Chapter 6
Gröbner Bases and Dimension Formulas for Ternary Partially Associative Operads

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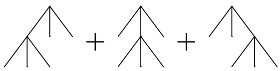
6.1 Introduction

We consider nonsymmetric operads in the category of $\mathbb{Z}$-graded vector spaces over a field of characteristic 0. The product is the tensor product (with Koszul signs) and the coproduct is the direct sum. Gröbner bases for operads were introduced by Dotsenko, Khoroshkin and Vallette [5, 6]; see also [2].

Let $\mathcal{LT}$ be the free nonsymmetric operad with one ternary operation $t = (\ast\ast\ast)$. Let α denote ternary partial associativity, which may be written as a tree polynomial, using partial compositions or as a nonassociative polynomial:

$\alpha =$ 

$$\begin{array}{l} t \circ_1 t + t \circ_2 t + t \circ_3 t, \\ ((\ast\ast\ast)\ast\ast) + (\ast(\ast\ast\ast)\ast) + (\ast\ast(\ast\ast\ast)). \end{array} \tag{6.1}$$

We compute a Gröbner basis for the ideal $\langle\alpha\rangle$ when t has even (homological) degree so that Koszul signs are irrelevant, and when t has odd degree so that Koszul signs are essential. We include details of the calculations to clarify the Gröbner basis algorithm for nonsymmetric operads. As an application, we calculate dimension formulas for the quotient operads. Similar results have been obtained independently in unpublished work of Vladimir Dotsenko.

For earlier work on partial associativity and its applications, see [1, 3, 7, 9–11, 13–15]. Recent results of Dotsenko, Shadrin and Vallette [8] have shown that

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the ternary partially associative operad with an odd generator arises naturally in the homology of the poset of interval partitions into intervals of odd length and in certain De Concini–Procesi models of subspace arrangements [4] over the real numbers.

6.2 Preliminaries

Definition 6.1 An *m-ary tree* is a rooted plane tree p in which every node has either no children (*leaf*) or m children (*internal node*). The *weight* $w(p)$ counts internal nodes; the *arity* $\ell(p) = 1 + w(p)(m-1)$ counts leaves indexed $1, \dots, \ell(p)$ from left to right. The *basic tree* t is the m-ary tree of weight 1. Set $[n] = \{1, \dots, n\}$.

Definition 6.2 If $n \equiv 1 \pmod{m-1}$ then $\mathcal{T}(n)$ is the set of m-ary trees of arity n, and $\mathcal{T}$ is the disjoint union of the $\mathcal{T}(n)$ for $n \geq 1$.

Definition 6.3 If $p, q \in \mathcal{T}$ then for $i \in [\ell(p)]$ the *partial composition* $p \circ_i q \in \mathcal{T}$ is obtained by identifying leaf i of p with the root of q.

Lemma 6.1 *Starting with t, every m-ary tree of weight w can be obtained by a sequence of $w-1$ partial compositions.*

Lemma 6.2 *Let p, q, r be m-ary trees. Partial composition satisfies* [2, p. 72]*:*

$$(p \circ_i q) \circ_j r = \begin{cases} p \circ_i (q \circ_{j-i+1} r), & i \leq j \leq i+\ell(q)-1; \\ (p \circ_{j-\ell(q)+1} r) \circ_i q, & i+\ell(q) \leq j \leq \ell(p)+\ell(q)-1; \\ (p \circ_j r) \circ_{i+\ell(r)-1} q, & 1 \leq j \leq i-1. \end{cases}$$

Lemma 6.3 *The set $\mathcal{T}$ with partial compositions is isomorphic to the free nonsymmetric (set) operad with one m-ary operation t.*

Definition 6.4 If $n \equiv 1 \pmod{m-1}$ then $\mathcal{LT}(n)$ is the vector space with basis $\mathcal{T}(n)$, and $\mathcal{LT}$ is the direct sum of $\mathcal{LT}(n)$ for $n \geq 1$. A *tree polynomial* of arity n is an element of $\mathcal{LT}(n)$. Partial composition in $\mathcal{T}$ extends bilinearly to $\mathcal{LT}$.

Lemma 6.4 *The vector space $\mathcal{LT}$ with partial compositions is isomorphic to the free nonsymmetric (vector) operad with one m-ary operation t.*

Definition 6.5 A *relation* of arity n is an element of $\mathcal{LT}(n) \setminus 0$. The *operad ideal* $\mathcal{I} = \langle f_1, \dots, f_k \rangle$ generated by relations $f_1, \dots, f_k$ is the intersection of all homogeneous subspaces $\mathcal{S} \subseteq \mathcal{LT}$ such that $f_1, \dots, f_k \in \mathcal{S}$, and for all $f \in \mathcal{S}(m)$, $g \in \mathcal{LT}(n)$ we have $f \circ_i g, g \circ_j f \in \mathcal{S}$ ($i \in [m]$, $j \in [n]$).

The following results come from [2, Sect. 3.4] and [6, Sects. 2.4, 3.1] with minor changes.

Definition 6.6 The *path sequence* of $p \in \mathcal{T}(n)$ is $\text{path}(p) = (a_1, \dots, a_n)$, where a_i is the length of the path from the root to the leaf i.

Lemma 6.5 *If* $p, q \in \mathcal{T}$ *then* $p = q$ *if and only if* $\text{path}(p) = \text{path}(q)$.

Definition 6.7 For $p, q \in \mathcal{T}(n)$ we write $p \prec q$ and say *p precedes q* in *path-lex order* if and only if $\text{path}(p) \prec \text{path}(q)$ in lex order on n-tuples of positive integers. If $f \in \mathcal{LT}(n)$ then its *leading monomial* $\ell m(f) \in \mathcal{T}(n)$ is the greatest monomial in path-lex order, and its *leading coefficient* $\ell c(f)$ is the coefficient of $\ell m(f)$.

Definition 6.8 If $p, q \in \mathcal{T}$ then q is *divisible* by p (written $p \mid q$) if p is a subtree of q: that is, $q = \cdots p \cdots$ where the dots denote sequences of partial compositions with parentheses. If $p \in \mathcal{T}(m)$, $q \in \mathcal{T}(n)$, $p \mid q$, and $f \in \mathcal{LT}(m)$ then we may replace p by f in q and use linearity and the same partial compositions to obtain the *substitution* of f for p in q:

$$M(q, p, f) = \cdots f \cdots \in \mathcal{LT}(n).$$

Definition 6.9 If $f, g \in \mathcal{LT}$ and $\ell m(g) \mid \ell m(f)$ then the *reduction of f by g* (which eliminates the leading term of f) is

$$R(f, g) = f - \frac{\ell c(f)}{\ell c(g)} M\big(\ell m(f), \ell m(g), g\big).$$

This extends to reduction of f by $g_1, \ldots, g_k$; see [2, Algorithm 3.4.2.16].

