Tail Asymptotics for the Waiting Time in an M/G/1/ROS Vacation Queue with Regularly-Varying Service

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Abstract. We study the asymptotic behavior of tail probability for the waiting time in the steady-state $M/G/1/ROS$ multiple-vacation queue with regularly-varying service time and vacation time distributions. Conditioning on the server being busy or on vacation, the asymptotic conditional tail probabilities are obtained explicitly. We also verify that the waiting-time tail for $M/G/1/ROS$ queue with multiple-vacation is asymptotically equivalent to that for the standard $M/G/1/ROS$ queue (without vacation), as long as the vacation time has a tail probability lighter than the service time.

Keywords: $M/G/1$ queue · Random order of service · Vacation · Waiting time · Tail asymptotics

1 Introduction

Triggered by the desire for measuring the quality of service (QoS) in modern communication networks (see, e.g., [14] and [15]), there has been much interest in studying the asymptotic behaviors for queues with heavy-tailed service time distributions. The tail asymptotics for queueing quantities, such as queue length and waiting time, is of fundamental importance due to the stringent QoS requirements often requiring these tail probabilities to be significantly small.

Queueing systems with vacations are a type of very important queueing systems, which find many applications in abroad range of areas, e.g., production, computer, and communication systems. A variety of queues with vacations have been extensively studied for more than 40 years. Literature reviews on vacation queues can be found in, e.g., the survey [8] and the book [17].

In this paper, we are interested in the asymptotic behavior of tail probability for the stationary waiting time in the $M/G/1$ queue with multiple-vacation and random order service (ROS) discipline. The customers are assumed to arrive

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according to a Poisson process with rate λ. The service time T_β is assumed to be i.i.d. r.v.'s having the distribution $F_\beta(t)$ with $F_\beta(0) = 0$ and mean $\beta_1 < \infty$. Each time a service is completed and the system is not empty, the next customer to be served is selected at random from all the customers waiting in the queue. Each time a busy period ends and system becomes empty, the server undergoes a vacation of random length of time T_α. Whenever the server returns from a vacation and finds one or more customers waiting, the server goes on serving a customer immediately, otherwise, on return from a vacation, the server finds no customer waiting, the server takes on a vacation again. The generic vacation time T_α is assumed to have the distribution $F_\alpha(t)$ with $F_\alpha(0) = 0$ and mean $\alpha_1 < \infty$. Besides, we use the notations $\alpha(s)$ and $\beta(s)$ to represent the Laplace-Stieltjes (LS) transforms of $F_\alpha(t)$ and $F_\beta(t)$, respectively. It is well known that the system is stable if and only if (iff) $\rho = \lambda\beta_1 < 1$, which is assumed to hold throughout this paper.

There are many references on asymptotic analysis for queueing systems with heavy-tailed distributions, e.g., Asmussen, Klüppelberg and Sigman [1], Boxma and Denisov [4], and more references can be found in two excellent surveys: Borst et al. [3], and Boxma and Zwart [6]. As far as the ROS discipline concerned, we refer readers to [5] and [12]. Under the assumption of regularly-varying service time distribution, Borst, et al. [5] and Kim, et al. [12] obtained asymptotic expressions for the waiting time distributions in the ordinary $M/G/1$ queue (without vacation) and the $M/G/1$ queue with retrials, respectively.

Our focus in this paper is to study the asymptoic behavior for the tail probability of the waiting time in the $M/G/1/ROS$ vacation queue with regularly-varying service time and vacation time distributions, which is one of typical and commonly used heavy-tailed distributions. Conditioning on the server being busy or on vacation, the asymptotic conditional tail probabilities are obtained explicitly. As a side product (Remark 2) of main results obtained in this paper, we verify that the waiting-time tail for $M/G/1/ROS$ queue with multiple vacation is asymptotically equivalent to that for the standard $M/G/1/ROS$ queue (without vacation), as long as the vacation time has a tail probability lighter than the service time.

The rest of the paper is organized as follows: Sect. 2 provides preliminaries to facilitate our analysis. In Sects. 3 and 4, we study the asymptotic behaviors for the conditional tail probabilities of waiting time conditioning on the server being busy and on vacation, respectively.

2 Preliminary

In this section, we present some definitions, notations and useful literature results, which will be used in later sections

Definition 1 (Bingham, Goldie and Teugels [2]). *A measurable function* $U : (0, \infty) \to (0, \infty)$ *is regularly varying at* ∞ *with index* $\sigma \in (-\infty, \infty)$ *(written* $U \in \mathcal{R}_\sigma$*) iff* $\lim_{t\to\infty} U(xt)/U(t) = x^\sigma$ *for all* $x > 0$. *If* $\sigma = 0$ *we call* U *slowly varying, i.e.,* $\lim_{t\to\infty} U(xt)/U(t) = 1$ *for all* $x > 0$.

