Quasi-isometry and Rigidity

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Abstract This is a brief exposition on the quasi-isometric rigidity of irreducible lattices in Lie groups. The basic notions in coarse geometry are recalled and illustrated. It is beyond the scope of these notes to go into the proofs of most of the results stated here. We shall be content with pointing the reader to standard references for detailed proofs. These notes are based on my talk in the *International Conference on Mathematics and its Analysis and Applications in Mathematical Modelling* held at Jadavpur University, Kolkata, in December 2017.

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1 Introduction

Let (X, d_X), (Y, d_Y) be metric spaces and let $\lambda \geq 1$, $\epsilon \geq 0$ be real numbers. A set-map $f : X \to Y$ is called a (λ, ϵ)-*quasi-isometric embedding* if the following condition holds: $-\epsilon + \lambda^{-1} d_X(x_0, x_1) \leq d_Y(f(x_0), f(x_1)) \leq \lambda d_X(x_0, x_1) + \epsilon$ for all $x_0, x_1 \in X$. If there exists a $C \geq 0$ such that each $y \in Y$ is at a distance at most C from the image $f(X)$, we say that f is C-dense. A quasi-isometric embedding which is C-dense for some $C \geq 0$, is called a (λ, ϵ, C)-*quasi-isometric equivalence* or, more briefly, a *quasi-isometry*. When $f : X \to Y$ is a (λ, ϵ)-quasi-isometric embedding with $\epsilon = 0$, then f is necessarily continuous. In general, however, f need not be so. When f is

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a (λ, ϵ, C) quasi-isometry, there exist $\mu \geq 1, \delta \geq 0, D$ and a $g : Y \to X$ which is a (μ, δ, D)-quasi-isometry such that $g \circ f$ and $f \circ g$ are a bounded distance away from id_X and id_Y respectively, that is, $||g \circ f - id_X|| := \sup_{x \in X} d_X(g(f(x)), x) < \infty$ and $||f \circ g - id_Y|| = \sup_{y \in Y} d_Y(f(g(y)), y) < \infty$. We say that f, g are quasi-inverses of each other and that X, Y are of the same quasi-isometry type; we write $X \sim_{\text{qi}} Y$. Being of the same quasi-isometry type is an 'equivalence relation' on the class of all metric spaces. A $(\lambda, 0, 0)$-quasi-isometry $f : (X, d_X) \to (Y, d_Y)$ is nothing but a bi-Lipschitz homeomorphism. Quite often, the specific values of λ, ϵ, C are not so important and so we usually omit explicit mention of them. As a first example, $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a $(1, 0, 1/2)$-isometry equivalence with $(1, 1, 0)$-quasi-inverse $\mathbb{R} \to \mathbb{Z}$ defined as $x \mapsto \lfloor x \rfloor$.

On the set of all quasi-isometry self-equivalences of (X, d_X), one has an equivalence relation where $f \sim g$ if $||f - g|| = \sup_{x \in X} d_X(f(x), g(x)) < \infty$ for quasi-isometries $f, g : X \to X$. The set of equivalence classes form a group $\mathcal{QI}(X)$, called the group of quasi-isometries of (X, d_X), where $[f] \cdot [g] = [f \circ g]$. The group of isometries will be denoted $\text{Isom}(X)$. One has a natural homomorphism $\text{Isom}(X) \to \mathcal{QI}(X)$ defined as $f \mapsto [f]$. In general, $\text{Isom}(X)$ and $\mathcal{QI}(X)$ are not closely related. For example, $\mathcal{QI}(\mathbb{S}^n)$ is trivial whereas $\text{Isom}(\mathbb{S}^n)$ is the orthogonal group $\text{O}(n+1)$. On the other hand, when $X = \mathbb{Z} \cup \{3/4\} \subset \mathbb{R}$ the group $\text{Isom}(X)$ is trivial whereas it can be seen $\mathcal{QI}(X) \cong \mathcal{QI}(\mathbb{R})$ contains the group $\text{GL}(1, \mathbb{R})$; indeed $\mathcal{QI}(\mathbb{R})$ is a rather large group. See [15].