Definition 6.10 If $p, q, r \in \mathcal{T}$ then we call p a *small common multiple* (SCM) of q and r if $q \mid p$, $r \mid p$, every node of p is a node of q or r (or both), and $\ell(p) < \ell(q) + \ell(r)$.

Definition 6.11 If f, g, h are monic tree polynomials and $\ell m(f)$ is an SCM of $\ell m(g)$, $\ell m(h)$ then the resulting *S-polynomial* is

$$S(f, g, h) = M\big(\ell m(f), \ell m(g), g\big) - M\big(\ell m(f), \ell m(h), h\big).$$

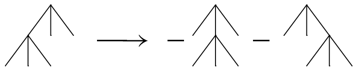
Definition 6.12 Let G be a finite set of relations and let $I = \langle G \rangle$. If for all $f \in I$ there exists $g \in G$ such that $\ell m(g) \mid \ell m(f)$ then we call G a *Gröbner basis* for I. We say G is *reduced* if $\ell m(g)$ is not divisible by $\ell m(h)$ for all $g, h \in G$.

Lemma 6.6 *Every operad ideal has a unique reduced Gröbner basis.*

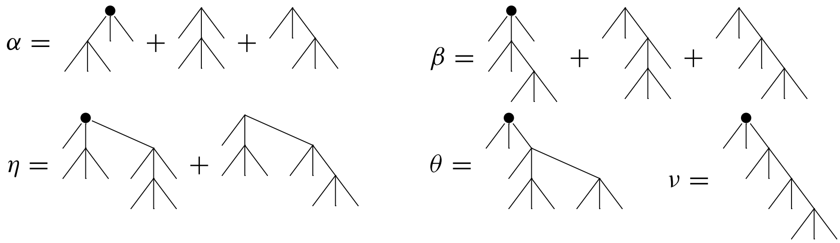
Theorem 6.1 *If* $I = \langle G \rangle$ *then* G *is a Gröbner basis for* I *if and only if for every SCM* f *of elements* $g, h \in G$ *the reduction of* $S(f, g, h)$ *by* G *is 0.*

6.3 Gröbner Bases and Dimension Formulas

In the rest of this paper, we consider a ternary operation ($m = 3$). We usually indicate the leading monomial of a tree polynomial by a bullet at the root, and write the terms of a tree polynomial from left to right in reverse path-lex order. The partially associative relation α corresponds to this rewrite rule:

$t \circ_1 t =$  $= -\, t \circ_2 t - t \circ_3 t \qquad (6.2)$

Theorem 6.2 *For the path-lex monomial order, the following tree polynomials form the reduced Gröbner basis for $\langle \alpha \rangle$ with an operation of even degree:*

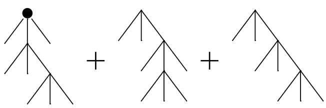


Proof The proof consists of Lemmas 6.7 to 6.13. □

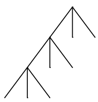
Remark 6.1 As nonassociative polynomials, the relations of Theorem 6.2 are

$((***)**) + (*(***)*) + (**(***)),$
$(*(**(***))*) + (**(*(***)*)) + (**(**(***))),$
$(*(***)(*(***)*)) + (*(***)(**(***))),\ (**(*(***)(***))),\ (**(**(**(***)))).$

Lemma 6.7 *There is only one SCM of $\ell m(\alpha)$ with itself; this produces reduced S-polynomial β, and the set $\{\alpha, \beta\}$ is self-reduced:*

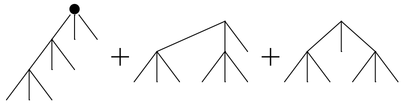
$\beta =$  $= t \circ_2 (t \circ_3 t) + t \circ_3 (t \circ_2 t) + t \circ_3 (t \circ_3 t).$

Proof We have $\ell m(\alpha) = t \circ_1 t$ and hence

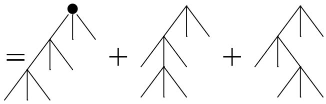
$\ell m(\alpha) \circ_1 t =$  $= t \circ_1 \ell m(\alpha).$

From this, we obtain these tree polynomials by substitution (Definition 6.8):

$$\alpha \circ_1 t = (t \circ_1 t) \circ_1 t + (t \circ_2 t) \circ_1 t + (t \circ_3 t) \circ_1 t$$

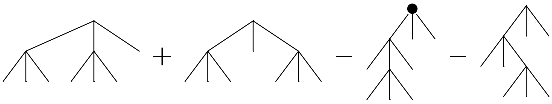
$=$ 

$$t \circ_1 \alpha = t \circ_1 (t \circ_1 t) + t \circ_1 (t \circ_2 t) + t \circ_1 (t \circ_3 t)$$

$=$ 

The difference is this (non-reduced) S-polynomial:

$$\alpha \circ_1 t - t \circ_1 \alpha = (t \circ_2 t) \circ_1 t + (t \circ_3 t) \circ_1 t - t \circ_1 (t \circ_2 t) - t \circ_1 (t \circ_3 t)$$

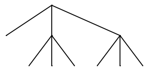
$=$ 

$$= (t \circ_1 t) \circ_4 t + (t \circ_1 t) \circ_5 t - (t \circ_1 t) \circ_2 t - (t \circ_1 t) \circ_3 t.$$

We have rewritten the partial compositions (Lemma 6.2). We apply rewrite rule (6.2) to the top subtree $\ell m(\alpha) = t \circ_1 t$ of each monomial (reduce using α):

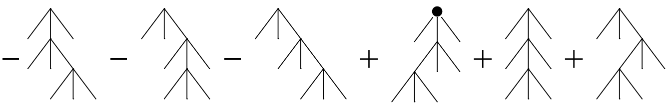
$$\begin{aligned} &- (t \circ_2 t) \circ_4 t - (t \circ_3 t) \circ_4 t - (t \circ_2 t) \circ_5 t - (t \circ_3 t) \circ_5 t \\ &+ (t \circ_2 t) \circ_2 t + (t \circ_3 t) \circ_2 t + (t \circ_2 t) \circ_3 t + (t \circ_3 t) \circ_3 t. \end{aligned}$$

