eq. (5.98), we have the set of coefficients for the nontrivial traveling wave solutions of Eq. (5.94) as given below:

Case 1:

$$c_0 = 0, c_1 = -\frac{k\sqrt{2M_1M_2}}{\sqrt{l\mu M_2 + (l+1)M_3}},$$

$$d_0 = 0, m = \frac{M_0 + 2M_1k^2p^2}{M_1k^2p^2}.$$

$$\Psi_{11}(X) = -\frac{kp\sqrt{2M_1M_2}}{\sqrt{l\mu M_2 + (l+1)M_3}}dn(pX|m),$$

$$\Phi_{11}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{11}^2(X),$$

$$\chi_{11}(X) = \frac{M_3}{M_2}\Psi_{11}^2(X),$$
$$\Psi_{12}(X) = -\frac{kp\sqrt{2M_1M_2}\sqrt{1-m}}{\sqrt{l\mu M_2 + (l+1)M_3}}nd(pX|m),$$
$$\Phi_{12}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{12}^2(X),$$
$$\chi_{12}(X) = \frac{M_3}{M_2}\Psi_{12}^2(X).$$

Case 2:

$$c_0 = 0, c_1 = \frac{k\sqrt{2M_1M_2}}{\sqrt{l\mu M_2 + (l+1)M_3}},$$

$$d_0 = 0, m = \frac{M_0 + 2M_1k^2p^2}{M_1k^2p^2}.$$

$$\Psi_{21}(X) = \frac{kp\sqrt{2M_1M_2}}{\sqrt{l\mu M_2 + (l+1)M_3}}dn(pX|m),$$

$$\Phi_{21}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{21}^2(X),$$

$$\chi_{21}(X) = \frac{M_3}{M_2}\Psi_{21}^2(X),$$

$$\Psi_{22}(X) = \frac{kp\sqrt{2M_1M_2}\sqrt{1-m}}{\sqrt{l\mu M_2 + (l+1)M_3}}nd(pX|m),$$

$$\Phi_{22}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{22}^2(X)$$

$$\chi_{22}(X) = \frac{M_3}{M_2}\Psi_{22}^2(X).$$

Case 3:

$$c_0 = 0, c_1 = -\frac{k\sqrt{M_1M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}},$$
$$d_0 = -\frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}}, m = \frac{2M_0 + M_1k^2p^2}{2M_1k^2p^2}.$$

$$\Psi_{31}(X) = -\frac{kp\sqrt{M_1M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}}dn(pX|m)$$
$$- p\sqrt{m}sn(pX|m)\frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{31}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{31}^2(X),$$

$$\chi_{31}(X) = \frac{M_3}{M_2}\Psi_{31}^2(X),$$

$$\Psi_{32}(X) = -\frac{kp\sqrt{M_1M_2}\sqrt{1-m}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}}dn(pX|m)$$
$$- p\sqrt{1-(1-m)nd^2(pX|m)}\frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{32}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{32}^2(X),$$

$$\chi_{32}(X) = \frac{M_3}{M_2}\Psi_{32}^2(X).$$

Case 4:

$$c_0 = 0, c_1 = \frac{k\sqrt{M_1M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}},$$

$$d_0 = -\frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}}, m = \frac{2M_0 + M_1k^2p^2}{2M_1k^2p^2}.$$

$$\Psi_{41}(X) = \frac{kp\sqrt{M_1M_2}}{\sqrt{2l\mu M_2 + (l+2)M_3}}dn(pX|m) - \frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{41}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{41}^2(X),$$

$$\chi_{41}(X) = \frac{M_3}{M_2}\Psi_{41}^2(X),$$

$$\Psi_{42}(X) = \frac{kp\sqrt{M_1M_2}\sqrt{1-m}}{\sqrt{2l\mu M_2 + (l+2)M_3}}nd(pX|m) - \frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{42}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{42}^2(X),$$

