On Bernstein–Chlodowsky Type Operators Preserving Exponential Functions

Firat Ozsarac, Ali Aral and Harun Karsli

Abstract In this paper, we introduce, analyze, and obtain some features of a new type of Bernstein–Chlodowsky operators using a different technique that is utilized as the classical Chlodowsky operators. These operators preserve the functions exp (μt) and exp $(2\mu t)$, $\mu > 0$. As a first result, the rate of convergence of the operator using an appropriately weighted modulus of continuity is obtained. Later, Quantitative-Voronovskaya type and Grüss–Voronovskaya type theorems for the new operators are presented. Then, we prove that the first derivative of the Bernstein–Chlodowsky operators applied to a function converges to the function itself. Finally, the variation detracting property of the operators is presented. It is proved that the variation seminorm property is preserved. Also, it is shown that the operators converge to $f/\exp_\mu$ in variation seminorm is valid if and only if the function is absolutely continuous.

Keywords Bernstein operators · Bernstein–Chlodowsky operators · Voronovskaya theorem · Generalized convexity

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1 Introduction

Recall that the classical Bernstein–Chlodowsky operator C_n defined from $C[0,\infty) \to C[0,\infty)$ is given by

$$C_n f(x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)\binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}, \; x \in [0, b_n] \tag{1.1}$$

where f is a function defined on $[0,\infty)$ and bounded on every finite interval $[0, b_n] \subset [0,\infty)$ with a certain rate, and b_n is a monotone increasing, positive and real sequence such that $\lim\limits_{n\to\infty} b_n = \infty$ and $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$.

The classical Bernstein–Chlodowsky polynomials were introduced by I. Chlodovsky in 1937 as a generalization of the Bernstein polynomials. Note that the case $b_n = 1, n \in \mathbb{N}$, in Eq. 1.1, defines an approximation to the function f on the interval $[0, 1]$ (or, suitably modified on any fixed finite interval $[-b, b]$).

For $b > 0$, let $M(b; f) := \sup\limits_{0\le t\le b} |f(t)|$. It is shown by Chlodowsky that when $f \in C[0,\infty)$ and $\lim\limits_{n\to\infty} M(b; f)\exp\left(-\frac{\sigma n}{b_n}\right) = 0$ for each $\sigma > 0$, then the classical Bernstein–Chlodowsky operator converges to $f(x)$ at each point where f is continuous. Chlodovsky also showed that the simultaneous convergence of the derivative $(C_n f)'(x)$ to $f'(x)$ at points x, where the derivative of $f(x)$ exists, a result taken up by Butzer [4, 5]. Due to these two former results, the classical Bernstein–Chlodowsky operators and their generalizations have been an increasing interest in the field of approximation theory.

During the paper, $\mu > 0$ is a fixed real parameter and $\exp_\mu$ represents the exponential function defined by $\exp_\mu(t) = e^{\mu t}$.

Herein, we consider a generalization of Bernstein–Chlodowsky operators of the form

$$\mathcal{C}_n f(x) = \sum_{k=0}^{n} \alpha_{n,k}(x) f\left(\frac{kb_n}{n}\right) p_{n,k}(a_n(x)), \; x \in [0, b_n] \tag{1.2}$$

$$\alpha_{n,k}(x) = e^{\mu x} e^{-\frac{\mu k b_n}{n}} \text{ and } p_{n,k}(x) = \binom{n}{k}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

with the property that

$$\mathcal{C}_n(\exp_\mu; x) = e^{\mu x}, \quad \mathcal{C}_n(\exp_\mu^2; x) = e^{2\mu x}. \tag{1.3}$$

Then, the operator $\mathcal{C}_n$ is more explicitly given by

$$\mathcal{C}_n f(x) = e^{\mu x}\left(e^{\frac{\mu b_n}{n}} - 1\right)^{-n} \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)\binom{n}{k} e^{-\frac{\mu k b_n}{n}}\left(e^{\frac{\mu x}{n}} - 1\right)^k\left(e^{\frac{\mu b_n}{n}} - e^{\frac{\mu x}{n}}\right)^{n-k}, \tag{1.4}$$

with

$$a_n(x) = b_n \frac{e^{\frac{\mu x}{n}} - 1}{e^{\frac{\mu b_n}{n}} - 1}.$$

Note that the connection of this operator with the classical Bernstein–Chlodowsky operator can be expressed as

$$\mathcal{C}_n f(x) = f_0(x)\, C_n(f/f_0)(a_n(x)), \qquad f_0(x) = e^{\mu x}. \tag{1.5}$$

Namely,

$$\begin{aligned} f_0(x)\, C_n\left(\frac{f}{f_0}\right)(a_n(x)) &= e^{\mu x} \sum_{k=0}^{n} \left(\frac{f}{f_0}\right)\left(\frac{kb_n}{n}\right)\binom{n}{k}\left(\frac{a_n(x)}{b_n}\right)^k \left(1 - \frac{a_n(x)}{b_n}\right)^{n-k} \\ &= e^{\mu x} \sum_{k=0}^{n} \frac{f\left(\frac{kb_n}{n}\right)}{f_0\left(\frac{kb_n}{n}\right)} p_{n,k}(a_n(x)) \\ &= e^{\mu x} \sum_{k=0}^{n} \frac{f\left(\frac{kb_n}{n}\right)}{e^{\mu \frac{kb_n}{n}}} p_{n,k}(a_n(x)) \\ &= e^{\mu x} \sum_{k=0}^{n} e^{-\mu \frac{kb_n}{n}} f\left(\frac{kb_n}{n}\right) p_{n,k}(a_n(x)) \\ &= \sum_{k=0}^{n} e^{\mu x} e^{-\mu \frac{kb_n}{n}} f\left(\frac{kb_n}{n}\right) p_{n,k}(a_n(x)) \\ &= \sum_{k=0}^{n} \alpha_{n,k}(x)\, f\left(\frac{kb_n}{n}\right) p_{n,k}(a_n(x)) \\ &= \mathcal{C}_n f(x). \end{aligned}$$

Also note that the Bernstein–Chlodovsky operators $\mathcal{C}_n$, based on functions defined on $[0, \infty)$, are bounded on every $[0, b_n] \subset [0, \infty)$ with a certain rate. Thus, they are a very natural polynomial process in approximating unbounded functions on the unbounded infinite interval $[0, \infty)$; but this approximation process is not so easy to handle.