Terms 3 and 6 cancel since both monomials represent the same tree:

$$(t \circ_2 t) \circ_5 t \ = \ (t \circ_3 t) \circ_2 t \ =$$ 

Six terms remain:

$$\begin{aligned} &- (t \circ_2 t) \circ_4 t - (t \circ_3 t) \circ_4 t - (t \circ_3 t) \circ_5 t \\ &\quad + (t \circ_2 t) \circ_2 t + (t \circ_2 t) \circ_3 t + (t \circ_3 t) \circ_3 t \end{aligned}$$

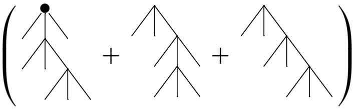
$=$ 

$$\begin{aligned} = \ &- t \circ_2 (t \circ_3 t) - t \circ_3 (t \circ_2 t) - t \circ_3 (t \circ_3 t) \\ &+ t \circ_2 (t \circ_1 t) + t \circ_2 (t \circ_2 t) + t \circ_3 (t \circ_1 t). \end{aligned}$$

In terms 4 and 6, we reduce the bottom subtree $\ell m(\alpha) = t \circ_1 t$ using α:

$$\begin{aligned} &- t \circ_2 (t \circ_3 t) - t \circ_3 (t \circ_2 t) - t \circ_3 (t \circ_3 t) - t \circ_2 (t \circ_2 t) \\ &- t \circ_2 (t \circ_3 t) + t \circ_2 (t \circ_2 t) - t \circ_3 (t \circ_2 t) - t \circ_3 (t \circ_3 t). \end{aligned}$$

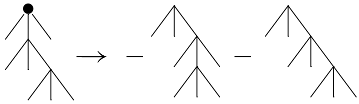
Terms 4 and 6 cancel and the others combine in pairs:

$$-2\big(t \circ_2 (t \circ_3 t) + t \circ_3 (t \circ_2 t) + t \circ_3 (t \circ_3 t)\big) = -2$$ 

No further reduction is possible. The monic form of this polynomial is β. □

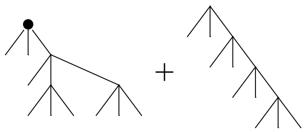
The relation β corresponds to this rewrite rule:

$$t \circ_2 (t \circ_3 t) = -t \circ_3 (t \circ_2 t) - t \circ_3 (t \circ_3 t)$$

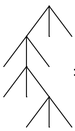
 (6.3)

We consider separately the four SCMs of $\ell m(\alpha) = t \circ_1 t$ and $\ell m(\beta) = t \circ_2 (t \circ_3 t)$.

Lemma 6.8 *Identifying the second t of $\ell m(\alpha) = t \circ_1 t$ with the first t of $\ell m(\beta) = t \circ_2 (t \circ_3 t)$ produces the reduced S-polynomial γ, and $\{\alpha, \beta, \gamma\}$ is self-reduced:*

$\gamma = 2$  $= 2(t \circ_3 (t \circ_2 t)) \circ_7 t + t \circ_3 (t \circ_3 (t \circ_3 t)).$

Proof We have the following equations:

$\ell m(\alpha) \circ_2 (t \circ_3 t) = (t \circ_1 t) \circ_2 (t \circ_3 t) =$  $= t \circ_1 (t \circ_2 (t \circ_3 t)) = t \circ_1 \ell m(\beta).$

We apply the same partial compositions to α and β:

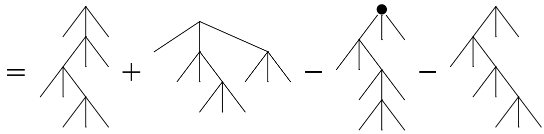
$$\begin{aligned} \alpha \circ_2 (t \circ_3 t) &= (t \circ_1 t) \circ_2 (t \circ_3 t) + (t \circ_2 t) \circ_2 (t \circ_3 t) + (t \circ_3 t) \circ_2 (t \circ_3 t), \\ t \circ_1 \beta &= t \circ_1 (t \circ_2 (t \circ_3 t)) + t \circ_1 (t \circ_3 (t \circ_2 t)) + t \circ_1 (t \circ_3 (t \circ_3 t)). \end{aligned}$$

Taking the difference, we obtain this (non-reduced) S-polynomial:

$$\begin{aligned} &(t \circ_1 t) \circ_2 (t \circ_3 t) + (t \circ_2 t) \circ_2 (t \circ_3 t) + (t \circ_3 t) \circ_2 (t \circ_3 t) \\ &- t \circ_1 (t \circ_2 (t \circ_3 t)) - t \circ_1 (t \circ_3 (t \circ_2 t)) - t \circ_1 (t \circ_3 (t \circ_3 t)). \end{aligned}$$

Terms 1 and 4 cancel, leaving

$$(t \circ_2 t) \circ_2 (t \circ_3 t) + (t \circ_3 t) \circ_2 (t \circ_3 t) - t \circ_1 (t \circ_3 (t \circ_2 t)) - t \circ_1 (t \circ_3 (t \circ_3 t))$$

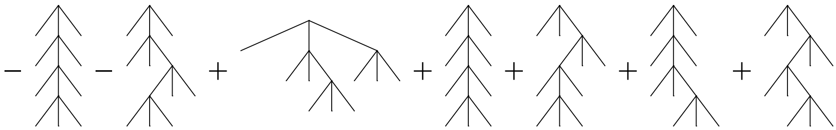


$$= t \circ_2 ((t \circ_1 t) \circ_3 t) + (t \circ_3 t) \circ_2 (t \circ_3 t) - (t \circ_1 t) \circ_3 (t \circ_2 t) - (t \circ_1 t) \circ_3 (t \circ_3 t).$$

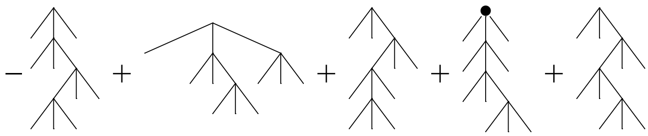
Terms 1, 3, 4 contain the subtree $\ell m(\alpha) = t \circ_1 t$, so we reduce them using α:

$$- t \circ_2 ((t \circ_2 t) \circ_3 t) - t \circ_2 ((t \circ_3 t) \circ_3 t) + (t \circ_3 t) \circ_2 (t \circ_3 t) + (t \circ_2 t) \circ_3 (t \circ_2 t) + (t \circ_3 t) \circ_3 (t \circ_2 t) + (t \circ_2 t) \circ_3 (t \circ_3 t) + (t \circ_3 t) \circ_3 (t \circ_3 t).$$

