Chapter 9
Actions on Alternating Matrices and Compound Matrices

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9.1 Introduction

In this note, we consider the action of $\mathrm{SL}_n(R)$ on $\mathrm{Alt}_n(R)$, the space of alternating matrices of order n over R, by conjugation: $V \mapsto \sigma V \sigma^t$, for $\sigma \in \mathrm{SL}_n(R)$, $V \in \mathrm{Alt}_n(R)$. We prove (See Theorem 9.2) that the matrix of the above linear transformation (associated to σ) is $\wedge^2\sigma$.

These results are well known to experts when R is a field, but we worked it, as we will need it, in a sequel, over any commutative ring R. (The book [5] gives some details.)

In the last section, we restrict to the case when $n = 4$. We show that by taking a suitable basis of $\mathrm{Alt}_4(R)$ we can get a map from $\mathrm{SL}_4(R)$ to $\mathrm{SO}_6(R)$. Moreover, this map induces an injection from $\mathrm{SL}_4(R)/\mathrm{E}_4(R)$ to $\mathrm{SO}_6(R)/\mathrm{EO}_6(R)$ (See Theorem 9.3). The case when $R = \mathbb{C}$ is proved in [1].

In some sense, this result is reminiscent to the Jose–Rao Theorem in [3, Theorem 4.14], when $n = 2$, where it was shown that

$$\mathrm{SUm}_r(R)/\mathrm{EUm}_r(R) \to \mathrm{SO}_{2(r+1)}(R)/\mathrm{EO}_{2(r+1)}(R)$$

is injective. (We refer the reader to [3] for details.)

In recent article [4], Jose–Rao have shown that for $v, w \in R^{r+1}$, $\sigma \in \mathrm{SL}_{r+1}(R)$, the Suslin matrix

$$S_r(v\sigma, w{\sigma^t}^{-1}) = AS_r(v, w)B,$$

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for some $A, B \in \mathrm{SL}_{2^r}(R)$, with AB, the Euler characteristic of σ. We may regard Theorem 9.2 as a prelude to this result; it signifies that the Suslin form brings out the Euler characteristic, whereas the alternating form only displays the initial $\wedge^1$ and $\wedge^2$.

9.2 Preliminaries

In this section, we recall a few definitions, state some results and fix some notations which will be used throughout this paper.

Let R be a commutative ring with 1. Let $\mathrm{M}_r(R)$ denote the set of all $r \times r$ matrices with entries in R.

Definition 9.1 The General Linear group $\mathrm{GL}_r(R)$ is defined as the group of $r \times r$ invertible matrices with entries in R.

Definition 9.2 The Special Linear group is denoted by $\mathrm{SL}_r(R)$ and is defined as $\mathrm{SL}_r(R) = \{\alpha \in \mathrm{GL}_r(R) : \det(\alpha) = 1\}$. It is a normal subgroup of $\mathrm{GL}_r(R)$.

Definition 9.3 The group of elementary matrices $\mathrm{E}_r(R)$ is a subgroup of $\mathrm{GL}_r(R)$ generated by matrices of the form $E_{ij}(\lambda) = I_r + \lambda e_{ij}$, where $\lambda \in R$, $i \neq j$ and $e_{ij} \in M_r(R)$ with ijth entry is 1 and all other entries are zero.

Note that $e_{ij}e_{rs} = \begin{cases} e_{is} & \text{if } j = r \\ 0 & \text{if } j \neq r \end{cases}$.

Following are some well-known properties of the elementary generators:

Lemma 9.1 *For $\lambda, \mu \in R$,*

(1) (*Splitting Property*) $E_{ij}(\lambda + \mu) = E_{ij}(\lambda)E_{ij}(\mu)$, $1 \leq i, j \leq r$, $i \neq j$.
(2) (*Steinberg relation*) $[E_{ij}(\lambda), E_{jk}(\mu)] = E_{ik}(\lambda\mu)$, $1 \leq i, j, k \leq r$, $i \neq j$, $i \neq k$, $j \neq k$.

