Chapter 2
New Approaches for Decomposition Method for the Solution of Differential Equations

2.1 Introduction

In many practical applications regarding the field of science and engineering, the physical systems are modeled by nonlinear partial differential equations (NLPDEs). These equations play a significant role in modeling problems in science and engineering. Many physical phenomena of the physical problems arising in various fields of science and engineering can be elegantly investigated by the NPDEs. Furthermore, NPDEs are widely used to describe complex phenomena in various fields of sciences, such as physics, biology, and chemistry and engineering. Because, in many of the cases exact solutions are very difficult or even impossible to obtain for NPDEs, the approximate analytical solutions are particularly important for the study of dynamic systems for analyzing their physical nature. In the case of approximate analytical solutions, the success of a certain approximation method depends on the nonlinearities that occur in the studied problem, and thus a general algorithm for the construction of such approximate solutions do not exist in the general cases. Various methods have been devised to find the exact and approximate solutions of nonlinear partial differential equations in order to impart a great deal of information for understanding physical phenomena arising in numerous scientific and engineering fields. The investigation of the analytical solutions of NPDEs plays a prominent role in the study of nonlinear physical phenomena.

In this chapter, the modified decomposition method has been implemented for solving a coupled Klein-Gordon Schrödinger equation. In this purpose, a system of coupled Klein-Gordon Schrödinger equation with appropriate initial values has been solved by using the modified decomposition method. The proposed method does not need linearization, weak nonlinearity assumptions or perturbation theory.

Spatially fractional order diffusion equations are generalizations of classical diffusion equations which are increasingly used in modeling practical superdiffusive problems in fluid flow, finance and other areas of application. This chapter presents the analytical solutions of space fractional diffusion equations by two-step Adomian

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decomposition method (TSADM). By using initial conditions, the explicit solutions of the equations have been presented in the closed form and then their solutions have been represented graphically. The solution procedures of a one-dimensional and a two-dimensional fractional diffusion equation are presented to show the application of the present technique. The solutions obtained by the standard decomposition method have been numerically evaluated and presented in the form of tables and then compared with those obtained by TSADM. After examining the results, it manifests that the present TSADM performs extremely well in terms of efficiency and simplicity.

This chapter also presents the new approach of the Adomian decomposition method (ADM) for the solution of space fractional diffusion equation with insulated ends. A typical example of special interest with fractional space derivative of order α, $1 < \alpha \leq 2$ is considered in the present analysis and solved by ADM after expressing the initial condition as Fourier series. The explicit solution of space fractional diffusion equation has been presented in the closed form and then the numerical solution has been represented graphically. The behaviour of Adomian solutions and the effects of different values of α are shown graphically.

2.2 Outline of the Present Study

The aim of the present chapter is to focus on the study of nonlinear partial differential equations (NLPDEs) that have particular applications appearing in engineering and applied sciences. The analytical approximate methods have been used for solving some specific nonlinear partial differential equations like coupled nonlinear Klein-Gordon-Schrödinger equations, space fractional diffusion equations on finite domain, space fractional diffusion equation with insulated ends, which have a wide variety of applications in physical models.

2.2.1 *Coupled Nonlinear Klein–Gordon–Schrödinger Equations*

The coupled nonlinear Klein–Gordon–Schrödinger (K-G-S) equations are considered in the following form:

$$\begin{aligned} &u_{tt} - u_{xx} + u - |v|^2 = 0 \\ &iv_t + v_{xx} + uv = 0. \end{aligned} \tag{2.1}$$

The modified decomposition method has been applied for solving coupled Klein-Gordon-Schrödinger equations which play an important role in modern physics.

Darwish and Fan [1] have been proposed an algebraic method to obtain the explicit exact solutions for coupled Klein-Gordon-Schrödinger (K-G-S) equations. Recently, the Jacobi elliptic function expansion method has been applied to obtain the solitary wave solutions for coupled K-G-S equations [2]. Hioe [3] has obtained periodic solitary waves for two coupled nonlinear Klein–Gordon and Schrödinger equations. Bao and Yang [4] have presented efficient, unconditionally stable and accurate numerical methods for approximations of the Klein-Gordon-Schrödinger equations. In order to determine the explicit series solutions of the coupled K-G-S equations, the notion of Adomian's decomposition method (in short ADM) [5, 6] has been used. Without the use of any linearization or transformation method, the ADM accurately computes the series solution. The ADM method which is of great interest to applied sciences [5–7], provides the solution in a rapidly convergent series with components that can be elegantly computed. The nonlinear equations are solved easily and elegantly without linearizing the problem by using the ADM [5, 6]. Large classes of linear and nonlinear differential equations, both ordinary as well as partial, can be solved by the Adomian decomposition method [5–41]. A reliable modification of Adomian decomposition method has been done by Wazwaz [42]. The decomposition method provides an effective procedure for analytical solution of a wide and general class of dynamical systems representing real physical problems [5–10, 12, 14–20, 23–25, 28–38, 40, 41]. This method efficiently works for initial-value or boundary-value problems and for linear or nonlinear, ordinary or partial differential equations and even for stochastic systems. Moreover, we have the advantage of a single global method for solving ordinary or partial differential equations as well as many types of other equations. Recently, the solution of the fractional differential equation has been obtained through the Adomian decomposition method by the researchers [38–40]. The method has features in common with many other methods, but it is distinctly different on close examinations, and one should not be misled by apparent simplicity into superficial conclusions [5, 6].

In the present chapter, the modified decomposition method (in short MDM) has been used to obtain the analytical approximate solutions of the coupled sine-Gordon equations (2.1).

2.2.2 *Space Fractional Diffusion Equations on Finite Domain*

Fractional diffusion equations are used to model problems in physics [43–45], finance [46–49], and hydrology [50–54]. Fractional space derivatives may be used to formulate anomalous dispersion models, where a particle plume spreads at a rate that is different than the classical Brownian motion model. When a fractional derivative of order $1 < \alpha < 2$ replaces the second derivative in a diffusion or dispersion model, it leads to a superdiffusive flow model. Nowadays, fractional diffusion equation plays important roles in modeling anomalous diffusion and subdiffusion systems, description of fractional random walk, the unification of diffusion and wave propagation phenomenon, see, e.g., the reviews in [43–58], and references therein.

A one-dimensional fractional diffusion equation has been considered as in [59]

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + q(x,t), \tag{2.2}$$

on a finite domain $x_L < x < x_R$ with $1 < \alpha \leq 2$. It is to be assumed that the diffusion coefficient (or diffusivity) $d(x) > 0$. We also assume an initial condition $u(x, t = 0) = s(x)$ for $x_L < x < x_R$ and Dirichlet boundary conditions of the form $u(x_L, t) = 0$ and $u(x_R, t) = b_R(t)$. Equation (2.2) uses a Riemann fractional derivative of order α.

Also, a two-dimensional fractional diffusion equation has been considered as in [60]

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{\alpha} u(x,y,t)}{\partial x^{\alpha}} + e(x,y)\frac{\partial^{\beta} u(x,y,t)}{\partial y^{\beta}} + q(x,y,t), \tag{2.3}$$

on a finite rectangular domain $x_L < x < x_H$ and $y_L < y < y_H$, with fractional orders $1 < \alpha \leq 2$ and $1 < \beta \leq 2$, where the diffusion coefficients $d(x,y) > 0$ and $e(x,y) > 0$. The 'forcing' function $q(x,y,t)$ can be used to represent sources and sinks. We will assume that this fractional diffusion equation has a unique and sufficiently smooth solution under the following initial and boundary conditions. Assume the initial condition $u(x,y,t=0) = f(x,y)$ for $x_L < x < x_H$, $y_L < y < y_H$, and Dirichlet boundary condition $u(x,y,t) = B(x,y,t)$ on the boundary (perimeter) of the rectangular region $x_L < x < x_H$, $y_L < y < y_H$, with the additional restriction that $B(x_L, y, t) = B(x, y_L, t) = 0$. In physical applications, this means that the left/lower boundary is set far away enough from an evolving plume that no significant concentrations reach that boundary. The classical dispersion equation in two dimensions is given by $\alpha = \beta = 2$. The values of $1 < \alpha < 2$, or $1 < \beta < 2$ model a super-diffusive process in that coordinate. Equation (2.3) also uses Riemann fractional derivatives of order α and β.

In this chapter, the new two-step Adomian Decomposition Method (ADM) [6] has been used to obtain the solutions of the fractional diffusion equations (2.2) and (2.3).

2.2.3 Space Fractional Diffusion Equation with Insulated Ends

The fractional differential equations appear more and more frequently in different research areas and engineering applications. Nowadays, fractional diffusion equation plays important roles in modeling anomalous diffusion and subdiffusion systems, description of fractional random walk, the unification of diffusion and wave propagation phenomenon, see, e.g. the reviews in [43, 44, 55–58, 61], and references therein.

In this chapter, the following space fractional diffusion equation with insulated ends has been considered [62]

$$\frac{\partial u(x,t)}{\partial t} = dD_x^\alpha u(x,t), \quad 0 < x < L, \quad t \geq 0, \quad 1 < \alpha \leq 2, \tag{2.4}$$

where d is the diffusion coefficient and D_x^α is Caputo fractional derivative of order α, which is defined as [63]

$$D_x^\alpha f(x) = \begin{cases} \frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}, & \alpha = m \in N \\ \frac{1}{\Gamma(m-\alpha)} \int\limits_0^x (x-\xi)^{m-\alpha-1} \frac{\mathrm{d}^m f(\xi)}{\mathrm{d}\xi^m} \mathrm{d}\xi, & m-1 < \alpha < m, \ m \in N. \end{cases} \tag{2.5}$$

We further consider the following Dirichlet's boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0, \quad t \geq 0, \tag{2.6}$$

and initial condition

$$u(x,0) = f(x), \quad 0 \leq x \leq L \tag{2.7}$$

In the present chapter, the Adomian decomposition method (ADM) [5, 6] with a simple variation has been used to obtain the analytical approximate solution of space fractional diffusion equation (2.4) with insulated ends.