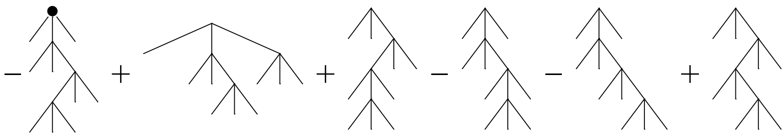
We write this polynomial in terms of trees:



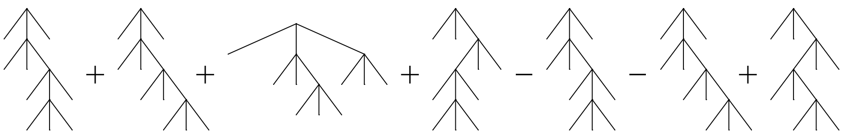
Terms 1 and 4 cancel, leaving



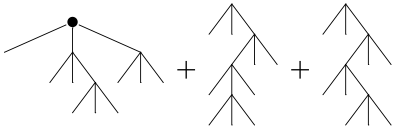
The leading monomial is divisible by $\ell m(\beta)$ but not $\ell m(\alpha)$; we reduce using β:



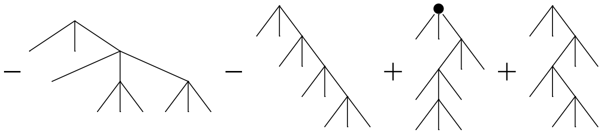
The leading monomial is divisible by α (bottom) and β (top). Using α gives



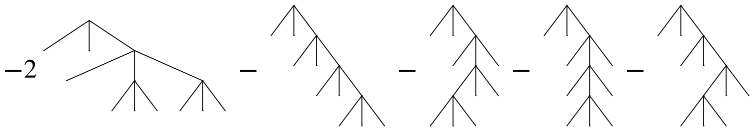
Terms 1, 5 and terms 2, 6 cancel, leaving



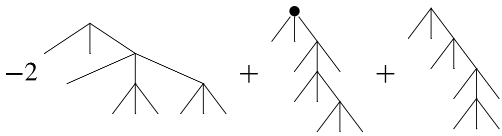
We reduce the leading monomial using β:



Terms 1, 2 cannot be reduced; terms 3, 4 can be reduced by α:

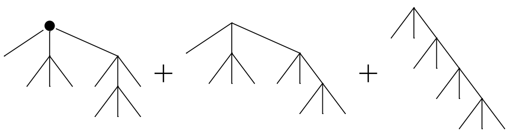


We reduce terms 3, 5 by α:



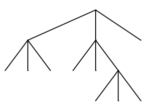
If we reduce term 2 using β, then two terms cancel and we obtain $-\gamma$. $\square$

Lemma 6.9 *Identifying the first t of $\ell m(\alpha) = t \circ_1 t$ and the first t of $\ell m(\beta) = t \circ_2 (t \circ_3 t)$ we obtain the S-polynomial δ, and $\{\alpha, \beta, \delta\}$ is self-reduced:*

$\delta =$ 

$$= (t \circ_3 (t \circ_2 t)) \circ_2 t) + (t \circ_3 (t \circ_3 t)) \circ_2 t) + (t \circ_3 (t \circ_3 (t \circ_3 t)).$$

Proof We have the equations

$\ell m(\alpha) \circ_4 (t \circ_3 t) = (t \circ_1 t) \circ_4 (t \circ_3 t) =$  $= (t \circ_2 (t \circ_3 t)) \circ_1 t = \ell m(\beta) \circ_1 t.$

We apply the same partial compositions to α and β:

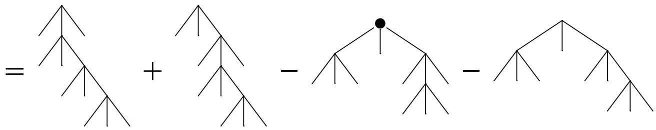
$$\alpha \circ_4 (t \circ_3 t) = (t \circ_1 t) \circ_4 (t \circ_3 t) + (t \circ_2 t) \circ_4 (t \circ_3 t) + (t \circ_3 t) \circ_4 (t \circ_3 t),$$
$$\beta \circ_1 t = (t \circ_2 (t \circ_3 t)) \circ_1 t + (t \circ_3 (t \circ_2 t)) \circ_1 t + (t \circ_3 (t \circ_3 t)) \circ_1 t.$$

The resulting S-polynomial is

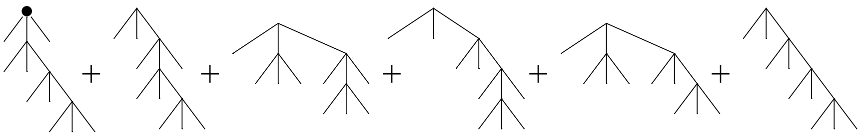
$$(t \circ_1 t) \circ_4 (t \circ_3 t) + (t \circ_2 t) \circ_4 (t \circ_3 t) + (t \circ_3 t) \circ_4 (t \circ_3 t)$$
$$- (t \circ_2 (t \circ_3 t)) \circ_1 t - (t \circ_3 (t \circ_2 t)) \circ_1 t - (t \circ_3 (t \circ_3 t)) \circ_1 t.$$

Terms 1, 4 cancel, leaving

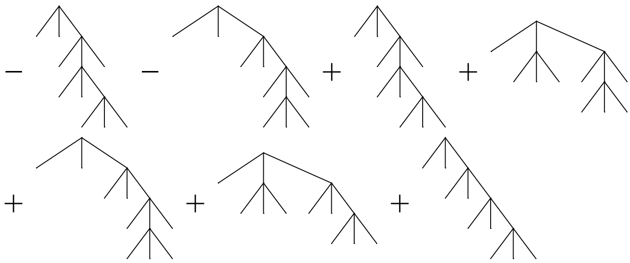
$$(t \circ_2 t) \circ_4 (t \circ_3 t) + (t \circ_3 t) \circ_4 (t \circ_3 t) - (t \circ_3 (t \circ_2 t)) \circ_1 t - (t \circ_3 (t \circ_3 t)) \circ_1 t$$



We reduce terms 3, 4 using α:

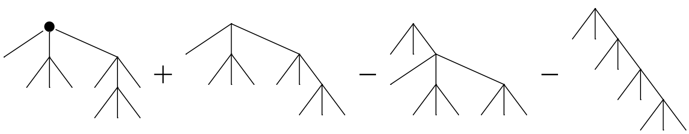


Reducing term 1 using β gives



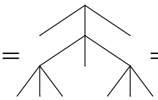
Terms 1, 3 and 2, 5 cancel; no further reduction is possible, producing δ. □