Remark 9.1 In view of the Steinberg relation, $\mathrm{E}_r(R)$ is generated by

$$\{E_{1i}(\lambda), E_{i1}(\mu) : 2 \leq i \leq r, \lambda, \mu \in R\}.$$

Note that $E_{ij}(\lambda), i \neq j, \lambda \in R$, is invertible with inverse $E_{ij}(-\lambda)$. In fact, $E_{ij}(\lambda)$ belongs to $\mathrm{SL}_r(R)$. Hence, $\mathrm{E}_r(R) \subseteq \mathrm{SL}_r(R) \subseteq \mathrm{GL}_r(R)$.

We now recall the notion of the compound matrix:

Definition 9.4 (*Minors of a matrix*) Given an $n \times m$ matrix $A = (a_{ij})$ over R, a minor of A is the determinant of a smaller matrix formed from its entries by selecting only some of the rows and columns. Let $K = \{k_1, k_2, \ldots, k_p\}$ and $L = \{l_1, l_2, \ldots, l_p\}$ be subsets of $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$, respectively. The indices are chosen such that $k_1 < k_2 < \cdots < k_p$ and $l_1 < l_2 < \cdots < l_p$. The pth-order minor defined by K and L is the determinant of the submatrix of A obtained by considering the rows $k_1, k_2, \ldots, k_p$ and columns $l_1, l_2, \ldots, l_p$ of A. We denote this submatrix as $A\begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}$ or $A(K \mid L)$.

Theorem 9.1 (The Cauchy–Binet formula) *Let A be an $m \times n$ matrix and B an $n \times m$ matrix over R, where $m \leq n$. Then the determinant of their product $C = AB$ can be written as a sum of products of minors of A and B, i.e.,*

$$|C| = \sum_{1 \leq k_1 < k_2 < \cdots < k_m \leq n} \det A \begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} \det B \begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

The sum is over the maximal (mth order) minors of A and the corresponding minor of B. In particular, $\det(AB) = \det(A)\det(B)$*, if A, B are $n \times n$ matrices.*

Definition 9.5 Suppose that A is an $m \times n$ matrix with entries from a ring R and $1 \leq r \leq \min\{m, n\}$. The rth compound matrix $C_r(A)$ or rth adjugate of A is the $\binom{m}{r} \times \binom{n}{r}$ matrix whose entries are the minors of order r, arranged in lexicographic order, i.e.

$$C_r(A) = \left(\det A \begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ j_1 & j_2 & \ldots & j_r \end{pmatrix} \right).$$

Lemma 9.2 (Properties, See [2, 5]) *Let A and B be $n \times n$ matrices over R and $r \leq n$. Then*

(i) $C_1(A) = A$.
(ii) $C_n(A) = \det(A)$.
(iii) $C_r(AB) = C_r(A)C_r(B)$.
(iv) $C_r(A^t) = (C_r(A))^t$.

9.3 Associated Linear Transformations

We shall always work over a commutative ring R with 1. In this section, we find the linear transformation of the action of $\mathrm{SL}_n(R)$ on the space of alternating matrices.

Definition 9.6 A matrix $A \in M_n(R)$ is said to be alternating if $a_{ij} = -a_{ji}$ and $a_{ii} = 0$, for $1 \leq i, j \leq n$.

Notation The space of all alternating $n \times n$ matrices over a commutative ring R will be denoted by $\mathrm{Alt}_n(R)$. It is clearly a free R-module of rank $1 + 2 + \cdots + (n-1) = \binom{n}{2}$ with basis $B_{ij} = e_{ij} - e_{ji}$, $1 \leq i < j \leq n$. □

One has the action of $\mathrm{SL}_n(R)$ on $\mathrm{Alt}_n(R)$ by

$$\begin{aligned}\mathrm{SL}_n(R) \times \mathrm{Alt}_n(R) &\to \mathrm{Alt}_n(R)\\ (\sigma, A) &\mapsto \sigma A \sigma^t.\end{aligned}$$

This action enables one to associate a linear transformation $T_\sigma : \mathrm{Alt}_n(R) \to \mathrm{Alt}_n(R)$ for $\sigma \in \mathrm{SL}_n(R)$, via $T_\sigma(A) = \sigma A \sigma^t$.

We input the next observation for completeness; which can be found in [5, pp. 399–400].

