



# Stochastic Duality and Orthogonal Polynomials

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*To Charles M. Newman on his 70<sup>th</sup> birthday*

**Abstract.** For a series of Markov processes we prove stochastic duality relations with duality functions given by orthogonal polynomials. This means that expectations with respect to the original process (which evolves the variable of the orthogonal polynomial) can be studied via expectations with respect to the dual process (which evolves the index of the polynomial). The set of processes include interacting particle systems, such as the exclusion process, the inclusion process and independent random walkers, as well as interacting diffusions and redistribution models of Kipnis–Marchioro–Presutti type. Duality functions are given in terms of classical orthogonal polynomials, both of discrete and continuous variable, and the measure in the orthogonality relation coincides with the process stationary measure.

**Keywords:** Interacting particle systems · Duality · Orthogonal polynomials · Exclusion process

## 1 Motivations and Main Results

### 1.1 Introduction

Duality theory is a powerful tool to deal with stochastic Markov processes by which information on a given process can be extracted from another process, its *dual*. The link between the two processes is provided by a set of so-called *duality functions*, i.e. a set of observables that are functions of both processes and whose expectations, with respect to the two randomness, can be placed in a precise relation (see Definition 1 below). It is the aim of this paper to enlarge the space of duality functions for a series of Markov processes that enjoy the stochastic

duality property. These novel duality functions are in turn connected to the polynomials that are orthogonal with respect to the inner product provided by the stationary measure of those processes. Thus the paper develops, via a series of examples, a connection between probabilistic objects (duality functions) and orthogonal polynomials. See [4] for a previous example of a process with duality functions in terms of Hermite polynomials. Moreover, in [10, 21, 41] it is shown how stochastic duality and orthogonal polynomials are connected from an algebraic perspective.

Before going to the statement of the results, it is worth to stress the importance of a duality relation. Duality theory has been used in several different contexts. Originally introduced for interacting particle systems in [45] and further developed in [36], the literature on stochastic duality covers nowadays a host of examples. Duality has been applied – among the others – to boundary driven and/or bulk driven models of transport [5, 27, 40, 46], Fourier’s law [23, 29], diffusive particle systems and their hydrodynamic limit [20, 36], asymmetric interacting particle systems scaling to KPZ equation [1, 6, 8, 9, 17, 19, 26], six vertex models [7, 18], multispecies particle models [2, 3, 33–35], correlation inequalities [25], mathematical population genetics [12, 38]. In all such different contexts it is used, in a way or another, the core simplification that duality provides. Namely, in the presence of a dual process, computation of  $k$ -point correlation functions of the original process is mapped into the study of the evolution of only  $k$  dual particles, thus substantially reducing the difficulty of the problem.

Besides applications, it is interesting to understand the mathematical structure behind duality. This goes back to classical works by Schütz and collaborators [43, 44], where the connection between stochastic duality and symmetries of quantum spin chains was pointed out. More recently, the works [13, 14, 24, 34, 35] further investigate this framework and provide an algebraic approach to Markov processes with duality starting from a Lie algebra in the symmetric case, and its quantum deformation in the asymmetric one. In this approach duality functions emerge as the intertwiners between two different representations. Therefore one has a constructive theory, in which duality functions arise from representation theory.

An interesting problem is to fully characterize the set of all duality functions. This question was first asked in [38] where it was defined the concept of duality space, i.e. the subspace of all measurable functions on the configuration product space of two Markov processes for which the duality relation (see Definition 1 below) holds. In [38] the dimension of this space is computed for some simple systems and, as far as we know, a general answer is not available in the general case. Although the algebraic approach recalled above yields a duality function from two representations of a (Lie) algebra, it is not clear a-priori if every duality function can be derived from this approach. In this paper we follow a different route. We shall show that duality functions can be placed in relation to orthogonal polynomials.

### 1.2 Results

We recall the classical definition of stochastic duality.

**Definition 1 (Duality of processes).** *Let  $X = (X_t)_{t \geq 0}$  and  $N = (N_t)_{t \geq 0}$  be two continuous time Markov processes with state spaces  $\Omega$  and  $\Omega^{dual}$ , respectively. We say that  $N$  is dual to  $X$  with duality function  $D : \Omega \times \Omega^{dual} \mapsto \mathbb{R}$  if*

$$\mathbb{E}_x[D(X_t, n)] = \mathbb{E}_n[D(x, N_t)], \tag{1}$$

for all  $x \in \Omega$ ,  $n \in \Omega^{dual}$  and  $t \geq 0$ . In (1)  $\mathbb{E}_x$  (respectively  $\mathbb{E}_n$ ) is the expectation w.r.t. the law of the  $X$  process initialized at  $x$  (respectively the  $N$  process initialized at  $n$ ). If  $X$  and  $N$  are the same process, we say that  $X$  is self-dual with self-duality function  $D$ .

Under suitable hypothesis (see [28]), the above definition is equivalent to the definition of duality between Markov generators.

**Definition 2 (Duality of generators).** *Let  $L$  and  $L^{dual}$  be generators of the two Markov processes  $X = (X_t)_{t \geq 0}$  and  $N = (N_t)_{t \geq 0}$ , respectively. We say that  $L^{dual}$  is dual to  $L$  with duality function  $D : \Omega \times \Omega^{dual} \rightarrow \mathbb{R}$  if*

$$[LD(\cdot, n)](x) = [L^{dual}D(x, \cdot)](n) \tag{2}$$

where we assume that both sides are well defined. In the case  $L = L^{dual}$  we shall say that the process is self-dual and the self-duality relation becomes

$$[LD(\cdot, n)](x) = [LD(x, \cdot)](n). \tag{3}$$

In (2) (resp. (3)) it is understood that  $L$  on the lhs acts on  $D$  as a function of the first variable  $x$ , while  $L^{dual}$  (resp.  $L$ ) on the rhs acts on  $D$  as a function of the second variable  $n$ . The Definition 2 is easier to work with, so we will always work under the assumption that the notion of duality (resp. self-duality) is the one in Eq. (2) (resp. (3)).

*Remark 1.* If  $D(x, n)$  is duality function between two processes and

$$c : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$$

is constant under the dynamics of the two processes then  $c(x, n)D(x, n)$  is also duality function. We will always consider duality functions modulo the quantity  $c(x, n)$ . For instance, the interacting particle systems studied in Sect. 3 conserve the total number of particles and thus  $c$  is an arbitrary function of such conserved quantity.

In this paper we shall prove that several Markov processes, for which a duality function is known from the algebraic approach, also admit a different duality function given in terms of polynomials that are orthogonal with respect to the process stationary measure.

The processes that we consider include discrete interacting particle systems (exclusion process, inclusion process and independent random walkers process) as well as interacting diffusions (Brownian momentum process, Brownian energy process) and redistribution models that are obtained via a thermalization limit (Kipnis–Marchioro–Presutti processes). Their generators, that are defined in Sects. 3 and 4, have an algebraic structure from which duality functions have been previously derived [24].

The orthogonal polynomials we use are some of those with hypergeometric structure. More precisely we consider classical orthogonal polynomials, both discrete and continuous, with the exception of discrete Hahn polynomials and continuous Jacobi polynomials. Through this paper we follow the definitions of orthogonal polynomials given in [39].

The added value of linking duality functions to orthogonal polynomials lies on the fact that they constitute an orthogonal basis of the associated Hilbert space. Often in applications [9, 13, 27] some quantity of interest are expressed in terms of duality functions, for instance the current in interacting particle systems. This is then used in the study of the asymptotic properties and relevant scaling limits. For these reasons it seems reasonable that having an orthogonal basis of polynomials should be useful in those analysis.

The following theorem collects the results of this paper, details and rigorous proofs can be found in Sect. 3 for self-duality and Sect. 4 for duality.