Lemma 6.10 *Identifying the first t of $\ell m(\alpha) = t \circ_1 t$ with the second t of $\ell m(\beta) = t \circ_2 (t \circ_3 t)$ we obtain the S-polynomial ϵ and $\{\alpha, \beta, \epsilon\}$ is self-reduced:*

$\epsilon =$ 

$$= ((t \circ_3 t) \circ_2 t) \circ_6 t + ((t \circ_3 t) \circ_2 t) \circ_7 t - (t \circ_3 (t \circ_3 t)) \circ_4 t - (t \circ_3 (t \circ_3 t)) \circ_7 t.$$

Proof We have the equations

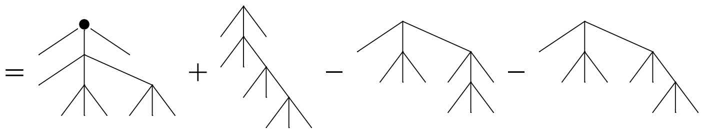
$t \circ_2 (\ell m(\alpha) \circ_5 t) = t \circ_2 ((t \circ_1 t) \circ_5 t) =$  $= (t \circ_2 (t \circ_3 t)) \circ_2 t = \ell m(\beta) \circ_2 t.$

The resulting S-polynomial $t \circ_2 (\alpha \circ_5 t) - \beta \circ_2 t$ is

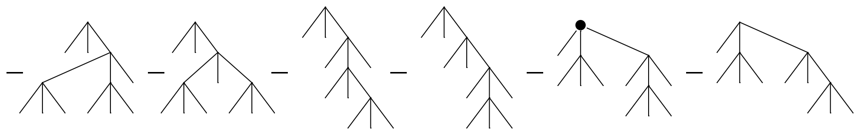
$$t \circ_2 ((t \circ_1 t) \circ_5 t) + t \circ_2 ((t \circ_2 t) \circ_5 t) + t \circ_2 ((t \circ_3 t) \circ_5 t)$$
$$- (t \circ_2 (t \circ_3 t)) \circ_2 t - (t \circ_3 (t \circ_2 t)) \circ_2 t - (t \circ_3 (t \circ_3 t)) \circ_2 t.$$

Terms 1, 4 cancel, leaving

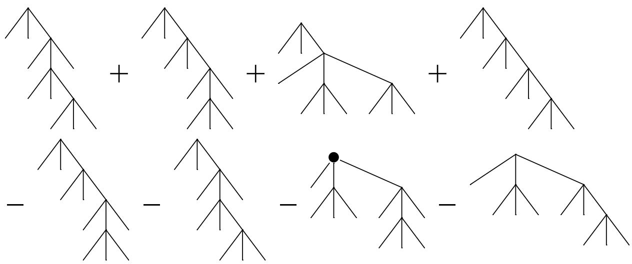
$$t \circ_2 ((t \circ_2 t) \circ_5 t) + t \circ_2 ((t \circ_3 t) \circ_5 t) - (t \circ_3 (t \circ_2 t)) \circ_2 t - (t \circ_3 (t \circ_3 t)) \circ_2 t$$



We reduce terms 1, 2 using β:

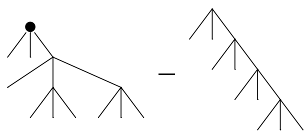


Reducing terms 1, 2 using α gives



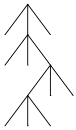
Terms 1, 6 and 2, 5 cancel. No further reduction is possible, giving $-\epsilon$. □

Lemma 6.11 *Identifying the first* t *of* $\ell m(\alpha) = t \circ_1 t$ *with the third* t *of* $\ell m(\beta) = t \circ_2 (t \circ_3 t)$ *we obtain new S-polynomial* ζ*, and* $\{\alpha, \beta, \zeta\}$ *is self-reduced:*

$\zeta =$  $= t \circ_3 ((t \circ_2 t) \circ_5 t) - t \circ_3 (t \circ_3 (t \circ_3 t)).$

Proof We have the equations

$$(t \circ_2 t) \circ_4 \ell m(\alpha) = (t \circ_2 t) \circ_4 (t \circ_1 t) =$$



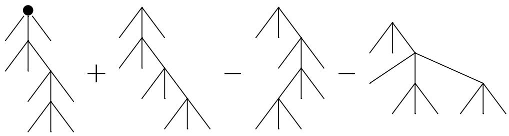
$$= (t \circ_2 (t \circ_3 t)) \circ_4 t = \ell m(\beta) \circ_4 t.$$

The resulting S-polynomial $(t \circ_2 t) \circ_4 \alpha - \beta \circ_4 t$ is

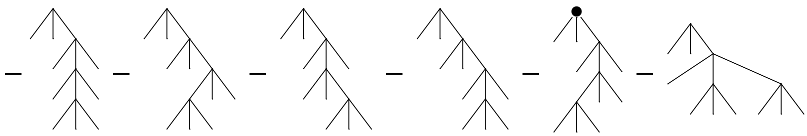
$$\begin{aligned} &(t \circ_2 t) \circ_4 (t \circ_1 t) + (t \circ_2 t) \circ_4 (t \circ_2 t) + (t \circ_2 t) \circ_4 (t \circ_3 t) \\ &- (t \circ_2 (t \circ_3 t)) \circ_4 t - (t \circ_3 (t \circ_2 t)) \circ_4 t - (t \circ_3 (t \circ_3 t)) \circ_4 t. \end{aligned}$$

Terms 1, 4 cancel, leaving

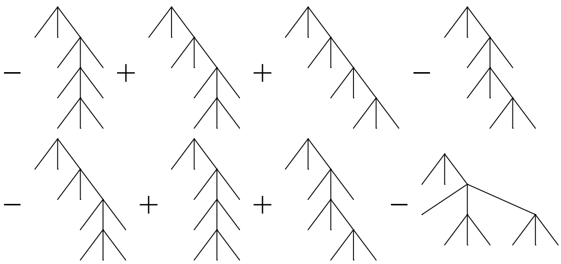
$$(t \circ_2 t) \circ_4 (t \circ_2 t) + (t \circ_2 t) \circ_4 (t \circ_3 t) - (t \circ_3 (t \circ_2 t)) \circ_4 t - (t \circ_3 (t \circ_3 t)) \circ_4 t$$