2.3 Analysis of Proposed Methods

In this section, the analysis of modified decomposition method (MDM), the new two-step Adomian Decomposition Method, and Adomian decomposition method with a simple variation have been presented for solving the above physical problems.

2.3.1 A Modified Decomposition Method for Coupled K-G-S Equations

The coupled K-G-S equations (2.1) can be written in the following operator form

$$\begin{aligned} L_{tt}u &= L_{xx}u - u + N(u,v) \\ L_t v &= iL_{xx}v + iM(u,v) \end{aligned} \tag{2.8}$$

where $L_t \equiv \frac{\partial}{\partial t}$, $L_{tt} \equiv \frac{\partial^2}{\partial t^2}$ and $L_{xx} \equiv \frac{\partial^2}{\partial x^2}$ symbolize the linear differential operators and the notations $N(u,v) = |v|^2$ and $M(u,v) = uv$ symbolize the nonlinear operators.

Applying the two-fold integration inverse operator $L_{tt}^{-1} \equiv \int_0^t \int_0^t (.)\mathrm{d}t\mathrm{d}t$ to the system (2.8) and using the specified initial conditions yields

$$\begin{aligned} u(x,t) &= u(x,0) + tu_t(x,0) + L_{tt}^{-1}L_{xx}u - L_{tt}^{-1}u + L_{tt}^{-1}N(u,v) \\ v(x,t) &= v(x,0) + iL_t^{-1}L_{xx}v + iL_t^{-1}M(u,v). \end{aligned} \tag{2.9}$$

The Adomian decomposition method [5, 6] assumes an infinite series of solutions for unknown function $u(x,t)$ and $v(x,t)$ given by

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t), \\ v(x,t) &= \sum_{n=0}^{\infty} v_n(x,t), \end{aligned} \tag{2.10}$$

and nonlinear operators $N(u,v) = |v|^2$ and $M(u,v) = uv$ by the infinite series of Adomain polynomials given by

$$\begin{aligned} N(u,v) &= \sum_{n=0}^{\infty} A_n(u_0,u_1,\ldots,u_n,v_0,v_1,\ldots,v_n); \\ M(u,v) &= \sum_{n=0}^{\infty} B_n(u_0,u_1,\ldots,u_n,v_0,v_1,\ldots,v_n), \end{aligned}$$

where A_n and B_n are the appropriate Adomian's polynomial which are generated according to algorithm determined in [5, 6]. For the nonlinear operator $N(u,v)$, these polynomials can be defined as

$$A_n(u_0,u_1,\ldots,u_n,v_0,v_1,\ldots,v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{2.11}$$

Similarly for the nonlinear operator $M(u,v)$,

$$B_n(u_0,u_1,\ldots,u_n,v_0,v_1,\ldots,v_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[M\left(\sum_{k=0}^{\infty}\lambda^k u_k, \sum_{k=0}^{\infty}\lambda^k v_k\right)\right]_{\lambda=0}, \quad n \geq 0. \tag{2.12}$$

These formulae are easy to set computer code to get as many polynomials as we need in the calculation of the numerical as well as explicit solutions. For the sake of convenience of the readers, we can give the first few Adomian polynomials for $N(u,v) = |v|^2$, $M(u,v) = uv$ of the nonlinearity as

$$\begin{aligned}
&A_0 = v_0\overline{v}_0,\\
&A_1 = v_1\overline{v}_0 + v_0\overline{v}_1,\\
&A_2 = v_2\overline{v}_0 + v_0\overline{v}_2 + v_1\overline{v}_1,\\
&\ldots\\
&\text{and}\\
&B_0 = u_0v_0,\\
&B_1 = u_1v_0 + u_0v_1,\\
&B_2 = u_2v_0 + u_0v_2 + u_1v_1,\\
&\ldots
\end{aligned}$$

and so on, the rest of the polynomials can be constructed in a similar manner.

Substituting the initial conditions into Eq. (2.9) and identifying the zeroth components u_0 and v_0, we then obtain the subsequent components by using the following recursive equations according to the standard ADM

$$\begin{aligned}
u_{n+1} &= L_{tt}^{-1}L_{xx}u_n - L_{tt}^{-1}u_n + L_{tt}^{-1}A_n, \quad n \geq 0,\\
v_{n+1} &= iL_t^{-1}L_{xx}v_n + iL_t^{-1}B_n, \quad n \geq 0.
\end{aligned} \tag{2.13}$$

Recently, Wazwaz [42] proposed that the construction of the zeroth component of the decomposition series can be defined in a slightly different way. In [42], he assumed that if the zeroth component $u_0 = g$ and the function g is possible to divide into two parts such as g_1 and g_2, the one can formulate the recursive algorithm for u_0 and general term u_{n+1} in a form of the modified recursive scheme as follows:

$$\begin{aligned}
u_0 &= g_1,\\
u_1 &= g_2 + L_{tt}^{-1}L_{xx}u_0 - L_{tt}^{-1}u_0 + L_{tt}^{-1}A_0,\\
u_{n+1} &= L_{tt}^{-1}L_{xx}u_n - L_{tt}^{-1}u_n + L_{tt}^{-1}A_n, \quad n \geq 1.
\end{aligned} \tag{2.14}$$

Similarly, if the zeroth component $v_0 = g'$ and the function g' is possible to divide into two parts such as g'_1 and g'_2, the one can formulate the recursive algorithm for v_0 and general term v_{n+1} in a form of the modified recursive scheme as follows:

$$\begin{aligned} v_0 &= g_1', \\ v_1 &= g_2' + iL_t^{-1}L_{xx}v_0 + iL_t^{-1}B_0, \\ v_{n+1} &= iL_t^{-1}L_{xx}v_n + iL_t^{-1}B_n, \quad n \geq 1. \end{aligned} \tag{2.15}$$

This type of modification is giving more flexibility to the ADM in order to solve complicate nonlinear differential equations. In many cases, the modified decomposition scheme avoids unnecessary computation especially in the calculation of the Adomian polynomials. The computation of these polynomials will be reduced very considerably by using the MDM.

It is worth noting that the zeroth components u_0 and v_0 are defined then the remaining components u_n and $v_n, \quad n \geq 1$ can be completely determined. As a result, the components $u_0, u_1, \ldots,$ and $v_0, v_1, \ldots,$ are identified and the series solutions thus entirely determined. However, in many cases, the exact solution in a closed form may be obtained.

The decomposition series solutions (2.10) generally converge very rapidly in real physical problems [6]. The rapidity of this convergence means that few terms are required. The convergence of this method has been rigorously established by Cherruault [64], Abbaoui and Cherruault [65, 66] and Himoun et al. [67]. The practical solutions will be the n-term approximations ϕ_n and ψ_n

$$\begin{aligned} \phi_n &= \sum_{i=0}^{n-1} u_i(x,t), \quad n \geq 1, \\ \psi_n &= \sum_{i=0}^{n-1} v_i(x,t), \quad n \geq 1. \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} \lim_{n\to\infty} \phi_n &= u(x,t), \\ \lim_{n\to\infty} \psi_n &= v(x,t). \end{aligned} \tag{2.17}$$

2.3.2 The Two-Step Adomian Decomposition Method

Equation (2.2) can be rewritten as

$$L_t u(x,t) = d(x) D_x^{\alpha} u(x,t) + q(x,t) \tag{2.18}$$

where $L_t \equiv \frac{\partial}{\partial t}$ which is an easily invertible linear operator, $D_x^{\alpha}(.)$ is the Riemann-Liouville derivative of order α.

The solution $u(x,t)$ of Eq. (2.18) is represented by the decomposition series

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.19}$$

Now, operating L_t^{-1} both sides of Eq. (2.18) and then substituting Eq. (2.19) yields

$$u(x,t) = u(x,0) + L_t^{-1}\left(d(x)D_x^{\alpha}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}(q(x,t)) \tag{2.20}$$

Each term of series (2.19) is given by the standard ADM recurrence relation

$$\begin{aligned} u_0 &= f, \\ u_{n+1} &= L_t^{-1}\big(d(x)D_x^{\alpha}u_n\big), \quad n \geq 0 \end{aligned} \tag{2.21}$$

where $f = u(x,0) + L_t^{-1}(q(x,t))$.

It is worth noting that once the zeroth component u_0 is defined, then the remaining components $u_n,\ n \geq 1$ can be completely determined; each term is computed by using the previous term. As a result, the components $u_0, u_1, \ldots$ are identified and the series solutions thus entirely determined. However, in many cases, the exact solution in a closed form may be obtained.

Without loss of generality let us assume that the zeroth component $u_0 = f$ and the function f is possible to divide into two parts such as f_1 and f_2, then one can formulate the recursive algorithm for u_0 and general term u_{n+1} in a form of the modified decomposition method (MDM) recursive scheme as follows:

$$\begin{aligned} u_0 &= f_1 \\ u_1 &= f_2 + L_t^{-1}\big(d(x)D_x^{\alpha}u_n\big) \\ u_{n+1} &= L_t^{-1}\big(d(x)D_x^{\alpha}u_n\big), \quad n \geq 1. \end{aligned} \tag{2.22}$$

Comparing the recursive scheme (2.21) of the standard Adomain method with the recursive scheme (2.22) of the modified technique leads to the conclusion that in Eq. (2.21) the zeroth component was defined by the function f, whereas in Eq. (2.22), the zeroth component u_0 is defined only by a part f_1 of f. The remaining part f_2 of f is added to the definition of the component u_1 in Eq. (2.22). Although the modified technique needs only a slight variation from the standard Adomian decomposition method, the results are promising in that it minimizes the size of calculations needed and will accelerate the convergence. The modification could lead to a promising approach for many applications in applied science.

