Chapter 18
Relative Controllability of Nonlinear Fractional Damped Delay Systems with Multiple Delays in Control

P. Suresh Kumar

Abstract This paper is concerned with the relative controllability of fractional damped dynamical systems with multiple delays in control for finite-dimensional spaces. Sufficient conditions for controllability are obtained using Schauder's fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffer matrix function. An example is provided to illustrate the theory.

Keywords Controllability · Fractional differential equations · Mittag-Leffler matrix function · Laplace transform

18.1 Introduction

Nowadays it is the realm of physicists and mathematicians who investigate the usefulness of non-integer order derivatives and integrals in different areas of physics and mathematics. It is a successful tool for describing complex quantum field dynamical systems, dissipation and long-range phenomena that cannot be well illustrated using ordinary differential operators. Many models are reformulated and expressed in terms of fractional differential equations so that their physical meaning will be incorporated in the mathematical models more realistically. In fact, fractional calculus attracts many physicists, biologists, engineers, and mathematicians for its interdisciplinary applications which are elegantly modeled with the help of fractional derivatives and it was conceptualized in connection with the infinitesimal calculus. Delay differential equations are often solved using numerical methods, asymptotic methods and graphical tools. Number of attempts have been made to find an analytical solution for delay differential equations by solving the characteristic equation under different conditions [16].

Controllability is one of the important qualitative aspects of a dynamical system. It is used to influence an object's behavior so as to accomplish the desired goal.

P. Suresh Kumar (✉)
Department of Mathematics, National Institute of Technology, Calicut 673601, Kerala, India
e-mail: sureshkumarp.maths@gmail.com

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Analysis of the control problems of fractional delay dynamical system is much more advanced. The control problems involving the delay in state variables are not developed much. Controllability of delay dynamical systems was studied by Wiess [19]. Chung [5] investigated the controllability of linear time-varying systems with delay. Controllability of nonlinear delay dynamical systems is studied by Dauer [6]. Klamka [8] addressed the constrained controllability of semilinear delayed systems. A sliding mode control for linear fractional systems with input and state delays is studied by Si-Ammour [14]. Balachandran et al. [1–4] investigated the controllability of damped dynamical systems with multiple delays in control. Controllability criteria for linear fractional differential systems with state delay and impulse are studied by Zhang et al. [20]. Wang [18] proposed a numerical method for delayed fractional-order differential equations. Explicit representations of solutions of linear delay systems are studied by Shu [13]. Morgado [9] analyzed and proposed numerical methods for fractional differential equations with delay. Recently controllability of a fractional delay dynamical systems and fractional systems with time-varying delays in control is studied by Joice Nirmala et al. [10, 11]. He et al. [7] addressed the controllability of fractional damped dynamical systems with delay in control. Suresh Kumar et al. [17] studied the controllability of nonlinear fractional Langevin delay systems by assuming the conditions $0 < \alpha, \beta \leq 1$ and $\alpha + \beta > 1$. In Caputo differential operators do not satisfy the semigroup property. We can apply the only fractional integral definition. Hence in the present manuscript, we consider $0 < \beta \leq 1 < \alpha \leq 2$. So, both the problems are different by formation in the fractional sense even though they are similar in the integer case. Moreover constrained controllability of fractional linear systems with delays in control is discussed by Sikora and Klamka [15]. Motivated by this, the main aim of the present article is to present controllability of nonlinear fractional damped delay dynamical systems with multiple delays in control of order $0 < \beta \leq 1 < \alpha \leq 2$.

In this paper, we discuss the controllability of linear fractional damped delay dynamical system by utilizing the solution representation. Further, sufficient conditions for the controllability of nonlinear fractional damped delay systems are established by using Schauder's fixed point theorem. Numerical examples with simulations are provided to illustrate the theory.

18.2 Preliminaries

In this section, we introduce the definitions and preliminary results from fractional calculus which are used throughout this paper.

Definition 18.1 The Caputo fractional derivative of order $\alpha \in \mathbb{C}$ with $1 < \alpha \leq 2$, for a suitable function f is defined as

$$^{C}D^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f^{(2)}(s)\mathrm{d}s.$$

For brevity, the Caputo fractional derivative $^{C}D^{\alpha}_{0+}$ is taken as $^{C}D^{\alpha}$.