**Theorem 1.** *For the processes listed below, the following duality relations hold true*

(i) **Self-duality**

<i>Process</i>	<i>Stationary measure</i>	<i>Duality function</i> <i><math>D(x, n)</math>: product of</i>
<i>Exclusion Process</i> <i>with up to <math>2j</math> particles,</i> <i>SEP(<math>j</math>)</i>	<i>Binomial(<math>2j, p</math>)</i>	$K_n(x) / \binom{2j}{n}$
<i>Inclusion Process</i> <i>with parameter <math>k</math>, SIP(<math>k</math>)</i>	<i>Negative Binomial (<math>2k, p</math>)</i>	$M_n(x) \frac{\Gamma(2k)}{\Gamma(2k+n)}$
<i>Independent Random</i> <i>Walkers, IRW</i>	<i>Poisson(<math>\lambda</math>)</i>	$C_n(x)$

where  $K_n(x)$  stands for Krawtchouk polynomials,  $M_n(x)$  for Meixner polynomials,  $C_n(x)$  for Charlier polynomials.

(ii) **Duality**

<i>Process</i>	<i>Stationary measure</i>	<i>Duality function</i> $D(x, n)$ : product of	<i>Dual Process</i>
<i>Brownian Momentum Process, BMP</i>	<i>Gaussian</i> (0, $\sigma^2$ )	$\frac{1}{(2n-1)!!} H_{2n}(x)$	<i>Inclusion process</i> with $k = 1/4$
<i>Brownian Energy Process</i> with parameter $k$ , <i>BEP</i> ( $k$ )	<i>Gamma</i> ( $2k, \theta$ )	$\frac{n!\Gamma(2k)}{\Gamma(2k+n)} L_n^{(2k-1)}(x)$	<i>Inclusion process</i> with parameter $k$
<i>Kipnis–Marchioro–Presutti</i> with parameter $k$ , <i>KMP</i> ( $k$ )	<i>Gamma</i> ( $2k, \theta$ )	$\frac{n!\Gamma(2k)}{\Gamma(2k+n)} L_n^{(2k-1)}(x)$	<i>dual-KMP</i> ( $k$ ) process with parameter $k$

where  $H_n(x)$  stands for Hermite polynomials and  $L_n^{(2k-1)}(x)$  for generalized Laguerre polynomials.

*Remark 2.* As can be inferred from the table, the duality function is not, in general, the orthogonal polynomial itself, but a suitable normalization. Moreover, for the process defined on multiple sites the duality functions exhibit a product structure where each factor is in terms of the relevant orthogonal polynomial.

**1.3 Comments**

*Old and New Dualities.* It is known that the six processes we consider satisfy the same (self-)duality relation described by the chart above with different (self-)duality functions. New and old (self-)duality functions can be related using the explicit form of the polynomial that appears in the new one via a relation of the following type:

$$D^{new}(x, n) = \sum_{k=0}^n d(k, n) D^{old}(x, k)$$

where  $d(k, n)$  also depends on the parameter of the stationary measure. If, for example, we consider the self-duality of the Independent Random Walker, where the new self-duality functions are given by the Charlier polynomials and the old self-duality functions are given by the falling factorial, then we have

$$D^{new}(x, n) = \sum_{k=0}^n \binom{n}{k} (-\lambda)^{n-k} \frac{x!}{(x-k)!} \quad D^{old}(x, n) = \frac{x!}{(x-n)!}$$

so that

$$D^{new}(x, n) = \sum_{k=0}^n \binom{n}{k} (-\lambda)^{n-k} D^{old}(x, k)$$

where  $\lambda$  is the Poisson distribution parameter.

*The Norm Choice.* A sequence of orthogonal polynomials  $\{p_n(x) : n \in \mathbb{N}\}$  on the interval  $(a, b)$  is defined by the choice of a measure  $\mu$  to be used in the scalar product and a choice of a norm  $d_n^2 > 0$

$$\langle p_n, p_m \rangle := \int_a^b p_n(x)p_m(x)d\mu(x) = \delta_{n,m}d_n^2.$$

The main result of this paper states that orthogonal polynomials with a properly chosen *normalization* furnish duality functions for the processes we consider. In our setting the probability measure  $\mu$  to be used is provided by the marginal stationary measure of the process. The appropriate norm has been found by making the ansatz that new duality functions can be obtained by the Gram–Schmidt orthogonalization procedure initialized with the old duality functions derived from the algebraic approach [24]. Namely, if one starts from a sequence  $p_n(x)$  of orthogonal polynomials with norm  $d_n^2$  and claims that  $D^{new}(x, n) = b_n p_n(x)$ , where  $b_n$  is the appropriate normalization, then the previous ansatz yields

$$b_n = \frac{\langle D^{old}(\cdot, n), p_n \rangle}{d_n^2}. \quad (4)$$

*General Strategy of the Proof.* Once a properly normalized orthogonal polynomial is identified as a candidate duality function, then the proof of the duality statement is obtained via explicit computations. Each proof heavily relies on the *hypergeometric structure* of the polynomials involved. The idea is to use three structural properties of the polynomials family  $p_n(x)$ , i.e. the differential or difference hypergeometric equation they satisfy, their three terms recurrence relation and the expression for the raising ladder operator. Those properties are then transported to the duality function using the proper rescaling  $b_n p_n(x)$ . This yields three identities for the duality function, that can be used in the expression of the process generator. Finally, algebraic manipulation allows to verify relation (2) for duality and (3) for self-duality.

*Stochastic Duality and Polynomials Duality.* Last, we point out that there exists a notion of duality within the context of orthogonal polynomials, see Definition 3.1 in [30]. For example, in case of discrete orthogonal polynomials, the polynomials self-duality can be described by the identity  $p_n(x) = p_x(n)$  where  $x$  and  $n$  take values in the same discrete set. It turns out that Charlier polynomials are self-dual, whereas Krawtchouk and Meixner polynomials, defined as in [39] see also Sect. 3, are not. However, it is possible to rescale them by a constant  $b_n$  so that they become self-dual. The constant  $b_n$  is indeed provided by (4). It is not clear if the two notions of stochastic duality and polynomials duality are in some relation one to the other. This will be investigated in a future work.

## 1.4 Paper Organization

The paper is organized as follows. Section 2 is devoted to a (non exhaustive) review of hypergeometric orthogonal polynomials, we recall their key properties

that are crucial in proving our results. We closely follow [39] and the expert reader might skip this part without being affected. The original results are presented in Sects. 3 and 4. In Sect. 3 we describe three interacting particle systems that are self-dual and prove the statement of Theorem 1, part (i). Section 4 is dedicated to the duality relations of two diffusion processes and a jump process obtained as thermalization limit of the previous ones and it contains the proof of Theorem 1, part (ii).

## 2 Preliminaries: Hypergeometric Orthogonal Polynomials

In this section we give a quick overview of the continuous and the discrete hypergeometric polynomials (see [16, 30, 39, 42]) by reviewing some of their structural properties that will be used in the following.

We start by recalling that the hypergeometric orthogonal polynomials arise from an hypergeometric equation, whose solution can be written in terms of an hypergeometric function  ${}_rF_s$ .

**Definition 3 (Hypergeometric function).** *The hypergeometric function is defined by the series*

$${}_rF_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{x^k}{k!} \tag{5}$$

where  $(a)_k$  denotes the Pochhammer symbol defined in terms of the Gamma function as

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

*Remark 3.* Whenever one of the numerator parameter  $a_j$  is a negative integer  $-n$ , the hypergeometric function  ${}_rF_s$  is a finite sum up to  $n$ , i.e. a polynomial in  $x$  of degree  $n$ .