$$\chi_{42}(X) = \frac{M_3}{M_2}\Psi_{42}^2(X).$$

Case 5:

$$c_0 = 0, c_1 = -\frac{k\sqrt{M_1 M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}},$$

$$d_0 = \frac{k\sqrt{M_1 M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}}, m = \frac{2M_0 + M_1 k^2 p^2}{2M_1 k^2 p^2}.$$

$$\Psi_{51}(X) = -\frac{kp\sqrt{M_1 M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}} dn(pX|m) + \frac{k\sqrt{M_1 M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{51}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{51}^2(X),$$

$$\chi_{51}(X) = \frac{M_3}{M_2}\Psi_{51}^2(X),$$

$$\Psi_{52}(X) = -\frac{kp\sqrt{M_1 M_2}\sqrt{1-m}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}} nd(pX|m) + \frac{k\sqrt{M_1 M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{52}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{52}^2(X),$$

$$\chi_{52}(X) = \frac{M_3}{M_2}\Psi_{52}^2(X).$$

Case 6:

$$c_0 = 0, c_1 = \frac{k\sqrt{M_1 M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}},$$

$$d_0 = \frac{k\sqrt{M_1 M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}}, m = \frac{2M_0 + M_1 k^2 p^2}{2M_1 k^2 p^2}.$$

$$\Psi_{61}(X) = \frac{kp\sqrt{M_1 M_2}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}} dn(pX|m) + \frac{k\sqrt{M_1 M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{61}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{61}^2(X),$$

$$\chi_{61}(X) = \frac{M_3}{M_2}\Psi_{61}^2(X),$$

$$\Psi_{62}(X) = \frac{kp\sqrt{M_1M_2}\sqrt{1-m}}{\sqrt{2l\mu M_2 + 2(l+1)M_3}} nd(pX|m) + \frac{k\sqrt{M_1M_2}}{\sqrt{-2l\mu M_2 - 2(l+1)M_3}},$$

$$\Phi_{62}(X) = \left(l\frac{M_3}{M_2} + \mu l\right)\Psi_{62}^2(X),$$

$$\chi_{62}(X) = \frac{M_3}{M_2}\Psi_{62}^2(X).$$

5.9 Conclusion

In this chapter, several traveling wave exact solutions of nonlinear fractional acoustic wave equations, namely the time fractional Burgers–Hopf and KZK equations have been successfully obtained by the first integral method with the help of fractional complex transform. The fractional complex transform can easily convert a fractional differential equation into its equivalent ordinary differential equation form. So, fractional complex transform has been efficiently used for solving fractional differential equations. Here, the fractional complex transform has been considered which is derived from the local fractional calculus defined on fractals.

The first integral method has been successfully employed to solve nonlinear fractional acoustic wave equations. The obtained solutions may be worthwhile for an explanation of some physical phenomena accurately. The present analysis indicates that the first integral method is effective and efficient for solving nonlinear fractional acoustic wave equations. The performance of this method is reliable, and it provides the exact traveling wave solutions. In this present analysis, the focused method clearly avoids linearization, discretization, and unrealistic assumptions, and therefore, it provides exact solutions efficiently and accurately.

Also, in this chapter, the new exact solutions of time fractional KdV-KZK equation have been obtained by classical Kudryashov and modified Kudryashov method, respectively, with the help of fractional complex transform. The fractional complex transform is employed in order to convert a fractional differential equation into its equivalent ordinary differential equation form. So, the fractional complex transform facilitates solving fractional differential equations. Two methods are successfully applied to solve nonlinear time fractional KdV-KZK equation. The new obtained exact solutions may be useful for the explanation of some physical phenomena accurately. The present analysis indicates that the focused methods are effective and efficient for analytically solving the time fractional KdV-KZK equation. It also demonstrates that performances of these methods are substantially influential and absolutely reliable for finding new exact solutions in terms of symmetric hyperbolic Fibonacci function solutions. In this present analysis, the discussed methods clearly avoid linearization, discretization, and unrealistic assumptions, and therefore, these methods provide exact solutions efficiently and accurately. To the best information of the author, new exact analytical solutions of the time fractional KdV-KZK equation are obtained for the first time in this respect.