We know that the classical Bernstein–Chlodowsky operators have the degree of exactness one, that is, they preserve the monomials 1 and x. On the other side, the operator (1.4) does not preserve 1 and x, but it satisfies the exponential moments (1.3) that play an important role in our calculations.

The aim of the present paper is to investigate the operators $\mathcal{C}_n$, $n \in \mathbb{N}$ in deeper to reveal, in addition to elementary properties, their advanced properties. Moreover, the development of the some theoretical results of the generalized operator is within the aim of the paper. After Voronovskaya type theorems for the generalized operator is stated , it is compared to the classical Bernstein–Chlodovsky operators in terms of effectiveness. For this purpose, the convergence of the derivative $(\mathcal{C}_n f)'(x)$ to $f'(x)$

is also considered. Finally, in the last section, the variation detracting property of the operators and variation seminorm property is stated. Moreover, it is proved that the operators converge to $f/\exp_\mu$ in variation seminorm is valid if and only if the function is absolutely continuous.

2 Preliminary Results

For the operator $\mathcal{C}_n, n \in \mathbb{N}$, we give here some of their properties and results. At first, we calculate all the moments of operator (1.4).

Lemma 1 *For each $n \in \mathbb{N}$ and $x \in [0, b_n]$, the following identities hold:*

$$\mathcal{C}_n e_0\left(x\right) = e^{\mu x - \mu b_n}\left(e^{\frac{\mu b_n}{n}} + 1 - e^{\frac{\mu x}{n}}\right)^n,$$

$$\mathcal{C}_n(\exp_\mu^3; x) = e^{\mu x}\left(e^{\frac{\mu x}{n}}\left(e^{\frac{\mu b_n}{n}} + 1\right) - e^{\frac{\mu b_n}{n}}\right)^n,$$

$$\mathcal{C}_n(\exp_\mu^4, x) = e^{\mu x}\left(e^{\frac{\mu b_n}{n}}\left(e^{\frac{\mu x}{n}} - 1\right)\left(e^{\frac{\mu b_n}{n}} + 1\right) + e^{\frac{\mu x}{n}}\right)^n.$$

Using *Mathematica*, we give two limits, which play an important role in both the uniform approximation of operator to functions and Voronoskaya type result.

For each $x \in (0, \infty)$, we shall consider the function $\exp_{\mu,x}$, defined for $t \in (0, \infty)$ by

$$\exp_{\mu,x}\left(t\right) = e^{\mu t} - e^{\mu x}.$$

Using Lemma 1 and (1.3) , one easily finds that

$$\begin{aligned}\mathcal{C}_n(\exp_{\mu,x}; x) &= \mathcal{C}_n(\exp_\mu; x) - e^{\mu x}\mathcal{C}_n e_0\left(x\right) \\ &= e^{\mu x}\left(1 - \mathcal{C}_n e_0\left(x\right)\right)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\mathcal{C}_n(\exp_{\mu,x}^2; x) &= \mathcal{C}_n(\exp_\mu^2; x) - 2e^{\mu x}\mathcal{C}_n(\exp_\mu; x) + e^{2\mu x}\mathcal{C}_n e_0\left(x\right) \\ &= e^{2\mu x}\left(\mathcal{C}_n e_0\left(x\right) - 1\right).\end{aligned} \tag{2.2}$$

Lemma 2 *For each $x \in [0, \infty)$, the following identities hold:*

$$\lim_{n\to\infty} \mathcal{C}_n e_0\left(x\right) = \lim_{n\to\infty} e^{\mu x - \mu b_n}\left(e^{\frac{\mu b_n}{n}} + 1 - e^{\frac{\mu x}{n}}\right)^n = 1, \tag{2.3}$$

$$\lim_{n\to\infty} n\left(\mathcal{C}_n e_0\left(x\right) - 1\right) = \lim_{n\to\infty} n\left(e^{\mu x - \mu b_n}\left(e^{\frac{\mu b_n}{n}} + 1 - e^{\frac{\mu x}{n}}\right)^n - 1\right) = \mu^2 x, \tag{2.4}$$

and

$$\lim_{n\to\infty} n^2 \mathcal{C}_n(\exp^4_{\mu,x}; x) = 0.$$

3 Quantitative Results

All concepts mentioned below can be found in [7] more generally. We denote by $C_\mu [0,\infty)$ the space of continous functions $f \in C[0,\infty)$ with the property that exists $M > 0$ such that $|f(x)| \leq Me^{\mu x}$, for every $x \in [0, b_n]$. This space endowed with norm

$$\|f\|_\mu = \sup_{x\in[0,b_n]} \frac{|f(x)|}{e^{\mu x}}.$$

Also,

$$C_\mu^k [0,\infty) := \left\{ f : f \in C_\mu [0,\infty) \text{ and } \lim_{x\to\infty} \frac{|f(x)|}{e^{\mu x}} = k,\ k \text{ is constant.} \right\}.$$

For $f \in C_\mu^k [0,\infty)$ we use the following modulus of continuity:

$$\Omega_\mu (f;\delta) = \sup_{\substack{x,t\in[0,b_n] \\ |e^{\mu t}-e^{\mu x}|\leq\delta}} \frac{|f(x)-f(t)|}{[|e^{\mu t}-e^{\mu x}|+1]\, e^{\mu x}}.$$

In [7], the authors proved the most general form of the following lemmas.

In the following, we give the main properties of the modulus of continuity.