**The Continuous Case.** Consider the hypergeometric differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0 \tag{6}$$

where  $\sigma(x)$  and  $\tau(x)$  are polynomials of at most second and first degree respectively and  $\lambda$  is a constant. A peculiarity of the hypergeometric equation is that, for all  $n$ ,  $y^{(n)}(x)$ , i.e. the  $n^{th}$  derivative of a solution  $y(x)$ , also solves an hypergeometric equation, namely

$$\sigma(x)y^{(n+2)}(x) + \tau_n(x)y^{(n+1)}(x) + \mu_n y^{(n)}(x) = 0 \tag{7}$$

with

$$\tau_n(x) = \tau(x) + n\sigma'(x) \tag{8}$$

and

$$\mu_n = \lambda + n\tau' + \frac{1}{2}n(n-1)\sigma''.$$

We concentrate on a specific family of solutions: for each  $n \in \mathbb{N}$ , let  $\mu_n = 0$ , so that

$$\lambda = \lambda_n = -n\tau' - \frac{1}{2}n(n-1)\sigma''$$

and Eq. (7) has a particular solution given by  $y^{(n)}(x)$  constant. This implies that  $y(x)$  is a polynomial of degree  $n$ , called *polynomial of hypergeometric type* (see Remark 3) and denoted by  $p_n(x)$ . In the following we will assume that those polynomials are of the form

$$p_n(x) = a_n x^n + b_n x^{n-1} + \dots \quad a_n \neq 0. \tag{9}$$

It is well known [39] that polynomials of hypergeometric type satisfy the orthogonality relation

$$\int_a^b p_n(x)p_m(x)\rho(x)dx = \delta_{n,m}d_n^2(x) \tag{10}$$

for some (possibly infinite) constants  $a$  and  $b$  and where the function  $\rho(x)$  satisfies the differential equation

$$(\sigma\rho)' = \tau\rho. \tag{11}$$

The sequence  $d_n^2$  can be written in terms of  $\sigma(x)$ ,  $\rho(x)$  and  $a_n$  as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1} (\lambda_n - \lambda_k)} \int_a^b (\sigma(x))^n \rho(x) dx.$$

As a consequence of the orthogonal property the polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{12}$$

where

$$\alpha_n = c_{n+1,n} \quad \beta_n = c_{n,n} \quad \gamma_n = c_{n-1,n}$$

with

$$c_{k,n} = \frac{1}{d_k^2} \int_a^b p_k(x) x p_n(x) \rho(x) dx.$$

The coefficients  $\alpha_n, \beta_n, \gamma_n$  can be expressed in terms of the squared norm  $d_n^2$  and the leading coefficients  $a_n, b_n$  in (9) as [39]

$$\alpha_n = \frac{a_n}{a_{n+1}} \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}.$$

Finally, we will use the raising operator  $R$  that, acting on the polynomials  $p_n(x)$ , provides the polynomials of degree  $n + 1$ . Such an operator is obtained from the Rodriguez formula, which provides an explicit form for polynomials of hypergeometric type

$$p_n(x) = \frac{B_n(\sigma^n(x)\rho(x))^{(n)}}{\rho(x)} \quad \text{with} \quad B_n = \frac{a_n}{\prod_{k=0}^{n-1} (\tau' + \frac{n+k-1}{2}\sigma'')}. \tag{13}$$



The expression of the raising operator (see Eq. 1.2.13 in [39]) reads

$$Rp_n(x) = r_n p_{n+1}(x) \tag{14}$$

where

$$Rp_n(x) = \lambda_n \tau_n(x) p_n(x) - n \sigma(x) \tau_n'(x) p_n'(x) \quad \text{and} \quad r_n = \lambda_n \frac{B_n}{B_{n+1}}.$$

Remark that the raising operator increases the degree of the polynomial by one, similarly to the so-called backward shift operator [30]. However the raising operator in (14) does not change the parameters involved in the function  $\rho$ , whereas the backward operator increases the degree and lowers the parameters [31].

**The Discrete Case.** Everything discussed for the continuous case has a discrete analog, where the derivatives are replaced by the discrete difference derivatives. In particular it is worth mentioning that

$$\Delta f(x) = f(x + 1) - f(x) \quad \text{and} \quad \nabla f(x) = f(x) - f(x - 1).$$

The corresponding hypergeometric differential Eq. (6) is the discrete hypergeometric difference equation

$$\sigma(x) \Delta \nabla y(x) + \tau(x) \Delta y(x) + \lambda y(x) = 0 \tag{15}$$

where  $\sigma(x)$  and  $\tau(x)$  are polynomials of second and first degree respectively,  $\lambda$  is a constant. The differential equation solved by the  $n^{th}$  discrete derivative of  $y(x)$ ,  $y^{(n)}(x) := \Delta^n y(x)$ , is the solution of another difference equation of hypergeometric type

$$\sigma(x) \Delta \nabla y^{(n)}(x) + \tau_n \Delta y^{(n)}(x) + \mu_n y^{(n)}(x) = 0 \tag{16}$$

with

$$\tau_n(x) = \tau(x + n) + \sigma(x + n) - \sigma(x)$$

and

$$\mu_n = \lambda + n \tau' + \frac{1}{2} n(n - 1) \sigma''.$$

If we impose  $\mu_n = 0$ , then

$$\lambda = \lambda_n = -n \tau' - \frac{1}{2} n(n - 1) \sigma''$$

and  $y^{(n)}(x)$  is a constant solution of Eq. (16). Under these conditions,  $y(x)$ , solution of (15), is a polynomial of degree  $n$ , called *discrete polynomial of hypergeometric type* (see Remark 3) and denoted by  $p_n(x)$ .

The derivation of the orthogonal property is done in a similar way than the one for the continuous case where the integral is replaced by a sum

$$\sum_{x=a}^{b-1} p_n(x) p_m(x) \rho(x) = \delta_{n,m} d_n^2$$

constants  $a$  and  $b$  can be either finite or infinite and the function  $\rho(x)$  is solution of

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

The sequence  $d_n^2$  can be written in terms of  $\sigma(x), \rho(x)$  and  $a_n$  as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1} (\lambda_n - \lambda_k)} \sum_{x=a}^{b-n-1} \left( \rho(x+n) \prod_{k=1}^n \sigma(x+k) \right)$$

As a consequence of the orthogonal property, the discrete polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{17}$$

where

$$\alpha_n = c_{n+1,n} \quad \beta_n = c_{n,n} \quad \gamma_n = c_{n-1,n}$$

with

$$c_{k,n} = \frac{1}{d_k^2} \sum_{x=a}^b p_k(x) x p_n(x).$$

The coefficients  $\alpha_n, \beta_n, \gamma_n$  can be expressed in terms of the squared norm  $d_n^2$  and the leading coefficients  $a_n, b_n$  in (9) as [39]

$$\alpha_n = \frac{a_n}{a_{n+1}} \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}.$$

The discrete Rodriguez formula

$$p_n(x) = \frac{B_n}{\rho(x)} \nabla^n \left[ \rho(x+n) \prod_{k=1}^n \sigma(x+k) \right] \text{ with } B_n = \frac{a_n}{\prod_{k=0}^{n-1} (\tau' + \frac{n+k-1}{2} \sigma')}$$

leads to an expression for the discrete raising operator  $R$  (see Eq. 2.2.10 in [39])

$$R p_n(x) = r_n p_{n+1}(x) \tag{18}$$

where

$$R p_n(x) = \left[ \lambda_n \tau_n(x) - n \tau'_n \sigma(x) \nabla \right] p_n(x) \quad r_n = \lambda_n \frac{B_n}{B_{n+1}}.$$

We remark again that the raising operator shouldn't be confused with the backward shift operator in [31] which changes the value of parameters of the distribution  $\rho$ .

### 3 Self-duality: Proof of Theorem 1 Part (i)

We consider first self-duality relations for some discrete interacting particle systems. In this case the dual process is an independent copy of the original one. Notice that, even if the original process and its dual are the same, a massive simplification occurs, namely the  $n$ -point correlation function of the original process can be expressed by duality in terms of only  $n$  dual particles. Thus a problem for many particles, possibly infinitely many in the infinite volume, may be studied via a finite number of dual walkers.

### 3.1 The Symmetric Exclusion Process, SEP( $j$ )

In the Symmetric Exclusion Process with parameter  $j \in \mathbb{N}/2$ , denoted by  $\text{SEP}(j)$ , each site can have at most  $2j$  particles, jumps occur at rate proportional to the number of particles in the departure site times the number of holes in the arrival site. The special case  $j = 1/2$  corresponds to the standard exclusion process with hard core exclusion, i.e. each site can be either full or empty [36]. We consider the setting where the spacial structure is given by the undirected connected graph  $G = (V, E)$  with  $N$  vertices and edge set  $E$ . A particle configuration is denoted by  $\mathbf{x} = (x_i)_{i \in V}$  where  $x_i \in \{0, \dots, 2j\}$  is interpreted as the particle number at vertex  $i$ . The process generator reads

$$L^{\text{SEP}(j)} f(\mathbf{x}) = \sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} x_i(2j - x_l) [f(\mathbf{x}^{i,l}) - f(\mathbf{x})] + (2j - x_i)x_l [f(\mathbf{x}^{l,i}) - f(\mathbf{x})]$$

where  $\mathbf{x}^{i,l}$  denotes the particle configuration obtained from the configuration  $\mathbf{x}$  by moving one particle from vertex  $i$  to vertex  $l$  and where  $f : \{0, 1, \dots, 2j\}^N \rightarrow \mathbb{R}$  is a function in the domain of the generator.