Definition 18.2 The Mittag-Leffler functions of various type are defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad (\alpha > 0, z \in \mathbb{C}).$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha, \beta > 0, z \in \mathbb{C}),$$

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)},$$

where $(\gamma)_n$ is a Pochhammer symbol which is defined as $\gamma(\gamma+1)\ldots(\gamma+n-1)$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. For an $n \times n$ matrix A

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta > 0,$$
$$E_{\alpha,1}(A) = E_\alpha(A) \text{ with } \beta = 1.$$

Definition 18.3 ([12]) The formal definition of the Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is given by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \mathrm{d}t, \quad s \in \mathbb{C}.$$

The convolution operator of two functions $f(t)$ and $g(t)$ given on $\mathbb{R}^+$ is defined for $x \in \mathbb{R}^+$ by the integral

$$(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s.$$

The Laplace transform of a convolution is given by

$$\mathscr{L}\{f(t) * g(t)\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}.$$

Let $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. The inverse Laplace transform of product of two functions $F(s)$ and $G(s)$ is defined by

$$\mathscr{L}^{-1}\{F(s)G(s)\} = \mathscr{L}^{-1}\{F(s)\} * \mathscr{L}^{-1}\{G(s)\}.$$

The Laplace transforms of Mittag-Leffler functions are defined as

$$\mathscr{L}[E_{\alpha,1}(\pm\lambda t^\alpha)](s) = \frac{s^{\alpha-1}}{(s^\alpha \mp \lambda)}, \quad Re(\alpha) > 0,$$

$$\mathscr{L}[t^{\beta-1}E_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0,$$
$$\mathscr{L}[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\lambda t^{\alpha})](s) = \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha} \mp \lambda)^{\gamma}}, \quad Re(\alpha) > 0, \quad Re(\beta) > 0.$$

18.3 Linear System with Multiple Delays in Control

Consider the linear fractional damped delay dynamical system with multiple delays of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) - A\,{}^{C}D^{\beta}x(t) &= Bx(t) + Cx(t-\tau) + \sum_{i=0}^{M} D_i u(h_i(t)), \quad t \in J : [0, T], \\ x(t) &= \phi(t), \quad -\tau < t \le 0, \\ x'(0) &= q_0, \end{aligned} \tag{18.1}$$

where $0 < \beta \le 1 < \alpha \le 2$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B and C are $n \times n$ matrices and D_i for $n \times m$ matrices for $i = 0, 1, 2, \ldots, M$. Assume the following conditions:

(H1) The functions $h_i : J \to \mathbb{R}, i = 0, 1, 2, \ldots M$ are twice differentiable and strictly increasing in J. Moreover

$$h_i(t) \le t, \quad \text{for } i = 0, 1, 2, \ldots M, \quad \text{for all} \quad t \in J, \tag{18.2}$$

(H2) Introduce the time lead functions $r_i(t) : [h_i(0), h_i(T)] \to [0, T], i = 0, 1, 2, \ldots M$, such that $r_i(h_i(t)) = t$ for $t \in J$. Further $h_0(t) = t$ and for $t = T$. The following inequality holds

$$\begin{aligned} h_M(T) \le h_{M-1}(T) \le \ldots h_{m+1}(T) \le 0 = h_m(T) < h_{m-1}(T) = \ldots \\ = h_1(T) = h_0(T) = T. \end{aligned} \tag{18.3}$$

(H3) Let $h > 0$ be given. For functions $u : [-h, T] \to \mathbb{R}^n$ and $t \in J$, we use the symbol u_t denote the function on $[-h, 0]$ defined by $u_t(s) = u(t + s)$, for $s \in [-h, 0)$.

The following definitions of complete state of the system (18.1) at time t and relative controllability are assumed.

Definition 18.4 The set $y(t) = \{x(t), u_t\}$ is the complete state of the system (18.1) at time t.

Definition 18.5 System (18.1) is said to be relatively controllable on $[0, T]$ if, for every complete state $y(t)$ and every $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on $[0, T]$ such that the solution of system (18.1) satisfies $x(T) = x_1$.