We will use $L(t)$ to represent a slowly varying function at ∞ (see Definition 1) and make the following basic assumptions on the service time T_β and the vacation time T_α:

A1. *The service time T_β has tail probability $P\{T_\beta > t\} \sim t^{-b}L(t)$ as $t \to \infty$, where $1 < b < 2$.*

A2. *The vacation time T_α has tail probability $P\{T_\alpha > t\} \sim \gamma P\{T_\beta > t\}$ as $t \to \infty$, where $\gamma \geq 0$.*

Remark 1. When $\gamma = 0$, Assumption A2 is to be interpreted as $P\{T_\alpha > t\} = o\big(P\{T_\beta > t\}\big)$, which means that the vacation time has a tail probability lighter than the service time. When $\gamma > 0$, two tail probabilities are asymptotically equivalent up to a prefactor γ.

Let T_π be the busy period of the standard $M/G/1$ queue with arrival rate λ and service time T_β. It is well known that $\pi_1 \overset{\text{def}}{=} E(T_\pi) = \beta_1/(1-\rho)$. By $\pi(s)$, we denote the LS transform of the probability distribution function of T_π. Under Assumption A1, the tail probability $P\{T_\pi > t\}$ is regularly varying according to de Meyer and Teugels [7]:

$$P\{T_\pi > t\} \sim \frac{1}{(1-\rho)^{b+1}} \cdot t^{-b}L(t) \quad \text{as } t \to \infty. \tag{1}$$

Let $F_\beta^{(e)}(t)$ be the so-called equilibrium distribution of $F_\beta(t)$, which is defined as $F_\beta^{(e)}(t) = \beta_1^{-1}\int_0^t (1 - F_\beta(x))dx$. Similarly, we define $F_\alpha^{(e)}(t) = \alpha_1^{-1}\int_0^t (1 - F_\alpha(x))dx$ and $F_\pi^{(e)}(t) = \pi_1^{-1}\int_0^t (1 - F_\pi(x))dx$. Denote by $\beta^{(e)}(s)$, $\alpha^{(e)}(s)$ and $\pi^{(e)}(s)$ the LS transforms of $F_\beta^{(e)}(t)$, $F_\alpha^{(e)}(t)$ and $F_\pi^{(e)}(t)$, respectively.

By Karamata's theorem (e.g., p. 28 in Bingham, Goldie and Teugels [2]) and Assumptions A1 and A2, we know that $1 - F_\beta^{(e)}(t) \sim c_\beta t^{-b+1}L(t)$, $1 - F_\alpha^{(e)}(t) \sim c_\alpha t^{-b+1}L(t)$ and $1 - F_\pi^{(e)}(t) \sim c_\pi t^{-b+1}L(t)$ as $t \to \infty$, where

$$c_\beta = \frac{1}{(b-1)\beta_1}, \tag{2}$$

$$c_\alpha = \frac{\gamma}{(b-1)\alpha_1}, \tag{3}$$

$$c_\pi = \frac{1}{(b-1)\pi_1} \cdot \frac{1}{(1-\rho)^{b+1}} = \frac{c_\beta}{(1-\rho)^b}. \tag{4}$$

Applying Theorem 8.1.6, p. 333 in Bingham et al. [2], we further obtain the asymptotic properties for LS transforms $\beta^{(e)}(s)$, $\alpha^{(e)}(s)$ and $\pi_0^{(e)}(s)$:

$$1 - \beta^{(e)}(s) = c_\beta c(b) s^{b-1} L(1/s)(1 + o(1)) \qquad s \downarrow 0, \tag{5}$$

$$1 - \alpha^{(e)}(s) = c_\alpha c(b) s^{b-1} L(1/s)(1 + o(1)) \qquad s \downarrow 0, \tag{6}$$

$$1 - \pi^{(e)}(s) = c_\pi c(b) s^{b-1} L(1/s)(1 + o(1)) \qquad s \downarrow 0, \tag{7}$$

where $c(b) = \Gamma(b-1)\Gamma(2-b)/\Gamma(b-1)$.