The decomposition series solution (2.19) generally converges very rapidly in real physical problems [5, 6]. Here also, the practical solution will be the n-term approximation ϕ_n

$$\phi_n = \sum_{i=0}^{n-1} u_i(x,t), \quad n \geq 1 \tag{2.23}$$

with

$$\lim_{n\to\infty} \phi_n = u(x,t). \tag{2.24}$$

Luo [68] presented the theoretical support of how the exact solution can be achieved by using only two iterations in the modified decomposition method. In detail, it is possible because all other components vanish if the zeroth component is equal to the exact solution.

Although the modified decomposition method may provide the exact solution by using two iterations only, the criterion of dividing the function f into two practical parts, and the case where f consists only of one term remains unsolved so far. The two-step Adomian decomposition method (TSADM) overcomes the difficulties arising in the modified decomposition method.

In the following, Luo [68] presents the two-step Adomian decomposition method. For the convenience of the reader, we consider the differential equation

$$Lu + Ru + Nu = g, \tag{2.25}$$

where L is the highest order derivative which is assumed to be easily invertible, R is a linear differential operator of order less than L, Nu represents the nonlinear terms, and g is the source term.

The main ideas of the two-step Adomian decomposition method are:

1. Applying the inverse operator L^{-1} to g, and using the given conditions we obtain

$$\varphi = \phi + L^{-1}g,$$

where the function ϕ represents the term arising from using the given conditions, all are assumed to be prescribed.
Let

$$\varphi = \sum_{i=0}^{m} \varphi_i, \tag{2.26}$$

where $\phi_0, \phi_1, \ldots, \phi_m$ are the terms arising from integrating the source term g and from using the given conditions. Based on this, we define $u_0 = \varphi_k + \ldots + \varphi_{k+s}$ where $k = 0, 1, \ldots, m$, $s = 0, 1, \ldots, m-k$. Then we verify that u_0 satisfies the original equation Eq. (2.25) and the given conditions by substitution, once the exact solution is obtained, we stop. Otherwise, we go to the following step two.

2. We set $u_0 = \varphi$ and continue with the standard Adomian recursive relation

$$u_{k+1} = -L^{-1}(Ru_k) - L^{-1}(A_k), \quad k \geq 0.$$

Similarly, for Eq. (2.3), we can obtain

$$\begin{aligned} u(x,y,t) = u(x,y,0) + L_t^{-1}\left(d(x,y)D_x^{\alpha}\left(\sum_{n=0}^{\infty} u_n\right)\right) \\ + L_t^{-1}\left(e(x,y)D_y^{\beta}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}(q(x,y,t)). \end{aligned} \tag{2.27}$$

Now, the standard Adomian decomposition method recurrence scheme is

$$\begin{aligned} u_0 &= f, \\ u_{n+1} &= L_t^{-1}\left(d(x,y)D_x^{\alpha}u_n\right) + L_t^{-1}\left(e(x,y)D_y^{\beta}u_n\right), \quad n \geq 0, \end{aligned} \tag{2.28}$$

where $f = u(x,y,0) + L_t^{-1}(q(x,y,t))$.

In the other hand, the modified decomposition method recursive scheme is as follows

$$\begin{aligned} u_0 &= f_1 \\ u_1 &= f_2 + L_t^{-1}\left(d(x,y)D_x^{\alpha}u_0\right) + L_t^{-1}\left(e(x,y)D_y^{\beta}u_0\right) \\ u_{n+1} &= L_t^{-1}\left(d(x,y)D_x^{\alpha}u_n\right) + L_t^{-1}\left(e(x,y)D_y^{\beta}u_n\right), \quad n \geq 1. \end{aligned} \tag{2.29}$$

Compared to the standard Adomian method and the modified method, we can see that the two-step Adomian method may provide the solution by using two iterations only.

2.3.3 ADM with a Simple Variation for Space Fractional Diffusion Model

Equation (2.4) can be written as

$$L_t u(x,t) = dD_x^{\alpha}u(x,t), \tag{2.30}$$

where $L_t \equiv \frac{\partial}{\partial t}$ which is an easily invertible linear operator, $D_x^{\alpha}(\cdot)$ is the Caputo fractional derivative of order α.

If $f(x)$ is a periodic function with period L, then the Fourier Cosine series of $f(x)$ in $[0, L]$ can be obtained as

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(\xi) \cos\left(\frac{n\pi\xi}{L}\right) \mathrm{d}\xi \cos\left(\frac{n\pi x}{L}\right). \tag{2.31}$$

The Fourier Cosine series is well adapted to functions whose first order derivatives are zero at the endpoints $x = 0$ and $x = L$ of the interval $[0, L]$, since all the basis functions $\cos\left(\frac{n\pi x}{L}\right)$ have this property.

Therefore, after considering the initial condition $u(x, 0) = f(x)$ as Fourier Cosine series, we can take

$$u(x, 0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(\xi) \cos\left(\frac{n\pi\xi}{L}\right) \mathrm{d}\xi \cos_\gamma\left(\frac{n\pi x}{L}\right), \tag{2.32}$$

where $\cos_\gamma\left(\frac{n\pi x}{L}\right)$ is the Generalized Cosine function defined in [69] and $\gamma = \alpha/2$, $\gamma \in \left(\frac{1}{2}, 1\right]$.

It is known that

$$D_x^\gamma \sin_\gamma x = \cos_\gamma x, \lim_{\gamma \to 1} \sin_\gamma x = \sin x$$

and

$$D_x^\gamma \sin_\gamma x = \cos_\gamma x,$$

where $\cos_\gamma x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\gamma}}{\Gamma(2n\gamma + 1)}$.

According to the Adomian decomposition method, we can write,

$$u(x, t) = u(x, 0) + L_t^{-1}(\mathrm{d}D_x^\alpha u(x, t)), \tag{2.33}$$

where

$$\begin{aligned} u_0 &= u(x, 0) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{L} \int_0^L f(\xi) \cos\left(\frac{n\pi\xi}{L}\right) \mathrm{d}\xi \cos_\gamma\left(\frac{n\pi x}{L}\right), \\ u_1 &= L_t^{-1}(\mathrm{d}D_x^\alpha u_0), \\ u_2 &= L_t^{-1}(\mathrm{d}D_x^\alpha u_1), \\ u_3 &= L_t^{-1}(\mathrm{d}D_x^\alpha u_2), \end{aligned}$$

and so on.

The decomposition series solution

$$u = \sum_{n=0}^{\infty} u_n,$$

generally converges very rapidly in real physical problems [6]. The practical solution will be the n-term approximation ϕ_n

$$\phi_n = \sum_{i=0}^{n-1} u_i(x,t), \quad n \geq 1 \tag{2.34}$$

with

$$\lim_{n\to\infty} \phi_n = u(x,t). \tag{2.35}$$

2.4 Solutions of Coupled Klein–Gordon–Schrödinger Equations

In this section, the modified decomposition method has been used for getting the analytical approximate solutions for the coupled K-G-S equations (2.1).

2.4.1 Implementation of MDM for Analytical Approximate Solutions of Coupled K-G-S Equations

We first consider the coupled K-G-S equations (2.1) with the initial conditions

$$\begin{aligned} &u(x,0) = 6B^2 \sec h^2(Bx), \quad u_t(x,0) = -12B^2 c \sec h^2(Bx)\tanh(Bx), \\ &v(x,0) = 3B \sec h^2(Bx) \mathrm{e}^{idx}, \end{aligned} \tag{2.36}$$

where $B(\geq 1/2)$, c and d are arbitrary constants.

Using (2.14) and (2.15) with (2.11) and (2.12) respectively and considering $c = \frac{\sqrt{4B^2-1}}{2}$, $d = -\frac{c}{2B}$ for the coupled K-G-S equations (2.1) and initial conditions (2.36) gives

$$
\begin{aligned}
u_0 &= 0,\\
u_1 &= u(x,0) + tu_t(x,0) + L_{tt}^{-1}L_{xx}u_0 - L_{tt}^{-1}u_0 + L_{tt}^{-1}A_0\\
&= 6B^2 \sec h^2(Bx) - 12B^2 ct \sec h^2(Bx)\tanh(Bx),\\
u_2 &= L_{tt}^{-1}L_{xx}u_1 - L_{tt}^{-1}u_1 + L_{tt}^{-1}A_1\\
&= t^2(-3B^2 \sec h^2(Bx) - 9B^4 \sec h^4(Bx) + 3B^4 \cosh(3Bx)\sec h^5(Bx))\\
&\quad + t^3(-2B^4 c \sec h^5(Bx)(-11\sinh(Bx) + \sinh(3Bx))\\
&\quad + 2B^2 c \sec h^2(Bx)\tanh(Bx)),
\end{aligned}
$$

and

$$
\begin{aligned}
v_0 &= 0,\\
v_1 &= v(x,0) + iL_t^{-1}L_{xx}v_0 + iL_t^{-1}B_0\\
&= 3B \sec h^2(Bx)\mathrm{e}^{idx},\\
v_2 &= iL_t^{-1}L_{xx}v_1 + iL_t^{-1}B_1\\
&= -3iB\mathrm{e}^{idx} t \sec h^2(Bx)(2B^2 \sec h^2(Bx) + (d + 2iB\tanh^2(Bx))^2)
\end{aligned}
$$

and so on, in this manner, the other components of the decomposition series can be easily obtained of which $u(x,t)$ and $v(x,t)$ were evaluated in the following series form

$$
\begin{aligned}
u(x,t) &= 6B^2 \sec h^2(Bx) - 12B^2 ct \sec h^2(Bx)\tanh(Bx)\\
&\quad + t^2(-3B^2 \sec h^2(Bx) - 9B^4 \sec h^4(Bx) + 3B^4\cosh(3Bx)\sec h^5(Bx))\\
&\quad + t^3(-2B^4 c \sec h^5(Bx)(-11\sinh(Bx) + \sinh(3Bx)) + 2B^2 c \sec h^2(Bx)\tanh(Bx)) + \cdots,\\
v(x,t) &= 3B \sec h^2(Bx)\mathrm{e}^{idx}
\end{aligned}
\tag{2.37}
$$

$$
-3iB\mathrm{e}^{idx} t \sec h^2(Bx)(2B^2 \sec h^2(Bx) + (d + 2iB\tanh^2(Bx))^2) + \cdots. \tag{2.38}
$$

follow immediately with the aid of *Mathematica* [70].