The notion of quasi-isometry captures the essential features of the *large scale* geometry of a metric space, that is, those features which are remain when "viewed from far away." For example, any two (non-empty) bounded metric spaces are quasi-isometrically equivalent to each other. Also if $B \subset X$ is a bounded subset of X, then X is quasi-isometrically equivalent to $X \setminus B$. More generally, if $Y \subset X$ is a C-dense subset of X, (i.e., if any $x \in X$ is at a distance at most C from a $y \in Y$) then X and Y are quasi-isometrically equivalent. Conversely, if the inclusion $Y \hookrightarrow X$ is a quasi-isometry, then Y is C-dense for some $C > 0$. *Coarse geometry* is the study of properties of metric spaces which remain invariant under quasi-isometric equivalence. An important problem in coarse geometry is the classification problem, which asks to classify metric spaces according to their quasi-isometry type. A part of this problem is to study invariants of quasi-isometry, which may be used to distinguish quasi-isometry types. Our aim here will be to give a brief exposition of the concept of quasi-isometric rigidity and to state the results concerning rigidity properties of lattices in semisimple Lie groups. I omit all the proofs and point the reader to relevant sources.

1.1 Groups as Geometric Objects

One of main objectives of coarse geometry is the study of (finitely generated) groups viewed as geometric objects via the word metric—a point of view that resulted in explosive growth of the subject since the seminal work of Gromov [8]. More precisely, suppose that Γ is finitely generated group and $S \subset \Gamma$ a finite generating set. One has the word metric defined as $d(\gamma_0, \gamma_1) = l_S(\gamma_0^{-1}\gamma_1)\ \forall \gamma_0, \gamma_1 \in S$; here $l_S(\gamma)$ denotes the length of γ as a word in $S \cup S^{-1}$, that is, $l_S(\gamma) = k$ where $k \geq 0$ is the smallest integer for which γ has an expression $\gamma = a_1 \cdots a_k, a_j \in S \cup S^{-1}, 1 \leq j \leq k$, with $l_S(1) := 0$. It turns out that, changing the generating set S to another finite generating set S' leads to another metric $d_{S'}$ but does not change the quasi-isometry type of (Γ, d_S). In fact, it is easily seen that d_S and $d_{S'}$ are bi-Lipschitz equivalent, i.e, the identity map $(\Gamma, d_S) \to (\Gamma, d_{S'})$ is bi-Lipschitz.

Recall that the Cayley graph $\mathcal{C} = \mathcal{C}(\Gamma, S)$ of (Γ, S) is a graph whose set of vertices is Γ and $(\gamma, \gamma') \in \Gamma \times \Gamma$ is an (oriented) edge whenever $\gamma^{-1}.\gamma'$ is in S. There is a natural metric on $\mathcal{C}$ in which each edge has length 1 and $d(\gamma, \gamma') = l_S(\gamma^{-1}\gamma')$. This metric is invariant under the natural Γ-action on the left of $\mathcal{C}$. The inclusion $\Gamma \hookrightarrow \mathcal{C}$ is 1-dense and hence is a quasi-isometry.

A metric space (X, d_X) is *proper* if closed all balls in X of finite radii are compact. (X, d_X) is a *length space* if $d_X(x_0, x_1) = \inf l(\sigma)$ where the infimum is taken over all (rectifiable) paths σ from x_0 to x_1 and $l(\sigma)$ denotes the length of σ. It is said to be a *geodesic metric space* if, for any $x_0, x_1 \in X$, there is a path $\sigma : [0, l] \to X$ from x_0, to x_1 such that $d_X(\sigma(t), \sigma(t')) = |t - t'|, \forall t, t' \in [0, l]$. Such a path is called a *geodesic*.

Suppose that Γ acts on a metric space (X, d_X). The action is *properly discontinuous* if, given any $x \in X$, there exists an open neighbourhood U of x such that $U \cap \gamma U \neq \emptyset$ for at most finitely many elements $\gamma \in \Gamma$. The action is said to be *cocompact* if $\Gamma\backslash X$ is compact; equivalently, there is a compact set $K \subset X$ such that $X = \cup_{\gamma \in \Gamma} \gamma K$.

The following theorem is often referred to as the fundamental theorem of coarse geometry. It was first proved by V.A. Efremovich in 1953 and by A. Švarc in 1955. It was later rediscovered by J. Milnor in 1968. It generalizes the above observation that a finitely generated group Γ (with word metric d_S) is quasi-isometric to its Cayley graph $\mathcal{C}(\Gamma, S)$. We refer the reader to [1, Chapter I.8, Prop. 8.19] for a proof.

