Q-Analogue of Generalized Bernstein–Kantorovich Operators

Ram Pratap and Naokant Deo

Abstract In the present article, we consider the q-analogue of generalized Bernstein–Kantorovich operators. For the proposed operators, we studied some convergence properties by using first- and second-order modulus of continuity.

Keywords Bernstein operators · Kantorovich operators · Modulus of continuity

2010 Mathematics Subject Classification 41A25 · 41A36

1 Introduction

In the year 1912, Bernstein [5] introduced the Bernstein operators and provided the constructive proof of Weierstrass theorem. Later, several researchers have generalized Bernstein operators using different parameters and studied various convergence properties. For more (see [6, 7, 16]).

Recently, Chen et al. [7] defined a family of Bernstein operators, for the functions $f \in [0, 1]$, α is fixed and $n \in \mathbb{N}$ are as follows:

$$B_n^{(\alpha)}(f; x) = \sum_{k=0}^{n} f_k p_{n,k}^{(\alpha)}(x), \tag{1.1}$$

where $f_k = f\left(\frac{k}{n}\right)$. For $n > 2$ the α-Bernstein polynomial $p_{n,k}^{(\alpha)}(x)$ of degree n is defined by

$$p_{1,0}^{(\alpha)}(x) = 1 - x, \;\; p_{1,1}^{(\alpha)}(x) = x,$$

R. Pratap (✉) · N. Deo
Department of Applied Mathematics, Delhi Technological University (Delhi College of Engineering), Bawana Road, Delhi 110042, India
e-mail: rampratapiitr@gmail.com

N. Deo
e-mail: naokantdeo@dce.ac.in

N. Deo et al. (eds.), *Mathematical Analysis I: Approximation Theory*, Springer Proceedings in Mathematics & Statistics 306,
https://doi.org/10.1007/978-981-15-1153-0_6

and

$$p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)\,x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right]x^{k-1}(1-x)^{n-k-1}, \qquad x \in [0,1].$$

For the first time in 1987, Bernstein operators based on q-integers were introduced by Lupas [12] and they are rational functions. Again in 1997, Phillips [14] introduced the q-Bernstein polynomials known as Phillips q-Bernstein operators. In past decade, linear positive operators based on q-integers is an active area of research. For more (see [4, 8, 11]).

Chai et al. [8] have considered the q-analouge of (1.1) is as follows:

$$B_{n,q}^{(\alpha)}(f;x) = \sum_{k=0}^{n} f_k\, p_{n,q,k}^{(\alpha)}(x), \tag{1.2}$$

where

$$p_{n,q,k}^{(\alpha)}(x) = \left(\begin{bmatrix} n-2 \\ k \end{bmatrix}_q (1-\alpha)\,x + \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q (1-\alpha)\,q^{n-k-2}\left(1-q^{n-k-1}x\right) + \begin{bmatrix} n \\ k \end{bmatrix}_q \alpha x\left(1-q^{n-k-1}x\right)\right)x^{k-1}(1-x)_q^{n-k-1},$$

$q \in (0,1]$ and $\mathrm{f}_k = f\left(\frac{[k]_q}{[n]_q}\right)$. For detailed explanation (see [3]).

Dhamija et al. [10] proposed the Kantorovich form of modified Szász–Mirakyan operators. Several researchers have also studied Kantorovich form of different linear positive operators and established local and global approximation results. More details (see [1, 2, 13, 15]).

Mohiuddine et al. [13] proposed the Kantorovich form of the operators (1.1), which is given as

$$K_n^{(\alpha)}(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int\limits_{k/(n+1)}^{(k+1)/(n+1)} f(t)dt, \tag{1.3}$$

where $p_{n,k}^{(\alpha)}(x)$ is defined in (1.1).

For $\alpha = 1$ and $q = 1$ the operators (1.4) reduces to Bernstein–Kantorovich operators.