Lemma 3 ([7]) *If* $f \in C_\mu [0,\infty)$ *and* $\lambda > 0$*, then*

$$\Omega_\mu (f;\lambda\delta) \leq (1+\lambda)(1+\delta)\,\Omega_\mu (f;\delta).$$

holds for every $\delta > 0$.

Lemma 4 ([7]) *For* $\delta > 0,\ f \in C_\mu [0,\infty)$ *and* $x, t \in [0, b_n]$, *the inequality*

$$|f(t) - f(x)| \leq 2e^{\mu x} (1+\delta)^2 \left(1 + \frac{\left(e^{\mu x}-e^{\mu t}\right)^2}{\delta^2}\right) \Omega_\mu (f;\delta)$$

holds.

Lemma 5 ([7]) *For any* $f \in C_\mu^k [0,\infty)$, *we have*

$$\lim_{\delta\to 0} \Omega_\mu (f;\delta) = 0.$$

Quantitative approximation theorems for sequences of linear positive operators play an important role not only in approximating functions, but also in estimating the error of the approximation. One of the most important convergence results in approximation theory is the Voronovskaya theorem. Roughly speaking, it is obtained to describe the rate of pointwise convergence.

Moreover, the other results presented in this paper are a quantitative-Voronovskaya type and a Grüss–Voronovskaya type theorems for the new operators. For more details, see [1]. Recently, Gal and Gonska obtained a Voronovskaya type theorem with the aid of Grüss inequality for Bernstein operators in [8] and called it Grüss–Voronovskaya type theorem. In this paper, we extend some of these results for our operators $\mathcal{C}_n$.

First, in the following theorem, we give quantitative type theorem for our operator $\mathcal{C}_n$:

Theorem 1 *For* $f \in C_\mu^k[0,\infty)$ *and* $x \in [0, b_n]$, *we have*

$$|\mathcal{C}_n f(x) - f(x)| \leq 8e^{\mu x}(1 + \mathcal{C}_n e_0(x))\left(1 + e^{\mu x}\right)\Omega_\mu\left(f; \sqrt{(\mathcal{C}_n e_0(x) - 1)}\right) + f(x)\,|(\mathcal{C}_n e_0(x) - 1)|\,.$$

Proof Suppose that $\delta < 1$. Using Lemma 3, 4 and (2.2), we have

$$\begin{aligned}|\mathcal{C}_n f(x) - f(x)| &\leq 2e^{\mu x}(1+\delta)^2\left(\mathcal{C}_n e_0(x) + \frac{1}{\delta^2}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\right)\Omega_\mu(f;\delta) + f(x)\,|(\mathcal{C}_n e_0(x) - 1)|\\ &\leq 8e^{\mu x}(1 + \mathcal{C}_n e_0(x))\,\Omega_\mu\left(f; \sqrt{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}\right) + f(x)\,|(\mathcal{C}_n e_0(x) - 1)|\\ &\leq 8e^{\mu x}(1 + \mathcal{C}_n e_0(x))\left(1 + e^{\mu x}\right)\Omega_\mu\left(f; \sqrt{(\mathcal{C}_n e_0(x) - 1)}\right) + f(x)\,|(\mathcal{C}_n e_0(x) - 1)|\,.\end{aligned}$$

□

We have that our operator has a different approach charecteristics

Remark 1 If in the previous theorem, we assume

$$\delta^2 = \lambda_n(x) := (\mathcal{C}_n e_0(x) - 1)\,,$$

then the estimate reads as

$$|\mathcal{C}_n f(x) - f(x)| \leq f(x)\,\lambda_n(x) + 8e^{\mu x}(1 + \mathcal{C}_n e_0(x))\left(1 + e^{\mu x}\right)\Omega_\mu\left(f; \sqrt{(\mathcal{C}_n e_0(x) - 1)}\right).$$

Hence, velocity of convergence of $\mathcal{C}_n f(x)$ to $f(x)$ is managed by the velocity of convergence of $\mathcal{C}_n e_0(x)$ to $e_0(x) = 1$, or equivalently, the one of $\lambda_n(x)$ to 0, and this is given by the undermentioned limit, that can be easily computed by elementary calculus.

$$\lim_{n\to\infty} n\left(\mathcal{C}_n e_0(x) - 1\right) = \lim_{n\to\infty} n\lambda_n(x)$$
$$= \lim_{n\to\infty} n\left(e^{\mu x-\mu b_n}\left(e^{\frac{\mu b_n}{n}} + 1 - e^{\frac{\mu x}{n}}\right)^n - 1\right) = \mu^2 x.$$

Now, we state quantitative-Voronovskaya type theorem for $\mathcal{C}_n$:

Theorem 2 *If* $f \in C_\mu^k[0,\infty)$ *and* $x \in (0, b_n)$*, then we get*

$$\left|\mathcal{C}_n f(x) - f(x) - (\mathcal{C}_n e_0(x) - 1)\left(f(x) - \frac{3}{2}\mu^{-1} f'(x) + \frac{1}{2}\mu^{-2} f''(x)\right)\right|$$
$$\leq 8e^{\mu x}\mathcal{C}_n\left(\exp_{\mu,x}^2; x\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right); \sqrt{\frac{\mathcal{C}_n\left(\exp_{\mu,x}^4; x\right)}{\mathcal{C}_n\left(\exp_{\mu,x}^2; x\right)}}\right).$$

Proof By Taylor's theorem, we have

$$f(t) = \left(f\circ\log_\mu\right)\left(e^{\mu t}\right)$$
$$= \left(f\circ\log_\mu\right)\left(e^{\mu x}\right) + \left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\exp_{\mu,x}(t) + \frac{1}{2}\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)\exp_{\mu,x}^2(t)$$
$$+ h(x,t)\exp_{\mu,x}^2(t),$$

where

$$h_x(t) := h(x,t) = \frac{\left(f\circ\log_\mu\right)''\left(\exp_\mu\right)(\xi) - \left(f\circ\log_\mu\right)''\left(\exp_\mu\right)(x)}{2}$$

with ξ a number between x and t. Applying the operator $\mathcal{C}_n$ to both side of above inequality, we get

$$\mathcal{C}_n f(x) = \mathcal{C}_n e_0(x) f(x) + \left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\mathcal{C}_n\left(\exp_{\mu,x}; x\right) + \frac{1}{2}\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)\mathcal{C}_n\left(\exp_{\mu,x}^2; x\right)$$
$$+ \mathcal{C}_n\left(h_x \exp_{\mu,x}^2; x\right).$$