Lemma 9.3 *Let $\sigma : R^n \longrightarrow R^m$ be a R-linear map. Then the matrix of the linear transformation $\wedge^r\sigma : \wedge^r R^n \longrightarrow \wedge^r R^m$ is $C_r(M(\sigma))$, where $M(\sigma)$ is the matrix of σ and $r \leq \min\{n, m\}$.*

Proof This is well-known to experts when R is a field. We compute it as follows:

Let $e_1, \ldots, e_n$ be a basis of R^n and $f_1, \ldots, f_m$ be a basis of R^m. Let us compute the matrix of $\wedge^r\sigma$ w.r.t. the standard basis $e_{i_1} \wedge \cdots \wedge e_{i_r}$ of $\wedge^r R^n$ and $f_{i_1} \wedge \cdots \wedge f_{i_r}$ of $\wedge^r R^m$ ordered lexicographically. Suppose $1 \leq i_1 < \cdots < i_r \leq n$ as usual. Then

$$\begin{aligned}\wedge^r(\sigma)(e_{i_1} \wedge \cdots \wedge e_{i_r}) &= \sigma(e_{i_1}) \wedge \cdots \wedge \sigma(e_{i_r})\\ &= \sum_{j_1=1}^{m} d_{j_1 i_1} f_j \wedge \cdots \wedge \sum_{j_r=1}^{m} d_{j_r i_r} f_j\\ &= \sum_{1 \leq j_1 < \cdots < j_r \leq n} \det A \begin{pmatrix} j_1\ j_2\ \ldots\ j_r \\ i_1\ i_2\ \ldots\ i_r \end{pmatrix} \left(f_{j_1} \wedge \cdots \wedge f_{j_r}\right),\end{aligned}$$

where A^t denotes the matrix of the linear transformation σ. □

Since $\wedge^r(\sigma \circ \tau) = \wedge^r(\sigma) \circ \wedge^r(\tau)$, it is clear from Lemma 9.3 that the multiplicative property of compound matrices hold, i.e.

$$C_r(AB) = C_r(A)C_r(B),$$

where A is an $m \times n$ matrix, B is an $n \times m$ matrix and $r \leq \min\{m, n\}$.

Let us compute the matrix associated to T_σ for $\sigma \in \mathrm{SL}_n(R)$. We prove that it is the matrix $\wedge^2\sigma$.

Theorem 9.2 *Let $\sigma \in \mathrm{SL}_n(R)$. Then the matrix of the linear transformation T_σ w.r.t. the basis $\{B_{ij} : 1 \leq i < j \leq n\}$ is the same as the matrix of the linear transformation $\wedge^2\sigma : \wedge^2 R^n \longrightarrow \wedge^2 R^n$; which is the compound matrix of order 2 associated to σ.*

Proof Let $\sigma = (a_{ij})$. For $1 \leq i < j \leq n$, by definition,

$$
\begin{aligned}
T_\sigma(B_{ij}) &= \sigma B_{ij}\sigma^t = \sigma(e_{ij} - e_{ji})\sigma^t = \sigma e_{ij}\sigma^t - \sigma e_{ji}\sigma^t \\
&= \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_{ri}a_{sj}B_{rs} - \sum_{r=1}^{n}\sum_{s=r+1}^{n} a_{rj}a_{si}B_{rs} \\
&= \sum_{r=1}^{n}\sum_{s=r+1}^{n} (a_{ri}a_{sj} - a_{rj}a_{si})B_{rs} \\
&= \sum_{r=1}^{n}\sum_{s=r+1}^{n} \det\sigma \begin{pmatrix} r & s \\ i & j \end{pmatrix} B_{rs}.
\end{aligned}
$$

Thus $[T_\sigma] = \left(\det\sigma \begin{pmatrix} i & j \\ r & s \end{pmatrix}\right) = C_2(\sigma)$. The rest follows via Lemma 9.3. □

The following Corollary gives the explicit form of $[T_{E_{1i}(\lambda)}]$, where $E_{1i}(\lambda) \in E_n(R)$. Since $E_{i1}(\lambda) = E_{1i}(\lambda)^t$, by Lemma 9.2(iv) one has, $[T_{E_{i1}(\lambda)}] = [T_{E_{1i}(\lambda)}]^t$.