2.4.2 Numerical Results and Discussion for Coupled K-G-S Equations

In this section, we analyze the numerical solutions for coupled K-G-S equations obtained by the modified decomposition method.

The numerical simulations using MDM

In the present numerical experiment, Eqs. (2.37) and (2.38) have been used to draw the graphs as shown in Figs. 2.1, 2.2, 2.3 and 2.4 respectively.

The numerical solutions of the coupled K-G-S equations (2.1) have been shown in Figs. 2.1, 2.2, 2.3 and 2.4 with the help of five-term and four-term approximations ϕ_5 and ψ_4 for the decomposition series solutions of $u(x,t)$ and $v(x,t)$ respectively. In the present numerical computation, we have assumed $B = 0.575$.

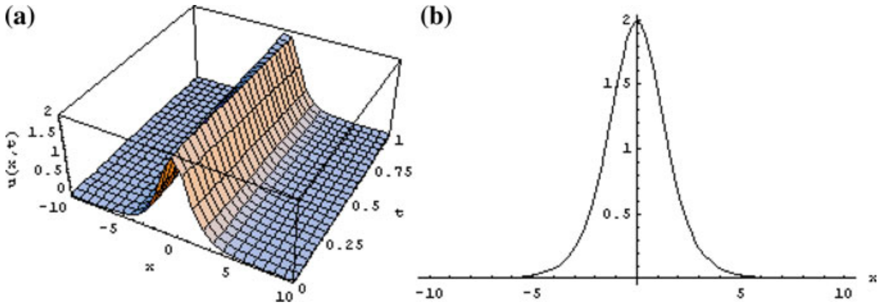

Fig. 2.1 **a** The decomposition method solution for $u(x,t)$, **b** Corresponding 2D solution for $u(x,t)$ when $t = 0$

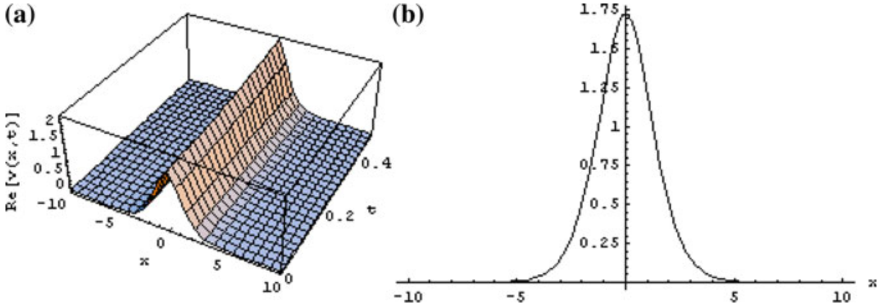

Fig. 2.2 **a** The decomposition method solution for $\mathrm{Re}(v(x,t))$, **b** Corresponding 2D solution for $\mathrm{Re}(v(x,t))$ when $t = 0$

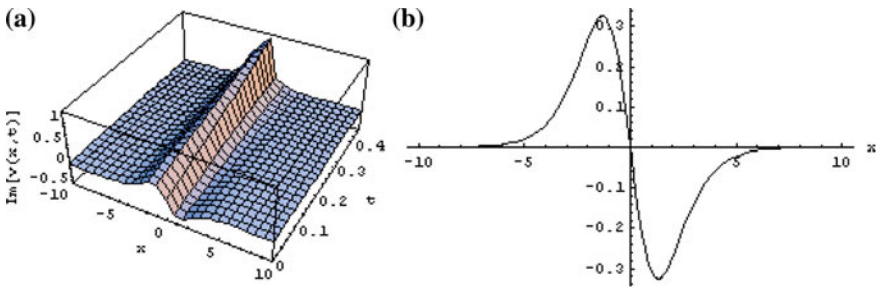

Fig. 2.3 **a** The decomposition method solution for $\mathrm{Im}(v(x,t))$, **b** Corresponding 2D solution for $\mathrm{Im}(v(x,t))$ when $t = 0$

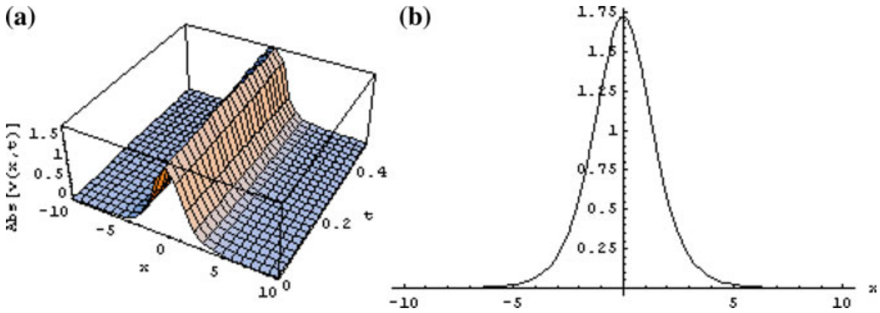

Fig. 2.4 **a** The decomposition method solution for $\text{Abs}(v(x,t))$, **b** Corresponding 2D solution for $\text{Abs}(v(x,t))$ when $t = 0$

2.5 Implementation of Two-Step Adomian Decomposition Method for Space Fractional Diffusion Equations on a Finite Domain

In this section, the new two-step Adomian decomposition method has been implemented for the solutions of one-dimensional and two-dimensional space fractional diffusion equations with finite domain respectively.

2.5.1 Solution of One-Dimensional Space Fractional Diffusion Equation

Let us consider the one-dimensional fractional diffusion equation (2.2), as taken in [59]

$$\frac{\partial u(x,t)}{\partial t} = d(x)\frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + q(x,t), \tag{2.39}$$

on a finite domain $0 < x < 1$, with the diffusion coefficient

$$d(x) = \Gamma(2.2)x^{2.8}/6 = 0.183634x^{2.8},$$

the source/sink function

$$q(x,t) = -(1+x)\mathrm{e}^{-t}x^3,$$

the initial condition

$$u(x,0) = x^3, \quad \text{for } 0 < x < 1$$

and the boundary conditions

$$u(0,t) = 0, u(1,t) = \mathrm{e}^{-t}, \quad \text{for } t > 0.$$

Now, Eq. (2.39) can be rewritten in operator form as

$$L_t u(x,t) = d(x) D_x^{1.8} u(x,t) + q(x,t), \tag{2.40}$$

where $L_t \equiv \frac{\partial}{\partial t}$ symbolizes the easily invertible linear differential operator, $D_x^{1.8}(.)$ is the Riemann–Liouville derivative of order 1.8.

Applying the one-fold integration inverse operator $L_t^{-1} \equiv \int_0^t (.)\mathrm{d}t$ to Eq. (2.40) and using the specified initial condition yields

$$\begin{aligned} u(x,t) &= u(x,0) + L_t^{-1}\left(d(x) D_x^{1.8}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}(q(x,t)) \\ &= \mathrm{e}^{-t}x^3 + \mathrm{e}^{-t}x^4 - x^4 + L_t^{-1}\left(d(x) D_x^{1.8}\left(\sum_{n=0}^{\infty} u_n\right)\right). \end{aligned} \tag{2.41}$$

The standard Adomian decomposition method:

$$\begin{aligned} u_0 &= \mathrm{e}^{-t}x^3 + \mathrm{e}^{-t}x^4 - x^4, \\ u_1 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}}{6} \frac{\partial^{1.8} u_0}{\partial x^{1.8}}\right) \\ &= (-\mathrm{e}^{-t} + 1)x^4 + \frac{4(-\mathrm{e}^{-t} + 1 - t)x^5}{2.2}, \\ u_2 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}}{6} \frac{\partial^{1.8} u_1}{\partial x^{1.8}}\right) \\ &= \frac{4(\mathrm{e}^{-t} + t - 1)x^5}{2.2} + \frac{80\left(\mathrm{e}^{-t} - \frac{t^2}{2!} + t - 1\right)x^6}{3.2 \times 2.2^2}, \\ u_3 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}}{6} \frac{\partial^{1.8} u_2}{\partial x^{1.8}}\right) \\ &= \frac{80\left(-\mathrm{e}^{-t} + \frac{t^2}{2!} - t + 1\right)x^6}{3.2 \times 2.2^2} + \frac{80\Gamma(6)\left(-\mathrm{e}^{-t} - \frac{t^3}{3!} + \frac{t^2}{2!} - t + 1\right)x^7}{4.2 \times 3.2^2 \times 2.2^3}, \end{aligned}$$

and so on.