Using Lemma 4 and the fact that $\left|e^{\mu\xi} - e^{\mu x}\right| \leq \left|e^{\mu t} - e^{\mu x}\right|$, then we can write

$$|h(x,t)| \leq e^{\mu x}(1+\delta)^2\left(1 + \frac{\left(e^{\mu\xi} - e^{\mu x}\right)^2}{\delta^2}\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right); \delta\right)$$
$$\leq e^{\mu x}(1+\delta)^2\left(1 + \frac{\left(e^{\mu t} - e^{\mu x}\right)^2}{\delta^2}\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right); \delta\right).$$

Suppose that $\delta < 1$. Thus, we can write

$$|h(x,t)| \leq 4e^{\mu x}\left(1+\frac{\left(e^{\mu t}-e^{\mu x}\right)^2}{\delta^2}\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right);\delta\right).$$

Multiplying this relation with $\exp^2_{\mu,x}$ and applying the operator $\mathcal{C}_n$, we get

$$\mathcal{C}_n\left(h_x\exp^2_{\mu,x};x\right) \leq 4e^{\mu x}\left(\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)+\frac{1}{\delta^2}\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right);\delta\right). \tag{3.1}$$

Using (2.1) and (2.2), we get

$$\begin{aligned}\mathcal{C}_n f(x)-f(x) = {} & f(x)\left(\mathcal{C}_n e_0(x)-1\right)+\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)e^{\mu x}\left(1-\mathcal{C}_n e_0(x)\right)\\ &+\frac{1}{2}\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)e^{2\mu x}\left(\mathcal{C}_n e_0(x)-1\right)\\ &+\mathcal{C}_n\left(h_x\exp^2_{\mu,x};x\right).\end{aligned}$$

We know that, since

$$\left(f\circ\tau^{-1}\right)'=\left(f'\circ\tau^{-1}\right)\left(\tau^{-1}\right)'$$

and

$$\left(\tau^{-1}\right)'(\tau(t))=\frac{1}{\tau'(t)},$$

we have

$$\left(f\circ\tau^{-1}\right)'(\tau(t))=\frac{f'(t)}{\tau'(t)}.$$

Also since

$$\left(f\circ\tau^{-1}\right)''=\left(f''\circ\tau^{-1}\right)\left(\left(\tau^{-1}\right)'\right)^2+\left(f'\circ\tau^{-1}\right)\left(\tau^{-1}\right)''$$

and

$$\frac{d}{dt}\left(\left(\tau^{-1}\right)'(\tau(t))\right)=\left(\tau^{-1}\right)''(\tau(t))\,\tau'(t)=-\frac{\tau''(t)}{\left(\tau'(t)\right)^2},$$

we get

$$\left(f\circ\tau^{-1}\right)''(\tau(t))=\frac{f''(t)}{\left(\tau'(t)\right)^2}-f'(t)\frac{\tau''(t)}{\left(\tau'(t)\right)^3}.$$

Therefore, since

$$\left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)=e^{-\mu x}\mu^{-1}f'(x)$$

and

$$\left(f\circ\log_\mu\right)''\left(e^{\mu x}\right)=e^{-2\mu x}\left(\mu^{-2}f''(x)-\mu^{-1}f'(x)\right),$$

we can write

$$\begin{aligned}&\left|\mathcal{C}_n f(x) - f(x) - (\mathcal{C}_n e_0(x) - 1)\left(f(x) - \frac{3}{2}\mu^{-1} f'(x) + \frac{1}{2}\mu^{-2} f''(x)\right)\right| \\ &\leq 4e^{\mu x}\left(\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right) + \frac{1}{\delta^2}\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)\right)\Omega_\mu\left(\left(f \circ \log_\mu\right)''\left(\exp_\mu\right); \delta\right) \\ &= 4e^{\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\left(1 + \frac{1}{\delta^2}\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}\right)\Omega_\mu\left(\left(f \circ \log_\mu\right)''\left(\exp_\mu\right); \delta\right).\end{aligned}$$

Choosing $\delta = \sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}}$, we have desired result. □

Later, we express quantitative-Grüss–Voronovskaya type theorem for $\mathcal{C}_n$:

Theorem 3 *If $f,g \in C^k_\mu[0,\infty)$, then for all $x \in [0, b_n]$ and $n \in \mathbb{N}$ we have*

$$\begin{aligned}&n\left|\mathcal{C}_n(fg)(x) - \mathcal{C}_n f(x)\mathcal{C}_n g(x) - xf'(x)g'(x)(\mathcal{C}_n e_0(x) - 1) + \mu^2 x f(x) g(x)(\mathcal{C}_n e_0(x) - 1)\right| \\ &\qquad \leq \mathcal{G}_n(\mathcal{C}_n, (fg); x) + \|f\|_\mu e^{\mu x}\mathcal{G}_n(\mathcal{C}_n, g; x) + \|g\|_\mu e^{\mu x}\mathcal{G}_n(\mathcal{C}_n, f; x) + nI_n(f) I_n(g),\end{aligned}$$

where

$$\mathcal{G}_n(\mathcal{C}_n, f; x) := 8e^{\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\Omega_\mu\left(\left(f \circ \log_\mu\right)''\left(\exp_\mu\right); \sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}}\right)$$

and

$$I_n(f) := \frac{\left\|\left(f \circ \log_\mu\right)''\right\|_\mu e^{\mu x}}{2}\left\{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right) + \sqrt{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}\right\} + 2\mu^{-1} f'(x)\left|1 - \mathcal{C}_n e_0(x)\right|.$$

Also, $\mathcal{G}_n(\mathcal{C}_n, g; x)$, $\mathcal{G}_n(\mathcal{C}_n, (fg); x)$, and $I_n(g)$ are the analogous one.

