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Sojourns in Probability Theory and Statistical Physics - III

Interacting Particle Systems and Random Walks, A Festschrift for Charles M. Newman

NYU SHANGHAI 上海纽约大学

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Preface

This three-volume set, entitled Sojourns in Probability Theory and Statistical Physics, constitutes a Festschrift for Chuck Newman on the occasion of his 70th birthday. In these coordinated volumes, Chuck's closest colleagues and collaborators pay tribute to the immense impact he has had on these two deeply intertwined fields of research. The papers published here include original research articles and survey articles, on topics gathered by theme as follows:

Volume 1: Spin Glasses and Statistical Mechanics
Volume 2: Brownian Web and Percolation
Volume 3: Interacting Particle Systems and Random Walks

Our colleague Vladas Sidoravicius conceived the idea for this Festschrift during the conference on Probability Theory and Statistical Physics that was hosted on 25–27 March 2016 by the NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai. This conference brought together more than 150 experts to discuss frontier research at the interface between these two fields, and it coincided with Chuck's 70th birthday. After the conference, Vladas approached various of Chuck's colleagues with invitations to contribute. Papers flowed in during the Fall of 2016 and the Spring of 2017. The Festschrift suffered delays in 2018, and then on 23 May 2019, Vladas passed away unexpectedly. Following discussions in June 2019 with NYU Shanghai and Springer Nature, we offered to assume editorial responsibility for bringing the volumes to completion.

We gratefully acknowledge Vladas's investment in these volumes, and we recognise that his presence in our community worldwide will be sorely missed. We offer our thanks to Julius Damarackas (NYU Shanghai) for his detailed preparation of the articles in these volumes.

Chuck has been one of the leaders in our profession for nearly 50 years. He has worked on a vast range of topics and has collaborated with and inspired at least three generations of mathematicians, sharing with them his deep insights into mathematics and statistical physics and his views on key developments, always leavened with his acute and captivating sense of humour. We wish him and his family many fruitful years to come.

July 2019

Federico Camia
Geoffrey Grimmett
Frank den Hollander
Daniel Stein

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Random Walk Among Mobile/Immobile Traps: A Short Review

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In celebration of Chuck's 70th Birthday

Abstract. There have been extensive studies of a random walk among a field of immobile traps (or obstacles), where one is interested in the probability of survival as well as the law of the random walk conditioned on its survival up to time t. In contrast, very little is known when the traps are mobile. We will briefly review the literature on the trapping problem with immobile traps, and then review some recent results on a model with mobile traps, where the traps are represented by a Poisson system of independent random walks on $\mathbb{Z}^d$. Some open questions will be given at the end.

Keywords: Trapping problem · Parabolic anderson model · Random walk in random potential

1 Introduction

The trapping problem, where particles diffuse in space with randomly located traps, has been studied extensively in the physics and mathematics literature. We refer the reader to the review article [17], which explains in detail the background for the trapping problem and some early results. Here we focus on a single particle diffusing on $\mathbb{Z}^d$ according to a random walk, with randomly located traps that may or may not be mobile. When the particle meets a trap, it is killed at a fixed rate $\gamma \in (0, \infty]$.

More precisely, let $X := (X(t))_{t\geq 0}$ be a random walk on $\mathbb{Z}^d$ with jump rate $\kappa \geq 0$. Let $\xi := (\xi(t, \cdot))_{t\geq 0}$ be a continuous time Markov process with values in $[0, \infty]^{\mathbb{Z}^d}$, which determines the trapping potential. Let $\gamma \in (0, \infty]$. If $X(t) = x$, then it is killed at rate $\gamma\xi(t, x)$. As we will further detail below, there are two

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fundamentally different regimes: The setting of *immobile* or *static traps*, where $\xi(t,\cdot)$ is constant in t (corresponding to the traps being realized at time 0 and not evolving in time), and the setting of *mobile traps*, where $\xi(t,\cdot)$ depends non-trivially on t. One can also consider the continuum model where X is a Brownian motion on $\mathbb{R}^d$ and $\xi(t,\cdot) \in [0,\infty]^{\mathbb{R}^d}$, which we will do when reviewing some classic results.

Denote by $\mathbb{P}^\xi$ and $\mathbb{E}^\xi$ the probability of and expectation with respect to ξ, and similarly by $\mathbb{P}_x^X$ and $\mathbb{E}_x^X$ the probability of and expectation with respect to X when starting at $x \in \mathbb{Z}^d$. The above model gives rise to the following quantities of interest.

Definition 1 (Survival Probabilities). *Conditional on the realization of* ξ, *the* quenched survival probability of X up to time t *is defined by*

$$Z_{\gamma,t}^\xi := \mathbb{E}_0^X\Big[\exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,\mathrm{d}s\Big\}\Big]. \tag{1}$$

The annealed survival probability up to time t *is defined by averaging over the trap configuration:*

$$Z_{\gamma,t} := \mathbb{E}^\xi\big[Z_{\gamma,t}^\xi\big] = \mathbb{E}^\xi\Big[\mathbb{E}_0^X\Big[\exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,\mathrm{d}s\Big\}\Big]\Big]. \tag{2}$$

The above expressions immediately lead to the following definition of path measures, which are the laws of the random walk X conditioned on survival up to time t.

Definition 2 (Path Measures). *We call the family of Gibbs measures*

$$P_{\gamma,t}^\xi(X\in\cdot) := \frac{\mathbb{E}_0^X\Big[\exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,\mathrm{d}s\Big\}\mathbb{1}_{X\in\cdot}\Big]}{Z_{\gamma,t}^\xi}, \quad t\geq 0, \tag{3}$$

on the space of càdlàg paths $D([0,t],\mathbb{Z}^d)$ *from* $[0,t]$ *to* $\mathbb{Z}^d$ *the* quenched path measures*.*

Similarly, the family

$$P_{\gamma,t}(X\in\cdot) := \frac{\mathbb{E}_0^X\Big[\mathbb{E}^\xi\Big[\exp\Big\{-\gamma\int_0^t \xi(s,X(s))\,\mathrm{d}s\Big\}\Big]\mathbb{1}_{X\in\cdot}\Big]}{\mathbb{E}^\xi[Z_{\gamma,t}^\xi]}, \quad t\geq 0, \tag{4}$$

will be called the annealed path measures*.*

The quenched and annealed survival probabilities and their respective path measures are the key objects of interest for the trapping problem.

The trapping problem is closely linked to the so-called parabolic Anderson model (PAM), which is the solution of the following parabolic equation with random potential ξ:

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \kappa\Delta u(t,x) - \gamma\,\xi(t,x)\,u(t,x), \qquad x\in\mathbb{Z}^d,\ t\geq 0,\\ u(0,x) &= 1,\end{aligned} \tag{5}$$

where γ, κ and ξ are as before, and the discrete Laplacian on $\mathbb{Z}^d$ is given by

$$\Delta f(x) = \frac{1}{2d} \sum_{\|y-x\|=1} (f(y) - f(x)).$$

By the Feynman–Kac formula, the solution u is given by

$$u(t,0) = \mathbb{E}_0^X \left[\exp \left\{ -\gamma \int_0^t \xi(t-s, X(s))\, \mathrm{d}s \right\} \right], \tag{6}$$

which differs from $Z^\xi_{\gamma,t}$ in (1) by a time reversal. Note that if $\xi(t,\cdot)$ does not depend on time, then $u(t,0) = Z^\xi_{\gamma,t}$, and more generally, if the law of $(\xi(s,\cdot))_{0\le s\le t}$ is invariant under time reversal, then

$$\begin{aligned}\mathbb{E}^\xi[u(t,0)] &= \mathbb{E}^\xi \mathbb{E}_0^X \left[\exp\left\{ -\gamma \int_0^t \xi(t-s, X(s))\, \mathrm{d}s \right\} \right] \\ &= \mathbb{E}_0^X \mathbb{E}^\xi \left[\exp\left\{ -\gamma \int_0^t \xi(s, X(s))\, \mathrm{d}s \right\} \right] = \mathbb{E}^\xi[Z^\gamma_{t,\xi}].\end{aligned}$$

Thus the study of the trapping problem is intimately linked to the study of the PAM, especially the case of immobile traps. A comprehensive account can be found in the recent monograph by König [19].

Historically, most studies of the trapping problem have focused on the case of immobile traps, for which we now have a very good understanding, see e.g. Sznitman [29] and [19]. Our goal is to review some of the recent results in this direction, where ξ is the occupation field of a Poisson system of independent random walks. These results provide first steps in the investigation, while much remains to be understood.

The trapping problem has connections to many other models of physical and mathematical interest, such as chemical reaction networks, random Schrödinger operators and Anderson localization, directed polymers in random environment, self-interacting random walks, branching random walks in random environment, etc. The literature is too vast to be surveyed here. The interested reader can consult [17] for motivations from the chemistry and physics literature and some early mathematical results, [29] for results on Brownian motion among Poisson obstacles, a continuum model with immobile traps, [8] for an overview of literature on mobile traps, and [19] for a comprehensive survey on the parabolic Anderson model, focusing mostly on immobile traps on the lattice $\mathbb{Z}^d$.

The rest of the paper is organized as follows. In Sect. 2, we briefly review the literature on the trapping problem with immobile traps, which has been the theme of the monographs [19,29]. We then review in Sect. 3 some recent results on the case of mobile traps [2,8]. Lastly in Sect. 4, we discuss some open questions.

2 Immobile Traps

In this section, we briefly review what is known for the trapping problem with immobile traps, where the trapping potential $\xi(t,x) = \xi(x)$ does not depend on time and $(\xi(x))_{x\in\mathbb{Z}^d}$ are i.i.d., which has been the subject of the recent monograph [19]. We focus here on the lattice setting, although historically, the first comprehensive mathematical results were obtained for the problem of *Brownian motion among Poisson obstacles* [10,29], where the random walk is replaced by a Brownian motion in $\mathbb{R}^d$, and the traps are balls whose centers follow a homogeneous Poisson point process on $\mathbb{R}^d$. As we will explain, the basic tools in the analysis of the trapping problem with i.i.d. immobile traps are large deviation theory and spectral techniques.

2.1 Annealed Asymptotics

We first consider the annealed survival probability $Z_{\gamma,t}$ (equivalently, $\mathbb{E}^\xi[u(t,0)]$ for the PAM). Using the fact that $(\xi(x))_{x\in\mathbb{Z}^d}$ are i.i.d., we can integrate out ξ to write

$$Z_{\gamma,t} = \mathbb{E}^\xi\Big[\mathbb{E}_0^X\Big[\exp\Big\{-\gamma\int_0^t \xi(X(s))\,\mathrm{d}s\Big\}\Big]\Big] = \mathbb{E}_0^X\Big[e^{\sum_{x\in\mathbb{Z}^d} H(\gamma L_t(x))}\Big], \tag{7}$$

where $L_t(x) := \int_0^t \mathbb{1}_{\{X_s=x\}}\,\mathrm{d}s$ is the local time of X, and

$$H(t) := \ln \mathbb{E}^\xi[e^{-t\xi(0)}]. \tag{8}$$

In the special case $\gamma=\infty$ and $\xi(x)$ are i.i.d. Bernoulli random variables with $\mathbb{P}(\xi(0)=0)=p$, the so-called *Bernoulli trap model with hard traps*, the expression simplifies to

$$Z_{\infty,t} = \mathbb{E}_0^X\Big[p^{|\mathrm{Range}_{s\in[0,t]}(X(s))|}\Big] = \mathbb{E}_0^X\big[e^{|\mathrm{Range}_{s\in[0,t]}(X(s))|\ln p}\big], \tag{9}$$

where $\mathrm{Range}_{s\in[0,t]}(X(s)) := \{X(s)\in\mathbb{Z}^d : s\in[0,t]\}$ is the range of X by time t. The asymptotic analysis of $Z_{\infty,t}$, and its continuum analogue, the Wiener sausage, were carried out by Donsker and Varadhan in a series of celebrated works [10,11] using large deviation techniques. The basic heuristics is that the random walk chooses to stay within a spatial window of scale $1\ll\alpha_t\ll\sqrt{t}$, and within that window, the random walk occupation time measure realizes an optimal profile. Using the large deviation principle for the random walk occupation time measure on spatial scale α_t, one can then optimize over the scale α_t and the occupation time profile to derive a variational representation for the asymptotics of $Z_{\infty,t}$. See e.g. [19, Sect. 4.2] and the references therein. Here we only sketch how to identify the optimal scale α_t.

In representation (9) there are two competing effects: the exponential factor which becomes large when $|\mathrm{Range}_t(X)|$ is small, and the probabilistic cost of a random walk having a small range. If, as the Faber–Krahn inequality suggests, we assume that the expectation in (9) is attained by $\mathrm{Range}_t(X)$ being roughly a ball of radius α_t, then $e^{|\mathrm{Range}_t(X)|\ln p}\approx e^{-c_1\alpha_t^d}$, while $\mathbb{P}_0^X(\sup_{0\le s\le t}\|X_s\|\le\alpha_t)\approx e^{-c_2 t/\alpha_t^2}$ due to Brownian scaling with c_1 and c_2 positive constants. Hence,

$$Z_{\infty,t} = \mathbb{E}_0^X\big[e^{|\mathrm{Range}_t(X)|\ln p}\big] \approx \exp\Big\{-\inf_{1\ll\alpha_t\ll\sqrt{t}}(c_1\alpha_t^d + c_2 t/\alpha_t^2)\Big\} = e^{-c_3 t^{\frac{d}{d+2}}}, \tag{10}$$

where we find that the optimal scale is $\alpha_t \approx t^{\frac{1}{d+2}}$.

Alternatively, we can first identify the optimal profile for the trapping potential ξ, which is to create a clearing free of traps in a ball of radius α_t around the origin, and the random walk in such a potential is then forced to stay within this region. The probability of the first event is of the order $e^{-c_1\alpha_t^d}$, while the probability of the second event is of the order $e^{-c_2 t/\alpha_t^2}$, which leads to the same optimal choice of $\alpha_t \approx t^{\frac{1}{d+2}}$.

For more general ξ and γ, one can still analyze (7) via large deviations of the random walk occupation time measure. Suppose that the random walk is confined to a box $[-R\alpha_t, R\alpha_t]^d$ for some $R > 0$ and scale $1 \ll \alpha_t \ll t^{1/d}$, and let $\widetilde{L}_t(y) := \frac{\alpha_t^d}{t} L_t(\lfloor \alpha_t y \rfloor)$, $y \in [-R, R]^d$, denote the rescaled occupation time measure. Furthermore, assume that the generating function H in (8) satisfies the assumption

$$\text{(H)}: \qquad \lim_{t\uparrow\infty} \frac{H(ty) - yH(t)}{\eta(t)} = \widehat{H}(y) \neq 0 \quad \text{for } y \neq 1, \tag{11}$$

for some $\widehat{H} : (0,\infty) \to \mathbb{R}$ and continuous $\eta : (0,\infty) \to (0,\infty)$ with

$$\lim_{t\to\infty} \eta(t)/t \in [0, \infty].$$

The assumption (H) essentially ensures an appropriate regularity in the right tail of the distribution of $-\xi(0)$. We can then rewrite (7) as

$$\begin{aligned}
Z_{\gamma,t} &= \mathbb{E}_0^X\Big[\exp\Big\{\eta\Big(\frac{\gamma t}{\alpha_t^d}\Big) \sum_{x\in\mathbb{Z}^d} \frac{H\Big(\frac{\gamma t}{\alpha_t^d}\widetilde{L}_t\big(\frac{x}{\alpha_t}\big)\Big) - \widetilde{L}_t\big(\frac{x}{\alpha_t}\big) H\Big(\frac{\gamma t}{\alpha_t^d}\Big)}{\eta\Big(\frac{\gamma t}{\alpha_t^d}\Big)}\Big\}\Big] e^{\alpha_t^d H\big(\frac{\gamma t}{\alpha_t^d}\big)} \\
&\approx e^{\alpha_t^d H\big(\frac{\gamma t}{\alpha_t^d}\big)} \mathbb{E}_0^X\Big[\exp\Big\{\eta\Big(\frac{\gamma t}{\alpha_t^d}\Big) \sum_{x\in\mathbb{Z}^d} \widehat{H}\Big(\widetilde{L}_t\big(\frac{x}{\alpha_t}\big)\Big)\Big\}\Big] \\
&\approx e^{\alpha_t^d H\big(\frac{\gamma t}{\alpha_t^d}\big)} \mathbb{E}_0^X\Big[\exp\Big\{\eta\Big(\frac{\gamma t}{\alpha_t^d}\Big)\alpha_t^d \int_{\mathbb{R}^d} \widehat{H}\big(\widetilde{L}_t(y)\big)\,\mathrm{d}y\Big\}\Big] \\
&= e^{\alpha_t^d H\big(\frac{\gamma t}{\alpha_t^d}\big)} \mathbb{E}_0^X\Big[\exp\Big\{\frac{t}{\alpha_t^2} \int_{\mathbb{R}^d} \widehat{H}\big(\widetilde{L}_t(y)\big)\,\mathrm{d}y\Big\}\Big],
\end{aligned} \tag{12}$$

where the scale α_t is chosen to satisfy

$$\eta\Big(\frac{\gamma t}{\alpha_t^d}\Big)\alpha_t^d = \frac{t}{\alpha_t^2}, \tag{13}$$

so that in (12), the exponential term is comparable to the large deviation probability of the random walk confined in a spatial window of scale α_t. Similar to the Bernoulli hard trap case, the asymptotics of $Z_{\gamma,t}$ can then be identified using the large deviation principle for the random walk occupation time measure on the scale α_t.

Alternatively, we can first identify the optimal profile $\psi(\alpha_t\cdot)$ for the trapping potential ξ on the spatial scale α_t. The survival probability of a random walk in such a potential then decays like $e^{\lambda_\psi t/\alpha_t^2}$, where λ_ψ is the principal eigenvalue of $\Delta - \gamma\psi$. In principle, the asymptotics of $Z_{\gamma,t}$ can then be identified by applying a large deviation principle for the potential ξ on the spatial scale α_t, and then optimizing over α_t and ψ. In practice, this approach has been difficult to implement, and the large deviation approach outlined above has been the standard route in the study of the PAM [19, Sect. 3.2].

We remark that the heuristics above applies when the scale α_t chosen as in (13) tends to infinity as t tends to infinity. However, there are also interesting cases where α_t remains bounded or even $\alpha_t = 1$. Under assumption (H) in (11), there are in fact four classes of potentials [16], each determined by the right tail probability of $-\xi(0)$:

(a) potentials with tails heavier than those of the double-exponential distribution;
(b) double-exponentially distributed potentials, i.e., $\mathbb{P}(-\xi(0) > r) = e^{-e^{r/\rho}}$ for some $\rho \in (0, \infty)$;
(c) so-called *almost bounded* potentials;
(d) bounded potentials.

What distinguishes the four classes are different scales α_t, with $\alpha_t = 1$ in case (a), also known as the single peak case; α_t stays bounded in case (b), where the potential follows the double exponential distribution; $1 \ll \alpha_t \ll t^\epsilon$ for any $\epsilon > 0$ in case (c); and $\alpha_t \to \infty$ faster than some power of t in case (d), which includes in particular the Bernoulli trap model, the discrete analogue of Brownian motion among Poisson obstacles. In the annealed setting, the potential ξ realizes an optimal profile on a so-called *intermittent island* of spatial scale α_t centered around the origin, and the walk then stays confined in that island. For further details, see [19, Chap. 3] and the references therein.

The heuristics sketched above also suggests what the annealed path measure $P_{\gamma,t}$ should look like, namely, the random walk X should fluctuate on the spatial scale α_t. If $\alpha_t \to \infty$, then after diffusively rescaling space-time by $(\alpha_t^{-1}, \alpha_t^{-2})$, we expect the random walk to converge to a Brownian motion h-transformed by the principal eigenfunction of $\Delta - \gamma\psi$, where $\psi(\alpha_t \cdot)$ is the optimal profile the potential ξ realizes on the intermittent island of scale α_t, and with Dirichlet boundary condition off the intermittent island. In practice, however, identifying the path behavior requires obtaining second-order asymptotics for the annealed survival probability $Z_{\gamma,t}$, and complete results have only been obtained in special cases.