Therefore, according to the decomposition method, the two-term approximation ϕ_2 is

$$\begin{aligned}\phi_2 &= u_0 + u_1 \\ &= e^{-t}x^3 + \frac{4(-e^{-t}+1-t)x^5}{2.2}.\end{aligned} \tag{2.42}$$

Therefore, the three-term approximation ϕ_3 is

$$\begin{aligned}\phi_3 &= u_0 + u_1 + u_2 \\ &= e^{-t}x^3 + \frac{80\left(e^{-t} - \frac{t^2}{2!} + t - 1\right)x^6}{3.2 \times 2.2^2}.\end{aligned} \tag{2.43}$$

Therefore, according to the decomposition method, the four-term approximation ϕ_4 is

$$\begin{aligned}\phi_4 &= u_0 + u_1 + u_2 + u_3 \\ &= e^{-t}x^3 + \frac{80\Gamma(6)\left(-e^{-t} - \frac{t^3}{3!} + \frac{t^2}{2!} - t + 1\right)x^7}{4.2 \times 3.2^2 \times 2.2^3}\end{aligned} \tag{2.44}$$

The TSADM:

Using the scheme (2.26) of TSADM, we have

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2,$$

where $\varphi_0 = e^{-t}x^3$, $\varphi_1 = e^{-t}x^4$, $\varphi_2 = -x^4$.

Clearly, φ_1 and φ_2 do not satisfy the initial condition $u(x, 0) = x^3$. By selecting $u_0 = \varphi_0$ and verifying that u_0 justifies Eq. (2.39) and satisfies the initial as well as boundary conditions, we obtain the following terms from the recursive scheme of MDM

$$\begin{aligned}u_0 &= e^{-t}x^3, \\ u_1 &= e^{-t}x^4 - x^4 + L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}}{6}\frac{\partial^{1.8}u_0}{\partial x^{1.8}}\right) \\ &= e^{-t}x^4 - x^4 - (e^{-t} - 1)x^4 \\ &= 0 \\ u_2 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}}{6}\frac{\partial^{1.8}u_1}{\partial x^{1.8}}\right) \\ &= 0\end{aligned}$$

and so on.

Therefore, the solution is

$$u(x,t) = \mathrm{e}^{-t}x^3 \tag{2.45}$$

The solution (2.45) can be verified through substitution in Eq. (2.39).

2.5.2 Solution of Two-Dimensional Space Fractional Diffusion Equation

Let us consider the two-dimensional fractional diffusion equation Eq. (1.2), as in [60]

$$\frac{\partial u(x,y,t)}{\partial t} = d(x,y)\frac{\partial^{1.8}u(x,y,t)}{\partial x^{1.8}} + e(x,y)\frac{\partial^{1.6}u(x,y,t)}{\partial y^{1.6}} + q(x,y,t), \tag{2.46}$$

on a finite rectangular domain $0 < x < 1$, $0 < y < 1$, for $0 \le t \le T_{\text{end}}$ with the diffusion coefficients

$$d(x,y) = \Gamma(2.2)x^{2.8}y/6,$$

and

$$e(x,y) = 2xy^{2.6}/\Gamma(4.6),$$

and the forcing function

$$q(x,y,t) = -(1+2xy)\mathrm{e}^{-t}x^3y^{3.6},$$

with the initial condition

$$u(x,y,0) = x^3y^{3.6},$$

and Dirichlet boundary conditions on the rectangle in the form $u(x,0,t) = u(0,y,t) = 0$, $u(x,1,t) = \mathrm{e}^{-t}x^3$, and $u(1,y,t) = \mathrm{e}^{-t}y^{3.6}$, for all $t \ge 0$.

Now, Eq. (2.46) can be rewritten in operator form as

$$L_t u(x,y,t) = d(x,y)D_x^{1.8}u(x,y,t) + e(x,y)D_y^{1.6}u(x,y,t) + q(x,y,t), \tag{2.47}$$

where $L_t \equiv \frac{\partial}{\partial t}$ symbolizes the easily invertible linear differential operator, $D_x^{1.8}(.)$ and $D_y^{1.6}(.)$ are the Riemann–Liouville derivatives of order 1.8 and 1.6 respectively.

Applying the one-fold integration inverse operator $L_t^{-1} \equiv \int_0^t (.)\mathrm{d}t$ to the Eq. (2.47) and using the specified initial condition yields

$$
\begin{aligned}
u(x,y,t) &= u(x,y,0) + L_t^{-1}\left(d(x,y)D_x^{1.8}\left(\sum_{n=0}^{\infty} u_n\right)\right) \\
&\quad + L_t^{-1}\left(e(x,y)D_y^{1.6}\left(\sum_{n=0}^{\infty} u_n\right)\right) + L_t^{-1}(q(x,y,t)) \\
&= x^3y^{3.6}\mathrm{e}^{-t} + 2x^4y^{4.6}\mathrm{e}^{-t} - 2x^4y^{4.6} + L_t^{-1}\left(d(x,y)D_x^{1.8}\left(\sum_{n=0}^{\infty} u_n\right)\right) \\
&\quad + L_t^{-1}\left(e(x,y)D_y^{1.6}\left(\sum_{n=0}^{\infty} u_n\right)\right)
\end{aligned}
\tag{2.48}
$$

The standard Adomian decomposition method:

$$
\begin{aligned}
u_0 &= x^3y^{3.6}\mathrm{e}^{-t} + 2x^4y^{4.6}\mathrm{e}^{-t} - 2x^4y^{4.6}, \\
u_1 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}y}{6}\frac{\partial^{1.8}u_0}{\partial x^{1.8}}\right) + L_t^{-1}\left(\frac{2xy^{2.6}}{\Gamma(4.6)}\frac{\partial^{1.6}u_0}{\partial y^{1.6}}\right) \\
&= 2x^4y^{4.6}(-\mathrm{e}^{-t}+1) + \left(\frac{8}{2.2} + \frac{2\times 4.6}{3}\right)x^5y^{5.6}(-\mathrm{e}^{-t}+1-t) \\
&= 2x^4y^{4.6}(-\mathrm{e}^{-t}+1) + \frac{1106}{165}x^5y^{5.6}(-\mathrm{e}^{-t}+1-t), \\
u_2 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}y}{6}\frac{\partial^{1.8}u_1}{\partial x^{1.8}}\right) + L_t^{-1}\left(\frac{2xy^{2.6}}{\Gamma(4.6)}\frac{\partial^{1.6}u_1}{\partial y^{1.6}}\right) \\
&= \frac{1106}{165}x^5y^{5.6}(\mathrm{e}^{-t}-1+t) + \frac{9101827}{272250}x^6y^{6.6}\left(\mathrm{e}^{-t}-1+t-\frac{t^2}{2}\right)
\end{aligned}
$$

and so on.

Therefore, according to the decomposition method, the three-term approximation ϕ_3 is

$$
\begin{aligned}
\phi_3 &= u_0 + u_1 + u_2 \\
&= x^3y^{3.6}\mathrm{e}^{-t} + \frac{9101827}{272250}x^6y^{6.6}\left(\mathrm{e}^{-t}-1+t-\frac{t^2}{2}\right)
\end{aligned}
\tag{2.49}
$$

The TSADM:

Using the scheme (2.26) of TSADM, we have

$$\varphi = \varphi_0 + \varphi_1 + \varphi_2,$$

where $\varphi_0 = x^3y^{3.6}\mathrm{e}^{-t}$, $\varphi_1 = 2x^4y^{4.6}\mathrm{e}^{-t}$, $\varphi_2 = -2x^4y^{4.6}$.

Clearly, φ_1 and φ_2 do not satisfy the initial condition $u(x,y,0) = x^3y^{3.6}$. By selecting $u_0 = \varphi_0$ and verifying that u_0 justifies Eq. (2.46) and satisfies the initial as well as boundary conditions, we obtain the following terms from the recursive scheme of MDM

$$\begin{aligned}
u_0 &= x^3 y^{3.6} \mathrm{e}^{-t}, \\
u_1 &= 2x^4 y^{4.6} \mathrm{e}^{-t} - 2x^4 y^{4.6} + L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}y}{6}\frac{\partial^{1.8} u_0}{\partial x^{1.8}}\right) + L_t^{-1}\left(\frac{2xy^{2.6}}{\Gamma(4.6)}\frac{\partial^{1.6} u_0}{\partial y^{1.6}}\right) \\
&= 2x^4 y^{4.6} \mathrm{e}^{-t} - 2x^4 y^{4.6} - 2(\mathrm{e}^{-t} - 1)x^4 y^{4.6} \\
&= 0, \\
u_2 &= L_t^{-1}\left(\frac{\Gamma(2.2)x^{2.8}y}{6}\frac{\partial^{1.8} u_1}{\partial x^{1.8}}\right) + L_t^{-1}\left(\frac{2xy^{2.6}}{\Gamma(4.6)}\frac{\partial^{1.6} u_1}{\partial y^{1.6}}\right) \\
&= 0
\end{aligned}$$

and so on.

Therefore, the solution is

$$u(x, y, t) = x^3 y^{3.6} \mathrm{e}^{-t}. \tag{2.50}$$

The solution (2.50) can be verified through substitution in Eq. (2.46).

2.5.3 Numerical Results and Discussion for Space Fractional Diffusion Equations

In this section, the numerical solutions for space fractional diffusion equations obtained by proposed new two-step Adomian decomposition method have been analyzed. Also, an analysis for the comparison of errors between TSADM solution and standard Adomian decomposition method solution has been presented here.

