The Nonemptiness of the Inner Core

Tomoki Inoue

Abstract We prove that if a non-transferable utility (NTU) game is cardinally balanced and if, at every individually rational and efficient payoff vector, every non-zero normal vector to the set of payoff vectors feasible for the grand coalition is strictly positive, then the inner core is nonempty. The condition on normal vectors is satisfied if the set of payoff vectors feasible for the grand coalition is non-leveled. An NTU game generated by an exchange economy where every consumer has a continuous, concave, and strongly monotone utility function satisfies our sufficient condition. Our proof relies on Qin's theorem on the nonemptiness of the inner core.

Keywords Inner core · Inhibitive set · Cardinal balancedness · NTU game

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1 Introduction

The inner core is a solution concept for a non-transferable utility (NTU) game. Players' utilities are not transferable, but an inner core payoff vector is stable even if players can transfer their utilities with some fictitious rates λ. Namely, at an inner core payoff vector, any coalition cannot improve upon the λ-weighted sum

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of utilities. This stability is stronger than the one required for the core. Thus, the inner core is a refinement of the core.

The inner core is of significance in the relations to (1) Walrasian equilibria for some economies, and to (2) the strictly inhibitive set of payoff vectors that are stable against randomized blocking plans. An exchange economy or a production economy generates a unique NTU game, but an NTU game may be generated by multiple economies. Billera [3] proposed a method how to induce a production economy from a totally cardinally balanced NTU game so that the economy actually generates the given NTU game. Qin [14] proved that the inner core of a totally cardinally balanced NTU game coincides with the set of utility vectors at Walrasian equilibria for Billera's induced production economy.[1] Inoue [9] constructed a coalition production economy generating a given NTU game and proved that the inner core coincides with the set of utility vectors at Walrasian equilibria for his induced coalition production economy.

When a payoff vector can be improved upon by multiple coalitions, it is not clear which coalition actually improves upon the payoff vector. Myerson [12, Section 9.8] considered the following scenario. A mediator who is not a player of a given game randomly chooses a coalition and payoffs to the members of the coalition. Every player accepts the mediator's randomized blocking plan if, conditional on being a member of a coalition, the expected payoff to him is better than the given payoff to him. The strictly inhibitive set is the set of payoff vectors that are feasible for the grand coalition and that are stable against randomized blocking plans. Qin [13] proved that the inner core is always a subset of the strictly inhibitive set and, in some classes of NTU games, these two sets coincide.

A sufficient condition for the inner core to be nonempty was given by Qin [15, Theorem 1]. Although his sufficient condition is mathematically general, it is not easy to check whether a given NTU game satisfies the condition. Accordingly, it is useful to give classes of NTU games satisfying Qin's sufficient condition. Qin [15, Corollary 1] proved that every cardinally balanced-with-slack NTU game satisfies his own sufficient condition. We give another class of NTU games satisfying Qin's sufficient condition: If an NTU game is cardinally balanced and if, at every individually rational and efficient payoff vector, every non-zero normal vector to the set of payoff vectors feasible for the grand coalition is strictly positive, then the NTU game satisfies Qin's sufficient condition for the inner core to be nonempty. The condition on normal vectors is met if the set of payoff vectors feasible for the grand coalition is non-leveled. Our class of NTU games with the nonempty inner core contains NTU games generated by exchange economies where every consumer has a continuous, concave, and strongly monotone utility function. de Clippel and Minelli [8] proved that the utility vector at a Walrasian equilibrium for such an exchange

[1]Furthermore, given an NTU game and given any payoff vector in the inner core, Qin [14] constructed a production economy generating the given NTU game such that the given payoff vector in the inner core is the utility vector at the unique Walrasian equilibrium for his induced production economy. Brangewitz and Gamp [6] extended this result from one point in the inner core to a closed subset with certain properties.

economy is always in the inner core and, therefore, the nonemptiness of the inner core follows from the well-known existence theorem of a Walrasian equilibrium. Hence, our theorem extends de Clippel and Minelli's class of NTU games with the nonempty inner core.

In the present paper, we prove our theorem on the nonemptiness of the inner core by applying Qin's [15] theorem. There are two other methods of proof of our theorem. The first way is due to Aubin [2, Theorem 2.1, Proposition 2.3]. Aubin adopted an abstract description of an NTU game. If we adopt a specified "representative function" of an NTU game, what Aubin called "an equilibrium for a representative function" turns out to be an inner core payoff vector. Since Aubin proved the existence of an equilibrium for a representative function, he indeed proved the nonemptiness of the inner core. At an inner core payoff vector, fictitious transfer rates λ of utilities must be strictly positive. Aubin first finds nonnegative fictitious transfer rates λ with certain properties and then provides a sufficient condition (condition (c) of Proposition 2 in Sect. 3) for λ to be strictly positive.[2] As discussed in Sect. 3, Aubin's sufficient condition for λ to be strictly positive can be slightly weakened. The second way is due to Inoue [11] who took two steps as well as Aubin [2]. Inoue [11] first gives a new coincidence theorem, a synthesis of Brouwer's fixed point theorem and a stronger separation theorem for convex sets due to Debreu and Schmeidler [7], and then, by applying the coincidence theorem, he obtains nonnegative fictitious transfer rates of utilities with certain properties. Inoue's [11] second step of the proof is the same as Aubin's [2].

The present paper is organized as follows. In Sect. 2, we give a precise description of an NTU game and the definition of the inner core. Also, we give Qin's sufficient condition for the inner core to be nonempty (Theorem 1). In Sect. 3, we provide characterizations of the efficient surface with strictly positive normal vectors. In Sect. 4, we prove the nonemptiness of the inner core (Theorem 2). In Sect. 5, we prove that our class of NTU games with the nonempty inner core contains de Clippel and Minelli's class of NTU games generated by exchange economies with continuous, concave, and strongly monotone utility functions. In Sect. 6, we give some remarks.

2 NTU Games and the Inner Core

We begin with some notation. We follow the notation of Qin [15]. Let $N = \{1, \ldots, n\}$ be a set with $n \geq 2$ elements and let $\mathbb{R}^N$ be the n-dimensional Euclidean space of vectors x with coordinates x_i indexed by $i \in N$. The inner product of x and y in $\mathbb{R}^N$ is denoted by $x \cdot y$, i.e., $x \cdot y = \sum_{i \in N} x_i y_i$. For $x, y \in \mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for every $i \in N$; $x \gg y$ if $x_i > y_i$ for every $i \in N$. The symbol 0 denotes

[2] Inoue [10, Appendix] summarizes Aubin's description of an NTU game and reproduces Aubin's method of proof.

the origin in $\mathbb{R}^N$ as well as the real number zero. Let $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \mid x \gg 0\}$. For a nonempty subset S of N, let $\mathbb{R}^S = \{x \in \mathbb{R}^N \mid x_i = 0 \text{ for every } i \in N \setminus S\}$, let $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for every } i \in S\}$, and let $e^S \in \mathbb{R}^N$ be the characteristic vector of S, i.e., $e^S_i = 1$ if $i \in S$ and 0 otherwise. For $x \in \mathbb{R}^N$, x^S denotes the projection of x to $\mathbb{R}^S$. Let $\Delta = \{\lambda \in \mathbb{R}^N_+ \mid \sum_{i \in N} \lambda_i = 1\}$, $\Delta^\circ = \{\lambda \in \Delta \mid \lambda \gg 0\}$, and, for every $m \geq n$, $\Delta_{1/m} = \{\lambda \in \Delta \mid \lambda_i \geq 1/m \text{ for every } i \in N\}$.

We regard N as the set of n players. Let $\mathscr{N}$ be the family of all nonempty subsets of N, i.e., $\mathscr{N} = \{S \subseteq N \mid S \neq \emptyset\}$. Elements in $\mathscr{N}$ are called *coalitions*. A *non-transferable utility game* (NTU game, for short) with n players is a correspondence $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ such that, for every $S \in \mathscr{N}$, $V(S)$ is a nonempty subset of $\mathbb{R}^S$ and satisfies $V(S) - \mathbb{R}^S_+ = V(S)$. An NTU game is *compactly generated* if, for every $S \in \mathscr{N}$, there exists a nonempty compact subset C_S of $\mathbb{R}^S$ with $V(S) = C_S - \mathbb{R}^S_+$. In the present paper, we consider only compactly generated NTU games V with $V(N)$ convex.

The core is the set of payoff vectors which is feasible for the grand coalition N and which cannot be improved upon by any coalition. By adopting a different notion of improvement by a coalition, we can define the inner core.

Definition 1

(1) The *core* $C(V)$ of NTU game V is the set of payoff vectors $u \in \mathbb{R}^N$ such that $u \in V(N)$ and there exists no $S \in \mathscr{N}$ and $u' \in V(S)$ with $u'_i > u_i$ for every $i \in S$.

(2) The *inner core* $IC(V)$ of NTU game V is the set of payoff vectors $u \in \mathbb{R}^N$ such that $u \in V(N)$ and there exists $\lambda \in \mathbb{R}^N_{++}$ such that, for every $S \in \mathscr{N}$ and every $u' \in V(S)$, $\lambda^S \cdot u \geq \lambda^S \cdot u'$ holds.