Proof For $x \in [0,\infty)$ and $n \in \mathbb{N}$, it is easily seen that we can write

$$\begin{aligned}&\mathcal{C}_n(fg)(x) - \mathcal{C}_n f(x)\mathcal{C}_n g(x) - xf'(x)g'(x)(\mathcal{C}_n e_0(x) - 1) + \mu^2 x g(x) f(x)(\mathcal{C}_n e_0(x) - 1) \\ &= \left[\mathcal{C}_n(fg)(x) - (fg)(x) - (\mathcal{C}_n e_0(x) - 1)\left(\mu^2 x(fg)(x) - \frac{3}{2}\mu x(fg)'(x) + \frac{1}{2}x(fg)''(x)\right)\right] \\ &\quad - f(x)\left[\mathcal{C}_n g(x) - g(x) - (\mathcal{C}_n e_0(x) - 1)\left(\mu^2 x g(x) - \frac{3}{2}\mu x g'(x) + \frac{1}{2}x g''(x)\right)\right] \\ &\quad - g(x)\left[\mathcal{C}_n f(x) - f(x) - (\mathcal{C}_n e_0(x) - 1)\left(\mu^2 x f(x) - \frac{3}{2}\mu x f'(x) + \frac{1}{2}x f''(x)\right)\right] \\ &\quad + [g(x) - \mathcal{C}_n g(x)][\mathcal{C}_n f(x) - f(x)] \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

So, we get

$$\left|\mathcal{C}_n(fg)(x) - \mathcal{C}_n f(x)\,\mathcal{C}_n g(x) - xf'(x)\,g'(x)\,(\mathcal{C}_n e_0(x) - 1) + \mu^2 x g(x)\, f(x)\,(\mathcal{C}_n e_0(x) - 1)\right|$$
$$\leq |I_1| + |I_2| + |I_3| + |I_4|.$$

By Theorem 2, we have the estimates

$$|I_1| \leq 8e^{\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\Omega_\mu\left(\left((fg)\circ\log_\mu\right)''\left(\exp_\mu\right); \sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}}\right),$$

$$|I_2| \leq \|f\|_\mu\, 8e^{2\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\Omega_\mu\left(\left(g\circ\log_\mu\right)''\left(\exp_\mu\right); \sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}}\right)$$

and

$$|I_3| \leq \|g\|_\mu\, 8e^{2\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right); \sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x}; x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right)}}\right).$$

On the other hand, since $f \in C^k_\mu[0,\infty)$ we write

$$\mathcal{C}_n(f;x) - f(x) = \left(f\circ\log_\mu\right)'\left(e^{\mu x}\right)\mathcal{C}_n\left(\exp_{\mu,x}; x\right) + \frac{1}{2}\mathcal{C}_n\left(\left(f\circ\log_\mu\right)''\left(e^{\mu\xi}\right)\exp^2_{\mu,x}; x\right)$$

and so we get

$$\begin{aligned}|\mathcal{C}_n(f;x) - f(x)| &\leq \mu^{-1}f'(x)\,|1-\mathcal{C}_n e_0(x)| + \frac{1}{2}\mathcal{C}_n\left(\left(f\circ\log_\mu\right)''\left(e^{\mu\xi}\right)\exp^2_{\mu,x}; x\right)\\ &\leq \mu^{-1}f'(x)\,|1-\mathcal{C}_n e_0(x)| + \left\|\left(f\circ\log_\mu\right)''\right\|_\mu \frac{1}{2}\mathcal{C}_n\left(e^{\mu\xi}\exp^2_{\mu,x}; x\right)\end{aligned}$$

where ξ is a number between t and x. If $t < \xi < x$, then $e^{\mu\xi} \leq e^{\mu x}$. In this case, we have

$$|\mathcal{C}_n(f;x) - f(x)| \leq \frac{\left\|\left(f\circ\log_\mu\right)''\right\|_\mu e^{\mu x}}{2}\mathcal{C}_n\left(\exp^2_{\mu,x}; x\right) + \mu^{-1}f'(x)\,|1-\mathcal{C}_n e_0(x)|$$

or if $x < \xi < t$, then $e^{\mu\xi} \leq e^{\mu t}$. In this case, with the help of Hölder's inequality, we get

$$\begin{aligned}
|\mathcal{C}_n(f;x)-f(x)| &\leq \frac{\left\|\left(f\circ\log_\mu\right)''\right\|_\mu}{2}\mathcal{C}_n\left(\exp_\mu \exp^2_{\mu,x};x\right)+\mu^{-1}f'(x)\left|1-\mathcal{C}_n e_0(x)\right| \\
&\leq \frac{\left\|\left(f\circ\log_\mu\right)''\right\|_\mu}{2}\mathcal{C}_n\left(\exp^2_\mu;x\right)^{\frac{1}{2}}\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)^{\frac{1}{2}}+\mu^{-1}f'(x)\left|1-\mathcal{C}_n e_0(x)\right| \\
&= \frac{\left\|\left(f\circ\log_\mu\right)''\right\|_\mu e^{\mu x}}{2}\sqrt{\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)}+\mu^{-1}f'(x)\left|1-\mathcal{C}_n e_0(x)\right|.
\end{aligned}$$

Hence, we gain for two cases of ξ that

$$\begin{aligned}
|\mathcal{C}_n(f;x)-f(x)| \leq{}& \frac{\left\|\left(f\circ\log_\mu\right)''\right\|_\mu e^{\mu x}}{2}\left\{\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)+\sqrt{\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)}\right\} \\
&+2\mu^{-1}f'(x)\left|1-\mathcal{C}_n e_0(x)\right| := I_n(f).
\end{aligned}$$

