Analytical Solution for Two-Dimensional Axisymmetric Thermoelastic Behavior in the Multilayer Composite Hollow Sphere

N. J. Wange, S. P. Pawar and M. N. Gaikwad

Abstract This article deals with an analytic solution of temperature distribution, displacement and stress distribution function for two-dimensional multilayered hollow spheres. The solution is obtained by using the separation of the variable method. Homogenous boundary conditions of the first or second kind can be applied on surfaces of $\theta =$ constant. However, homogeneous boundary conditions of the third kind (convection) are used in the r-direction. Under prescribed conditions, the temperature distribution, displacement and thermal stresses in the sphere are to be analyzed under the steady-state temperature field. The layers of the multilayer sphere are homogeneous and isotropic.

Keywords Composite hollow sphere · Temperature distribution · Displacement and thermal stresses · Steady state

1 Introduction

The temperature effect on thermal stresses in the composite regions consisting of several layers have numerous applications in engineering, Technology, manufacturing fields, etc. The increasing use of multilayer composite materials in engineering applications has resulted in considerable research activity in recent years. The use of composite materials of multilayer type has been tremendous in many engineering

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fields such as aerospace, automobiles, chemical and energy, civil and infrastructure, sports and recreation, biomedical engineering, and so on.

An understanding of thermally induced stresses in multilayer isotropic bodies is essential for a comprehensive study of their response due to exposure to a temperature field, which may, in turn, occur in service or during the manufacturing stages.

The Laplace transform technique is used by Carslaw and Jaeger [1], they discussed the infinite composite of two different medium and obtained the temperature distribution. They solved the transient boundary value problem of heat conduction in solids consisting of many parallel layers. In practice, if the number of layers is more than two, the inverse of the Laplace transform becomes quite difficult. The Adjoin-solution technique which has been introduced by Goodman [2] provides a method of solution to a large class of heat conduction problems in composite slabs from the solution but one adjoin problem. The primary disadvantage of the Adjoin-solution method is that only the solutions of the boundaries (i.e., interface) of the layers can be determined. Recently Tittle [3] introduced a technique for orthogonal expansion of functions over a one-dimensional multilayer region. The method essentially is an extension of Sturm–Liouville problem to the case of one-dimensional multilayer region and it has the advantage on other analytic methods is that its application to the solution of the boundary value problem of heat conduction is relatively simple. Bulavin and Kashcheev [4] used the method of separation of variables and orthogonal expansion of functions over a one-dimensional multilayer region to solve the transient heat conduction problem involving distributed volume heat sources in a multilayer region.

Recently, Vollbrech [5] discussed the stress in cylindrical and spherical walls subjected to internal pressure stationary heat flow. Kandil [6] has studied the effect of steady-state temperature and pressure gradient on the compound cylinder under high pressure and temperature. Ghosn and Sabbaghian [7] investigated a one-dimensional axisymmetric quasi-static coupled thermoelasticity problem. The solution technique uses the Laplace transform. The inversion to the real domain is obtained by means of Cauchy's theorem of residues. Sherif and Anwar [8] discussed the problem of infinitely long elastic circular cylinder whose inner and outer surfaces are subjected to a known temperature and are traction free. They have neglected both the inertia term and relaxation effects. Chen and Yang [9] discussed the thermal response of one dimensional quasi-static coupled thermoelastic problem of an infinitely long cylinder composed of two different materials. They applied the Laplace transform with respect to time and used the Fourier series and matrix operation to obtain the solution. Jane and Lee [10] considered the solution by using the Laplace transform and the finite difference method. The cylinder was composed of multilayer of different materials. They obtained a solution for the temperature and thermal stress distributions in a transient state. Lee [11] studied the one dimensional quasi-static coupled thermoelastic problem of the multilayered sphere with time-dependent boundary conditions is considered. The medium is without body forces and heat generation. Laplace transform and finite difference methods are used to obtain the solution of a wide range of transient thermal stresses.

The study of all above-cited papers and other referred literature on multilayer composites with different geometries reveals that results appearing with complexities such as space and time-dependent properties. In most of the articles, the authors have discussed the heat conduction problem only. In view of these findings, there is a need to quantify the conclusions regarding the effect of temperature asymmetry in fundamental problems where these multilayer composites are homogeneous and isotropic. Pawar et al. [12] discussed the problem where the temperature and thermal stresses are discussed under surface temperature asymmetry and heat generation and obtained analytic solution. Recently, Pawar et al. [13] discussed the problem where the steady temperature distribution and stress distribution function for one-dimensional three-layered sphere subjected to asymmetric surface temperature and internal heat generation is presented. The solutions are obtained and the effects on thermal stresses due to heat generation and surface temperature asymmetry parameter in the sphere are analyzed.

This article deals with an analytic solution of temperature distribution, displacement, and stress distribution function for two-dimensional multilayered steady-state hollow spheres. The solution is obtained by using the separation of the variable method, The radial and tangential displacement inside the sphere is discussed with the help of Goodier's displacement potential and Boussinesq harmonic functions.

2 Formulation of the Problem

This work deals with the two-dimensional axisymmetric thermoelastic problem of multilayer composite hollow sphere using the quasi-static approach. Composite n-layer sphere contains an inner region $r_0 \leq r \leq r_1$, middle region $r_1 \leq r \leq r_{n-1}$, and an outer region $r_{n-1} \leq r \leq r_n$ which are in perfect thermal contact. Homogeneous boundary conditions of the first and second kind are applied to the angular surfaces of $\theta = 0$ and $\theta = \varpi$. Third kind boundary condition set on the inner radial surface $i = 1, r = r_0$ and outer $i = n, r = r_n$ radial surfaces. Under these more realistic prescribed conditions the temperature distribution, displacement and thermal stresses in the sphere are to be analyzed under the steady-state temperature field. The layers of the multilayer sphere are homogeneous and isotropic, $k^{(i)}$, $(i = 1, 2, 3)$ are the thermal conductivities of material of these layers.