It is easy to verify that the reversible (and thus stationary) measure of the process for every  $p \in (0, 1)$  is given by the homogeneous product measure with marginals the Binomial distribution with parameters  $2j > 0$  and  $p \in (0, 1)$ , i.e. with probability mass function

$$\rho(x) = \binom{2j}{x} p^x (1 - p)^{2j-x}, \quad x \in \{0, 1, \dots, 2j\}. \tag{19}$$

The orthogonal polynomials with respect to the Binomial distribution are the Krawtchouk polynomials  $K_n(x)$  with parameter  $2j$  [32]. These polynomials are obtained by choosing in Eq. (15)

$$\sigma(x) = x \quad \tau(x) = \frac{2jp - x}{1 - p} \quad \lambda_n = \frac{n}{1 - p}.$$

Equivalently, Krawtchouk polynomials are polynomial solutions of the finite difference equation

$$x[K_n(x + 1) - 2K_n(x) + K_n(x - 1)] + \frac{2jp - x}{1 - p} [K_n(x + 1) - K_n(x)] + \frac{n}{1 - p} K_n(x) = 0. \tag{20}$$

They satisfy the orthogonality relation

$$\sum_{x=0}^{2j} K_n(x) K_m(x) \rho(x) = \delta_{n,m} d_n^2$$

with  $\rho$  as in (19) and norm in  $\ell^2(\{0, 1, \dots, 2j\}, \rho)$  given by

$$d_n^2 = \binom{2j}{n} p^n (1 - p)^n.$$

As a consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = n + 1 \quad \beta_n = n + 2jp - 2np \quad \gamma_n = p(1 - p)(2j - n + 1)$$

and the three-point recurrence then becomes

$$xK_n(x) = (n + 1)K_{n+1}(x) + (n + 2jp - 2np)K_n(x) + p(1 - p)(2j - n + 1)K_{n-1}(x). \tag{21}$$

Furthermore, the raising operator in (18) provides the relation

$$xK_n(x - 1) + \frac{p}{1 - p}(n + x - 2j)K_n(x) = \frac{n - 1}{1 - p}K_{n+1}(x). \tag{22}$$

**Self-Duality for SEP(j).** The Krawtchouk polynomials  $K_n(x)$  do not satisfy a self-duality relation in the sense of Definition 2. However, the following theorem shows that, with a proper normalization, it is possible to find a duality function related to them.

**Theorem 2.** *The SEP(j) is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N \frac{n_i!(2j - n_i)!}{2j!} K_{n_i}(x_i) \tag{23}$$

where  $K_n(x)$  denotes the Krawtchouk polynomial of degree  $n$ .

*Remark 4.* Since the Krawtchouk polynomials can be rewritten in terms of hypergeometric functions, the self-duality function turns out to be

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N {}_2F_1 \left( \begin{matrix} -n_i, -x_i \\ -2j \end{matrix} \middle| \frac{1}{p} \right)$$

where  ${}_2F_0$  is the hypergeometric function defined in (5).

*Proof.* We need to verify the self-duality relation in Eq. (3). Since the generator of the process is a sum of terms acting on two variables only, we shall verify the self-duality relation for two sites, say 1 and 2. We start by writing the action of the SEP(j) generator working on the duality function for these two sites:

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) &= x_1(2j - x_2) [D_{n_1}(x_1 - 1) D_{n_2}(x_2 + 1) \\ &\quad - D_{n_1}(x_1) D_{n_2}(x_2)] \\ &\quad + (2j - x_1)x_2 [D_{n_1}(x_1 + 1) D_{n_2}(x_2 - 1) \\ &\quad - D_{n_1}(x_1) D_{n_2}(x_2)] \end{aligned}$$

rewriting this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) &= x_1 D_{n_1}(x_1 - 1) (2j - x_2) D_{n_2}(x_2 + 1) \\ &\quad - x_1 D_{n_1}(x_1) (2j - x_2) D_{n_2}(x_2) \\ &\quad + (2j - x_1) D_{n_1}(x_1 + 1) x_2 D_{n_2}(x_2 - 1) \\ &\quad - (2j - x_1) D_{n_1}(x_1) x_2 D_{n_2}(x_2) \end{aligned} \tag{24}$$

we see that we need an expression for the following terms:

$$xD_n(x), \quad xD_n(x-1), \quad (2j-x)D_n(x+1). \tag{25}$$

To get those we first write the difference Eq. (20), the recurrence relation (21) and the raising operator Eq. (22) in terms of  $D_n(x)$ . This is possible using Eq. (23) that provides  $D_n(x)$  as a suitable normalization of  $K_n(x)$ , i.e.

$$D_n(x) = \frac{n!(2j-n)!}{2j!} K_n(x).$$

Then the first term in (25) is simply obtained from the normalized recurrence relation, whereas the second and third terms are provided by simple algebraic manipulation of the normalized raising operator equation and the normalized difference equation. We get

$$\begin{aligned} xD_n(x) &= -p(2j-n)D_{n+1}(x) + (n+2pj-2pn)D_n(x) \\ &\quad - n(1-p)D_{n-1}(x) \\ xD_n(x-1) &= -p(2j-n)D_{n+1}(x) + p(2j-2n)D_n(x) \\ &\quad + npD_{n-1}(x) \\ (2j-x)D_n(x+1) &= p(2j-n)D_{n+1}(x) + (1-p)(2j-2n)D_n(x) \\ &\quad - \frac{n}{p}(1-p)^2D_{n-1}(x). \end{aligned}$$

These expressions can now be inserted into (24), which then reads:

$$\begin{aligned} &L^{SEP(j)}D_{n_1}(x_1)D_{n_2}(x_2) \\ &= [p(n_1-2j)D_{n_1+1}(x_1) + p(2j-2n_1)D_{n_1}(x_1) + pn_1D_{n_1-1}(x_1)] \\ &\quad \times \left[ p(2j-n_2)D_{n_2+1}(x_2) + (1-p)(2j-2n_2)D_{n_2}(x_2) \right. \\ &\quad \left. - \frac{n_2}{p}(1-p)^2D_{n_2-1}(x_2) \right] \\ &+ [p(2j-n_1)D_{n_1+1}(x_1) - (n_1+2jp-2pn_1)D_{n_1}(x_1) \\ &\quad + (1-p)n_1D_{n_1-1}(x_1)] \\ &\times [p(2j-n_2)D_{n_2+1}(x_2) - (n_2+2pj-2pn_2)D_{n_2}(x_2) \\ &\quad + n_2(1-p)D_{n_2-1}(x_2) + 2jD_{n_2}(x_2)] \\ &+ [p(n_2-2j)D_{n_2+1}(x_2) + p(2j-2n_2)D_{n_2}(x_2) + pn_2D_{n_2-1}(x_2)] \\ &\times \left[ p(2j-n_1)D_{n_1+1}(x_1) + (1-p)(2j-2n_1)D_{n_1}(x_1) \right. \\ &\quad \left. - \frac{n_1}{p}(1-p)^2D_{n_1-1}(x_1) \right] \\ &+ [p(2j-n_2)D_{n_2+1}(x_2) - (n_2+2pj-2pn_2)D_{n_2}(x_2) + (1-p)n_2D_{n_2-1}(x_2)] \\ &\times [p(2j-n_1)D_{n_1+1}(x_1) - (n_1+2pj-2pn_1)D_{n_1}(x_1) \\ &\quad + n_1(1-p)D_{n_1-1}(x_1) + 2jD_{n_1}(x_1)]. \end{aligned}$$

Working out the algebra, substantial simplifications are revealed in the above expression. A long but straightforward computation shows that only products of polynomials with degree  $n_1 + n_2$  survive. In particular, after simplifications, one is left with

$$\begin{aligned} &L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) \\ &= n_1(2j - n_2) [D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \\ &\quad + (2j - n_1)n_2 [D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \end{aligned}$$

and the theorem is proved. □

### 3.2 The Symmetric Inclusion Process, SIP( $k$ )

The Symmetric Inclusion Process with parameter  $k > 0$ , denoted by SIP( $k$ ), is a Markov jump process with unbounded state space where each site can have an arbitrary number of particles. Again, we define the process on an undirected connected  $G = (V, E)$  with  $|V| = N$ . Jumps occur at rate proportional to the number of particles in the departure and the arrival sites, as the generator describes:

$$L^{SIP(k)} f(\mathbf{x}) = \sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} x_i(2k+x_l) [f(\mathbf{x}^{i,l}) - f(\mathbf{x})] + x_l(2k+x_i) [f(\mathbf{x}^{l,i}) - f(\mathbf{x})].$$