Indeed, for the continuum model of a Brownian motion $(X(s))_{s\in[0,\infty)}$ among Poisson obstacles in $d = 1$, Schmock [27] (hard obstacles) and Sethuraman [30] (soft obstacles) have shown that under the annealed path measure $P_{\gamma,t}$, the law of $(t^{-\frac{1}{3}} X(s \cdot t^{\frac{2}{3}}))_{s\in[0,\infty)}$ converges weakly to a mixture of the laws of so-called Brownian taboo processes, i.e., a mixture of Brownian motions conditioned to stay inside a randomly centered interval of length $(\pi^2/\nu)^{\frac{1}{3}}$, and the distribution of the random center can also be determined explicitly (here ν is the Poisson intensity of the traps). In particular, this result implies that typical annealed fluctuations for the Brownian motion among Poisson obstacles are of the order $t^{\frac{1}{3}}$ in $d = 1$. Similar convergence results have been proved by Sznitman in dimension 2 [28], and as observed by Povel in [24], can also be established in dimensions $d \geq 3$, where Brownian motion is confined to a ball with radius of the order $t^{\frac{1}{d+2}}$. For the Bernoulli trap model in dimension 2, path confinement to a ball with radius of the order $t^{\frac{1}{4}}$ has been proved by Bolthausen in [3].

In the lattice setting, when the potential $(\xi(x))_{x\in\mathbb{Z}^d}$ is i.i.d. with double exponential distribution (class (b) above), the intermittent islands are bounded in size, and it is known that the rescaled occupation time measure $\widetilde{L}_t(x) = L_t(x)/t$, $x \in \mathbb{Z}^d$, converges to a deterministic measure with a random shift, and the distribution of the random shift can also be identified explicitly [19, Theorem 7.2]. The convergence of the annealed path measure should follow by the same proof techniques, although it does not seem to have been formulated explicitly in the literature. For potentials in class (a) above, where the intermittent island is a single site, the same result should hold, and the quenched setting leads to more interesting results (see e.g. the survey [20]).

2.2 Quenched Asymptotics

In the quenched setting, the random walk X must seek out regions in space that are favorable for its survival. The heuristics is that, one may first confine the random walk to a box Λ_t of spatial scale $t(\ln t)^2$, and within Λ_t, there are intermittent islands of an optimal spatial scale $\widetilde{\alpha}_t$, on which ξ is close to $\min_{x\in\Lambda_t}\xi(x)$ and takes on an optimal deterministic profile that favors the random walk's survival. The random walk then seeks out an optimal intermittent island and stays there until time t, where the choice of the island depends on the balance between the cost for the random walk to get there in a short time and the probability of surviving on the island until time t.

For instance, for Brownian motion among Poisson obstacles (or the Bernoulli trap model), the intermittent islands should be balls containing no obstacles, and it is then easy to see that within Λ_t, the largest such intermittent islands should have spatial scale of the order $\widetilde{\alpha}_t = (\ln t)^{1/d}$, which suggests that the quenched survival probability $Z^{\xi}_{\gamma,t}$ should decay like $e^{-ct/(\ln t)^{2/d}}$. For i.i.d. potential ξ on $\mathbb{Z}^d$ satisfying assumption (H) in (11), the scale $\widetilde{\alpha}_t$ for the intermittent islands can be determined similarly by considering the following:

(a) the large deviation probability of ξ achieving a non-trivial profile on the intermittent island after suitable centering and scaling;
(b) a balance between this large deviation probability and the probability that the random walk survives on the intermittent island until time t;
(c) and the requirement that there should be of order one such islands in Λ_t.

It turns out that $\widetilde{\alpha}_t = \alpha_{\beta(t)}$, where $\beta(t)$ is another scale satisfying

$$\frac{\beta(t)}{\alpha^2_{\beta(t)}} = \ln(t^d). \tag{14}$$

See e.g. [19, Sect. 5.1] for further details. Note that if $\alpha_t = t^\epsilon$ for some $\epsilon \in [0, 1/2)$, then $\beta(t) = (d \ln t)^{\frac{1}{1-2\epsilon}}$. This implies that the size of the quenched intermittent islands is of the same order as the size of the annealed intermittent islands when the latter is bounded, and is much smaller when the latter grows to infinity. The heuristic picture above leads to a sharp lower bound on $Z^\xi_{\gamma,t}$. The challenge is to obtain the matching upper bound, as well as to identify finer asymptotics to draw conclusions about the quenched path measure $P^\xi_{\gamma,t}$.

The main tools are spectral techniques. The exponential rate of decay of the quenched survival probability $Z^\xi_{\gamma,t}$ is given by the principal eigenvalue of $\Delta - \gamma\xi$ on the box Λ_t. Different techniques have been developed to bound the principal eigenvalue. For Brownian motion among Poisson obstacles, Sznitman developed the *method of enlargement of obstacles* (MEO), which resulted in the monograph [29]. The basic idea is to enlarge the obstacles to reduce the combinatorial complexity, without significantly altering the principal eigenvalue of $\Delta - \gamma\xi$ in Λ_t. We refer to the review [18] and the monograph [29] for further details. The MEO was adapted to the lattice setting in [1]. In the lattice setting, a different approach has been developed in the context of the PAM. One divides Λ_t into microboxes of scale $\widetilde{\alpha}_t$. The principal Dirichlet eigenvalues of $\Delta - \gamma\xi$ on the microboxes are i.i.d. random variables, and spectral domain decomposition techniques allow one to control the principal eigenvalue on Λ_t in terms of the principal eigenvalues on the microboxes. This reduces the analysis to an extreme value problem, where the random walk optimizes over the choice of the microbox to go to and the cost of getting there (see [19, Sects. 4.4, 6.3]). We note that there is also a simple way to transfer the annealed asymptotics for the survival probability to the quenched asymptotics using the so-called Liftshitz tail, as shown by Fukushima in [13].

As the heuristics suggest, under the quenched measure, the random walk should seek out one of the optimal intermittent islands and stay there until time t. This has been partially verified. For Brownian motion among Poisson obstacles, Sznitman [29] used the MEO to show that the quenched survival probability decays asymptotically as $e^{-(C+o(1))t/(\ln t)^{2/d}}$. The expected picture is that there are $t^{o(1)}$ many intermittent islands, with size $t^{o(1)}$ and mutual distance $t^{1-o(1)}$ to each other and to the origin, and the time when the Brownian motion first enters one of the intermittent islands is $o(t)$. This has been rigorously verified in dimension 1 (see e.g. [29, Chapter 6]). But in higher dimensions, there is still a lack of control on the time of entering the intermittent islands. Much more precise control on the size and number of intermittent islands have recently been obtained by Ding and Xu for the Bernoulli hard trap model [12]: there are at most $(\log n)^a$ many intermittent islands, with size at most $(\log n)^b$ for some $a, b > 0$, and their distance to the origin is at least $n/(\log n)^a$. For the PAM on $\mathbb{Z}^d$ with i.i.d. potential $(\xi(x))_{x\in\mathbb{Z}^d}$ satisfying assumption (H) in (11), sharp asymptotics for the quenched survival probability has been obtained in all cases (see [19, Theorem 5.1] and the references therein). When the trapping potential $(\xi(x))_{x\in\mathbb{Z}^d}$ are i.i.d. with double exponential distribution, the eigenvalue order statistics has been successfully analyzed in [5], the quenched path measure has been shown to concentrate on a single island at a distance of the order $t/(\ln t \ln\ln\ln t)$ from the origin, and the scaling limit of the quenched path measure has been obtained (see [6] or [19, Theorem 6.5]). Similar results have been obtained for potentials with heavier tails, for which the intermittent islands consists of single sites (see [19, Sec. 6.4] or [20]).

3 Mobile Traps

There have been few mathematical results on the trapping problem with mobile traps. Redig [26] considered trapping potential ξ generated by a reversible Markov process, such as a Poisson field of independent random walks or the symmetric exclusion process in equilibrium, and he obtained exponential upper bounds on the annealed survival probability using spectral techniques for the process of traps viewed from the random walk. When ξ is generated by a single mobile trap, i.e., $\xi(t,x) = \delta_x(Y_t)$ for a simple symmetric random walk Y on $\mathbb{Z}^d$, Schnitzler and Wolff [31] have computed the asymptotics of the decay of the annealed survival probability via explicit calculations. The large deviation and spectral techniques outlined in Sect. 2 for immobile traps largely fail for mobile traps, and new techniques need to be developed.

Recently, we investigated further the model where ξ is generated by a Poisson field of independent random walks, previously considered in [26]. More precisely, given $\nu > 0$, let $(N_y)_{y\in\mathbb{Z}^d}$ be a family of i.i.d. Poisson random variables with mean ν. We then start a family of independent random walks $(Y^{j,y})_{y\in\mathbb{Z}^d,\, 1\leq j\leq N_y}$ on $\mathbb{Z}^d$, each with jump rate $\rho \geq 0$ and $Y^{j,y} := (Y^{j,y}_t)_{t\geq 0}$ denotes the path of the j-th trap starting from y at time 0. For $t \geq 0$ and $x \in \mathbb{Z}^d$, we then define

$$\xi(t,x) := \sum_{y\in\mathbb{Z}^d,\, 1\leq j\leq N_y} \delta_x(Y^{j,y}_t) \tag{15}$$

which counts the number of traps at site x at time t. When $\rho = 0$, we recover the case of immobile traps.

For trapping potential ξ defined as in (15), there are no systematic tools except the Poisson structure of the traps. The only known results so far are: (1) The quenched survival probability decays at a well-defined exponential rate in all dimensions, while the annealed survival probability decays sub-exponentially in dimensions $d = 1$ and 2 and exponentially in $d \geq 3$ [8]; (2) The random walk is sub-diffusive under the annealed path measure in dimension $d = 1$ [2]. We will review these results in Sects. 3.1 and 3.2 below. We will assume for the rest of this section that ξ is defined as in (15).

We remark that in the physics literature, recent studies of the trapping problem with mobile traps have focused on the annealed survival probability [21,22], and the particle and trap motion may also be sub-diffusive (see e.g. [4,32], and also [7] for some mathematical results). In the recent mathematics literature, continuum analogues of the results in [8] on the survival probability were also obtained in [25], where the traps move as a Poisson collection of independent Brownian motions in $\mathbb{R}^d$. Further extensions to Lévy trap motion were carried out in [9].

3.1 Decay of Survival Probabilities

The precise rate of decay of the annealed and quenched survival probabilities were determined in [8]. Recall that γ, κ, ρ and ν are respectively the rate of killing per trap, the jump rate of the random walk X, the jump rate of the traps, and the density of the traps.

Theorem 1 [Quenched survival probability]. *Assume that $d \geq 1$, $\gamma > 0$, $\kappa \geq 0$, $\rho > 0$ and $\nu > 0$, and the traps as well as the walk X follow independent simple symmetric random walks. Then there exists $\lambda^{\mathrm{q}}_{d,\gamma,\kappa,\rho,\nu}$ deterministic such that $\mathbb{P}^\xi$-a.s.,*

$$Z^\xi_{\gamma,t} = \exp\big\{ - \lambda^{\mathrm{q}}_{d,\gamma,\kappa,\rho,\nu}\, t(1+o(1))\big\} \quad \text{as } t \to \infty. \tag{16}$$

Furthermore, $0 < \lambda^{\mathrm{q}}_{d,\gamma,\kappa,\rho,\nu} \leq \gamma\nu + \kappa$.

The proof of Theorem 1 is based on the sub-additive ergodic theorem [8].

Theorem 2 [Annealed survival probability]. *Assume that $\gamma \in (0,\infty]$, $\kappa \geq 0$, $\rho > 0$ and $\nu > 0$, and the traps as well as the walk X follow independent simple symmetric random walks. Then*

$$Z_{\gamma,t} = \mathbb{E}^\xi[Z^\xi_{\gamma,t}] \;=\; \begin{cases} \exp\Big\{ - \nu\sqrt{\dfrac{8\rho t}{\pi}}(1+o(1))\Big\}, & d=1,\\ \exp\Big\{ - \nu\pi\rho\dfrac{t}{\ln t}(1+o(1))\Big\}, & d=2,\\ \exp\Big\{ - \lambda^{\mathrm{a}}_{d,\gamma,\kappa,\rho,\nu}\, t(1+o(1))\Big\}, & d\geq 3. \end{cases} \tag{17}$$

Furthermore,

$$\lambda^{\mathrm{a}}_{d,\gamma,\kappa,\rho,\nu} \geq \lambda^{\mathrm{a}}_{d,\gamma,0,\rho,\nu} = \frac{\nu\gamma}{\Big(1+\frac{\gamma G_d(0)}{\rho}\Big)},$$

where $G_d(0) := \int_0^\infty p_t(0)\,\mathrm{d}t$ is the Green function of a simple symmetric random walk on $\mathbb{Z}^d$ with jump rate 1 and transition kernel $p_t(\cdot)$.

Note that in dimensions $d=1$ and 2, the annealed survival probability decays sub-exponentially, and the pre-factor in front of the decay rate is independent of $\gamma \in (0,\infty]$ and $\kappa \geq 0$.

Although Theorems 1 and 2 were proved in [8] for traps and X following simple symmetric random walks, they can be easily extended to more general symmetric random walks with mean zero and finite variance.

We sketch below the proof of Theorem 2 and the main tools used in [8]. The first step is to integrate out the Poisson random field ξ and derive suitable representations for the annealed survival probability $Z_{\gamma,t}$.

Representation of $Z_{\gamma,t}$ in terms of random walk range: Given the Poisson field ξ defined as in (15), we can integrate out ξ to obtain

$$
\begin{aligned}
Z_{\gamma,t} = \mathbb{E}^{\xi}[Z^{\xi}_{\gamma,t}] &= \mathbb{E}^X_0 \mathbb{E}^{\xi}\Big[\exp\Big\{-\gamma \int_0^t \xi(s, X(s))\,\mathrm{d}s\Big\}\Big] \\
&= \mathbb{E}^X_0\Big[\exp\Big\{\nu \sum_{y\in\mathbb{Z}^d} (v_X(t,y) - 1)\Big\}\Big],
\end{aligned} \tag{18}
$$

where conditional on X,

$$
v_X(t,y) = \mathbb{E}^Y_y\left[\exp\left\{-\gamma \int_0^t \delta_0(Y(s) - X(s))\,\mathrm{d}s\right\}\right] \tag{19}
$$

with $\mathbb{E}^Y_y[\cdot]$ denoting expectation with respect to the motion of a trap starting at y.

When $\gamma = \infty$, we can interpret

$$
v_X(t,y) = \mathbb{P}^Y_y\big(Y(s) \neq X(s)\,\forall\, s \in [0,t]\big),
$$

which leads to

$$
\begin{aligned}
Z_{\infty,t} &= \mathbb{E}^X_0\Big[\exp\Big\{-\nu \sum_{y\in\mathbb{Z}^d} \mathbb{P}^Y_y\big(Y(s) - X(s) = 0 \text{ for some } s \in [0,t]\big)\Big\}\Big] \\
&= \mathbb{E}^X_0\Big[\exp\Big\{-\nu \sum_{y\in\mathbb{Z}^d} \mathbb{P}^Y_0\big(Y(s) - X(s) = -y \text{ for some } s \in [0,t]\big)\Big\}\Big] \\
&= \mathbb{E}^X_0\Big[\exp\Big\{-\nu \mathbb{E}^Y_0\big[\big|\mathrm{Range}_{s\in[0,t]}(Y(s) - X(s))\big|\big]\Big\}\Big],
\end{aligned} \tag{20}
$$

where we recall

$$
\mathrm{Range}_{s\in[0,t]}(Y(s) - X(s)) = \{Y(s) - X(s) \in \mathbb{Z} \,:\, s \in [0,t]\}. \tag{21}
$$

from (9).

When $\gamma \in (0,\infty)$, we can give a similar representation of $Z_{\gamma,t}$ in terms of the range of $Y - X$. More precisely, let $\mathcal{T} := \{T_1, T_2, \ldots\} \subset [0,\infty)$ be an independent Poisson point process on $[0,\infty)$ with intensity γ, and define

$$
\mathrm{SoftRange}_{s\in[0,t]}(Y(s) - X(s)) := \{Y(T_k) - X(T_k) : k \in \mathbb{N}, T_k \in [0,t]\}. \tag{22}
$$

Then we have

$$
Z_{\gamma,t} = \mathbb{E}^X_0\Big[\exp\Big\{-\nu \mathbb{E}^{Y,\mathcal{T}}_0\big[\big|\mathrm{SoftRange}_{s\in[0,t]}(Y(s) - X(s))\big|\big]\Big\}\Big], \tag{23}
$$

where $\mathbb{E}^{Y,\mathcal{T}}_0$ denotes expectation with respect to both Y and the Poisson point process $\mathcal{T}$.

Alternative representation of $Z_{\gamma,t}$: We now derive an alternative representation of $Z_{\gamma,t}$ which will turn out useful below e.g. for showing the existence of an asymptotic exponential decay of the survival probability. For this purpose, note that the time-reversed process $\widetilde{\xi}(s,\cdot) := \xi(t-s,\cdot)$, $s \in [0,t]$, is also the occupation field of a Poisson system of random walks, where each walk follows the law of $\widetilde{Y} := -Y$, which is the same as that of Y by symmetry. We can then write

$$Z_{\gamma,t} = \mathbb{E}_0^X \mathbb{E}^\xi \left[\exp\left\{-\gamma \int_0^t \widetilde{\xi}(t-s, X(s))\,\mathrm{d}s\right\}\right] = \mathbb{E}_0^X \left[\exp\left\{\nu \sum_{y\in\mathbb{Z}^d} (\widetilde{v}_X(t,y)-1)\right\}\right], \tag{24}$$

where

$$\widetilde{v}_X(t,y) = \mathbb{E}_y^{\widetilde{Y}} \left[\exp\left\{-\gamma \int_0^t \delta_0(\widetilde{Y}(t-s) - X(s))\,\mathrm{d}s\right\}\right]. \tag{25}$$

Let $\widetilde{L}$ denote the generator of $\widetilde{Y}$. Then by the Feynman–Kac formula,

$$(\widetilde{v}_X(t,y))_{t\geq 0, y\in\mathbb{Z}^d}$$

solves the equation

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{v}_X(t,y) &= \widetilde{L}\widetilde{v}_X(t,y) - \gamma\delta_{X(t)}(y)\,\widetilde{v}_X(t,y), \qquad y\in\mathbb{Z}^d,\ t\geq 0, \\ \widetilde{v}_X(0,\cdot) &\equiv 1, \end{aligned} \tag{26}$$

which implies that $\Sigma_X(t) := \sum_{y\in\mathbb{Z}^d}(\widetilde{v}_X(t,y)-1)$ is the solution of the equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Sigma_X(t) &= -\gamma\widetilde{v}_X(t,X(t)), \\ \Sigma_X(0) &= 0. \end{aligned} \tag{27}$$

Hence, $\Sigma_X(t) = -\gamma\int_0^t \widetilde{v}_X(s,X(s))\,\mathrm{d}s$, which leads to the alternative representation

$$Z_{\gamma,t} = \mathbb{E}_0^X \left[\exp\left\{-\nu\gamma \int_0^t \widetilde{v}_X(s,X(s))\,\mathrm{d}s\right\}\right]. \tag{28}$$

Existence of $\lim_{t\to\infty}\frac{1}{t}\ln Z_{\gamma,t}$**:** From the representation (28), it is easily seen that $\ln Z_{\gamma,t}$ is super-additive, which implies the existence of $\lim_{t\to\infty}\frac{1}{t}\ln Z_{\gamma,t}$. Indeed, for $t_1, t_2 > 0$,

$$\begin{aligned} Z_{\gamma,t_1+t_2} &= \mathbb{E}_0^X \Bigg[\exp\left\{-\nu\gamma\int_0^{t_1} \widetilde{v}_X(s,X(s))\,\mathrm{d}s\right\} \\ &\qquad \times \exp\left\{-\nu\gamma\int_{t_1}^{t_1+t_2} \widetilde{v}_X(s,X(s))\,\mathrm{d}s\right\}\Bigg] \\ &\geq \mathbb{E}_0^X \Bigg[\exp\left\{-\nu\gamma\int_0^{t_1} \widetilde{v}_X(s,X(s))\,\mathrm{d}s\right\} \\ &\qquad \times \exp\left\{-\nu\gamma\int_0^{t_2} \widetilde{v}_{\theta_{t_1}X}(s,(\theta_{t_1}X)(s))\,\mathrm{d}s\right\}\Bigg] \\ &= Z_{\gamma,t_1}Z_{\gamma,t_2}, \end{aligned}$$

where $\theta_{t_1} X := ((\theta_{t_1} X)(s))_{s \geq 0} = (X(t_1 + s) - X(t_1))_{s \geq 0}$, we used the independence of $(X(s))_{0 \leq s \leq t_1}$ and $((\theta_{t_1} X)(s))_{0 \leq s \leq t_2}$, and the fact that for $s > t_1$,

$$
\begin{aligned}
\widetilde{v}_X(s, X(s)) &= \mathbb{E}^{\widetilde{Y}}_{X(s)} \left[\exp\left\{ -\gamma \int_0^s \delta_0(\widetilde{Y}(r) - X(s-r))\, \mathrm{d}r \right\} \right] \\
&\leq \mathbb{E}^{\widetilde{Y}}_{X(s)} \left[\exp\left\{ -\gamma \int_0^{s-t_1} \delta_0(\widetilde{Y}(r) - X(s-r))\, \mathrm{d}r \right\} \right] \\
&= \widetilde{v}_{\theta_{t_1} X}(s - t_1, (\theta_{t_1} X)(s - t_1)).
\end{aligned}
$$

It then follows by super-additivity that $\lim_{t \to \infty} \frac{1}{t} \ln Z_{\gamma,t}$ exists, although the rate is zero in dimensions 1 and 2.

