Improved Nonparametric Estimation Using Partially Ordered Sets

Ehsan Zamanzade and Xinlei Wang

Abstract Ranked set sampling (RSS) is a cost efficient design that has been widely used in agriculture, forestry, ecological and environmental sciences. Frey (Environmental and Ecological Statistics 19(3):309–326, 2012) proposed a sampling scheme based on to allow for partially ordered sets. This scheme permits a ranker to declare ties and then record the tie structure for potential use in statistical analysis. We first introduce two nonparametric maximum likelihood estimators (MLEs) of the population cumulative distribution function (CDF) that incorporate the information for partially ordered sets. We compare the proposed MLEs with the standard nonparametric MLE of the CDF (without utilizing tie information) via Monte Carlo simulation. Motivated by good performance of the new CDF estimators, we further derive two mean estimators for partially ordered sets. Our numerical results from both simulation and real data show that the proposed estimators outperform their competitors provided that the quality of ranking is not low.

Keywords Imperfect ranking · Nonparametric maximum likelihood estimation · Ranked set sampling · Relative efficiency · Ranking ties

1 Introduction

Ranked set sampling (RSS), proposed by McIntyre (1952), is an appropriate sampling technique for use in situations where ranking sample units in a small set is much easier or cheaper than obtaining their precise values. Ranking can be done by personal

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judgment, eye inspection or using available values of a concomitant variable, and need not to be completely accurate (perfect). The ranking information is then used by the researcher to draw a more representative sample from the population of interest, and therefore statistical inference based on a ranked set sample should be more efficient than a simple random sample of the same size. RSS has been found to be useful in various fields including agriculture, forestry, ecological and environmental sciences, biology, medicine and so on.

To draw a balanced ranked set sample of size $N \equiv n \times m$ with set size m, one draws nm simple random samples (sets) of size m. The units in each set of size m are then ranked in an increasing magnitude without referring to their precise values. Then from the first n sets of size m, the sample units with rank 1 are selected for actual quantification, from the second n sets of size m, the sample units with rank 2 are selected for actual quantification and so on. For balanced RSS, n is the number of measured sample units with rank i for all $i = 1, \ldots, m$, and is referred to as the number of cycles. By contrast, in unbalanced RSS, one needs to determine a vector $\boldsymbol{n} = (n_1, \ldots, n_m)$ in a way that n_i is the number of measured sample units with rank i and $N \equiv \sum_{i=1}^{m} n_i$ is the total sample size of the ranked set sample.

Virtually, all standard statistical problems have been well addressed in the RSS literature including estimation of the population mean (Takahasi and Wakimoto 1968; Wang et al. 2006, 2008; Frey 2011), population variance (Stokes 1980; MacEachern et al. 2002; Perron and Sinha 2004), cumulative distribution function (CDF) (Stokes and Sager 1988; Kvam and Samaniego 1994; Huang 1997; Duembgen and Zamanzade 2018), population proportion (Chen et al. 2006, 2007; Zamanzade and Mahdizadeh 2017; Zamanzade and Wang 2017), mean difference (Wang et al. 2016, 2017), distribution-free confidence intervals (Frey 2007), reliability estimation (Mahdizadeh and Zamanzade 2018) and perfect ranking tests (Frey et al. 2007; Zamanzade et al. 2012; Frey 2017).

In RSS, the researcher is required to provide a unique rank for each unit in the set of size m. However, there are situations in which the researcher is not sure about how to rank two or more sample units in the set and he must break the ties at random to implement RSS. Such situations occur frequently in studies in agriculture, forestry, ecological and environmental sciences. To alleviate this difficulty, Frey (2012) proposed a new variation of RSS (denoted hereafter by RSS-t) in which the researcher is allowed to declare ties as he wishes. When implementing RSS-t, the researcher breaks the ties at random, but he also records the tie structure to be used in the estimation process. Let $\left\{ X_{[i]j}, i = 1, \ldots, m, j = 1, \ldots, n \right\}$ be a balanced ranked set sample of size $N = mn$, where $X_{[i]j}$ is the jth unit with rank i. RSS-t includes not only the $X_{[i]j}$ values, but also the indicator variables $I_{[i]jk}$, where $I_{[i]jk}$ is one when the jth sample unit with rank i is tied for rank k in its own comparison set, for $i, k \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. Note that $\sum_{k=1}^{m} I_{[i]jk}$ is always at least one because the sample unit with rank i is always tied for itself. Frey (2012) then developed several mean estimators for RSS-t samples and discussed two models which allow ties in rankings: discrete perceived size (DPS) and tied-if-close (TIC) models. Frey (2012) compared mean estimators with/without utilizing the tie

information under the DPS model via Monte Carlo simulation and concluded that using tie information would improve efficiency in estimating the population mean.

We describe three nonparametric maximum likelihood estimators (MLEs) of the population CDF based on RSS-t: one is the standard MLE of the CDF proposed by Kvam and Samaniego (1994) which ignores tie information, and the other two are novel which incorporate tie information into the estimation process. We then compare these three likelihood-based estimators: via Monte Carlo simulation. Motivated by the observation that the two new MLEs of the CDF have better performance than the standard one, we further propose new mean estimators based on these CDF estimators, to make use of the tie information. We compare the proposed mean estimators with those in Frey (2012) using both simulated data and real data of body fat percentage, which consistently shows that the new estimators beat their competitors as long as the quality of ranking is good. Note that to estimate a population CDF $F(t)$, a typical approach is to use the relationship $F(t) = E[I(X \leq t)]$, where $I(\cdot)$ denotes the indicator function and the mean of $I(X \leq t)$ is estimated from the sample. In this paper, we adopt the approach that constructs a nonparametric maximum likelihood estimator of the CDF and then finds the mean estimator using $E(X) = \int_{-\infty}^{+\infty} t \, dF(t)$. As will be shown in Sect. 3.2, the mean estimators resulting from the NPMLEs of the CDF can perform substantially better than the existing mean estimators when ranking quality is good.

2 CDF Estimation

2.1 Nonparametric Maximum Likelihood Estimators for RSS-t

Maximum likelihood estimation of CDF based on a balanced ranked set sample was developed by Kvam and Samaniego (1994), and its asymptotic behavior was studied by Huang (1997) and Duembgen and Zamanzade (2018). Let $\{X_{(i)j}, i = 1, \ldots, m, j = 1, \ldots, n\}$ be a balanced ranked set sample of size $N = mn$ from a population with CDF $F(t)$, obtained under the assumption of perfect ranking. Then, $X_{(i)1}, \ldots, X_{(i)n}$ are independently and identically distributed, following the distribution of the ith order statistic in a sample of size m from the original population. That is, the CDF of $X_{(i)j}$ is given by $F_i(t) = P(X_{(i)j} \leq t) = B_i(F(t))$ where

$$B_i(F(t)) = \sum_{r=i}^{m} \binom{m}{r} (F(t))^r (1 - F(t))^{m-r} = \int_0^{F(t)} m \binom{m-1}{i-1} (F(t))^{i-1} (1 - F(t))^{m-i}$$

is the CDF of the beta distribution with parameters i and $m + 1 - i$, evaluated at the point $F(t)$. Therefore, the log-likelihood function of $F(t)$ based on RSS can be written as

$$L\left(F(t)\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ I\left(X_{(i)j} \leq t\right) \log\left(B_i\left(F(t)\right)\right) + \left(1 - I\left(X_{(i)j} \leq t\right)\right) \log\left(1 - B_i\left(F(t)\right)\right) \right\},$$

where $I(\cdot)$ is the indicator function. Log-concavity of $B_i(F(t))$ in $F(t)$ follows from the fact that a beta distribution with parameters α, $\beta \geq 1$ has an increasing hazard rate (Crowder 2008) and therefore $L(F(t))$ is a concave function in $F(t)$ as well. Thus, the MLE of the population CDF based on RSS is well defined and can be obtained by $\hat{F}_{NM0}(t) = \arg \max_{F(t) \in [0,1]} L(F(t))$. One can use $\hat{F}_{NM0}(t)$ for estimating $F(t)$ based on data from RSS-t by simply ignoring the tie structure. Obviously, $\hat{F}_{NM0}(t)$ is not the true MLE of $F(t)$ unless the rankings are perfect, with no ties allowed.