Here the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. The solution of the system (18.1) can be written [11] as

$$\begin{aligned} x(t) = {} & X_{\alpha-\beta}(t)\phi(0) - AX_{\alpha-\beta,\alpha-\beta+1}(t)\phi(0) + tX_{\alpha-\beta,2}(t)q_0 \\ & + C\int_{-\tau}^{0}(t-s-\tau)^{\alpha-1}X_{\alpha-\beta,\alpha}(t-s-\tau)\phi(s)\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1}X_{\alpha-\beta,\alpha}(t-s)\sum_{i=0}^{M} D_i u(h_i(s))\mathrm{d}s. \end{aligned} \tag{18.4}$$

Using the time lead functions $r_i(t)$, we have

$$x(t) = x_L(t;\phi) + \sum_{i=0}^{M}\int_{h_i(0)}^{h_i(t)}(t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u(s)\mathrm{d}s,$$

where

$$\begin{aligned} x_L(t;\phi) = {} & X_{\alpha-\beta}(t)\phi(0) - AX_{\alpha-\beta,\alpha-\beta+1}(t)\phi(0) + tX_{\alpha-\beta,2}(t)q_0 \\ & + C\int_{-\tau}^{0}(t-s-\tau)^{\alpha-1}X_{\alpha-\beta,\alpha}(t-s-\tau)\phi(s)\mathrm{d}s. \end{aligned}$$

By using the inequality (18.3) we get

$$\begin{aligned} x(t) = {} & x_L(t;\phi) + \sum_{i=0}^{m}\int_{h_i(0)}^{0}(t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u_0(s)\mathrm{d}s \\ & + \sum_{i=0}^{m}\int_0^t (t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u(s)\mathrm{d}s \\ & + \sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(t)}(t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u_0(s)\mathrm{d}s. \end{aligned}$$

For simplicity, let us write the solution as

$$x(t) = x_L(t;\phi) + G(t) + \sum_{i=0}^{M}\int_0^t (t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u(s)\mathrm{d}s, \tag{18.5}$$

where

$$G(t) = \sum_{i=0}^{m}\int_{h_i(0)}^{0}(t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u_0(s)\mathrm{d}s$$

$$+\sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(t)}(t-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(t-r_i(s))D_i\dot{r}_i(s)u_0(s)\mathrm{d}s.$$

Now let us define the controllability Grammian matrix by

$$W=\sum_{i=0}^{m}\int_0^T(T-r_i(s))^{2(\alpha-1)}(X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s))(X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s))^*\mathrm{d}s.$$

Theorem 18.1 *The linear system (18.1) is relatively controllable on* $[0, T]$ *if and only if the controllability Grammian matrix is positive definite for some* $T>0$.

Proof Assume that W is positive definite. Define the control function by

$$u(t)=(T-r_i(t))^{\alpha-1}(X_{\alpha-\beta,\alpha}(T-r_i(t))D_i\dot{r}_i(t))^*W^{-1}[x_1-x_L(T;\phi)-G(T)], \tag{18.6}$$

where the complete state $y(0)$ and the vector $x_1\in\mathbb{R}^n$ are chosen arbitrary. Taking $t=T$ in (18.5) and by using (18.6), we have $x(T)=x_1$. Then

$$y^*Wy=0,$$

that is,

$$y^*\left[\sum_{i=0}^{m}\int_0^T(T-r_i(s))^{2(\alpha-1)}(X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s))(X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s))^*\mathrm{d}s\right]y=0,$$

which implies

$$y^*\sum_{i=0}^{m}(T-r_i(s))^{\alpha-1}(X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s))=0,\ \text{ on }[0,T].$$

Consider the zero initial function $\phi=0$ and $u_0=0$ on $[-h,0]$ and the final point $x_1=y$. Since the system is controllable there exists a control $u(t)$ on J that steers the response to $x_1=y$. For $\phi=0$, $x_L(T,\phi)=0$, $G(t)=0$. On the other hand

$$y=x_L(T)=\sum_{i=0}^{m}\int_0^T(T-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s)u(s)\mathrm{d}s.$$

Then

$$y^*y=\sum_{i=0}^{m}\int_0^T y^*(T-r_i(s))^{\alpha-1}X_{\alpha-\beta,\alpha}(T-r_i(s))D_i\dot{r}_i(s)u(s)\mathrm{d}s=0.$$

This contradicts for $y\neq 0$. Hence W is nonsingular.

18.4 Nonlinear Systems with Multiple Delays in Control

Consider the nonlinear fractional damped delay dynamical system with multiple delays in control of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) - A\,{}^{C}D^{\beta}x(t) &= Bx(t) + Cx(t-\tau) + \sum_{i=0}^{M} D_i u(h_i(t)) + f(t, x(t), x(t-\tau), u(t)),\\ x(t) &= \phi(t),\\ x'(0) &= q_0, \quad -\tau < t \le 0, \end{aligned} \tag{18.7}$$

where $0 < \beta \le 1 < \alpha \le 2$, $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is a control vector, A, B, C are $n \times n$ matrices, D_i for $i = 0, 1, 2, \ldots M$, are $n \times m$ matrices and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. Further we impose the following assumption:

Let Q be the Banach space of continuous $\mathbb{R}^n \times \mathbb{R}^m$ valued functions defined on the interval J with the norm

$$\|(x, u)\| = \|x\| + \|u\|,$$

where $\|x\| = \sup\{x(t) : t \in J\}$ and $\|u\| = \sup\{u(t) : t \in J\}$. That is $Q = C_n(J) \times C_m(J)$, where $C_n(J)$ is the Banach space of continuous $\mathbb{R}^n$ valued functions defined on the interval J with the sup norm.