Below we sketch a proof of the precise sub-exponential rates of decay of $Z_{\gamma,t}$ in dimensions 1 and 2. We refer the reader to [8] for details and proof of the case when $d = 3$.

Lower bound on $Z_{\gamma,t}$ for $d = 1, 2$: A lower bound is achieved by clearing a ball of radius R_t around the origin where there are no traps up to time t, and then force the random walk X to stay inside this ball up to time t. Optimizing the choice of R_t then gives the desired lower bound which, surprisingly, turns out to be sharp in dimensions 1 and 2.

Let $B_{R_t} := \{x \in \mathbb{Z}^d : \|x\|_\infty \leq R_t\}$ denote the L_∞ ball of radius R_t centered around the origin, with R_t to be optimized over later. Let E_t denote the event that there are no traps in B_{R_t} at time 0, F_t the event that no traps starting from outside B_{R_t} at time 0 will enter B_{R_t} before time t, and G_t the event that the random walk X with $X(0) = 0$ does not leave B_{R_t} before time t. Then we have

$$
Z_{\gamma,t} \geq \mathbb{P}(E_t \cap F_t \cap G_t) = \mathbb{P}(E_t)\mathbb{P}(F_t)\mathbb{P}(G_t). \tag{29}
$$

Since E_t is the event that $\xi(0, x) = 0$ for all $x \in B_{R_t}$, where $\xi(0, x)$ are i.i.d. Poisson with mean ν, we have

$$
\mathbb{P}(E_t) = e^{-\nu(2R_t+1)^d}. \tag{30}
$$

To estimate $\mathbb{P}(G_t)$, note that for $1 \ll R_t \ll \sqrt{t}$, we can approximate X by a Brownian motion and rescale space-time by $(1/R_t, 1/R_t^2)$ to obtain the estimate

$$
\mathbb{P}(G_t) \geq e^{-c_1 t / R_t^2} \tag{31}
$$

for some $c_1 > 0$.

To estimate $\mathbb{P}(F_t)$, note that F_t is the event that for each $y \notin B_{R_t}$, no trap starting at y will enter B_{R_t} before time t. Since the number of traps starting from each $y \in \mathbb{Z}^d$ are i.i.d. Poisson with mean ν, we obtain

$$
\mathbb{P}(F_t) = \exp\Big\{ -\nu \sum_{y \notin B_{R_t}} \mathbb{P}^Y_y(\tau_{B_{R_t}} \leq t) \Big\}, \tag{32}
$$

where $\tau_{B_{R_t}}$ is the stopping time when Y enters B_{R_t}.

In dimension 1, since we have assumed for simplicity that Y makes nearest-neighbor jumps, we note that $\mathbb{P}(F_t)$ in fact does not depend on the choice of R_t. Furthermore, when $R_t = 0$, $\mathbb{P}(F_t)$ is easily seen as the annealed survival probability for the trapping problem with instant killing by traps ($\gamma = \infty$) and immobile X ($\kappa = 0$). The asymptotics of the annealed survival probability was obtained in [8, Sect. 2.2], with

$$\mathbb{P}(F_t) = e^{-(1+o(1))\nu\sqrt{8\rho t/\pi}}. \tag{33}$$

We can then combine the estimates for $\mathbb{P}(E_t)$, $\mathbb{P}(F_t)$ and $\mathbb{P}(G_t)$ and optimize over R_t to obtain the lower bound

$$Z_{\gamma,t} \geq e^{-(1+o(1))\nu\sqrt{8\rho t/\pi}} e^{-ct^{1/3}}, \tag{34}$$

where the leading order asymptotics is determined by that of $\mathbb{P}(F_t)$, and the second order asymptotics comes from $\mathbb{P}(E_t)\mathbb{P}(G_t)$, with the optimal choice of R_t being a constant multiple of $t^{1/3}$. This lower bound strategy strongly suggests that the fluctuation of the random walk X under the path measure $P_{\gamma,t}$ will be of the order $t^{1/3}$.

In dimension 2, it was shown in [8, Section 2.3] that if $R_t \ll t^\epsilon$ for all $\epsilon > 0$, then $\mathbb{P}(F_t)$ has the same leading order asymptotics as the annealed survival probability of the trapping problem with $\gamma = \infty$ and $\kappa = 0$, which coincides with the asymptotics for $Z_{\gamma,t}$ stated in (17). To obtain the desired lower bound on $Z_{\gamma,t}$, we can then choose any R_t satisfying $\sqrt{\ln t} \ll R_t \ll t^\epsilon$ for any $\epsilon > 0$, which ensures that $\mathbb{P}(E_t)\mathbb{P}(G_t)$ gives lower order contributions. This suggests that under the path measure $P_{\gamma,t}$, X fluctuates on a scale between $\sqrt{\ln t}$ and t^ϵ for any $\epsilon > 0$. However, to identify the optimal choice of R_t, we would need to obtain more precise estimates on how $\mathbb{P}(F_t)$ depends on R_t.

Upper bound on $Z_{\gamma,t}$ for $d = 1, 2$: The key ingredient is what has been named in the physics literature as the *Pascal principle*, which asserts that in (18), if we condition on the random walk trajectory X, then the annealed survival probability

$$Z^X_{\gamma,t} := \mathbb{E}^\xi\left[\exp\left\{-\gamma\int_0^t \xi(s, X(s))\,\mathrm{d}s\right\}\right] = \exp\left\{\nu\sum_{y\in\mathbb{Z}^d}(v_X(t,y) - 1)\right\} \tag{35}$$

is maximized when $X \equiv 0$, provided that the trap motion Y is a symmetric random walk. Therefore

$$Z_{\gamma,t} = \mathbb{E}^X_0[Z^X_{\gamma,t}] \leq Z^{X\equiv 0}_{\gamma,t}, \tag{36}$$

which has the same leading order asymptotic decay as the case $\gamma = \infty$ and $\kappa = 0$ that appeared in the lower bound.

For discrete time random walks under suitable symmetry assumptions, the Pascal principle was proved by Moreau, Oshanin, Bénichou and Coppey in [21,22], which can then be easily extended to general continuous time symmetric random walks [8]. An interesting corollary is that if Y is a continuous time symmetric random walk, then for any deterministic path X on $\mathbb{Z}^d$ with locally finitely many jumps, we have

$$\mathbb{E}_0^Y[|\mathrm{Range}_{s\in[0,t]}(Y(s))|] \le \mathbb{E}_0^Y[|\mathrm{Range}_{s\in[0,t]}(Y(s)+X(s))|]. \tag{37}$$

I.e., the expected range of a symmetric random walk can only be increased under deterministic perturbations.

3.2 Path Measures

The only known result so far on the path measures is the following sub-diffusive bound under the annealed path measure $P_{\gamma,t}$ in dimension one, recently proved in [2].

Theorem 3 (Sub-diffusivity in dimension one). *Let X and the trap motion Y follow continuous time random walks on $\mathbb{Z}$ with jump rates $\kappa, \rho > 0$ and non-degenerate jump kernels p_X and p_Y respectively. Assume that p_X has mean zero and p_Y is symmetric, with*

$$\sum_{x\in\mathbb{Z}} e^{\lambda^*|x|} p_X(x) < \infty \qquad \text{and} \qquad \sum_{x\in\mathbb{Z}} e^{\lambda^*|x|} p_Y(x) < \infty \tag{38}$$

for some $\lambda^* > 0$. Then there exists $\alpha > 0$ such that for all $\epsilon > 0$,

$$P_{\gamma,t}\big(\|X\|_t \in (\alpha t^{\frac{1}{3}}, t^{\frac{11}{24}+\epsilon})\big) \to 1 \qquad \text{as } t \to \infty. \tag{39}$$

where $\|X\|_t := \sup_{s\in[0,t]} |X_s|$.

Since $\frac{11}{24} < \frac{1}{2}$, this result shows that under the annealed path measure $P_{\gamma,t}$, X is sub-diffusive.

Remark 1. Very recently, Öz [23] improved the upper bound $t^{\frac{11}{24}+\epsilon}$ in (39) to $ct^{\frac{5}{11}}$ for the continuum analogue model of a Brownian motion among a Poisson field of moving traps.

We sketch below the proof strategy followed in [2].

Simple random walk, $\gamma = \infty$: Let us assume for simplicity that X and Y are continuous time simple symmetric random walks, and $\gamma = \infty$. By (4),

$$P_{\infty,t}(X \in \cdot) = \frac{\mathbb{E}_0^X\big[Z_{\infty,t}^X \, \mathbb{1}_{X\in\cdot}\big]}{Z_{\infty,t}}, \tag{40}$$

where $Z_{\infty,t} = \mathbb{E}_0^X[Z_{\infty,t}^X]$ with $Z_{\gamma,t}^X$ defined in (35), and by (18) and (20),

$$Z_{\infty,t}^X = \exp\Big\{ -\nu \mathbb{E}_0^Y\big[\big|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))\big|\big]\Big\}. \tag{41}$$

As shown in the lower bound for the annealed survival probability in (29)–(34),

$$Z_{\infty,t} \ge Z_{\infty,t}^{X\equiv 0} e^{-ct^{1/3}}. \tag{42}$$

Therefore, from (40) we obtain

$$
\begin{aligned}
&P_{\infty,t}(X\in\cdot)\\
&\le e^{ct^{1/3}}\mathbb{E}_0^X\left[Z^X_{\infty,t}/Z^{X\equiv 0}_{\infty,t}\,\mathbb{1}_{X\in\cdot}\right] &(43)\\
&= e^{ct^{1/3}}\mathbb{E}_0^X\left[e^{-\nu\left(\mathbb{E}_0^Y\left[\left|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))\right|-\left|\mathrm{Range}_{s\in[0,t]}Y(s)\right|\right]\right)}\,\mathbb{1}_{X\in\cdot}\right]. &(44)
\end{aligned}
$$

Note that by the corollary of the Pascal principle (37), the exponent in the exponential is negative, and hence

$$
P_{\infty,t}(X\in\cdot)\le e^{ct^{1/3}}\mathbb{P}_0^X(X\in\cdot). \tag{45}
$$

This implies that

$$
P_{\infty,t}(\|X\|_t\le\alpha t^{1/3})\le e^{ct^{1/3}}\mathbb{P}_0^X(\|X\|_t\le\alpha t^{1/3})\to 0 \quad \text{as } \to\infty, \tag{46}
$$

if α is sufficiently small, as can be easily seen if X is replaced by a Brownian motion. This proves the lower bound on the fluctuations.

Proving the sub-diffusive upper bound on X in (39) requires finding lower bounds on the differences of the ranges in (44) for typical realizations of X. Using the fact that X and Y make nearest-neighbor jumps and Y is symmetric, we have

$$
\begin{aligned}
&\mathbb{E}_0^Y\left[\left|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))\right|\right]\\
&=1+\mathbb{E}_0^Y\big[\sup_{s\in[0,t]}(Y(s)-X(s))-\inf_{s\in[0,t]}(Y(s)-X(s))\big]\\
&=1+\mathbb{E}_0^Y\big[\sup_{s\in[0,t]}(Y(s)-X(s))+\sup_{s\in[0,t]}(X(s)-Y(s))\big]\\
&=1+\mathbb{E}_0^Y\big[\sup_{s\in[0,t]}(Y(s)-X(s))+\sup_{s\in[0,t]}(Y(s)+X(s))\big].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathbb{E}_0^Y\left[\left|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))\right|-\left|\mathrm{Range}_{s\in[0,t]}Y(s)\right|\right]\\
&=\mathbb{E}_0^Y\big[\sup_{s\in[0,t]}(Y(s)-X(s))+\sup_{s\in[0,t]}(Y(s)+X(s))-2\sup_{s\in[0,t]}Y(s)\big]. \qquad (47)
\end{aligned}
$$

We will show that for any $\epsilon>0$, uniformly in X with $\|X\|_t\ge t^{\frac{11}{24}+\epsilon}$,

$$
\text{The above expectation is bounded from below by } Ct^{\frac{1}{3}+\epsilon}, \tag{48}
$$

which by (44) then implies

$$
P_{\infty,t}(\|X\|_t\ge t^{\frac{11}{24}+\epsilon})\le e^{ct^{\frac{1}{3}}-Ct^{\frac{1}{3}+\epsilon}}\to 0 \quad \text{as } t\to\infty. \tag{49}
$$

To bound the expectation in (47), first note that we always have

$$
\sup_{s\in[0,t]}(Y(s)-X(s))+\sup_{s\in[0,t]}(Y(s)+X(s))-2\sup_{s\in[0,t]}Y(s)\ge 0,
$$

in contrast to $|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))|-|\mathrm{Range}_{s\in[0,t]}Y(s)|$. Therefore we can restrict to a suitable subset of trajectories of Y to obtain a lower bound.

Let σ_X be the first time when X achieves its global maximum in $[0,t]$, and $\tau_X \in [0,t]$ the first time when X achieves its global minimum in $[0,t]$. If $\|X\|_t \geq t^{\frac{11}{24}+\epsilon}$, then one of the two sets

$$S := \{s \in [0,t] : X(\sigma_X) - X(s) \geq t^{\frac{11}{24}+\epsilon}/2\}$$

and

$$T := \{s \in [0,t] : X(s) - X(\tau_X) \geq t^{\frac{11}{24}+\epsilon}/2\}$$

must have Lebesgue measure at least $t/2$. Assume w.l.o.g. that $|S| \geq t/2$. Then a lower bound on (47) can be obtained by restricting Y to the set of trajectories

$$\mathcal{F} := \Big\{\sigma_Y \in S \cap [t/8, 7t/8] \quad \text{and} \quad Y(\sigma_Y) - Y(\sigma_X) \leq t^{\frac{11}{24}+\epsilon}/4\Big\}.$$

Indeed, on this event,

$$\begin{aligned}
&\sup_{s\in[0,t]}(Y(s)-X(s)) + \sup_{s\in[0,t]}(Y(s)+X(s)) - 2\sup_{s\in[0,t]}Y(s) \\
\geq\ & Y(\sigma_Y) - X(\sigma_Y) + Y(\sigma_X) + X(\sigma_X) - 2Y(\sigma_Y) \\
=\ & (X(\sigma_X) - X(\sigma_Y)) - (Y(\sigma_Y) - Y(\sigma_X)) \\
\geq\ & \frac{1}{4} t^{\frac{11}{24}+\epsilon}.
\end{aligned} \tag{50}$$

On the other hand, it is easily seen that there exists $\alpha > 0$ such that uniformly in t large,

$$\mathbb{P}_0^Y(\sigma_Y \in S \cap [t/8, 7t/8]) \geq \alpha. \tag{51}$$

Furthermore, conditioned on σ_Y, $(Y(\sigma_Y) - Y(\sigma_Y + s))_{s\in[0,t-\sigma_Y]}$ and $(Y(\sigma_Y) - Y(\sigma_Y - s))_{s\in[0,\sigma_Y]}$ are two independent random walks conditioned respectively to not hit zero or go below zero. Such conditioned random walks are comparable to 3-dimensional Bessel processes. Assume w.l.o.g. that $\sigma_X > \sigma_Y$, then we have the rough estimate

$$\begin{aligned}
\mathbb{P}_0^Y(Y(\sigma_Y) - Y(\sigma_X) \leq t^{\frac{11}{24}+\epsilon}/4\,|\sigma_Y) &\geq C\,\mathbb{P}_0^B(|B(\sigma_X - \sigma_Y)| \leq t^{\frac{11}{24}+\epsilon}/4) \\
&\geq C' \frac{(t^{\frac{11}{24}+\epsilon})^3}{t^{\frac{3}{2}}} = C' t^{-\frac{1}{8}+3\epsilon},
\end{aligned} \tag{52}$$

where B is a 3-dimensional Brownian motion starting from 0 and its Euclidean norm, $|B_t|$, is a 3-dimensional Bessel process, while in the last inequality, we used the local central limit theorem for B and $\sigma_X - \sigma_Y \leq t$. Combining (50)–(52), we then obtain

$$\begin{aligned}
&\mathbb{E}_0^Y\Big[\sup_{s\in[0,t]}(Y(s)-X(s)) + \sup_{s\in[0,t]}(Y(s)+X(s)) - 2\sup_{s\in[0,t]}Y(s)\Big] \\
\geq\ & C'\alpha t^{-\frac{1}{8}+3\epsilon}\frac{t^{\frac{11}{24}+\epsilon}}{4} = C'' t^{\frac{1}{3}+4\epsilon},
\end{aligned} \tag{53}$$

which then implies (48) and hence (49). The above heuristic calculations were made rigorous in [2].

Simple random walk, $\gamma < \infty$: When $\gamma < \infty$, the representation (41) for $Z^X_{\infty,t}$ should be replaced by

$$Z^X_{\gamma,t} = \exp\Big\{ -\nu \mathbb{E}_0^{Y,\mathcal{T}}\big[\big|\mathrm{SoftRange}_{s\in[0,t]}(Y(s)-X(s))\big|\big]\Big\} \tag{54}$$

as can be seen from (23). The difficulty then lies in controlling the difference

$$\begin{aligned} F_{\gamma,t}(Y-X) &:= \big[\big|\mathrm{Range}_{s\in[0,t]}(Y(s)-X(s))\big|\big] \\ &\quad - \mathbb{E}^{\mathcal{T}}\big[\big|\mathrm{SoftRange}_{s\in[0,t]}(Y(s)-X(s))\big|\big] \\ &= \sum_{x\in\mathbb{Z}} e^{-\gamma L_t^{Y-X}(x)} \mathbb{1}_{L_t^{Y-X}(x)>0}, \end{aligned}$$

where $L_t^{Y-X}(x) := \int_0^t \mathbb{1}_{\{Y(s)-X(s)=x\}} \mathrm{d}s$ is the local time of $Y-X$ at x. In [2], this control is achieved by proving that

$$\sup_{t\geq e} \mathbb{E}_0^Y\Big[\exp\Big\{\frac{c}{\ln t} F_{\gamma,t}(Y)\Big\}\Big] < \infty, \tag{55}$$

which also leads to interesting bounds on the set of thin points of Y, i.e., the set of x where the local time $L_t^Y(x)$ is positive but unusually small.

Non-simple random walks: When X and Y satisfy the assumptions in Theorem 3, but are not necessarily simple random walks, the arguments outlined above can still be salvaged, provided we can control the difference

$$\begin{aligned} G_t(Y) &:= \sup_{s\in[0,t]} Y(s) - \inf_{s\in[0,t]} Y(s) + 1 - |\mathrm{Range}_{s\in[0,t]}(Y(s))| \\ &= \sum_{\inf_{s\in[0,t]} Y(s) \leq x \leq \sup_{s\in[0,t]} Y(s)} \mathbb{1}_{L_t^Y(x)=0}, \end{aligned}$$

which is the total size of the holes in the range of Y. In [2], this control is achieved by proving

$$\sup_{t\geq e} \mathbb{E}_0^Y\Big[\exp\Big\{\frac{c}{\ln t} G_t(Y)\Big\}\Big] < \infty. \tag{56}$$

The proof of (55) and (56) in [2] in fact follow the same line of arguments.

4 Some Open Questions

The large deviation and spectral techniques that have been successful for the trapping problem with immobile traps largely fail for mobile traps. As a result, many questions remain unanswered for the trapping problem with mobile traps, even when the traps are just a Poisson system of independent random walks. We list below some natural open questions.

Open questions:

(1) In dimension 1, we conjecture that under the annealed path measure $P_{\gamma,t}$, X fluctuates on the scale $t^{1/3}$, which is based on the lower bound strategy for survival in Sect. 3.1. In fact, we saw that the probability that traps from outside the ball of radius R_t do not enter it before time t does not depend on the choice of R_t, which leaves only the interplay between the cost of clearing the ball of traps at time 0 and the cost of the forcing the walk to stay within the ball up to time t. This is also what happens in the case of immobile traps, which leads us to conjecture that not only the fluctuation is on the same scale of $t^{1/3}$ [27,30], but also the rescaled paths converge to the same limit.