$=$ 

We use β to reduce terms 1, 2:



We use α to reduce terms 2, 5:



Terms 1, 6 and 2, 5 and 4, 7 cancel. No further reductions are possible, and the monic form of the last polynomial is ζ. □

Lemma 6.12 *The polynomials γ, δ, ϵ, ζ span a subspace with basis η, θ, ν.*

Proof It is easy to see that

$$\eta = \frac{1}{3}(\gamma + \delta + 2\epsilon), \qquad \theta = \frac{1}{3}(2\gamma - \delta + \epsilon), \qquad \nu = -\frac{1}{3}(\gamma - 2\delta + 2\epsilon),$$

and that these three polynomials form a basis of span$(\gamma, \delta, \epsilon, \zeta)$. □

Lemma 6.13 *Every S-polynomial obtained from α, β, η, θ, ν reduces to 0.*

Proof If either f or g is a monomial then clearly every S-polynomial obtained from f and g reduces to 0. We have already considered S-polynomials from α and β; the other cases are α, η and β, η and η, η with many subcases. We give details for the most difficult subcase and leave the others as exercises. These calculations can be simplified using the triangle lemma for nonsymmetric operads [2, Prop. 3.5.3.2].

We identify the second t of $\ell m(\alpha)$ with the first t of $\ell m(\eta)$ and obtain this SCM:

$$\ell m(\alpha) = t \circ_1 t, \quad \ell m(\eta) = (t \circ_2 t) \circ_5 (t \circ_2 t), \quad (\ell m(\alpha) \circ_2 t) \circ_5 (t \circ_2 t) = t \circ_1 \ell m(\eta).$$

To save space, we switch to nonassociative notation. We obtain the S-polynomial

$$\begin{aligned}&(\alpha \circ_2 t) \circ_5 (t \circ_2 t) - t \circ_1 \eta = \\ &(\ast((\ast\ast\ast)(\ast(\ast\ast\ast)\ast)\ast)\ast) + (\ast(\ast\ast\ast)((\ast(\ast\ast\ast)\ast)\ast\ast)) - ((\ast(\ast\ast\ast)(\ast\ast(\ast\ast\ast))\ast\ast).\end{aligned}$$

Rewrite rules (6.2) and (6.3) have this form; the letters represent submonomials:

$$((vwx)yz) \longmapsto -(v(wxy)z) - (vw(xyz)), \tag{6.4}$$

$$(t(uv(wxy))z) \longmapsto -(tu(v(wxy)z)) - (tu(vw(xyz))). \tag{6.5}$$

When we apply (6.4) or (6.5), we use bars to indicate the submonomials. To begin we reduce all three monomials in the S-polynomial using α and obtain

$$\begin{aligned}&(\ast((\bar{\ast}\bar{\ast}\bar{\ast})\overline{(\ast(\ast\ast\ast)\ast)}\bar{\ast})\ast) + (\ast(\ast\ast\ast)((\bar{\ast}\overline{(\ast\ast\ast)}\bar{\ast})\bar{\ast}\bar{\ast})) - ((\bar{\ast}\overline{(\ast\ast\ast)}\,\overline{(\ast\ast(\ast\ast\ast))}\bar{\ast}\bar{\ast}) = \\ &-(\ast(\ast(\ast\ast(\ast(\ast\ast\ast)\ast))\ast)\ast) - (\ast(\ast\ast(\ast(\ast(\ast\ast\ast)\ast)\ast))\ast) - (\ast(\ast\ast\ast)(\ast((\bar{\ast}\bar{\ast}\bar{\ast})\bar{\ast}\bar{\ast})\ast)) \\ &-(\ast(\ast\ast\ast)(\ast(\ast\ast\ast)(\ast\ast\ast))) + (\ast((\bar{\ast}\bar{\ast}\bar{\ast})\overline{(\ast\ast(\ast\ast\ast))}\bar{\ast})\ast) + (\ast(\ast\ast\ast)((\bar{\ast}\bar{\ast}\overline{(\ast\ast\ast)})\bar{\ast}\bar{\ast})).\end{aligned}$$

Terms 3, 5, 6 reduce using α as indicated; term 4 is $\theta \circ_2 t$ and reduces to 0:

$$\begin{aligned}&-(\ast(\bar{\ast}(\bar{\ast}\bar{\ast}(\bar{\ast}\overline{(\ast\ast\ast)})\bar{\ast})\bar{\ast})\ast) - (\bar{\ast}(\bar{\ast}\bar{\ast}(\bar{\ast}\overline{(\ast(\ast\ast\ast)\ast)}\bar{\ast}))\bar{\ast}) + (\ast(\ast\ast\ast)(\ast(\ast(\ast\ast\ast)\ast)\ast)) \\ &+(\ast(\ast\ast\ast)(\bar{\ast}(\bar{\ast}\bar{\ast}(\bar{\ast}\bar{\ast}\bar{\ast}))\bar{\ast})) - (\ast(\bar{\ast}(\bar{\ast}\bar{\ast}(\bar{\ast}\bar{\ast}\overline{(\ast\ast\ast)}))\bar{\ast})\ast) - (\bar{\ast}(\bar{\ast}\bar{\ast}(\bar{\ast}\overline{(\ast\ast(\ast\ast\ast))})\bar{\ast})\bar{\ast}) \\ &-(\ast(\ast\ast\ast)(\ast(\ast(\ast\ast\ast)\ast)\ast)) - (\ast(\ast\ast\ast)(\ast\ast(\ast\ast\ast)\ast)\ast)).\end{aligned}$$

Terms 3, 7 cancel, and terms 1, 2, 4, 5, 6 reduce using β as indicated:

$$
\begin{aligned}
&(\bar{*}(\bar{*}\bar{*}(\bar{*}(\overline{*(***)*}))\bar{*})\bar{*}) + (\bar{*}(\bar{*}\bar{*}(\bar{*}\bar{*}(\overline{(***)**})))\bar{*}) + (**(*(*(*(***)*)*)*)) \\
&+ (**(**((*(***)*)**)) - (*(***)(**(*(***)*))) - (*(***)(**(**(***)))) \\
&+ (\bar{*}(\bar{*}\bar{*}(\bar{*}(\overline{**(***)})\bar{*}))\bar{*}) + (\bar{*}(\bar{*}\bar{*}(\bar{*}\bar{*}(\overline{*(***)*}))\bar{*}) + (**(*(*(**(***))*))*) \\
&+ (**(**((**(***))**))) - (*(***)(**((***)**))).
\end{aligned}
$$