By definition, we have $IC(V) \subseteq C(V)$. The vector $\lambda \in \mathbb{R}^N_{++}$ in the definition of the inner core represents fictitious transfer rates of utilities among players. Note that $u \in IC(V)$ if and only if $u \in V(N) \cap C(V_\lambda)$ for some $\lambda \in \mathbb{R}^N_{++}$, where $V_\lambda : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ is the λ-transfer game defined by

$$V_\lambda(S) = \left\{u \in \mathbb{R}^S \;\middle|\; \lambda \cdot u \leq \lambda \cdot u' \text{ for some } u' \in V(S)\right\} \quad \text{for every } S \in \mathscr{N}.$$

Note also that we can restrict the space of fictitious transfer rates to Δ°.

For $\beta \in \mathbb{R}^N_+$, let

$$\Gamma(\beta) = \left\{\gamma = (\gamma_S)_{S \in \mathscr{N}} \;\middle|\; \gamma_S \geq 0 \text{ for every } S \in \mathscr{N} \text{ and } \sum_{S \in \mathscr{N}} \gamma_S\, e^S = \beta\right\}$$

and let $\widehat{\Gamma}(\beta) = \{\gamma \in \Gamma(\beta) \mid \gamma_N = 0\}$. Note that $\Gamma(e^N)$ is the set of balancing vectors of weights. An NTU game V is *cardinally balanced* if, for every $\gamma \in \Gamma(e^N)$,

$$\sum_{S \in \mathscr{N}} \gamma_S\, V(S) \subseteq V(N).$$

This notion of balancedness is stronger than the balancedness due to Scarf [16]. Scarf's ordinal balancedness is sufficient for the nonemptiness of the core. For our theorem (Theorem 2) on the nonemptiness of the inner core, this stronger balancedness is assumed.

Before we give a condition equivalent to the cardinal balancedness, we introduce some notation. Let $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. For every $S \in \mathscr{N}$, let C_S be a compact subset of $\mathbb{R}^S$ generating $V(S)$, i.e., $V(S) = C_S - \mathbb{R}^S_+$, and let C_N be also convex. For every $\lambda \in \mathbb{R}^N_+$ and every $S \in \mathscr{N}$, define

$$v_\lambda(S) = \max\{\lambda \cdot u \mid u \in V(S)\} = \max\{\lambda \cdot u \mid u \in C_S\}\ .$$

Note that, by Berge's maximum theorem, $\mathbb{R}^N_+ \ni \lambda \mapsto v_\lambda(S) \in \mathbb{R}$ is continuous. For every $i \in N$, let $b_i \in \mathbb{R}$ be the utility level that player i can achieve by himself, i.e,

$$b_i = \max\{u_i \in \mathbb{R} \mid u \in V(\{i\})\}\ .$$

Define a correspondence $B : \mathbb{R}^N_{++} \twoheadrightarrow \mathbb{R}^N$ by

$$B(\lambda) = \{u \in V(N) \mid \lambda \cdot u = v_\lambda(N)\} = \{u \in C_N \mid \lambda \cdot u = v_\lambda(N)\}\ .$$

Note that $B(\lambda)$ is positively homogeneous of degree zero.

The following proposition gives a condition equivalent to the cardinal balancedness.

Proposition 1 *Let $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. Then, V is cardinally balanced if and only if, for every $\lambda \in \Delta^\circ$ and every $\gamma \in \widehat{\Gamma}(e^N)$, $\sum_{S \in \mathscr{N}} \gamma_S\, v_\lambda(S) \le v_\lambda(N)$.*

This equivalence is due to Shapley (see Qin [15, Proposition 1]).

For $m \ge n$, define continuous functions $p^m : \Delta \to \mathbb{R}^N_{++}$ and $\beta^m : \Delta \to \mathbb{R}^N_+ \setminus \{0\}$ by

$$p^m_j(\lambda) = \begin{cases} \lambda_j & \text{if } \lambda_j \ge 1/m, \\ 1/m & \text{if } \lambda_j < 1/m, \end{cases}$$

and

$$\beta^m_j(\lambda) = \frac{\lambda_j}{p^m_j(\lambda)} = \begin{cases} 1 & \text{if } \lambda_j \ge 1/m, \\ m\lambda_j & \text{if } \lambda_j < 1/m. \end{cases}$$

We are now ready to give Qin's theorem [15, Theorem 1] on the nonemptiness of the inner core. Inoue [11] gives another proof to the theorem.

Theorem 1 (Qin) *Let $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. If there exists $m \geq n$ such that*

(i) *for every $\lambda \in \Delta_{1/m}$ and every $\gamma \in \widehat{\Gamma}(e^N)$, $\sum_{S \in \mathscr{N}} \gamma_S\, v_\lambda(S) \leq v_\lambda(N)$, and*
(ii) *for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and every $\gamma \in \widehat{\Gamma}(\beta^m(\lambda))$, there exists $u \in B(p^m(\lambda))$ such that $\sum_{S \in \mathscr{N}} \gamma_S\, v_{p^m(\lambda)}(S) \leq \lambda \cdot u$,*

then the inner core $IC(V)$ of V is nonempty.

By Proposition 1, condition (i) is weaker than the cardinal balancedness. Qin [15, Corollary 1] gives a class of NTU games that satisfy both conditions (i) and (ii) for some m: If a compactly generated NTU game V with $V(N)$ convex is *cardinally balanced with slack*, i.e., for every $\gamma \in \widehat{\Gamma}(e^N)$, $\sum_{S \in \mathscr{N}} \gamma_S\, V(S)$ is a subset of the interior of $V(N)$, then V satisfies conditions (i) and (ii) for some $m \geq n$, and, therefore, the inner core of V is nonempty.

Theorem 2 in Sect. 4 gives another class of NTU games with the nonempty inner core.

3 Characterization of Efficient Surface with Strictly Positive Normal Vectors

In the next section, we prove the nonemptiness of the inner core for an NTU game V where the normal cone to $V(N)$ at any individually rational and efficient payoff vector is a subset of $\mathbb{R}^N_{++} \cup \{0\}$. In this section, we provide characterizations of the set of individually rational and efficient payoff vectors with the above mentioned property.

Let $V(N)$ be a nonempty, closed, convex subset of $\mathbb{R}^N$ generated by a compact set $C_N \subseteq \mathbb{R}^N$, i.e., $V(N) = C_N - \mathbb{R}^N_+$. Let $b \in \mathbb{R}^N$ be such that $\{x \in V(N) \mid x \geq b\} \neq \emptyset$. Define

$$\begin{aligned}&\mathrm{Eff}(V(N), b)\\ &= \left\{x \in V(N) \;\middle|\; x \geq b,\ \text{there exists no } x' \in V(N) \text{ with } x' \geq x \text{ and } x' \neq x\right\}\end{aligned}$$

and

$$\mathrm{Eff_w}(V(N), b) = \left\{x \in V(N) \;\middle|\; x \geq b,\ \text{there exists no } x' \in V(N) \text{ with } x' \gg x\right\}.$$

The set $\mathrm{Eff}(V(N), b)$ (resp. $\mathrm{Eff_w}(V(N), b)$) is the set of individually rational and efficient payoff vectors (resp. individually rational and weakly efficient payoff vectors).

Remark 1

(1) $\emptyset \neq \mathrm{Eff}(V(N), b) \subseteq \mathrm{Eff_w}(V(N), b)$.
(2) $\mathrm{Eff_w}(V(N), b)$ is compact.

The nonemptiness of $\mathrm{Eff}(V(N), b)$ in statement (1) follows from the assumption that $V(N)$ is compactly generated and $\{x \in V(N) \mid x \geq b\} \neq \emptyset$. Statement (2) can be easily shown.

The following lemma gives a characterization of $\mathrm{Eff_w}(V(N), b)$ by normal vectors to $V(N)$.[3]

Lemma 1

$$\begin{aligned}&\mathrm{Eff_w}(V(N), b)\\&= \{x \in V(N) \mid x \geq b,\\&\qquad \text{there exists } \lambda \in \Delta \text{ such that, for every } y \in V(N),\ \lambda \cdot x \geq \lambda \cdot y\}.\end{aligned}$$

Proof Let $x \in \mathrm{Eff_w}(V(N), b)$. Then, $x \in V(N)$ and $x \geq b$. Furthermore, $V(N) \cap (\{x\} + \mathbb{R}^N_{++}) = \emptyset$. By the separation theorem for convex sets, there exists $\lambda \in \mathbb{R}^N \setminus \{0\}$ such that, for every $y \in V(N)$ and every $z \in \{x\} + \mathbb{R}^N_{++}$, $\lambda \cdot z \geq \lambda \cdot y$. Then, we have $\lambda \in \mathbb{R}^N_{+} \setminus \{0\}$. By normalization, we may assume that $\lambda \in \Delta$. Since the inner product is continuous, we have $\lambda \cdot x \geq \lambda \cdot y$ for every $y \in V(N)$.

We next prove the inverse inclusion. Let $x \in V(N)$ be such that $x \geq b$ and there exists $\lambda \in \Delta$ such that, for every $y \in V(N)$, $\lambda \cdot x \geq \lambda \cdot y$. Suppose, to the contrary, that $x \notin \mathrm{Eff_w}(V(N), b)$. Then, there exists $x' \in V(N)$ with $x' \gg x$. Thus, we have $\lambda \cdot x' > \lambda \cdot x$, a contradiction. Hence, we have $x \in \mathrm{Eff_w}(V(N), b)$. □

The following proposition gives a characterization of the individually rational and efficient surface such that every non-zero normal vector to $V(N)$ at every point of the surface is strictly positive.