(2) In dimension 2, the lower bound strategy for the annealed survival probability suggests that under the annealed path measure $P_{\gamma,t}$, X fluctuates on a scale R_t with $\sqrt{\ln t} \ll R_t \ll t^{\epsilon}$ for all $\epsilon > 0$. The correct scale of R_t still needs to be determined. In dimension 1, we heavily used the fact that the range of a random walk is essentially captured by its maximum and minimum. This is no longer applicable in dimensions $d \geq 2$, and new techniques need to be developed.

(3) In dimensions $d \geq 3$, if we follow the same lower bound strategy for the annealed survival probability, then it is not difficult to see that the optimal size of the ball which is free of traps should be of order one instead of diverging as in dimensions 1 and 2. This seems to suggest that under the annealed path measure $P_{\gamma,t}$, X should be localized near the origin. On the other hand, it is also reasonable to expect that the interaction between the traps and the random walk X will create a clearing around X in a time of order 1, and random fluctuations will then cause the clearing and X to undergo diffusive motion, leading to a central limit theorem and invariance principle for X under $P_{\gamma,t}$. We conjecture that the second scenario is what actually happens.

(4) What can we say about the quenched path measure $P^{\xi}_{\gamma,t}$? For Brownian motion among Poisson obstacles, Sznitman [29] has shown that the particle X seeks out pockets of space free of traps. There is a balance between the cost of going further away from the starting point to find larger pockets and the benefit of surviving in a larger pocket, which leads to super-diffusive motion under the quenched path measure (as pointed out to us by R. Fukushima, this follows from results in [29]; and see [12] for the Bernoulli hard trap model). When the traps are mobile, these pockets of space free of traps are destroyed quickly and become much more rare. Would X still be super-diffusive under the quenched path measure?

(5) In our analysis, we heavily used the fact that the random trapping potential ξ is a Poisson field. One could also consider ξ generated by other interacting particle systems, such as the exclusion process or the voter model as Gärtner et al. considered in the context of the parabolic Anderson model [14,15]. Their analysis has focused on the exponential asymptotics of the quenched and annealed solution of the PAM. The path behavior of the corresponding

trapping problem remains open. The behavior is expected to be similar when ξ is the occupation field of either the symmetric exclusion process or the Poisson system of independent random walks, because the two particle systems have the same large scale space-time fluctuations.

(6) It is natural to also consider the case when the random walk moves with a deterministic drift. A naive survival strategy of creating a ball centered at the origin free of traps, then there is an exponential cost for the random walk to stay within that region, while leaving the region will also incur an exponential cost depending on γ, the rate of killing when the walk meets a trap. Depending the strength of the drift relative to γ, it is conceivable that there is a transition in the path behavior as the drift varies, such that when the drift is small, the random walk is localized near the origin just as in the case of zero drift, and when the drift becomes sufficiently large, the random walk becomes delocalized. This is also a question of active interest for the PAM with immobile traps (see e.g. [19, Sec. 7.10], [29, Sec. 5.4 and 7.3]).

(7) In Theorem 2 on the asymptotics of the annealed survival probability, we note that in dimensions 1 and 2, the precise rate of decay of $Z_{\gamma,t}$ does not depend on the killing rate γ. The heuristics is that as long as $\gamma > 0$, the random walk X would not see the traps under the annealed path measure. This suggests sending $\gamma = \gamma_t \to 0$ as $t \to \infty$ in order to see non-trivial dependence on the killing rate. In dimension 1, the correct rate is $\gamma_t = \widetilde{\gamma}/\sqrt{t}$, and in dimension 2, $\gamma_t = \widetilde{\gamma}/\ln t$. Decaying killing rate has been considered in the immobile trap case, see [19, Sec. 7.3.3] and the references therein.

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Random Walks in Cooling Random Environments

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To Chuck Newman on the occasion of his 70th birthday

Abstract. We propose a model of a one-dimensional random walk in dynamic random environment that interpolates between two classical settings: (I) the random environment is sampled at time zero only; (II) the random environment is resampled at every unit of time. In our model the random environment is resampled along an increasing sequence of deterministic times. We consider the annealed version of the model, and look at three growth regimes for the resampling times: (R1) linear; (R2) polynomial; (R3) exponential. We prove weak laws of large numbers and central limit theorems. We list some open problems and conjecture the presence of a crossover for the scaling behaviour in regimes (R2) and (R3).

Keywords: Random walk · Dynamic random environment · Resampling times · Law of large numbers · Central limit theorem

1 Introduction, Model, Main Theorems and Discussion

1.1 Background and Outline

Models for particles moving in media with impurities pose many challenges. In mathematical terms, the medium with impurities is represented by a random environment on a lattice and the particle motion is represented by a random walk on this lattice with transition probabilities that are determined by the environment.

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Static Random Environment. A model that has been studied extensively in the literature is that of *Random Walk in Random Environment* (RWRE), where the random environment is *static* (see Zeitouni [13] for an overview). In one dimension this model exhibits striking features that make the presence of a random environment particularly interesting. Namely, there are regions in the lattice where the random walk remains *trapped* for a very long time. The presence of these traps leads to a local slow down of the random walk in comparison to a homogeneous random walk. This, in turn, is responsible for a non-trivial limiting speed, as well as for anomalous scaling behaviour (see Sect. 1.2 below). For instance, under proper assumptions on the random environment, the random walk can be transient yet sub-ballistic in time, or it can be non-diffusive in time with non-Gaussian fluctuations. To derive such results and to characterise associated limit distributions, a key approach has been to represent the environment by a potential function: a deep valley in the potential function corresponds to a region in the lattice where the random walk gets trapped for a very long time.

Dynamic Random Environment. If, instead, we consider a random walk in a random environment that itself evolves over time, according to a prescribed *dynamic* rule, then the random environment is still inhomogeneous, but the dynamics dissolves existing traps and creates new traps. Depending on the choice of the dynamics, the random walk behaviour can be similar to that in the static model, i.e., show some form of *localisation*, or it can be similar to that of a homogeneous random walk, in which case we speak of *homogenisation*. The simplest model of a dynamic random environment is given by an i.i.d. field of spatial random variables that is resampled in an i.i.d. fashion after every step of the random walk. This model has been studied in several papers. Clearly, under the so-called annealed measure homogenisation occurs: the independence of the random environment in space and time causes the random walk to behave like a homogeneous random walk, for which a standard law of large numbers and a standard central limit theorem hold.

Cooling Random Environment. In the present paper we introduce a new model, which we call *Random Walk in Cooling Random Environment* (RWCRE). This model interpolates between the two settings mentioned above: start at time zero with an i.i.d. random environment and resample it along an increasing sequence of deterministic times (the name "cooling" is chosen here because the static model is sometimes called "frozen"). If the resampling times increase rapidly enough, then we expect to see a behaviour close to that of the static model, via some form of *localisation*. Conversely, if the resampling times increase slowly enough, then we expect *homogenisation* to be dominant. The main goal of our paper is to start making this rough heuristics precise by proving a few basic results for RWCRE in a few cooling regimes. From a mathematical point of view, the analysis of RWCRE reduces to the study of sums of independent random variables distributed according to the static model, viewed at successive time scales. In Sect. 1.5 we will point out what type of results for the static model, *currently still unavailable*, would be needed to pursue a more detailed analysis of RWCRE.

Outline. In Sect. 1.2 we recall some basic facts about the one-dimensional RWRE model. In Sect. 1.3 we define our model with a *cooling* random environment and introduce the three cooling regimes we are considering. In Sect. 1.4 we state three theorems and make a remark about more general cooling regimes. In Sect. 1.5 we mention a few open problems and state a conjecture. Sections 2–4 are devoted to the proofs of our theorems. Appendix A proves a variant of a Toeplitz lemma that is needed along the way. Appendix B recalls the central limit theorem for sums of independent random variables. Appendix C strengthens a well-known convergence in distribution property of recurrent RWRE, which is needed for one of our theorems.

1.2 RWRE

Throughout the paper we use the notation $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ with $\mathbb{N} = \{1, 2, \dots\}$. The classical one-dimensional random walk in random environment (RWRE) is defined as follows. Let $\omega = \{\omega(x)\colon\ x \in \mathbb{Z}\}$ be an i.i.d. sequence with probability distribution

$$\mu = \alpha^{\mathbb{Z}}$$

for some probability distribution α on $(0,1)$. The random walk in the *space* environment ω is the Markov process $Z = (Z_n)_{n\in\mathbb{N}_0}$ starting at $Z_0 = 0$ with transition probabilities

$$P^\omega(Z_{n+1} = x + e \mid Z_n = x) = \begin{cases} \omega(x), & \text{if } e = 1,\\ 1-\omega(x), & \text{if } e = -1, \end{cases} \qquad n \in \mathbb{N}_0.$$

The properties of Z are well understood, both under the *quenched* law $P^\omega(\cdot)$ and the *annealed* law

$$\mathbb{P}_\mu(\cdot) = \int_{(0,1)^{\mathbb{Z}}} P^\omega(\cdot)\, \mu(\mathrm{d}\omega). \tag{1}$$

Let $\langle \cdot \rangle$ denote expectation w.r.t. α. Abbreviate

$$\rho(0) = \frac{1-\omega(0)}{\omega(0)}. \tag{2}$$

Without loss of generality we make the following assumption on α:

$$\langle \log \rho(0) \rangle \le 0. \tag{3}$$

This assumption guarantees that Z has a preference to move to the right. In what follows we recall some key results for RWRE on $\mathbb{Z}$. For a general overview, we refer the reader to Zeitouni [13].

The following result due to Solomon [12] characterises recurrence versus transience and asymptotic speed.

Proposition 1 (Recurrence, transience and speed RWRE). *Let α be any distribution on $(0,1)$ satisfying (3). Then the following hold:*

– *Z is recurrent if and only if $\langle \log \rho(0) \rangle = 0$.*
– *If $\langle \log \rho(0) \rangle < 0$ and $\langle \rho(0) \rangle \geq 1$, then Z is transient to the right with zero speed:*

$$\lim_{n\to\infty} \frac{Z_n}{n} = 0 \quad \mathbb{P}_\mu\text{-a.s.}$$

– *If $\langle \log \rho(0) \rangle < 0$ and $\langle \rho(0) \rangle < 1$, then Z is transient to the right with positive speed:*

$$\lim_{n\to\infty} \frac{Z_n}{n} = v_\mu = \frac{1 - \langle \rho(0) \rangle}{1 + \langle \rho(0) \rangle} > 0 \quad \mathbb{P}_\mu\text{-a.s.}$$

The scaling limits in the different regimes have been studied in a number of papers, both under the quenched and the annealed law. While the results are the same for the law of large numbers, they are in general different for the scaling limits. Under the quenched law only partial results are available (see Peterson [8] for a summary of what is known). For this reason we are forced to restrict ourselves to the annealed law.

In the recurrent case the proper scaling was identified by Sinai [11] and the limit law by Kesten [6]. The next proposition summarises their results.

Proposition 2 (Scaling limit RWRE: recurrent case). *Let α be any probability distribution on $(0,1)$ satisfying $\langle \log \rho(0) \rangle = 0$ and $\sigma_\mu^2 = \langle \log^2 \rho(0) \rangle \in (0,\infty)$. Then, under the annealed law $\mathbb{P}_\mu$, the sequence of random variables*

$$\frac{Z_n}{\sigma_\mu^2 \log^2 n}, \quad n \in \mathbb{N}, \tag{4}$$

converges in distribution to a random variable V on $\mathbb{R}$ that is independent of α. The law of V has a density $p(x)$, $x \in \mathbb{R}$, with respect to the Lebesgue measure that is given by

$$p(x) = \frac{2}{\pi} \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{2k+1} \exp\left[-\frac{(2k+1)^2\pi^2}{8}|x|\right], \quad x \in \mathbb{R}. \tag{5}$$

For later use we need to show that the sequence in (4) converges to V in L^p for every $p > 0$. This is done in Appendix C. Note that the law of V is symmetric with variance $\sigma_V^2 \in (0,\infty)$.

The scaling in the (annealed) transient case was first studied by Kesten, Kozlov and Spitzer [7]. In order to state their results, we need some more notation. Given $s, b > 0$, denote by $L_{s,b}$ the s-stable distribution with scaling parameter b, centred at 0 and totally skewed to the right. In formulas, $L_{s,b}$ lives on $\mathbb{R}$ and is identified by its characteristic function

$$\hat{L}_{s,b}(u) = \int_{\mathbb{R}} \mathrm{e}^{iux} L_{s,b}(\mathrm{d}x) = \exp\left[-b|u|^s \left(1 - i\frac{u}{|u|} g_s(u)\right)\right], \qquad u \in \mathbb{R}, \tag{6}$$

with

$$g_s(u) = \begin{cases} \tan(\frac{s\pi}{2}), & s \neq 1, \\ \frac{2}{\pi}\log|u|, & s = 1. \end{cases} \quad u \in \mathbb{R}.$$

Write $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \mathrm{d}y$, $x \in \mathbb{R}$, to denote the standard normal distribution.

Proposition 3 (Scaling limit RWRE: transient case). *Let α be any probability distribution on $(0,1)$ satisfying $\langle \log \rho(0) \rangle < 0$ such that the support of the distribution of $\log \rho(0)$ is non-lattice. Let $s \in (0,\infty)$ be the unique root of the equation*

$$\langle \rho(0)^s \rangle = 1, \tag{7}$$

and suppose that $\langle \rho(0)^s \log \rho(0) \rangle < \infty$. Then, under the annealed law $\mathbb{P}_\mu$, the following hold:

- *If $s \in (0,1)$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_\mu \left(\frac{Z_n}{n^s} \leq x \right) = [1 - L_{s,b}(x^{-1/s})]\, \mathbb{1}_{\{x>0\}}. \tag{8}$$

- *If $s = 1$, then there exist $b > 0$ and $\{\delta_\alpha(n)\}_{n\in\mathbb{N}}$, satisfying $\delta_\alpha(n) = [1 + o(1)]\, n/b \log n$ as $n \to \infty$, such that*

$$\lim_{n\to\infty} \mathbb{P}_\mu \left(\frac{Z_n - \delta_\alpha(n)}{n/\log^2 n} \leq x \right) = 1 - L_{1,b}(-b^2 x), \ x \in \mathbb{R}. \tag{9}$$

- *If $s \in (1,2)$, then there exist $b > 0$ and $c = c(b) > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_\mu \left(\frac{Z_n - v_\mu n}{b n^{1/s}} \leq x \right) = 1 - L_{s,c}(-x), \ x \in \mathbb{R}. \tag{10}$$

- *If $s = 2$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_\mu \left(\frac{Z_n - v_\mu n}{b\sqrt{n \log n}} \leq x \right) = \phi(x), \ x \in \mathbb{R}.$$

- *If $s \in (2,\infty)$, then there exists a $b > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}_\mu \left(\frac{Z_n - v_\mu n}{b\sqrt{n}} \leq x \right) = \phi(x), \ x \in \mathbb{R}.$$

In (9) and (10), the limiting laws are stable laws that are totally skewed to the left, i.e., their characteristic function is as in (6) but with a $+$ sign in the term with the imaginary factor i in the exponential. In (8), the limiting law is an inverse stable law, sometimes referred to as the Mittag–Leffler distribution.

1.3 RWCRE

In the present paper we look at a model where ω is updated along a growing sequence of deterministic times. To that end, let $\tau \colon \mathbb{N}_0 \to \mathbb{N}_0$ be a strictly increasing map such that $\tau(0) = 0$ and $\tau(k) \geq k$ for $k \in \mathbb{N}$ (see Fig. 1). Define a sequence of random environments $\Omega = (\omega_n)_{n\in\mathbb{N}_0}$ as follows:

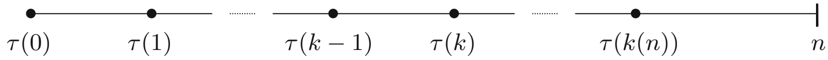

Fig. 1. Resampling times $\tau(k)$, $0 \le k \le k(n)$, prior to time n.

► At each time $\tau(k)$, $k \in \mathbb{N}_0$, the environment $\omega_{\tau(k)}$ is freshly resampled from $\mu = \alpha^{\mathbb{Z}}$ and does not change during the time interval $[\tau(k), \tau(k+1))$.

The random walk in the *space-time* environment Ω is the Markov process $X = (X_n)_{n \in \mathbb{N}_0}$ starting at $X_0 = 0$ with transition probabilities

$$P^{\Omega}(X_{n+1} = x + e \mid X_n = x) = \begin{cases} \omega_n(x), & \text{if } e = 1, \\ 1 - \omega_n(x), & \text{if } e = -1, \end{cases} \quad n \in \mathbb{N}_0.$$

We call X the *random walk in cooling random environment* (RWCRE) with *resampling rule* α and *cooling rule* τ. Our goal will be to investigate the behavior of X under the *annealed law*

$$\mathbb{P}_{\mathbb{Q}}(\cdot) = \int_{(\mathbb{R}^{\mathbb{Z}})^{\mathbb{N}_0}} P^{\Omega}(\cdot)\, \mathbb{Q}(\mathrm{d}\Omega), \tag{11}$$

where $\mathbb{Q} = \mathbb{Q}_{\alpha,\tau}$ denotes the law of Ω.

Note that if $\tau(1) = \infty$, then RWCRE reduces to RWRE, i.e., X has the same distribution as Z. On the other hand, if $\tau(k) = k$, $k \in \mathbb{N}$, then ω_n is freshly sampled from μ for all $n \in \mathbb{N}$ and RWCRE reduces to what is often referred to as random walk in an i.i.d. space-time random environment. Under the annealed law, the latter is a homogeneous random walk and is trivial. Under the quenched law it is non-trivial and has been investigated in a series of papers, e.g. Boldrighini, Minlos, Pellegrinotti and Zhizhina [4], and Rassoul-Agha and Seppalainen [10]. For any other choice of τ the model has not been considered before.

In this paper we will focus on *three different growth regimes* for $\tau(k)$ as $k \to \infty$:

(R1) *No cooling:* $\tau(k) \sim Ak$ for some $A \in (1, \infty)$.
(R2) *Slow cooling:* $\tau(k) \sim Bk^{\beta}$ for some $B \in (0, \infty)$ and $\beta \in (1, \infty)$.
(R3) *Fast cooling:* $\log \tau(k) \sim Ck$ for some $C \in (0, \infty)$.

Let

$$T_k = \tau(k) - \tau(k-1), \quad k \in \mathbb{N},$$

be the increments of the resampling times of the random environment. We assume that $k \mapsto T_k$ is sufficiently regular so that

$$\begin{array}{ll} T_k \sim \beta B k^{\beta - 1} & \text{in regime (R2)}, \\ \log T_k \sim Ck & \text{in regime (R3)}. \end{array} \tag{12}$$

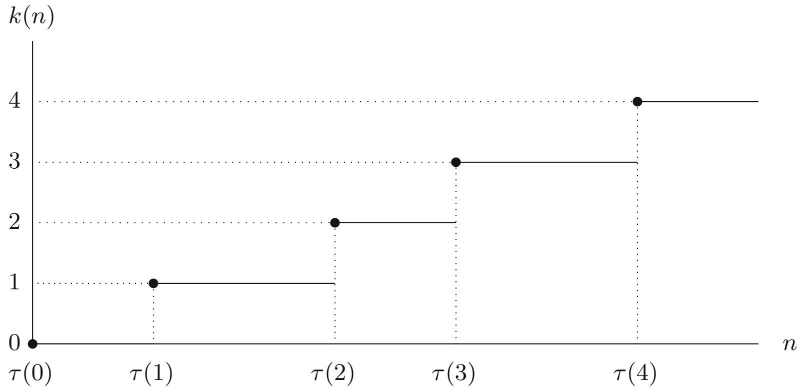

Fig. 2. Plot of $n \mapsto k(n)$.

For instance, in regime (R2) this regularity holds as soon as $k \mapsto T_k$ is ultimately non-decreasing (Bingham, Goldie and Teugels [3, Sections 1.2 and 1.4]).

Let

$$Y_k = X_{\tau(k)} - X_{\tau(k-1)}, \quad k \in \mathbb{N},$$

be the increments of the random walk between the resampling times. Our starting point will be the relation

$$X_n = \sum_{k=1}^{k(n)} Y_k + \bar{Y}^n, \tag{13}$$

where (see Fig. 2)

$$k(n) = \max\{k \in \mathbb{N}\colon\ \tau(k) \le n\} \tag{14}$$

is the last resampling prior to time n, Y_k is distributed as Z_{T_k} in environment $\omega_{\tau(k-1)}$, while $\bar{Y}^n$ is a boundary term that is distributed as $Z_{\bar{T}^n}$ in environment $\omega_{\tau(k(n))}$ with

$$\bar{T}^n = n - \tau(k(n)) \tag{15}$$

the remainder time after the last resampling. Note that all terms in (13) are independent.