Corollary 9.1 *Let $A = E_{1i}(\lambda) \in E_n(R)$, $\lambda \in R$. Let $\alpha = \{i_1, i_2\}$, $\beta = \{j_1, j_2\}$, where $1 \le i_1 < i_2 \le n$ and $1 \le j_1 < j_2 \le n$. Then the $(\alpha\beta)$th entry $\det A(\alpha|\beta)$ of $\wedge^2 A$ is given by*

$$
\det A(\alpha|\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ (-1)^r\lambda & \text{if } |\alpha \cap \beta| = 1,\ 1 \in \alpha,\ i \in \beta \text{ and } 1, i \notin \alpha \cap \beta \\ 0 & \text{otherwise}, \end{cases}
$$

where r is the number of integers in $\alpha \cap \beta$ between 1 and i.

Proof Clearly if $\alpha = \beta$, then $\det A(\alpha|\beta) = 1$ as the submatrix $A(\alpha|\beta) = A\begin{pmatrix} i_1 & i_2 \\ j_1 & j_2 \end{pmatrix}$ is either I_2 or an upper triangular matrix $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$.

If $1 \in \alpha, i \in \beta$ and $1, i \notin \alpha \cap \beta$, then for $r \in \alpha \cap \beta$, $A(\alpha|\beta)$ is of the form $A\begin{pmatrix} 1 & r \\ r & i \end{pmatrix}$ if $1 < r < i$ and is of the form $A\begin{pmatrix} 1 & r \\ i & r \end{pmatrix}$ if $i < r \le n$. Note that if $A = (a_{ij})$, then

$$
a_{jk} = \begin{cases} 1 & \text{if } j = k \\ \lambda & \text{if } j = 1, k = i\,. \\ 0 & \text{otherwise}\,. \end{cases}
$$

Thus if $1 < r < i$, $A\begin{pmatrix} 1 & r \\ r & i \end{pmatrix} = \begin{pmatrix} a_{1r} & a_{1i} \\ a_{rr} & a_{ri} \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$ and hence $\det A(\alpha|\beta) = -\lambda$. Also if $i < r \le n$, $A\begin{pmatrix} 1 & r \\ i & r \end{pmatrix} = \begin{pmatrix} a_{1i} & a_{1r} \\ a_{ri} & a_{rr} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ and hence $\det A(\alpha|\beta) = \lambda$. All other entries of $\wedge^2 A$ contains either a zero row or a zero column. □

9.4 The 4 × 4 Case

L. N. Vaserstein studied the case when $n = 4$ in [6]. We consider the Vaserstein space $V = \mathrm{Alt}_4(R)$ of dimension 6.

Definition 9.7 Let π denote the permutation $(1\ r+1)\cdots(r\ 2r)$ corresponding to the form $\begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$. The elementary orthogonal matrices over R are defined by

$$oe_{ij}(\lambda) = I_{2r} + \lambda e_{ij} - \lambda e_{\pi(j)\pi(i)}, \text{ if } i \neq \pi(j),$$

where $1 \leq i, j \leq 2r$ and $\lambda \in R$.

Definition 9.8 The elementary orthogonal group $\mathrm{EO}_{2r}(R)$ is the subgroup of $\mathrm{SO}_{2r}(R)$ generated by the matrices $oe_{ij}(\lambda)$, where $1 \leq i < j \leq 2r$, $i \neq \pi(j)$ and $\lambda \in R$.

It is observed that the matrix $[T_\sigma]$ w.r.t. the basis $\{B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}\}$, where $\sigma = E_{1i}$ or $E_{i1}, 2 \leq i \leq 4$ are not orthogonal w.r.t. the standard form $\begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$. However, we have the following lemma.

Lemma 9.4 *With respect to the ordered basis* $\{B_{12}, B_{13}, B_{14}, B_{34}, -B_{24}, B_{23}\}$, *the matrix* $[T_{E_{1i}(\lambda)}]$ *and* $[T_{E_{i1}(\lambda)}]$, $2 \leq i \leq 4$ *are elementary orthogonal w.r.t. the standard form.*