Proposition 2 *The following three conditions are equivalent.*

(a) *Let* $x \in \mathrm{Eff}(V(N), b)$ *and* $\lambda \in \mathbb{R}^N \setminus \{0\}$ *be such that* $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$. *Then,* $\lambda \gg 0$. *Namely, for every* $x \in \mathrm{Eff}(V(N), b)$, *the normal cone to* $V(N)$ *at* x *is a subset of* $\mathbb{R}^N_{++} \cup \{0\}$.[4]
(b) *There exists* $b' \in \mathbb{R}^N$ *such that* $b' \ll b$ *and* $\mathrm{Eff}(V(N), b') = \mathrm{Eff_w}(V(N), b')$.
(c) *For every* $x \in V(N)$ *with* $x \geq b$ *and every* $S \in \mathcal{N}$ *with* $S \subsetneq N$, *there exist* $\varepsilon, \delta > 0$ *such that* $x - \varepsilon e^S + \delta e^{N \setminus S} \in V(N)$.

Condition (b) holds if $V(N)$ is *non-leveled*, i.e., $x = y$ whenever x and y are on the boundary of $V(N)$ and $x \geq y$. Condition (c) is due to Aubin [2, Proposition 2.3]. Although $\mathrm{Eff}(V(N), b)$ need not be closed (see Arrow et al. [1] for such an example), condition (a) implies that $\mathrm{Eff}(V(N), b)$ is closed as shown by the next lemma.

[3]For a convex subset A of $\mathbb{R}^N$ and $x \in A$, $\lambda \in \mathbb{R}^N$ is *normal* to A at x if $\lambda \cdot x \geq \lambda \cdot y$ for every $y \in A$.

[4]For a convex subset A of $\mathbb{R}^N$ and $x \in A$, the *normal cone* to A at x is the set of all vectors $\lambda \in \mathbb{R}^N$ normal to A at x.

Lemma 2 *Condition* (a) *of Proposition* 2 *implies that*

$$\mathrm{Eff}(V(N), b) = \mathrm{Eff}_{\mathrm{w}}(V(N), b).$$

Therefore, under condition (a), $\mathrm{Eff}(V(N), b)$ *is compact.*

***Proof of Lemma* 2** Suppose, to the contrary, that $\mathrm{Eff}(V(N), b) \subsetneq \mathrm{Eff}_{\mathrm{w}}(V(N), b)$. Then, there exists $x \in \mathrm{Eff}_{\mathrm{w}}(V(N), b)$ with $x \notin \mathrm{Eff}(V(N), b)$. By Lemma 1, there exists $\lambda \in \Delta$ such that, for every $y \in V(N)$, $\lambda \cdot x \geq \lambda \cdot y$.

Claim 1 $\lambda_i = 0$ for some $i \in N$.

***Proof of Claim* 1** Suppose, to the contrary, that $\lambda \gg 0$. Since $x \in V(N)$, $x \geq b$, and $x \notin \mathrm{Eff}(V(N), b)$, there exists $x' \in V(N)$ with $x' \geq x$ and $x' \neq x$. Therefore, we have $\lambda \cdot x' > \lambda \cdot x$, a contradiction. Thus, $\lambda_i = 0$ for some $i \in N$. □

Define $S = \{i \in N \mid \lambda_i = 0\}$. By Claim 1 above, we have $S \neq \emptyset$. Define

$$A = \left\{ y \in V(N) \;\middle|\; y^{N \setminus S} = x^{N \setminus S},\ y^S \geq x^S \right\}.$$

Since A is nonempty and compact, A has a maximal element with respect to $\geq$, i.e., there exists $\bar{y} \in A$ such that there exists no $y' \in A$ with $y' \geq \bar{y}$ and $y' \neq \bar{y}$. Note that, for every $x' \in V(N)$ with $x' \geq x$, we have $x' \in A$, since $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$.

Claim 2 $\bar{y} \in \mathrm{Eff}(V(N), b)$.

***Proof of Claim* 2** Suppose, to the contrary, that $\bar{y} \notin \mathrm{Eff}(V(N), b)$. Then, there exists $y' \in V(N)$ with $y' \geq \bar{y}$ and $y' \neq \bar{y}$. Since $y' \geq \bar{y} \geq x$, we have $y' \in A$. This contradicts that $\bar{y}$ is a maximal element in A. Thus, $\bar{y} \in \mathrm{Eff}(V(N), b)$. □

Since $\bar{y}^{N \setminus S} = x^{N \setminus S}$, we have $\lambda \cdot \bar{y} = \lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$. Thus, by condition (a), we have $\lambda \gg 0$, which contradicts that $\lambda_i = 0$ for every $i \in S$. Therefore, $\mathrm{Eff}(V(N), b) = \mathrm{Eff}_{\mathrm{w}}(V(N), b)$. The compactness of $\mathrm{Eff}(V(N), b)$ follows from Remark 1. This completes the proof of Lemma 2. □

Remark 2 By Lemma 2, we can replace $\mathrm{Eff}(V(N), b)$ by $\mathrm{Eff}_{\mathrm{w}}(V(N), b)$ in condition (a) of Proposition 2.

We are now ready to prove Proposition 2.

***Proof of Proposition* 2** (a) ⇒ (b): Suppose, to the contrary, that for every $r \in \mathbb{N}$,

$$\mathrm{Eff}(V(N), b - 1/r\, e^N) \subsetneq \mathrm{Eff}_{\mathrm{w}}(V(N), b - 1/r\, e^N).$$

Then, for every $r \in \mathbb{N}$, there exists $x^r \in \mathrm{Eff}_{\mathrm{w}}(V(N), b - 1/r\, e^N)$ such that $x^r \notin \mathrm{Eff}(V(N), b - 1/r\, e^N)$. By Lemma 1, for every $r \in \mathbb{N}$, there exists $\lambda^r \in \Delta$ with $\lambda^r \cdot x^r = \max_{y \in V(N)} \lambda^r \cdot y$. For every $r \in \mathbb{N}$, define $I_r = \{i \in N \mid \lambda_i^r = 0\}$. Since

$x^r \notin \mathrm{Eff}(V(N), b - 1/r\, e^N)$, we have $I_r \neq \emptyset$ for every $r \in \mathbb{N}$.[5] Since N has finitely many nonempty subsets, there exists $S \in \mathscr{N}$ such that $I_r = S$ for infinitely many r. By passing to a subsequence if necessary, we may assume that $I_r = S$ for every r. Since both sequences $(x^r)_r$ and $(\lambda^r)_r$ are bounded, by passing to further subsequences if necessary, we have

$$x^r \to x \quad \text{and} \quad \lambda^r \to \lambda \in \Delta.$$

Then, $x \in V(N)$ and $x \geq b$. Thus, we have also that $x \in \mathrm{Eff_w}(V(N), b)$. By Lemma 2, $x \in \mathrm{Eff}(V(N), b)$. Since $\lambda_i^r = 0$ for every $i \in S$ and every r, we have $\lambda_i = 0$ for every $i \in S$. Since $\tilde{\lambda} \mapsto \max_{y \in C_N} \tilde{\lambda} \cdot y$ is continuous, from

$$\lambda^r \cdot x^r = \max_{y \in V(N)} \lambda^r \cdot y = \max_{y \in C_N} \lambda^r \cdot y,$$

it follows that

$$\lambda \cdot x = \max_{y \in C_N} \lambda \cdot y = \max_{y \in V(N)} \lambda \cdot y.$$

Since $\lambda_i = 0$ for every $i \in S$, we have a contradiction.
(b) $\Rightarrow$ (c): Let $x \in V(N)$ with $x \geq b$ and let $\emptyset \neq S \subsetneq N$. Since $b' \ll b$, there exists $\varepsilon > 0$ such that $x - \varepsilon\, e^S \geq b'$. Since $x - \varepsilon\, e^S \notin \mathrm{Eff}(V(N), b') = \mathrm{Eff_w}(V(N), b')$ and $V(N) - \mathbb{R}^N_+ = V(N)$, there exists $\delta > 0$ such that $x - \varepsilon\, e^S + \delta\, e^N \in V(N)$. Since $x - \varepsilon\, e^S + \delta\, e^{N \setminus S} \leq x - \varepsilon\, e^S + \delta\, e^N$, we have $x - \varepsilon\, e^S + \delta\, e^{N \setminus S} \in V(N)$.
(c) $\Rightarrow$ (a): Let $x \in \mathrm{Eff}(V(N), b)$ and $\lambda \in \mathbb{R}^N \setminus \{0\}$ be such that $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$. Since $V(N) = C_N - \mathbb{R}^N_+$, we have $\lambda \in \mathbb{R}^N_+$. Suppose, to the contrary, that $\lambda_i = 0$ for some $i \in N$. By condition (c), there exist $\varepsilon, \delta > 0$ such that $x - \varepsilon\, e^{\{i\}} + \delta\, e^{N \setminus \{i\}} \in V(N)$. Since

$$\lambda \cdot \left(x - \varepsilon\, e^{\{i\}} + \delta\, e^{N \setminus \{i\}} \right) = \lambda \cdot x + \delta \sum_{j \in N \setminus \{i\}} \lambda_j > \lambda \cdot x,$$

we have a contradiction. Thus, $\lambda \gg 0$. □

Remark 3 If a compactly generated NTU game V satisfies one of the conditions of Proposition 2, its inner core $IC(V)$ is closed.