1.4 Main Theorems

We are now ready to state our results under the annealed measure $\mathbb{P}_{\mathbb{Q}}$ in the three cooling regimes (R1)–(R3).

• **No cooling.** Regime (R1) in essence corresponds to the situation where the increments of the resampling times do not diverge, i.e., $\lim_{k\to\infty} T_k \neq \infty$. To analyze this regime, we assume that the empirical measure of the increments of the resampling times

$$L_n = \frac{1}{k(n)} \sum_{k=1}^{k(n)} \delta_{\tau(k)-\tau(k-1)}$$

has a non-degenerate L^1-limit, i.e.,

$$\lim_{n\to\infty} \sum_{\ell\in\mathbb{N}} \ell\,|L_n(\ell) - \nu(\ell)| = 0$$

for some $\nu \in \mathcal{M}_1(\mathbb{N})$. Since $\sum_{\ell\in\mathbb{N}} \ell L_n(\ell) = \tau(k(n))/k(n)$ with $\tau(k(n)) \sim n$ and $k(n) \sim n/A$, it follows that $\sum_{\ell\in\mathbb{N}} \ell\nu(\ell) = A$. Abbreviate

$$v_\nu = \frac{1}{A}\sum_{\ell\in\mathbb{N}} \nu(\ell)\,\mathbb{E}_\mu(Z_\ell), \quad \sigma_\nu^2 = \frac{1}{A}\sum_{\ell\in\mathbb{N}} \nu(\ell)\,\mathbb{V}\mathrm{ar}_\mu(Z_\ell).$$

Write $w-\lim_{n\to\infty}$ to denote convergence in distribution.

Theorem 1 (No cooling: Strong LLN and CLT). *In regime* (R1) *the following hold.*

(1) *Strong law of large numbers:*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_\nu \quad \mathbb{P}_\mathbb{Q}\text{-}a.s.$$

(2) *Central limit theorem:*

$$w - \lim_{n\to\infty} \frac{X_n - v_\nu n}{\sigma_\nu\sqrt{n}} = \mathcal{N}(0,1) \qquad \text{under the law } \mathbb{P}_\mathbb{Q}, \tag{16}$$

provided

$$\begin{aligned} &\text{(i) } \bar{T}^n = o(\sqrt{n}),\\ &\text{(ii) } \textstyle\sum_{\ell\in\mathbb{N}} \ell\,|L_n(\ell) - \nu(\ell)| = o\left(1/\sqrt{n}\,\right). \end{aligned} \tag{17}$$

(It is possible to weaken (17), but we will not pursue this further.)

• **Cooling.** Regimes (R2) and (R3) is essence correspond to the situation where the increments of the resampling times diverge, i.e., $\lim_{k\to\infty} T_k = \infty$. We have a weak LLN under the latter condition only, which we refer to as *cooling*.

Theorem 2 (Cooling: Weak LLN). *Let α be as in Proposition 1. If the cooling rule τ is such that*

$$\lim_{k\to\infty} T_k = \infty, \tag{18}$$

then

$$w - \lim_{n\to\infty} \frac{X_n}{n} = v_\mu \qquad \text{under the law } \mathbb{P}_\mathbb{Q}.$$

For regimes (R2) and (R3) we derive Gaussian fluctuations for recurrent RWRE:

Theorem 3 (Slow and fast cooling: Gaussian fluctuations for recurrent RWRE). *Let* α *be as in Proposition 2. In regimes* (R2) *and* (R3)*,*

$$w - \lim_{n\to\infty} \frac{X_n - \mathbb{E}_{\mathbb{Q}}(X_n)}{\sqrt{\chi_n(\tau)}} = \mathcal{N}(0,1) \qquad \textit{under the law } \mathbb{P}_{\mathbb{Q}}$$

with

$$\chi_n(\tau) = \begin{cases} (\sigma_\mu^2 \sigma_V)^2 \left(\frac{\beta-1}{\beta}\right)^4 \left(\frac{n}{B}\right)^{1/\beta} \log^4 n, & \textit{in regime } (\mathrm{R2}), \\ (\sigma_\mu^2 \sigma_V)^2 \left(\frac{1}{5C^5}\right) \log^5 n, & \textit{in regime } (\mathrm{R3}), \end{cases} \tag{19}$$

with σ_μ^2 *the variance of the random variable* $\log \rho(0)$ *in* (2) *and* $\sigma_V^2 \in (0,\infty)$ *is the variance of the random variable* V *in* (5)*.*

Note that the scaling in (19) depends on the parameters (B,β) and C in the regimes (R2) and (R3), respectively, as well as on the law μ of the static random environment.

1.5 Discussion, Open Problems and a Conjecture

Preliminary Results. The results presented in Sect. 1.4 are modest and are only a first step in the direction of understanding the effect of the cooling of the random environment. The weak LLN in Theorem 2 holds as soon as the cooling is *effective*, i.e., the increments of the resampling times diverge (as assumed in (18)). In particular, the asymptotic speed v_μ is the same as for RWRE. This is markedly different from Theorem 1, where homogenisation occurs (defined in Sect. 1.1), but with an averaged speed v_ν that is different from v_μ. The CLT in Theorem 3 can be extended to more general cooling regimes than (R2) and (R3), but fails when the cooling becomes too rapid.

Future Targets. In future work we will show that also the strong LLN holds in the effective cooling regime. The derivation is more technical and requires that we distinguish between different choices of the parameter s defined in (7), which controls the scaling behaviour of RWRE (recall Proposition 3). Below we discuss what scaling to expect for RWCRE and what properties of the static model in Sect. 1.2 would be needed to prove this scaling. Essentially, we need *rate of convergence* properties of RWRE, as well as control on the *fluctuations of the resampling times.*

Scaling Limits in the Recurrent Case. Suppose that α is as in Proposition 2. Theorem 3 establishes Gaussian fluctuations for the position of the random walk around its mean, of an order that depends on the cooling rule τ, given by (19). From our knowledge of the static model (recall Proposition 2), we can only conclude that $\mathbb{E}_\mu(Z_n) = o(\log^2 n)$. Suppose that, under suitable conditions on α, we could show that $\mathbb{E}_\mu(Z_n) = o(n^{-1/2} \log n)$ (one example is when α is symmetric w.r.t. $\frac{1}{2}$, in which case $\mathbb{E}_\mu(Z_n) = 0$ for all $n \in \mathbb{N}_0$). Then, as we will see in Sect. 4, in the slow cooling regime (R2) this extra information would imply for RWCRE that $\mathbb{E}_{\mathbb{Q}}(X_n) = o(n^{1/2\beta} \log^2 n)$, in which case Theorem 3 would say that $X_n/n^{1/2\beta}\log^2 n$ converges in distribution to a Gaussian random variable. In other words, the cooling rule would have the effect of strongly *homogenising* the random environment, and the limiting Kesten distribution in (5) would be washed out. For the recurrent case it is reasonable to expect the presence of a *crossover* from a Gaussian distribution to the Kesten distribution as the cooling gets slower.

A similar picture should hold in the fast cooling regime (R3). In fact, it would be natural to look at even faster cooling regimes, namely, *super-exponential cooling*, in order to see whether, after an appropriate scaling, the limiting distribution is the same as in the static model. For double-exponential cooling like $\tau(k) = \mathrm{e}^{\mathrm{e}^k}$ the Lyapunov condition in Lemma 2, on which the proof of Theorem 3 is based, is no longer satisfied.

Scaling Limits in the Transient Case. It is natural to expect that a similar *crossover* also appears in the transient case, from the Gaussian distribution to stable law distributions. The general philosophy is the same as in the recurrent case: the faster the cooling, the closer RWCRE is to the static model and therefore the weaker the homogenisation.

We conjecture the following scenario for the scaling limits of the *centred* position of the random walk (i.e., $X_n - v_\mu n$, with v_μ the speed in Theorem 2):

Transient Scaling	(R1)	(R2)	(R3)
$s \in (0,2)$	*CLT*	$\exists\, \beta_c = \beta_c(s)$: $\beta < \beta_c$: *Homogenisation* $\beta > \beta_c$: *Static Law*	*Static Law*
$s \in (2,\infty)$	*CLT*	*CLT*	*CLT*

In this table, $s \in (0,\infty)$ is the parameter in Proposition 3, β is the exponent in regime (R2), and:

- *CLT* means that the centred position divided by $\sqrt{n}$ converges in distribution to a Gaussian.
- *Homogenisation* means that the centred position divided by a factor that is different from $\sqrt{n}$ (and depends on the cooling rule τ) convergences in distribution to a Gaussian (compare with Theorem 3).
- *Static Law* means that the centred position divided by a factor that is different from $\sqrt{n}$ (and depends on the cooling rule τ) has the same limit distribution as in the static model (compare with Proposition 3).

The items under (R2) and (R3) are conjectured, the items under (R1) are proven (compare with Theorem 1).

The most interesting feature in the above table is that, in regime (R2) and for $s \in (0,2)$, we conjecture the existence of a critical exponent $\beta_c = \beta_c(s)$ above which there is Gaussian behaviour and below which there is stable law behaviour. This is motivated by the fact that, for $s \in (0, 2)$, fluctuations are of polynomial order in the static model (see Proposition 3). Hence when the cooling rule is also of polynomial order, as in regime (R2), there is a competition between "localisation between resamplings" and "homogenisation through resamplings".

Role of the Static Model. To establish the CLTs in the second row of the table, for $s \in (2, \infty)$, a deeper understanding of the static model is required. In fact, to prove Gaussian behaviour in regimes (R2) and (R3) we may try to check a Lyapunov condition, as we do in the proof of Theorem 3 (see Lemma 2 below). However, as in the proof of Theorem 3, L^p convergence for some $p > 2$ of $Z_n/\sqrt{n}$ would be needed. As far as we know, this is not available in the literature. An alternative approach, which would be natural for all regimes and would make rigorous the picture sketched in the above table, is to check convergence of the characteristic functions of the random variables $X_n - v_\mu n$ properly scaled. The advantage of this approach is that, due to the independence of the summands in (13), this characteristic function factorises into a product of characteristic functions of the centred static model, on different time scales. However, we would need suitable rate-of-convergence results for these static characteristic functions, which are also not available.

In the table we did not include the case $s = 2$, nor did we comment on the distinction between $s \in (0, 1)$, $s = 1$ and $s \in (1, 2)$, for which the centring appears to be delicate (compare with Proposition 3).

Hitting Times. To derive the static scaling limits in Proposition 3, Kesten, Kozlov and Spitzer [7] (see Peterson [8] for later work) first derive the scaling limits for the hitting times $\sigma_x = \inf\{n \in \mathbb{N}_0 \colon Z_n = x\}$, $x \in \mathbb{N}$, and afterwards pass to the scaling limits for the positions Z_n, $n \in \mathbb{N}_0$ via a simple inversion argument. This suggests that a further approach might be to look at the hitting times associated with our RWCRE model. However, a decomposition into a sum of independent random variables, in the flavour of (13), would no longer hold for σ_x, $x \in \mathbb{N}_0$. Consequently, in this approach it seems even more difficult to exploit what is known about the static model.

Remark 1. Since the completion of this paper, two follow-up papers have appeared [1,2] in which some of the open problems have been settled.

2 No Cooling: Proof of Theorem 1

Throughout the sequel we use the same symbol $\mathbb{P}$ for the annealed law $\mathbb{P}_\mu$ of RWRE in (1) and the annealed law $\mathbb{P}_\mathbb{Q}$ of RWCRE in (11). At all times it will be clear which of the two applies: no confusion is possible because they are linked via (13).

Proof. The proof uses the Lyapunov condition in Lemma 2.
(1) Rewrite (13) as

$$\begin{aligned}\frac{X_n}{n} &= \frac{1}{n}\sum_{\ell\in\mathbb{N}}\sum_{k=1}^{k(n)} Y_k\,\mathbb{1}_{\{\tau(k)-\tau(k-1)=\ell\}} + \frac{1}{n}\,\bar{Y}^n \\ &\hat{=} \frac{k(n)}{n}\sum_{\ell\in\mathbb{N}} L_n(\ell)\left(\frac{1}{k(n)L_n(\ell)}\sum_{m=1}^{k(n)L_n(\ell)} Z_\ell^{(m)}\right) + \frac{1}{n}\,Z_{\bar{T}^n},\end{aligned}$$

where $Z_\ell^{(m)}$, $m\in\mathbb{N}$, are independent copies of Z_ℓ and $\hat{=}$ denotes equality in distribution. Since $\tau(k)\sim Ak$, we have $k(n)\sim n/A$ and $\bar{T}^n = o(n)$. Moreover, since $|Z_m|\le m$ for all $m\in\mathbb{N}_0$, we have

$$\lim_{n\to\infty}\frac{1}{n}\,Z_{\bar{T}^n} = 0 \quad \mathbb{P}\text{-a.s.}, \qquad \lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N} Z_\ell^{(m)} = \mathbb{E}(Z_\ell) \quad \mathbb{P}\text{-a.s.},$$

and hence the claim follows by dominated convergence.
(2) Lemma 2 implies that, subject to the Lyapunov condition,

$$w-\lim_{n\to\infty}\frac{X_n-a_n}{\sqrt{b_n}} = \mathcal{N}(0,1) \qquad \text{under the law } \mathbb{P}$$

with

$$a_n = \sum_{k=1}^{k(n)}\mathbb{E}(Y_k) + \mathbb{E}(\bar{Y}^n), \qquad b_n = \sum_{k=1}^{k(n)}\mathbb{V}\text{ar}(Y_k) + \mathbb{V}\text{ar}(\bar{Y}^n).$$

We need to show that $a_n = v_\nu n + o(\sqrt{n})$ and $b_n \sim \sigma_\nu^2 n$, and verify the Lyapunov condition.

To compute b_n, we note that $\mathbb{V}\text{ar}(\bar{Y}^n) \le (\bar{T}^n)^2 = o(n)$ by Assumption (i) in (17), and we write

$$\begin{aligned}\sum_{k=1}^{k(n)}\mathbb{V}\text{ar}(Y_k) &= k(n)\sum_{\ell\in\mathbb{N}} L_n(\ell)\,\mathbb{V}\text{ar}(Z_\ell) \\ &= k(n)\sum_{\ell\in\mathbb{N}}\nu(\ell)\,\mathbb{V}\text{ar}(Z_\ell) + k(n)\sum_{\ell\in\mathbb{N}}[L_n(\ell)-\nu(\ell)]\,\mathbb{V}\text{ar}(Z_\ell).\end{aligned}$$

The first term in the right-hand side equals $k(n)\,A\sigma_\nu^2 \sim \sigma_\nu^2 n$, which is the square of the denominator in (16). The second term is $k(n)\,o(1) = o(n)$, because $\mathbb{V}\text{ar}(Z_\ell)\le\ell$ and L_n converges to ν in mean. Hence $b_n\sim\sigma_\nu^2 n$.

To compute a_n, we note that $|\mathbb{E}(\bar{Y}^n)| \le \bar{T}^n = o(\sqrt{n})$ by Assumption (i) in (17), and we write

$$\begin{aligned}\sum_{k=1}^{k(n)}\mathbb{E}(Y_k) &= k(n)\sum_{\ell\in\mathbb{N}} L_n(\ell)\,\mathbb{E}(Z_\ell) \\ &= k(n)\sum_{\ell\in\mathbb{N}}\nu(\ell)\,\mathbb{E}(Z_\ell) + k(n)\sum_{\ell\in\mathbb{N}}[L_n(\ell)-\nu(\ell)]\,\mathbb{E}(Z_\ell).\end{aligned}$$

The first term in the right-hand side equals $k(n)\,Av_\nu \sim v_\nu n$, which is the numerator in (16). By Assumption (ii) in (17), the second term is $k(n)\,o(1/\sqrt{n})$ because $|\mathbb{E}(Z_\ell)| \leq \ell$. Hence $a_n = v_\nu n + o(\sqrt{n})$.

It remains to verify the Lyapunov condition. To do so, we first show that

$$\max_{1\leq k\leq k(n)} T_k = o(\sqrt{n}). \tag{20}$$

Define

$$M_n = \operatorname{argmax}_{1\leq k\leq k(n)} T_k,$$

i.e., the index for which the gap between two successive resampling times is maximal. If there are several such indices, then we pick the largest one. We estimate

$$\begin{aligned}\frac{\max_{1\leq k\leq k(n)} T_k}{\sqrt{n}} &\leq \frac{\tau(M_n)-\tau(M_n-1)}{\sqrt{\tau(M_n)}}\\ &\leq \frac{1+\big[\big(\tau(M_n)-1\big)-\tau\big(k\big(\tau(M_n)-1\big)\big)\big]}{\sqrt{\tau(M_n)-1}}\\ &= \frac{1+\bar{T}^{\tau(M_n)-1}}{\sqrt{\tau(M_n)-1}},\end{aligned} \tag{21}$$

where in the first inequality we use that $\tau(M_n) \leq n$ and in the second inequality that $\tau(M_n-1) = \tau(k(\tau(M_n))-1) = \tau(k(\tau(M_n)-1))$ because $m = k(\tau(m))$, $m \in \mathbb{N}_0$ (recall Fig. 2). By Assumption (i) in (17), the right-hand side of (21) tends to zero when $\lim_{n\to\infty} M_n = \infty$. If the latter fails, then $n \mapsto \max_{1\leq k\leq k(n)} T_k$ is bounded, and the left-hand side of (21) still tends to zero.

Armed with (20) we prove the Lyapunov condition as follows. Abbreviate $\ell(n) = \bar{T}^n \vee \max_{1\leq k\leq k(n)} T_k = o(\sqrt{n})$. We have $|Y_k| \leq \ell(n)$, $1 \leq k \leq k(n)$, and $|\bar{Y}^n| \leq \ell(n)$. This enables us to estimate

$$\begin{aligned}&\frac{1}{b_n^{(2+\delta)/2}}\left[\sum_{k=1}^{k(n)} \mathbb{E}\left(|Y_k-\mathbb{E}(Y_k)|^{2+\delta}\right) + \mathbb{E}\left(|\bar{Y}^n-\mathbb{E}(\bar{Y}^n)|^{2+\delta}\right)\right]\\ &\leq \frac{[2\ell(n)]^\delta}{b_n^{(2+\delta)/2}}\left[\sum_{k=1}^{k(n)} \mathbb{E}\left(|Y_k-E(Y_k)|^{2}\right) + \mathbb{E}\left(|\bar{Y}^n-\mathbb{E}(\bar{Y}^n)|^{2}\right)\right] = \left(\frac{2\ell(n)}{\sqrt{b_n}}\right)^\delta,\end{aligned}$$

which tends to zero because $b_n \sim \sigma_\nu^2 n$ and $\ell(n) = o(\sqrt{n})$. □

3 Cooling: Proof of Theorem 2

Proof. We need to show that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|n^{-1}X_n - v_\mu\right| > \epsilon\right) = 0 \qquad \forall\, \epsilon > 0.$$

To that end, we rewrite the decomposition in (13) (recall (14)–(15)) as

$$X_n = \sum_{k\in\mathbb{N}} \gamma_{k,n} \frac{Y_k}{T_k} + \gamma_n \frac{\bar{Y}^n}{\bar{T}^n}$$

with (see Fig. 3)

$$\gamma_{k,n} = \frac{T_k}{n} \mathbb{1}_{\{1\le k\le k(n)\}}, \qquad \gamma_n = \frac{\bar{T}^n}{n}. \tag{22}$$

Note that $\sum_{k\in\mathbb{N}} \gamma_{k,n} + \gamma_n = 1$ and $\lim_{n\to\infty} \gamma_{k,n} = 0$ for all $k \in \mathbb{N}$.

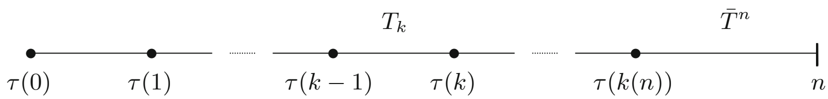

Fig. 3. Resampling times and increments (recall Fig. 1).

To deal with the representation of X_n in (3), we need the following variant of a Toeplitz lemma, adapted to our problem, the proof of which is given in Appendix A.