In what follows, we propose two novel likelihood-based CDF estimators which incorporate tie information from RSS-t into the estimation process. Suppose that $X_{[i]j}$ is tied for two or more units in the set of size m. Then, the CDF of $X_{[i]j}$, say $F_{[i]j,T}(t)$ ("T" in the subscript stands for "tie"), is a mixture of CDFs of beta distributions evaluated at the point $F(t)$, given by

$$F_{[i]j,T}(t) = \frac{\sum_{k=1}^{m} I_{[i]jk} B_k\left(F(t)\right)}{\sum_{k=1}^{m} I_{[i]jk}}.$$

Therefore, the log-likelihood function of $F(t)$ based on data from RSS-t can be written as

$$L_1\left(F(t)\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \left\{ I\left(X_{[i]j} \leq t\right) \log\left(F_{[i]j,T}(t)\right) + \left(1 - I\left(X_{[i]j} \leq t\right)\right) \log\left(1 - F_{[i]j,T}(t)\right) \right\},$$

and the MLE of $F(t)$ based on RSS-t can be obtained as $\hat{F}_{NM1}(t) = \arg \max_{F(t) \in [0,1]} L_1(F(t))$. To guarantee the existence of a unique maximizer of $L_1(F(t))$, we need to show the concavity of $L_1(F(t))$ in $F(t)$. This requires the log-concavity of $F_{[i]j,T}(t)$, which follows from Theorem 2 in Mu (2015).

Another way of incorporating the tie information into the likelihood function is to use a splitting strategy proposed by MacEachern et al. (2004), in which each tied unit is split among the strata corresponding to the ranks for which the unit is tied. Thus, if a sample unit is tied for r different ranks, then we assign it to each of those r strata with equal weight $\frac{1}{r}$. This splitting strategy leads to the following pseudo-likelihood function:

$$L_2\left(F(t)\right) = \sum_{i=1}^{m} \left\{ n_i' \hat{F}_{i,sp}\left(t\right) \log\left(F_i\left(t\right)\right) + n_i' \left(1 - \hat{F}_{i,sp}\left(t\right)\right) \log\left(1 - F_i\left(t\right)\right) \right\},$$

where

$$n_i' = \sum_{l=1}^{m} \sum_{j=1}^{n} \frac{I_{[l]ji}}{\sum_{k=1}^{m} I_{[l]jk}} \tag{1}$$

and

$$\hat{F}_{i,sp}\left(t\right) = \frac{1}{n_i'} \sum_{l=1}^{m} \sum_{j=1}^{n} \frac{I_{[l]ji}}{\sum_{k=1}^{m} I_{[l]jk}} I\left(X_{[l]j} \le t\right)$$

for $i = 1, \ldots, m$. The corresponding MLE of $F(x)$ based on pseudo-likelihood is then given by $\hat{F}_{NM2}(t) = \arg\max_{F(t) \in [0,1]} L_2\left(F(t)\right)$.

2.2 Comparison

We now compare the performance of ML-type estimators of the population CDF using RSS-t samples via simulation in which we consider different parent distributions, models for generating ties, quality of ranking and varying design parameters. We assume that ranking is done using a perceptual linear ranking model (Dell and Clutter 1972; Fligner and MacEachern 2006), which assumes the ranking of the variable of interest X in each set of size m is done via a concomitant variable Y, satisfying

$$Y = \rho\left(\frac{X - \mu_x}{\sigma_x}\right) + \sqrt{1 - \rho^2}Z,$$

where μ_x is the mean of X, σ_x is the standard deviation of X, Z is a random variable following the standard normal distribution and the parameter ρ controls the quality of ranking.

We consider two classes of models for generating ranking ties: discrete perceived size (DPS) and tied-if-close (TIC), as proposed by Frey (2012). The DPS model discretizes the values of the concomitant variable Y by rounding Y/c to the largest integer greater than or equal to Y/c. The TIC model declares the ith and jth units to be tied if $|Y_i - Y_j| < c$. Since the transitivity in TIC model is also required, the ith and jth units may be still declared tied even if $|Y_i - Y_j| > c$ as long as there is at least one unit in the set that bridges the gap. In either model, $c > 0$ is a user-chosen model parameter. Frey (2012) pointed out that both DPS and TIC models may show certain undesirable behavior when the parameters of m and c are changed. For DPS models, increasing the value of c does not necessarily lead to more ties in each set, and for TIC models, the number of ties among the units in the set can increase if we

add an additional unit to it. Frey (2012) further discussed the differences between two classes of models, but only evaluated the mean estimators under the DPS model and normal distribution.

In the first simulation study, we compare the overall performance of CDF estimators on the real line via the mean integrated square error (MISE), defined as $MISE(\hat{F}) = E\left[\int_{-\infty}^{+\infty}\{\hat{F}(t) - F(t)\}^2 dt\right]$. The relative efficiency (RE) of $\hat{F}_{NMi}$ to $\hat{F}_{NM0}$ is defined as the ratio of their MISEs, i.e., $MISE(\hat{F}_{NM0})/MISE(\hat{F}_{NMi})$ for $i = 1, 2$. We set $N \in \{15, 30\}$, $m \in \{3, 5\}$, $\rho \in \{0, 0.5, 0.8, 1\}$, and for each combination of (N, m, ρ), we generate 10,000 RSS-t samples from standard normal ($N(0, 1)$), standard exponential ($Exp(1)$) and standard uniform ($U(0, 1)$) distributions under both DPS and TIC models, respectively. For the DPS model, we set $c \in \{0.5, 1, 2, 4\}$, and for the TIC model, $c \in \{0.25, 0.5, 1, 2\}$. The REs are estimated based on the 10,000 RSS-t samples for each setting. It is worth mentioning that in both DPS and TIC models, whenever ties occur in the ranking process, we assume that the researcher is not aware of actual values of tied units, as is typical in practice, and so he/she selects one of the tied units at random.

Table 1 presents RE values of $\hat{F}_{NM1}$ to $\hat{F}_{NM0}$ under the DPS model. We observe that the efficiency gain using $\hat{F}_{NM1}$ instead of $\hat{F}_{NM0}$ can be as large as 40% for the case of perfect ranking $\rho = 1$, while the efficiency loss is never more than 10% in the case of completely random ranking. When the quality of ranking is fairly good ($\rho \geq 0.8$), the RE never falls below one; when $\rho = 0.5$, $\hat{F}_{NM1}$ is still more efficient than $\hat{F}_{NM0}$ for standard normal and standard exponential distributions, but slightly less efficient than $\hat{F}_{NM0}$ if the population distribution is standard uniform and $N = 15$. This indicates that using tie information, $\hat{F}_{NM1}$ improves the overall performance of CDF estimation as long as the quality of ranking is not bad. Also, note that when $RE > 1$, the RE is generally an increasing function of c.

The general patterns of the estimated REs of $\hat{F}_{NM1}$ versus $\hat{F}_{NM0}$ under the TIC model in Table 2 are similar to those of Table 1, except for two major differences: First, the efficiency gain using $\hat{F}_{NM1}$ instead of $\hat{F}_{NM0}$ can be as large as 66% for the standard exponential distribution and the efficiency loss can be as large as 16%. Second, the REs for the standard uniform distribution are generally lower and fall below one in many cases when $\rho \leq 0.8$.

The estimated REs of $\hat{F}_{NM2}$ versus $\hat{F}_{NM0}$ under the DPS model are presented in Table 3. We observe that although the REs are generally lower than those in Table 1 for $\rho \geq 0.5$, $\hat{F}_{NM2}$ is more robust to ranking errors as compared to $\hat{F}_{NM1}$. The RE values in Table 3 never fall below one when $\rho \geq 0.5$, and even for $\rho = 0$, the maximum efficiency loss over $\hat{F}_{NM0}$ never exceeds 4%.

We also observe from Table 4 that although the REs are generally lower than those in Table 2, $\hat{F}_{NM2}$ is more robust to ranking errors under the TIC model as well and the maximum of efficiency loss using $\hat{F}_{NM2}$ instead of $\hat{F}_{NM0}$ never exceeds 4%.