Theorem 1.1 (Švarc-Milnor Lemma) *Suppose that a group* Γ *acts properly discontinuously and cocompactly by isometries on a proper length metric space* (X, d_X). *Then* Γ *is a finitely generated group. If* $S \subset \Gamma$ *is any finite generating set and if* $x_0 \in X$ *is arbitrary, then the map* $\gamma \mapsto \gamma.x_0$ *is a quasi-isometry* $(\Gamma, d_S) \to (X, d_X)$. □

Example 1.2 (i) Let Γ be a group generated by a finite set $S \subset \Gamma$. If $\Lambda \subset \Gamma$ is a finite index subgroup, then $\Gamma \sim_{qi} \Lambda$. This follows from Theorem 1.1 by restricting the action of Γ on $\mathcal{C}(\Gamma, S)$ to Λ. Also if N is a finite subgroup of Γ, then $\Gamma \sim_{qi} \Gamma/N =: \bar{\Gamma}$. To see this, we assume, as we may, that $N \setminus \{1\} \subset S$ and that no two distinct elements of $S \setminus N$ are in the same coset γN. Then it is readily seen that $l_{\bar{S}}(\bar{\gamma}) \leq l_S(\gamma) \leq l_{\bar{S}}(\bar{\gamma}) + 1$ where $\bar{\gamma}$ denotes $\gamma N \in \bar{\Gamma}$ and $\bar{S} := \{\bar{s} \mid s \in S \setminus N\} \subset \bar{\Gamma}$. It follows that the canonical quotient map $\eta : (\Gamma, d_S) \to (\bar{\Gamma}, d_{\bar{S}})$ is a (λ, ϵ, C)-quasi-isometry where $\lambda = 1, \epsilon = 1, C = 0$. Alternatively one may apply Švarc-Milnor lemma (Theorem 1.1) to the action of Γ (via the quotient map $\Gamma \to \bar{\Gamma}$) on the Cayley graph $\mathcal{C}(\bar{\Gamma}, \bar{S})$ where $\bar{S}$ is any finite generating set of $\bar{\Gamma}$. Note that the Γ-action is proper since N is finite.

(ii) Two groups Γ_0 and Γ_1 are said to be *commensurable* if there exists a group Γ such that Γ is isomorphic to a finite index subgroup of Γ_i for $i = 0, 1$. Since intersection of two finite index subgroups is again of finite index, commensurability is an equivalence relation. For example, any free group F_n of rank $n \geq 2$ may be realised as a finite index subgroup of F_2. Thus any two non-abelian free groups of finite rank are commensurable. As another example, let G_0 and G_1 be finite groups with $o(G_0) \geq 2$, $o(G_1) \geq 3$, then their free product $G := G_0 * G_1$ contains a non-abelian free group F of finite rank such that F has finite index in G. Thus G is commensurable with F_2.

We say that Γ_0 and Γ_1 are *weakly commensurable* if there exist finite normal subgroups $N_i \subset \Gamma_i, i = 0, 1$, such that Γ_0/N_0 and Γ_1/N_1 are commensurable. Weak commensurability is an equivalence relation. This follows from two observations: (a) the normal subgroup $NN' \subset \Gamma$ is finite whenever N, N' are finite normal subgroups a group Γ, and, (b) commensurability is an equivalence relation. From (i), we see that weakly commensurable groups are quasi-isometrically equivalent (with respect to any word metrics). As an application, recall that $\mathrm{PSL}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})/\pm I$ is isomorphic to a free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. It follows that SL(2) is weakly commensurable with F_2 and so $\mathrm{SL}(2, \mathbb{Z}) \sim_{\mathrm{qi}} F_n$ for any $n \geq 2$.

(iii) Suppose that S_g is a closed connected oriented surface of genus g. If $g \geq 2$, then one has a finite covering projection $S_g \to S_2$. It follows that the fundamental group $\Gamma_g := \pi_1(S_g)$ is a finite index subgroup of $\pi_1(S_2)$. So $\Gamma_g \sim_{\mathrm{qi}} \Gamma_2$ for $g \geq 2$. On the other hand, $S_g = \mathcal{H}/\Gamma_g$ when $g \geq 2$ where $\mathcal{H}$ is the Poincaré upper half space and Γ_g acts freely and properly discontinuously via isometries on $\mathcal{H}$. Applying Švarc-Milnor lemma, we see that $\Gamma_g \sim_{\mathrm{qi}} \mathcal{H}$ for ≥ 2. In the case of the torus $S_1 = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2/\mathbb{Z}^2$, and, again by the Švarc-Milnor lemma, $\Gamma_1 \sim_{\mathrm{qi}} \mathbb{Z}^2 \sim_{\mathrm{qi}} \mathbb{R}^2$.