3 Heat Conduction Equation

Assume two-dimensional steady-state radial temperature field the heat conduction equation in the ith layer of the composite is given as [14].

$$\frac{\partial^2 T^{(i)}(r,\theta)}{\partial r^2} + \frac{2}{r}\frac{\partial T^{(i)}(r,\theta)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial T^{(i)}(r,\theta)}{\partial\theta}\right) = 0 \tag{1}$$

$$r_0 \le r \le r_n, r_{i-1} \le r \le r_i, 1 \le i \le n, 0 \le \theta \le \varpi, \varpi < \pi$$

Boundary, interface and initial conditions as
For the inner surface of the first layer ($i = 1$)

$$k^{(1)}\frac{\partial T^{(1)}(r_0,\theta)}{\partial r} + h^{(1)}T^{(1)}(r_0,\theta) = 0 \tag{2}$$

For outer surface of the nth layer ($i = n$)

$$k^{(n)}\frac{\partial T^{(n)}(r_n,\theta)}{\partial r} + h^{(n)}T^{(n)}(r_n,\theta) = 0 \tag{3}$$

For $\theta = 0$ surface ($i = 1, 2, 3 \ldots n$)

$$T^{(i)}(r,\theta = 0) = 0 \text{ or } \frac{\partial T^{(i)}(r,\theta = 0)}{\partial\theta} = 0 \tag{4}$$

For $\theta = \varpi$ surfaces ($i = 1, 2, 3 \ldots n$)

$$T^{(i)}(r,\theta = \varpi) = 0 \text{ or } \frac{\partial T^{(i)}(r,\theta = \varpi)}{\partial\theta} = 0 \tag{5}$$

For Inner interface of the ith layer $i = 2, 3 \ldots n$

$$T^{(i)}(r_{i-1},\theta) = T^{(i-1)}(r_{i-1},\theta) \tag{6}$$

$$k^{(i)}\frac{\partial T^{(i)}(r_{i-1},\theta)}{\partial r} = k^{(i-1)}\frac{\partial T^{(i-1)}(r_{i-1},\theta)}{\partial r} \tag{7}$$

For the outer interface of the ith layer $i = 1, 2, 3 \ldots n - 1$

$$T^{(i)}(r_i,\theta) = T^{(i+1)}(r_i,\theta) \tag{8}$$

$$k^{(i)}\frac{\partial T^{(i)}(r_i,\theta,t)}{\partial r} = k^{(i+1)}\frac{\partial T^{(i+1)}(r_i,\theta,t)}{\partial r} \tag{9}$$

Initial condition is

$$T^{(i)}(r_i,\theta) = f^{(i)}(r,\theta) \tag{10}$$

4 Thermoelastic Problem

Two-dimensional problem of a spherical body, It is assumed that the body is deformed symmetrically with respect to the coordinate axis z. Making use of the spherical coordinate system (r, θ, ϕ), the force equilibrium equations in the directions of r and θ as [N. Noda].

$$\frac{\partial \sigma_{rr}^{(i)}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}^{(i)}}{\partial \theta} + \frac{1}{r}\left(2\sigma_{rr}^{(i)} - \sigma_{\theta\theta}^{(i)} - \sigma_{\phi\phi}^{(i)} + \sigma_{\theta r}^{(i)} \cot\theta\right) + F_r^{(i)} = 0 \tag{11}$$

$$\frac{\partial \sigma_{r\theta}^{(i)}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}^{(i)}}{\partial \theta} + \frac{1}{r}\left[\left(\sigma_{\theta\theta}^{(i)} - \sigma_{\phi\phi}^{(i)}\right)\cot\theta + 3\sigma_{r\theta}^{(i)}\right] + F_r^{(i)} = 0 \tag{12}$$

The constitutive equations in the spherical coordinate system, or the generalized Hooke’s law as

$$\varepsilon_{rr}^{(i)} = \frac{1}{E^{(i)}}\left[\sigma_{rr}^{(i)} - \nu^{(i)}\left(\sigma_{\theta\theta}^{(i)} + \sigma_{\phi\phi}^{(i)}\right)\right] + \alpha^{(i)}\tau^{(i)} = \frac{1}{2G^{(i)}}\left(\sigma_{rr}^{(i)} - \frac{\nu^{(i)}}{1+\nu^{(i)}}\Theta^{(i)}\right) + \alpha^{(i)}\tau^{(i)} \tag{13}$$

$$\varepsilon_{\theta\theta}^{(i)} = \frac{1}{E^{(i)}}\left[\sigma_{\theta\theta}^{(i)} - \nu^{(i)}\left(\sigma_{rr}^{(i)} + \sigma_{\phi\phi}^{(i)}\right)\right] + \alpha^{(i)}\tau^{(i)} = \frac{1}{2G^{(i)}}\left(\sigma_{\theta\theta}^{(i)} - \frac{\nu^{(i)}}{1+\nu^{(i)}}\Theta^{(i)}\right) + \alpha^{(i)}\tau^{(i)} \tag{14}$$

$$\varepsilon_{\phi\phi}^{(i)} = \frac{1}{E^{(i)}}\left[\sigma_{\phi\phi}^{(i)} - \nu^{(i)}\left(\sigma_{rr}^{(i)} + \sigma_{\theta\theta}^{(i)}\right)\right] + \alpha^{(i)}\tau^{(i)} = \frac{1}{2G^{(i)}}\left(\sigma_{\phi\phi}^{(i)} - \frac{\nu^{(i)}}{1+\nu^{(i)}}\Theta^{(i)}\right) + \alpha^{(i)}\tau^{(i)} \tag{15}$$

$$\varepsilon_{r\theta}^{(i)} = \frac{1}{2G^{(i)}}\sigma_{r\theta}^{(i)} \tag{16}$$

where $\Theta^{(i)} = \sigma_{rr}^{(i)} + \sigma_{\theta\theta}^{(i)} + \sigma_{\phi\phi}^{(i)}$. Alternative forms are

$$\begin{aligned}
\sigma_{rr}^{(i)} &= 2\mu^{(i)}\varepsilon_{rr}^{(i)} + \lambda^{(i)}e^{(i)} - \beta^{(i)}\tau^{(i)} \\
\sigma_{\theta\theta}^{(i)} &= 2\mu^{(i)}\varepsilon_{\theta\theta}^{(i)} + \lambda^{(i)}e^{(i)} - \beta^{(i)}\tau^{(i)} \\
\sigma_{\phi\phi}^{(i)} &= 2\mu^{(i)}\varepsilon_{\phi\phi}^{(i)} + \lambda^{(i)}e^{(i)} - \beta^{(i)}\tau^{(i)} \\
\sigma_{r\theta}^{(i)} &= 2\mu^{(i)}\varepsilon_{r\theta}^{(i)}
\end{aligned} \tag{17}$$

where $e^{(i)} = \varepsilon_{rr}^{(i)} + \varepsilon_{\theta\theta}^{(i)} + \varepsilon_{\phi\phi}^{(i)}$