Terms 1, 2, 7, 8 reduce using β as indicated; term 6 is $\nu \circ_2 t$ and reduces to 0; omitting terms which cancel, we obtain

$$
\begin{aligned}
&(**(*(**((\bar{*}\bar{*}\bar{*})\bar{*}\bar{*}))*)) + (**(**(*((\bar{*}\bar{*}\bar{*})\bar{*}\bar{*})*))) - (*(***)(**(*(***)*))) \\
&- (**(*(*(**(***))*)*)) - (**(*(**(*(***)*))*)) - (**(**(*(*(***))*)*)) \\
&+ (**(**((**(***))**))) - (*(***)(**((\bar{*}\bar{*}\bar{*})\bar{*}\bar{*}))).
\end{aligned}
$$

Terms 1, 2, 8 reduce using α as indicated; omitting terms which cancel, we obtain

$$
\begin{aligned}
&-2(**(\bar{*}(\bar{*}\bar{*}(\bar{*}(\overline{***})\bar{*}))\bar{*})) - 2(**(**(*(*(***)*)*))) - (**(\bar{*}(\bar{*}\bar{*}(\bar{*}\bar{*}(\overline{***})))\bar{*})) \\
&- (**(**(\bar{*}(\bar{*}\bar{*}(\bar{*}\bar{*}\bar{*}))\bar{*}))).
\end{aligned}
$$

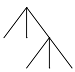
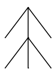
Terms 1, 3, 4 reduce using β as indicated. Some terms cancel, and others reduce to 0 using ν, leaving the single term $(**(**(\bar{*}(\bar{*}\bar{*}(\bar{*}\bar{*}\bar{*}))\bar{*})))$. We reduce using β and then both terms reduce to 0 using ν. □

We use Theorem 6.2 to calculate the dimensions of the homogeneous components of the ternary partially associative operad $\mathcal{TPA} = \mathcal{LT}/\langle\alpha\rangle$ with an operation of even (homological) degree. Theorem 6.3 below implies the conditional result of Goze and Remm [11, Theorem 15]; our proof using Gröbner bases is much simpler. For a more general conjecture, see [2, Conjecture 10.3.2.6, case 6].

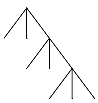
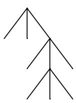
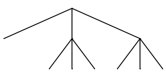
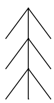
Lemma 6.14 *For* $n = 1, 3, 5, 7$ *we have*

$$
\dim \mathcal{TPA}(1) = \dim \mathcal{TPA}(3) = 1, \qquad \dim \mathcal{TPA}(5) = 2, \qquad \dim \mathcal{TPA}(7) = 4.
$$

For $\mathcal{TPA}(5)$ *a monomial basis in increasing path-lex order is*

$T_1 =$  $T_2 =$ 

For $\mathcal{TPA}(7)$ *a monomial basis in increasing path-lex order is*

$T_3 =$  $T_4 =$  $T_5 =$  $T_6 =$ 

Proof The case $n = 1$ is trivial, and for $n = 3$ we have only the basic tree t. For $n = 5$, the monomial $t \circ_1 t$ reduces by α, leaving only $T_1 = t \circ_2 t$ and $T_2 = t \circ_3 t$.

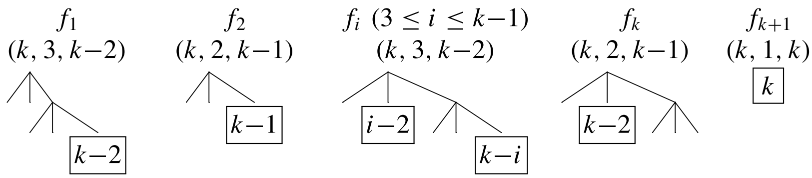
For $n = 7$, we have (i) $T_1 \circ_i t$: if $i = 1, 3$ the result reduces by α, and if $i = 2, 4, 5$ we obtain T_5, T_4, T_3; (ii) $t \circ_i T_1$: if $i = 1, 2$ the result reduces by α, β, and if $i = 3$ we obtain T_3; (iii) $T_2 \circ_i t$: if $i = 1, 2$ the result reduces by α, if $i = 3$ we obtain T_6, if $i = 4$ the result reduces by β, and if $i = 5$ we obtain T_5; (iv) $t \circ_i T_2$: if $i = 1$ the result reduces by α, if $i = 2, 3$ we obtain T_6, T_4. Clearly $T_3, \ldots, T_6$ cannot be reduced using α or β, which proves linear independence. □

Theorem 6.3 *For weight $k \geq 3$ we have* $\dim \mathcal{TPA}(2k+1) = k+1$.

Proof Let M_0 be the tree with one vertex, set $M_1 = t$, and for $\ell \geq 2$ set

$$M_\ell = t \circ_2 (t \circ_2 (t \circ_2 \cdots (t \circ_2 t) \cdots)) \qquad (\ell \text{ copies of } t).$$

Consider the following $k+1$ monomials of weight k in increasing path-lex order; to save space we write $\boxed{\ell} = M_\ell$:



We say a leaf is left (middle, right) if it is the left (middle, right) child of its parent. The ordered triples above give the number of left (middle, right) leaves. We have $f_1 = t \circ_3 (t \circ_3 M_{k-2})$, $f_2 = t \circ_3 M_{k-1}$, and

$$f_i = (t \circ_3 (t \circ_3 M_{k-i})) \circ_2 M_{i-2} \quad (3 \leq i \leq k).$$

For $3 \leq i \leq k-1$, we obtain f_{i+1} from f_i by moving the bottom t of the right–right subtree to the middle subtree. We will show that $f_1, \ldots, f_{k+1}$ form a basis of $\mathcal{TPA}(2k+1)$. For linear independence, we simply observe that no f_i $(1 \leq i \leq k)$ can be reduced using any Gröbner basis element α, β, η, θ, ν.

To prove that $f_1, \ldots, f_{k+1}$ span $\mathcal{TPA}(2k+1)$ we use induction on $k \geq 3$. Basis: Lemma 6.14 gives $f_1 = T_3$, $f_2 = T_4$, $f_3 = T_5$, $f_4 = T_6$. Induction: Assume that $f_1, \ldots, f_{k+1}$ span $\mathcal{TPA}(2k+1)$ and write $f'_1, \ldots, f'_{k+2}$ for the monomials of weight $k+1$. For each f_i in $\mathcal{TPA}(2k+1)$ we obtain monomials of weight $k+1$ in two ways:

(1) $t \circ_j f_i$ for $j \in [3], i \in [k+1]$; (2) $f_i \circ_j t$ for $i \in [k+1]$, $j \in [2k+1]$.