To examine the point-wise performance of the CDF estimators on the real line, we perform another simulation study in which the CDF estimators are compared via their mean square errors (MSEs) at various points. For a given point t, we define the relative efficiency of $\hat{F}_{NMi}(t)$ to $\hat{F}_{NM0}(t)$ as $RE(t) = MSE(\hat{F}_{NM0}(t))/MSE(\hat{F}_{NMi}(t))$ for

Table 1 Estimating the population CDF under the DPS model: simulated relative efficiencies (defined as ratio of $MISEs$) of $\hat{F}_{NM1}$ versus $\hat{F}_{NM0}$

ρ	m	N	c	$N(0,1)$				$Exp(1)$				$U(0,1)$			
				$\frac{1}{2}$	1	2	4	$\frac{1}{2}$	1	2	4	$\frac{1}{2}$	1	2	4
1	3	15		1.03	1.08	1.18	1.27	1.04	1.05	1.10	1.19	1.05	1.11	1.22	1.22
		30		1.03	1.08	1.20	1.32	1.03	1.01	1.10	1.21	1.05	1.11	1.25	1.25
	5	15		1.06	1.11	1.21	1.32	1.05	1.05	1.11	1.23	1.07	1.15	1.24	1.24
		30		1.06	1.12	1.25	1.40	1.03	1.03	1.11	1.27	1.07	1.15	1.25	1.25
0.8	3	15		1.01	1.04	1.09	1.14	1.01	1.01	1.04	1.10	1.02	1.04	1.08	1.10
		30		1.02	1.05	1.13	1.19	1.00	1.01	1.05	1.16	1.02	1.05	1.11	1.11
	5	15		1.02	1.05	1.11	1.16	1.00	1.02	1.04	1.13	1.02	1.04	1.07	1.06
		30		1.03	1.07	1.19	1.24	1.01	1.03	1.08	1.20	1.03	1.06	1.11	1.10
0.5	3	15		1.00	1.01	1.02	1.04	1.00	1.00	1.03	1.07	0.99	0.98	0.97	0.98
		30		1.01	1.03	1.08	1.10	1.01	1.02	1.06	1.12	1.00	1.00	1.01	1.00
	5	15		1.00	1.01	1.04	1.04	1.00	1.01	1.03	1.09	0.99	0.98	0.98	0.96
		30		1.03	1.06	1.12	1.15	1.02	1.04	1.11	1.20	1.01	1.02	1.02	1.02
0	3	15		0.98	0.97	0.95	0.96	0.99	0.97	0.97	0.98	0.96	0.94	0.91	0.90
		30		1.00	1.00	1.02	1.03	1.01	1.03	1.05	1.05	0.98	0.97	0.94	0.95
	5	15		0.99	0.97	0.96	0.97	0.99	0.99	0.98	0.99	0.97	0.93	0.90	0.90
		30		1.01	1.02	1.04	1.06	1.02	1.03	1.07	1.09	0.99	0.99	0.97	0.97

Table 2 Estimating the population CDF under the TIC model: simulated relative efficiencies (defined as ratio of $MISEs$) of $\hat{F}_{NM1}$ versus $\hat{F}_{NM0}$

ρ	m	N	c	$N(0,1)$				$Exp(1)$				$U(0,1)$			
				$\frac{1}{4}$	$\frac{1}{2}$	1	2	$\frac{1}{4}$	$\frac{1}{2}$	1	2	$\frac{1}{4}$	$\frac{1}{2}$	1	2
1	3	15		1.03	1.09	1.15	1.12	1.03	1.05	1.06	1.07	1.21	1.09	0.97	0.97
		30		1.04	1.09	1.18	1.21	1.03	1.05	1.06	1.14	1.24	1.14	1.07	1.07
	5	15		1.07	1.17	1.25	1.26	1.06	1.10	1.11	1.21	1.30	1.07	1.02	1.02
		30		1.07	1.18	1.35	1.55	1.05	1.09	1.18	1.42	1.36	1.24	1.23	1.23
0.8	3	15		1.02	1.05	1.11	1.14	1.00	1.02	1.05	1.10	0.97	0.96	0.96	0.97
		30		1.02	1.06	1.15	1.25	1.01	1.02	1.07	1.19	0.99	0.99	1.03	1.06
	5	15		1.03	1.08	1.22	1.28	1.00	1.03	1.13	1.27	0.96	0.93	1.00	1.03
		30		1.03	1.11	1.37	1.58	1.01	1.05	1.23	1.52	1.00	1.02	1.15	1.22
0.5	3	15		1.00	1.01	1.05	1.13	0.99	1.00	1.04	1.14	0.97	0.94	0.90	0.95
		30		1.02	1.04	1.10	1.25	1.01	1.03	1.09	1.26	0.98	0.96	0.96	1.04
	5	15		1.00	1.02	1.13	1.27	1.00	1.00	1.10	1.33	0.96	0.90	0.88	1.01
		30		1.03	1.08	1.28	1.59	1.01	1.07	1.26	1.66	0.99	0.98	1.00	1.20
0	3	15		0.98	0.96	0.96	1.06	0.99	0.97	0.98	1.11	0.96	0.93	0.88	0.91
		30		1.00	1.00	1.04	1.17	1.00	1.02	1.06	1.24	0.98	0.96	0.94	1.00
	5	15		0.98	0.96	0.96	1.21	0.99	0.97	0.99	1.29	0.96	0.90	0.84	0.97
		30		1.01	1.03	1.11	1.50	1.02	1.05	1.17	1.62	0.98	0.96	0.94	1.14

Table 3 Estimating the population CDF under the DPS model: simulated relative efficiencies (defined as ratio of $MISEs$) of $\hat{F}_{NM2}$ versus $\hat{F}_{NM0}$

ρ	m	N	c	$N(0,1)$				$Exp(1)$				$U(0,1)$			
				$\frac{1}{2}$	1	2	4	$\frac{1}{2}$	1	2	4	$\frac{1}{2}$	1	2	4
1	3	15		1.05	1.10	1.15	1.15	1.08	1.15	1.16	1.16	1.06	1.12	1.19	1.19
		30		1.05	1.11	1.15	1.16	1.09	1.14	1.17	1.16	1.07	1.12	1.21	1.21
	5	15		1.07	1.14	1.17	1.16	1.10	1.16	1.19	1.18	1.08	1.16	1.24	1.24
		30		1.08	1.15	1.18	1.16	1.10	1.17	1.18	1.18	1.09	1.17	1.23	1.23
0.8	3	15		1.02	1.05	1.07	1.08	1.03	1.05	1.08	1.07	1.03	1.06	1.11	1.13
		30		1.02	1.05	1.09	1.09	1.02	1.05	1.07	1.09	1.03	1.07	1.12	1.12
	5	15		1.03	1.06	1.09	1.08	1.02	1.06	1.08	1.08	1.03	1.07	1.13	1.13
		30		1.03	1.06	1.10	1.09	1.02	1.05	1.07	1.08	1.04	1.07	1.13	1.13
0.5	3	15		1.00	1.01	1.02	1.02	1.00	1.01	1.03	1.03	1.01	1.02	1.03	1.04
		30		1.00	1.01	1.02	1.02	1.01	1.01	1.02	1.02	1.00	1.01	1.03	1.02
	5	15		1.00	1.00	1.02	1.01	1.00	1.01	1.02	1.02	1.01	1.01	1.04	1.03
		30		1.00	1.00	1.02	1.02	1.00	1.01	1.02	1.02	1.00	1.01	1.02	1.03
0	3	15		0.98	0.97	0.96	0.96	0.98	0.97	0.96	0.96	0.98	0.97	0.96	0.96
		30		0.98	0.97	0.97	0.97	0.99	0.98	0.97	0.97	0.98	0.97	0.95	0.96
	5	15		0.98	0.97	0.96	0.96	0.98	0.97	0.96	0.96	0.98	0.97	0.95	0.96
		30		0.99	0.97	0.97	0.97	0.99	0.98	0.97	0.98	0.98	0.97	0.96	0.96