The Jacobi elliptic function method has been also used to determine the exact solutions of time fractional (2 + 1)-dimensional Davey–Stewartson equation and new integrable Davey–Stewartson-type equation. In both problems, with the help of fractional complex transform, the Davey–Stewartson system was first transformed into a system of nonlinear ordinary differential equations, which were then solved to obtain the exact solutions. Here also, the fractional complex transform has been considered which is derived from the local fractional calculus defined on fractals. The proposed method is more general than the dn-function method [64] and may be applied to other nonlinear evolution equations. Several classes of traveling wave solutions of the fractional Davey–Stewartson equation have been derived from the solitary wave solutions in Jacobi elliptic functions. Using this proposed method, some new solitary wave solutions and double-periodic solutions have been obtained. This method can also be used for many other nonlinear evolution equations or coupled ones. To the best information of the author, these solitary wave solutions of the fractional Davey–Stewartson equation are new exact solutions which are not reported earlier. Being concise and powerful, this current method can also be extended to solve many other fractional partial differential equations arising in mathematical physics.

References

1. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
2. Debnath, L.: Integral transforms and their applications. CRC Press, Boca Raton (1995)
3. Sabatier, J., Agrawal, O.P., Tenreiro Machado, J.A.: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht, The Netherlands (2007)
4. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Das, S.: Functional Fractional Calculus. Springer, New York (2011)
6. Ortigueira, M.D.: Fractional Calculus for Scientists and Engineers, Lecture Notes in Electrical Engineering, vol. 84. Springer, Netherlands (2011)
7. Prieur, F., Vilenskiy, G., Holm, S.: A more fundamental approach to the derivation of nonlinear acoustic wave equations with fractional loss operators (L). J. Acoust. Soc. Am. **132** (4), 2169–2172 (2012)