Detailed balance is satisfied by a product measure with marginals given by identical Negative Binomial distributions with parameters  $2k > 0$  and  $0 < p < 1$ , i.e. with probability mass function

$$\rho(x) = \binom{2k+x-1}{x} p^x (1-p)^{2k}, \quad x \in \{0, 1, \dots\}. \tag{26}$$

The polynomials that are orthogonal with respect to the Negative Binomial distribution are the Meixner polynomials  $M_n(x)$  with parameter  $2k$ , first introduced in [37]. Choosing in (15)

$$\sigma(x) = x \quad \tau(x) = 2kp - x(1-p) \quad \lambda_n = n(1-p)$$

we have that the Meixner polynomials are solution of the difference equation

$$\begin{aligned} &x [M_n(x+1) - 2M_n(x) + M_n(x-1)] \\ &+ (2kp - x + xp) [M_n(x+1) - M_n(x)] + n(1-p)M_n(x) = 0. \end{aligned} \tag{27}$$

They satisfy the orthogonal relation

$$\sum_{x=0}^{\infty} M_n(x) M_m(x) \rho(x) = \delta_{m,n} d_n^2$$

with  $\rho$  as in (26) and norm in  $\ell^2(\mathbb{N}_0, \rho)$  given by

$$d_n^2 = \frac{n! \Gamma(2k+n)}{p^n \Gamma(2k)}$$

where  $\Gamma(x)$  is the Gamma function. As consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = \frac{p}{p-1} \quad \beta_n = \frac{n+pn+2kp}{1-p} \quad \gamma_n = \frac{n(n-1+2k)}{p-1}$$

which then becomes

$$xM_n(x) = \frac{p}{p-1}M_{n+1}(x) + \frac{n+pn+2kp}{1-p}M_n(x) + \frac{n(n-1+2k)}{p-1}M_{n-1}(x). \tag{28}$$

Furthermore the raising operator in Eq. (18) provides the identity

$$[p(n+2k+x)]M_n(x) - xM_n(x-1) = pM_{n+1}(x). \tag{29}$$

**Self-duality for SIP(k).** In analogy with the result for the Exclusion process it is possible to find a duality function for the Symmetric Inclusion Process in terms of the Meixner polynomials.

**Theorem 3.** *The SIP(k) is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N \frac{\Gamma(2k)}{\Gamma(2k+n_i)} M_{n_i}(x_i)$$

where  $M_n(x)$  is the Meixner polynomial of degree  $n$ .

*Remark 5.* The self-duality function can be rewritten in terms of hypergeometric function as

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N {}_2F_1 \left( \begin{matrix} -n_i, -x_i \\ 2k \end{matrix} \middle| 1 - \frac{1}{p} \right).$$

*Proof.* As was done for the Exclusion Process we verify the self-duality relation in Eq. (3) for two sites, say 1 and 2. The action of the  $SIP(k)$  generator working on the self-duality function for two sites is given by

$$\begin{aligned} &L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\ &= x_1(2k+x_2) [D_{n_1}(x_1-1) D_{n_2}(x_2+1) - D_{n_1}(x_1) D_{n_2}(x_2)] \\ &\quad + (2k+x_1)x_2 [D_{n_1}(x_1+1) D_{n_2}(x_2-1) - D_{n_1}(x_1) D_{n_2}(x_2)]. \end{aligned}$$

We rewrite this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) &= x_1 D_{n_1}(x_1-1) (2k+x_2) D_{n_2}(x_2+1) \\ &\quad - x_1 D_{n_1}(x_1) (2k+x_2) D_{n_2}(x_2) \\ &\quad + (2k+x_1) D_{n_1}(x_1+1) x_2 D_{n_2}(x_2-1) \\ &\quad - (2k+x_1) D_{n_1}(x_1) x_2 D_{n_2}(x_2) \end{aligned} \tag{30}$$

so that we now need an expression for the following terms:

$$xD_n(x), \quad xD_n(x - 1), \quad (2k + x)D_n(x + 1). \tag{31}$$

To get those, we first write the difference Eq. (27), the recurrence relation (28) and the raising operator Eq. (29) in terms of  $D_n(x)$  using

$$D_n(x) = \frac{\Gamma(2k)}{\Gamma(2k + n)}M_n(x).$$

Then the first term in (31) is simply obtained from the normalized recurrence relation, whereas the second and third terms are provided by simple algebraic manipulation of the normalized raising operator equation and the normalized difference equation. We have,

$$\begin{aligned} xD_n(x) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{n+p(n+2k)}{p-1}D_n(x) \\ &\quad + \frac{n}{p-1}D_{n-1}(x) \\ xD_n(x-1) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{p}{p-1}(2k+2n)D_n(x) \\ &\quad + \frac{p}{p-1}nD_{n-1}(x) \\ (2k+x)D_n(x+1) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{1}{p-1}(2k+2n)D_n(x) \\ &\quad + \frac{1}{p-1}nD_{n-1}(x). \end{aligned}$$

These relations allow us to expand the SIP( $k$ ) generator in Eq. (30) as

$$\begin{aligned} &L^{SIP}D_{n_1}(x_1)D_{n_2}(x_2) \\ &= \left[ \frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{p(2k+2n_1)}{p-1}D_{n_1}(x_1) + \frac{pn_1}{p-1}D_{n_1-1}(x_1) \right] \\ &\quad \times \left[ \frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{2k+2n_2}{p-1}D_{n_2}(x_2) + \frac{n_2}{p-1}D_{n_2-1}(x_2) \right] \\ &\quad - \left[ \frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1}D_{n_1}(x_1) + \frac{n_1}{p-1}D_{n_1-1}(x_1) \right] \\ &\quad \times \left[ \frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1}D_{n_2}(x_2) \right. \\ &\quad \left. + \frac{n_2}{p-1}D_{n_2-1}(x_2) + 2kD_{n_2}(x_2) \right] \\ &\quad + \left[ \frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{p(2k+2n_2)}{p-1}D_{n_2}(x_2) + \frac{pn_2}{p-1}D_{n_2-1}(x_2) \right] \end{aligned}$$



$$\begin{aligned} & \times \left[ \frac{p(2k+n_1)}{p-1} D_{n_1+1}(x_1) - \frac{2k+2n_1}{p-1} D_{n_1}(x_1) + \frac{n_1}{p-1} D_{n_1-1}(x_1) \right] \\ & - \left[ \frac{p(2k+n_2)}{p-1} D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1} D_{n_2}(x_2) + \frac{n_2}{p-1} D_{n_2-1}(x_2) \right] \\ & \times \left[ \frac{p(2k+n_1)}{p-1} D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1} D_{n_1}(x_1) \right. \\ & \left. + \frac{n_1}{p-1} D_{n_1-1}(x_1) + 2k D_{n_1}(x_1) \right]. \end{aligned}$$

At this point it is sufficient to notice that the coefficients of products of polynomials with degree different than  $n_1 + n_2$  are all zero, so that we are left with

$$\begin{aligned} & L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\ & = n_1(2k+n_2) [D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \\ & \quad + (2k+n_1)n_2 [D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \end{aligned}$$

and the theorem is proved. □

### 3.3 The Independent Random Walker, IRW

The Symmetric Independent Random Walkers, denoted IRW, is one of the simplest, yet non-trivial particle system studied in the literature. It consists of independent particles that perform a symmetric continuous time random walk at rate 1. The generator, defined on the undirected connected graph  $G = (V, E)$  with  $N$  vertices and edge set  $E$ , is given by

$$L^{IRW} f(\mathbf{x}) = \sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} x_i [f(\mathbf{x}^{i,l}) - f(\mathbf{x})] + x_l [f(\mathbf{x}^{l,i}) - f(\mathbf{x})]. \tag{32}$$