Proof Let $b \in \mathbb{R}^N$ be such that $b_i = \max\{u_i \in \mathbb{R} \mid u \in V(\{i\})\}$ for every $i \in N$. Let $(x^r)_r$ be a sequence in $IC(V)$ such that $x^r \to x$. Since $x^r \in \mathrm{Eff}(V(N), b)$ for every r and since, by Lemma 2, $\mathrm{Eff}(V(N), b)$ is closed, we have $x \in \mathrm{Eff}(V(N), b)$. For every r, let $\lambda^r \in \Delta^\circ$ be such that, for every $S \in \mathscr{N}$, $\lambda^{r,S} \cdot x^r \geq v_{\lambda^r}(S)$. Since $(\lambda^r)_r$ is a bounded sequence, by passing to a subsequence if necessary, we have $\lambda^r \to \lambda \in \Delta$. Since $\Delta \ni \tilde{\lambda} \mapsto v_{\tilde{\lambda}}(S) \in \mathbb{R}$ is continuous, we have $\lambda^S \cdot x \geq v_\lambda(S)$

[5]This can be shown by the same argument as Claim 1 in the proof of Lemma 2.

for every $S \in \mathscr{N}$. Thus, in particular, $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$. By condition (a) of Proposition 2, we have $\lambda \in \Delta^\circ$. Therefore, we have $x \in IC(V)$. Hence, $IC(V)$ is closed. □

The following example illustrates that condition (a), (b), or (c) of Proposition 2 can be weakened for the nonemptiness of the inner core.

Example 1 Let $N = \{1, 2\}$ and let $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ be such that $b_i = \max\{u_i \in \mathbb{R} \mid u \in V(\{i\})\} = 1$ for $i \in N$ and

$$V(N) = \left\{ u \in \mathbb{R}^N \mid u_1 + u_2 \leq 3,\ u_1 \leq 2,\ u_2 \leq 3 \right\}.$$

Then, V is compactly generated and $V(N)$ is convex. Note that $x := (2, 1) \in \mathrm{Eff}(V(N), b)$ (see Fig. 1). The vector $\lambda := (1, 0)$ is normal to $V(N)$ at x. Thus, condition (a) of Proposition 2 is violated. In this example, however, $V(N)$ can be extended to $V'(N)$ such that the pair $(V'(N), b)$ satisfies condition (a) of Proposition 2 and the inner core of the extended NTU game V' is *not* larger than the inner core of V. Therefore, we can weaken the conditions of Proposition 2 sufficient for the nonemptiness of the inner core of V.

For example, define

$$V'(N) = \left\{ u \in \mathbb{R}^N \mid u_1 + u_2 \leq 3,\ u_1 \leq 3,\ u_2 \leq 3 \right\}.$$

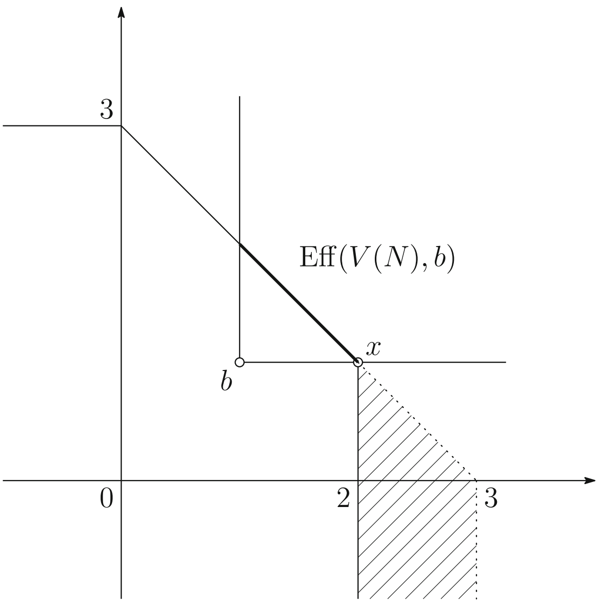

Fig. 1 V of Example 1

The shaded region in Fig. 1 is extended. The pair $(V'(N), b)$ satisfies condition (a) of Proposition 2. Any payoff vector y in the extended region is not in the inner core of the new game V', because it violates the individual rationality. Thus, the inner core $IC(V')$ of V' is not larger than the inner core $IC(V)$ of V. By the extension of $V(N)$, since the normal cone to the set of payoff vectors feasible for N at x becomes smaller,[6] $IC(V')$ can be smaller than $IC(V)$.[7] In this example, however, both inner cores are the same.

Condition (d) of the following proposition represents that, for some extension $V'(N)$ of $V(N)$, every non-zero normal vector to $V'(N)$ at every payoff vector in $\mathrm{Eff}(V'(N), b)$ is strictly positive.

Proposition 3 *The following two conditions are equivalent.*

(d) *There exists a nonempty, closed, convex subset $V'(N)$ of $\mathbb{R}^N$ such that $V(N) \subseteq V'(N)$, $V'(N)$ is generated by a compact set C'_N, $\{x \in V(N) \mid x \geq b\} = \{x \in V'(N) \mid x \geq b\}$, and condition* (a) *of Proposition* 2 *holds for* $(V'(N), b)$*, i.e.,*

$$\left[x \in \mathrm{Eff}(V'(N), b),\ \ \lambda \in \mathbb{R}^N \setminus \{0\},\ \ \textit{and}\ \lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y \right] \textit{ implies } \lambda \gg 0.$$

(e) *There exists a compact subset K of Δ° such that for every $x \in \mathrm{Eff_w}(V(N), b)$, there exists $\lambda \in K$ with $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$.*

Clearly, condition (a) of Proposition 2 implies condition (d). Thus, by Propositions 2 and 3, we have

$$\text{(a)} \;\Leftrightarrow\; \text{(b)} \;\Leftrightarrow\; \text{(c)} \;\Rightarrow\; \text{(d)} \;\Leftrightarrow\; \text{(e)}.$$

***Proof of Proposition* 3** (d) $\Rightarrow$ (e): Define a correspondence $\xi : \mathrm{Eff}(V'(N), b) \twoheadrightarrow \Delta$ by

$$\xi(x) = \left\{ \lambda \in \Delta \;\middle|\; \lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y \right\}.$$

Then, by Lemma 1, ξ is nonempty-valued. By Lemma 2, condition (d) implies that $\mathrm{Eff}(V'(N), b) = \mathrm{Eff_w}(V'(N), b)$ and this common set is compact. It is clear that ξ is compact-valued and upper hemi-continuous. Thus, $\xi(\mathrm{Eff}(V'(N), b))$ is compact.

Let $K = \xi(\mathrm{Eff}(V'(N), b))$. By condition (d), we have $K \subseteq \Delta^\circ$. From $\{x \in V(N) \mid x \geq b\} = \{x \in V'(N) \mid x \geq b\}$, it follows that

$$\mathrm{Eff_w}(V(N), b) = \mathrm{Eff_w}(V'(N), b) = \mathrm{Eff}(V'(N), b).$$

[6]In this example, $(2/3, 1/3)$ is normal to $V(N)$ at x, but this vector is not normal to $V'(N)$ at x.
[7]For an example of $IC(V') \subsetneq IC(V)$, see Example 2.

Therefore, for every $x \in \mathrm{Eff_w}(V(N), b)$, there exists $\lambda \in K$ with

$$\lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y \geq \max_{y \in V(N)} \lambda \cdot y.$$

Since $x \in V(N)$, we have $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$.
(e) $\Rightarrow$ (d): By condition (e), for every $x \in \mathrm{Eff_w}(V(N), b)$, we choose and fix a $\lambda^x \in K$ with $\lambda^x \cdot x = \max_{y \in V(N)} \lambda^x \cdot y$. Let $b' \in \mathbb{R}^N$ be such that $b' \ll b$ and $b' \leq y$ for every $y \in C_N$, where C_N is a compact set generating $V(N)$. Define

$$C'_N = \left\{ y \in \mathbb{R}^N \;\middle|\; y \geq b' \text{ and, for every } x \in \mathrm{Eff_w}(V(N), b),\ \lambda^x \cdot y \leq \lambda^x \cdot x \right\}.$$

Then, $C_N \subseteq C'_N$ and thus C'_N is nonempty. We have also that C'_N is compact and convex. Define $V'(N) = C'_N - \mathbb{R}^N_+$. Then, $V'(N)$ is nonempty, closed, convex, and satisfies $V(N) \subseteq V'(N)$. Therefore, $\{y \in V(N) \mid y \geq b\} \subseteq \{y \in V'(N) \mid y \geq b\}$.

Claim 3 $\{y \in V'(N) \mid y \geq b\} \subseteq \{y \in V(N) \mid y \geq b\}$.

Proof of Claim 3 Let $\bar{y} \in V'(N)$ be such that $\bar{y} \geq b$. Suppose, to the contrary, that $\bar{y} \notin V(N)$. Let $\bar{z} \in \mathbb{R}^N$ be the closest point in $\{y \in V(N) \mid y \geq b\}$ from $\bar{y}$, i.e., $\bar{z} \in V(N)$, $\bar{z} \geq b$, and

$$\|\bar{y} - \bar{z}\| = \min \{\|\bar{y} - z\| \mid z \in V(N),\ z \geq b\},$$

where $\|\cdot\|$ stands for the Euclidean norm. Since $\{y \in V(N) \mid y \geq b\}$ is nonempty, compact, and convex, $\bar{z}$ is uniquely determined.