Similar to the linear system, the solution of nonlinear system (18.7) using time lead function $r_i(t)$ is given as

$$\begin{aligned} x(t) = x_L(t;\phi) + G(t) &+ \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) u(s)\mathrm{d}s\\ &+ \int_0^t (t-s)^{\alpha-1} X_{\alpha-\beta,\alpha}(t-s) f(s, x(s), x(s-\tau), u(s))\mathrm{d}s. \end{aligned} \tag{18.8}$$

Theorem 18.2 *Let the continuous function f satisfy the condition*

$$\lim_{|p|\to\infty} \frac{|f(t,p)|}{|p|} = 0 \tag{18.9}$$

uniformly in $t \in J$ and suppose that the system (18.1) is relatively controllable on J. Then the system (18.7) is relatively controllable on J.

Proof Let $\phi(t)$ be continuous on $[-\tau, 0]$ and let $x_1 \in \mathbb{R}^n$. Let Q be the Banach space of all continuous functions

$$(x, u) : [-\tau, T] \times [0, T] \to \mathbb{R}^n \times \mathbb{R}^m,$$

with the norm

$$\|(x,u)\| = \|x\| + \|u\|,$$

where $\|x\| = \{\sup|x(t)| \text{ for } t \in [-\tau, T]\}$ and $\|u\| = \{\sup|u(t)| \text{ for } t \in [0, T]\}$.

The solution of (18.7) using time lead function $r_i(t)$ is given by

$$x(t) = x_L(t;\phi) + G(t) + \sum_{i=0}^{M}\int_0^t (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) u(s)\mathrm{d}s$$
$$+ \int_0^t (t-s)^{\alpha-1} X_{\alpha-\beta,\alpha}(t-s) f(s, x(s), x(s-\tau), u(s))\mathrm{d}s. \qquad (18.10)$$

Let us assume

$$a_i = \sup\|X_{\alpha-\beta,\alpha}(T - r_i(s))\|, b_i = \|\dot{r}_i(s)\|, i = 0, 1, 2, \ldots, M, \upsilon = \sup\|u_0(s)\|,$$
$$\vartheta = \sup\|X_{\alpha-\beta,\alpha}(T - s)\|, \mu = \sum_{i=0}^{m} a_i b_i \|D_i\| N_i + \sum_{i=m+1}^{M} a_i b_i \|D_i\| M_i,$$
$$c_1 = 4[a_i b_i \|D_i^*\|] \|W^{-1}\| \upsilon(\alpha-\beta)^{-1} T^{\alpha-\beta}, d_1 = 4[a_i b_i \|D_i^*\|] \|W^{-1}\| [|x_1 + \gamma + \mu|],$$
$$a = \max\{b(\alpha-\beta)^{-1} T^{\alpha-\beta}\|D_i\|, 1\}, b = \sum_{i=0}^{m} a_i b_i L_i, c_2 = 4\vartheta(\alpha-\beta)^{-1} T^{\alpha-\beta}, d_2 = 4[\gamma + \upsilon\mu],$$
$$N_i = \int_{h_(0)}^{0} (T - r_i(s))^{\alpha-1}\mathrm{d}s, M_i = \int_{h_(0)}^{h_i(T)} (T - r_i(s))^{\alpha-1}\mathrm{d}s,$$
$$L_i = \int_0^T (T - r_i(s))^{\alpha-1}\mathrm{d}s, c = \max\{c_1, c_2\}, d = \max\{d_1, d_2\},$$