The group $\Gamma_g, g > 1$, is not quasi-isometric to $\mathbb{Z}^n$ for any n. The fact that Γ_g contains a non-abelian free group F implies that the number $b_k(\Gamma_g)$ of elements in a ball of radius k (with respect to a word metric) grows exponentially in k, whereas it grows at a polynomial rate in the case of $\mathbb{Z}^n$. It is known that the growth rate of the function $k \to b_k(\Gamma)$ of a finitely generated group (Γ, d_S) is a quasi-isometric invariant. This proves our assertion. See [1, Chapter I.8] for details.

It has been shown that if Γ is a finitely generated group which is virtually nilpotent, then Γ has polynomial growth. A major landmark result, whose proof due to Gromov [8] greatly influenced the development of geometric group theory, is the converse: *A finitely generated group* Γ *is virtually nilpotent if it has polynomial growth.*

2 Quasi-isometric Rigidity

When two finitely generated groups are quasi-isometrically equivalent, one would like to know how closely they are related as *algebraic* objects. Since weakly commensurable groups are quasi-isometric, one may ask whether it is possible to *recover* the group, up to weak commensurability, from its quasi-isomorphism type. This leads to the notions of quasi-isometric rigidity. We refer the reader to [1, Chapter I.8] and [7, §8.6] for detailed discussions on this topic.

There are several variants of rigidity, we consider only two: one version captures the idea that a rigid group is one with the property that any group quasi-isometric to it should be weakly commensurable to it. Another version is based on the idea that a rigid group is one which has a relatively 'small' quasi-isometry group.

Definition 2.1 (i) We say that a finitely generated group Γ with a word metric is *quasi-isometrically rigid.* if any finitely generated group quasi-isometric to Γ is weakly commensurable to Γ. (ii) Let $\mathcal{G}$ be a class of finitely generated groups which is closed under weak commensurability. We say that $\mathcal{G}$ is *quasi-isometrically rigid* if a finitely generated group Λ is quasi-isometric to a $\Gamma \in \mathcal{G}$, then $\Lambda \in \mathcal{G}$.

Obviously, the trivial group is quasi-isometrically rigid. For a non-trivial example, Bridson and Gersten [2] have shown that if a group is quasi-isometrically equivalent $\mathbb{Z}^n$, then it is virtually $\mathbb{Z}^n$; see [13] for a more general result. Thus $\mathbb{Z}^n$, $n \in \mathbb{N}$, is quasi-isometrically rigid. By applying a theorem of Stallings on the structure of groups with infinitely many ends, it can be shown that any group quasi-isometric to a finitely generated (non-abelian) free group is virtually free.

Example 2.2 (i) The class of all finitely presented groups is quasi-isometrically rigid; see [1, Chapter I.8, Prop. 8.24]. (ii) The class of all finitely generated virtually nilpotent groups is quasi-isometrically rigid, since, by Gromov's polynomial growth theorem, any such group is virtually nilpotent. (iii) If Γ is rigid, then the class $\mathcal{G}(\Gamma)$ of all groups which are weakly commensurable with Γ is quasi-isometrically rigid.

Let Γ be a finitely generated group and let $\mathcal{A}(\Gamma)$ be the set of all isomorphisms $\phi : H_0 \to H_1$ where H_0, H_1 are arbitrary finite index subgroups of Γ. One has an equivalence relation on $\mathcal{A}(\Gamma)$ defined as $\phi \sim \psi$ where $\psi : H_0' \to H_1'$ if $\phi|_K = \psi|_K$ for some finite index subgroup $K \subset H_0 \cap H_1$. The equivalence classes are called virtual automorphisms of Γ and the set $\mathcal{A}(\Gamma)/\sim$ is denoted $\mathrm{Vaut}(\Gamma)$. If Γ has no finite index subgroup, then $\mathrm{Vaut}(\Gamma) = \mathrm{Aut}(\Gamma)$. When Γ is residually finite, the group $\mathrm{Vaut}(\Gamma)$ is particularly interesting. For example, it is not difficult to show that $\mathrm{Vaut}(\mathbb{Z}^n) \cong \mathrm{GL}(n; \mathbb{Q})$. Note that any isomorphism $\phi : H_0 \to H_1$ defines an element of $\mathcal{QI}(\Gamma)$ leading to a well-defined homomorphism $\mathrm{Vaut}(\Gamma) \to \mathcal{QI}(\Gamma)$. In general this homomorphism is not surjective. For example, the linear action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ yields an embedding of $\mathrm{GL}(n, \mathbb{R})$ into $\mathcal{QI}(\mathbb{R}^n) \cong \mathcal{QI}(\mathbb{Z}^n)$ whereas $\mathrm{Vaut}(\mathbb{Z}^n) = \mathrm{GL}(n, \mathbb{Q})$.