Table 4 Estimating the population CDF under the TIC model: simulated relative efficiencies (defined as ratio of $MISEs$) of $\hat{F}_{NM2}$ versus $\hat{F}_{NM0}$

| ρ	m	N	c	$N(0,1)$				$Exp(1)$				$U(0,1)$			
				$\frac{1}{4}$	$\frac{1}{2}$	1	2	$\frac{1}{4}$	$\frac{1}{2}$	1	2	$\frac{1}{4}$	$\frac{1}{2}$	1	2
1	3	15		1.05	1.12	1.18	1.09	1.08	1.14	1.18	1.11	1.21	1.15	1.04	1.04
		30		1.06	1.12	1.19	1.09	1.08	1.15	1.18	1.10	1.23	1.15	1.03	1.03
	5	15		1.09	1.19	1.21	1.07	1.12	1.20	1.19	1.10	1.28	1.10	1.05	1.05
		30		1.09	1.20	1.21	1.06	1.13	1.21	1.19	1.09	1.29	1.08	1.04	1.04
0.8	3	15		1.03	1.07	1.11	1.08	1.02	1.07	1.11	1.09	1.00	1.02	1.04	1.04
		30		1.03	1.07	1.12	1.07	1.03	1.06	1.12	1.08	1.00	1.01	1.03	1.03
	5	15		1.03	1.09	1.14	1.06	1.03	1.09	1.14	1.09	1.00	1.01	1.06	1.05
		30		1.03	1.09	1.14	1.06	1.03	1.09	1.14	1.08	0.99	1.02	1.04	1.04
0.5	3	15		1.00	1.02	1.04	1.06	1.00	1.02	1.05	1.06	0.99	0.98	0.98	1.03
		30		1.01	1.02	1.04	1.05	1.00	1.02	1.04	1.05	0.98	0.98	0.98	1.02
	5	15		1.00	1.01	1.06	1.05	1.00	1.02	1.06	1.05	0.98	0.97	1.00	1.05
		30		1.00	1.01	1.05	1.05	1.00	1.02	1.06	1.05	0.98	0.98	0.99	1.04
0	3	15		0.98	0.96	0.97	1.01	0.98	0.96	0.97	1.02	0.98	0.97	0.96	1.01
		30		0.98	0.97	0.97	1.01	0.98	0.97	0.97	1.01	0.98	0.96	0.96	1.01
	5	15		0.98	0.97	0.97	1.04	0.98	0.96	0.97	1.03	0.98	0.96	0.97	1.04
		30		0.98	0.97	0.98	1.04	0.99	0.97	0.98	1.03	0.98	0.96	0.96	1.03
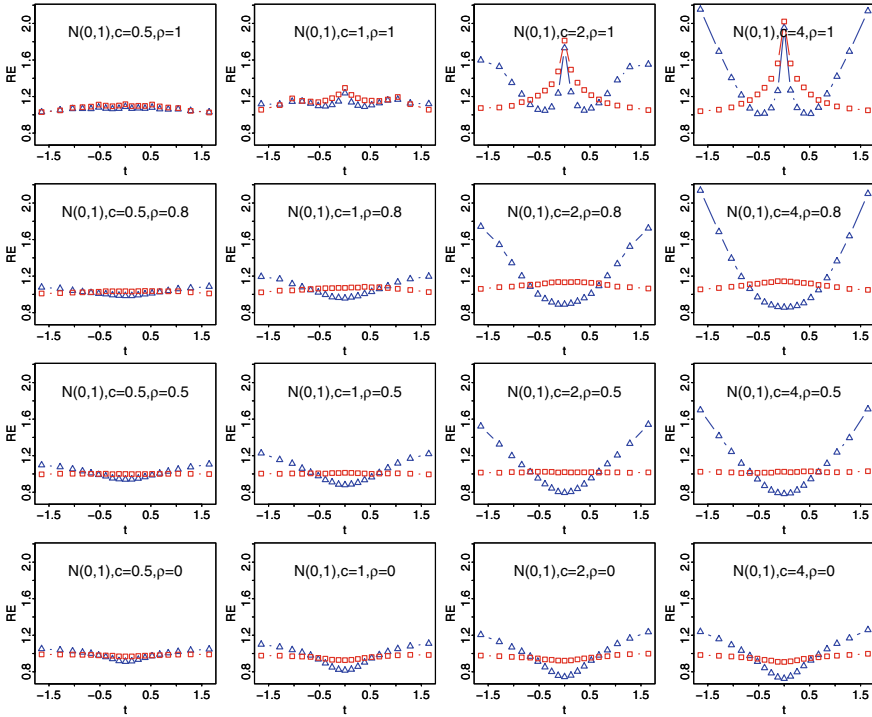
Fig. 1 Estimating the population CDF under the DPS model: simulated relative efficiencies (defined as ratio of MSEs) of $\hat{F}_{NM1}(t)$ versus $\hat{F}_{NM0}(t)$ (represented by $\triangle$ and blue color) and $\hat{F}_{NM2}(t)$ versus $\hat{F}_{NM0}(t)$ (represented by $\square$ and red color) as a function of t when the population distribution is $N(0, 1)$ for $\rho \in \{0, 0.5, 0.8, 1\}$ and $c \in \{0.5, 1, 2, 4\}$

$i = 1, 2$, where $RE(t) > 1$ indicates that $\hat{F}(t)$ is more efficient than $\hat{F}_{NM0}(t)$ at the point t. We set $(N, m) = (30, 5)$, $\rho \in \{0, 0.5, 0.8, 1\}$, $c \in \{0.5, 1, 2, 4\}$ for the DPS model and $c \in \{0.25, 0.5, 1, 2\}$ for the TIC model, $t \in \{Q_{0.05}, Q_{0.1}, \ldots, Q_{0.95}\}$ where Q_p is the pth quantile of the population distribution. For each combination of (ρ, c, t), we estimate $RE(t)$ using 10,000 RSS-t samples randomly generated under each tie-generating model, where the population distribution is set to $N(0, 1)$, and Fig. 1 shows results of the DPS model. We observe that when $c = 0.5$, the performance between $\hat{F}_{NM1}$ and $\hat{F}_{NM2}$ is almost identical but as the value of c increases, the difference between their performances becomes more distinguishable. It is interesting to note that $RE(t)$ of $\hat{F}_{NM1}$ as a function of t has a "W" ("U") shape roughly when the ranking is perfect (imperfect), and its $RE(t)$ falls below one for values of t around zero when the ranking is not perfect. However, the relative efficiency of $\hat{F}_{NM2}$ has an approximate "∧" shape and more stable patterns than $\hat{F}_{NM1}$ when the quality of ranking varies; it rarely falls below one. This is consistent with what we observe in Table 3. The relative efficiency of $\hat{F}_{NM2}$ is higher (lower) than $\hat{F}_{NM1}$ when the values of t are near the center (tails).
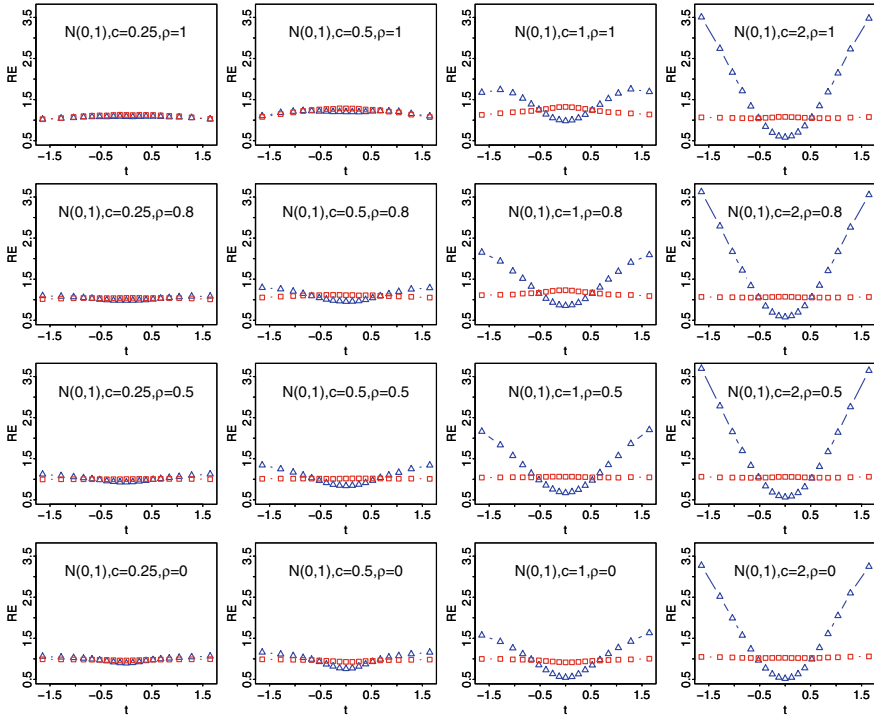
Fig. 2 Estimating the population CDF under the TIC model: simulated relative efficiencies (defined as ratio of MSEs) of $\hat{F}_{NM1}(t)$ versus $\hat{F}_{NM0}(t)$ (represented by $\triangle$ and blue color) and $\hat{F}_{NM2}(t)$ versus $\hat{F}_{NM0}(t)$ (represented by $\square$ and red color) as a function of t when the population distribution is $N(0, 1)$ for $\rho \in \{0, 0.5, 0.8, 1\}$ and $c \in \{0.5, 1, 2, 4\}$

The results under the TIC model can be found in Fig. 2. Here, the RE curves of $\hat{F}_{NM1}$ and $\hat{F}_{NM2}$ are almost identical for $c = 0.25$ and 0.5 but become more distinguishable as the value of c increases. For $c = 1$ and 2, the RE of $\hat{F}_{NM1}$ has a roughly "U" shape and is quite robust to ranking errors for $c = 2$. Again, this is consistent with what we observe in Table 2.