We prove that $\bar{y} - \bar{z} \in \mathbb{R}^N_+ \setminus \{0\}$. Since $\bar{y} \notin V(N)$ and $\bar{z} \in V(N)$, we have $\bar{y} - \bar{z} \neq 0$. Suppose, to the contrary, that $\bar{y}_i < \bar{z}_i$ for some $i \in N$. Define $\widehat{z} \in \mathbb{R}^N$ by

$$\widehat{z}_j = \begin{cases} \bar{z}_j & \text{if } j \neq i, \\ \bar{y}_i & \text{if } j = i. \end{cases}$$

Since $\bar{z} \geq b$ and $\bar{y} \geq b$, we have $\widehat{z} \geq b$. We have also that $\widehat{z} \in \{\bar{z}\} - \mathbb{R}^N_+ \subseteq V(N) - \mathbb{R}^N_+ = V(N)$. Since $\|\bar{y} - \widehat{z}\| < \|\bar{y} - \bar{z}\|$, we have a contradiction. Hence, $\bar{y} - \bar{z} \in \mathbb{R}^N_+ \setminus \{0\}$.

We next prove that $\bar{z} \in \mathrm{Eff_w}(V(N), b)$. Suppose, to the contrary, that $\bar{z} \notin \mathrm{Eff_w}(V(N), b)$. Then, there exists $z' \in V(N)$ with $z' \gg \bar{z}$. Since $\bar{y} - \bar{z} \in \mathbb{R}^N_+ \setminus \{0\}$, $\bar{y}_k > \bar{z}_k$ for some $k \in N$. Define $\tilde{z} \in \mathbb{R}^N$ by

$$\tilde{z}_j = \begin{cases} \bar{z}_j & \text{if } j \neq k, \\ \min\{\bar{y}_k, z'_k\} & \text{if } j = k. \end{cases}$$

Since $z' \gg \bar{z} \geq b$ and $\bar{y} \geq b$, we have $\tilde{z} \geq b$. We have also that $\tilde{z} \in \{z'\} - \mathbb{R}^N_+ \subseteq V(N) - \mathbb{R}^N_+ = V(N)$. Since $\|\bar{y} - \tilde{z}\| < \|\bar{y} - \bar{z}\|$, we have a contradiction. Hence, $\bar{z} \in \mathrm{Eff_w}(V(N), b)$.

From $\bar{z} \in \mathrm{Eff_w}(V(N), b)$, it follows that $\lambda^{\bar{z}} \in K \subseteq \Delta^\circ$ and $\lambda^{\bar{z}} \cdot \bar{z} = \max_{y \in V(N)} \lambda^{\bar{z}} \cdot y$. Since $\lambda^{\bar{z}} \gg 0$ and $\bar{y} - \bar{z} \in \mathbb{R}^N_+ \setminus \{0\}$, we have $\lambda^{\bar{z}} \cdot \bar{y} > \lambda^{\bar{z}} \cdot \bar{z}$. Thus, by the definition of $V'(N)$, we have $\bar{y} \notin V'(N)$, which is a contradiction. Therefore, we have proven that $\{y \in V'(N) \mid y \geq b\} \subseteq \{y \in V(N) \mid y \geq b\}$. □

It remains to prove that condition (a) of Proposition 2 holds for $(V'(N), b)$. Let $x \in \mathrm{Eff}(V'(N), b)$ and $\lambda \in \mathbb{R}^N \setminus \{0\}$ be such that $\lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y$. Since $V'(N) - \mathbb{R}^N_+ = V'(N)$, we have $\lambda \in \mathbb{R}^N_+$. Suppose, to the contrary, that $\lambda_i = 0$ for some $i \in N$. Since $b' \ll b$, we have $x - \alpha\, e^{\{i\}} \geq b'$ for sufficiently small $\alpha > 0$. Since $K \subseteq \Delta^\circ$ is compact, there exists $\delta > 0$ such that for every $\lambda' \in K$ and every $j \in N$, $\lambda'_j > \delta$ holds. Thus, for every $z \in \mathrm{Eff_w}(V(N), b)$, we have $\lambda^z \cdot \left(\alpha\, e^{\{i\}}\right) = \alpha \lambda^z_i > \alpha\delta$. For every $z \in \mathrm{Eff_w}(V(N), b)$, since $x \in V'(N)$, we have $\lambda^z \cdot z \geq \lambda^z \cdot x$. Let $l \in N$ be such that $\lambda_l > 0$. Then, for every $z \in \mathrm{Eff_w}(V(N), b)$,

$$\begin{aligned}\lambda^z \cdot \left(x - \alpha\, e^{\{i\}} + \alpha\delta\, e^{\{l\}}\right) &\leq \lambda^z \cdot z + \lambda^z \cdot \left(-\alpha\, e^{\{i\}} + \alpha\delta\, e^{\{l\}}\right) \\ &< \lambda^z \cdot z - \alpha\delta + \alpha\delta = \lambda^z \cdot z.\end{aligned}$$

Since $x - \alpha\, e^{\{i\}} + \alpha\delta\, e^{\{l\}} \geq b'$, we have

$$x - \alpha\, e^{\{i\}} + \alpha\delta\, e^{\{l\}} \in C'_N \subseteq V'(N).$$

We have also that

$$\lambda \cdot \left(x - \alpha\, e^{\{i\}} + \alpha\delta\, e^{\{l\}}\right) = \lambda \cdot x + \alpha\delta\lambda_l > \lambda \cdot x,$$

which contradicts that $\lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y$. Therefore, $\lambda \gg 0$. This completes the proof of Proposition 3. □

The following example inspired by Qin [13, Example 2] illustrates that (1) $IC(V')$ can be strictly smaller than $IC(V)$ and (2) the inner core $IC(V)$ of V need not be closed when V satisfies condition (d) or (e) of Proposition 3. Recall that if V satisfies one of the conditions of Proposition 2, $IC(V)$ is closed (see Remark 3).

Example 2 Let $N = \{1, 2, 3\}$. An NTU game $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ is given by

$$\begin{aligned}V(\{1,2\}) &= \{u \in \mathbb{R}^{\{1,2\}}_+ \mid u_1^2 + u_2^2 \leq 4/25\} - \mathbb{R}^{\{1,2\}}_+, \\ V(N) &= \{u \in \mathbb{R}^N \mid u \cdot e^N = 1 \text{ and } u \geq 0\} - \mathbb{R}^N_+, \text{ and} \\ V(S) &= \{(0,0,0)\} - \mathbb{R}^S_+ \quad \text{for any other coalition } S.\end{aligned}$$

Then, V is compactly generated, $V(N)$ is convex, and $b = 0 \in \mathbb{R}^N$.

Note that, for every $t \in (2/5, 1]$, we have $x(t) := (0, t, 1-t) \in IC(V)$ as shown below. Let $t \in (2/5, 1]$. Then, $x(t) \in V(N)$. For $\lambda := (\lambda_1, (1-\lambda_1)/2, (1-\lambda_1)/2) \in$

$\mathbb{R}^N$ with $\lambda_1 \in (0, 1/3)$, let

$$z = \left(\frac{4\lambda_1}{5\sqrt{1 - 2\lambda_1 + 5\lambda_1^2}},\ \frac{2(1 - \lambda_1)}{5\sqrt{1 - 2\lambda_1 + 5\lambda_1^2}},\ 0 \right) \in V(\{1, 2\}).$$

Then, $v_\lambda(\{1, 2\}) = \lambda^{\{1,2\}} \cdot z$. Thus,

$$\lambda^{\{1,2\}} \cdot x(t) - v_\lambda(\{1, 2\}) = \lambda^{\{1,2\}} \cdot x(t) - \lambda^{\{1,2\}} \cdot z = \frac{1 - \lambda_1}{2} \cdot t - \frac{\sqrt{1 - 2\lambda_1 + 5\lambda_1^2}}{5}.$$

For sufficiently small $\lambda_1 > 0$, e.g., $\lambda_1 = (t - 2/5)/2$, we have

$$\frac{1 - \lambda_1}{2} \cdot t - \frac{\sqrt{1 - 2\lambda_1 + 5\lambda_1^2}}{5} > 0.$$

Thus, $v_\lambda(\{1, 2\}) \leq \lambda^{\{1,2\}} \cdot x(t)$ for sufficiently small $\lambda_1 > 0$. Since $0 < \lambda_1 < 1/3$, we have $v_\lambda(N) = (1 - \lambda_1)/2 = \lambda \cdot x(t)$. Since $v_\lambda(S) = 0$ for any other coalition S, it is clear that $v_\lambda(S) = 0 \leq \lambda^S \cdot x(t)$. Thus, for every $t \in (2/5, 1]$, $x(t) \in IC(V)$ holds.

Let $x^* := (0, 1/2, 1/2) \in IC(V)$. Note that $x^* \in \mathrm{Eff}(V(N), 0)$ and the vector $(0, 1/2, 1/2) \in \Delta$ is normal to $V(N)$ at x^*. Thus, the pair $(V(N), 0)$ does not satisfy condition (a) of Proposition 2. Define $V' : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ by

$$V'(N) = \left\{ u \in \mathbb{R}^N \,\middle|\, u \cdot e^N = 1 \quad \text{and} \quad u \geq -1/5\, e^N \right\} - \mathbb{R}^N_+$$

and $V'(S) = V(S)$ for every $S \in \mathcal{N} \setminus \{N\}$. Then, the pair $(V'(N), 0)$ satisfies condition (d) of Proposition 3.

We prove that $x^* \notin IC(V')$. Since $\mu := (1/3, 1/3, 1/3) \in \Delta$ is a unique normal vector to $V'(N)$ at x^* and since

$$\max_{u \in V'(\{1,2\})} \mu^{\{1,2\}} \cdot u = \frac{1}{3} \cdot \frac{\sqrt{2}}{5} + \frac{1}{3} \cdot \frac{\sqrt{2}}{5} = \frac{2\sqrt{2}}{15} > \frac{1}{6} = \mu^{\{1,2\}} \cdot x^*,$$

we have $x^* \notin IC(V')$. Hence, $IC(V') \subsetneq IC(V)$.