Let W be the waiting time of a generic customer, W_b and W_v be two r.v.s whose probability distributions coincide with the conditional probability distributions of W given that the generic customer finds the server busy and on vacation upon its arrival, respectively. Precisely, $P\{W_b \le t\} = P\{W \le t|\text{busy}\}$ and $P\{W_v \le t\} = P\{W \le t|\text{vacation}\}$. Therefore, the probability distributions $P\{W_b \le t\}$ and $P\{W_v \le t\}$ have the LS transforms $W_b(s) \stackrel{\text{def}}{=} E(e^{-sW}|\text{busy})$ and $W_v(s) \stackrel{\text{def}}{=} E(e^{-sW}|\text{vacation})$, respectively. Our starting point for tail asymptotic analysis on $P\{W_b > t\}$ and $P\{W_v > t\}$ is based on the expressions for $W_b(s)$ and $W_v(s)$, which can be found in [17].

$$W_b(s) = \frac{1-\rho}{\alpha_1 \rho s} \int_{\pi(s)}^{1} \frac{\big[1-\alpha(\lambda-\lambda u)\big]\big[\beta(\lambda-\lambda u)-\beta(s+\lambda-\lambda u)\big]}{\big[\beta(\lambda-\lambda u)-u\big]\big[u-\beta(s+\lambda-\lambda u)\big]} \cdot \exp\{-G(s,u)\}\, du, \tag{8}$$

$$W_v(s) = \frac{1}{\alpha_1 s} \int_{\pi(s)}^{1} \frac{\alpha(\lambda-\lambda u)-\alpha(s+\lambda-\lambda u)}{u-\beta(s+\lambda-\lambda u)} \cdot \exp\{-G(s,u)\}\, du, \tag{9}$$

where

$$G(s,u) = \int_u^1 \frac{1}{v-\beta(s+\lambda-\lambda v)} dv. \tag{10}$$

3 Asymptotic Tail Probability of W_b

In this section, we are going to derive the asymptotic expression of $P\{W_b > t\}$ as $t \to \infty$ based on $W_b(s)$, the LS transform of distribution function $P\{W_b \le t\}$. Let us rewrite (8) as follows

$$W_b(s) = \frac{1-\rho}{\rho s} \int_{\pi(s)}^{1} \alpha^{(e)}(\lambda-\lambda u) \cdot \Big[\frac{\lambda-\lambda u}{\beta(\lambda-\lambda u)-u} + \frac{\lambda-\lambda u}{u-\beta(s+\lambda-\lambda u)}\Big] \cdot \exp\{-G(s,u)\}\, du. \tag{11}$$

Setting $u = u(t) = 1 - st/\lambda$ in (11), and noting that

$$\frac{\lambda-\lambda\pi(s)}{s} = \frac{\rho}{1-\rho} \cdot \frac{1-\pi(s)}{sE(T_\pi)} = \frac{\rho}{1-\rho}\pi^{(e)}(s), \tag{12}$$

we get that

$$W_b(s) = \frac{1-\rho}{\rho} \int_0^{\frac{\rho}{1-\rho}\pi^{(e)}(s)} \alpha^{(e)}(st)\Big[\frac{st}{st-\lambda+\lambda\beta(st)} + \frac{st}{-st+\lambda-\lambda\beta(s+st)}\Big] \cdot \exp\{-G(s, 1-st/\lambda)\}dt. \tag{13}$$

It follows from (10) that

$$G(s,1-st/\lambda) = \int_{1-(st/\lambda)}^{1} \frac{1}{v-\beta(s+\lambda-\lambda v)}dv = \int_0^1 \frac{st}{-stw+\lambda-\lambda\beta(s+stw)}dw = \int_0^1 \frac{1}{D(s,t,w)}dw, \tag{14}$$

where

$$D(s,t,w) = \frac{-stw+\lambda-\lambda\beta(s+stw)}{st} = -w+\rho(w+1/t)\beta^{(e)}(s+stw). \tag{15}$$

Let

$$B_1(s,t) = \frac{st}{st-\lambda+\lambda\beta(st)} = \frac{1}{1-\rho\beta^{(e)}(st)}, \tag{16}$$

$$B_2(s,t) = \frac{st}{-st+\lambda-\lambda\beta(s+st)} = \frac{1}{D(s,t,1)}, \tag{17}$$

$$C(s,t) = \exp\left\{-\int_0^1 \frac{1}{D(s,t,w)}dw\right\}, \tag{18}$$

$$g(s,t) = \alpha^{(e)}(st)\big(B_1(s,t)+B_2(s,t)\big)C(s,t). \tag{19}$$

Then we can further rewrite (13) as

$$W_b(s) = \frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}\pi^{(e)}(s)} g(s,t)dt = W_{b_1}(s) - W_{b_2}(s) \tag{20}$$

where

$$W_{b_1}(s) = \frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}} g(s,t)dt, \qquad W_{b_2}(s) = \frac{1-\rho}{\rho}\int_{\frac{\rho}{1-\rho}\pi^{(e)}(s)}^{\frac{\rho}{1-\rho}} g(s,t)dt. \tag{21}$$

In following subsections, we are going to discuss the asymptotic properties for $W_{b_1}(s)$ and $W_{b_2}(s)$ as $s \downarrow 0$, which will be used to obtain the asymptotics of $P\{W_b > t\}$ as $t \to \infty$ later. For this purpose, we start with studying the asymptotic behaviors for $B_1(s,t)$, $B_2(s,t)$ and $C(s,t)$ as $s \downarrow 0$.