8. Zabolotskaya, E.A., Khokhlov, R.V.: Quasi-plane waves in the nonlinear acoustics of confined beams. Sov. Phys. Acoust. **15**, 35–40 (1969)
9. Kuznetsov, V.P.: Equations of nonlinear acoustics. Sov. Phys. Acoust. **16**, 467–470 (1971)
10. Saha Ray, S.: On Haar wavelet operational matrix of general order and its application for the numerical solution of fractional Bagley Torvik equation. Appl. Math. Comput. **218**, 5239–5248 (2012)
11. Saha Ray, S., Bera, R.K.: Analytical solution of a dynamic system containing fractional derivative of order one-half by Adomian decomposition method. Trans. ASME J. Appl. Mech. **72**(2), 290–295 (2005)
12. Diethelm, K., Ford, N.J., Freed, A.D.: A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**(1–4), 3–22 (2002)
13. Liu, F., Anh, V., Turner, I.: Numerical solution of the space fractional Fokker-Planck equation. J. Comput. Appl. Math. **166**(1), 209–219 (2004)
14. Saha, Ray S.: A new approach for the application of adomian decomposition method for the solution of fractional space diffusion equation with insulated ends. Appl. Math. Comput. **202** (2), 544–549 (2008)
15. Ertürk, V.S., Momani, S., Odibat, Z.: Application of generalized differential transform method to multi-order fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **13**(8), 1642–1654 (2008)
16. Tong, B., He, Y., Wei, L., Zhang, X.: A generalized fractional sub-equation method for fractional differential equations with variable coefficients. Phys. Lett. A **376**(38–39), 2588–2590 (2012)
17. Liu, F., Zhuang, P., Burrage, K.: Numerical methods and analysis for a class of fractional advection-dispersion models. Comput. Math Appl. **64**(10), 2990–3007 (2012)
18. Zheng, B.: (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics. Commun. Theor. Phys. **58**(5), 623–630 (2012)
19. Kadem, A., Kılıçman, A.: The approximate solution of fractional Fredholm integro-differential equations by variational iteration and homotopy perturbation methods. Abstr. Appl. Anal., Article ID 486193 (2012)
20. Gupta, A.K., Saha, Ray S.: Numerical treatment for the solution of fractional fifth order Sawada-Kotera equation using second kind Chebyshev wavelet method. Appl. Math. Model. **39**(17), 5121–5130 (2015)
21. Sahoo, S., Saha, Ray S.: New approach to find exact solutions of time-fractional Kuramoto-Sivashinsky equation. Physica A **434**, 240–245 (2015)
22. Saha, Ray S., Sahoo, S.: Improved fractional sub-equation method for (3 + 1)-dimensional generalized fractional KdV-Zakharov-Kuznetsov equations. Comput. Math Appl. **70**(2), 158–166 (2015)
23. Raslan, K.R.: The first integral method for solving some important nonlinear partial differential equations. Nonlinear Dynam. **53**, 281–286 (2008)
24. Abbasbandy, S., Shirzadi, A.: The first integral method for modified Benjamin–Bona–Mahony equation. Commun. Nonlinear Sci. Numer. Simul. **15**, 1759–1764 (2010)
25. Lu, B.: The first integral method for some time fractional differential equations. J. Math. Anal. Appl. **395**, 684–693 (2012)
26. Jafari, H., Soltani, R., Khalique, C.M., Baleanu, D.: Exact solutions of two nonlinear partial differential equations by using the first integral method. Bound. Value Probl. **2013**, 117 (2013)
27. Bekir, A., Güner, Ö., Ünsal, Ö.: The first integral method for exact solutions of nonlinear fractional differential equations. J. Comput. Nonlinear Dyn. **10**(021020), 1–5 (2015)
28. Saha Ray, S.: Exact solutions for time-fractional diffusion-wave equations by decomposition method. Phys. Scr. **75**(1), Article number 008, 53–61 (2007)
29. Ortigueira, M.D., Tenreiro Machado, J.A.: Fractional calculus applications in signals and systems. Sig. Process. **86**(10), 2503–2504 (2006)
30. Gan W.S.: Analytical Solutions of the KdV-KZK Equation. In: André M.P. (ed.), Acoustical Imaging, vol. 28, pp. 445–452 (2007)