The components of strain for an axisymmetric deformation in the spherical coordinate system are

$$
\begin{aligned}
\varepsilon_{rr}^{(i)} &= \frac{\partial u_r^{(i)}}{\partial r} \\
\varepsilon_{\theta\theta}^{(i)} &= \frac{u_r^{(i)}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(i)}}{\partial \theta} \\
\varepsilon_{\phi\phi}^{(i)} &= \frac{u_r^{(i)}}{r} + \cot\theta \frac{u_\theta^{(i)}}{r} \\
\varepsilon_{r\theta}^{(i)} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r^{(i)}}{\partial \theta} + \frac{\partial u_\theta^{(i)}}{\partial r} - \frac{u_\theta^{(i)}}{r}\right) \\
e^{(i)} &= \frac{\partial u_r^{(i)}}{\partial \theta} + 2\frac{u_r^{(i)}}{r} + \cot\theta \frac{u_\theta^{(i)}}{r} + \frac{1}{r}\frac{\partial u_\theta^{(i)}}{\partial \theta}
\end{aligned} \tag{18}
$$

Substituting Eqs. (17) and (18) into Eqs. (11) and (12) Navier's equations of thermoelasticity for axisymmetric problems may be expressed as

$$
\left(\lambda^{(i)} + 2\mu^{(i)}\right)\frac{\partial e^{(i)}}{\partial r} - \frac{2\mu^{(i)}}{r\sin\theta}\frac{\partial(\omega_\phi^{(i)}\sin\theta)}{\partial\theta} - \beta^{(i)}\frac{\partial\tau^{(i)}}{\partial r} + F_r^{(i)} = 0 \tag{19}
$$

$$
\left(\lambda^{(i)} + 2\mu^{(i)}\right)\frac{1}{r}\frac{\partial e^{(i)}}{\partial \theta} - \frac{2\mu^{(i)}}{r}\frac{\partial(r\omega_\phi^{(i)})}{\partial r} - \beta^{(i)}\frac{1}{r}\frac{\partial\tau^{(i)}}{\partial \theta} + F_\theta^{(i)} = 0 \tag{20}
$$

where $\omega_\phi^{(i)} = \frac{1}{2r}\left[\frac{\partial(ru_\theta^{(i)})}{\partial r} - \frac{\partial u_r^{(i)}}{\partial \theta}\right]$

The solution of the Navier's equations without the body force for the axisymmetric problem in the spherical coordinate system can be expressed by the Goodier's thermoelastic displacement potential $\Phi^{(i)}$ and the Boussinesq harmonic functions $\varphi^{(i)}$ and $\psi^{(i)}$

$$
\begin{aligned}
u_r^{(i)} &= \frac{\partial \Phi^{(i)}}{\partial r} + \frac{\partial \varphi^{(i)}}{\partial r} + r\cos\theta\frac{\partial\psi^{(i)}}{\partial r} - (3-4\nu^{(i)})\psi^{(i)}\cos\theta \\
u_r^{(i)} &= \frac{\partial \Phi^{(i)}}{\partial r} + \left[\frac{\partial \varphi^{(i)}}{\partial r} + r\cos\theta\frac{\partial\psi^{(i)}}{\partial r} - (3-4\nu^{(i)})\psi^{(i)}\cos\theta\right] \\
u_r^{(i)} &= \overline{u_r^{(i)}} + \overline{\overline{u_r^{(i)}}}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
u_\theta^{(i)} &= \frac{1}{r}\frac{\partial \Phi^{(i)}}{\partial \theta} + \frac{1}{r}\frac{\partial \varphi^{(i)}}{\partial \theta} + \cos\theta\frac{\partial\psi^{(i)}}{\partial \theta} - (3-4\nu^{(i)})\psi^{(i)}\sin\theta \\
u_\theta^{(i)} &= \frac{1}{r}\frac{\partial \Phi^{(i)}}{\partial \theta} + \left[\frac{1}{r}\frac{\partial \varphi^{(i)}}{\partial \theta} + \cos\theta\frac{\partial\psi^{(i)}}{\partial \theta} - (3-4\nu^{(i)})\psi^{(i)}\sin\theta\right] \\
u_\theta^{(i)} &= \overline{u_\theta^{(i)}} + \overline{\overline{u_\theta^{(i)}}}
\end{aligned} \tag{22}
$$

where the component with a single bar is displacement with respect to $\Phi^{(i)}$ and component with double bar is displacement with respect to $\varphi^{(i)}$ and $\psi^{(i)}$. $G^{(i)}$, $\nu^{(i)}$ and $a_t^{(i)}$ are the shear modulus, Poisson's ratio, coefficient of thermal expansion for the material of multilayer hollow sphere. Functions must satisfy the equations as

$$\nabla^2\Phi^{(i)} = K^{(i)}\tau^{(i)}, \nabla^2\varphi^{(i)} = 0, \quad \nabla^2\psi^{(i)} = 0 \tag{23}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial\theta}$$

$$K^{(i)} = \frac{1+\nu^{(i)}}{1-\nu^{(i)}}a_t^{(i)}$$

The strain components are represented as

$$\begin{aligned}
\varepsilon_{rr}^{(i)} &= \frac{\partial^2\Phi^{(i)}}{\partial r^2} + \frac{\partial^2\varphi^{(i)}}{\partial r^2} + r\cos\theta\frac{\partial^2\psi^{(i)}}{\partial r^2} - 2(1-2\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r}\\
\varepsilon_{\theta\theta}^{(i)} &= \frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi^{(i)}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi^{(i)}}{\partial\theta^2} + \cos\theta\frac{\partial\psi^{(i)}}{\partial r}\\
&\quad + \frac{1}{r}\cos\theta\frac{\partial^2\psi^{(i)}}{\partial\theta^2} + 2(1-2\nu^{(i)})\frac{1}{r}\sin\theta\frac{\partial\psi^{(i)}}{\partial r}\\
\varepsilon_{\phi\phi}^{(i)} &= \frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\Phi^{(i)}}{\partial\theta} + \frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\varphi^{(i)}}{\partial\theta}\\
&\quad + \cos\theta\frac{\partial\psi^{(i)}}{\partial r} + \frac{\cos^2\theta}{\sin\theta}\frac{1}{r}\frac{\partial\psi^{(i)}}{\partial\theta}\\
\varepsilon_{r\theta}^{(i)} &= \frac{\partial^2}{\partial r\partial\theta}\left(\frac{\Phi^{(i)}}{r}\right) + \frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi^{(i)}}{r}\right) + (1-2\nu^{(i)})\sin\theta\frac{\partial\psi^{(i)}}{\partial r}\\
&\quad + \cos\theta\frac{\partial^2\psi^{(i)}}{\partial r\partial\theta} - 2(1-\nu^{(i)})\frac{1}{r}\cos\theta\frac{\partial\psi^{(i)}}{\partial\theta}
\end{aligned}$$