3.1 Asymptotic Properties for $B_1(s,t)$, $B_2(s,t)$ and $C(s,t)$ as $s \downarrow 0$

It follows from (16) and (5) that

$$B_1(s,t) = \frac{1}{1-\rho+\rho(1-\beta^{(e)}(st))} = \frac{1}{1-\rho}\cdot\frac{1}{1+\dfrac{\rho}{1-\rho}\beta_0^{(e)}(st)} = \frac{1}{1-\rho} - \frac{c_\beta c(b)\rho}{(1-\rho)^2}(st)^{b-1}L(1/s)(1+o(1)). \tag{22}$$

where we have used the fact that $1/(1-x) = 1 + x + x^2 + \cdots$ for $|x| < 1$.

It follows from (15) and (5) that

$$\begin{aligned} D(s,t,w) &= -w + \rho w + \rho/t - \rho(w+1/t)\beta_0^{(e)}(s+stw) \\ &= (-w+\rho w+\rho/t)\left[1 - \frac{\rho(w+1/t)\beta_0^{(e)}(s+stw)}{-w+\rho w+\rho/t}\right] \\ &= (-w+\rho w+\rho/t) \\ &\quad \cdot\left[1 - c_\beta c(b)\rho\frac{(1/t)(1+tw)^b}{-w+\rho w+\rho/t}\cdot s^{b-1}L(1/s)(1+o(1))\right], \end{aligned} \tag{23}$$

which implies that

$$\begin{aligned} \frac{1}{D(s,t,w)} &= \frac{1}{-w+\rho w+\rho/t} \\ &\quad + c_\beta c(b)\rho\frac{(1/t)(1+tw)^b}{(-w+\rho w+\rho/t)^2}\cdot s^{b-1}L(1/s)(1+o(1)). \end{aligned} \tag{24}$$

Therefore, by (18),

$$C(s,t) = \varphi(t)\cdot\exp\left\{-c_\beta c(b)\rho H(t)s^{b-1}L(1/s)(1+o(1))\right\}, \tag{25}$$

where

$$\varphi(t) = \exp\left\{-\int_0^1 \frac{1}{-w+\rho w+\rho/t}dw\right\} = \left(\frac{-t+\rho t+\rho}{\rho}\right)^{\frac{1}{1-\rho}}, \tag{26}$$

$$H(t) = \int_0^1 \frac{(1/t)(1+tw)^b}{(-w+\rho w+\rho/t)^2}dw. \tag{27}$$

Note that the fact that $e^{-x} = 1 - x + (-x)^2/2! + \cdots$. Then (25) yields

$$C(s,t) = \varphi(t)\left[1 - c_\beta c(b)\rho H(t)s^{b-1}L(1/s)(1+o(1))\right]. \tag{28}$$

In addition, by (17) and (24), we get

$$B_2(s,t) = \frac{1}{-1+\rho+\rho/t} + c_\beta c(b)\rho\frac{(1/t)(1+t)^b}{(-1+\rho+\rho/t)^2}s^{b-1}L(1/s)(1+o(1)). \tag{29}$$

3.2 Asymptotic Property for $W_{b_1}(s)$ as $s \downarrow 0$

It follows from (22) and (29) that

$$B_1(s,t) + B_2(s,t) = \psi(t) - c_\beta c(b)\rho K(t)s^{b-1}L(1/s)(1+o(1)), \tag{30}$$

where

$$\psi(t) = \frac{1}{1-\rho} \cdot \frac{\rho}{-t+\rho t+\rho}. \tag{31}$$

$$K(t) = K_0(t) - K_1(t) \tag{32}$$

$$K_0(t) = \frac{t^{b-1}}{(1-\rho)^2}, \tag{33}$$

$$K_1(t) = \frac{(1/t)(1+t)^b}{(-1+\rho+\rho/t)^2}. \tag{34}$$

By (19), (6), (28) and (30), we know

$$g(s,t) = \varphi(t)\psi(t) - \Big[c_\alpha c(b) t^{b-1} \varphi(t)\psi(t) + c_\beta c(b) \rho \varphi(t)\big(K(t) + \psi(t)H(t)\big)\Big] \cdot s^{b-1} L(1/s)(1+o(1)). \tag{35}$$