The reversible invariant measure is provided by a product of Poisson distributions with parameter  $\lambda > 0$ , i.e. with probability mass function

$$\rho(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x \in \mathbb{N}_0. \tag{33}$$

The orthogonal polynomials with respect the Poisson distribution are the Charlier polynomials  $C_n(x)$  [15] with parameter  $\lambda$ . Choosing in Eq. (15)

$$\sigma(x) = x \quad \tau(x) = \lambda - x \quad \lambda_n = n$$

we obtain the difference equation whose solution is  $C_n(x)$

$$x [C_n(x+1) - 2C_n(x) + C_n(x-1)] + (\lambda - x) [C_n(x+1) - C_n(x)] + nC_n(x) = 0. \tag{34}$$

The orthogonal relation satisfied by Charlier polynomials is

$$\sum_{x=0}^{\infty} C_n(x)C_m(x)\rho(x) = \delta_{m,n}d_n^2$$

where  $\rho$  is given in (33) and the norm in  $\ell^2(\mathbb{N}_0, \rho)$  is

$$d_n^2 = n!\lambda^{-n}$$

As consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = -\lambda \quad \beta_n = n + \lambda \quad \gamma_n = -n$$

which then becomes

$$xC_n(x) = -\lambda C_{n+1}(x) + (n + \lambda)C_n(x) - nC_{n-1}(x). \tag{35}$$

Furthermore, the raising operator in Eq. (18) provides

$$\lambda C_n(x) - xC_n(x - 1) = \lambda C_{n+1}(x). \tag{36}$$

**Self-duality of IRW.** As the following theorem shows, the self-duality relation is given by the Charlier polynomials themselves.

**Theorem 4.** *The IRW is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N C_{n_i}(x_i) \tag{37}$$

where  $C_n(x)$  is the Charlier polynomial of degree  $n$ .

*Remark 6.* Reading the Charlier polynomial as hypergeometric function, the duality function then becomes

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N {}_2F_0 \left( \begin{matrix} -n_i, -x_i \\ - \end{matrix} \middle| -\frac{1}{\lambda} \right).$$

*Proof.* It is clear from (37) that the difference equations, the recurrence relations and the raising operator for  $D_n(x)$  are respectively (34), (35) and (36), that we rewrite as:

$$\begin{aligned} D_n(x + 1) &= D_n(x) - \frac{n}{\lambda} D_{n-1}(x) \\ xD_n(x) &= -\lambda D_{n+1}(x) + (n + \lambda)D_n(x) - nD_{n-1}(x) \\ xD_n(x - 1) &= \lambda D_n(x) - \lambda D_{n+1}(x). \end{aligned}$$

As done before, we use the two particles IRW generator in (32) and the three equations above to check that the self-duality relation holds. We have

$$\begin{aligned}
 &L^{IRW} D_{n_1}(x_1)D_{n_2}(x_2) \\
 &= x_1 D_{n_1}(x_1 - 1)D_{n_2}(x_2 + 1) - x_1 D_{n_1}(x_1)D_{n_2}(x_2) \\
 &\quad + D_{n_1}(x_1 + 1)x_2 D_{n_2}(x_2 - 1) - D_{n_1}(x_1)x_2 D_{n_2}(x_2) \\
 &= [\lambda D_{n_1}(x_1) - \lambda D_{n_1+1}(x_1)] \left[ D_{n_2}(x_2) - \frac{n_2}{\lambda} D_{n_2-1}(x_2) \right] \\
 &\quad - [-\lambda D_{n_1+1}(x_1) + (n_1 + \lambda) D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)] [D_{n_2}(x_2)] \\
 &\quad + \left[ D_{n_1}(x_1) - \frac{n_1}{\lambda} D_{n_1+1}(x_2) \right] [\lambda D_{n_2}(x_2) - \lambda D_{n_2+1}(x_2)] \\
 &\quad - [D_{n_1}(x_1)] [-\lambda D_{n_2+1}(x_2) + (n_2 + \lambda) D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)].
 \end{aligned}$$

After computing the products and suitable simplifications we get

$$\begin{aligned}
 L^{IRW} D_{n_1}(x_1)D_{n_2}(x_2) &= n_1 [D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)] \\
 &\quad + n_2 [D_{n_1+1}(x_1)D_{n_2-1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)].
 \end{aligned}$$

□

## 4 Duality: Proof of Theorem 1 Part (ii)

In this last section we show two examples of duality: the initial process is an interacting diffusion, while the dual one is a jump process, which, in particular, turns out to be the SIP process introduced in Sect. 3.2. We also show an example of duality for a redistribution model of Kipnis–Marchioro–Presutti type.

### 4.1 The Brownian Momentum Proces, BMP

The Brownian Momentum Process (BMP) is a Markov diffusion process introduced in [22]. On the undirected connected graph  $G = (V, E)$  with  $N$  vertices and edge set  $E$ , the generator reads

$$L^{BMP} f(\mathbf{x}) = \sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} \left( x_i \frac{\partial f}{\partial x_l}(\mathbf{x}) - x_l \frac{\partial f}{\partial x_i}(\mathbf{x}) \right)^2 \tag{38}$$

where  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is a function in the domain of the generator. A configuration is denoted by  $\mathbf{x} = (x_i)_{i \in V}$  where  $x_i \in \mathbb{R}$  has to be interpreted as a particle momentum. A peculiarity of this process regards its conservation law: if the process is started from the configuration  $\mathbf{x}$  then  $\|\mathbf{x}\|_2^2 = \sum_{i=1}^N x_i^2$  is constant during the evolution, i.e. the total kinetic energy is conserved.

The stationary reversible measure of the BMP process is given by a family of product measures with marginals given by independent centered Gaussian random variables with variance  $\sigma^2 > 0$ . Without loss of generality we will identify

a duality function related to Hermite polynomials in the case the variance equals  $1/2$ . The case with a generic value of the variance would be treated in a similar way by using generalized Hermite polynomials. Choosing in (6)

$$\sigma(x) = 1 \quad \tau(x) = -2x \quad \lambda_n = 2n$$

we acquire the differential equation satisfied by the Hermite polynomials  $H_n(x)$  [39]

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{39}$$

The orthogonal relation they satisfy is

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)\rho(x)dx = \delta_{m,n}d_n^2$$

with density function

$$\rho(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$$

and norm in  $L^2(\mathbb{R}, \rho)$  given by

$$d_n^2 = 2^n n!$$

As consequence of the orthogonality they satisfy the recurrence relation (12) with

$$\alpha_n = \frac{1}{2} \quad \beta_n = 0 \quad \gamma_n = n$$

which then can be written as

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{40}$$

Furthermore the raising operator in Eq. (14) provides

$$2xH_n(x) - H_n'(x) = H_{n+1}(x). \tag{41}$$

**Duality Between BMP and SIP( $\frac{1}{4}$ ).** The following theorem has a similar version in [4] and it states the duality result involving the Hermite polynomials.

**Theorem 5.** *The BMP process is dual to the SIP( $\frac{1}{4}$ ) process through duality function*

$$D_{\mathbf{n}}(x) = \prod_{i=1}^N \frac{1}{(2n_i - 1)!!} H_{2n_i}(x_i)$$

where  $H_{2n}(x)$  is the Hermite polynomial of degree  $2n$ .