We finally prove that $IC(V)$ is not closed. Let $y := (0, 2/5, 3/5)$. For every $\lambda \in \Delta^\circ$, we have $v_\lambda(\{1, 2\}) > \lambda^{\{1,2\}} \cdot y$. Thus, $y \notin IC(V)$. Since $x(t) \in IC(V)$ for every $t \in (2/5, 1]$ and $x(t) \to y$ as $t \to 2/5$, $IC(V)$ is not closed. Therefore, even if an NTU game satisfies condition (d) or (e) of Proposition 3, its inner core need not be closed.

4 Nonemptiness of the Inner Core

We give the main result.

Theorem 2 *Let* $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ *be a compactly generated NTU game with* $V(N)$ *convex. If* V *is cardinally balanced and if* V *satisfies condition* (d) *or* (e) *of Proposition* 3, *then the inner core* $IC(V)$ *of* V *is nonempty.*

We prove this theorem by applying Theorem 1.

***Proof of Theorem* 2** Note first that the inner core satisfies the following covariance property.

Lemma 3 *Let* $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ *be an NTU game and let* $a \in \mathbb{R}^N$. *Define an NTU game* $V + a : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ *by*

$$(V + a)(S) = V(S) + \{a^S\} \quad \textit{for every } S \in \mathcal{N}.$$

Then, $IC(V + a) = IC(V) + \{a\}$.

This can be easily shown, so we omit the proof.

Since, by Proposition 3, conditions (d) and (e) are equivalent, we may assume that V satisfies condition (e). Let $b' \in \mathbb{R}^N$ be such that $b' \ll b$ and, for every $S \in \mathcal{N}$ and every $y \in C_S$, $b' \leq y$, where C_S is a compact subset of $\mathbb{R}^S$ with $V(S) = C_S - \mathbb{R}^S_+$. We define V' as in the proof of (e) $\Rightarrow$ (d). By condition (e), for every $x \in \mathrm{Eff_w}(V(N), b)$, we choose and fix a $\lambda^x \in K$ with $\lambda^x \cdot x = \max_{y \in V(N)} \lambda^x \cdot y$. Define

$$C'_N = \{y \in \mathbb{R}^N \mid y \geq b' \text{ and, for every } x \in \mathrm{Eff_w}(V(N), b), \lambda^x \cdot y \leq \lambda^x \cdot x\},$$

$V'(N) = C'_N - \mathbb{R}^N_+$, and $V'(S) = V(S)$ for every $S \in \mathcal{N}$ with $S \subsetneq N$. By the proof of (e) $\Rightarrow$ (d) of Proposition 3, $V'(N)$ satisfies all the properties of condition (d). Define an NTU game $\widehat{V}$ by $\widehat{V} = V' - b'$. Then, $\widehat{V}(N)$ is convex, $\widehat{V}$ is cardinally balanced,[8] and, for every $S \in \mathcal{N}$, $\widehat{V}(S)$ is generated by a compact subset of $\mathbb{R}^S_+$. Since $IC(\widehat{V} + b') = IC(V') \subseteq IC(V)$, by Lemma 3, it suffices to prove that $IC(\widehat{V}) \neq \emptyset$. We will prove that $\widehat{V}$ satisfies all the conditions of Theorem 1. Since $\widehat{V}$ is cardinally balanced, by Proposition 1, condition (i) of Theorem 1 is met for $\widehat{V}$ and for every $m \geq n$. It remains to prove that condition (ii) is met for $\widehat{V}$ and for some $m \geq n$, i.e., there exists $m \geq n$ such that, for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and every $\gamma \in \widehat{\Gamma}(\beta^m(\lambda))$, there exists $y \in B(p^m(\lambda))$ with $\sum_{S \in \mathcal{N}} \gamma_S v_{p^m(\lambda)}(S) \leq \lambda \cdot y$, where $B(p^m(\lambda))$ and $v_{p^m(\lambda)}$ are defined for NTU game $\widehat{V}$.

Since, by the definition of b', for every $S \in \mathcal{N} \setminus \{N\}$ and every $y \in C_S$, $y \geq b'$ holds and since, for every $y \in C'_N$, $y \geq b'$ holds, for every $m \geq n$, every $\lambda \in$

[8]Since V is cardinally balanced and $V(N) \subseteq V'(N)$, V' is cardinally balanced. Thus, $\widehat{V} = V' - b'$ also is cardinally balanced.

$\Delta^\circ \setminus \Delta_{1/m}$, and every $S \in \mathcal{N}$, we have $v_{p^m(\lambda)}(S) \geq 0$. Since $0 \leq \beta^m(\lambda) \leq e^N$ for every $\lambda \in \Delta$, $\widehat{V}$ is cardinally balanced, and $v_{p^m(\lambda)}(S) \geq 0$ for every $S \in \mathcal{N}$, we have, for every $m \geq n$, every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$, every $\gamma \in \widehat{\Gamma}(\beta^m(\lambda))$, and every $y \in B(p^m(\lambda))$,

$$\sum_{S \in \mathcal{N}} \gamma_S \, v_{p^m(\lambda)}(S) \leq v_{p^m(\lambda)}(N) = p^m(\lambda) \cdot y.$$

Thus, it suffices to prove that there exists $m \geq n$ such that, for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and every $y \in B(p^m(\lambda))$, $p^m(\lambda) \cdot y \leq \lambda \cdot y$ holds.

Since $K \subseteq \Delta^\circ$ is compact, there exists $k \geq n$ with $K \subseteq \Delta_{1/k}$. Let $m \in \mathbb{N}$ be such that $m > (n-1)(k-n+1)+1$.

Claim 4 $m > k$.

Proof of Claim 4 Since $k \geq n \geq 2$, we have

$$\begin{aligned} m - k &> (n-1)(k-n+1) + 1 - k = k(n-2) - (n-1)^2 + 1 \\ &\geq n(n-2) - (n-1)^2 + 1 = 0. \end{aligned}$$

Hence, $m > k$. □

Claim 5 Let $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and $x \in \mathrm{Eff_w}(\widehat{V}(N), 0)$ be such that

$$\lambda \cdot x = \max_{y \in \widehat{V}(N)} \lambda \cdot y.$$

Then, for every $i \in N$ with $\lambda_i < 1/m$, $x_i = 0$ holds.

Proof of Claim 5 Suppose, to the contrary, that $x_i > 0$ for some $i \in N$ with $\lambda_i < 1/m$. Since $\lambda \in \Delta$ and $\lambda_i < 1/m$, there exists $j \in N \setminus \{i\}$ such that

$$\lambda_j > \frac{1 - \frac{1}{m}}{n-1} = \frac{m-1}{m(n-1)}.$$

Define

$$E = \left\{ y \in \mathbb{R}^N_+ \;\middle|\; y^{N \setminus \{i,j\}} = x^{N \setminus \{i,j\}}, \; \mu \cdot y \leq \mu \cdot x \quad \text{for every } \mu \in \Delta_{1/k} \right\}.$$

We first prove that $E \subseteq C'_N - \{b'\}$. Let $y \in E$. Since $x \in \widehat{V}(N) = C'_N - \{b'\} - \mathbb{R}^N_+$, we have $x + b' \in C'_N - \mathbb{R}^N_+$. Hence, by the definition of C'_N, for every $z \in \mathrm{Eff_w}(V(N), b)$, $\lambda^z \cdot (x + b') \leq \lambda^z \cdot z$ holds. Since $\lambda^z \in K \subseteq \Delta_{1/k}$,

$$\lambda^z \cdot (y + b') \leq \lambda^z \cdot (x + b') \leq \lambda^z \cdot z.$$

Since $y \geq 0$, we have $y + b' \geq b'$. Thus, $y + b' \in C'_N$. Hence, $y \in C'_N - \{b'\}$ and $E \subseteq C'_N - \{b'\}$.

For every $l \in N$, define $\mu^{k,l} \in \Delta_{1/k}$ by $\mu^{k,l}_t = 1/k$ for $t \in N \setminus \{l\}$ and $\mu^{k,l}_l = 1 - (n-1)/k$. Then, $\{\mu^{k,1}, \ldots, \mu^{k,n}\}$ is the set of all extreme points of $\Delta_{1/k}$. Therefore,

$$\begin{aligned} E &= \left\{ y \in \mathbb{R}^N_+ \,\middle|\, y^{N\setminus\{i,j\}} = x^{N\setminus\{i,j\}},\ \mu^{k,l} \cdot y \leq \mu^{k,l} \cdot x \quad \text{for every } l \in N \right\} \\ &= \left\{ y \in \mathbb{R}^N_+ \,\middle|\, y^{N\setminus\{i,j\}} = x^{N\setminus\{i,j\}},\ \mu^{k,i} \cdot y \leq \mu^{k,i} \cdot x,\ \mu^{k,j} \cdot y \leq \mu^{k,j} \cdot x \right\} \\ &= \{ y \in \mathbb{R}^N_+ \mid y^{N\setminus\{i,j\}} = x^{N\setminus\{i,j\}},\ (k-n+1)y_i + y_j \leq (k-n+1)x_i + x_j, \\ &\qquad y_i + (k-n+1)y_j \leq x_i + (k-n+1)x_j \}. \end{aligned}$$

Define $z^* \in \mathbb{R}^N$ by

$$\begin{aligned} &z^{*N\setminus\{i,j\}} = x^{N\setminus\{i,j\}}, \\ &z^*_i = 0, \\ &z^*_j = x_j + \tfrac{1}{k-n+1} x_i \end{aligned}$$