Recalling the expression of $W_{b_1}(s)$ in (21), along with (35), we can write

$$W_{b_1}(s) = \frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)dt - \Big[c_\alpha c(b)\frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}} t^{b-1}\varphi(t)\psi(t)dt + c_\beta c(b)(1-\rho)\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\Big(K(t)+\psi(t)H(t)\Big)dt\Big] \cdot s^{b-1}L(1/s)(1+o(1)). \tag{36}$$

In the following, we are going to calculate the integrals in (36). By (26) and (31),

$$\varphi(t)\psi(t) = \frac{1}{1-\rho}\Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{1}{1-\rho}-1}, \tag{37}$$

hence

$$\frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)dt = \frac{1}{1-\rho}\int_0^1 (1-x)^{\frac{1}{1-\rho}-1}dx \;=\; 1, \tag{38}$$

$$\frac{1-\rho}{\rho}\int_0^{\frac{\rho}{1-\rho}} t^{b-1}\varphi(t)\psi(t)dt = \frac{1}{\rho}\Big(\frac{\rho}{1-\rho}\Big)^b \int_0^1 (1-x)^{\frac{1}{1-\rho}-1}x^{b-1}dx. \tag{39}$$

By (26) and (33),

$$\int_0^{\frac{\rho}{1-\rho}} \varphi(t)K_0(t)dt = \frac{1}{(1-\rho)^2}\int_0^{\frac{\rho}{1-\rho}} \Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{1}{1-\rho}} \cdot t^{b-1}dt = \frac{1}{(1-\rho)^2}\Big(\frac{\rho}{1-\rho}\Big)^b\int_0^1 (1-x)^{\frac{1}{1-\rho}} \cdot x^{b-1}dx. \tag{40}$$

By (26) and (34),

$$\int_0^{\frac{\rho}{1-\rho}} \varphi(t)K_1(t)dt = \int_0^{\frac{\rho}{1-\rho}} \Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{1}{1-\rho}} \cdot \frac{t(1+t)^b}{(-t+\rho t+\rho)^2}dt = \frac{1}{(1-\rho)^2}\int_0^1 (1-x)^{\frac{1}{1-\rho}-2} \cdot x\Big(1+\frac{\rho}{1-\rho}x\Big)^b dx. \tag{41}$$

By (37) and (27),

$$
\begin{aligned}
&\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)H(t)dt \\
&= \int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)\cdot\Big(\int_0^t \frac{(1+y)^b}{(-y+\rho y+\rho)^2}dy\Big)dt \\
&= \frac{1}{(1-\rho)}\int_0^{\frac{\rho}{1-\rho}} \frac{(1+y)^b}{(-y+\rho y+\rho)^2}\Big[\int_y^{\frac{\rho}{1-\rho}} \Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{\rho}{1-\rho}} dt\Big]dy \\
&= \frac{1}{(1-\rho)\rho}\int_0^{\frac{\rho}{1-\rho}} \Big(\frac{-y+\rho y+\rho}{\rho}\Big)^{\frac{\rho}{1-\rho}-1}(1+y)^b dy \\
&= \frac{1}{(1-\rho)^2}\int_0^1 (1-x)^{\frac{1}{1-\rho}-2}\Big(1+\frac{\rho}{1-\rho}x\Big)^b dx.
\end{aligned}
\tag{42}
$$

Noting that $K(t)$ is given in (32) and substituting (38)–(42) into (36), we obtain

$$
W_{b_1}(s) = 1 - d_b c(b)\cdot s^{b-1}L(1/s)(1+o(1)),
\tag{43}
$$

where

$$
\begin{aligned}
d_b = \Big(\frac{c_\alpha}{\rho}+\frac{c_\beta}{1-\rho}\Big)\Big(\frac{\rho}{1-\rho}\Big)^b \int_0^1 \Big(1-x\Big)^{\frac{1}{1-\rho}}\cdot x^{b-1}dx \\
+\frac{c_\beta}{1-\rho}\int_0^1 \Big(1-x\Big)^{\frac{1}{1-\rho}-1}\Big(1+\frac{\rho}{1-\rho}x\Big)^b dx.
\end{aligned}
\tag{44}
$$