*Remark 7.* Reading the Hermite polynomial as hypergeometric function, the duality function then becomes

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N \frac{1}{(2n_i - 1)!!} (2x_i)^{2n_i} {}_2F_0 \left( \begin{matrix} -n_i, (-2n_i + 1)/2 \\ - \end{matrix} \middle| -\frac{1}{x_i^2} \right).$$

*Proof.* Although the proof in [4] can be easily adapted to our case, we show here an alternative proof that follows our general strategy of using the structural properties of Hermite polynomials. It is sufficient, as before, to show the duality relation in Eq. (1) for sites 1 and 2. The action of the BMP generator on duality function reads

$$\begin{aligned} &L^{BMP} D_{n_1}(x_1)D_{n_2}(x_2) \\ &= (x_1\partial_{x_2} - x_2\partial_{x_1})^2 D_{n_1}(x_1)D_{n_2}(x_2) \\ &= x_1^2 D_{n_1}(x_1)D_{n_2}''(x_2) + D_{n_1}''(x_1)x_2^2 D_{n_2}(x_2) - x_1 D_{n_1}'(x_1)D_{n_2}(x_2) \\ &\quad - D_{n_1}(x_1)x_2 D_{n_2}'(x_2) - 2x_1 D_{n_1}'(x_1)x_2 D_{n_2}'(x_2) \end{aligned}$$

where we use  $\partial_{x_i} = \frac{\partial}{\partial x_i}$ . We now need the recurrence relation and the raising operator appropriately rewritten in term of the duality function in order to get suitable expression for

$$x^2 D_n(x), \quad D_n''(x), \quad x D_n'(x).$$

This can be done using

$$D_n(x) = \frac{1}{(2n-1)!!} H_{2n}(x)$$

so that

$$x^2 D_n(x) = \frac{1}{4}(2n+1)D_{n+1}(x) + \left(2n + \frac{1}{2}\right) D_n(x) + 2nD_{n-1}(x) \tag{42}$$

$$D_n''(x) = 8nD_{n-1}(x) \tag{43}$$

$$x D_n'(x) = 2nD_n(x) + 4nD_{n-1}(x) \tag{44}$$

where (42) is obtained from iterating twice the recurrence relation in (40), for Eq. (44) we combined (40) and (41) and then (43) is found from the differential Eq. (39) using (44). Proceeding with the substitution into the generator we find

$$\begin{aligned} &L^{BMP} D_{n_1}(x_1)D_{n_2}(x_2) \\ &= \left(\frac{1}{4}(2n_1+1)D_{n_1+1}(x_1) + \left(2n_1 + \frac{1}{2}\right) D_{n_1}(x_1) + 2n_1D_{n_1-1}(x_1)\right) \\ &\quad \times 8n_2D_{n_2-1}(x_2) \\ &\quad + 8n_1D_{n_1-1}(x_1) \\ &\quad \times \left(\frac{1}{4}(2n_2+1)D_{n_2+1}(x_2) + \left(2n_2 + \frac{1}{2}\right) D_{n_2}(x_2) + 2n_2D_{n_2-1}(x_2)\right) \\ &\quad - (2n_1D_{n_1}(x_1) + 4n_1D_{n_1-1}(x_1)) D_{n_2}(x_2) \\ &\quad - D_{n_1}(x_1) (2n_1D_{n_2}(x_2) + 4n_2D_{n_2-1}(x_2)) \\ &\quad - 2(2n_1D_{n_1}(x_1) + 4n_1D_{n_1-1}(x_1)) (2n_2D_{n_2}(x_2) + 4n_2D_{n_2-1}(x_2)) \end{aligned}$$

Finally, after appropriate simplification of the terms whose degree is different from  $n_1 + n_2$ , we get

$$\begin{aligned} &L^{BMP} D_{n_1}(x_1)D_{n_2}(x_2) \\ &= (2n_1 + 1)2n_2 [D_{n_1+1}(x_1)D_{n_2-1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)] \\ &\quad + 2n_1(2n_2 + 1) [D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{2n_1}(x_1)D_{n_2}(x_2)] \\ &= L^{SIP} D_{n_1}(x_1)D_{n_2}(x_2) \end{aligned}$$

which proves the theorem. □

### 4.2 The Brownian Energy Process, BEP(k)

We now introduce a process, known as Brownian Energy Process with parameter  $k$ , BEP( $k$ ) in short notation, whose generator is

$$\begin{aligned} L^{BEP(k)} f(\mathbf{x}) = &\sum_{1 \leq i < l \leq N(i,l) \in E} \left[ x_i x_j \left( \frac{\partial}{\partial x_i} f(\mathbf{x}) - \frac{\partial}{\partial x_j} f(\mathbf{x}) \right)^2 \right. \\ &\left. + 2k(x_i - x_j) \left( \frac{\partial}{\partial x_i} f(\mathbf{x}) - \frac{\partial}{\partial x_j} f(\mathbf{x}) \right) \right]. \end{aligned} \quad (45)$$

where  $f : \mathbb{R}^N \rightarrow \mathbb{R}$  is in the domain of the generator and  $\mathbf{x} = (x_i)_{i \in V}$  denotes a configuration of the process with  $x_i \in \mathbb{R}^+$  interpreted as a particle energy. The generator in (45) describes the evolution of particle system that exchange their (kinetic) energies. It is easy to verify that the total energy of the system  $\sum_{i=1}^N x_i$  is conserved by the dynamic.

In [24] it was shown that the BEP( $k$ ) can be obtained from the BMP process once  $4k \in \mathbb{N}$  vertical copies of the graph  $G$  are introduced. Under these circumstances denoting  $z_{i,\alpha}$  the momentum of the  $i^{th}$  particle at the  $\alpha^{th}$  level, the kinetic energy per (vertical) site is

$$x_i = \sum_{\alpha=1}^{4k} z_{i,\alpha}^2.$$

If we use the above change of variable in the generator of such BMP process on the ladder graph with  $4k$  layers, the generator of the BEP( $k$ ) is revealed.

The stationary measure of the BEP( $k$ ) process is given by a product of independent Gamma distribution with shape parameter  $2k$  and scale parameter  $\theta$ , i.e. with Lebesgue probability mass function

$$\rho(x) = \frac{x^{2k-1} e^{-\frac{x}{\theta}}}{\Gamma(2k)\theta^k}. \quad (46)$$

Without loss of generality we can set  $\theta = 1$  so that the polynomials orthogonal with respect to the Gamma distribution are the generalized Laguerre polynomials  $L_n^{(2k-1)}(x)$  [39].

Choosing

$$\sigma(x) = x \quad \tau(x) = 2k - x \quad \lambda_n = n$$

the differential equation whose generalized Laguerre polynomials are solution becomes

$$x \frac{d^2}{dx^2} L_n^{(2k-1)}(x) + (2k - x) \frac{d}{dx} L_n^{(2k-1)}(x) + n L_n^{(2k-1)}(x) = 0 \tag{47}$$

The orthogonal relation they satisfy is

$$\int_0^{+\infty} L_n^{(2k-1)}(x) L_m^{(2k-1)}(x) \rho(x) dx = \delta_{m,n} d_n^2$$

with mass function as in (46) with  $\theta = 1$  and norm in  $L^2(\mathbb{R}^+, \rho)$  given by

$$d_n^2 = \frac{\Gamma(n + 2k)}{n! \Gamma(2k)}.$$

As consequence of the orthogonality, they satisfy the recurrence relation (12) with

$$\alpha_n = -(n + 1) \quad \beta_n = 2n + 2k \quad \gamma_n = -(n + 2k - 1)$$

which then can be written as

$$x L_n^{(2k-1)}(x) = -(n + 1) L_{n+1}^{(2k-1)}(x) + (2n + 2k) L_n^{(2k-1)}(x) - (n + 2k - 1) L_{n-1}^{(2k-1)}(x) \tag{48}$$

Furthermore, the raising operator in Eq. (14) is given by

$$(2k - x + n) L_n^{(2k-1)}(x) + x \frac{d}{dx} L_n^{(2k-1)}(x) = (n + 1) L_{n+1}^{(2k-1)}(x). \tag{49}$$

**Duality Between BEP(k) and SIP(k).** The duality relation for the Brownian energy process with parameter  $k$  is stated below.

**Theorem 6.** *The BEP(k) process and the SIP(k) process are dual via*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N \frac{n_i! \Gamma(2k)}{\Gamma(2k + n_i)} L_{n_i}^{(2k-1)}(x_i) \tag{50}$$

where  $L_n^{(2k-1)}(x)$  is the generalized Laguerre polynomial of degree  $n$ .