3   Mean Estimation

3.1   New Nonparametric Estimators Based on MLEs of the CDF

Let $\{X_{[i]j}, i = 1, \ldots, m, j = 1, \ldots, n\}$ be a balanced ranked set sample of size $N = mn$ from a population with CDF $F(t)$, in which some sample units are tied with some others. We develop several mean estimators based on the ML-type estimators

of $F(t)$ described in Sect. 2, using the fact that the population mean can be written as a function of $F(t)$, namely

$$E(X) = \int_{-\infty}^{+\infty} t \, dF(t). \tag{2}$$

If we replace $F(t)$ in Eq. (2) with any of the MLEs based on data from RSS-t, i.e., $\hat{F}_{NM0}(t)$, $\hat{F}_{NM1}(t)$ and $\hat{F}_{NM2}(t)$, then we can obtain the corresponding ML-based nonparametric estimators of the population mean, denoted by $\hat{\mu}_{NM0}, \hat{\mu}_{NM1}$ and $\hat{\mu}_{NM2}$, respectively.

3.2 Comparison

Below, we compare the ML-based nonparametric estimators of the population mean with the estimators proposed by Frey (2012). For this purpose, we set $N \in \{15, 30\}$, $m \in \{3, 5\}$, $\rho \in \{0, 0.5, 0.8, 1\}$, and then for each combination of (N, m, ρ), we generate 10,000 RSS-t samples under both DPS and TIC models where the population distribution is standard normal (i.e., $N(0, 1)$), standard exponential (i.e., $Exp(1)$) and standard uniform (i.e., $U(0, 1)$), respectively, $c \in \{0.5, 1, \ldots, 3.5, 4\}$ for the DPS model and $c \in \{0.25, 0.5, \ldots, 1.5, 2\}$ for the TIC model. The competing estimators are listed below.

- The standard mean estimator in RSS which ignores tie information and has the form $\hat{\mu}_{st} = \frac{1}{nm} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{[i]j}$.
- The mean estimator based on splitting each tied unit among the strata corresponding to the ranks for which the units were tied. This estimator has the form $\hat{\mu}_{sp} = \frac{1}{m} \sum_{i=1}^{m} \bar{X}'_{[i]}$, where

$$\bar{X}'_{[i]} = \frac{1}{n'_i} \sum_{l=1}^{m} \sum_{j=1}^{n} \frac{I_{[l]ji}}{\sum_{k=1}^{m} I_{[l]jk}} X_{[l]j},$$

and n'_i is given by Eq. (1).

- The isotonized version of $\hat{\mu}_{sp}$, denoted by $\hat{\mu}_{iso}$. This estimator is obtained using the fact that if the judgment strata are stochastically ordered, then $\mu_{[1]} \leq \cdots \leq \mu_{[m]}$, where $\mu_{[i]}$ is the true mean of the ith stratum. However, this constraint may be violated by their estimates $\bar{X}'_{[1]}, \ldots, \bar{X}'_{[m]}$. Thus, one natural way to improve $\hat{\mu}_{sp}$ is to isotonize the estimates $\bar{X}'_{[1]}, \ldots, \bar{X}'_{[m]}$ using the weighted sample sizes $n'_1, \ldots, n'_m$, and the resulting estimates follow the order constraint. The isotonized version of $\bar{X}'_{[1]}, \ldots, \bar{X}'_{[m]}$ can be given by

$$\bar{X}'_{[i], iso^-} = \min_{1 \leq r \leq i} \; \max_{i \leq s \leq m} \frac{\sum_{l=r}^{s} n'_l \bar{X}'_{[l]}}{\sum_{l=r}^{s} n'_l},$$

or

$$\bar{X}'_{[i],iso^+} = \max_{i \le s \le m} \; \min_{1 \le r \le i} \frac{\sum_{l=r}^{s} n'_l \bar{X}'_{[l]}}{\sum_{l=r}^{s} n'_l},$$

for $i = 1, \ldots, m$ (see Eqs. (4) and (5) in Wang et al. 2012). As pointed out by Wang et al. (2012), when $n'_i > 0$ for all $i \in \{1, \ldots, m\}$ (i.e., no empty strata exist), $\bar{X}'_{[i],iso^-} = \bar{X}'_{[i],iso^+}$ holds and so we denote the isotonized version of $\bar{X}'_{[1]}, \ldots, \bar{X}'_{[m]}$ by $\bar{X}'_{[1],iso}, \ldots, \bar{X}'_{[m],iso}$. These isotonized estimates can be computed using the pool adjacent violator algorithm (PAVA) (see Chap. 1 of Robertson et al. 1988), and the resulting mean estimator is given by $\hat{\mu}_{iso} = \frac{1}{m} \sum_{i=1}^{m} \bar{X}'_{[i],iso}$.

Remark 1 Frey (2012) described the isotonized version of $\hat{\mu}_{st}$, say $\hat{\mu}_{st,iso}$, and the Rao-Blackwellized (RB) versions of $\hat{\mu}_{st}$ and $\hat{\mu}_{st,iso}$ (note that the RB versions of $\hat{\mu}_{sp}$ and $\hat{\mu}_{iso}$ do not produce new estimators). However, these three lead to new estimators only in unbalanced RSS-t. Since we focus on balanced RSS-t in this paper, we drop them from our comparison set.

In order to compare different mean estimators, we define the relative efficiency of each estimator $\hat{\mu}$ versus $\hat{\mu}_{st}$ by $RE = MSE(\hat{\mu}_{st})/MSE(\hat{\mu})$. Again, an RE value larger than one indicates the preference of $\hat{\mu}$ over $\hat{\mu}_{st}$, and thus it shows utilizing tie information improves efficiency of mean estimation. Here, we only report the simulated REs from settings with $N = 30$ in Figs. 3 and 4 for the DPS model and in Figs. 5 and 6 for the TIC model. Results from settings with $N = 15$ for both models are not reported for brevity, because the sample size N does not have much impact on performance patterns of these estimators. We further note that although $\hat{\mu}_{iso}$ has higher REs than $\hat{\mu}_{sp}$ in all considered cases, their RE values are very close so that they are hardly distinguishable from each other. Thus, we omit $\hat{\mu}_{sp}$ in our discussion below.