Proof By Lemma 9.3, w.r.t. the basis $B_1 = \{B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}\}$, the matrix of $T_{E_{12}(\lambda)}$ is the compound matrix of order 2 associated to $A = E_{12}(\lambda)$. By Corollary 9.1, $\det A(\{1,3\},\{2,3\}) = \lambda$ and $\det A(\{1,4\},\{2,4\}) = \lambda$ and all other $\det A(\alpha|\beta) = 0$ if $\alpha \neq \beta$. If $\alpha = \beta$, then $\det A(\alpha|\beta) = 1$. Thus (24)th and (35)th entry of $[T_{E_{12}(\lambda)}]_{B_1}$ are λ. Hence we have $[T_{E_{12}(\lambda)}]_{B_1} = E_{24}(\lambda)E_{35}(\lambda)$. Then w.r.t. the basis $B_2 = \{B_{12}, B_{13}, B_{14}, B_{34}, -B_{24}, B_{23}\}$, the matrix $[T_{E_{12}(\lambda)}]_{B_2} = E_{26}(\lambda)E_{35}(-\lambda)$ which is by definition $oe_{26}(\lambda)$ w.r.t. the permutation $\pi = (14)(25)(36)$. Similarly w.r.t. the basis B_2 one has

$$\left[T_{E_{13}(\lambda)}\right]_{B_2} = E_{34}(\lambda)E_{16}(-\lambda) = oe_{34}(\lambda).$$
$$\left[T_{E_{14}(\lambda)}\right]_{B_2} = E_{15}(\lambda)E_{24}(-\lambda) = oe_{15}(\lambda).$$
$$\left[T_{E_{21}(\lambda)}\right]_{B_2} = E_{62}(\lambda)E_{53}(-\lambda) = oe_{62}(\lambda).$$
$$\left[T_{E_{31}(\lambda)}\right]_{B_2} = E_{43}(\lambda)E_{61}(-\lambda) = oe_{43}(\lambda).$$
$$\left[T_{E_{41}(\lambda)}\right]_{B_2} = E_{51}(\lambda)E_{42}(-\lambda) = oe_{51}(\lambda).$$

Hence the result. □

In general one has the following.

Proposition 9.1 *Let $\sigma \in \mathrm{SL}_4(R)$. Then the matrix of the linear transformation T_σ on the Vaserstein space V w.r.t. the ordered basis $\{B_{12}, B_{13}, B_{14}, B_{34}, -B_{24}, B_{23}\}$ is an orthogonal matrix w.r.t. the standard form.*

Proof Let $\widetilde{\psi}_3 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$. Let β be the matrix of T_σ. We show that β is in the orthogonal group of $\widetilde{\psi}_3$.

Let $\mathfrak{p}$ be a prime ideal of R. It suffices to show that $\beta_\mathfrak{p}$ is in the orthogonal group of $\widetilde{\psi}_3$, for all prime ideals $\mathfrak{p}$ of R. (Note that of $T_{\sigma_\mathfrak{p}}$ is the same as the matrix of $(T_\sigma)_\mathfrak{p}$.)

As $R_\mathfrak{p}$ is a local ring, $\mathrm{SL}_r(R_\mathfrak{p}) = \mathrm{E}_r(R_\mathfrak{p})$, for all $r \geq 2$. Hence, $\sigma_\mathfrak{p}$ is an elementary matrix, i.e. it is a product of elementary generators $\varepsilon_1, \ldots, \varepsilon_k$, for some k. We may assume that ε_i is of type $E_{1i}(x)$ or $E_{i1}(x)$, for some i, and arbitrary $x \in R$.

Now, $T_{\sigma_\mathfrak{p}} = \prod T_{\varepsilon_k}$. By Lemma 9.4, the matrix of each T_{ε_j} is an elementary orthogonal matrix w.r.t. the ordered basis $\{B_{12}, B_{13}, B_{14}, B_{34}, -B_{24}, B_{23}\}$. Hence, so is $T_{\sigma_\mathfrak{p}}$, for all prime ideals $\mathfrak{p}$ of R. □

But one has the following:

Remark 9.2 Let $\sigma \in \mathrm{SL}_4(R)$. Then the matrix of the linear transformation T_σ on the Vaserstein space V w.r.t. the ordered basis $\{B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}\}$ is an orthogonal matrix with respect to the form $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$, where $\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Proof Let A and B denote the matrices of T_σ w.r.t. the bases

$$B_1 = \{B_{12}, B_{13}, B_{14}, B_{34}, -B_{24}, B_{23}\} \text{ and } B_2 = \{B_{12}, B_{13}, B_{14}, B_{23}, B_{24}, B_{34}\},$$

respectively. Let P denote the transition matrix from B_1 to B_2. Then clearly $P = I_3 \perp \alpha$ and $P^{-1}AP = B$. Note that $P^{-1} = P^T = P$. Hence $P^{-1}A^tP = (P^{-1}AP)^t = B^t$. By Proposition 9.1, A is orthogonal w.r.t. the standard form $\widetilde{\psi}_3 = \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix}$. Thus we have

$$A\widetilde{\psi}_3A^t = \widetilde{\psi}_3 \Rightarrow P^{-1}(A\widetilde{\psi}_3A^t)P = P^{-1}\widetilde{\psi}_3P \Rightarrow B(P^{-1}\widetilde{\psi}_3P)B^t = P^{-1}\widetilde{\psi}_3P,$$

which means B is orthogonal w.r.t. the form $P^{-1}\widetilde{\psi}_3P = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$. □

9.5 Injectivity

In this section, we show that we can obtain a map from $\mathrm{SL}_4(R) \to \mathrm{SO}_6(R)$ and this map induces an injection $\dfrac{\mathrm{SL}_4(R)}{\mathrm{E}_4(R)} \hookrightarrow \dfrac{\mathrm{SO}_6(R)}{\mathrm{EO}_6(R)}$.

Proposition 9.2 *The map $\varphi : \mathrm{E}_4(R) \to \mathrm{EO}_6(R)$ is defined as $\varphi(\sigma) = [T_\sigma]$ is surjective.*

Proof Note that $\mathrm{EO}_6(R)$ is generated by the elementary orthogonal matrices $oe_{12}(\lambda)$, $oe_{21}(\lambda)$, $oe_{13}(\lambda)$, $oe_{31}(\lambda)$, $oe_{23}(\lambda)$, $oe_{32}(\lambda)$, $oe_{24}(\lambda)$, $oe_{42}(\lambda)$, $oe_{34}(\lambda)$, $oe_{43}(\lambda)$, $oe_{35}(\lambda)$ and $oe_{53}(\lambda)$. By the same argument as that of Lemma 9.4, one has

$$\begin{aligned}
\left[T_{E_{23}(\lambda)}\right] &= oe_{12}(\lambda), & \left[T_{E_{32}(\lambda)}\right] &= oe_{21}(\lambda), & \left[T_{E_{24}(\lambda)}\right] &= oe_{13}(\lambda),\\
\left[T_{E_{42}(\lambda)}\right] &= oe_{31}(\lambda), & \left[T_{E_{34}(\lambda)}\right] &= oe_{23}(\lambda), & \left[T_{E_{43}(\lambda)}\right] &= oe_{32}(\lambda),\\
\left[T_{E_{14}(-\lambda)}\right] &= oe_{24}(\lambda), & \left[T_{E_{41}(-\lambda)}\right] &= oe_{42}(\lambda), & \left[T_{E_{13}(\lambda)}\right] &= oe_{34}(\lambda),\\
\left[T_{E_{31}(\lambda)}\right] &= oe_{43}(\lambda), & \left[T_{E_{12}(-\lambda)}\right] &= oe_{35}(\lambda), & \left[T_{E_{21}(-\lambda)}\right] &= oe_{53}(\lambda).
\end{aligned}$$

Hence φ is surjective. □

Lemma 9.5 *Let u be a unit in R with $u^2 = 1$. Then $uI_4 \in \mathrm{E}_4(R)$.*

Proof This follows from Whitehead's lemma. Explicitly, if

$$\alpha_1 = \begin{pmatrix} I_2 & (1-u)I_2 \\ 0 & I_2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} I_2 & 0 \\ -I_2 & I_2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} I_2 & 0 \\ uI_2 & I_2 \end{pmatrix},$$

then clearly $\alpha_1, \alpha_2, \alpha_3 \in \mathrm{E}_4(R)$ and the direct computation shows $uI_4 = \alpha_1\alpha_2\alpha_1\alpha_3$. Hence the result. □