The stress components in the spherical coordinate system are represented in terms of three functions $\Phi^{(i)}$, $\varphi^{(i)}$ and $\psi^{(i)}$.

$$\begin{aligned}
\sigma_{rr}^{(i)} &= 2G^{(i)}\left[\frac{\partial^2\Phi^{(i)}}{\partial r^2} - K^{(i)}\tau^{(i)}\right]\\
&\quad + 2G^{(i)}\left[\frac{\partial^2\varphi^{(i)}}{\partial r^2} + r\cos\theta\frac{\partial^2\psi^{(i)}}{\partial r^2} - 2(1-\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r} + 2\nu^{(i)}\frac{1}{r}\sin\theta\frac{\partial\psi^{(i)}}{\partial\theta}\right]\\
\sigma_{rr}^{(i)} &= \overline{\sigma_{rr}^{(i)}} + \overline{\overline{\sigma_{rr}^{(i)}}}
\end{aligned} \tag{24}$$

$$\sigma_{\theta\theta}^{(i)} = 2G^{(i)}\left[\begin{array}{r}\frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi^{(i)}}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi^{(i)}}{\partial\theta^2} + (1-2\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r}\\ +\frac{1}{r}\cos\theta\frac{\partial^2\psi^{(i)}}{\partial\theta^2} + 2(1-\nu^{(i)})\frac{1}{r}\sin\theta\frac{\partial\psi^{(i)}}{\partial\theta} - K^{(i)}\tau^{(i)}\end{array}\right]$$

$$\overline{\sigma_{\theta\theta}^{(i)}} = 2G^{(i)}\left[\frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi^{(i)}}{\partial\theta^2} - K^{(i)}\tau^{(i)}\right]$$

$$\overline{\overline{\sigma_{\theta\theta}^{(i)}}} = 2G^{(i)}\left[\frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi^{(i)}}{\partial\theta^2} + (1-2\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial^2\psi^{(i)}}{\partial\theta^2} + 2(1-\nu^{(i)})\frac{1}{r}\sin\theta\frac{\partial\psi^{(i)}}{\partial\theta}\right]$$

$$\sigma_{\theta\theta}^{(i)} = \overline{\sigma_{\theta\theta}^{(i)}} + \overline{\overline{\sigma_{\theta\theta}^{(i)}}} \tag{25}$$

$$\sigma_{\phi\phi}^{(i)} = 2G^{(i)}\left[\begin{array}{l}\frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\Phi^{(i)}}{\partial\theta} + \frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\varphi^{(i)}}{\partial\theta} + (1-2\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r}\\ +\frac{1}{r}(\cos\theta\cot\theta + 2\nu^{(i)}\sin\theta)\frac{\partial\psi^{(i)}}{\partial\theta} - K^{(i)}\tau^{(i)}\end{array}\right]$$

$$\overline{\sigma_{\phi\phi}^{(i)}} = 2G^{(i)}\left[\frac{1}{r}\frac{\partial\Phi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\Phi^{(i)}}{\partial\theta} - K^{(i)}\tau^{(i)}\right]$$

$$\overline{\overline{\sigma_{\phi\phi}^{(i)}}} = 2G^{(i)}\left[\frac{1}{r}\frac{\partial\varphi^{(i)}}{\partial r} + \cot\theta\frac{1}{r^2}\frac{\partial\varphi^{(i)}}{\partial\theta} + (1-2\nu^{(i)})\cos\theta\frac{\partial\psi^{(i)}}{\partial r} + \frac{1}{r}(\cos\theta\cot\theta + 2\nu^{(i)}\sin\theta)\frac{\partial\psi^{(i)}}{\partial\theta}\right]$$

$$\sigma_{\phi\phi}^{(i)} = \overline{\sigma_{\phi\phi}^{(i)}} + \overline{\overline{\sigma_{\phi\phi}^{(i)}}} \tag{26}$$

$$\overline{\sigma_{r\theta}^{(i)}} = 2G^{(i)}\left[\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\Phi^{(i)}}{r}\right)\right]$$

$$\overline{\overline{\sigma_{r\theta}^{(i)}}} = 2G^{(i)}\left[\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi^{(i)}}{r}\right) + (1-2\nu^{(i)})\sin\theta\frac{\partial\psi^{(i)}}{\partial r} + \cos\theta\frac{\partial^2\psi^{(i)}}{\partial r\partial\theta} - 2(1-\nu^{(i)})\frac{1}{r}\cos\theta\frac{\partial\psi^{(i)}}{\partial\theta}\right]$$

$$\sigma_{r\theta}^{(i)} = 2G^{(i)}\left[\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\Phi^{(i)}}{r}\right)\right] + 2G^{(i)}\left[\frac{\partial^2}{\partial r\partial\theta}\left(\frac{\varphi^{(i)}}{r}\right) + (1-2\nu^{(i)})\sin\theta\frac{\partial\psi^{(i)}}{\partial r} + \cos\theta\frac{\partial^2\psi^{(i)}}{\partial r\partial\theta} - 2(1-\nu^{(i)})\frac{1}{r}\cos\theta\frac{\partial\psi^{(i)}}{\partial\theta}\right]$$

$$\sigma_{r\theta}^{(i)} = \overline{\sigma_{r\theta}^{(i)}} + \overline{\overline{\sigma_{r\theta}^{(i)}}} \tag{27}$$

The mechanical boundary conditions on the traction free surfaces $r = r_0$ and $r = r_n$ are

$$\sigma_{rr}^{(i)} = 0 \quad \text{and} \quad \sigma_{r\theta}^{(i)} = 0 \tag{28}$$