31. Kudryashov, N.A.: On "new travelling wave solutions" of the KdV and the KdV–Burgers equations. Commun. Nonlinear Sci. Numer. Simul. **14**, 1891–1900 (2009)
32. Ryabov, P.N., Sinelshchikov, D.I., Kochanov, M.B.: Application of the Kudryashov method for finding exact solutions of the high order nonlinear evolution equations. Appl. Math. Comput. **218**, 3965–3972 (2011)
33. Kudryashov, N.A.: One method for finding exact solutions of nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**(6), 2248–2253 (2012)
34. Kabir, M.M., Khajeh, A., Aghdam, E.A., Koma, A.Y.: Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations. Math. Methods Appl. Sci. **34**, 244–246 (2011)
35. Ege, S.M., Misirli, E.: The modified Kudryashov method for solving some fractional-order nonlinear equations. Adv. Differ. Equ. **2014**, 135 (2014)
36. Taghizadeh, N., Mirzazadeh, M., Mahmoodirad, A.: Application of Kudryashov method for high-order nonlinear Schrödinger equation. Indian J. Phys. **87**(8), 781–785 (2013)
37. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations. Wiley, New York (1993)
38. Tavazoei, M.S., Haeri, M.: Describing function based methods for predicting chaos in a class of fractional order differential equations. Nonlinear Dyn. **57**(3), 363–373 (2009)
39. Caputo, M.: Elasticità e dissipazione. Zanichelli, Bologna (1969)
40. Suarez, L.E., Shokooh, A.: "An eigenvector expansion method for the solution of motion containing fractional derivatives. ASME J. Appl. Mech. **64**, 629–635 (1997)
41. Torvik, P.J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials. ASME J. Appl. Mech. **51**, 294–298 (1984)
42. Khater A.H., Hassan M.M.: Travelling and periodic wave solutions of some nonlinear wave equations. Z. Naturforsch., **59a**, 389–396 (2004)
43. Rozanova-Pierrat, A.: Mathematical analysis of Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation, p. 6. Prepint of Laboratory Jaques-Louis Lions, Paris (2006)
44. Bakhvalov, N.S., Zhileikin, Y.M., Zabolotskaya, E.A.: Nonlinear theory of sound beams. American Institute of Physics, New York (1987)
45. Hamilton, M.F., Blackstock, D.T.: Nonlinear acoustics. Academic Press, San Diego (1997)
46. Pinton, G., Trahey, G.: Numerical solutions of the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation satisfying the Rankine-Hugoniot condition. J. Acoust. Soc. Am. **120**, 3109 (2006)
47. Davey, A., Stewartson, K.: On three-dimensional packets of surface waves. Proceeding R. Soc. Lond. A **338**, 101–110 (1974)
48. Shehata, A.R., Kamal, E.M., Kareem, H.A.: Solutions of the space-time fractional of some nonlinear systems of partial differential equations using modified kudryashov method. Int. J. Pure Appl. Math. **101**(4), 477–487 (2015)
49. El-Wakil, S.A., Abdou, M.A., Elhanbaly A.: New solitons and periodic wave solutions for nonlinear evolution equations. Phys. Lett. A, **353**(1), 40–47 (2006)
50. Fan, E., Zhang, J.: Applications of the Jacobi elliptic function method to special-type nonlinear equations. Phys. Lett. A **305**(6), 383–392 (2002)
51. Bekir, A., Cevikel, A.C.: New solitons and periodic solutions for nonlinear physical models in mathematical physics. Nonlinear Anal.: R. World Appl. **11**(4), 3275–3285 (2010)
52. Maccari, A.: A new integrable Davey–Stewartson-type equation. J. Math. Phys. **40**(8), 3971–3977 (1999)
53. Konopelchenko, B., Dubrovsky, V.: Some new integrable nonlinear evolution equations in 2 + 1 dimensions. Phys. Lett. A **102**(1–2), 15–17 (1984)
54. Feng, Z.S.: The first integral method to study the Burgers-KdV equation. J. Phys. A: Math. Gen. **35**, 343–349 (2002)
55. Su, W.H., Yang, X.J., Jafari, H., Baleanu, D.: Fractional complex transform method for wave equations on Cantor sets within local fractional differential operator. Adv. Differ. Equ. **97**, 1–8 (2013)

56. Yang, X.J., Baleanu, D., Srivastava, H.M.: Local Fractional Integral Transforms and Their Applications. Academic Press, Elsevier, London (2015)
57. He, J.H., Elagan, S.K., Li, Z.B.: Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Phys. Lett. A **376**(4), 257–259 (2012)
58. Güner, O., Bekir, A., Cevikel, A.C.: A variety of exact solutions for the time fractional Cahn-Allen equation. Eur. Phys. J. Plus **130**, 146 (2015). https://doi.org/10.1140/epjp/i2015-15146-9
59. Ding, T.R., Li, C.Z.: Ordinary Differential Equations. Peking University Press, Peking (1996)
60. Bourbaki, N.: Commutative Algebra. Addison-Wesley, Paris (1972)
61. Stakhov, A., Rozin, B.: On a new class of hyperbolic functions. Chaos, Solitons Fractals **23**, 379–389 (2005)
62. Pandir, Y.: Symmetric fibonacci function solutions of some nonlinear partial differential equations. Appl. Math. Inf. Sci. **8**(5), 2237–2241 (2014)
63. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1965)
64. Khater, A.H., Hassan, M.M., Temsah, R.S.: Cnoidal wave solutions for a class of fifth-order KdV equations. Math. Comput. Simulation **70**(4), 221–226 (2005)