*Remark 8.* The factor  $\Gamma(2k)$  in (50) is not crucial to assess a duality relation, but it allows to write the duality function as the hypergeometric function

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N {}_1F_1 \left( \begin{matrix} -n_i \\ 2k \end{matrix} \middle| x_i \right).$$

*Proof.* As in the previous cases we notice that the proof can be shown for sites 1 and 2 only, in which case the generator of the BEP acts on

$$\begin{aligned} &L^{BEP(k)}D_{n_1}(x_1)D_{n_2}(x_2) \\ &= \left[ x_1x_2(\partial_{x_1} - \partial_{x_2})^2 - 2k(x_1 - x_1)(\partial_{x_1} - \partial_{x_2}) \right] D_{n_1}(x_1)D_{n_2}(x_2) \\ &= (x_1\partial_{x_1}^2 + 2k\partial_{x_1})D_{n_1}(x_1)x_2D_{n_2}(x_2) + x_1D_{n_1}(x_1)(x_2\partial_{x_2}^2 + 2k\partial_{x_2})D_{n_2}(x_2) \\ &\quad - x_1\partial_{x_1}D_{n_1(x_1)}(x_2\partial_{x_2} + 2k)D_{n_2}(x_2) - (x_1\partial_{x_1} + 2k)D_{n_1}(x_1)x_2\partial_{x_2}D_{n_2}(x_2) \end{aligned}$$

We seek an expression for

$$x\partial_x^2D_n + 2k\partial_xD_n, \quad xD_n, \quad x\partial_xD_n$$

that can easily be obtained rewriting (47), (48) and (49) for the duality function, using

$$D_n(x) = \frac{n! \Gamma(2k)}{\Gamma(2k + n)} L_n^{(2k-1)}(x)$$

so that, after simple manipulation

$$xD_n''(x) + (2k - x)D_n'(x) + nD_n(x) = 0 \tag{51}$$

$$xD_n(x) = -(n + 2k)D_{n+1}(x) + (2n + 2k)D_n(x) - nD_{n-1}(x) \tag{52}$$

$$xD_n'(x) = nD_n(x) - nD_{n-1}(x). \tag{53}$$

Note that plugging (53) into the difference Eq. (51), we get

$$xD_n''(x) + 2kD_n'(x) = -nD_{n-1}(x).$$

Let's now use these information to write explicitly the BEP(k) generator.

$$\begin{aligned} &L^{BEP(k)}D_{n_1}(x_1)D_{n_2}(x_2) \\ &= [-n_1D_{n_1-1}(x_1)] [-(2k + n_2)D_{n_2+1}(x_2) + (2n_2 + 2k)D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)] \\ &\quad + [-(2k + n_1)D_{n_1+1}(x_1) + (2n_1 + 2k)D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)] [-n_2D_{n_2-1}(x_2)] \\ &\quad - [n_1D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)] [(n_2 + 2k)D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)] \\ &\quad - [(n_1 + 2k)D_{n_1}(x_1) - n_1D_{n_1-1}(x_1)] [n_2D_{n_2}(x_2) - n_2D_{n_2-1}(x_2)]. \end{aligned}$$

Expanding products in the above expression we find

$$\begin{aligned} &L^{BEP(k)}D_{n_1}(x_1)D_{n_2}(x_2) \\ &= n_1D_{n_1-1}(x_1)(n_2 + 2k)D_{n_2+1}(x_2) + (n_1 + 2k)D_{n_1+1}(x_1)n_2D_{n_2-1}(x_2) \\ &\quad + n_1D_{n_1}(x_1)(n_2 + 2k)D_{n_2}(x_2) + (n_1 + 2k)D_{n_1+1}(x_1)n_2D_{n_2-1}(x_2) \\ &\quad + D_{n_1-1}(x_1)D_{n_2}(x_2) [-n_1(2n_2 + 2k) + n_1(n_2 + 2k) + n_1n_2] \\ &\quad + D_{n_1}(x_1)D_{n_2-1}(x_2) [-(2n_1 + 2k)n_1 + (n_1 + 2k)n_2 + n_1n_2] \\ &\quad + D_{n_1-1}(x_1)D_{n_2-1}(x_2) [n_1n_2 + n_1n_2 - n_1n_2 - n_1n_2]. \end{aligned}$$



Noticing that the coefficients of the last three lines are zeros, we finally get

$$\begin{aligned}
 &L^{BEP(k)} D_{n_1}(x_1)D_{n_2}(x_2) \\
 &= n_1(n_2 + 2k) [D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{n_1}(x_1)D_{n_2-1}(x_2)] \\
 &\quad + (n_1 + 2k)n_2 [D_{n_1+1}(x_1)D_{n_2-2}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)] \\
 &= L^{SIP(k)} D_{n_1}(x_1)D_{n_2}(x_2)
 \end{aligned} \tag{54}$$

where  $L^{SIP(k)}$  works on the dual variables  $(n_1, n_2)$ . □

### 4.3 The Kipnis–Marchioro–Presutti Process, KMP(k)

The KMP model was first introduced by Kipnis, Marchioro and Presutti [29] in 1982 as a model of heat conduction that was solved by using a dual process. It is a stochastic model where a continuous non-negative variable (interpreted as energy) is uniformly redistributed among two random particles on a lattice at Poisson random times. A general version with parameter  $k$ , that we shall call  $KMP(k)$  was defined in [11], by considering a redistribution rule where a fraction  $p$  of the total energy is assigned to one particle and the remaining fraction  $(1 - p)$  to the other particle, with  $p$  a  $Beta(2k, 2k)$  distributed random variable. Thus the case  $k = 1/2$  corresponds to the original KMP model. In [11] it was shown that  $KMP(k)$  is in turn related to the Brownian Energy Process with parameter  $k$ , as it can be obtained from the  $BEP(k)$  via a procedure called “instantaneous thermalization”. If the spatial setting remains as described in the previous sections, the generator of the  $KMP(k)$  process is

$$\begin{aligned}
 L^{KMP(k)} f(\mathbf{x}) = &\sum_{1 \leq i < l \leq N, (i,l) \in E} \int_0^1 [f(x_1, \dots, x_{i-1}, p(x_i + x_{i+1}), \\
 &(1 - p)(x_i + x_{i+1}), x_{i+2}, \dots, x_N) - f(\mathbf{x})] \nu_{2k}(p) dp
 \end{aligned}$$

where  $\nu_{2k}(p)$  is the density function of the Beta distribution with parameters  $(2k, 2k)$ , i.e.

$$\nu_{2k}(p) = \frac{p^{2k-1}(1-p)^{2k-1}\Gamma(4k)}{\Gamma(2k)\Gamma(2k)}, \quad p \in (0, 1).$$

The dual process of  $KMP(k)$  [11] is generated by

$$\begin{aligned}
 L^{\text{dual-KMP}(k)} f(\mathbf{n}) = &\sum_{\substack{1 \leq i < l \leq N \\ (i,l) \in E}} \sum_{r=0}^{n_i+n_{i+1}} [f(n_1, \dots, n_{i-1}, r, n_i + n_{i+1} - r, \\
 &n_{i+2}, \dots, x_N) - f(\mathbf{n})] \mu_{2k}(r \mid n_i + n_{i+1})
 \end{aligned}$$

where  $\mu_{2k}(r|C)$  is the mass density function of the Beta Binomial distribution with parameter  $(C, 2k, 2k)$ , i.e.

$$\mu_{2k}(r \mid C) = \frac{\binom{2k+r-1}{r} \binom{2k+C-r-1}{C-r}}{\binom{4k+C-1}{C}}, \quad r \in \{0, 1, \dots, C\}.$$

This generator is the result of a thermalized limit of the SIP( $k$ ) [11]. Our last theorem is stated below.

**Theorem 7.** *The KMP( $k$ ) process duality relation with its dual is established via duality function*

$$D_n(\mathbf{x}) = \prod_{i=1}^N \frac{n_i! \Gamma(2k)}{\Gamma(2k + n_i)} L_{n_i}^{(2k-1)}(x_i) \quad (55)$$

where  $L_n^{(2k-1)}(x)$  is the generalized Laguerre polynomial of degree  $n$ .

*Proof.* As expected, the duality function is the same as the one for the BEP( $k$ ) and SIP( $k$ ) duality relation. This shouldn't surprise since BEP( $k$ ) and SIP( $k$ ) are dual through duality function (55) and the thermalization limit doesn't affect the duality property. Indeed, considering two graph vertices, one has from [11]

$$L^{KMP(k)} f(x_1, x_2) = \lim_{t \rightarrow \infty} (e^{tL^{BEP(k)}} - I) f(x_1, x_2)$$

and

$$L^{\text{dual-KMP}(k)} f(n_1, n_2) = \lim_{t \rightarrow \infty} (e^{tL^{SIP(k)}} - I) f(n_1, n_2).$$

Thus, combining the previous two equations and (54), the claim follows.  $\square$

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