Results Under the DPS Model

From comparing the results in Fig. 3 with those in Fig. 4, we observe that the performance patterns for $m = 3$ are very similar to those for $m = 5$, except that the RE values are generally higher (lower) for $m = 5$ than those of $m = 3$ if they are larger (smaller) than one. It is interesting to see the best estimator depends on the quality of ranking, value of c and the population distribution. If $N(0, 1)$ is the population distribution, then $\hat{\mu}_{iso}$ is the best estimator provided that the quality of ranking is not very good ($\rho \le 0.5$). But for $\rho \ge 0.8$, $\hat{\mu}_{NM1}$ or $\hat{\mu}_{NM2}$ is the best estimator depending on whether c is larger or smaller than 2. This pattern also holds for $Exp(1)$, in which $\hat{\mu}_{iso}$ is the winner for $\rho \le 0.5$, but it is beaten by $\hat{\mu}_{NM1}$ and/or $\hat{\mu}_{NM2}$ for $\rho \ge 0.8$. For $U(0, 1)$, any of the three ML-type mean estimators can be the best, depending on the value of ρ. If the ranking is completely random, then the standard ML-type estimator $\hat{\mu}_{NM0}$ that does not utilize the tie information is the best. If the ranking is imperfect but better than random, then $\hat{\mu}_{NM2}$ is the best and for perfect ranking case ($\rho = 1$), $\hat{\mu}_{NM1}$ is the winner except for $c = 1.5$ in which $\hat{\mu}_{NM2}$ is slightly better.
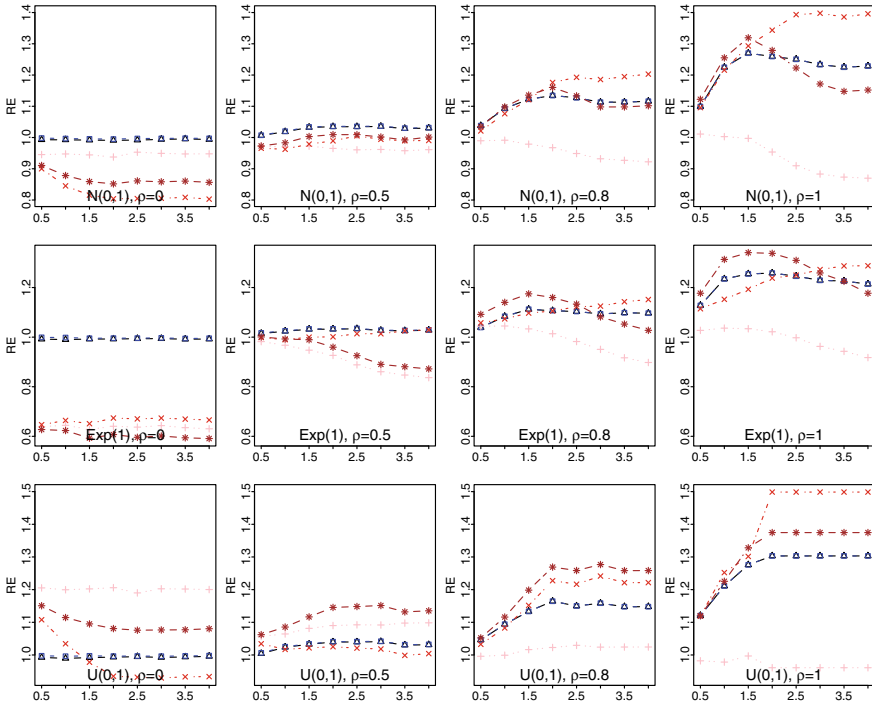
Fig. 3 Relative efficiency of $\hat{\mu}_{sp}$ (represented by $\triangle$ and black color), $\hat{\mu}_{iso}$ (represented by $\square$ and blue color), $\hat{\mu}_{NM0}$ (represented by $+$ and pink color), $\hat{\mu}_{NM1}$ (represented by $\times$ and red color) and $\hat{\mu}_{NM2}$ (represented by $*$ and brown color) to $\hat{\mu}_{st}$ as a function of c under the DPS model for $(N, m) = (30, 3)$, $\rho \in \{0, 0.5, 0.8, 1\}$, when the population distribution is standard normal, standard exponential and standard uniform. Note that the existing mean estimators $\hat{\mu}_{sp}$ and $\hat{\mu}_{iso}$ are represented by symbols with closed shapes $\triangle$ and $\square$. By contrast, all the three ML-type mean estimators are represented by symbols with open shapes, with $+$ for $\hat{\mu}_{NM0}$, $\times$ for $\hat{\mu}_{NM1}$ and $*$ for $\hat{\mu}_{NM2}$ (sort of from simple to more complex shapes); they are also represented by red or like colors (pink, red and brown) from light to dark

Results Under the TIC Model

From Figs. 5 and 6, we find that again the REs are generally increasing or decreasing in m while the other parameters are fixed, depending on whether they are larger or smaller than one and the pattern of REs is almost the same for $m = 3$ and $m = 5$. For standard normal and exponential distributions, $\hat{\mu}_{iso}$ is the best estimator in most cases when $\rho \leq 0.5$, but it is overtaken by $\hat{\mu}_{NM1}$ and/or $\hat{\mu}_{NM2}$ if the quality of ranking is fairly good ($\rho \geq 0.8$). For the standard uniform distribution, the winner always belongs to the set of ML-type estimators. If $\rho = 0$, then $\hat{\mu}_{NM0}$ is the best estimator, followed by $\hat{\mu}_{NM2}$. This pattern also holds for $\rho = 0.5$ except for the cases of $m = 5$ and $c \geq 1.5$ in which $\hat{\mu}_{NM2}$ beats $\hat{\mu}_{NM0}$. For $\rho \geq 0.8$, $\hat{\mu}_{NM2}$ is the best estimator
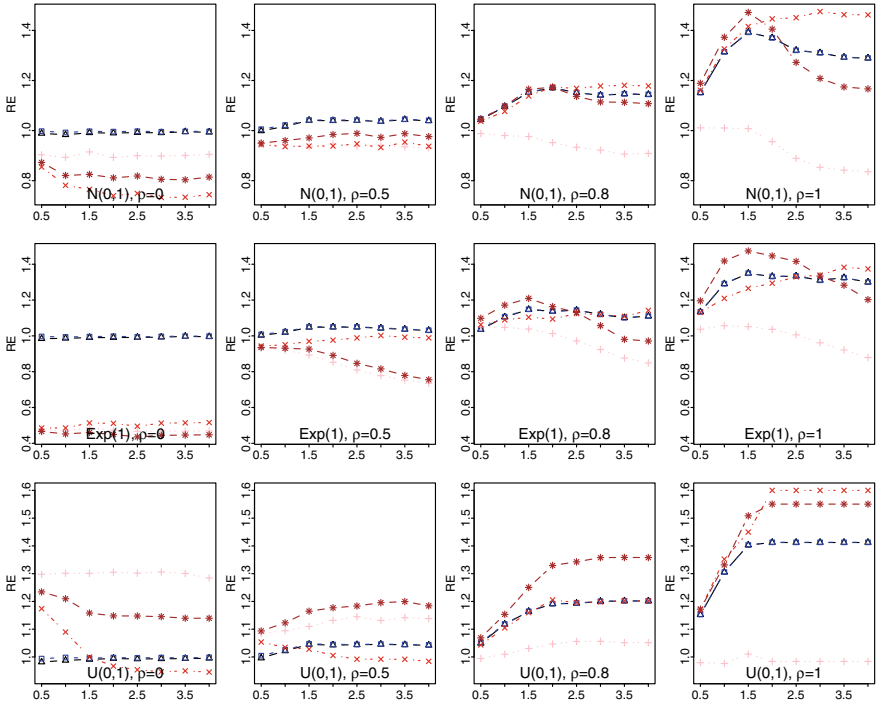
Fig. 4 Relative efficiency of $\hat{\mu}_{sp}$ (represented by $\triangle$ and black color), $\hat{\mu}_{iso}$ (represented by $\square$ and blue color), $\hat{\mu}_{NM0}$ (represented by $+$ and pink color), $\hat{\mu}_{NM1}$ (represented by $\times$ and red color) and $\hat{\mu}_{NM2}$ (represented by $*$ and brown color) to $\hat{\mu}_{st}$ as a function of c under the DPS model for $(N, m) = (30, 5)$, $\rho \in \{0, 0.5, 0.8, 1\}$, when the population distribution is standard normal, standard exponential and standard uniform. Note that the existing mean estimators $\hat{\mu}_{sp}$ and $\hat{\mu}_{iso}$ are represented by symbols with closed shapes $\triangle$ and $\square$. By contrast, all the three ML-type mean estimators are represented by symbols with open shapes, with $+$ for $\hat{\mu}_{NM0}$, $\times$ for $\hat{\mu}_{NM1}$ and $*$ for $\hat{\mu}_{NM2}$ (sort of from simple to more complex shapes); they are also represented by red or like colors (pink, red and brown) from light to dark

except for the case that $c = 0.25$ and $\rho = 1$ in which $\hat{\mu}_{NM1}$ is slightly better than $\hat{\mu}_{NM2}$. In terms of the overall performance of the estimators under the TIC model, we recommend using $\hat{\mu}_{NM2}$ if the quality of ranking is fairly good ($\rho \geq 0.8$).