3.3 Asymptotic Property for $W_{b_2}(s)$ as $s \downarrow 0$

Recall (21). By the integration middle value theorem, there exists $\xi(s)\in(0,1)$ such that

$$
W_{b_2}(s) = g\big(s,h(s)\big)\Big(1-\pi^{(e)}(s)\Big),
\tag{45}
$$

where $h(s) = \frac{\rho}{1-\rho}\Big[\pi^{(e)}(s)+\xi(s)(1-\pi^{(e)}(s))\Big]$. It follows from (7) that

$$
h(s) = \frac{\rho}{1-\rho}\Big(1+O(1)\cdot s^{b-1}L(1/s)\Big).
\tag{46}
$$

Next, we will prove that $g\big(s,h(s)\big)=o(1)$. By (35), we have

$$
\begin{aligned}
g\big(s,h(s)\big) = \varphi(h(s))\psi(h(s)) - \Big[c_\alpha c(b)(h(s))^{b-1}\varphi(h(s))\psi(h(s)) \\
+c_\beta c(b)\rho\varphi(h(s))\Big(K(h(s))+\psi(h(s))H(h(s))\Big)\Big] \\
\cdot s^{b-1}L(1/s)(1+o(1)).
\end{aligned}
\tag{47}
$$

By (46), we know that $-h(s)+\rho h(s)+\rho = O(1)\cdot s^{b-1}L(1/s)$, which together with (26) and (37) leads to

$$\varphi(h(s)) = O(1)\cdot s^{(b-1)/(1-\rho)}\big(L(1/s)\big)^{\frac{1}{1-\rho}}, \tag{48}$$

$$\varphi\big(h(s)\big)\psi\big(h(s)\big) = O(1)\cdot s^{(b-1)(\frac{1}{1-\rho}-1)}\big(L(1/s)\big)^{\frac{1}{1-\rho}-1}. \tag{49}$$

Because $\lim_{s\to 0} h(s) = \rho/(1-\rho)$, it follows from (33), (34) and (27) that

$$K_0(h(s)) = \frac{(h(s))^{b-1}}{(1-\rho)^2} = O(1), \tag{50}$$

$$K_1(h(s)) = \frac{(1/h(s))(1+h(s))^b}{(-1+\rho+\rho/h(s))^2} = O(1)\cdot\Big(\frac{1}{s^{b-1}L(1/s)}\Big)^2, \tag{51}$$

$$H(h(s)) = \int_0^1 \frac{(1/h(s))(1+h(s)w)^b}{(-w+\rho w+\rho/h(s))^2}dw = O(1). \tag{52}$$

Further, by (48)–(52),

$$\varphi(h(s))K(h(s))s^{b-1}L(1/s) = o(1), \tag{53}$$

$$\varphi\big(h(s)\big)\psi\big(h(s)\big)H(h(s))s^{b-1}L(1/s) = o(1). \tag{54}$$

It follows from (47), (49), (53) and (54) that $g\big(s,h(s)\big) = o(1)$, which, together with (45) and (7), results in

$$W_{b_2}(s) = o(1)\cdot s^{b-1}L(1/s). \tag{55}$$

3.4 Tail Probability Asymptotics for W_b

By (20), (43) and (55),

$$W_b(s) = 1 - c(b)d_b s^{b-1}L(1/s)(1+o(1)). \tag{56}$$

Applying Theorem 8.1.6, p. 333 in Bingham et al. [2], we obtain

$$P\{W_b > t\} = P\{W > t|\text{busy}\} \sim d_b t^{-b+1}L(t) \qquad \text{as } t\to\infty. \tag{57}$$

where d_b is given in (44).

4 Asymptotic Tail Probability of W_v

In this section, we are going to derive the asymptotic expression of $P\{W_v > t\}$ as $t\to\infty$ based on $W_v(s)$ given in (9). Let

$$A(s,t) = \frac{\alpha(st)-\alpha(s+st)}{\alpha_1 st} \;=\; (1+1/t)\alpha^{(e)}(s+st) - \alpha^{(e)}(st), \tag{58}$$

$$g_v(s,t) = A(s,t)B_2(s,t)C(s,t). \tag{59}$$

Similar to derivation of (20), we get that

$$W_v(s) = \int_0^{\frac{\rho}{1-\rho}\pi^{(e)}(s)} g_v(s,t)dt \;=\; W_{v1}(s) - W_{v2}(s), \tag{60}$$

where

$$W_{v1}(s) = \int_0^{\frac{\rho}{1-\rho}} g_v(s,t)dt, \qquad W_{v2}(s) = \int_{\frac{\rho}{1-\rho}\pi^{(e)}(s)}^{\frac{\rho}{1-\rho}} g_v(s,t)dt. \tag{61}$$

In following subsections, we will discuss the asymptotic behavior of $W_{v_1}(s)$ and $W_{v_2}(s)$ as $s \downarrow 0$, which will be used later to obtain the asymptotics of $P\{W_v > t\}$ as $t \to \infty$.