Assuming the interface conditions as $i = 1, 2, 3, \ldots n-1$

$$\begin{aligned}\sigma_{rr}^{(i)}(r_i) &= \sigma_{rr}^{(i+1)}(r_i)\\ \sigma_{r\theta}^{(i)}(r_i) &= \sigma_{r\theta}^{(i+1)}(r_i)\\ \sigma_{\theta\theta}^{(i)}(r_i) &= \sigma_{\theta\theta}^{(i+1)}(r_i)\end{aligned} \tag{29}$$

5 Solutions

Heat Conduction
The separation of variables method is used to solve the problem.

$$T^{(i)}(r,\theta) = R^{(i)}(r)\theta^{(i)}(\theta) \tag{30}$$

$$\begin{gathered}\frac{d^2R^{(i)}(r)}{dr^2}\Theta^{(i)}(\theta) + \frac{2}{r}\frac{dR^{(i)}(r)}{dr}\Theta^{(i)}(\theta) + \frac{1}{r^2}R^{(i)}(r)\frac{d^2\Theta^{(i)}(\theta)}{d\theta^2} + \frac{\cot\theta}{r^2}R^{(i)}(r)\frac{d\Theta^{(i)}(\theta)}{d\theta} = 0\\ -r^2\left[\frac{R^{(i)''}(r)}{R^{(i)}(r)} + \frac{2}{r}\frac{R^{(i)'}(r)}{R^{(i)}(r)}\right] = \frac{\Theta^{(i)''}(\theta)}{\Theta^{(i)}(\theta)} + \cot\theta\frac{\Theta^{(i)'}(\theta)}{\Theta^{(i)}(\theta)} = -m(1+m)\end{gathered} \tag{31}$$

where $m(1+m) = \left(\lambda_m^{(i)}\right)^2$

$$\begin{gathered}r^2R^{(i)''}(r) + 2rR^{(i)'}(r) - m(1+m)R^{(i)}(r) = 0\\ R_m^{(i)}(r) = a_m^{(i)}r^m + b_m^{(i)}r^{-(1+m)}\end{gathered} \tag{32}$$

Application of the interface conditions (8)–(11) and boundary conditions (2)–(3) to the transverse eigenfunction $R_m^{(i)}(r)$. The matrix (2n × 2n) are as follows:

$$\begin{bmatrix} c_{1in} & c_{2in} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ x_{11} & x_{12} & x_{13} & x_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ y_{11} & y_{12} & y_{13} & y_{14} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & x_{i1} & x_{i2} & x_{i3} & x_{i4} & \dots & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \dots & y_{i1} & y_{i2} & y_{i3} & y_{i4} & \dots & 0 & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & x_{i-1,1} & x_{i-1,2} & x_{i-1,3} & x_{i-1,4}\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & y_{i-1,1} & y_{i-1,2} & y_{i-1,3} & y_{i-1,4}\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & c_{1out} & c_{2out} \end{bmatrix} \begin{bmatrix} a_m^{(1)}\\ b_m^{(1)}\\ \dots\\ \dots\\ a_m^{(i)}\\ b_m^{(i)}\\ \dots\\ \dots\\ a_m^{(n)}\\ b_m^{(n)} \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ \dots\\ \dots\\ 0\\ 0\\ \dots\\ \dots\\ 0\\ 0 \end{bmatrix} \tag{33}$$

where

$$\begin{aligned}
&c_{1in} = k^{(1)}mr_1^{m-1} + h^{(1)}r_1^m\\
&c_{2in} = -k^{(1)}(1+m)r_1^{-m-2} + h^{(1)}r_1^{-1-m}\\
&x_{i1} = r_i^m \quad x_{i2} = r_i^{-1-m} \quad x_{i3} = -r_i^m \quad x_{i4} = -r_i^{-1-m}\\
&y_{i1} = k^{(i)}mr_i^{m-1} \quad y_{i2} = k^{(i)}(-1-m)r_i^{-m-2} \quad y_{i3} = -k^{(i+1)}mr_i^{m-1}\\
&y_{i4} = k^{(i+1)}(1+m)r_i^{-m-2}\\
&c_{1out} = k^{(n)}mr_n^{m-1} + h^{(n)}r_n^m \quad c_{2out} = -k^{(n)}(1+m)r_n^{-m-2} + h^{(n)}r_n^{-1-m}
\end{aligned}$$

For heat flux to be continuous at the layer interfaces for all values of t.

$$\alpha^{(i)}\left(\lambda_m^{(i)}\right)^2 = \alpha^{(1)}\left(\lambda_m^{(1)}\right)^2, \quad \mathrm{i} = 1, 2, \dots n \tag{34}$$

In the above matrix equation, $\lambda_m^{(i)}$(i $\neq$ 1) maybe written in terms of $\lambda_m^{(1)}$ using the above equation. Subsequently, transverse eigen condition can be obtained by setting the determinant of the $(2n \times 2n)$ coefficient matrix equal to zero. And after that eigenvalue determined the constants $a_m^{(i)}$ and $b_m^{(i)}$.

The equation in $\Theta^{(i)}(\theta)$ can be written as

$$\frac{\Theta^{(i)''}(\theta)}{\Theta^{(i)}(\theta)} + \cot\theta\frac{\Theta^{(i)'}(\theta)}{\Theta^{(i)}(\theta)} = -m(1+m)$$
$$\Theta^{(i)''} + \cot\theta\Theta^{(i)'} + m(m+1)\Theta^{(i)} = 0 \tag{35}$$

By change of variable using $\mu = \cos\theta$

$$\Theta^{(i)'} = -\sin\theta\frac{d\Theta^{(i)}}{d\mu} \tag{36}$$

$$\Theta^{(i)''} = -\cos\theta\frac{d\Theta^{(i)}}{d\mu} + \sin^2\theta\frac{d^2\Theta^{(i)}}{d\mu^2} \tag{37}$$

Substituting Eqs. (36) and (37) in Eq. (35).