4 An Empirical Study

In this section, we use a real dataset to compare the performance of different mean estimators for RSS-t samples. It contains measurements of body fat percentage along with several body circumference measurements for 252 men, available at http://lib. stat.cmu.edu/datasets/bodyfat. The histogram of the body fat percentage along with
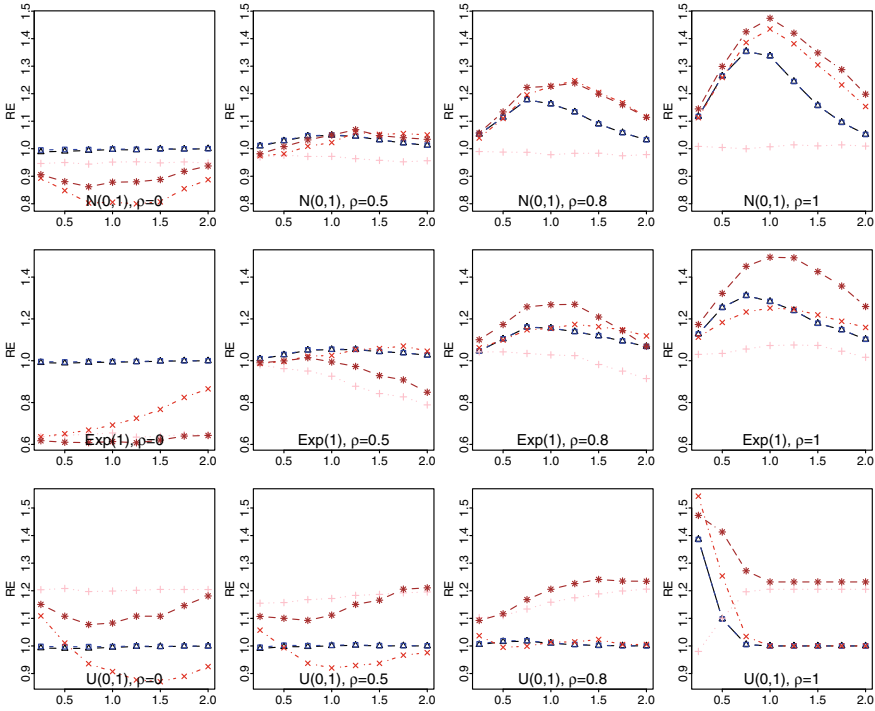
Fig. 5 Relative efficiency of $\hat{\mu}_{sp}$ (represented by $\triangle$ and black color), $\hat{\mu}_{iso}$ (represented by $\square$ and blue color), $\hat{\mu}_{NM0}$ (represented by $+$ and pink color), $\hat{\mu}_{NM1}$ (represented by $\times$, and red color) and $\hat{\mu}_{NM2}$ (represented by $*$ and brown color), to $\hat{\mu}_{st}$ as a function of c under the TIC model for $(N, m) = (30, 3)$, $\rho \in \{0, 0.5, 0.8, 1\}$, when the population distribution is standard normal, standard exponential and standard uniform, respectively. Note that the existing mean estimators $\hat{\mu}_{sp}$ and $\hat{\mu}_{iso}$ are represented by symbols with closed shapes $\triangle$ and $\square$. By contrast, all the three ML-type mean estimators are represented by symbols with open shapes, with $+$ for $\hat{\mu}_{NM0}$, $\times$ for $\hat{\mu}_{NM1}$ and $*$ for $\hat{\mu}_{NM2}$ (sort of from simple to more complex shapes); they are also represented by red or like colors (pink, red and brown) from light to dark

a fitted normal curve is presented in Fig. 7. Although it is slightly right-skewed, the distribution can be roughly approximated by a normal distribution.

We treat the body fat dataset as our hypothetical population and suppose that we are interested in estimating the population mean of the body fat percentage whose true value is $\mu = 19.15$. To draw an RSS-t sample from this population, we assume that ranking is done using standardized values of the concomitant variables, including abdomen circumference, weight and age, under both DPS and TIC models where the parameter c is set as before. The correlation coefficients between the variable of interest and the three concomitant variables are 0.81, 0.61 and 0.29, respectively. So, cases of fairly good ranking ($\rho = 0.81$), moderate ranking ($\rho = 0.61$) and poor ranking ($\rho = 0.29$) are all considered in this study. We also use the standardized values of body fat percentage for ranking, and so the case of perfect ranking ($\rho = 1$)
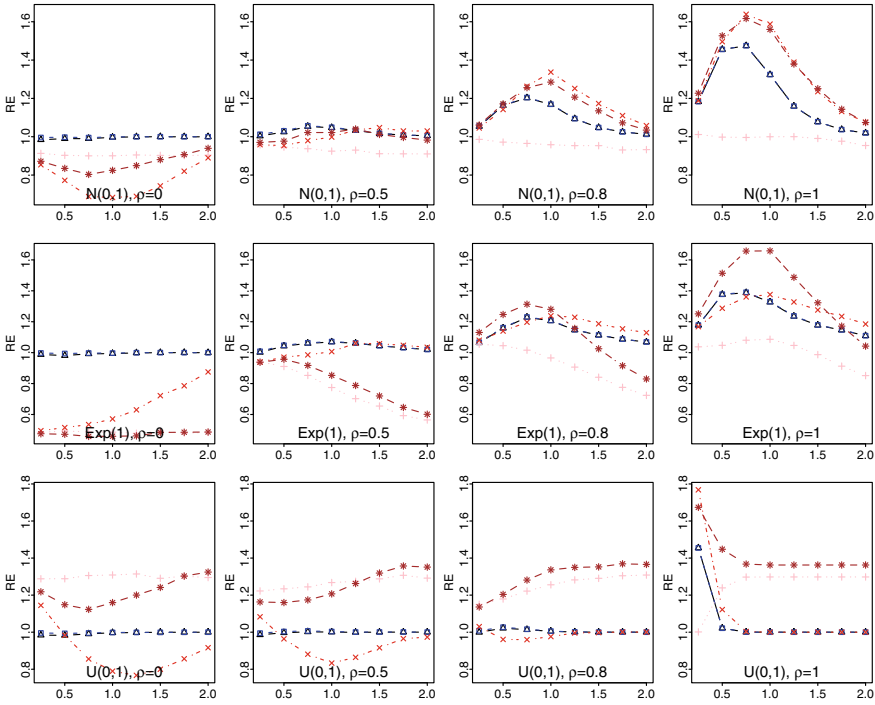
Fig. 6 Relative efficiency of $\hat{\mu}_{sp}$ (represented by $\triangle$ and black color), $\hat{\mu}_{iso}$ (represented by $\square$ and blue color), $\hat{\mu}_{NM0}$ (represented by $+$ and pink color), $\hat{\mu}_{NM1}$ (represented by $\times$ and red color) and $\hat{\mu}_{NM2}$ (represented by $*$ and brown color) to $\hat{\mu}_{st}$ as a function of c under the TIC model for $(N, m) = (30, 5)$, $\rho \in \{0, 0.5, 0.8, 1\}$, when the population distribution is standard normal, standard exponential and standard uniform, respectively. Note that the existing mean estimators $\hat{\mu}_{sp}$ and $\hat{\mu}_{iso}$ are represented by symbols with closed shapes $\triangle$ and $\square$. By contrast, all the three ML-type mean estimators are represented by symbols with open shapes, with "$+$" for $\hat{\mu}_{NM0}$, $\times$ for $\hat{\mu}_{NM1}$ and $*$ for $\hat{\mu}_{NM2}$ (sort of from simple to more complex shapes); they are also represented by red or like colors (pink, red and brown) from light to dark

is included as well. We set $N = 30$, $m \in \{3, 5\}$; for each combination of (N, m), we draw 10,000 RSS-t samples of size N with replacement from the given population and compute the relative efficiency as defined in Sect. 3. The estimated REs are reported in Tables 5 and 6 for DPS and TIC models, respectively. Clearly, the RE of each estimator decreases as the correlation coefficient ρ decreases in each setting.

For the DPS model, we can see from Table 5 that if $RE > 1$, then it generally increases as the value of m goes from 3 to 5. One of the ML-type estimators is the best estimator which one depends on the quality of ranking and the value of c. When the ranking is perfect ($\rho = 1$), $\hat{\mu}_{NM2}$ is the winner for $c \leq 1$ and it is overtaken by $\hat{\mu}_{NM1}$ for $c \geq 2$. It is interesting to observe that as the quality of ranking decreases, the span of c in which $\hat{\mu}_{NM2}$ beats $\hat{\mu}_{NM1}$ becomes wider and for $\rho \leq 0.61$, $\hat{\mu}_{NM2}$
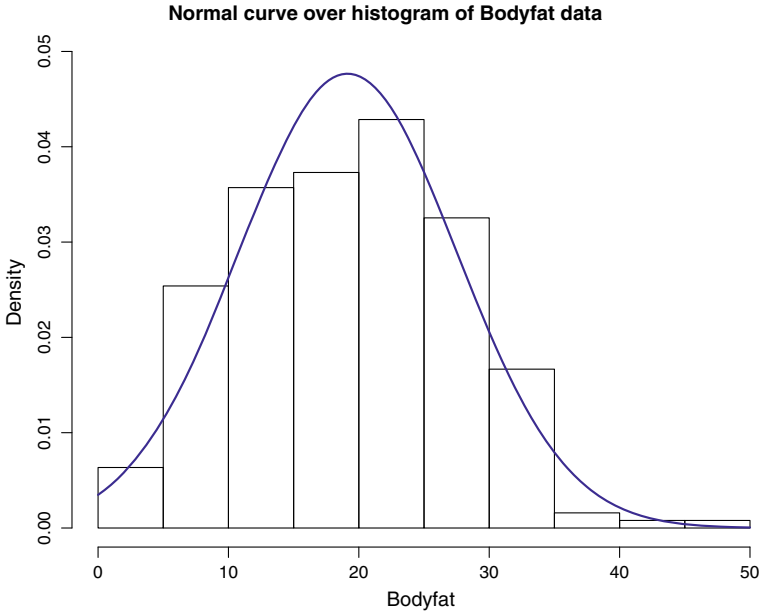
Fig. 7 Histogram of the body fat percentage along with a fitted normal curve