and

$$\sup = \{\sup|f(t, x(t), x(t-\tau), u(t))|, t \in J\}.$$

Define $\Psi : Q \to Q$ by

$$\Psi(x,u) = (z, v),$$

where

$$v(t) = (T - r_i(t))^{\alpha-1}(x_{\alpha-\beta,\alpha}(T - r_i(t))(D_i)^* \dot{r}_i(t))^* W^{-1}\Bigg[x_1 - x_L(T;\phi)$$
$$- \sum_{i=0}^{m}\int_{h_i(0)}^{0} (T - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(T - r_i(s)) D_i \dot{r}_i(s) u_0(s)\mathrm{d}s$$
$$+ \sum_{i=m+1}^{M}\int_0^T (T - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(T - r_i(s)) D_i \dot{r}_i(s) u_0(s)\mathrm{d}s$$
$$- \int_0^T (T-s)^{\alpha-1} X_{\alpha-\beta,\alpha}(T-s) f(s, x(s), x(s-\tau), u(s))\mathrm{d}s\Bigg],$$

and

$$z(t) = x_L(t;\phi) - \sum_{i=0}^{m} \int_{h_i(0)}^{0} (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) u_0(s) \mathrm{d}s$$
$$+ \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) v(s) \mathrm{d}s$$
$$+ \sum_{i=m+1}^{M} \int_{\alpha_0}^{t} (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) u_0(s) \mathrm{d}s$$
$$+ \int_{0}^{t} (t - s)^{\alpha-1} X_{\alpha-\beta,\alpha}(t - s) f(s, x(s), x(s - \tau), u(s)) \mathrm{d}s.$$

Then

$$|v(t)| \le \|D_i^*\| a_i b_i \|w^{-1}\| [\|x_1\| + \gamma + \mu] + a_i b_i \|D_i^*\| \|W^{-1}\| \vartheta (\alpha - \beta)^{-1} T^{\alpha-\beta},$$
$$\le \frac{1}{4a}(d + c \sup |f|)$$

and

$$|z(t)| \le \gamma + \upsilon\mu + \left(\sum_{i=0}^{m} a_i b_i \|D_i\| L_i \alpha^{-1} T^{\alpha-\beta} \right) v(s) + \vartheta (\alpha - \beta)^{-1} T^{\alpha-\beta} \sup |f|,$$
$$\le \frac{d}{2} + \frac{c}{2} \sup |f|.$$

Further P maps

$$Q(r) = \left\{ (z, v) \in Q : \|z\| \le \frac{r}{2} \text{ and } \|v\| \le \frac{r}{2} \right\}$$

into itself and has a fixed point by the Schauder's fixed point theorem such that $P(z, v) = (z, v) = (x, u)$. Hence we have

$$x(t) = x_L(t;\phi) + G(t) + \sum_{i=0}^{M} \int_{0}^{t} (t - r_i(s))^{\alpha-1} X_{\alpha-\beta,\alpha}(t - r_i(s)) D_i \dot{r}_i(s) u(s) \mathrm{d}s$$
$$+ \int_{0}^{t} (t - s)^{\alpha-1} X_{\alpha-\beta,\alpha}(t - s) f(s, x(s), x(s - \tau), u(s)) \mathrm{d}s. \tag{18.11}$$

for $t \in J$ and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$ and

$$x(T) = x_1.$$

Hence the system (18.7) is relatively controllable on J.

18.5 Example

Example 18.1 Consider the nonlinear fractional damped delay dynamical system

$$
\begin{aligned}
{}^{C}D^{\alpha}x(t) - A^{C}D^{\beta}x(t) &= Bx(t) + Cx(t-1) + D_0u(t) + D_1u(t-1) + f(t, x(t), x(t-1), u(t)),\\
x(t) &= \phi(t),\\
x'(0) &= q_0, \quad -1 < t \leq 0,
\end{aligned}
\tag{18.12}
$$