Case 1: If $j = 1$ then $t \circ_1 f_i$ reduces by α. If $j = 2$ then $t \circ_2 f_i$ reduces by β for $i \in [k]$, and $t \circ_2 f_{k+1} = M_{k+1} = f'_{k+2}$. If $j = 3$ then $t \circ_3 f_1$ reduces using ν, $t \circ_3 f_2 = f'_1$, $t \circ_3 f_i$ reduces using θ for $i \in [k]$, and $t \circ_3 f_{k+1} = f'_2$.

Case 2 has three subcases depending on where we attach t. If we attach to a left leaf of f_i then the result reduces by α. If we attach to a right leaf then for f_1 the result reduces by ν or β, for $f_2, \ldots, f_k$ the result reduces by β or θ, and for f_{k+1} either we obtain f'_{k+1} or the result reduces by β. If we attach to a middle leaf of f_1 then we obtain either f'_3 or f'_1 or the result reduces by θ. If we attach to a middle leaf of f_2 then we obtain either f'_3 or f'_2. If we attach to a middle leaf of f_i for $3 \le i \le k$ then we obtain f'_j for $3 \le j \le k+1$ or the result reduces by θ. If we attach to the middle leaf of f_{k+1} then we obtain f'_{k+2}. □

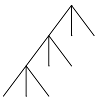
We now assume that the ternary operation t has odd (homological) degree. Thus every tree has even or odd parity depending the number of internal nodes. We write $|f| \in \{0, 1\}$ for the parity of f. We must include Koszul signs in the relations for partial compositions: transposing two odd elements introduces a minus sign.

Lemma 6.15 ([12, Def. 1.1]) *If $p, q, r \in \mathcal{T}$ then*

$$(p \circ_i q) \circ_j r = \begin{cases} p \circ_i (q \circ_{j-i+1} r) & i \le j \le i + \ell(q) - 1 \\ (-1)^{|q||r|} (p \circ_{j-\ell(q)+1} r) \circ_i q & i + \ell(q) \le j \le \ell(p) + \ell(q) - 1 \\ (-1)^{|q||r|} (p \circ_j r) \circ_{i+\ell(r)-1} q & 1 \le j \le i - 1 \end{cases}$$

Theorem 6.4 *The relation α is a Gröbner basis for $\langle \alpha \rangle$ in the free nonsymmetric operad with a ternary operation of odd homological degree.*

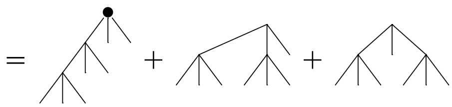
Proof The first few steps are identical to those for an even operation. The leading monomial $\ell m(\alpha) = t \circ_1 t$ overlaps with itself in one way to produce this SCM:



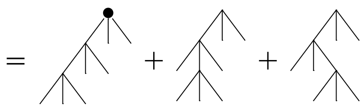
$$= \ell m(\alpha) \circ_1 t \; = \; t \circ_1 \ell m(\alpha).$$

We apply the same partial compositions to α instead of $\ell m(\alpha)$:

$$\alpha \circ_1 t = (t \circ_1 t) \circ_1 t + (t \circ_2 t) \circ_1 t + (t \circ_3 t) \circ_1 t$$

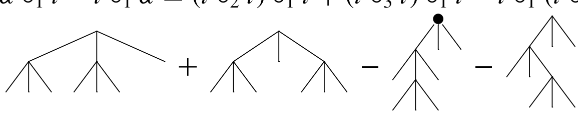


$$t \circ_1 \alpha = t \circ_1 (t \circ_1 t) + t \circ_1 (t \circ_2 t) + t \circ_1 (t \circ_3 t)$$



The difference is this (non-reduced) S-polynomial:

$$\alpha \circ_1 t - t \circ_1 \alpha = (t \circ_2 t) \circ_1 t + (t \circ_3 t) \circ_1 t - t \circ_1 (t \circ_2 t) - t \circ_1 (t \circ_3 t) =$$



(6.6)

Lemma 6.15 (case 3), $p = q = r = t$, with $i, j = 2, 1$ and $i, j = 3, 1$ gives

$$(t \circ_2 t) \circ_1 t = -(t \circ_1 t) \circ_4 t, \qquad (t \circ_3 t) \circ_1 t = -(t \circ_1 t) \circ_5 t.$$

Lemma 6.15 (case 1), $p = q = r = t$, with $i, j = 2, 2$ and $i, j = 1, 3$ gives

$$-t \circ_1 (t \circ_2 t) = -(t \circ_1 t) \circ_2 t. \qquad -t \circ_1 (t \circ_3 t) = -(t \circ_1 t) \circ_3 t.$$

Therefore (6.6) equals

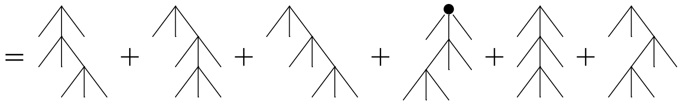
$$-(t \circ_1 t) \circ_4 t - (t \circ_1 t) \circ_5 t - (t \circ_1 t) \circ_2 t - (t \circ_1 t) \circ_3 t.$$

We reduce each monomial using α and obtain

$$\begin{aligned}&(t \circ_2 t) \circ_4 t + (t \circ_3 t) \circ_4 t + (t \circ_2 t) \circ_5 t + (t \circ_3 t) \circ_5 t \\ &+ (t \circ_2 t) \circ_2 t + (t \circ_3 t) \circ_2 t + (t \circ_2 t) \circ_3 t + (t \circ_3 t) \circ_3 t.\end{aligned}$$

Terms 3, 6 cancel by Lemma 6.15 (case 2), $(t \circ_2 t) \circ_5 t = -(t \circ_3 t) \circ_2 t$, leaving

$$\begin{aligned}&(t \circ_2 t) \circ_4 t + (t \circ_3 t) \circ_4 t + (t \circ_3 t) \circ_5 t \\ &\quad + (t \circ_2 t) \circ_2 t + (t \circ_2 t) \circ_3 t + (t \circ_3 t) \circ_3 t\end{aligned}$$



We reduce terms 4, 6 using α; this cancels terms 1, 5 and terms 2, 3. □

Theorem 6.5 *For an odd operation, the dimension of the ternary partially associative operad is the binary Catalan number (in the weight grading).*

Proof Relation α shows that any left subtree reduces, so the dimension for weight w is the number of binary trees of weight w. □

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