(See Fig. 2). Then, $z^* \in E \subseteq C'_N - \{b'\} \subseteq \widehat{V}(N)$. Since $\lambda_j > \frac{m-1}{m(n-1)}$ and $m > (n-1)(k-n+1)+1$, we have

$$\frac{\lambda_j}{k-n+1} > \frac{m-1}{m(n-1)(k-n+1)} > \frac{m-1}{m(m-1)} = \frac{1}{m} > \lambda_i.$$

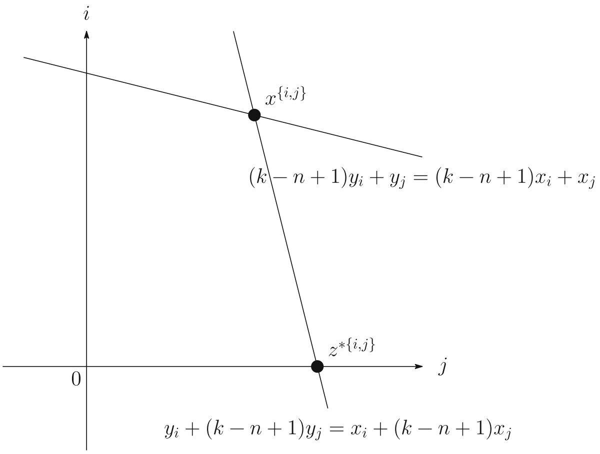

Fig. 2 Definition of $z^{*\{i,j\}}$

Therefore, since $x_i > 0$, we have

$$\begin{aligned}\lambda \cdot z^* &= \lambda \cdot x^{N\setminus\{i,j\}} + \lambda_j x_j + \frac{\lambda_j}{k-n+1} x_i \\ &> \lambda \cdot x^{N\setminus\{i,j\}} + \lambda_j x_j + \lambda_i x_i \\ &= \lambda \cdot x = \max_{y \in \widehat{V}(N)} \lambda \cdot y \geq \lambda \cdot z^*,\end{aligned}$$

a contradiction. Therefore, $x_i = 0$ for every $i \in N$ with $\lambda_i < 1/m$. □

Define $\sigma^m : \Delta \to \Delta_{1/(m+n-1)}$ by

$$\sigma_j^m(\lambda) = \frac{p_j^m(\lambda)}{p^m(\lambda) \cdot e^N}.$$

Since

$$\begin{aligned}B(p^m(\lambda)) &= \left\{ y \in \widehat{V}(N) \;\middle|\; p^m(\lambda) \cdot y = \max_{z \in \widehat{V}(N)} p^m(\lambda) \cdot z \right\} \\ &= \left\{ y \in \widehat{V}(N) \;\middle|\; \sigma^m(\lambda) \cdot y = \max_{z \in \widehat{V}(N)} \sigma^m(\lambda) \cdot z \right\} \\ &= \left\{ y \in C'_N - \{b'\} \;\middle|\; \sigma^m(\lambda) \cdot y = \max_{z \in \widehat{V}(N)} \sigma^m(\lambda) \cdot z \right\},\end{aligned}$$

by Lemma 1 and the definition of b', we have $B(p^m(\lambda)) \subseteq \mathrm{Eff_w}(\widehat{V}(N), 0)$ for every $\lambda \in \Delta$. Let $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and let $i \in N$ with $\lambda_i < 1/m$. Then, $p_i^m(\lambda) = 1/m$ and $p^m(\lambda) \cdot e^N > 1$. Thus, $\sigma_i^m(\lambda) < 1/m$. By Claim 5, for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$, every $y \in B(p^m(\lambda))$, and every $i \in N$ with $\lambda_i < 1/m$, $y_i = 0$ holds. Therefore, for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$ and every $y \in B(p^m(\lambda))$,

$$\begin{aligned}p^m(\lambda) \cdot y &= \sum_{j:\lambda_j \geq 1/m} p_j^m(\lambda) y_j + \sum_{j:\lambda_j < 1/m} p_j^m(\lambda) y_j \\ &= \sum_{j:\lambda_j \geq 1/m} p_j^m(\lambda) y_j = \sum_{j:\lambda_j \geq 1/m} \lambda_j y_j = \lambda \cdot y.\end{aligned}$$

Hence, condition (ii) of Theorem 1 holds for $\widehat{V}$ and m. Thus, $IC(\widehat{V}) \neq \emptyset$. As we mentioned before, this implies that $IC(V) \neq \emptyset$. □

Qin [15, Example 1] exemplifies that, if a cardinally balanced NTU game does not satisfy condition (d) or (e) of Proposition 3, then its inner core can be empty. We give another example of a totally cardinally balanced NTU game V with every $V(S)$

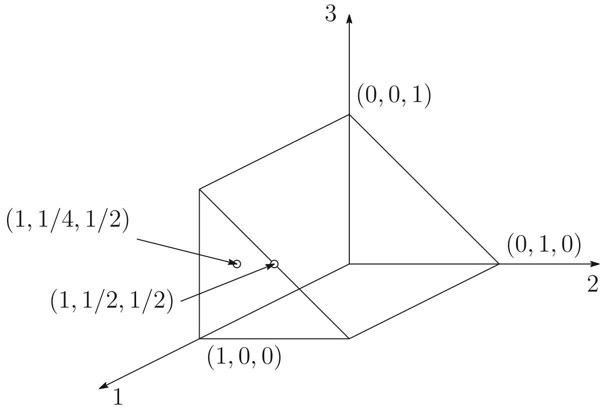

Fig. 3 $V(N)$ of Example 3

polyhedral such that the inner core $IC(V)$ of V is empty.[9] Billera and Bixby [4] proved that any totally cardinally balanced NTU game with every $V(S)$ polyhedral can be generated by an exchange economy where agents' consumption sets have the form $[0, 1]^l$ and their utility functions are concave and continuous. Therefore, by the following example, the nonemptiness of the inner core is irrelevant to such representation of NTU games. It should be worth mentioning that in the first step of the proof of Billera and Bixby's representation, the inner core plays an essential role. Actually, $\bar{x}$ in the proof of Lemma 3.2 of Billera and Bixby [4] is an inner core payoff vector.

Example 3 Let $N = \{1, 2, 3\}$. An NTU game $V : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ is given by

$$V(\{i\}) = \{(0, 0, 0)\} - \mathbb{R}_+^{\{i\}} \quad \text{for every } i \in N,$$
$$V(\{1, 2\}) = \{(1, 1/2, 0)\} - \mathbb{R}_+^{\{1,2\}},$$
$$V(\{1, 3\}) = \{(1, 0, 1)\} - \mathbb{R}_+^{\{1,3\}},$$
$$V(\{2, 3\}) = \{(0, 0, 0)\} - \mathbb{R}_+^{\{2,3\}}, \text{ and}$$
$$V(N) = \{x \in \mathbb{R}^N \mid x_i \leq 1 \text{ for every } i \in N \text{ and } x_2 + x_3 \leq 1\}.$$

Then, $b = 0 \in \mathbb{R}^N$. Figure 3 depicts $V(N) \cap \mathbb{R}_+^N$. Note that every $V(S)$ is a polyhedron. For a balancing vector γ with $\gamma_{\{1,2\}} = \gamma_{\{1,3\}} = \gamma_{\{2,3\}} = 1/2$ and

[9] An NTU game V is *totally cardinally balanced* if every subgame of V is cardinally balanced, i.e., for every $S \in \mathscr{N}$ and every $\gamma \in \Gamma(e^S)$, $\sum_{T \in \mathscr{N}} \gamma_T V(T) \subseteq V(S)$ holds.

$\gamma_S = 0$ otherwise, we have

$$\frac{1}{2}\left(1, \frac{1}{2}, 0\right) + \frac{1}{2}(1, 0, 1) + \frac{1}{2}(0, 0, 0) = \left(1, \frac{1}{4}, \frac{1}{2}\right) \in V(N).$$

For any other balancing vector γ' of weights, we can show that $\sum_{S \in \mathcal{N}} \gamma'_S V(S) \subseteq V(N)$. Hence, V is cardinally balanced. Moreover, any subgame of V is cardinally balanced and, therefore, V is totally cardinally balanced. Since $(1, 0, 0)$ is normal to $V(N)$ at $(1, 1/2, 1/2) \in \mathrm{Eff}(V(N), 0)$ and since $(1, 1/2, 1/2) \gg b$, there exists no $V'(N)$ satisfying condition (d) of Proposition 3.

We prove that $IC(V) = \emptyset$. Let $\lambda \in \mathbb{R}^N_{++}$. Since $\lambda^{\{1,3\}} \cdot (1, 0, 1) > \lambda^{\{1,3\}} \cdot x$ for every $x \in V(N)$ with $x \geq 0$ and $x \neq (1, 0, 1)$, payoff vector $(1, 0, 1)$ is a unique candidate of an element of the inner core. Since $(1, 1/2, 0) \in V(\{1, 2\})$ and

$$\lambda^{\{1,2\}} \cdot \left(1, \frac{1}{2}, 0\right) = \lambda_1 + \frac{1}{2}\lambda_2 > \lambda_1 = \lambda^{\{1,2\}} \cdot (1, 0, 1),$$

payoff vector $(1, 0, 1)$ is not in the inner core. Therefore, $IC(V) = \emptyset$.