The solution of the above problem (18.12) using Laplace transform we get

$$
\begin{aligned}
x(t) = \sum_{n=0}^{[t]} \Big[& B^n (t-n)^{\alpha n} E^{n+1}_{\alpha-\beta,\alpha n+1}(A(t-n)^{\alpha-\beta}) + C^n (t-n)^{\alpha n} E^{n+1}_{\alpha-\beta,\alpha n+1}(A(t-n)^{\alpha-\beta}) \Big] \phi(0)\\
& -A \sum_{n=0}^{[t]} \Big[B^n (t-n)^{\alpha n+\alpha-\beta} E^{n+1}_{\alpha-\beta,\alpha n+\alpha-\beta+1}(A(t-n)^{\alpha-\beta})\\
& +C^n (t-n)^{\alpha n+\alpha-\beta} E^{n+1}_{\alpha-\beta,\alpha n+\alpha-\beta+1}(A(t-n)^{\alpha-\beta}) \Big] \phi(0)\\
& + \sum_{n=0}^{[t]} \Big[B^n (t-n)^{\alpha n+1} E^{n+1}_{\alpha-\beta,\alpha n+2}(A(t-n)^{\alpha-\beta}\\
& +C^n (t-n)^{\alpha n+1} E^{n+1}_{\alpha-\beta,\alpha n+2}(A(t-n)^{\alpha-\beta}) \Big] y_0\\
& +C \sum_{n=0}^{[t]} B^n \int_{-1}^{0} (t-s-n-1)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta}) \phi(s) \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
& +C^n \int_{-1}^{0} (t-s-n-1)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta}) \phi(s) \mathrm{d}s\\
& + \sum_{n=0}^{[t]} \Big[B^n \int_0^{t-n} (t-s-n)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta})\\
& +C^n \int_0^{t-n} (t-s-n)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta}) \Big] D\dot{r}_i u(s) \mathrm{d}s\\
& + \sum_{n=0}^{[t]} \Big[B^n \int_0^{t-n} (t-s-n)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta})\\
& +C^n \int_0^{t-n} (t-s-n)^{\alpha n+\alpha-1} E^{n+1}_{\alpha-\beta,\alpha}(A(t-n)^{\alpha-\beta}) \Big] f(s, x(s), x(s-1), u(s) \mathrm{d}s,
\end{aligned}
$$

where $[\cdot]$ is the greatest integer function. Now consider the controllability on $[0, 1]$. Here $[t]$=0; and let $\alpha = \frac{3}{2}$, β=$\frac{1}{2}$, $h = 1$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$, $x(t) = \phi(t) \in \mathbb{R}^2$ and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ with initial conditions $\phi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and final condition $x(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $f(t, x(t), x(t-1), u(t)) = \frac{x(t)+x(t-1)}{x^2(t)+x^2(t-1)+u(t)}$. By applying Laplace transform on both sides of the equation, we get the solution as therefore the solution of (18.12) on [0,1] is

$$\begin{aligned} x(t) = & 2E_{\alpha-\beta}(At^{\alpha-\beta})\phi(0) - 2t^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})\phi(0) + 2tE_{\alpha-\beta,2}(At^{\alpha-\beta})y_0 \\ & +2C\int_{-1}^{0}(t-s-1)^{\alpha-\beta}E_{\alpha-\beta,\alpha}(A(t-s-1)^{\alpha-\beta})\phi(s)\mathrm{ds} \\ & +2\int_0^t (t-r_i(s))^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})D\dot{r}_i u(s)\mathrm{ds} \\ & +2\int_0^t (t-r_i(s))^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})f(s, x(s), x(s-1), u(s)\mathrm{ds}, \end{aligned}$$

and on further simplification

$$\begin{aligned} x(t) = & 2E_{\alpha-\beta}(At^{\alpha-\beta})\phi(0) - 2t^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})\phi(0) + 2tE_{\alpha-\beta,2}(At^{\alpha-\beta})y_0 \\ & +2t^{\alpha-1}(t)^{\alpha-\beta}E_{\alpha-\beta,\alpha}(A(t)^{\alpha-\beta})\phi(0) \\ & +2\int_0^t (t-r_i(s))^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})D\dot{r}_i u(s)\mathrm{ds} \\ & +2\int_0^t (t-r_i(s))^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-r_i(s))^{\alpha-\beta})f(s, x(s), x(s-1), u(s)\mathrm{ds}, \end{aligned}$$

By simple matrix calculation, we have the controllability Grammian matrix as

$$W = \begin{pmatrix} 26.6369 & -19.9353 \\ -19.9353 & 57.6070 \end{pmatrix} > 0,$$

which is positive definite. Hence the system (18.12) is controllable on [0, 1]. Therefore, the linear system of (18.12) is controllable on [0, 1]. And the nonlinear function $f(t, x(t), x(t-1), u(t))$ satisfies the hypothesis of Theorem (18.2) and hence the nonlinear system (18.12) steering from the initial point $\phi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to a desire state $x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ during [0, 1]. Hence the nonlinear system (18.12) is relatively controllable.

18.6 Conclusion

This paper deals with the relative controllability of nonlinear fractional damped delay systems with multiple delays in control. In [10, 11] the authors have studied the problem of order $0 < \alpha \leq 1$. In this paper, we considered two different orders α

and β which satisfy $0 < \beta \leq 1 < \alpha \leq 2$. Sufficient conditions for the controllability results are established using Schauder's fixed point theorem. Also, the controllability of nonlinear fractional damped delay system with multiple delays in control are discussed. An example is provided to illustrate the theory.

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