$$-\cos\theta\frac{d\Theta^{(i)}}{d\mu} + \sin^2\theta\frac{d^2\Theta^{(i)}}{d\mu^2} + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta\frac{d\Theta^{(i)}}{d\mu}\right) + m(m+1)\Theta^{(i)} = 0 \tag{38}$$

$$(1-\mu^2)\frac{d^2\Theta^{(i)}}{d\mu^2} - 2\mu\frac{d\Theta^{(i)}}{d\mu} + m(m+1)\Theta^{(i)} = 0$$
$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta^{(i)}}{d\mu}\right] + m(m+1)\Theta^{(i)} = 0 \tag{39}$$

$$\Theta_m^{(i)}(\mu) = c_1P_m^{(i)}(\mu) + c_2Q_m^{(i)}(\mu)$$
$$\Theta_m^{(i)}(\mu) = c_1P_m^{(i)}(\cos\theta) + c_2Q_m^{(i)}(\cos\theta) \tag{40}$$

By using boundary conditions

$$Q_m^{(i)}(\cos 0) = Q_m^{(i)}(1) = \infty$$

Hence $c_2 = 0$

$$\Theta_m^{(i)}(\theta) = c_1P_m^{(i)}(\cos\theta)$$

Hence

$$c_1 \neq 0, \Theta_m^{(i)}(\theta) = P_m^{(i)}(\cos\theta) \tag{41}$$

It is assumed that the solution of the problem is in the form of a series expansion of the derived eigenfunctions as follows such that

$$T^{(i)}(r,\theta)=\sum_{m=0}^{\infty}R_m^{(i)}(r)\Theta_m^{(i)}(\theta)$$
$$T^{(i)}(r,\theta)=\sum_{m=0}^{\infty}\left(a_m^{(i)}r^m+b_m^{(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta) \tag{42}$$

Initial temperature is $f^{(i)}$

6 Displacement Components

Displacement component corresponding to $\Phi^{(i)}$ and $\tau^{(i)}$

Therefore, the temperature change $\tau^{(i)}$ obtained as

$$\tau^{(i)}=T^{(i)}(r,\theta)-f^{(i)}$$
$$\tau^{(i)}=\sum_{m=0}^{\infty}\left(a_m^{(i)}r^m+b_m^{(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta)-f^{(i)} \tag{43}$$

The Goodier's thermoelastic displacement potential $\Phi^{(i)}$ satisfying the first of Eq. (25) is obtained as [15]

$$\Phi^{(i)}=K^{(i)}\left\{-\frac{1}{6}f^{(i)}r^2+\sum_{m=0}^{\infty}\left[\frac{1}{2(2m+3)}a_m^{(i)}r^{m+2}-\frac{1}{2(2m-1)}b_m^{(i)}r^{-m+1}\right]P_m^{(i)}(\cos\theta)\right\} \tag{44}$$

Displacement component corresponding to $\varphi^{(i)}$ and $\psi^{(i)}$

We have considered the problem of axisymmetric thermoelastic deformation in a hollow multilayer sphere. In this the coordinate variable θ is defined within the interval $0\le\theta\le\pi$. The Legendre functions $P_m^{(i)}(\cos\theta)$ is considered as the fundamental solution of the Boussinesq harmonic functions $\varphi^{(i)}$ and $\psi^{(i)}$ for the axisymmetric case in the spherical coordinate system for $m=0,1,2,3\ldots$ thus the displacement functions $\varphi^{(i)}$ and $\psi^{(i)}$ are represented by the series forms as [Noda].

$$\varphi_m^{(i)}=\sum_{m=0}^{\infty}\left(c_{1m}^{\prime(i)}r^m+c_{2m}^{\prime(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta) \tag{45}$$

$$\psi_m^{(i)}=\sum_{m=0}^{\infty}\left(d_{1m}^{\prime(i)}r^m+d_{2m}^{\prime(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta) \tag{46}$$

The displacements corresponding to the Goodier's thermoelastic potential functions $\Phi^{(i)}$ and the displacement functions $\varphi^{(i)}$ and $\psi^{(i)}$.

$$\overline{u_r^{(i)}} = K^{(i)}\left\{-\frac{1}{3}f^{(i)}r + \sum_{m=0}^{\infty}\left[\frac{m+2}{2(2m+3)}a_{im}r^{m+1} + \frac{m-1}{2(2m-1)}b_{im}r^{-m}\right]P_{im}(\cos\theta)\right\} \tag{47}$$

$$\begin{aligned}
\overline{\overline{u_r^{(i)}}} &= \frac{\partial}{\partial r}\left(\sum_{m=0}^{\infty}\left(c_{1m}^{'(i)}r^m + c_{2m}^{'(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta)\right) + r\cos\theta\frac{\partial}{\partial r}\left(\sum_{m=0}^{\infty}\left(d_{1m}^{'(i)}r^m + d_{2m}^{'(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta)\right) \\
&\quad - (3-4\nu^{(i)})\cos\theta\left(\sum_{m=0}^{\infty}\left(d_{1m}^{'(i)}r^m + d_{2m}^{'(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta)\right) \\
\overline{\overline{u_r^{(i)}}} &= \sum_{m=0}^{\infty}\left(c_{1m}^{'(i)}mr^{m-1} + c_{2m}^{'(i)}(-m-1)r^{-m-2}\right)P_m^{(i)}(\cos\theta) \\
&\quad + r\cos\theta\sum_{m=0}^{\infty}\left(d_{1m}^{'(i)}mr^{m-1} + d_{2m}^{'(i)}(-m-1)r^{-m-2}\right)P_m^{(i)}(\cos\theta) \\
&\quad - (3-4\nu^{(i)})\cos\theta\left(\sum_{m=0}^{\infty}\left(d_{1m}^{'(i)}r^m + d_{2m}^{'(i)}r^{-m-1}\right)P_m^{(i)}(\cos\theta)\right) \\
\overline{\overline{u_r^{(i)}}} &= \sum_{m=0}^{\infty}\left(c_{1m}^{'(i)}mr^{m-1} - c_{2m}^{'(i)}(m+1)r^{-m-2}\right)P_m^{(i)}(\cos\theta) \\
&\quad + \sum_{m=0}^{\infty}\left[(m-3+4\nu^{(i)})d_{1m}^{'(i)}r^m - (m+4-4\nu^{(i)})d_{2m}^{'(i)}r^{-m-1}\right] \\
&\quad \times \frac{1}{2m+1}\left[(m+1)P_{m+1}^{(i)}(\cos\theta) + mP_{m-1}^{(i)}(\cos\theta)\right]
\end{aligned} \tag{48}$$

where $\cos\theta P_m^{(i)}(\cos\theta) = \frac{1}{2m+1}\left[(m+1)P_{m+1}^{(i)}(\cos\theta) + mP_{m-1}^{(i)}(\cos\theta)\right]$