is superior to $\hat{\mu}_{NM1}$ for all considered values of c. This can be justified by the fact that $\hat{\mu}_{NM1}$ is obtained under the assumption of perfect ranking. When the quality of ranking is poor but better than random ($\rho = 0.29$), all estimators except for $\hat{\mu}_{NM1}$ are better than or comparable to $\hat{\mu}_{st}$ and $\hat{\mu}_{NM0}$ is slightly better than the others.

Table 6 shows that for the TIC model, although RE often increases as m increases if $RE > 1$, this is not true for some cases including those with $c = 2$. If the quality of ranking is not low ($\rho \geq 0.61$), then $\hat{\mu}_{NM2}$ is the best mean estimator except for a couple of cases, in which only $\hat{\mu}_{NM1}$ is slightly better than $\hat{\mu}_{NM2}$. When the quality of ranking is poor ($\rho = 0.29$), all estimators except for $\hat{\mu}_{iso}$ and $\hat{\mu}_{NM1}$ have comparable performance; $\hat{\mu}_{iso}$ is slightly better, and $\hat{\mu}_{NM1}$ is a bit worse than the others.

Overall, when the quality of ranking is not inferior (i.e., $\rho \geq 0.61$), either $\hat{\mu}_{NM1}$ or $\hat{\mu}_{NM2}$ has the highest relative efficiency and their gains in RE over the other estimators become more noticeable for large values of c. This is consistent with what we have observed in Sect. 3 for normal data.

Table 5 Estimating the population mean of body fat data under the DPS model: simulated relative efficiencies (defined as ratio of $MSEs$) of each mean estimator to standard mean estimator for RSS-t samples. The winner of the mean estimators is boldfaced in each setting

	c	$m=3$					$m=5$				
		$\hat{\mu}_{sp}$	$\hat{\mu}_{iso}$	$\hat{\mu}_{NM0}$	$\hat{\mu}_{NM1}$	$\hat{\mu}_{NM2}$	$\hat{\mu}_{sp}$	$\hat{\mu}_{iso}$	$\hat{\mu}_{NM0}$	$\hat{\mu}_{NM1}$	$\hat{\mu}_{NM2}$
Body fat ($\rho=1$)	0.5	1.12	1.12	1.00	1.12	1.13	1.16	1.16	1.00	1.17	**1.19**
	1	1.24	1.24	1.00	1.25	1.27	1.33	1.33	1.00	1.35	**1.38**
	2	1.27	1.27	0.95	**1.40**	1.31	1.39	1.39	0.96	**1.52**	1.46
	4	1.26	1.26	0.89	**1.45**	1.22	1.33	1.33	0.87	**1.51**	1.26
Abdomen ($\rho=0.81$)	0.5	1.06	1.06	1.00	1.05	**1.07**	1.07	1.07	1.00	1.07	**1.09**
	1	1.11	1.11	1.00	1.11	**1.13**	1.14	1.14	1.01	1.16	**1.19**
	2	1.15	1.15	1.00	1.21	**1.22**	1.19	1.19	1.00	1.23	**1.27**
	4	1.14	1.14	0.96	**1.24**	1.18	1.16	1.16	0.96	**1.22**	1.19
Weight ($\rho=0.61$)	0.5	**1.02**	**1.02**	0.99	1.00	**1.02**	**1.02**	**1.02**	0.99	1.00	**1.02**
	1	1.05	1.05	1.01	1.03	**1.08**	1.05	1.06	1.02	1.02	**1.09**
	2	1.05	1.06	1.01	1.04	**1.10**	1.06	1.06	1.02	0.99	**1.09**
	4	1.05	1.05	1.00	1.05	**1.09**	1.06	1.06	1.00	0.99	**1.08**
Age ($\rho=0.29$)	0.5	**1.00**	**1.00**	**1.00**	1.00	0.99	0.99	**1.00**	**1.00**	0.97	0.98
	1	1.00	1.01	**1.02**	0.98	1.00	1.00	1.01	**1.01**	0.98	1.00
	2	1.01	1.01	**1.04**	0.95	1.01	1.00	1.01	**1.05**	0.91	1.00
	4	1.00	1.00	**1.04**	0.93	1.00	1.00	1.01	**1.06**	0.91	1.00

Table 6 Estimating the population mean of body fat data under the TIC model: simulated relative efficiencies (defined as ratio of $MSEs$) of each mean estimator to standard mean estimator for RSS-t samples. The winner of the mean estimators is boldfaced in each setting

	c	$m=3$					$m=5$				
		$\hat{\mu}_{sp}$	$\hat{\mu}_{iso}$	$\hat{\mu}_{NM0}$	$\hat{\mu}_{NM1}$	$\hat{\mu}_{NM2}$	$\hat{\mu}_{sp}$	$\hat{\mu}_{iso}$	$\hat{\mu}_{NM0}$	$\hat{\mu}_{NM1}$	$\hat{\mu}_{NM2}$
Body fat ($\rho=1$)	0.25	1.11	1.11	1.00	1.11	**1.12**	1.19	1.19	1.00	1.20	**1.22**
	0.5	1.26	1.26	0.99	1.28	**1.29**	1.48	1.48	0.98	**1.53**	1.52
	1	1.35	1.35	1.00	**1.48**	**1.48**	1.32	1.32	1.00	**1.62**	1.58
	2	1.05	1.05	1.06	1.15	**1.26**	1.02	1.02	1.06	1.07	**1.19**
Abdomen ($\rho=0.81$)	0.25	1.05	1.05	1.01	1.05	**1.07**	1.06	1.07	1.01	1.08	**1.11**
	0.5	1.15	1.15	1.01	1.15	**1.19**	1.18	1.18	1.01	1.20	**1.25**
	1	1.17	1.17	1.01	1.26	**1.31**	1.16	1.16	1.03	1.32	**1.37**
	2	1.03	1.03	1.05	1.11	**1.20**	1.01	1.01	1.03	1.06	**1.14**
Weight ($\rho=0.61$)	0.25	1.03	1.03	1.00	1.02	**1.04**	1.03	**1.04**	0.99	1.00	**1.04**
	0.5	1.07	1.07	1.00	1.04	**1.09**	1.08	1.08	0.99	1.04	**1.10**
	1	1.08	1.08	1.02	1.11	**1.17**	1.07	1.07	1.00	1.13	**1.17**
	2	1.01	1.01	1.01	1.07	**1.11**	1.01	1.01	0.99	1.03	**1.08**
Age ($\rho=0.29$)	0.25	1.00	1.00	**1.01**	0.98	0.99	0.99	**1.00**	**1.00**	0.96	0.98
	0.5	**1.01**	**1.01**	1.00	0.96	0.99	1.00	**1.01**	0.99	0.94	0.97
	1	1.01	**1.02**	0.99	0.97	1.00	1.01	**1.02**	0.96	0.92	0.99
	2	1.00	1.00	1.00	1.00	**1.03**	1.00	1.00	0.99	0.99	**1.04**

5   Discussion

We have developed two novel ML-type estimators of the population CDF for RSS samples with tie information available and then used them for constructing new mean estimators. Using Monte Carlo simulation and a real dataset, we have shown that in many situations, the new estimators perform better than their competitors in the literature.

In this paper, we focused on balanced RSS in which tie information is recorded. It would be interesting to investigate the performance of different estimators when tie information is available in unbalanced RSS with empty strata and judgment post-stratification sampling with empty strata which the different versions of isotonized estimators are not identical anymore.

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