Motivated from the above stated work, we consider the q-analogue of the operators (1.3) as follows:

$$K_{n,q}^{(\alpha)}(f;x) = [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q t, \tag{1.4}$$

and $p_{n,q,k}^{(\alpha)}(x)$ is given in (1.2).

In this paper, we estimated the moments of the proposed operators and discuss the rate of convergence using modulus of continuity.

2 Basic Results

In this section, we prove some auxiliary result to prove our main results.

Lemma 2.1 *From [8], we have* $B_{n,q}^{(\alpha)}(1;x) = 1$, $B_{n,q}^{(\alpha)}(t;x) = x$ *and*

$$B_{n,q}^{(\alpha)}(t^2;x) = x^2 + \frac{x(1-x)}{[n]_q} + \frac{(1-\alpha)q^{n-1}[2]_q x(1-x)}{[n]_q^2}.$$

Lemma 2.2 *(i)* $K_{n,q}^{(\alpha)}(1;x) = 1$;
(ii) $K_{n,q}^{(\alpha)}(t;x) = \frac{2q[n]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}$;
(iii) $K_{n,q}^{(\alpha)}(t^2;x) = \frac{3q^2[n]_q^2}{[3]_q[n+1]_q^2}x^2 + \frac{3q^2}{[3]_q[n+1]_q^2}\left([n]_q + (1-\alpha)q^{n-1}[2]_q\right)x(1-x) + \frac{3q[n]_q x}{[3]_q[n+1]_q^2} + \frac{1}{[3]_q[n+1]_q^2}$.

Proof From [15], $\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} 1 d_q t = \frac{1}{[n+1]_q}$, $\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t d_q t = \frac{2q[k]_q}{[2]_q[n+1]_q^2} + \frac{1}{[2]_q[n+1]_q^2}$ and

$$\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t^2 d_q t = \frac{3q^2\,[k]_q^2}{[3]_q\,[n+1]_q^3} + \frac{3q[k]_q}{[3]_q\,[n+1]_q^3} + \frac{1}{[3]_q\,[n+1]_q^3}.$$

It is easy to say that $K_{n,q}^{(\alpha)}(1;x) = 1$.

For $f(t) = t$ and using Lemma 2.1, we have

$$\begin{aligned}K_{n,q}^{(\alpha)}(t;x) &= [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} t d_q t \\ &= [n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \left(\frac{2q[k]_q}{[2]_q[n+1]_q^2} + \frac{1}{[2]_q[n+1]_q^2} \right) \\ &= \frac{[n]_q}{[n+1]_q} \left(\frac{2q}{[2]_q} \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \frac{[k]_q}{[n]_q} + \frac{1}{[2]_q[n]_q} \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \right) \\ &= \frac{2q[n]_q x + 1}{[2]_q[n+1]_q}.\end{aligned}$$

Similarly, for $f(t) = t^2$, we can estimate. So here we skip. □

Lemma 2.3 *The central moments for the operators (1.4) are as follows:*

(i) $K_{n,q}^{(\alpha)}(t-x;x) = \frac{2q[n]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}$;

(ii) $K_{n,q}^{(\alpha)}((t-x)^2;x) = \left(\frac{3q^2[n]_q^2}{[3]_q[n+1]_q^2} - \frac{4q[n]_q}{[2]_q[n+1]_q} + 1\right)x^2$
$+ \frac{3q^2}{[3]_q[n+1]_q}\left([n]_q + [2]_q(1-\alpha)q^{n-1}\right)x(1-x) + \left(\frac{3q[n]_q}{[3]_q[n+1]_q^2} - \frac{2}{[3]_q[n+1]_q}\right)x$
$+ \frac{1}{[3]_q[n+1]_q^2}$.