It can be seen from expression (48) that this is not suitable for solving practical boundary value problem because it contains three kinds Legendre functions with different orders $n-1, n, n+1$ under the summation signs. Two solve this problem we introduce new unknown constants given by

$$\begin{aligned}
c_{1m}^{'(i)} &= c_{1m}^{(i)} - (m-4+4v^{(i)})d_{1m-2}^{(i)} \\
c_{2m}^{'(i)} &= c_{2m}^{(i)} - (m+5-4v^{(i)})d_{2m+2}^{(i)} \\
d_{1m}^{'(i)} &= (2m+1)d_{1m-1}^{(i)} \\
d_{2m}^{'(i)} &= (2m+1)d_{2m+1}^{(i)}
\end{aligned}$$

The function $\varphi^{(i)}$ and $\psi^{(i)}$ reduce to

$$\varphi^{(i)} = \sum_{m=0}^{\infty}\left\{\left[c_{1,m}^{(i)} - (m-4+4\nu^{(i)})d_{1,m-2}^{(i)}\right]r^m + \left[c_{2,m}^{(i)} - (m+5-4\nu^{(i)})d_{2,m+2}^{(i)}\right]r^{-m-1}\right\}P_m^{(i)}(\cos\theta) \tag{49}$$

$$\psi^{(i)} = \sum_{m=0}^{\infty}\left[(2m+1)d_{1,m-1}^{(i)}r^m + (2m+1)d_{2,m+1}^{(i)}r^{-m-1}\right]P_m^{(i)}(\cos\theta) \tag{50}$$

Using new unknown constants (48) can be expressed as

$$\overline{\overline{u_r^{(i)}}} = \sum_{m=0}^{\infty}\left[mc_{1,m}^{(i)}r^{m-1} - (m+1)c_{2,m}^{(i)}r^{-m-2} + (m+1)(m-2+4\nu^{(i)})d_{1,m}^{(i)}r^{m+1} - m(m+3-4\nu^{(i)})d_{2,m}^{(i)}r^{-m}\right]P_m^{(i)}(\cos\theta)$$

Similarly the tangential displacement with respect to $\varphi^{(i)}$ and $\psi^{(i)}$

$$\overline{\overline{u_\theta^{(i)}}} = -\left(1-\mu^2\right)^{1/2}\sum_{m=1}^{\infty}\left[\begin{array}{l} c_{1,m}^{(i)}r^{m-1} + c_{2,m}^{(i)}r^{-m-2} + (m+5-4\nu^{(i)})d_{1,m}^{(i)}r^{m+1} \\ +(m-4+4\nu^{(i)})d_{2,m}^{(i)}r^{-m} \end{array}\right]\frac{m+1}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right]$$

The radial displacement and tangential displacement is obtained as

$$\begin{aligned} u_r^{(i)} &= K^{(i)}\left\{-\frac{1}{3}f^{(i)}r + \sum_{m=0}^{\infty}\left[\frac{m+2}{2(2m+3)}a_m^{(i)}r^{m+1} + \frac{m-1}{2(2m-1)}b_m^{(i)}r^{-m}\right]P_m^{(i)}(\cos\theta)\right\} \\ &\quad + \sum_{m=0}^{\infty}\left[\begin{array}{l} mc_{1,m}^{(i)}r^{m-1} - (m+1)c_{2,m}^{(i)}r^{-m-2} + (m+1)(m-2+4\nu^{(i)})d_{1,m}^{(i)}r^{m+1} \\ -m(m+3-4\nu^{(i)})d_{2,m}^{(i)}r^{-m} \end{array}\right]P_m^{(i)}(\cos\theta) \end{aligned} \tag{51}$$

$$\begin{aligned} u_\theta^{(i)} &= -K^{(i)}\left(1-\mu^2\right)^{1/2}\sum_{m=1}^{\infty}\left[\frac{1}{2(2m+3)}a_m^{(i)}r^{m+1} - \frac{1}{2(2m-1)}b_m^{(i)}r^{-m}\right]\frac{m+1}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] \\ &\quad - \left(1-\mu^2\right)^{1/2}\sum_{m=1}^{\infty}\left[\begin{array}{l} c_{1,m}^{(i)}r^{m-1} + c_{2,m}^{(i)}r^{-m-2} + (m+5-4\nu^{(i)})d_{1,m}^{(i)}r^{m+1} \\ +(m-4+4\nu^{(i)})d_{2,m}^{(i)}r^{-m} \end{array}\right]\frac{m+1}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] \end{aligned} \tag{52}$$

7 Thermal Stresses

The stress components corresponding to $\Phi^{(i)}$ and $\tau^{(i)}$

$$\overline{\sigma_{rr}^{(i)}} = 2G^{(i)}K^{(i)}\left\{\frac{2}{3}f^{(i)} + \sum_{m=0}^{\infty}\left[\frac{m^2-m-4}{2(2m+3)}a_m^{(i)}r^m - \frac{m^2+3m-2}{2(2m-1)}b_m^{(i)}r^{-m-1}\right]P_m^{(i)}(\cos\theta)\right\} \tag{53}$$

The stress components corresponding to $\varphi^{(i)}$ and $\psi^{(i)}$

$$\overline{\overline{\sigma_{rr}^{(i)}}} = 2G^{(i)}\sum_{m=0}^{\infty}\left[\begin{array}{l} m(m-1)c_{1,m}^{(i)}r^{m-2} - (m+1)(m+2)c_{2,m}^{(i)}r^{-m-3} \\ +(m+1)(m^2-m-2+2\nu^{(i)})d_{1,m}^{(i)}r^m + m(m^2+3m-2\nu^{(i)})d_{2,m}^{(i)}r^{-m-1} \end{array}\right]P_m^{(i)}(\cos\theta) \tag{54}$$