Lemma 1. *Let* $(\gamma_{k,n})_{k,n\in\mathbb{N}}$ *and* $(\gamma_n)_{n\in\mathbb{N}}$ *be as in* (22). *Let* $(z_k)_{k\in\mathbb{N}}$ *be a real-valued sequence such that* $\lim_{k\to\infty} z_k = z^*$ *for some* $z^* \in \mathbb{R}$. *Then*

$$\lim_{n\to\infty} \left(\sum_{k\in\mathbb{N}} \gamma_{k,n} z_k + \gamma_n z_{\bar{T}^n} \right) = z^*.$$

With the help of (3), we may write

$$n^{-1}X_n - v_\mu = \left[\sum_{k\in\mathbb{N}} \gamma_{k,n} C_k + \gamma_n \bar{C}^n \right] + \left[\sum_{k\in\mathbb{N}} \gamma_{k,n} \left(v_k - v_\mu\right) + \gamma_n \left(\bar{v}^n - v_\mu\right) \right], \tag{23}$$

with

$$C_k = \frac{Y_k - \mathbb{E}[Y_k]}{T_k}, \qquad \bar{C}^n = \frac{\bar{Y}^n - \mathbb{E}[\bar{Y}^n]}{\bar{T}^n}, \qquad v_k = \frac{\mathbb{E}[Z_{T_k}]}{T_k}, \qquad \bar{v}^n = \frac{\mathbb{E}[Z_{\bar{T}^n}]}{\bar{T}^n}.$$

Next, note that $\lim_{k\to\infty} v_k = v_\mu$ because $\lim_{k\to\infty} T_k = \infty$. Applying Lemma 1 with $z_k = v_k - v_\mu$ and $z^* = 0$, we see that the second term between square brackets in (23) tends to zero. Therefore, it suffices to show that the first term between square brackets in (23) tends to zero in probability.

Estimate

$$\mathbb{P}\left(\left| \sum_{k\in\mathbb{N}} \gamma_{k,n} C_k + \gamma_n \bar{C}^n \right| > \epsilon \right) \le \frac{1}{\epsilon} \left(\sum_{k\in\mathbb{N}} \gamma_{k,n} \mathbb{E}\left[|C_k|\right] + \gamma_n \mathbb{E}\left[|\bar{C}^n|\right] \right). \tag{24}$$

Since $|C_k| \leq 2$, $k \in \mathbb{N}$, we have $\mathbb{E}[|C_k|] \leq 2\,\mathbb{P}(|C_k| > \delta) + \delta$, $k \in \mathbb{N}$, for any $\delta > 0$. On the other hand, with the help of Proposition 1 we get

$$\lim_{k\to\infty} \mathbb{P}(|C_k| > \delta) = \lim_{k\to\infty} \mathbb{P}\left(\left|\frac{Z_{T_k} - \mathbb{E}[Z_{T_k}]}{T_k}\right| > \delta\right) = 0, \;\; \delta > 0,$$

and hence

$$\lim_{k\to\infty} \mathbb{E}[|C_k|] = 0. \tag{25}$$

Applying Lemma 1 with $z_k = \mathbb{E}[|C_k|]$ and $z^* = 0$, and using (25), we see that the right-hand side of (24) vanishes as $n \to \infty$ for any $\epsilon > 0$. □

4 Slow and Fast Cooling: Proof of Theorem 3

The proof again uses the Lyapunov condition in Lemma 2.

Proof. For an arbitrary cooling rule τ, set

$$\chi_n(\tau) = \sum_{k=1}^{k(n)} \mathbb{V}\mathrm{ar}(Y_k) + \mathbb{V}\mathrm{ar}(\bar{Y}^n),$$

and

$$\chi_n(\tau;p) = \sum_{k=1}^{k(n)} \mathbb{E}(|Y_k - \mathbb{E}(Y_k)|^p) + \mathbb{E}(|\bar{Y}^n - \mathbb{E}(\bar{Y}^n)|^p), \;\; p > 2.$$

In view of Lemma 2, it suffices to show that, in regimes (R2) and (R3),

$$\lim_{n\to\infty} \frac{\chi_n(\tau;p)}{\chi_n(\tau)^{p/2}} = 0, \;\; p > 2. \tag{26}$$

By Proposition 2, we have $\mathbb{E}(Z_n) = o(\log^2 n)$, $n \to \infty$. By Proposition 4 below we further have

$$\mathbb{E}(|Z_n - \mathbb{E}(Z_n)|^2) \sim \varSigma^2 \log^4 n, \quad \mathbb{E}(|Z_n - \mathbb{E}(Z_n)|^p) = O(\log^{2p} n), \quad p > 2, \tag{27}$$

where $\varSigma = \sigma_\mu^2 \sigma_V$. Consequently, using (13) and (27) we get that, for an arbitrary cooling rule τ,

$$\chi_n(\tau) \sim \varSigma^2 \sum_{k=1}^{k(n)} \log^4 T_k + \varSigma^2 \log^4 \bar{T}^n,$$

$$\chi_n(\tau;p) = \sum_{k=1}^{k(n)} O(\log^{2p} T_k) + O\big(\log^{2p} \bar{T}^n\big), \quad p > 2.$$

By (12), in regime (R2) we have $T_k \sim \beta B k^{\beta-1}$, $k \to \infty$, and $\bar{T}^n \sim (n/B)^{1/\beta}$, $n \to \infty$, while in regime (R3), we have $\log T_k \sim Ck$, $k \to \infty$, and $\bar{T}^n \sim (1/C)\log n$, $n \to \infty$. Thus we see that, in both regimes, (26) holds and $\chi_n(\tau)$ scales as in (19). Hence the claim follows from Lemma 2 below. □

A Toeplitz Lemma

In this appendix we prove Lemma 1.

Proof. Estimate

$$\left|\sum_{k\in\mathbb{N}}\gamma_{k,n}z_k+\gamma_n z_{\bar{T}^n}-z^*\right|\leq\sum_{k\in\mathbb{N}}\gamma_{k,n}\left|z_k-z^*\right|+\gamma_n\left|z_{\bar{T}^n}-z^*\right|. \tag{28}$$

Hence it suffices to shows that, for n large enough, the right-hand side is smaller than an arbitrary $\epsilon>0$. To this end, pick $\epsilon_1>0$ (which will be fixed at the end), and choose $N_0=N_0(\epsilon_1)$ such that $|z_k-z^*|<\epsilon_1$ for $k>N_0$. Note further that, in view of (22), we can choose $N_1=N_1(\epsilon_1)$ such that, for $n>N_1$,

$$\sum_{k=1}^{N_0}\gamma_{k,n}\left|z_k-z^*\right|<\epsilon_1, \tag{29}$$

and

$$\gamma_n=\frac{\bar{T}^n}{n}>\epsilon_1\quad\Longrightarrow\quad\bar{T}^n>N_0. \tag{30}$$

Write the right-hand side of (28) as

$$\sum_{k=1}^{N_0}\gamma_{k,n}\left|z_k-z^*\right|+\sum_{k>N_0}\gamma_{k,n}\left|z_k-z^*\right|+\gamma_n\left|z_{\bar{T}^n}-z^*\right|. \tag{31}$$

The first two terms in (31) are bounded from above by ϵ_1 for $n>\max\{N_0,N_1\}$. Indeed, for the first term this is due to (29), while for the second term it is true because $\sum_{k\in\mathbb{N}}\gamma_{k,n}\leq 1$ and $|z_k-z^*|<\epsilon_1$ for $k>N_0$. For the third term we note that either $\gamma_n=\bar{T}^n/n>\epsilon_1$, in which case (30) together with $\gamma_n\leq 1$ guarantees that $\gamma_n\left|z_{\bar{T}^n}-z^*\right|<\epsilon_1$, or $\gamma_n\leq\epsilon_1$, in which case $\gamma_n\left|z_{\bar{T}^n}-z^*\right|<K\epsilon_1$ with $K=\sup_{k\in\mathbb{N}}|z_k-z^*|<\infty$. We conclude that (31) is bounded from above by $(2+\max\{1,K\})\epsilon_1$. Now we let ϵ_1 be such that $(2+\max\{1,K\})\epsilon_1<\epsilon$, to get the claim. □

B Central Limit Theorem

Lemma 2 (Lindeberg and Lyapunov condition, Petrov [9, Theorem 22]**).** *Let $U=(U_k)_{k\in\mathbb{N}}$ be a sequence of independent random variables (at least one of which has a non-degenerate distribution). Let $m_k=\mathbb{E}(U_k)$ and $\sigma_k^2=\mathbb{V}\mathrm{ar}(U_k)$. Define*

$$\chi_n=\sum_{k=1}^{n}\sigma_k^2.$$

Then the Lindeberg condition

$$\lim_{n\to\infty}\frac{1}{\chi_n}\sum_{k=1}^{n}\mathbb{E}\left((U_k-m_k)^2\,1_{|U_k-m_k|\geq\epsilon\sqrt{\chi_n}}\right)=0$$

implies that

$$w - \lim_{n\to\infty} \frac{1}{\sqrt{\chi_n}} \sum_{k=1}^{n} (U_k - m_k) = \mathcal{N}(0,1).$$

Moreover, the Lindeberg condition is implied by the Lyapunov condition

$$\lim_{n\to\infty} \frac{1}{\chi_n^{p/2}} \sum_{k=1}^{n} \mathbb{E}\left(|U_k - m_k|^p\right) = 0 \quad \textit{for some } p > 2.$$

C RWRE: L^p Convergence Under Recurrence

The authors are grateful to Zhan Shi for suggesting the proof of Proposition 4 below.

We begin by observing that all the moments of the limiting random variable V in Theorem 2 are finite.

Lemma 3. *Let V be the random variable with density function (5). Let P denote its law. Then $E\left(V^p\right) < \infty$ for all $p > 0$ with $E\left(V^{2k}\right) = 0$ for $k \in \mathbb{N}$.*

Proof. For $k \in \mathbb{N}$, it follows from (5) that $E(V^{2k}) = 0$. For arbitrary $p > 0$, compute

$$E\left(|V|^p\right) = \frac{4}{\pi} \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{2k+1} \int_0^\infty x^p \exp\left(-\frac{(2k+1)^2\pi^2}{8} x\right) \mathrm{d}x. \tag{32}$$

Since $b^q \int_0^\infty x^{q-1}\,\mathrm{e}^{-bx}\,\mathrm{d}x = \Gamma(q)$, the integral in (32) equals

$$\frac{8^{p+1}\Gamma(p+1)}{(2k+1)^{2(p+1)}\pi^{2(p+1)}}.$$

Therefore

$$E\left(|V|^p\right) = \frac{4\Gamma(p+1)8^{p+1}}{\pi^{2p+3}} \sum_{k\in\mathbb{N}_0} \frac{(-1)^k}{(2k+1)^{2p+3}},$$

which is finite for all $p > 0$. □

Proposition 4. *The convergence in Proposition 2 holds in L^p for all $p > 0$.*

Proof. The proof comes in 3 Steps.

1. As shown by Sinai [11],

$$w - \lim_{n\to\infty} \frac{Z_n - b_n}{\log^2 n} = 0 \quad \text{under the law } \mathbb{P},$$

where b_n is the bottom of the valley of height $\log n$ containing the origin for the potential process $(U(x))_{x\in\mathbb{Z}}$ given by

$$U(x) = \begin{cases} \sum_{y=1}^{x} \log\rho(y), & x \in \mathbb{N}, \\ 0, & x = 0, \\ -\sum_{y=x}^{-1} \log\rho(y), & x \in -\mathbb{N}, \end{cases}$$

with $\rho(y) = (1-\bar{\omega}(y))/\bar{\omega}(y)$. This process depends on the environment ω only, and

$$w - \lim_{n\to\infty} \frac{b_n}{\log^2 n} = V \quad \text{under the law } \alpha^{\mathbb{Z}}.$$

We will prove the claim by showing that, for all $p > 0$,

$$\sup_{n\geq 3} \mathbb{E}_{\alpha^{\mathbb{Z}}}\Big(\Big|\frac{b_n}{\log^2 n}\Big|^p\Big) < \infty, \qquad \sup_{n\geq 3} \mathbb{E}\Big(\Big|\frac{Z_n}{\log^2 n}\Big|^p\Big) < \infty. \tag{33}$$

To simplify the proof we may assume that there is a *reflecting barrier at the origin*, in which case b_n and Z_n take values in $\mathbb{N}_0$. This restriction is harmless because without reflecting barrier we can estimate $|b_n| \leq \max\{b_n^+, -b_n^-\}$ and $|Z_n| \leq \max\{Z_n^+, -Z_n^-\}$ in distribution for two independent copies of b_n and Z_n with reflecting barrier to the right, respectively, to the left.

2. To prove the first half of (33) with reflecting barrier, define

$$H(r) = \inf\{x \in \mathbb{N}_0\colon\ |U(x)| \geq r\}, \qquad r \geq 0.$$

Then

$$b_n \leq H(\log n). \tag{34}$$

We have

$$\mathbb{E}_{\alpha^{\mathbb{Z}}}\Big(\Big|\frac{H(\log n)}{\log^2 n}\Big|^p\Big) = \int_0^\infty p\lambda^{p-1}\, \mathbb{P}\big(H(\log n) > \lambda \log^2 n\big)\, \mathrm{d}\lambda.$$

Since $\int_0^1 p\lambda^{p-1}\mathrm{d}\lambda = 1$, we need only care about $\lambda \geq 1$. To that end, note that

$$\big\{H(\log n) > \lambda \log^2 n\big\} = \Big\{\max_{0\leq x\leq \lambda \log^2 n} |U(x)| < \log n\Big\}$$

and

$$\alpha^{\mathbb{Z}}\left(\max_{0\leq x\leq \lambda\log^2 n} |U(x)| < \log n\right) \leq \hat{P}\left(\max_{0\leq t\leq \lambda N} \sigma|W(t)| < \sqrt{N}\right),\ N = \log^2 n,$$

where σ^2 is the variance of $\rho(0)$ and $(W(t))_{t\geq 0}$ is standard Brownian motion on $\mathbb{R}$ with law $\hat{P}$. But there exists a $c > 0$ (depending on σ) such that

$$\hat{P}\Big(\max_{0\leq t\leq \lambda N} \sigma|W(t)| < \sqrt{N}\Big) = \hat{P}\Big(\max_{0\leq t\leq \lambda} \sigma|W(t)| < 1\Big) \leq \mathrm{e}^{-c\lambda},\ \lambda \geq 1. \tag{35}$$

Combining (34)–(35), we get the first half of (33).

3. To prove the second half of (33), write

$$\mathbb{E}\Big(\Big|\frac{Z_n}{\log^2 n}\Big|^p\Big) = \int_0^\infty p\lambda^{p-1}\, \mathbb{P}\big(Z_n > \lambda \log^2 n\big)\, \mathrm{d}\lambda.$$

Again we need only care about $\lambda \geq 1$. As in Step 2, we have

$$\sup_{n\geq 3} \int_1^\infty p\lambda^{p-1}\, \alpha^{\mathbb{Z}}\big(H(\lambda^{1/3}\log n) > \lambda \log^2 n\big)\, \mathrm{d}\lambda < \infty.$$

It therefore remains to check that

$$\sup_{n\geq 3}\int_1^\infty p\lambda^{p-1}\,\mathbb{P}(\mathcal{E}_{\lambda,n})\,\mathrm{d}\lambda<\infty \tag{36}$$

with

$$\mathcal{E}_{\lambda,n}=\Big\{Z_n>\lambda\log^2 n,\; H(\lambda^{1/3}\log n)\leq\lambda\log^2 n\Big\}.$$

To that end, for $x\in\mathbb{N}_0$, let $T(x)=\inf\{n\in\mathbb{N}_0\colon\; Z_n=x\}$. On the event $\mathcal{E}_{\lambda,n}$ we have $T(H(\lambda^{1/3}\log n))\leq n$. Therefore, by Golosov [5, Lemma 7],

$$\mathbb{P}(T(x)\leq n\mid\omega)\leq n\,\exp\Big(-\max_{0\leq y<x}[U(x-1)-U(y)]\Big),\; x\in\mathbb{N},\, n\in\mathbb{N},$$

which is bounded from above by $n\,\mathrm{e}^{-U(x-1)}$. Picking $x=H(\lambda^{1/3}\log n)$, we obtain

$$\mathbb{P}(\mathcal{E}_{\lambda,n}\mid\omega)\leq n\,\mathrm{e}^{-U(H(\lambda^{1/3}\log n)-1)},$$

which is approximately $n\,\mathrm{e}^{-\lambda^{1/3}\log n}$ because $U(H(x))$ is approximately x. (The undershoot at x can be neglected because it has finite first moment, by our assumption that $\sigma<\infty$.) Taking the expectation over ω, we get

$$\mathbb{P}(\mathcal{E}_{\lambda,n})\leq n^{-(\lambda^{1/3}-1)}.$$

This implies (36), and hence we have proved the second half of (33). □

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Constructive Euler Hydrodynamics for One-Dimensional Attractive Particle Systems

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To Chuck Newman, Friend, Colleague, and Mentor

Abstract. We review a (constructive) approach first introduced in [6] and further developed in [7–9,38] for hydrodynamic limits of asymmetric attractive particle systems, in a weak or in a strong (that is, almost sure) sense, in an homogeneous or in a quenched disordered setting.

Keywords: Hydrodynamics · Attractive particle system · Nonexplicit invariant measures · Nonconvex or nonconcave flux · Entropy solution · Glimm scheme

1 Introduction

Among the most studied conservative interacting particle systems (IPS) are the simple exclusion and the zero-range processes. They are *attractive processes*, and possess a one-parameter family of product extremal invariant and translation invariant probability measures, that we denote by $\{\nu_\alpha\}_\alpha$, where α represents the mean density of particles per site: for simple exclusion $\alpha \in [0, 1]$, and for zero-range $\alpha \in [0, +\infty)$ (see [36, Chapter VIII], and [1]). Both belong to a more general class of systems with similar properties, called *misanthropes* processes [19].

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Hydrodynamic limit [21,33,45] is a law of large numbers for the time evolution (usually described by a limiting PDE, called the hydrodynamic equation) of empirical density fields in interacting particle systems. Most usual IPS can be divided into two groups, diffusive and hyperbolic. In the first group, which contains for instance the symmetric or mean-zero asymmetric simple exclusion process, the macroscopic→microscopic space-time scaling is $(x, t) \mapsto (Nx, N^2 t)$ with $N \to \infty$, and the limiting PDE is a diffusive equation. In the second group, which contains for instance the nonzero mean asymmetric simple exclusion process, the scaling is $(x, t) \mapsto (Nx, Nt)$, and the limiting PDE is of Euler type. In both groups the PDE often exhibits nonlinearity, either via the diffusion coefficient in the first group, or via the flux function in the second one. This raises special difficulties in the hyperbolic case, due to shocks and non-uniqueness for the solution of the PDE, in which case the natural problem is to establish convergence to the so-called *entropy solution* [44].

In most known results, only a weak law of large numbers is established. In this description one need not have an explicit construction of the dynamics: the limit is shown in probability with respect to the law of the process, which is characterized in an abstract way by its Markov generator and Hille–Yosida's theorem [36]. Nevertheless, when simulating particle systems, one naturally uses a pathwise construction of the process on a Poisson space-time random graph (the so-called *graphical construction*). In this description the dynamics is deterministically driven by a random space-time measure which tells when and where the configuration has a chance of being modified. It is of special interest to show that the hydrodynamic limit holds for almost every realization of the space-time measure, as this means a single simulation is enough to approximate solutions of the limiting PDE.

We are interested here in the hydrodynamic behavior of a class of asymmetric particle systems of $\mathbb{Z}$, which arise as a natural generalization of the asymmetric exclusion process. For such processes, hydrodynamic limit is given by the entropy solutions to a scalar conservation law of the form

$$\partial_t u(x, t) + \partial_x G(u(x, t)) = 0 \tag{1}$$

where $u(., .)$ is the density field and G is the *macroscopic flux*. The latter is given for the asymmetric exclusion process by $G(u) = \gamma u(1 - u)$, where γ is the mean drift of a particle. Because there is a single conserved quantity (*i.e.* mass) for the particle system, and an ergodic equilibrium measure for each density value, (1) can be guessed through heuristic arguments if one takes for granted that the system is in *local equilibrium*. The macroscopic flux G is obtained by an equilibrium expectation of a microscopic flux which can be written down explicitly from the dynamics. A rigorous proof of the hydrodynamic limit turns out to be a difficult problem, mainly because of the non-existence of strong solutions for (1) and the non-uniqueness of weak solutions. Since the conservation law is not sufficient to pick a single solution, the so-called entropy weak solution must be characterized by additional properties; one must then look for related properties of the particle system to establish its convergence to the entropy solution.