Proposition 9.3 *Let $\alpha \in \mathrm{M}_4(R)$ such that $\alpha A \alpha^t = A$ for all $A \in \mathrm{Alt}_4(R)$. Then $\alpha = uI_4$, where $u^2 = 1$.*

Proof Let $\alpha = (\alpha_{ij})_{4\times 4}$. Consider the generators $\{B_{ij} : 1 \le i < j \le 4\}$ of $\mathrm{Alt}_4(R)$. From $\alpha B_{1i}\alpha^t = B_{1i}$, $2 \le i \le 3$, one has

$$\alpha_{11}\alpha_{ki} - \alpha_{1i}\alpha_{k1} = 0, \quad i+1 \le k \le 4, \tag{9.1}$$
$$\alpha_{i1}\alpha_{ki} - \alpha_{ii}\alpha_{k1} = 0, \quad i+1 \le k \le 4, \tag{9.2}$$
$$\alpha_{11}\alpha_{ii} - \alpha_{1i}\alpha_{i1} = 1. \tag{9.3}$$

Now (9.1) $\times\alpha_{ii}-$ (9.2) $\times\alpha_{1i} \Rightarrow (\alpha_{11}\alpha_{ii} - \alpha_{1i}\alpha_{i1})\alpha_{ki} = 0$. Thus by (9.3),

$$\alpha_{ki} = 0, \quad i+1 \le k \le 4.$$

Also (9.1) $\times\alpha_{i1}-$ (9.2) $\times\alpha_{11} \Rightarrow (\alpha_{11}\alpha_{ii} - \alpha_{1i}\alpha_{i1})\alpha_{k1} = 0$. Again by (9.3), $\alpha_{k1} = 0$ for $k = 3, 4$.

Now we show that $\alpha_{21} = 0$. Consider $\alpha B_{13}\alpha^t = B_{13}$, we get

$$\alpha_{11}\alpha_{23} - \alpha_{13}\alpha_{21} = 0, \tag{9.4}$$
$$\alpha_{21}\alpha_{33} - \alpha_{23}\alpha_{31} = 0, \tag{9.5}$$
$$\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31} = 1. \tag{9.6}$$

Now (9.4) $\times\alpha_{31}-$ (9.5) $\times\alpha_{11} \Rightarrow (\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31})\alpha_{21} = 0$. Thus by (9.6), $\alpha_{21} = 0$. Hence

$$\alpha_{ij} = 0 \text{ for } 1 \leq j < i \leq 4. \tag{9.7}$$

Similarly using $\alpha B_{i4} \alpha^t = B_{i4}$, $1 \leq i \leq 3$, one can show that

$$\alpha_{ij} = 0 \text{ for } 1 \leq i < j \leq 4. \tag{9.8}$$

From (9.7) and (9.8), $\alpha_{ij} = 0, \quad \forall \ i \neq j$.

Now from (9.3) and the relations obtained from $\alpha B_{i4} \alpha^t = B_{i4}$, $1 \leq i \leq 3$ one get, $\alpha_{11}\alpha_{22} = \alpha_{11}\alpha_{33} = \alpha_{11}\alpha_{44} = \alpha_{22}\alpha_{44} = 1$ and hence $\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{44} = u$, where $u \in R$ with $u^2 = 1$. Hence the result. □

Theorem 9.3 *One has an injective homomorphism*

$$\overline{\varphi} : \frac{\mathrm{SL}_4(R)}{\mathrm{E}_4(R)} \hookrightarrow \frac{\mathrm{SO}_6(R)}{\mathrm{EO}_6(R)}$$

($\overline{\varphi}$ *is induced by the homomorphism* $\varphi : \mathrm{SL}_4(R) \to \mathrm{SO}_6(R)$*).*

Proof Let $\alpha \in \mathrm{SL}_4(R)$ with $[T_\alpha] = I_6$. Then $\alpha V \alpha^t = V$, for all $V \in \mathrm{Alt}_4(R)$. Thus by Proposition 9.3, $\alpha = uI_4$ with $u^2 = 1$. By Lemma 9.5, $\alpha \in \mathrm{E}_4(R)$. Hence $\frac{\mathrm{SL}_4(R)}{\mathrm{E}_4(R)} \hookrightarrow \frac{\mathrm{SO}_6(R)}{\mathrm{EO}_6(R)}$. □

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