A similar reasoning yields $|\mathcal{C}_n(g;x)-g(x)|\leq I_n(g)$. Therefore we get

$$\begin{aligned}
& n\left|\mathcal{C}_n(fg)(x)-\mathcal{C}_n f(x)\,\mathcal{C}_n g(x)-xf'(x)\,g'(x)\left(\mathcal{C}_n e_0(x)-1\right)\right. \\
&\qquad \left.+\mu^2 xg(x)\,f(x)\left(\mathcal{C}_n e_0(x)-1\right)\right| \\
&\leq 8e^{\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)\Omega_\mu\left(\left((fg)\circ\log_\mu\right)''\left(\exp_\mu\right);\sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)}}\right) \\
&\quad +\|f\|_\mu\,8e^{2\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)\Omega_\mu\left(\left(g\circ\log_\mu\right)''\left(\exp_\mu\right);\sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)}}\right) \\
&\quad +\|g\|_\mu\,8e^{2\mu x}\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)\Omega_\mu\left(\left(f\circ\log_\mu\right)''\left(\exp_\mu\right);\sqrt{\frac{\mathcal{C}_n\left(\exp^4_{\mu,x};x\right)}{\mathcal{C}_n\left(\exp^2_{\mu,x};x\right)}}\right) \\
&\quad +nI_n(f)\,I_n(g),
\end{aligned}$$

as desired. □

Theorem 4 *For each $n\in\mathbb{N}$ and $x\in[0,\infty)$, we have*

$$\lim_{n\to\infty}\left(\frac{\mathcal{C}_n f}{\exp_\mu}\right)'(x)=\left(\frac{f}{\exp_\mu}\right)'(x).$$

Proof Using (1.5), we obtain

$$\left(\frac{\mathcal{C}_n f}{\exp_\mu}\right)'(a_n(x)) = \left(C_n\left(\frac{f}{\exp_\mu}\right)(x)\right)'$$
$$= \left[\sum_{k=0}^{n}\left(\frac{f}{\exp_\mu}\right)\left(\frac{kb_n}{n}\right)p_{n,k}(a_n(x))\right]'$$
$$= \frac{a_n'(x)}{a_n(x)\left(1-\frac{a_n(x)}{b_n}\right)}\sum_{k=0}^{n}\left(\frac{f}{\exp_\mu}\right)\left(\frac{kb_n}{n}\right)$$
$$\times p_{n,k}(a_n(x))\frac{n}{b_n}\left(\frac{kb_n}{n}-a_n(x)\right). \tag{3.2}$$

First, we take into account the case $x = 0$.

From (3.2), we have

$$\left(\frac{\mathcal{C}_n f}{\exp_\mu}\right)'(a_n(x)) = -\left(\frac{f}{\exp_\mu}\right)(0)\,na_n'(x)\left(1-\frac{a_n(x)}{b_n}\right)^{n-1}$$
$$+\left(\frac{f}{\exp_\mu}\right)\left(\frac{b_n}{n}\right)na_n'(x)\left(1-n\frac{a_n(x)}{b_n}\right)\left(1-\frac{a_n(x)}{b_n}\right)^{n-2}$$
$$\sum_{k=2}^{n}\left(\frac{f}{\exp_\mu}\right)\left(\frac{kb_n}{n}\right)\binom{n}{k}a_n'(x)\left(k-n\frac{a_n(x)}{b_n}\right)\left(\frac{a_n(x)}{b_n}\right)^{k-1}$$
$$\times\left(1-\frac{a_n(x)}{b_n}\right)^{n-k-1}.$$

For $x = 0$, because of $a_n(x) = 0$, we get

$$\left(\frac{\mathcal{C}_n f}{\exp_\mu}\right)'(0) = -na_n'(x)\left(\frac{f}{\exp_\mu}\right)(0) + na_n'(x)\left(\frac{f}{\exp_\mu}\right)\left(\frac{b_n}{n}\right)$$
$$= a_n'(x)\frac{\left(\frac{f}{\exp_\mu}\right)\left(\frac{b_n}{n}\right)-\left(\frac{f}{\exp_\mu}\right)(0)}{\frac{1}{n}-0}.$$

If the limit of both sides is taken above equality, then we obtain

$$\lim_{n\to\infty}\left(\frac{\mathcal{C}_n f}{\exp_\mu}\right)'(0) = \left(\frac{f}{\exp_\mu}\right)'(0).$$

Now, let's $x > 0$.

We consider the following function:

$$\lambda_x(t) = \frac{\left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu t}\right) - \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu x}\right)}{e^{\mu t} - e^{\mu x}} - \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)'\left(e^{\mu x}\right).$$

In that case, $\lim_{t \to x} \lambda_x(t) = 0$. We get

$$\left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu t}\right) = \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu x}\right) + \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)'\left(e^{\mu x}\right)\left(e^{\mu t} - e^{\mu x}\right)$$
$$+\lambda_x(t)\left(e^{\mu t} - e^{\mu x}\right).$$

If $\frac{kb_n}{n}$ is changed instead of t, then we have

$$\left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu \frac{kb_n}{n}}\right) = \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)\left(e^{\mu x}\right) + \left(\frac{f}{\exp_\mu} \circ \log_\mu\right)'\left(e^{\mu x}\right)\left(e^{\mu \frac{kb_n}{n}} - e^{\mu x}\right)$$
$$+\lambda_x(t)\left(e^{\mu \frac{kb_n}{n}} - e^{\mu x}\right).$$