The derivation of hyperbolic equations of the form (1) as hydrodynamic limits began with the seminal paper [41], which established a strong law of large numbers for the totally asymmetric simple exclusion process on $\mathbb{Z}$, starting with 1's to the left of the origin and 0's to the right. This result was extended by [15] and [2] to nonzero mean exclusion process starting from product Bernoulli distributions with arbitrary densities λ to the left and ρ to the right (the so-called *Riemann initial condition*). The Bernoulli distribution at time 0 is related to the fact that uniform Bernoulli measures are invariant for the process. For the one-dimensional totally asymmetric (nearest-neighbor) K-exclusion process, a particular misanthropes process without explicit invariant measures, a strong hydrodynamic limit was established in [43], starting from arbitrary initial profiles, by means of the so-called *variational coupling*, that is a microscopic version of the Lax–Hopf formula. These were the only strong laws available before the series of works reviewed here. A common feature of these works is the use of subadditive ergodic theorem to exhibit some a.s. limit, which is then identified by additional arguments.

On the other hand, many *weak* laws of large numbers were established for attractive particle systems. A first series of results treated systems with product invariant measures and product initial distributions. In [3], for a particular zero-range model, a weak law was deduced from conservation of local equilibrium under Riemann initial condition. It was then extended in [4] to the misanthropes process of [19] under an additional convexity assumption on the flux function. These were substantially generalized (using Kružkov's entropy inequalities, see [34]) in [39] to multidimensional attractive systems with product invariant measures for arbitrary Cauchy data, without any convexity requirement on the flux. In [40], using an abstract characterization of the evolution semigroup associated with the limiting equation, hydrodynamic limit was established for the one-dimensional nearest-neighbor K-exclusion process.

The above results are concerned with translation-invariant particle dynamics. We are also interested in hydrodynamic limits of particle systems in random environment, leading to homogenization effects, where an effective diffusion matrix or flux function is expected to capture the effect of inhomogeneity. Hydrodynamic limit in random environment has been widely addressed and robust methods have been developed in the diffusive case. In the hyperbolic setting, the few available results in random environment (prior to [9]) depended on particular features of the investigated models. In [16], the authors prove, for the asymmetric zero-range process with site disorder on $\mathbb{Z}^d$, a quenched hydrodynamic limit given by a hyperbolic conservation law with an effective homogenized flux function. To this end, they use in particular the existence of explicit product invariant measures for the disordered zero-range process below some critical value of the density value. In [42], extension to the supercritical case is carried out in the totally asymmetric case with constant jump rate. In [43], the author establishes a quenched hydrodynamic limit for the totally asymmetric nearest-neighbor K-exclusion process on $\mathbb{Z}$ with i.i.d site disorder, for which explicit invariant measures are not known. The last two results rely on variational coupling. However, the simple exclusion process beyond the totally asymmetric nearest-neighbor case, or more complex models with state-dependent jump rates, remain outside the scope of this approach.

In this paper, we review successive stages [6–9,38] of a *constructive* approach to hydrodynamic limits given by equations of the type (1), which ultimately led us in [9] to a very general hydrodynamic limit result for attractive particle systems in one dimension in ergodic random environment. We shall detail our method in the setting of [9]. However, we will first explain our approach and advances along the progression of papers, and quote results for each one, since they are interesting in their own. We hope this could be helpful for a reader looking for hydrodynamics of a specific model: according to the available knowledge on this model, this reader could derive either a weak, or a strong (without disorder or with a quenched disorder) hydrodynamic limit.

Our motivation for [6] was to prove with a constructive method hydrodynamics of one-dimensional attractive dynamics with product invariant measures, but without a concave/convex flux, in view of examples of k-step exclusion processes and misanthropes processes. We initiated for that a "resurrection" of the approach of [4], and we introduced a variational formula for the entropy solution of the hydrodynamic equation in the Riemann case, and an approximation scheme to go from a Riemann to a general initial profile. Our method is based on an interplay of macroscopic properties for the conservation law and analogous microscopic properties for the particle system. The next stage, achieved in [7], was to derive hydrodynamics (which was still a weak law) for attractive processes without explicit invariant measures. In the same setting, we then obtained almost sure hydrodynamics in [8], relying on a graphical representation of the dynamics. The latter result, apart from its own interest, proved to be an essential step to obtain quenched hydrodynamics in the disordered case [9]. For this last paper, we also relied on [38], which improves an essential property we use, macroscopic stability.

Let us mention that we also worked [12,13] on the hydrodynamic behavior of the disordered asymmetric zero-range process. This model falls outside the scope of the present paper because it exhibits a phase transition with a critical density above which no invariant measure exists. In the supercritical regime, the hydrodynamic limit cannot be established by local equilibrium arguments, and condensation may occur locally on a finer scale than the hydrodynamic one. Other issues related to this model have been studied recently in [10,11].

This review paper is organized as follows. In Sect. 2, after giving general notation and definitions, we introduce the two basic models we originally worked with, the misanthropes process and the k-step exclusion process. Then we describe informally the results, and the main ideas involved in [4] which was our starting point, and in each of the papers [6–9,38]. Section 3 contains our main results, stated (for convenience) for the misanthropes process. We then aim at explaining how these results are proved. In Sect. 4, we first give a self-contained introduction to scalar conservation laws, with the main definitions and results important for our purposes; then we explain the derivation of our variational formula in the Riemann case (illustrated by an example of 2-step exclusion process), and finally our approximation scheme to solve the general Cauchy problem. In Sect. 5, we outline the most important steps of our proof of hydrodynamic limit in a quenched disordered setting: again, we first deal with the Riemann problem, then with the Cauchy problem. Finally, in Sect. 6, we define a general framework which enables to describe a class of models possessing the necessary properties to derive hydrodynamics, and study a few examples.

2 Notation and Preliminaries

Throughout this paper $\mathbb{N} = \{1, 2, ...\}$ will denote the set of natural numbers, $\mathbb{Z}^+ = \{0, 1, 2, ...\}$ the set of non-negative integers, and $\mathbb{R}^{+*} = \mathbb{R}^+ \setminus \{0\}$ the set of positive real numbers. The integer part $\lfloor x \rfloor \in \mathbb{Z}$ of $x \in \mathbb{R}$ is uniquely defined by $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

The set of environments (or disorder) is a probability space $(\mathbf{A}, \mathcal{F}_\mathbf{A}, Q)$, where $\mathbf{A}$ is a compact metric space and $\mathcal{F}_\mathbf{A}$ its Borel σ-field. On $\mathbf{A}$ we have a group of space shifts $(\tau_x : x \in \mathbb{Z})$, with respect to which Q is ergodic.

We consider particle configurations (denoted by greek letters $\eta, \xi \ldots$) on $\mathbb{Z}$ with at most K (but always finitely many) particles per site, for some given $K \in \mathbb{N} \cup \{+\infty\}$. Thus the state space, which will be denoted by $\mathbf{X}$, is either $\mathbb{N}^\mathbb{Z}$ in the case $K = +\infty$, or $\{0, \cdots, K\}^\mathbb{Z}$ for $K \in \mathbb{N}$. For $x \in \mathbb{Z}$ and $\eta \in \mathbf{X}$, $\eta(x)$ denotes the number of particles on site x. This state space is endowed with the product topology, which makes it a metrisable space, compact when $\mathbf{X} = \{0, \cdots, K\}^\mathbb{Z}$.

A function f defined on $\mathbf{A} \times \mathbf{X}$ (resp. g on $\mathbf{A} \times \mathbf{X}^2$, h on $\mathbf{X}$) is called *local* if there is a finite subset Λ of $\mathbb{Z}$ such that $f(\alpha, \eta)$ depends only on α and $(\eta(x), x \in \Lambda)$ (resp. $g(\alpha, \eta, \xi)$ depends only on α and $(\eta(x), \xi(x), x \in \Lambda)$, $h(\eta)$ depends only on $(\eta(x), x \in \Lambda)$). We denote again by τ_x either the spatial translation operator on the real line for $x \in \mathbb{R}$, defined by $\tau_x y = x + y$, or its restriction to $x \in \mathbb{Z}$. By extension, if f is a function defined on $\mathbb{Z}$ (resp. $\mathbb{R}$), we set $\tau_x f = f \circ \tau_x$ for $x \in \mathbb{Z}$ (resp. $\mathbb{R}$). In the sequel this will be applied to different types of functions: particle configurations $\eta \in \mathbf{X}$, disorder configurations $\alpha \in \mathbf{A}$, or joint disorder-particle configurations $(\alpha, \eta) \in \mathbf{A} \times \mathbf{X}$. In the latter case, unless mentioned explicitly, τ_x applies simultaneously to both components.

If τ_x acts on some set and μ is a measure on this set, $\tau_x \mu = \mu \circ \tau_x^{-1}$. We let $\mathcal{M}^+(\mathbb{R})$ denote the set of nonnegative measures on $\mathbb{R}$ equipped with the metrizable topology of vague convergence, defined by convergence on continuous test functions with compact support. The set of probability measures on $\mathbf{X}$ is denoted by $\mathcal{P}(\mathbf{X})$. If η is an $\mathbf{X}$-valued random variable and $\nu \in \mathcal{P}(\mathbf{X})$, we write $\eta \sim \nu$ to specify that η has distribution ν. Similarly, for $\alpha \in \mathbf{A}, Q \in \mathcal{P}(\mathbf{A})$, $\alpha \sim Q$ means that α has distribution Q.

2.1 Preliminary Definitions

Let us introduce briefly the various notions we shall use in this review, in view of the next section, where we informally tell the content of each of our papers. We shall be more precise in the following sections. Reference books are [33, 36].

The Process. We work with a conservative (i.e. involving only particle jumps but no creation/annihilation), attractive (see (2) for its definition below) Feller process $(\eta_t)_{t\geq 0}$ with state space $\mathbf{X}$. When this process evolves in a random environment $\alpha \in \mathbf{A}$, we denote its generator by L_α and its semigroup by $(S_\alpha(t), t \geq 0)$. Otherwise we denote them by L and $S(t)$. In the absence of disorder, we denote by $\mathcal{S}$ the set of translation invariant probability measures on $\mathbf{X}$, by $\mathcal{I}$ the set of invariant probability measures for the process $(\eta_t)_{t\geq 0}$, and by $(\mathcal{I} \cap \mathcal{S})_e$ the set of extremal invariant and translation invariant probability measures for $(\eta_t)_{t\geq 0}$. In the disordered case, $\mathcal{S}$ will denote the set of translation invariant probability measures on $\mathbf{A} \times \mathbf{X}$, see Proposition 2.

A sequence $(\nu_n, n \in \mathbb{N})$ of probability measures on $\mathbf{X}$ converges weakly to some $\nu \in \mathcal{P}(\mathbf{X})$, if and only if $\lim_{n\to\infty} \int f\, d\nu_n = \int f\, d\nu$ for every continuous function f on $\mathbf{X}$. The topology of weak convergence is metrizable and makes $\mathcal{P}(\mathbf{X})$ compact when $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$.

We equip $\mathbf{X}$ with the *coordinatewise order*, defined for $\eta, \xi \in \mathbf{X}$ by $\eta \leq \xi$ if and only if $\eta(x) \leq \xi(x)$ for all $x \in \mathbb{Z}$. A partial stochastic order is defined on $\mathcal{P}(\mathbf{X})$; namely, for $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$, we write $\mu_1 \leq \mu_2$ if the following equivalent conditions hold (see *e.g.* [32,36,46]):

(i) For every non-decreasing nonnegative function f on $\mathbf{X}$, $\int f\, d\mu_1 \leq \int f\, d\mu_2$.
(ii) There exists a coupling measure $\overline{\mu}$ on $\mathbf{X} \times \mathbf{X}$ with marginals μ_1 and μ_2, such that $\overline{\mu}\{(\eta, \xi) : \eta \leq \xi\} = 1$.

The process $(\eta_t)_{t\geq 0}$ is *attractive* if its semigroup acts monotonically on probability measures, that is: for any $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$,

$$\mu_1 \leq \mu_2 \Rightarrow \forall t \in \mathbb{R}^+,\ \mu_1 S_\alpha(t) \leq \mu_2 S_\alpha(t) \tag{2}$$

Hydrodynamic Limits. Let $N \in \mathbb{N}$ be the *scaling parameter* for the hydrodynamic limit, that is, the inverse of the macroscopic distance between two consecutive sites. The empirical measure of a configuration η viewed on scale N is given by

$$\pi^N(\eta)(dx) = N^{-1} \sum_{y\in\mathbb{Z}} \eta(y)\delta_{y/N}(dx) \in \mathcal{M}^+(\mathbb{R})$$

where, for $x \in \mathbb{R}$, δ_x denotes the Dirac measure at x, and $\mathcal{M}^+(\mathbb{R})$ denotes the space of Radon measures on $\mathbb{R}$. This space will be endowed with the metrizable topology of vague convergence, defined by convergence against the set $C_K^0(\mathbb{R})$ of continuous test functions on $\mathbb{R}$ with compact support. Let d_v be a distance associated with this topology, and $\pi_.$, $\pi'_.$ be two mappings from $[0, +\infty)$ to $\mathcal{M}^+(\mathbb{R})$. We set

$$D_T(\pi_., \pi'_.) := \operatorname{ess}\sup_{t\in[0,T]} d_v(\pi_t, \pi'_t)$$

$$D(\pi_., \pi'_.) := \sum_{n=0}^{+\infty} 2^{-n} \min[1, D_n(\pi_., \pi'_.)]$$

A sequence $(\pi^n_.)_{n\in\mathbb{N}}$ of random $\mathcal{M}^+(\mathbb{R})$-valued paths is said to converge locally uniformly in probability to a random $\mathcal{M}^+(\mathbb{R})$-valued path $\pi_.$ if, for every $\varepsilon > 0$,

$$\lim_{n\to+\infty} \mu^n \left(D(\pi^n_., \pi_.) > \varepsilon \right) = 0$$

where μ^n denotes the law of $\pi^n_.$.

Let us now recall, in the context of scalar conservation laws, standard definitions in hydrodynamic limit theory. Recall that $K \in \mathbb{Z}^+ \cup \{+\infty\}$ bounds the number of particles per site. Macroscopically, the set of possible particle densities will be $[0, K] \cap \mathbb{R}$. Let $G : [0, K] \cap \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function, called the *flux*. It is a.e. differentiable, and its derivative G' is an (essentially) uniformly bounded function. We consider the scalar conservation law

$$\partial_t u + \partial_x [G(u)] = 0 \tag{3}$$

where $u = u(x, t)$ is some $[0, K] \cap \mathbb{R}$-valued density field defined on $\mathbb{R} \times \mathbb{R}^+$. We denote by $L^{\infty,K}(\mathbb{R})$ the set of bounded Borel functions from $\mathbb{R}$ to $[0, K] \cap \mathbb{R}$.

Definition 1. *Let $(\eta^N)_{N\geq 0}$ be a sequence of* $\mathbf{X}$*-valued random variables, and $u_0 \in L^{\infty,K}(\mathbb{R})$. We say that the sequence $(\eta^N)_{N\geq 0}$ has:*

(i) weak density profile $u_0(.)$, *if $\pi^N(\eta^N) \to u_0(.)$ in probability with respect to the topology of vague convergence, that is equivalent to: for all $\varepsilon > 0$ and test function $\psi \in C^0_K(\mathbb{R})$,*

$$\lim_{N\to\infty} \mu^N \left(\left| \int_{\mathbb{R}} \psi(x)\pi^N(\eta^N)(dx) - \int_{\mathbb{R}} \psi(x)u_0(x)dx \right| > \varepsilon \right) = 0$$

where μ^N denotes the law of η^N.

(ii) strong density profile $u_0(.)$, *if the random variables are defined on a common probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, and $\pi^N(\eta^N) \to u_0(.)$ $\mathbb{P}_0$-almost surely with respect to the topology of vague convergence, that is equivalent to: for all test function $\psi \in C^0_K(\mathbb{R})$,*

$$\mathbb{P}_0 \left(\lim_{N\to\infty} \int_{\mathbb{R}} \psi(x)\pi^N(\eta^N)(dx) = \int_{\mathbb{R}} \psi(x)u_0(x)dx \right) = 1,$$

We consider hydrodynamic limits under hyperbolic time scaling, that is Nt, since we work with asymmetric dynamics.

Definition 2. *The sequence $(\eta^N_t, t \geq 0)_{N\geq 0}$ has* hydrodynamic limit *(resp.* a.s. hydrodynamic limit*) $u(.,.)$ if: for all $t \geq 0$, $(\eta^N_{Nt})_N$ has weak (resp. strong) density profile $u(.,t)$ where $u(.,t)$ is the* weak entropy solution *of* (3) *with initial condition $u_0(.)$, for an appropriately defined* macroscopic flux function G, *where u_0 is the density profile of the sequence $(\eta^N_0)_N$ in the sense of Definition 1.*

2.2 Our Motivations and Approach

Most results on hydrodynamics deal with dynamics with product invariant measures; in the most familiar cases, the flux function appearing in the hydrodynamic equation is convex/concave [33]. But for many usual examples, the first or the second statement is not true.

Reference Examples. We present the attractive misanthropes process on one hand, and the k-step exclusion process on the other hand: these two classical examples will illustrate our purposes along this review. In these basic examples, we take $\alpha \in \mathbf{A} = [c, 1/c]^{\mathbb{Z}}$ (for a constant $0 < c < 1$) as the space of environments; this corresponds to site disorder. We also consider those models without disorder, which corresponds to $\alpha(x) \equiv 1$. However, our approach applies to a much broader class of models and environments, as will be explained in Sect. 6.

The misanthropes process was introduced in [19] (without disorder). It has state space either $\mathbf{X} = \mathbb{N}^{\mathbb{Z}}$ or $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$ ($K \in \mathbb{N}$), and $b : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ is the jump rate function. A particle present on $x \in \mathbb{Z}$ chooses $y \in \mathbb{Z}$ with probability $p(y-x)$, where $p(.)$ (the particles' jump kernel) is an asymmetric probability measure on $\mathbb{Z}$, and jumps to y at rate $\alpha(x)b(\eta(x), \eta(y))$. We assume the following:

(M1) $b(0,.) = 0$, with a K-exclusion rule when $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$: $b(., K) = 0$;
(M2) Attractiveness: b is nondecreasing (nonincreasing) in its first (second) argument.
(M3) b is a bounded function.
(M4) p has a finite first moment, that is, $\sum_{z\in\mathbb{Z}} |z|\, p(z) < +\infty$.

The quenched disordered process has generator

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x)p(y-x)b(\eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta)\right] \tag{4}$$

where $\eta^{x,y}$ denotes the new state after a particle has jumped from x to y (that is $\eta^{x,y}(x) = \eta(x) - 1$, $\eta^{x,y}(y) = \eta(y) + 1$, $\eta^{x,y}(z) = \eta(z)$ otherwise).
There are two well-known particular cases of attractive misanthropes processes: the *simple exclusion process* [36] corresponds to

$$\mathbf{X} = \{0,1\}^{\mathbb{Z}} \quad \text{with} \quad b(\eta(x), \eta(y)) = \eta(x)(1 - \eta(y));$$

the *zero-range process* [1] corresponds to

$$\mathbf{X} = \mathbb{N}^{\mathbb{Z}} \quad \text{with} \quad b(\eta(x), \eta(y)) = g(\eta(x)),$$

for a non-decreasing function $g : \mathbb{Z}^+ \to \mathbb{R}^+$ (in [1] it is not necessarily bounded).

Let us now restrict ourselves to the model without disorder. For the simple exclusion and zero-range processes, $(\mathcal{I} \cap \mathcal{S})_e$ is a one-parameter family of product probability measures. The flux function is convex/concave for simple exclusion, but not necessarily for zero-range. However, in the general set-up of misanthropes processes, unless the rate function b satisfies additional algebraic conditions (see [19,23]), the model does not have product invariant measures; even when this is the case, the flux function is not necessarily convex/concave. We refer the reader to [6,23] for examples of misanthropes processes with product invariant measures. Note also that a misanthropes process with product invariant measures generally loses this property if disorder is introduced, with the sole known exception of the zero-range process [16,22].

The *k-step exclusion process* ($k \in \mathbb{N}$) was introduced in [29] (without disorder). Its state space is $\mathbf{X} := \{0,1\}^{\mathbb{Z}}$. Let $p(.)$ be a probability distribution on $\mathbb{Z}$, and $\{X_n\}_{n\in\mathbb{N}}$ denote a random walk on $\mathbb{Z}$ with jump distribution $p(.)$. We denote by $\mathbf{P}^x$ the law of the random walk starting from x; expectation with respect to this law is denoted by $\mathbf{E}^x$. The k-step exclusion process with jump distribution $p(.)$ has generator

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x)c(x,y,\eta)\left[f\left(\eta^{x,y}\right) - f(\eta)\right] \qquad \text{with} \tag{5}$$

$$c(x,y,\eta) = \eta(x)(1-\eta(y))\mathbf{E}^x\left[\prod_{i=1}^{\sigma_y - 1}\eta(X_i), \sigma_y \le \sigma_x, \sigma_y \le k\right]$$

where $\sigma_y = \inf\{n \ge 1 : X_n = y\}$ is the first (non zero) arrival time to site y of the walk starting at site x. In words if a particle at site x wants to jump it may go to the first empty site encountered before returning to site x following the walk $\{X_n\}_{n\in\mathbb{N}}$ (starting at x) provided it takes less than k attempts; otherwise the movement is cancelled. When $k = 1$, we recover the simple exclusion process. The k-step exclusion is an attractive process.