5 NTU Games Generated by Exchange Economies

de Clippel and Minelli [8, Proposition 1] proved that, in an exchange economy where every agent has a continuous, concave, and strongly monotone utility function, the utility vector at any Walrasian allocation is always in the inner core. Since there exists a Walrasian equilibrium for such an exchange economy, the inner core is nonempty. In this section, we prove that an NTU game generated by such an exchange economy satisfies condition (a) of Proposition 2. Thus, by Theorem 2, its inner core is nonempty. Therefore, our class of NTU games with the nonempty inner core contains de Clippel and Minelli's [8] class.

An *exchange economy* with n consumers is a list of the commodity space $\mathbb{R}^L$, where L is a finite set of commodities, and consumers' characteristics $(u^i, \omega^i)_{i \in N}$ such that, for every consumer i, utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous, concave, and strongly monotone, and endowment vector ω^i is in $\mathbb{R}^L_+ \setminus \{0\}$. An exchange economy is denoted by $\mathscr{E} = (\mathbb{R}^L, (u^i, \omega^i)_{i \in N})$. For every coalition $S \in \mathcal{N}$, let $F_{\mathscr{E}}(S)$ be the set of feasible S-allocations, i.e.,

$$F_{\mathscr{E}}(S) = \left\{ (z^i)_{i \in S} \;\middle|\; z^i \in \mathbb{R}^L_+ \quad \text{for every } i \in S \text{ and } \quad \sum_{i \in S} \left(z^i - \omega^i\right) = 0 \right\}.$$

A feasible N-allocation $(z^i)_{i \in N}$ is a *Walrasian allocation* for $\mathscr{E}$ if there exists a price vector $p \in \mathbb{R}^L$ such that, for every $i \in N$, $p \cdot z^i \leq p \cdot \omega^i$ and $u^i(z^i) \geq u^i(y)$ for every $y \in \mathbb{R}^L_+$ with $p \cdot y \leq p \cdot \omega^i$. Let $W(\mathscr{E})$ be the set of all Walrasian allocations for $\mathscr{E}$.

A feasible N-allocation $(z^i)_{i \in N}$ is an *inner core allocation* for $\mathscr{E}$ if there exists $\lambda \in \mathbb{R}^N_{++}$ such that, for every $S \in \mathscr{N}$ and every $(y^i)_{i \in S} \in F_{\mathscr{E}}(S)$, $\sum_{i \in S} \lambda_i u^i(z^i) \geq \sum_{i \in S} \lambda_i u^i(y^i)$ holds. Let $IC(\mathscr{E})$ be the set of all inner core allocations for $\mathscr{E}$.

An exchange economy $\mathscr{E} = (\mathbb{R}^L, (u^i, \omega^i)_{i \in N})$ generates an NTU game $V_{\mathscr{E}} : \mathscr{N} \twoheadrightarrow \mathbb{R}^N$ by defining

$$V_{\mathscr{E}}(S) = \left\{ v \in \mathbb{R}^S \;\middle|\; \text{there exists } (z^i)_{i \in S} \in F_{\mathscr{E}}(S) \text{ such that, for every } i \in S, v_i \leq u^i(z^i) \right\}.$$

Since $F_{\mathscr{E}}(S)$ is nonempty, compact, and convex for every $S \in \mathscr{N}$, NTU game $V_{\mathscr{E}}$ is compactly convexly generated. Note that, for every $i \in N$,

$$b_i = \max\{v_i \in \mathbb{R} \mid v \in V_{\mathscr{E}}(\{i\})\} = u^i(\omega^i),$$

and $IC(V_{\mathscr{E}}) = \{(u^i(z^i))_{i \in N} \mid (z^i)_{i \in N} \in IC(\mathscr{E})\}$.

Qin [13, Remark 2] states the following without proof. For completeness, we give its proof.

Proposition 4 *Let $\mathscr{E} = (\mathbb{R}^L, (u^i, \omega^i)_{i \in N})$ be an exchange economy where, for every $i \in N$, $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous, concave, and strongly monotone, and $\omega^i \in \mathbb{R}^L_+ \setminus \{0\}$. Then, NTU game $V_{\mathscr{E}}$ generated by $\mathscr{E}$ satisfies condition* (a) *of Proposition 2, i.e., if $x \in \mathrm{Eff}(V_{\mathscr{E}}(N), b)$ and $\lambda \in \mathbb{R}^N \setminus \{0\}$ satisfy $\lambda \cdot x = \max_{y \in V_{\mathscr{E}}(N)} \lambda \cdot y$, then $\lambda \gg 0$.*

Proof Let $x \in \mathrm{Eff}(V_{\mathscr{E}}(N), b)$ and $\lambda \in \mathbb{R}^N \setminus \{0\}$ be such that $\lambda \cdot x = \max_{y \in V_{\mathscr{E}}(N)} \lambda \cdot y$. Since $V_{\mathscr{E}}(N) - \mathbb{R}^N_+ = V_{\mathscr{E}}(N)$, we have $\lambda \in \mathbb{R}^N_+$. Suppose, to the contrary, that there exists $j \in N$ with $\lambda_j = 0$. Since $\lambda \in \mathbb{R}^N_+ \setminus \{0\}$, there exists $k \in N$ with $\lambda_k > 0$. From $x \in V_{\mathscr{E}}(N)$, it follows that there exists a feasible N-allocation $(z^i)_{i \in N}$ such that $u^i(z^i) \geq x_i$ for every $i \in N$. Since $u^j(z^j) \geq x_j \geq b_j = u^j(\omega^j)$, $\omega^j \in \mathbb{R}^L_+ \setminus \{0\}$, and u^j is strongly monotone, we have $z^j \in \mathbb{R}^L_+ \setminus \{0\}$. Define an allocation $(y^i)_{i \in N}$ by

$$y^i = \begin{cases} z^i & \text{if } i \in N \setminus \{j, k\}, \\ z^k + z^j & \text{if } i = k, \\ 0 & \text{if } i = j. \end{cases}$$

Then, $(y^i)_{i \in N}$ is a feasible N-allocation. Therefore, $u := (u^i(y^i))_{i \in N} \in V_{\mathscr{E}}(N)$. Since $\lambda_j = 0$, $\lambda_k > 0$, and $u^k(z^k + z^j) > u^k(z^k)$, we have

$$\begin{aligned} \lambda \cdot u = \sum_{i \in N} \lambda_i u^i(y^i) &= \sum_{i \in N \setminus \{j,k\}} \lambda_i u^i(z^i) + \lambda_k u^k(z^k + z^j) + \lambda_j u^j(0) \\ &> \sum_{i \in N \setminus \{j,k\}} \lambda_i u^i(z^i) + \lambda_k u^k(z^k) + \lambda_j u^j(z^j) \geq \lambda \cdot x. \end{aligned}$$

This contradicts that $\lambda \cdot x = \max_{y \in V_{\mathscr{E}}(N)} \lambda \cdot y$. Therefore, $\lambda \gg 0$ and hence $V_{\mathscr{E}}$ satisfies condition (a). □

Since an exchange economy with the properties in Proposition 4 generates a cardinally balanced NTU game,[10] by Theorem 2, we have the following.

Corollary 1 *Let $\mathscr{E} = (\mathbb{R}^L, (u^i, \omega^i)_{i \in N})$ be an exchange economy where, for every $i \in N$, $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ is continuous, concave, and strongly monotone, and $\omega^i \in \mathbb{R}^L_+ \setminus \{0\}$. Then, the inner core $IC(V_{\mathscr{E}})$ of NTU game $V_{\mathscr{E}}$ generated by $\mathscr{E}$ is nonempty.*

We can show this corollary without relying on Theorem 2. de Clippel and Minelli [8, Proposition 1] proved the inclusion $W(\mathscr{E}) \subseteq IC(\mathscr{E})$ for an exchange economy $\mathscr{E}$ satisfying the properties in Corollary 1. Since there exists a Walrasian equilibrium for $\mathscr{E}$, the inner core of $V_{\mathscr{E}}$ is nonempty.

6 Concluding Remarks

The inner core has a relation to the *strictly inhibitive set*, the set of payoff vectors stable against randomized blocking plans (see Myerson [12, Section 9.8] and Qin [13]). For a compactly generated NTU game, its inner core is a subset of the strictly inhibitive set (Qin [13, Theorem 2]). Thus, our Theorem 2 gives a sufficient condition for the nonemptiness of the strictly inhibitive set. Furthermore, if $V(S)$ is convex for every $S \in \mathscr{N}$ and if V satisfies one of the conditions of Proposition 2, then the inner core coincides with the strictly inhibitive set (Qin [13, Theorem 4]).

By Proposition 4, an exchange economy with continuous, concave, and strongly monotone utility functions generates an NTU game satisfying the cardinal balancedness and condition (a) of Proposition 2. Since different economies can generate the same NTU game, exchange economies without the properties in Proposition 4 or production economies may generate NTU games satisfying condition (d) or (e) of Proposition 3. Billera [3] proved that every compactly generated, totally cardinally balanced NTU game V with every $V(S)$ convex can be generated by a production economy where every consumer has a upper semi-continuous and concave utility function on a compact convex consumption set and has his own compact convex production set. Billera's induced production economy can be converted to an exchange economy (see Billera and Bixby [5]). Inoue [9] proved that every compactly generated NTU game can be generated by a coalition production economy. In both induced economies due to Billera [3] and due to Inoue [9], the inner core coincides with the set of utility vectors at Walrasian allocations (see Qin [14] and Inoue [9], respectively). Thus, if an NTU game satisfies all the assumptions of Theorem 2, then there exists a Walrasian equilibrium for both Billera's and Inoue's induced economies.

[10] This can be shown by the same method as Billera and Bixby [4, Theorem 2.1].

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