Proof Using linearity property of the operators (1.4) and Lemma 2.2, we get the required results. □

Lemma 2.4 *Let* $0 < q < 1$ *and* $c \in [0, qd]$, $d > 0$. *Then the inequality*

$$\int_c^d |t-x|\, d_q t \leq \left(\int_c^d (t-x)^2 d_q t\right)^{\frac{1}{2}} \left(\int_c^d d_q t\right)^{\frac{1}{2}}.$$

Proof For the proof of the Lemma (see [15]). □

3 Main Results

Let $C[0,1]$ be the space of all continuous functions on $[0,1]$ with sup-norm $\|f\| := \sup_{x\in[0,1]} |f(x)|$. Let $f \in C[0,1]$ and $\delta > 0$. Then the modulus of continuity $\omega(f,\delta)$ is given as:

$$\omega(f,\delta) = \sup_{\substack{|v-w|\leq\delta \\ v,w\in[0,1]}} |f(v) - f(w)|.$$

It is well-known $\lim_{\delta\to 0}\omega(f;\delta)=0$. For $f\in C[0,1]$ and $x,t\in[0,1]$, we have

$$|f(t)-f(x)|\leq\omega(f;\delta)\left(1+\frac{|t-x|}{\delta}\right) \tag{3.1}$$

For $f\in C[0,1]$ the Peetre K-functional is given by

$$K_2(f;\delta)=\inf_{g\in W^2}\left\{|f-g|+\delta\left\|g''\right\|\right\},$$

where $\delta>0$ and $W^2=\left\{g\in C[0,1]:g',g''\in C[0,1]\right\}$. In [9], there exists an absolute constant $\lambda>0$, such that

$$K_2(f;\delta)\leq\lambda\omega_2(f;\sqrt{\delta}). \tag{3.2}$$

and the second-order modulus of continuity $\omega_2(.;\delta)$ for $f\in C[0,1]$ as follows:

$$\omega_2(f;\delta)=\sup_{0<h\leq\delta}\sup_{x,x+h,x+2h\in[0,1]}|f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 3.1 *For $0<q\leq 1$, $q=\{q_n\}$ be a sequence converging to 1 as $n\to\infty$. Then, for all $f\in C[0,1]$ and $\alpha\in[0,1]$, it implies $K_{n,q}^{(\alpha)}(f;x)$ converges to $f(x)$ uniformly on $[0,1]$ for sufficiently large n.*

Proof From Lemma 2.2, $\lim_{n\to\infty}q_n=1$, we have $\lim_{n\to\infty}K_{n,q}^{(\alpha)}(1;x)=1$, $\lim_{n\to\infty}K_{n,q}^{(\alpha)}(t;x)=x$ and $\lim_{n\to\infty}K_{n,q}^{(\alpha)}(t^2;x)=x^2$. Then by Bohaman–Korovkin theorem $\lim_{n\to\infty}K_{n,q}^{(\alpha)}(f(t);x)=f(x)$ converges uniformly on $[0,1]$. □

Theorem 3.2 *For $f\in C[0,1]$, $q\in(0,1)$ and $\alpha\in[0,1]$, we have*

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\lambda\omega_2\left(f;\sqrt{\mu_{n,2}^q(x)+{\mu_{n,1}^q}^2(x)}\right)+\omega\left(f;\omega_{n,1}^q(x)\right),$$

where $\mu_{n,2}^q(x)$ and $\mu_{n,1}^q(x)$ are second- and first-central moments of the operators (1.4).

Proof We define an auxiliary operators

$$\hat{K}_{n,q}^{(\alpha)}(f;x)=K_{n,q}^{(\alpha)}(f;x)-f\left(\frac{2q[n+1]_qx+1}{[2]_q[n+1]_q}\right)+f(x). \tag{3.3}$$