4.1 Asymptotic Property for $W_{v_1}(s)$ as $s \downarrow 0$

It follows from (6) and (58) that

$$A(s,t) = \frac{1}{t}\Big[1 - c_\alpha c(b)\big((1+t)^b - t^b\big)s^{b-1}L(1/s)(1+o(1))\Big], \tag{62}$$

which, together with (29), implies that

$$\begin{aligned} A(s,t)B_2(s,t) &= \frac{1}{-t+\rho t+\rho}\Big[1 - \Big(c_\alpha c(b)\big((1+t)^b - t^b\big) - c_\beta c(b)\rho\frac{(1+t)^b}{-t+\rho t+\rho}\Big) \\ &\qquad \cdot s^{b-1}L(1/s)(1+o(1))\Big] \\ &= \frac{1-\rho}{\rho}\psi(t)\Big[1 - c(b)\rho\Big(c_\alpha R_1(t) - c_\beta R_2(t)\Big) \\ &\qquad \cdot s^{b-1}L(1/s)(1+o(1))\Big]. \end{aligned} \tag{63}$$

where

$$R_1(t) = \frac{(1+t)^b - t^b}{\rho}, \tag{64}$$

$$R_2(t) = \frac{(1+t)^b}{-t+\rho t+\rho}. \tag{65}$$

By (63) and (28), we obtain

$$\begin{aligned} g_v(s,t) = \varphi(t)\psi(t)\Big[\frac{1-\rho}{\rho} - c(b)(1-\rho)\Big(c_\beta H(t) + c_\alpha R_1(t) - c_\beta R_2(t)\Big) \\ \cdot s^{b-1}L(1/s)(1+o(1))\Big]. \end{aligned} \tag{66}$$

Recalling the expression of $W_{v_1}(s)$ given in (61), along with (38) and (66), we can write

$$\begin{aligned} W_{v_1}(s) = 1 - c(b)(1-\rho)\Big[\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)\Big(c_\beta H(t) + c_\alpha R_1(t) - c_\beta R_2(t)\Big)dt\Big] \\ \cdot s^{b-1}L(1/s)(1+o(1)). \end{aligned} \tag{67}$$

Next, let us calculate the integrals in (67). By (37) and (64),

$$\begin{aligned}
&\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)R_1(t)dt \\
&= \frac{1}{\rho(1-\rho)} \int_0^{\frac{\rho}{1-\rho}} \Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{1}{1-\rho}-1} \cdot \Big[(1+t)^b - t^b\Big]dt \\
&= \frac{1}{(1-\rho)^2} \int_0^1 (1-x)^{\frac{1}{1-\rho}-1} \Big[\Big(1+\frac{\rho x}{1-\rho}\Big)^b - \Big(\frac{\rho x}{1-\rho}\Big)^b\Big]dx. \qquad (68)
\end{aligned}$$

By (37) and (65),

$$\begin{aligned}
\int_0^{\frac{\rho}{1-\rho}} \varphi(t)\psi(t)R_2(t)dt &= \frac{1}{1-\rho} \int_0^{\frac{\rho}{1-\rho}} \Big(\frac{-t+\rho t+\rho}{\rho}\Big)^{\frac{1}{1-\rho}-1} \cdot \frac{(1+t)^b}{-t+\rho t+\rho}dt \\
&= \frac{1}{(1-\rho)^2} \int_0^1 (1-x)^{\frac{1}{1-\rho}-2} \cdot \Big(1+\frac{\rho}{1-\rho}x\Big)^b dx. \qquad (69)
\end{aligned}$$

Substituting (68), (69) and (42) into (67), we obtain

$$W_{v_1}(s) = 1 - c(b)d_v \cdot s^{b-1}L(1/s)(1+o(1)), \qquad (70)$$

where

$$d_v = \frac{c_\alpha}{1-\rho} \int_0^1 (1-x)^{\frac{1}{1-\rho}-1} \Big[\Big(1+\frac{\rho x}{1-\rho}\Big)^b - \Big(\frac{\rho x}{1-\rho}\Big)^b\Big]dx. \qquad (71)$$

4.2 Asymptotic Property for $W_{v_2}(s)$ as $s \downarrow 0$

Recall (61). By the integration middle value theorem, there exists $\xi_v(s) \in (0,1)$ such that

$$W_{v_2}(s) = g_v\big(s, h_v(s)\big)\Big(1 - \pi^{(e)}(s)\Big), \qquad (72)$$

where $h_v(s) = \frac{\rho}{1-\rho}\Big[\pi^{(e)}(s) + \xi_v(s)(1-\pi^{(e)}(s))\Big]$. Further, by (7),

$$h_v(s) = \frac{\rho}{1-\rho}\Big[1 + O(1) \cdot s^{b-1}L(1/s)\Big]. \qquad (73)$$