Definition 2.3 (i) A finitely generated group Γ is said to be *strongly quasi-isometrically rigid* if the natural homomorphism $\mathrm{Vaut}(\Gamma) \to \mathcal{QI}(\Gamma)$ is a surjection. (ii) A metric space X is quasi-isometrically rigid if $\mathrm{Isom}(X) \to \mathcal{QI}(X)$ is an isomorphism of groups (Sec §3.4 [9]).

The infinite cyclic group is quasi-isometrically rigid but not strongly.

Before proceeding further, we recall some standard notions concerning lattices in semisimple Lie groups. The reader is referred to [14, 19] for detailed expositions.

2.1 Lattices in Semisimple Lie Group

Let G be a connected Lie group. One says that G is *simple* if it has no connected normal subgroup. G is called *semisimple* if G is an almost direct product $G_1 \cdots G_k$ where G_i are simple normal subgroups of G. Here almost direct product means that $G_i \cap G_j$ is a finite normal subgroup of G. When G is semisimple, $G_i \cap G_j$ is in fact contained in the centre of G. We will assume that G is a connected non-compact semisimple Lie group and that it has finite centre, denoted $Z(G)$. Let K be a maximal compact subgroup of G and then $X := G/K$ is connected and admits a G-invariant Riemannian metric. It is a globally symmetric space of non-compact type. Our assumption that G is non-compact implies that X has non-positive sectional curvature.

The real rank of a linear semisimple Lie group $G \subset \mathrm{GL}(N)$ may be defined as the maximum number d such that G has a diagonalizable subgroup isomorphic to $(\mathbb{R}^\times)^d$. When G is not linear, (example a non-trivial cover of $\mathrm{SL}(2, \mathbb{R})$) one defines its real rank to be that of $G/Z(G)$ (which is always linear). For example, the real rank of $\mathrm{SL}(n, \mathbb{R})$ equals $n-1$ whereas the real rank of $\mathrm{SO}_0(p, q)$ is $\min p, q$. (Recall that $\mathrm{SO}_0(p, q)$ is the identity component of all linear transformations of $\mathbb{R}^{p+q}$ which preserve the quadratic form $\beta(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$.)

A discrete subgroup Γ of G is called a *lattice* if the homogeneous space $\Gamma\backslash G$ carries a G-invariant measure with respect to which its volume is finite. For example, it is known that $\mathrm{SL}(n, \mathbb{Z}) \subset \mathrm{SL}(n, \mathbb{R})$ is a lattice. A lattice $\Gamma \subset G$ is *uniform* if $\Gamma\backslash G$ is compact; otherwise it is *nonuniform*. One says that a lattice $\Gamma \subset G$ is *reducible* if it contains infinite subgroups Γ_0, Γ_1 which generate a subgroup Λ isomorphic to an almost direct product $\Gamma_0 \cdot \Gamma_1$ such that Λ has finite index in Γ. We say that Γ is *irreducible* if it is not reducible. It can be shown that if Γ is a lattice in a non-compact semisimple Lie group G with finite centre and finitely many components, then it is weakly commensurable with a lattice in a connected Lie group with trivial centre.

Let Γ be an irreducible nonuniform lattice in a connected semisimple Lie group G. Then the centre $Z(\Gamma)$ of Γ is finite and $\Gamma/Z(\Gamma)$ is a lattice in $G/Z(G)$. Define the commensurator of Γ, $\mathrm{Comm}(\Gamma)$, to be the group $\{g \in G \mid \Gamma$ is commensurable with $g\Gamma g^{-1}\}$. It is clear that $\Gamma \subset \mathrm{Comm}(\Gamma)$.

If G has trivial centre and no compact factors and if Γ is an irreducible lattice, then: either Γ is non-arithmetic and the group $\mathrm{Comm}(\Gamma)$ is also a lattice in G, or, Γ is an 'arithmetic lattice' and $\mathrm{Comm}(\Gamma)$ is dense in G. This result due to Margulis. In the latter case, under a further hypothesis (namely that G *equals* the $\mathbb{R}$-points of a $\mathbb{Q}$-algebraic group), $\mathrm{Comm}(\Gamma)$ is the rational points $G(\mathbb{Q})$ of G. Thus, in this case, the group $\mathrm{Comm}(\Gamma)$ is a countable dense subgroup of G. See [19] for these results and for the definition of arithmetic lattices.