For the operators $\hat{K}_{n,q}^{(\alpha)}(.;x)$, we get

$$\hat{K}_{n,q}^{(\alpha)}(t-x;x)=0. \tag{3.4}$$

Suppose, $g \in W^2, x, t \in [0, 1]$. Then by Tylor's expansion, we have

$$g(t) = g(x) + (t - x)g'(x) + \int_x^t (t - u)g''(u)du.$$

Applying $\hat{K}_{n,q}^{(\alpha)}(.; x)$ in above equation, we have

$$\hat{K}_{n,q}^{(\alpha)}(g; x) = g(x) + \hat{K}_{n,q}^{(\alpha)}\left(\int_x^t (t - u)g''(u)du; x\right).$$

Therefore,

$$\begin{aligned}\left|\hat{K}_{n,q}^{(\alpha)}(g; x) - g(x)\right| &\leq \left|K_{n,q}^{(\alpha)}\left(\int_x^t (t - u)g''(u)du; x\right)\right| \\ &+ \left|\left(\int_x^{\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}} \left(\frac{2q[n + 1]_q x + 1}{[2]_q[n + 1]_q} - x\right) g''(u)du; x\right)\right| \\ &\leq K_{n,q}^{(\alpha)}\left(\int_x^t |t - x|\, g''(u)du; x\right) \\ &+ \left|\left(\int_x^{\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}} \left|\frac{2q[n + 1]_q x + 1}{[2]_q[n + 1]_q} - u\right| \left|g''(x)\right| du; x\right)\right| \\ &\leq \left[K_{n,q}^{(\alpha)}((t - x)^2; x) + \left(\frac{2q[n + 1]_q x + 1}{[2]_q[n + 1]_q} - x\right)^2\right] \|g''\|. \end{aligned} \tag{3.5}$$

From (3.3), we have

$$\left|K_{n,q}^{(\alpha)}(f; x) \leq \|f\|\right| K_{n,q}^{(\alpha)}(1; x) + 2\|f\| = 3\|f\|. \tag{3.6}$$

From (3.3), (3.5) and (3.6), we have

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| &\le \left|K_{n,q}^{(\alpha)}(f-g;x)\right|+|f-g| \\ &\quad+\left|f\left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}\right)-f(x)\right| \\ &\le 4\,\|f-g\|+\left(\mu_{n,2}^{q}(x)+{\mu_{n,1}^{q}}^{2}(x)\right) \\ &\quad+\left|f\left(\frac{2q[n+1]_q x+1}{[2]_q[n+1]_q}\right)-f(x)\right|\end{aligned}$$

Now taking infimum on the right-hand side of the above inequality over $g \in W^2$, we get

$$\le 4K_2\left(f;\mu_{n,2}^{q}(x)+{\mu_{n,1}^{q}}^{2}(x)\right)+\omega\left(f;\mu_{n,1}^{q}(x)\right)$$

From (3.2), we get

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \le \lambda\omega_2\left(f;\sqrt{\mu_{n,2}^{q}(x)+{\mu_{n,1}^{q}}^{2}(x)}\right)+\omega\left(f;\omega_{n,1}^{q}(x)\right).$$

Hence, this is our required result. □

Theorem 3.3 *Let* $q_n \in (0,1)$ *be a sequence converging to* 1 *and* α *is fixed. Then for* $f \in C[0,1]$, *we have*

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \le 2\omega(f;\delta_n(x)),$$

where $\delta_n(x)=\left(K_{n,q}^{(\alpha)}((t-x)^2;x)\right)^{\frac{1}{2}}$.

Proof For nondecreasing function $f \in C[0,1]$. Using linearity and monotonicity of $K_{n,q}^{(\alpha)}$, we have

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| &\le K_{n,q}^{(\alpha)}\left(|f(t)-f(x)|\,;x\right) \\ &\le \omega(f;\delta)\left(1+\frac{1}{\delta}K_{n,q}^{(\alpha)}\left(|t-x|\,;x\right)\right)\end{aligned}$$

Applying Lemma 2.4 with $c=\frac{q[k]_q}{[n+1]_q}$ and $d=\frac{[k+1]_q}{[n+1]_q}$, we get