The numerical simulations using TSADM

In this present numerical experiment, Eqs. (2.45) and (2.50) have been used to draw the graphs as shown in Figs. 2.5 and 2.6 respectively. Figure 2.5 shows the 3D surface solution $u(x, t)$ for one-dimensional fractional diffusion equation. On the other hand, Fig. 2.6 shows the 3D surface solution $u(x, y, t)$ for two-dimensional fractional diffusion equation.

Comparison of errors between TSADM solution and standard Adomian decomposition method solution

In this present analysis, the solutions of the two-step Adomian decomposition method have been compared with that obtained by standard Adomian decomposition method. Here we demonstrate the absolute errors by taking different values of x and t. Comparison results in Tables 2.1, 2.2, 2.3 and 2.4 exhibit that there is a good agreement between TSADM and standard Adomian decomposition method solutions.

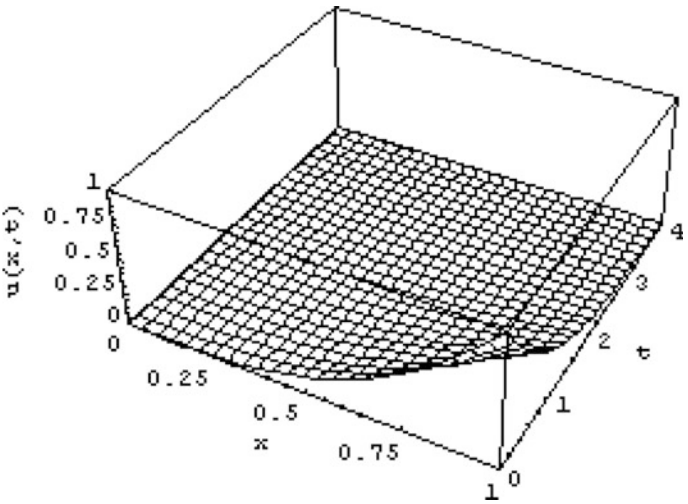

Fig. 2.5 Three dimensional surface solution $u(x, t)$ of one-dimensional fractional diffusion Eq. (2.39)

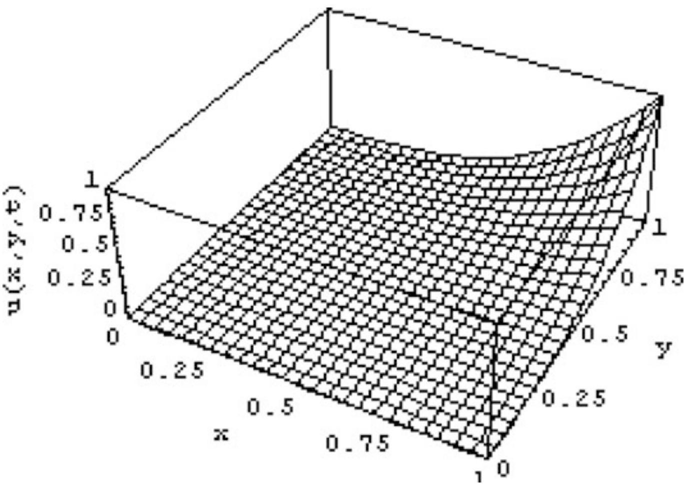

Fig. 2.6 Three dimensional surface solution $u(x, y, t)$ of two-dimensional fractional diffusion Eq. (2.46) at time $t = 1$

From Tables 2.1, 2.2 and 2.3, it can be observed that the standard Adomian decomposition method solution converges very slowly to the exact solution. On the other hand, TSADM requires only two iterations to achieve the exact solution. Therefore, TSADM is more effective and promising compared to standard Adomian decomposition method.

Table 2.1 Comparison between TSADM solution and standard Adomian decomposition method solution ϕ_2

(x, t)	Two-step Adomain decomposition method solution (exact solution $u(x,t) = e^{-t}x^3$)	Standard Adomian decomposition method two term solution ϕ_2	Absolute error $\lvert u - \phi_2 \rvert$
(0, 0)	0	0	0
(0.25, 0)	0.015625	0.015625	0
(0.5, 0)	0.125	0.125	0
(0.75, 0)	0.421875	0.421875	0
(1, 0)	1	1	0
(0, 1)	0	0	0
(0.25, 1)	0.00574812	0.00509492	0.000653195
(0.5, 1)	0.0459849	0.0250827	0.0209022
(0.75, 1)	0.155199	−0.00352725	0.158726
(0, 2)	0	0	0
(0.25, 2)	0.00211461	0.0000987486	0.00201587
(0.5, 2)	0.0169169	−0.0475908	0.0645077
(0.75, 2)	0.0570946	−0.432761	0.489855
(0, 3)	0	0	0
(0.25, 3)	0.000777923	−0.00286161	0.00363954
(0.5, 3)	0.00622338	−0.110242	0.116465

Table 2.2 Comparison between TSADM solution and standard Adomian decomposition method solution ϕ_3

(x, t)	Two-step Adomain decomposition method solution (exact solution $u(x,t) = e^{-t}x^3$)	Standard Adomian decomposition method three term solution ϕ_3	Absolute error $\lvert u - \phi_3 \rvert$
(0, 0)	0	0	0
(0.25, 0)	0.015625	0.015625	0
(0.5, 0)	0.125	0.125	0
(0.75, 0)	0.421875	0.421875	0
(1, 0)	1	1	0
(0, 1)	0	0	0
(0.25, 1)	0.00574812	0.0055815	0.000166612
(0.5, 1)	0.0459849	0.0353218	0.0106631
(0.75, 1)	0.155199	0.0337393	0.12146
(0, 2)	0	0	0
(0.25, 2)	0.00211461	0.00102422	0.00109039
(0.5, 2)	0.0169169	−0.0528681	0.0697851
(0, 3)	0	0	0
(0.25, 3)	0.000777923	−0.00231194	0.00308986
(0.5, 3)	0.00622338	−0.191528	0.197751

Table 2.3 Comparison between TSADM solution and standard Adomian decomposition method solution ϕ_4

(x, t)	Two-step Adomain decomposition method solution (exact solution $u(x,t) = e^{-t}x^3$)	Standard Adomian decomposition method four term solution ϕ_4	Absolute error $\lvert u - \phi_4 \rvert$
(0, 0)	0	0	0
(0.25, 0)	0.015625	0.015625	0
(0.5, 0)	0.125	0.125	0
(0.75, 0)	0.421875	0.421875	0
(1, 0)	1	1	0
(0, 1)	0	0	0
(0.25, 1)	0.00574812	0.00570392	0.0000442011
(0.5, 1)	0.0459849	0.0403272	0.00565774
(0.75, 1)	0.155199	0.0585313	0.0966678
(0, 2)	0	0	0
(0.25, 2)	0.00211461	0.00151496	0.000599653
(0.5, 2)	0.0169169	−0.0598386	0.0767556
(0, 3)	0	0	0
(0.25, 3)	0.000777923	−0.00184474	0.00262266
(0.5, 3)	0.00622338	−0.329478	0.335701

Table 2.4 Comparison between TSADM solution and standard Adomian decomposition method solution ϕ_3

$(x, y, t = 1)$	Two-step Adomain decomposition method solution (exact solution $u(x,y,t = 1) = x^3y^3$)	Standard Adomian decomposition method three term solution ϕ_3	Absolute error $\lvert u - \phi_3 \rvert$
(0, 0.25, 1)	0	0	0
(0.25, 0.25, 1)	0.000039094	0.0000389794	0.0000001146
(0.5, 0.25, 1)	0.000312752	0.000305417	0.000007335
(0.75, 0.25, 1)	0.00105554	0.000971995	0.0000835416
(1, 0.25, 1)	0.00250201	0.00203262	0.000469391
(0, 0.5, 1)	0	0	0
(0.25, 0.5, 1)	0.000474043	0.000462926	0.0000111166
(0.5, 0.5, 1)	0.00379234	0.00308088	0.000711464
(0.75, 0.5, 1)	0.0127992	0.00469513	0.00810402
(1, 0.5, 1)	0.0303387	−0.015195	0.0455337
(0, 0.75, 1)	0	0	0
(0.25, 0.75, 1)	0.00204054	0.00187904	0.000161501
(0.5, 0.75, 1)	0.0163244	0.00598829	0.0103361
(0.75, 0.75, 1)	0.0550947	−0.0626396	0.117734
(0, 1, 1)	0	0	0
(0.25, 1, 1)	0.00574812	0.00466974	0.00107838
(0.5, 1, 1)	0.0459849	−0.0230313	0.0690162

From Table 2.4, it can be observed that the absolute errors for TSADM solution and standard Adomian decomposition method solution ϕ_3 are very small for small values of x and y. But as the values of x and y increase the absolute errors also increase.

2.6 Solution of Space Fractional Diffusion Equation with Insulated Ends

In this section, a variation of Adomian decomposition method has been proposed for getting analytical approximate solution of space fractional diffusion equation with insulated ends.

2.6.1 Implementation of the Present Method

Let us consider initial conditions

$$u(x,0) = x^2, \quad 0 \le x \le \pi \tag{2.51}$$

and boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(\pi,t)}{\partial x} = 0, \quad t \ge 0 \tag{2.52}$$

for Eq. (2.4), as taken in [62].

We see that $f(x) = x^2$ is a periodic function with period π. The Fourier sine series of $f(x)$ in [0, π] can be obtained as

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx. \tag{2.53}$$

Therefore, after considering $f(x)$ as Fourier Cosine series, we can take

$$u(x,0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_\gamma nx, \tag{2.54}$$

where $\cos_\gamma nx$ is the Generalized Cosine function and $\gamma = \alpha/2$, $\gamma \in (\frac{1}{2}, 1]$.