Next, we will prove that $g_v\big(s, h_v(s)\big) = o(1)$. By (66), we have

$$\begin{aligned}
&g_v\big(s, h_v(s)\big) \\
&= \varphi(h_v(s))\psi(h_v(s))\Big[\frac{1-\rho}{\rho} - c(b)(1-\rho) \\
&\quad \cdot\Big(c_\beta H(h_v(s)) + c_\alpha R_1(h_v(s)) - c_\beta R_2(h_v(s))\Big)s^{b-1}L(1/s)(1+o(1))\Big]. \qquad (74)
\end{aligned}$$

Immediately, by (73), we know that $-h_v(s) + \rho h_v(s) + \rho = O(1) \cdot s^{b-1}L(1/s)$, which together with (37) leads to

$$\varphi\big(h_v(s)\big)\psi\big(h_v(s)\big) = O(1) \cdot s^{(b-1)(\frac{1}{1-\rho}-1)}\big(L(1/s)\big)^{\frac{1}{1-\rho}-1}. \qquad (75)$$

Because $\lim_{s\to 0} h_v(s) = \rho/(1-\rho)$, it follows from (64) and (65) that

$$R_1(h_v(s)) = \frac{(1+h_v(s))^b - (h_v(s))^b}{\rho} \;=\; O(1), \tag{76}$$

$$R_2(h_v(s)) = \frac{(1+h_v(s))^b}{-h_v(s)+\rho h_v(s)+\rho} \;=\; O(1)\cdot\Big(\frac{1}{s^{b-1}L(1/s)}\Big), \tag{77}$$

$$H(h_v(s)) = \int_0^1 \frac{(1/h_v(s))(1+h_v(s)w)^b}{(-w+\rho w+\rho/h_v(s))^2}dw \;=\; O(1). \tag{78}$$

Further, by (75) and (77),

$$\varphi(h_v(s))\psi(h_v(s))R_2(h_v(s))s^{b-1}L(1/s) = o(1). \tag{79}$$

It follows from (74)–(76) and (78)–(79) that $g_v\big(s, h_v(s)\big) = o(1)$, which, together with (72) and (7), results in

$$W_{v_2}(s) = o(1)\cdot s^{b-1}L(1/s). \tag{80}$$

4.3 Tail Probability Asymptotics for W_v and W

By (60), (70) and (80)

$$W_v(s) = 1 - c(b)d_v\cdot s^{b-1}L(1/s)(1+o(1)). \tag{81}$$

Applying Theorem 8.1.6, p. 333 in Bingham et al. [2], we obtain

$$P\{W>t|\text{vacation}\} = P\{W_v>t\} \sim d_v\cdot t^{-b+1}L(t) \qquad \text{as } t\to\infty. \tag{82}$$

where d_v is given in (71).

Note that $P\{W>t\} = \rho P\{W_b>t\} + (1-\rho)P\{W_v>t\}$. By (57) and (82), we have

$$P\{W>t\} \sim \Big(\rho d_b + (1-\rho)d_v\Big)\cdot t^{-b+1}L(t) \qquad \text{as } t\to\infty. \tag{83}$$

A special case: $\gamma = 0$**.**
This is the case when the vacation time T_α has a tail lighter than the service time T_β, in which $c_\alpha = 0$. Thus, by (83), (44) and (71),

$$P\{W>t\} \sim \frac{\rho}{1-\rho}c_\beta c_W\cdot t^{-b+1}L(t) \qquad \text{as } t\to\infty. \tag{84}$$

where

$$\begin{aligned} c_W = \Big(\frac{\rho}{1-\rho}\Big)^b \int_0^1 \Big(1-x\Big)^{\frac{1}{1-\rho}}\cdot x^{b-1}dx \\ + \int_0^1 \Big(1-x\Big)^{\frac{1}{1-\rho}-1}\Big(1+\frac{\rho}{1-\rho}x\Big)^b dx. \end{aligned} \tag{85}$$

Remark 2. In [5], Boxma et al. (2004) have shown that the asymptotic result (84), along with (85) is true for the standard $M/G/1/ROS$ queue (without vacation). As one of side products in this paper, we have verified that such a result is still valid even for the $M/G/1/ROS$ queue with multiple vacations, as long as the vacation time has a tail probability lighter than the service time.

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