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \le \omega(f;x)\left\{1+\frac{[n+1]_q}{\delta}\sum_{k=0}^{n}p_{n,q,k}^{(\alpha)}(x)\left(\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}}(t-x)^2 d_q t\right)^{\frac{1}{2}}\right.$$

$$\times \left(\int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q t \right)^{\frac{1}{2}} \Bigg\}$$

Using Hӧlder's inequality for sums, we have

$$= \omega(f;x) \left\{ 1 + \frac{1}{\delta} \left([n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} (t-x)^2 d_q t \right)^{\frac{1}{2}} \right.$$

$$\left. \times \left([n+1]_q \sum_{k=0}^{n} p_{n,q,k}^{(\alpha)}(x) \int_{\frac{q[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} d_q t \right)^{\frac{1}{2}} \right\}$$

$$= \omega(f;x) \left\{ 1 + \frac{1}{\delta} \left(K_{n,q}^{(\alpha)}((t-x)^2;x) \right)^{\frac{1}{2}} \right\}.$$

By choosing $\delta = \delta_n(x)$, we get the required result. □

References

1. O. Agratini, Kantorovich-type operators preserving affine functions. Hacet. J. Math. Stat. **45**(6), 1657–1663 (2016)
2. T. Acar, A.M. Acu, N. Manav, Approximation of functions by genuine Bernstein-Durrmeyer type operators. J. Math. Inequal. **12**(4), 975–987 (2018)
3. A. Aral, V. Gupta, R.P. Agrawal, *Applications of q-Calculus in Operator Theory* (Springer, New York, 1993)
4. P.N. Agrawal, V. Gupta, A.S. Kumar, On q-analogue of Bernstein-Schurer-Stancu operators. Appl. Math. Comput. **219**(14), 7754–7764 (2013)
5. S.N. Bernstein, Demonstration of the Weierstrass theorem based on the calculation of probabilities. Common Soc. Math. Charkow Ser. 2t **13**, 1–2 (1912)
6. Q.B. Cai, B.Y. Lian, G. Zho, Approximation properties of Lambda-Bernstein operators. J. Inequal. Appl. **61** (2018). https://doi.org/10.1186/s13660-018-1653-7
7. X. Chen, J. Tan, Z. Liu, J. Xie, Approximation of functions by a new family of generalized Bernstein operators. J. Math. Anal. Appl. **450**, 244–261 (2017)
8. Q.B. Cai, X.W. Xu, Shape-preserving properties of a new family of generalized Bernstein operators. J. Inequal. Appl. (2018)
9. R.A. Devore, G.G. Lorentz, *Constructive Approximation* (Springer, Berlin, 1993)
10. M. Dhamija, R. Pratap, N. Deo, Approximation by Kantorovich form of modified Szász-Mirakyan operators. Appl. Math. Comput. **317**, 109–120 (2018)
11. V. Gupta, T.M. Rassias, H. Sharma, q-Durrmeyer operators based on Pólya distribution. J. Nonlinear Sci. Appl. **9**, 1497–1504 (2016)

12. A. Lupas, A q-analogue of the Bernstein operator, *Seminar on Numerical and Statistical Calculus*, vol. 9 (University of Cluj-Napoca, 1987)
13. S.A. Mohiuddine, T. Acar, A. Alotaibi, Construction of a new family of Bernstein-Kantorovich operators. Math. Methods Appl. Sci. 1–11 (2017). https://doi.org/10.1002/mma.4559
14. G.M. Phillips, Bernstein polynomials based on the q-integers. Ann. Numer. Math. **4**, 511–518 (1997)
15. M. Örkcü, O. Doğru, q-Szasz-Mirakyan-Kantorovich operators preserving test functions. Appl. Math. Lett. **24**, 1588–1593 (2011)
16. X. Zeng, F. Cheng, On the rate of Bernstein type operators. J. Approx. Theory **109**(2), 242–256 (2001)