From Eq. (2.33), the following terms can be obtained

$$u_0 = u(x,0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_\gamma nx,$$
$$u_1 = L_t^{-1}(dD_x^\alpha u_0) = \frac{-td}{1!}\sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n n^{2\gamma} \cos_\gamma nx,$$
$$u_2 = L_t^{-1}(dD_x^\alpha u_1) = \frac{t^2d^2}{2!}\sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n n^{4\gamma} \cos_\gamma nx,$$
$$u_3 = L_t^{-1}(dD_x^\alpha u_2) = -\frac{t^3d^3}{3!}\sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n n^{6\gamma} \cos_\gamma nx$$

and so on.

Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_\gamma nx\left(1 - \frac{tdn^{2\gamma}}{1!} + \frac{t^2d^2n^{4\gamma}}{2!} - \frac{t^3d^3n^{6\gamma}}{3!} + \cdots\right) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_\gamma nx\, E_1(-tdn^{2\gamma}), \end{aligned}$$

where $E_\lambda(z)$ is the Mittag-Leffler function in one parameter.

$$\begin{aligned} &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_\gamma nx\, \mathrm{e}^{-tdn^{2\gamma}} \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos_{\alpha/2} nx\, \mathrm{e}^{-tdn^{\alpha}} \end{aligned} \tag{2.55}$$

The solution (2.55) can be verified through substitution in Eq. (2.4).

2.6.2 Numerical Results and Discussion

In this section, the numerical solutions of the space fractional diffusion equation with insulated ends obtained by the proposed method have been analyzed.

The numerical simulations for the proposed method

In this present numerical experiment, Eq. (2.55) has been used to draw the graphs as shown in Figs. 2.7, 2.8 and 2.9 for different fractional order values of α respectively. In this numerical analysis, we assume $d = 0.4$ for Eq. (2.4).

Figures 2.7, 2.8 and 2.9 show anomalous diffusion behaviour. These figures exhibit slow diffusion at the beginning and fast diffusion later. From these figures, it is also observed that diffusion behaviour increases as α increases.

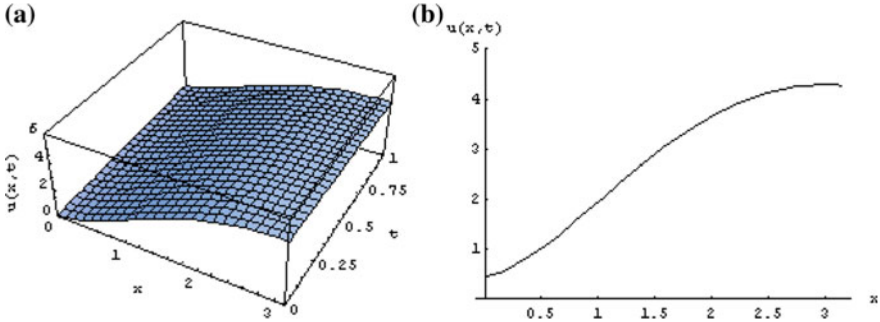

Fig. 2.7 **a** The 3D surface solution, **b** The corresponding 2D solution at $t = 0.5$, $d = 0.4$ and $\alpha = 1.5$

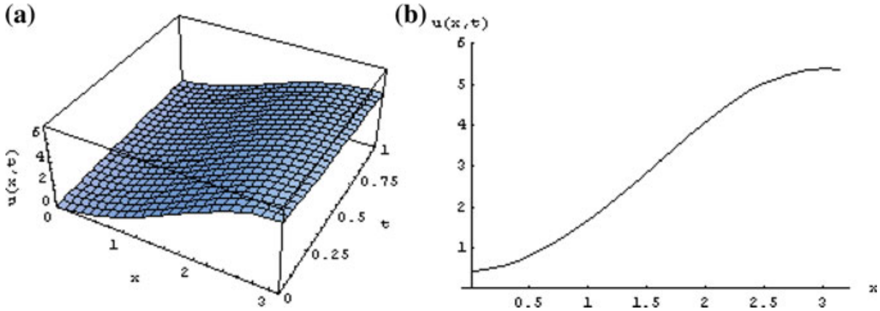

Fig. 2.8 **a** The 3D surface solution, **b** The corresponding 2D solution at $t = 0.5$, $d = 0.4$ and $\alpha = 1.75$

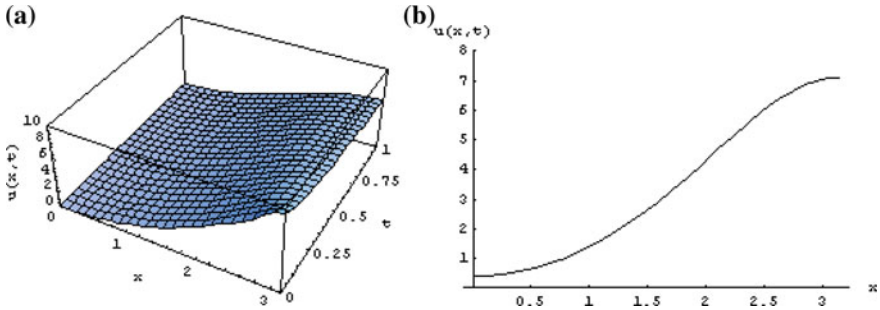

Fig. 2.9 **a** The 3D surface solution, **b** The corresponding 2D solution at $t = 0.5$, $d = 0.4$ and $\alpha = 2$

2.7 Conclusion

In the present chapter, the modified decomposition method has been used for finding the solutions for the coupled K-G-S equations with initial conditions. The approximate solutions to the coupled K-G-S equations have been calculated by using the MDM without any need of transformation techniques and linearization of the equations. Additionally, it does not need any discretization method to get numerical solutions. This proposed method thus eliminates the difficulties of massive computational work.

This chapter includes an analytical scheme to obtain the solutions of the one dimensional and two-dimensional fractional diffusion equations. Two typical examples have been discussed as illustrations. In this work, it has been established that TSADM is well suited to solve the fractional diffusion equation. TSADM proceeds in two steps. The first step consists of verifying that the zeroth component of the series solution includes the exact solution. Once the exact solution is obtained, we stop. Otherwise, we continue with the standard Adomian recursion relation in the second step.

In this chapter, TSADM has been applied for the solutions of space fractional diffusion equations. The TSADM may provide the solution by using two iterations only if compared with the standard Adomian method and the modified decomposition method. Moreover, the TSADM overcomes the difficulties arising in the modified decomposition method as discussed earlier. A comparison study between the TSADM and the standard decomposition method is conducted to illustrate the efficiency of the TSADM and the results obtained indicate that the TSADM is more feasible and effective.

This chapter also presents an analytical scheme to obtain the solution of space fractional diffusion equation with insulated ends by ADM with a simple variation. In the present analysis, a new approach of Adomian decomposition method has been successfully applied after expressing the initial condition as a Fourier series. The physical significance of the solution has been also presented graphically. The present work demonstrates that this proposed technique is well suited to solve the space fractional diffusion equation with insulated ends.

The proposed methods are straightforward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, these methods do not change the problem into a convenient one for the use of linear theory. Therefore, they provide more realistic series solutions that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Moreover, no linearization or perturbation is required. It can avoid the difficulty of finding the inverse of the Laplace Transform and can reduce the labour of perturbation method. It is quite obvious to see that these methods are quite accurate, easy and efficient technique for solving fractional partial differential equations arising in physical problems.

As mentioned, the proposed methods avoid linearization and physically unrealistic assumptions. Furthermore, as the present methods do not require discretization of the variables, i.e., time and space, it is not affected by computational round off errors and one is not faced with the necessity of large computer memory and time. Consequently, the computational size will be reduced.

References

1. Darwish, A., Fan, E.G.: A series of new explicit exact solutions for the coupled Klein-Gordon-Schrödinger equations. Chaos. Soliton. Fractals. **20**, 609–617 (2004)
2. Liu, S., Fu, Z., Liu, S., Wang, Z.: The periodic solutions for a class of coupled nonlinear Klein-Gordon equations. Phys. Lett. A **323**, 415–420 (2004)
3. Hioe, F.T.: Periodic solitary waves for two coupled nonlinear Klein-Gordon and Schrödinger equations. J. Phys. A: Math. Gen. **36**, 7307–7330 (2003)
4. Bao, W., Yang, L.: Efficient and accurate numerical methods for the Klein-Gordon-Schrödinger equations. http://web.cz3.nus.edu.sg/~bao/PS/KGS.pdf
5. Adomian, G.: Nonlinear stochastic systems theory and applications to physics. Kluwer Academic publishers, Netherlands (1989)
6. Adomian, G.: Solving frontier problems of physics: the decomposition method. Kluwer Academic Publishers, Boston (1994)
7. Wazwaz, A.M.: Partial differential equations: methods and applications. Balkema, Lisse, The Netherlands (2002)
8. Adomian, G.: An analytical solution of the stochastic Navier-Stokes system. Found. Phys. **21** (7), 831–843 (1991)
9. Adomian, G., Rach, R.: Linear and nonlinear Schrödinger equations. Found. Phys. **21**, 983–991 (1991)
10. Adomian, G.: Solution of physical problems by decomposition. Comput. Math. Appl. **27**(9–10), 145–154 (1994)
11. Adomian, G.: Solutions of nonlinear P.D.E. Appl. Math. Lett. **11**(3), 121–123 (1998)
12. Wazwaz, A.M.: The decomposition method applied to systems of partial differential equations and to the reaction—diffusion Brusselator model. Appl. Math. Comput. **110**(2–3), 251–264 (2000)
13. Wazwaz, A.M.: Approximate solutions to boundary value problems of higher order by the modified decomposition method. Comput. Math Appl. **40**(6–7), 679–691 (2000)
14. Wazwaz, A.M.: The modified decomposition method applied to unsteady flow of gas through a porous medium. Appl. Math. Comput. **118**(2–3), 123–132 (2001)
15. Wazwaz, A.M.: Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method. Chaos. Solitons. Fractals. **12**(8), 1549–1556 (2001)
16. Wazwaz, A.M.: Construction of solitary wave solutions and rational solutions for the KdV equation by Adomian decomposition method. Chaos. Solitons. Fractals. **12**(12), 2283–2293 (2001)
17. Wazwaz, A.M.: A computational approach to soliton solutions of the Kadomtsev-Petviashvili equation. Appl. Math. Comput. **123**(2), 205–217 (2001)
18. Wazwaz, A.M., Gorguis, A.: An analytic study of fisher's equation by using Adomian decomposition method. Appl. Math. Comput. **154**(3), 609–620 (2004)
19. Wazwaz, A.M.: Analytical solution for the time-dependent Emden-Fowler type of equations by Adomian decomposition method. Appl. Math. Comput. **166**(3), 638–651 (2005)
20. Mavoungou, T., Cherruault, Y.: Solving frontier problems of physics by decomposition method: a new approach. Kybernetes **27**(9), 1053–1061 (1998)