If this equality is written in (3.2), then we attain

$$\begin{aligned}
\left(\frac{C_n f}{\exp_\mu}\right)'(x) &= \frac{a_n'(x)}{a_n(x)\left(1 - \frac{a_n(x)}{b_n}\right)}\left[\left(\frac{f}{\exp_\mu}\right)(x)\sum_{k=0}^{n} p_{n,k}(a_n(x))\frac{n}{b_n}\left(\frac{kb_n}{n} - a_n(x)\right)\right. \\
&\quad + \frac{n}{b_n}\frac{\left(\frac{f}{\exp_\mu}\right)'(x)}{\mu e^{\mu x}}\sum_{k=0}^{n}\left(e^{\mu \frac{kb_n}{n}} - e^{\mu x}\right)p_{n,k}(a_n(x))\left(\frac{kb_n}{n} - a_n(x)\right) \\
&\quad \left. + \frac{n}{b_n}\sum_{k=0}^{n}\lambda_x(t)\left(e^{\mu \frac{kb_n}{n}} - e^{\mu x}\right)p_{n,k}(a_n(x))\left(\frac{kb_n}{n} - a_n(x)\right)\right] \\
&= \frac{a_n'(x)}{a_n(x)\left(1 - \frac{a_n(x)}{b_n}\right)}\frac{n}{b_n}\left[\left(\frac{f}{\exp_\mu}\right)(x)\, C_n(t - a_n(x); a_n(x))\right. \\
&\quad + \frac{\left(\frac{f}{\exp_\mu}\right)'(x)}{\mu e^{\mu x}} C_n\left(\left(e^{\mu t} - e^{\mu x}\right)(t - a_n(x)); a_n(x)\right) \\
&\quad \left. + C_n\left(\lambda_x(t)\left(e^{\mu t} - e^{\mu x}\right)(t - a_n(x)); a_n(x)\right)\right].
\end{aligned}$$

We know

$$C_n(t - a_n(x); a_n(x)) = 0.$$

We can write

$$\begin{aligned}
C_n\left(\left(e^{\mu t} - e^{\mu x}\right)(t - a_n(x)); a_n(x)\right) &= C_n\left(te^{\mu t} - te^{\mu x} - a_n(x)e^{\mu t} + a_n(x)e^{\mu x}; a_n(x)\right) \\
&= C_n\left(te^{\mu t}; a_n(x)\right) - e^{\mu x} C_n(t; a_n(x)) \\
&\quad - a_n(x)\, C_n\left(e^{\mu t}; a_n(x)\right) + a_n(x)\, e^{\mu x} C_n(1; a_n(x)).
\end{aligned}$$

Because

$$C_n\left(te^{\mu t};a_n(x)\right)=a_n(x)\,e^{\frac{\mu b_n+\mu x(n-1)}{n}},$$

$$C_n\left(e^{\mu t};a_n(x)\right)=e^{\mu x},$$

$$\lim_{n\to\infty}a_n(x)=x$$

and

$$\lim_{n\to\infty}a_n'(x)=\lim_{n\to\infty}b_n\frac{\frac{\mu}{n}e^{\frac{\mu x}{n}}}{e^{\frac{\mu b_n}{n}}-1}=1,$$

we have

$$\lim_{n\to\infty}\frac{a_n'(x)}{a_n(x)\left(1-\frac{a_n(x)}{b_n}\right)}\frac{n}{b_n}\frac{1}{\mu e^{\mu x}}C_n\left(\left(e^{\mu t}-e^{\mu x}\right)(t-a_n(x));a_n(x)\right)=1.$$

Now, we use Hölder inequality:

$$0\le C_n\left(\lambda_x(t)\left(e^{\mu t}-e^{\mu x}\right)(t-a_n(x));a_n(x)\right)\le$$
$$\left(C_n\left(\lambda_x^2(t);a_n(x)\right)\right)^{\frac{1}{2}}\left(C_n\left(\left(e^{\mu t}-e^{\mu x}\right)^2;a_n(x)\right)\right)^{\frac{1}{2}}\left(C_n\left((t-a_n(x))^2;a_n(x)\right)\right)^{\frac{1}{2}}.$$

From Korovkin theorem, we know

$$\lim_{n\to\infty}C_n\left(\lambda_x^2(t);a_n(x)\right)=\lambda_x^2(x)=0.$$

As

$$\lim_{n\to\infty}C_n\left(\left(e^{\mu t}-e^{\mu x}\right)^2;a_n(x)\right)=0$$

and

$$\lim_{n\to\infty}C_n\left((t-a_n(x))^2;a_n(x)\right)=0,$$

we obtain desired result. □

4 Variation Detracting Property of Bernstein–Chlodowsky Operators

The first study about the variation detracting property and the convergence in variation of a sequence of linear positive operators was come out by Lorentz (1953). He proved that B_n have

$$V_{[0,1]}[B_n f] \leq V_{[0,1]}[f]$$

and it is called the variation detracting property.

The main purpose of this section is to confirm the variation detracting property and convergence in the variation seminorm for the Bernstein–Chlodowsky operators. We firstly give the definitions related to variation detracting property.

Definition 1 ([11]) The least upper bound of the set of all possible sums V is called the total variation of the function $f(x)$ on $[a, b]$ and is designated by $V_{[a,b]}[f]$.

Definition 2 ([2]) The class of all functions of bounded variation on I is called BV space and denoted by $BV(I)$. This space can be endowed both with seminorm $|.|_{BV(I)}$ and with a norm, $\|.\|_{BV(I)}$, where

$$|f|_{BV(I)} := V_I[f] \quad , \quad \|f\|_{BV(I)} := V_I[f] + |f(a)|,$$

$f \in BV(I)$, a being any fixed point of I.

Definition 3 ([3]) Let $I \subseteq \mathbb{R}$ be a fixed integral, and $V_I[f]$ the total variation of the function $f : I \to \mathbb{R}$. The class of all bounded functions of bounded variation on I endowed with the seminorm

$$\|f\|_{TV(I)} := V_I[f]$$

is called TV space and is denoted by $TV(I)$.