Restriction of the conjugation by an element $g \in \mathrm{Comm}(\Gamma)$ to $\Gamma \cap g\Gamma g^{-1}$ yields a well-defined element of $\mathcal{QI}(\Gamma)$ since $\Gamma \cap g\Gamma g^{-1} \hookrightarrow \Gamma$ is a quasi-isometry. This leads to a homomorphism $\mathrm{Comm}(\Gamma) \to \mathcal{QI}(\Gamma)$.

We are ready state the result concerning quasi-isometry group of lattices in semisimple Lie groups. The final result is the outcome work of several mathematicians. The beautiful survey article by Farb [5] outlines not only the proofs, but also the history of the problem including description of the contributions to this problem of the mathematicians whose work we have merely cited.

Theorem 2.4 (Schwartz *[16, 17],* Eskin *[3],* Farb and Schwartz *[6]*) *Suppose that* Γ *is an irreducible nonuniform lattice in a semisimple Lie group G with trivial centre and without compact factors. If G is not locally isomorphic to* $SL(2, \mathbb{R})$*, then* $\mathcal{QI}(\Gamma)$ *is isomorphic to* $Comm(\Gamma)$.

When G is locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$, then any nonuniform lattice Γ is virtually free. Thus $\mathcal{QI}(\Gamma) \cong \mathcal{QI}(F_2)$ and the conclusion of the above theorem fails. Denote by ∂F_2 the end space of the Cayley graph $\mathcal{C}(F_2)$. It is homeomorphic to the Cantor space. It is known that $\mathcal{QI}(F_2)$ is a certain group of homeomorphisms of the Cantor space known as the group of *quasisymmetric homeomorphisms*.

When Γ is uniform, it is quasi-isometric to $X = G/K$ by Švarc-Milnor lemma. It follows that $\mathcal{QI}(\Gamma) \cong \mathcal{QI}(X)$. It turns out that when G has real rank at least 2, $\mathrm{Isom}(X) \to \mathcal{QI}(X)$ is an isomorphism. This is also true when X is a quaternionic hyperbolic space $\mathrm{Sp}(n, 1)/K$ or the Cayley hyperbolic plane F_4/K.

Theorem 2.5 (Mostow *[12],* Tukia *[18],* Koryani and Reimann *[11],* Kleiner and Leeb *[10],* Eskin and Farb *[4],* Pansu *[13]*) *Suppose that* Γ *is a uniform lattice in a non-compact simple Lie group G such that either the real rank of G is at least* 2 *or* $X = G/K$ *is either a quaternionic hyperbolic the Cayley hyperbolic plane. Then* $\mathcal{QI}(\Gamma) \cong Isom(X)$.

We have left out irreducible uniform lattices in the rank 1 groups locally isomorphic to $\mathrm{SO}_0(n, 1)$, $n \geq 3$, or to $\mathrm{SU}(n, 1)$, $n \geq 2$. In these cases, the corresponding symmetric spaces X are real and complex hyperbolic spaces, denoted $\mathcal{H}^n$ and $\mathbb{C}\mathcal{H}^n$ respectively. Note that if Γ is any such lattice, then $\Gamma \sim_{\mathrm{qi}} X$ by the Švarc-Milnor lemma. Associated to X is its 'boundary' ∂X, which is homeomorphic to the sphere $\mathbb{S}^{n-1}$ or $\mathbb{S}^{2n-1}$ according as X is the real or complex hyperbolic n-space. It turns out that any quasi-isometry of X induces a homeomorphism of ∂X leading to a homomorphism $\mathcal{QI}(\Gamma) \cong \mathcal{QI}(X) \to \mathrm{Homeo}(\partial X)$. It is known that this is a monomorphism and the image is a certain group known as the group of *quasi-conformal group of homeomorphisms* of ∂X.

Finally we end this note with the following theorem:

Theorem 2.6 (Quasi-isometric rigidity for irreducible lattices) *Let G be a connected semisimple Lie group with trivial centre and without compact factors and let* Γ *be an irreducible lattice in G. If* Λ *is any finitely generated group that is quasi-isometric to* Γ*, then there exists a finite normal subgroup* $F \subset \Lambda$ *such that* Λ/F *is isomorphic to a lattice* Γ' *in G.*

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