21. Abbaoui, K., Cherruault, Y.: The decomposition method applied to the Cauchy problem. Kybernetes **28**, 103–108 (1999)
22. Khelifa, S., Cherruault, Y.: Approximation of the solution for a class of first order p.d.e. by Adomian method. Kybernetes **31**(3/4), 577–595 (2002)
23. Inc, M., Cherruault, Y.: A new approach to travelling wave solution of a fourth-order semilinear diffusion equation. Kybernetes **32**(9–10), 1492–1503 (2003)
24. Inc, M., Cherruault, Y., Abbaoui, K.: A computational approach to the wave equations: an application of the decomposition method. Kybernetes **33**(1), 80–97 (2004)
25. Inc, M., Cherruault, Y.: A reliable approach to the Korteweg-de Vries equation: an application of the decomposition method. Kybernetes **34**(7–8), 951–959 (2005)
26. Inc, M., Cherruault, Y.: A reliable method for obtaining approximate solutions of linear and nonlinear Volterra-Fredholm integro-differential equations. Kybernetes **34**(7–8), 1034–1048 (2005)
27. Kaya, D., Yokus, A.: A numerical comparison of partial solutions in the decomposition method for linear and nonlinear partial differential equations. Math. Comp. Simul. **60**(6), 507–512 (2002)
28. Kaya, D.: A numerical solution of the Sine-Gordon equation using the modified decomposition method. Appl. Math. Comp. **143**, 309–317 (2003)
29. Kaya, D., El-Sayed, S.M.: On a generalized fifth order KdV equations. Phys. Lett. A **310**(1), 44–51 (2003)
30. Kaya, D., El-Sayed, S.M.: An application of the decomposition method for the generalized KdV and RLW equations. Chaos. Solitons. Fractals. **17**(5), 869–877 (2003)
31. Kaya, D.: An explicit and numerical solutions of some fifth-order KdV equation by decomposition method. Appl. Math. Comp. **144**(2–3), 353–363 (2003)
32. Kaya, D.: Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation. Appl. Math. Comp. **147**, 69–78 (2004)
33. Kaya, D.: A numerical simulation of solitary-wave solutions of the generalized regularized long-wave equation. Appl. Math. Comp. **149**(3), 833–841 (2004)
34. Kaya, D., Inan, I.E.: A numerical application of the decomposition method for the combined KdV–MKdV equation. Appl. Math. Comput. **168**(2), 915–926 (2005)
35. Geyikli, T., Kaya, D.: An application for a modified KdV equation by the decomposition method and finite element method. Appl. Math. Comput. **169**(2), 971–981 (2005)
36. El-Sayed, S.M., Kaya, D.: A numerical solution and an exact explicit solution of the NLS equation. Appl. Math. Comput. **172**(2), 1315–1322 (2006)
37. Kaya, D.: The exact and numerical solitary-wave solutions for generalized modified Boussinesq equation. Phys. Lett. A **348**(3–6), 244–250 (2006)
38. Ray, S.S., Bera, R.K.: Analytical solution of a dynamic system containing fractional derivative of order one-half by Adomian decomposition method. Trans. ASME J. Appl. Mech. **72**(2), 290–295 (2005)
39. Ray, S.S., Bera, R.K.: An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method. Appl. Math. Comput. **167**(1), 561–571 (2005)
40. Ray, S.S., Bera, R.K.: Analytical solution of the Bagley Torvik equation by Adomian decomposition method. Appl. Math. Comput. **168**(1), 398–410 (2005)
41. Ray, S.S.: A numerical solution of the coupled Sine-Gordon equation using the modified decomposition method. Appl. Math. Comput. **175**(2), 1046–1054 (2006)
42. Wazwaz, A.M.: A Reliable Modification of Adomian decomposition method. Appl. Math. Comp. **102**(1), 77–86 (1999)
43. Metzler, R., Barkai, E., Klafter, J.: Anomalous diffusion and relaxation close to thermal equilibrium: a fractional Fokker-Planck equation approach. Phys. Rev. Lett. **82**(18), 3563–3567 (1999)
44. Metzler, R., Klafter, J.: The random walk's guide to Anomalous diffusion: a fractional dynamics approach. **339**(1), 1–77 (2000)

45. Metzler, R., Klafter, J.: The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. J. Phys. A **37**, R161–R208 (2004)
46. Gorenflo, R., Mainardi, F., Scalas, E., Raberto, M.: Fractional calculus and continuous time finance. III. The diffusion limit. Mathematical finance (Konstanz, 2000), Trends Math., Birkhuser, Basel. 171–180 (2001)
47. Mainardi, F., Raberto, M., Gorenflo, R., Scalas, E.: Fractional calculus and continuous-time finance II: the waiting-time distribution. Phys. A **287**, 468–481 (2000)
48. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Phys. A **284**, 376–384 (2000)
49. Raberto, M., Scalas, E., Mainardi, F.: Waiting-times and returns in high-frequency financial data: an empirical study. Phys. A **314**, 749–755 (2002)
50. Benson, D.A., Wheatcraft, S., Meerschaert, M.M.: Application of a fractional advection–dispersion equation. Water Resour. Res **36**, 1403–1412 (2000)
51. Baeumer, B., Meerschaert, M.M., Benson, D.A., Wheatcraft, S.W.: Subordinated advection—dispersion equation for contaminant transport. Water Resour. Res. **37**, 1543–1550 (2001)
52. Benson, D.A., Schumer, R., Meerschaert, M.M., Wheatcraft, S.W.: Fractional dispersion, Lévy motions, and the MADE tracer tests. Transp. Porous Media **42**, 211–240 (2001)
53. Schumer, R., Benson, D.A., Meerschaert, M.M., Wheatcraft, S.W.: Eulerian derivation of the fractional advection-dispersion equation. J. Contam. Hydrol. **48**, 9–88 (2001)
54. Schumer, R., Benson, D.A., Meerschaert, M.M., Baeumer, B.: Multiscaling fractional advection-dispersion equations and their solutions. Water Resour. Res. **39**, 1022–1032 (2003)
55. Mainardi, F.: Fractional calculus: some basic problems in continuum and statistical mechanics. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and fractional calculus in continuum mechanics, pp. 291–348. Springer, Vienna (1997)
56. Mainardi, F., Pagnini, G.: The weight functions as solutions of the time-fractional diffusion equation. Appl. Math. Comput. **141**, 51–62 (2003)
57. Agrawal, O.P.: Solution for a fractional diffusion-wave equation defined in a bounded domain. Nonlinear Dyn. **29**, 145–155 (2002)
58. Schneider, W.R., Wyss, W.: Fractional diffusion and wave equations. J. Math. Phys. **30**, 134–144 (1989)
59. Meerschaert, M.M., Scheffler, H., Tadjeran, C.: Finite difference methods for two-dimensional fractional dispersion equation. J. Comput. Phys. **211**, 249–261 (2006)
60. Tadjeran, C., Meerschaert, M.M., Scheffler, H.: A second-order accurate numerical approximation for the fractional diffusion equation. J. Comput. Phys. **213**, 205–213 (2006)
61. Mainardi, F., Luchko, Y., Pagnini, G.: The fundamental solution of the space-time fractional diffusion equation. Fractional Calc. Appl Anal. **4**(2), 153–192 (2001)
62. Shen, S., Liu, F.: Error Analysis of an explicit finite difference approximation for the space fractional diffusion equation with insulated ends. ANZIAM J., **46**(E), C871-C887 (2005)
63. Podlubny, I.: Fractional differential equations. Academic Press, New York (1999)
64. Cherruault, Y.: Convergence of Adomian's method. Kybernetes **18**, 31–38 (1989)
65. Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Comput. Math. Applic. **28**(5), 103–109 (1994)
66. Abbaoui, K., Cherruault, Y.: New Ideas for proving convergence of decomposition methods. Comput. Math. Applic. **29**, 103–108 (1995)
67. Himoun, N., Abbaoui, K., Cherruault, Y.: New Results of Convergence of Adomian's Method. Kybernetes **28**(4–5), 423–429 (1999)
68. Luo, X.G.: A two-step Adomian decomposition method. Appl. Math. Comput. **170**(1), 570–583 (2005)
69. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional integrals and derivatives: theory and applications. Taylor and Francis, London (1993)
70. Wolfram, S.: Mathematica for Windows, Version 5.0, Wolfram Research, (2003)