$$\begin{aligned} \sigma_{rr}^{(i)} &= 2G^{(i)}K^{(i)}\left\{\frac{2}{3}f^{(i)} + \sum_{m=0}^{\infty}\left[\frac{m^2-m-4}{2(2m+3)}a_m^{(i)}r^m - \frac{m^2+3m-2}{2(2m-1)}b_m^{(i)}r^{-m-1}\right]P_m^{(i)}(\cos\theta)\right\} \\ &\quad + 2G^{(i)}\sum_{m=0}^{\infty}\left[\begin{array}{l} m(m-1)c_{1,m}^{(i)}r^{m-2} - (m+1)(m+2)c_{2,m}^{(i)}r^{-m-3} \\ +(m+1)(m^2-m-2+2\nu^{(i)})d_{1,m}^{(i)}r^m + m(m^2+3m-2\nu^{(i)})d_{2,m}^{(i)}r^{-m-1} \end{array}\right]P_m^{(i)}(\cos\theta) \end{aligned} \tag{55}$$

$$\sigma_{\theta\theta}^{(i)} = 2G^{(i)}K^{(i)}\left\{\frac{2}{3}f^{(i)} + \sum_{m=0}^{\infty}\left[\begin{array}{l}\frac{1}{2(2m+3)}a_m^{(i)}r^m \\ \times\left\{(m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] - (m+1)^2 P_m^{(i)}(\mu)\right\} \\ +\frac{1}{2(2m-1)}b_m^{(i)}r^{-m-1} \\ \times\left\{-(m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] + (m-1)^2 P_m^{(i)}(\mu)\right\}\end{array}\right]\right\}$$
$$- 2G^{(i)}\sum_{m=0}^{\infty}\left[\begin{array}{l} m^2 c_{1,m}^{(i)}r^{m-2} + (m+1)^2 c_{2,m}^{(i)}r^{-m-3} + (m+1)(m^2+4m+2+2\nu^{(i)})d_{1,m}^{(i)}r^m \\ +m(m^2-2m-1+2\nu^{(i)})d_{2,m}^{(i)}r^{-m-1}\end{array}\right]P_m^{(i)}(\cos\theta)$$
$$+ 2G^{(i)}\sum_{m=1}^{\infty}\left[\begin{array}{l} c_{1,m}^{(i)}r^{m-2} + c_{2,m}^{(i)}r^{-m-3} + (m+5-4\nu^{(i)})d_{1,m}^{(i)}r^m \\ +(m-4+4\nu^{(i)})d_{2,m}^{(i)}r^{-m-1}\end{array}\right](m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] \tag{56}$$

$$\sigma_{\phi\phi}^{(i)} = 2G^{(i)}K^{(i)}\left\{\frac{2}{3}f^{(i)} + \sum_{m=0}^{\infty}\left[\begin{array}{l}\frac{1}{2(2m+3)}a_m^{(i)}r^m \\ \times\left\{-(m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] - (3m+4) P_m^{(i)}(\mu)\right\} \\ +\frac{1}{2(2m-1)}b_m^{(i)}r^{-m-1} \\ \times\left\{(m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] - (3m-1) P_m^{(i)}(\mu)\right\}\end{array}\right]\right\}$$
$$+ 2G^{(i)}\sum_{m=0}^{\infty}\left[\begin{array}{l} mc_{1,m}^{(i)}r^{m-2} - (m+1)c_{2,m}^{(i)}r^{-m-3} + (m+1)\left[m-2-2\nu^{(i)}(2m+1)\right]d_{1,m}^{(i)}r^m \\ -m\left[m+3-2\nu^{(i)}(2m+1)\right]d_{2,m}^{(i)}r^{-m-1}\end{array}\right]P_m^{(i)}(\cos\theta)$$
$$- 2G^{(i)}\sum_{m=1}^{\infty}\left[\begin{array}{l} c_{1,m}^{(i)}r^{m-2} + c_{2,m}^{(i)}r^{-m-3} + (m+5-4\nu^{(i)})d_{1,m}^{(i)}r^m \\ +(m-4+4\nu^{(i)})d_{2,m}^{(i)}r^{-m-1}\end{array}\right](m+1)\frac{\mu}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right] \tag{57}$$

$$\sigma_{r\theta}^{(i)} = -2G^{(i)}K^{(i)}\left(1-\mu^2\right)^{1/2}\sum_{m=1}^{\infty}\left[\frac{m+1}{2(2m+3)}a_m^{(i)}r^m + \frac{m}{2(2m-1)}b_m^{(i)}r^{-m-1}\right]\times\frac{m+1}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right]$$
$$- 2G^{(i)}\left(1-\mu^2\right)^{1/2}\sum_{m=1}^{\infty}\begin{array}{l}\left[\begin{array}{l}(m-1)c_{1,m}^{(i)}r^{m-2} - (m+2)c_{2,m}^{(i)}r^{-m-3} \\ +(m^2+2m-1+2\nu^{(i)})d_{1,m}^{(i)}r^m - (m^2-2+2\nu^{(i)})d_{2,m}^{(i)}r^{-m-1}\end{array}\right] \\ \times\frac{m+1}{1-\mu^2}\left[\mu P_m^{(i)}(\mu) - P_{m+1}^{(i)}(\mu)\right]\end{array} \tag{58}$$

unknown constants are determined by using boundary and interface conditions $\sigma_{rr}^{(i)} = 0$ and $\sigma_{r\theta}^{(i)} = 0$ at $r = r_0$ and $r = r_n$.

Assuming the interface conditions as $i = 1, 2, 3, \ldots n-1$.

$$\sigma_{rr}^{(i)}(r_i) = \sigma_{rr}^{(i+1)}(r_i)$$
$$\sigma_{r\theta}^{(i)}(r_i) = \sigma_{r\theta}^{(i+1)}(r_i)$$
$$\sigma_{\theta\theta}^{(i)}(r_i) = \sigma_{\theta\theta}^{(i+1)}(r_i)$$

8 Conclusion

In this paper, the exact analytical solutions are presented for displacement and thermal stresses with two-dimensional steady-state temperature distribution in the multilayer hollow sphere. The temperature distribution is obtained by solving the heat conduction equation by separation of variable method and using the condition of continuity at the interface to get required eigenvalues in the solution. Each layer of the spherical sphere is considered as isotropic and homogeneous. The components of displacement and thermal stress function has been discussed with the help of Goodier's displacement potential and Boussinesq harmonic functions as Noda et al. [16]. On determining temperature distribution function from the heat conduction equation, it is used as a known function. Furthermore, we have investigated the results on the basis of assumed boundary conditions and approach was purely mathematical. Obtained results are considered to be useful in the design of the smart multilayer spherical vessels. Also, results may be used in industrial furnace, nuclear reactors, chemical industry, turbines, spacecraft where multilayer materials are highly used.

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