Definition 4 ([11]) Let $f(x)$ be a finite function defined on the closed interval $[a, b]$. Suppose that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \sum_{k=1}^{n} \{f(b_k) - f(a_k)\} \right| < \epsilon$$

for all numbers $a_1, b_1, \ldots, a_n, b_n$ such that $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and

$$\sum_{k=1}^{n} (b_k - a_k) < \delta.$$

Then the function $f(x)$ is said to be absolutely continuous. The class of all absolutely continuous function on $[a, b]$ is denoted by $AC[a, b]$.

Now, we give the variation detracting property of the Bernstein–Chlodowsky operators:

Theorem 5 *If* $f \in TV[0, b_n]$, *then* $V_{[0,b_n]}\left[\frac{\mathcal{C}_n f}{\exp_\mu}\right] \leq V_{[0,b_n]}\left[\frac{f}{\exp_\mu}\right]$.

Proof As $\frac{\mathcal{C}_n f}{\exp_\mu}$ polynomials are differentiable and their derivatives are integrable, by [9, 10], the equality

$$\left\| \frac{\mathcal{C}_n f}{\exp_\mu} \right\|_{TV[0,b_n]} = V_{[0,b_n]} \left[\frac{\mathcal{C}_n f}{\exp_\mu} \right] = \int_0^{b_n} \left| \frac{d}{dx} \frac{\mathcal{C}_n}{\exp_\mu} (f; x) \right| dx$$

is implemented. From (1.5), we can write

$$\begin{aligned} V_{[0,b_n]} \left[\frac{\mathcal{C}_n f}{\exp_\mu} \right] &= \int_0^{b_n} \left| \frac{d}{dx} \frac{\mathcal{C}_n}{\exp_\mu} (f; x) \right| dx \\ &= \int_0^{b_n} \left| \frac{d}{dx} \left[\frac{C_n}{\exp_\mu} (f; a_n (x)) \right] \right| dx. \end{aligned}$$

By Theorem 3.13 in [6], we get

$$\begin{aligned} V_{[0,b_n]} \left[\frac{\mathcal{C}_n f}{\exp_\mu} \right] &= \int_0^{b_n} \left| \frac{n}{b_n} \sum_{k=0}^{n-1} p_{n-1,k} (a_n (x)) \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| a_n^{'} (x) \, dx \\ &\leq \frac{n}{b_n} \sum_{k=0}^{n-1} \int_0^{b_n} \left| p_{n-1,k} (a_n (x)) \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| a_n^{'} (x) \, dx \\ &= \frac{n}{b_n} \sum_{k=0}^{n-1} \left| \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| \int_0^{b_n} p_{n-1,k} (a_n (x)) a_n^{'} (x) \, dx. \end{aligned}$$

If $\frac{a_n(x)}{b_n} = y$ is changed, then we have

$$V_{[0,b_n]} \left[\frac{\mathcal{C}_n f}{\exp_\mu} \right] \leq n \sum_{k=0}^{n-1} \binom{n-1}{k} \left| \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| \int_0^1 y^k (1-y)^{n-k-1} \, dy.$$

Now, let's consider the integral on the left side of the inequality. From definition of Beta function, we obtain

$$\begin{aligned} V_{[0,b_n]} \left[\frac{\mathcal{C}_n f}{\exp_\mu} \right] &\leq n \sum_{k=0}^{n-1} \binom{n-1}{k} \left| \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| \frac{1}{n\binom{n-1}{k}} \\ &= \sum_{k=0}^{n-1} \left| \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right| \end{aligned}$$

$$\leq \sup \sum_{k=0}^{n-1} \left| \Delta_{\frac{b_n}{n}} \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right|$$
$$= \sup \sum_{k=0}^{n-1} \left| \frac{f}{\exp_\mu} \left(\frac{k+1}{n} b_n \right) - \frac{f}{\exp_\mu} \left(\frac{k}{n} b_n \right) \right|$$
$$= V_{[0,b_n]} \left[\frac{f}{\exp_\mu} \right] = \left\| \frac{f}{\exp_\mu} \right\|_{TV[0,b_n]} .$$

□

Theorem 6 *Let* $f \in TV\,[0, b_n]$. *There holds*

$$\lim_{n\to\infty} \left\| \frac{\mathcal{C}_n f}{\exp_\mu} - \frac{f}{\exp_\mu} \right\|_{TV[0,\infty)} = 0 \iff \frac{f}{\exp_\mu} \in AC\,[0, b_n]\,.$$

Proof Since $\frac{f}{\exp_\mu}$ and $\frac{\mathcal{C}_n f}{\exp_\mu} \in AC\,[0, b_n]$, then $\frac{\mathcal{C}_n f}{\exp_\mu} - \frac{f}{\exp_\mu} \in AC\,[0, b_n]$. By Theorem 3.13 and Remark 3.20 in [6], it is written

$$\lim_{n\to\infty} \left\| \frac{\mathcal{C}_n f}{\exp_\mu} - \frac{f}{\exp_\mu} \right\|_{TV[0,\infty)} = \lim_{n\to\infty} \int_0^\infty \left| \left(\frac{\mathcal{C}_n f}{\exp_\mu} \right)' (x) - \left(\frac{f}{\exp_\mu} \right)' (x) \right| dx.$$

From Theorem 4, it can be seen easily that $\left(\frac{\mathcal{C}_n f}{\exp_\mu} \right)' (x) \longrightarrow \left(\frac{f}{\exp_\mu} \right)' (x)$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \left\| \frac{\mathcal{C}_n f}{\exp_\mu} - \frac{f}{\exp_\mu} \right\|_{TV[0,\infty)} = 0.$$

Conversely, let $\lim_{n\to\infty} \left\| \frac{\mathcal{C}_n f}{\exp_\mu} - \frac{f}{\exp_\mu} \right\|_{TV[0,\infty)} = 0$. This means that $\frac{\mathcal{C}_n f}{\exp_\mu} \longrightarrow \frac{f}{\exp_\mu}$ in TV space. Therefore $\frac{f}{\exp_\mu}$ is in AC because of AC is closed. □

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