Let us now restrict ourselves to the model without disorder. Then $(\mathcal{I}\cap\mathcal{S})_e$ is a one-parameter family of product Bernoulli measures. In the totally asymmetric nearest-neighbor case, $c(x,y,\eta) = 1$ if $\eta(x) = 1$, $y - x \in \{1,\ldots,k\}$ and y is the first nonoccupied site to the right of x; otherwise $c(x,y,\eta) = 0$. The flux function belongs to $\mathcal{C}^2(\mathbb{R})$, it has one inflexion point, thus it is neither convex nor concave. Besides, flux functions with arbitrarily many inflexion points can be constructed by superposition of different k-step exclusion processes with different kernels and different values of k [6].

A Constructive Approach to Hydrodynamics. To overcome the difficulties to derive hydrodynamics raised by the above examples, our starting point was the constructive approach introduced in [4]. There, the authors proved the *conservation of local equilibrium* for the one-dimensional zero-range process with a concave macroscopic flux function G *in the Riemann case* (in a translation invariant setting, G is the mean flux of particles through the origin), that is

$$\forall t > 0, \quad \eta^N_{Nt} \xrightarrow{\mathcal{L}} \nu_{u(t,x)} \tag{6}$$

where ν_ρ is the product invariant measure of the zero-range process with mean density ρ, and $u(.,.)$ is the *entropy solution* of the conservation law

$$\partial_t u + \partial_x[G(u)] = 0; \quad u(x,0) = R_{\lambda,\rho}(x) = \lambda\mathbf{1}_{\{x<0\}} + \rho\mathbf{1}_{\{x\ge 0\}}$$

One can show (see [33]) that (6) implies the hydrodynamic limit in the sense of Definition 2. Let us begin by explaining (informally) their method. They first show in [4, Lemma 3.1] that a weak Cesaro limit of (the measure of) the process is an invariant and translation invariant measure, thus a convex combination of elements of $(\mathcal{I}\cap\mathcal{S})_e$, the one-parameter family of extremal invariant and translation invariant probability measures for the dynamics. Then they compute in [4, Lemma 3.2] the (Cesaro) limiting density inside a macroscopic box, thanks to the explicit knowledge of the product measures elements of $(\mathcal{I}\cap\mathcal{S})_e$. They prove next in [4, Lemma 3.3 and Theorem 2.10] that the above convex combination is in fact the Dirac measure concentrated on the solution of the hydrodynamic equation, thanks to the concavity of their flux function. They conclude by proving that the Cesaro limit implies the weak limit via monotonicity arguments, in [4, Propositions 3.4 and 3.5]. Their proof is valid for misanthropes processes with product invariant measures and a concave macroscopic flux.

In [6], we derive by a constructive method the hydrodynamic behavior of attractive processes with finite range irreducible jumps, and for which the set $(\mathcal{I}\cap\mathcal{S})_e$ consists in a one-parameter family of explicit product measures but the flux is not necessarily convex or concave. Our approach relies on (i) an explicit construction of Riemann solutions without assuming convexity of the macroscopic flux, and (ii) a general result which proves that the hydrodynamic limit for Riemann initial profiles implies the same for general initial profiles.

For point (i), we rely on the (parts of) the proofs in [4] based only on attractiveness and on the knowledge of the product measures composing $(\mathcal{I}\cap\mathcal{S})_e$, and we provide a new approach otherwise. Instead of the convexity assumption on the flux, which belongs here to $\mathcal{C}^2(\mathbb{R})$, we prove that the solution of the hydrodynamic equation is given by a variational formula, whose index set is an interval, namely the set of values of the parameter of the elements of $(\mathcal{I}\cap\mathcal{S})_e$. Knowing $(\mathcal{I}\cap\mathcal{S})_e$ explicitly enables us to deal with dynamics with the non compact state space $\mathbb{N}^{\mathbb{Z}}$.

Point (ii) is based on an approximation scheme inspired by Glimm's scheme for hyperbolic systems of conservation laws (see [44]). Among our tools are the *finite propagation property* and the *macroscopic stability* of the dynamics. The latter property is due to [17]; both require finite range transitions.

We illustrate our results on variations of our above reference examples.

While the results and examples of [6] include the case $K = +\infty$, in our subsequent works, for reasons explained below, we considered $K < +\infty$, thus $\mathbf{X} = \{0, \cdots, K\}^{\mathbb{Z}}$, which will be assumed from now on. Under this additional assumption, in [7], we extend the hydrodynamics result of [6] to dynamics without explicit invariant measures. Indeed, thanks to monotonicity, we prove that $(\mathcal{I}\cap\mathcal{S})_e$ is still a one-parameter family of probability measures, for which the set $\mathcal{R}$ of values of the parameter is a priori not an interval anymore, but a closed subset of $[0, K]$:

Proposition 1 ([7, Proposition 3.1]). *Assume p satisfies the irreducibility assumption and $\alpha := 1$*

$$\forall z \in \mathbb{Z}, \quad \sum_{n=1}^{+\infty} [p^{*n}(z) + p^{*n}(-z)] > 0 \tag{7}$$

where p^{*n} denotes the n-th convolution power of the kernel p, that is the law of the sum of n independent p-distributed random variables. Then there exists a closed subset $\mathcal{R}$ of $[0, K]$, containing 0 and K, such that

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^\rho : \rho \in \mathcal{R}\}$$

where the probability measures ν^ρ on $\mathbf{X}$ satisfy the following properties:

$$\lim_{l \to +\infty} (2l+1)^{-1} \sum_{x=-l}^{l} \eta(x) = \rho, \quad \nu^\rho\text{- a.s.}$$

and

$$\rho \leq \rho' \Rightarrow \nu^\rho \leq \nu^{\rho'}$$

Following the same general scheme as in [6], but with additional difficulties, we then obtain the following main result.

Theorem 1 ([7, Theorem 2.2]). *Assume $p(.)$ satisfies the irreducibility assumption (7) $\alpha := 1$. Then there exists a Lipschitz-continuous function $G : [0, K] \to \mathbb{R}^+$ such that the following holds. Let $u_0 \in L^{\infty,K}(\mathbb{R})$, and $(\eta^N_.)_N$ be any sequence of processes with generator (4), such that the sequence $(\eta^N_0)_N$ has density profile $u_0(.)$. Then, the sequence $(\eta^N_{N.})_N$ has hydrodynamic limit given by $u(.,.)$, the entropy solution to (3) with initial condition $u_0(.)$.*

The drawback is that we have (and it will also be the case in the following papers) to restrict ourselves to dynamics with compact state space to prove hydrodynamics with general initial data. This is necessary to define the macroscopic flux outside $\mathcal{R}$, by a linear interpolation; this makes this flux Lipschitz continuous, a minimal requirement to define entropy solutions. We have to consider a $\mathcal{R}$-valued Riemann problem, for which we prove conservation of local equilibrium. Then we use an averaging argument to prove hydrodynamics (in the absence of product invariant measures, the passage from local equilibrium to hydrodynamics is no longer a consequence of [33]). For general initial profiles, we have to refine the approximation procedure of [6]: we go first to $\mathcal{R}$-valued entropy solutions, then to arbitrary entropy solutions.

In [8], by a refinement of our method, we obtain a *strong* (that is, an almost sure) hydrodynamic limit, when starting from an arbitrary initial profile. By almost sure, we mean that we construct the process with generator (4) on an explicit probability space defined as the product $(\Omega_0 \times \Omega, \mathcal{F}_0 \otimes \mathcal{F}, \mathbb{P}_0 \otimes \mathbb{P})$, where $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ is a probability space used to construct random initial states, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a Poisson space used to construct the evolution from a given state.

Theorem 2 ([8, Theorem 2.1]). *Assume $p(.)$ has finite first moment and satisfies the irreducibility assumption (7). Then the following holds, where G is the same function as in Theorem 1. Let $(\eta_0^N,\, N \in \mathbb{N})$ be any sequence of $\mathbf{X}$-valued random variables on a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-}a.s.$$

for some measurable $[0, K]$-valued profile $u_0(.)$. Then the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}^N)(dx) = u(.,t)dx$$

holds uniformly on all bounded time intervals, where $(x,t) \mapsto u(x,t)$ denotes the unique entropy solution to (3) with initial condition $u_0(.)$.

Our constructive approach requires new ideas since the sub-additive ergodic theorem (central to the few previous existing proofs for strong hydrodynamics) is no longer effective in our setting. We work with the graphical representation of the dynamics, on which we couple an arbitrary number of processes, thanks to the *complete monotonicity property* of the dynamics. To solve the $\mathcal{R}$-valued Riemann problem, we combine proofs of almost sure analogues of the results of [4], rephrased for currents which become our centerpiece, with a space-time ergodic theorem for particle systems and large deviation results for the empirical measure. In the approximation steps, new error analysis is necessary: in particular, we have to do an explicit time discretization (vs. the "instantaneous limit" of [6,7]), we need estimates uniform in time, and each approximation step requires a control with exponential bounds.

In [38] we derive the macroscopic stability property when the particles' jump kernel $p(.)$ has a finite first moment and a positive mean. We also extend under those hypotheses the ergodic theorem for densities due to [40] that we use in [8]. Finally, we prove the finite propagation property when $p(.)$ has a finite third moment. This enables us to get rid of the finite range assumption on p required so far, and to extend the strong hydrodynamic result of [8] when the particles' jump kernel has a finite third moment and a positive mean.

In [9], we derive, thanks to the tools introduced in [8], a quenched strong hydrodynamic limit for bounded attractive particle systems on $\mathbb{Z}$ evolving in a random ergodic environment. (This result, which contains Theorems 1 and 2 above, is stated later on in this paper as Theorem 3). Our method is robust with respect to the model and disorder (we are not restricted to site or bond disorder). We introduce a general framework to describe the rates of the dynamics, which applies to a large class of models. To overcome the difficulty of the simultaneous loss of translation invariance and lack of knowledge of explicit invariant measures for the disordered system, we study a joint disorder-particle process, which is translation invariant. We characterize its extremal invariant and translation invariant measures, and prove its strong hydrodynamic limit. This implies the quenched hydrodynamic result we look for.

We illustrate our results on various examples.

3 Main Results

The construction of interacting particle systems is done either analytically, through generators and semi-groups (we refer to [36] for systems with compact state space, and to [1,23,37] otherwise), or through a graphical representation. Whereas the former is sufficient to derive hydrodynamic limits in a weak sense, which is done in [6,7], the latter is necessary to derive strong hydrodynamic limits, which is done in [8,9]. First, we explain in Subsect. 3.1 the graphical construction, then in Subsect. 3.2 we detail our results from [9] on invariant measures for the dynamics and hydrodynamic limits.

For simplicity, we restrict ourselves in this section to the misanthropes process with site disorder, which corresponds to the generator (4). However, considering only the necessary properties of the misanthropes process required to prove our hydrodynamic results, it is possible to deal with more general models including the k-step exclusion process, by embedding them in a global framework, in which the dynamics is viewed as a random transformation of the configuration; the latter simultaneously defines the graphical construction *and* generator. More general forms of random environments than site disorder can also be considered. In Subsect. 3.3 we list the above required properties of misanthropes processes, and we defer the study of the k-step exclusion process and more general models to Sect. 6.

3.1 Graphical Construction

This subsection is based on [8, Section 2.1]. We now describe the graphical construction (that is the pathwise construction on a Poisson space) of the system given by (4), which uses a Harris-like representation ([30,31]; see for instance [2,11,25,47] for details and justifications). This enables us to define the evolution from arbitrarily many different initial configurations simultaneously on the same probability space, in a way that depends monotonically on these initial configurations.

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of measures ω on $\mathbb{R}^+ \times \mathbb{Z}^2 \times [0,1]$ of the form

$$\omega(dt, dx, dz, du) = \sum_{m \in \mathbb{N}} \delta_{(t_m, x_m, z_m, u_m)}$$

where $\delta_{(\cdot)}$ denotes Dirac measure, and $(t_m, x_m, z_m, u_m)_{m \geq 0}$ are pairwise distinct and form a locally finite set. The σ-field $\mathcal{F}$ is generated by the mappings $\omega \mapsto \omega(S)$ for Borel sets S. The probability measure $\mathbb{P}$ on Ω is the one that makes ω a Poisson process with intensity

$$m(dt, dx, dz, du) = ||b||_\infty \lambda_{\mathbb{R}^+}(dt) \times \lambda_{\mathbb{Z}}(dx) \times p(dz) \times \lambda_{[0,1]}(du)$$

where λ denotes either the Lebesgue or the counting measure. We denote by $\mathbb{E}$ the corresponding expectation. Thanks to assumption (*M4*), we can proceed as in [2,25] (for a construction with a weaker assumption we refer to [11,47]): for $\mathbb{P}$-a.e. ω, there exists a unique mapping

$$(\alpha, \eta_0, t) \in \mathbf{A} \times \mathbf{X} \times \mathbb{R}^+ \mapsto \eta_t = \eta_t(\alpha, \eta_0, \omega) \in \mathbf{X} \tag{8}$$

satisfying: (a) $t \mapsto \eta_t(\alpha, \eta_0, \omega)$ is right-continuous; (b) $\eta_0(\alpha, \eta_0, \omega) = \eta_0$; (c) for $t \in \mathbb{R}^+$, $(x, z) \in \mathbb{Z}^2$, $\eta_t = \eta_{t^-}^{x,x+z}$ if

$$\exists u \in [0,1] : \omega\{(t, x, z, u)\} = 1 \text{ and } u \leq \alpha(x) \frac{b(\eta_{t^-}(x), \eta_{t^-}(x+z))}{||b||_\infty} \tag{9}$$

and (d) for all $s, t \in \mathbb{R}^{+*}$ and $x \in \mathbb{Z}$,

$$\omega\{[s,t] \times Z_x \times (0,1)\} = 0 \Rightarrow \forall v \in [s,t], \eta_v(x) = \eta_s(x) \tag{10}$$

where

$$Z_x := \left\{(y, z) \in \mathbb{Z}^2 : y = x \text{ or } y + z = x\right\}$$

In short, (9) tells how the state of the system can be modified by an "ω-event", and (10) says that the system cannot be modified outside ω-events.

Thanks to assumption (*M2*), we have that

$$(\alpha, \eta_0, t) \mapsto \eta_t(\alpha, \eta_0, \omega) \text{ is nondecreasing w.r.t. } \eta_0 \tag{11}$$

Property (11) implies (2), that is, attractiveness. But it is more powerful: it implies the *complete monotonicity* property [20,24], that is, existence of a monotone Markov coupling for an *arbitrary* number of processes with generator (4), which is necessary in our proof of strong hydrodynamics for general initial profiles. The coupled process can be defined by a Markov generator, as in [19] for two components, that is

$$\begin{aligned}
&\overline{L}_\alpha f(\eta, \xi) \\
&= \sum_{x,y \in \mathbb{Z}:\, x \neq y} \Big\{ \alpha(x) p(y-x) [b(\eta(x), \eta(y)) \wedge b(\xi(x), \xi(y))] \left[f(\eta^{x,y}, \xi^{x,y}) - f(\eta, \xi)\right] \\
&\quad + \alpha(x) p(y-x) [b(\eta(x), \eta(y)) - b(\xi(x), \xi(y))]^+ \left[f(\eta^{x,y}, \xi) - f(\eta, \xi)\right] \\
&\quad + \alpha(x) p(y-x) [b(\xi(x), \xi(y)) - b(\eta(x), \eta(y))]^+ \left[f(\eta, \xi^{x,y}) - f(\eta, \xi)\right] \Big\}
\end{aligned} \tag{12}$$

One may further introduce an "initial" probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$, large enough to construct random initial configurations $\eta_0 = \eta_0(\omega_0)$ for $\omega_0 \in \Omega_0$. The general process with random initial configurations is constructed on the enlarged space $(\widetilde{\Omega} = \Omega_0 \times \Omega, \widetilde{\mathcal{F}} = \sigma(\mathcal{F}_0 \times \mathcal{F}), \widetilde{\mathbb{P}} = \mathbb{P}_0 \otimes \mathbb{P})$ by setting

$$\eta_t(\alpha, \widetilde{\omega}) = \eta_t(\alpha, \eta_0(\omega_0), \omega)$$

for $\widetilde{\omega} = (\omega_0, \omega) \in \widetilde{\Omega}$. One can show (see for instance [11,25,47]) that this defines a Feller process with generator (4): that is for any $t \in \mathbb{R}^+$ and $f \in C(\mathbf{X})$ (the set of continuous functions on $\mathbf{X}$), $S_\alpha(t) f \in C(\mathbf{X})$ where $S_\alpha(t) f(\eta_0) =$

$\mathbb{E}[f(\eta_t(\alpha, \eta_0, \omega))]$. If η_0 has distribution μ_0, then the process thus constructed is Feller with generator (4) and initial distribution μ_0.

We define on Ω the *space-time shift* θ_{x_0,t_0}: for any $\omega \in \Omega$, for any (t, x, z, u)

$$(t, x, z, u) \in \theta_{x_0,t_0}\omega \text{ if and only if } (t_0 + t, x_0 + x, z, u) \in \omega \tag{13}$$

where $(t, x, z, u) \in \omega$ means $\omega\{(t, x, z, u)\} = 1$. By its very definition, the mapping introduced in (8) enjoys the following properties, for all $s, t \geq 0$, $x \in \mathbb{Z}$ and $(\eta, \omega) \in \mathbf{X} \times \Omega$:

$$\eta_s(\alpha, \eta_t(\alpha, \eta, \omega), \theta_{0,t}\omega) = \eta_{t+s}(\alpha, \eta, \omega) \tag{14}$$

which implies Markov property, and

$$\tau_x \eta_t(\alpha, \eta, \omega) = \eta_t(\tau_x \alpha, \tau_x \eta, \theta_{x,0}\omega)$$

which yields the *commutation property*

$$L_\alpha \tau_x = \tau_x L_{\tau_x \alpha} \tag{15}$$

3.2 Hydrodynamic Limit and Invariant Measures

This section is based on [9, Sections 2, 3]. A central issue in interacting particle systems, and more generally in the theory of Markov processes, is the characterization of invariant measures [36]. Besides, this characterization plays a crucial role in the derivation of hydrodynamic limits [33]. We detail here our results on these two questions.

Hydrodynamic Limit. We first state the strong hydrodynamic behavior of the process with quenched site disorder.

Theorem 3 ([9, Theorem 2.1]). *Assume $K < +\infty$, $p(.)$ has finite third moment, and satisfies the irreducibility assumption (7). Let Q be an ergodic probability distribution on $\mathbf{A}$. Then there exists a Lipschitz-continuous function G^Q on $[0, K]$ defined in (23) and (24)–(25) below (depending only on $p(.)$, $b(.,.)$ and Q) such that the following holds. Let $(\eta_0^N, N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued random variables on a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx \quad \mathbb{P}_0\text{-}a.s.$$

for some measurable $[0, K]$-valued profile $u_0(.)$. Then for Q-a.e. $\alpha \in \mathbf{A}$, the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N(\eta_{Nt}(\alpha, \eta_0^N(\omega_0), \omega))(dx) = u(., t)dx$$

holds uniformly on all bounded time intervals, where $(x, t) \mapsto u(x, t)$ denotes the unique entropy solution with initial condition u_0 to the conservation law

$$\partial_t u + \partial_x [G^Q(u)] = 0 \tag{16}$$

The $\mathbb{P}$-almost sure convergence in Theorem 3 refers to the graphical construction of Subsect. 3.1, and is stronger than the usual notion of hydrodynamic limit, which is a convergence in probability (cf. Definition 2). The strong hydrodynamic limit implies the weak one [8]. This leads to the following weaker but more usual statement, which also has the advantage of not depending on a particular construction of the process.

Theorem 4. *Under assumptions and notations of Theorem 3, there exists a subset* $\mathbf{A}'$ *of* $\mathbf{A}$*, with* Q*-probability* 1*, such that the following holds for every* $\alpha \in \mathbf{A}'$*. For any* $u_0 \in L^\infty(\mathbb{R})$*, and any sequence* $\eta_.^N = (\eta_t^N)_{t\geq 0}$ *of processes with generator* (4) *satisfying the convergence in probability*

$$\lim_{N\to\infty} \pi^N(\eta_0^N)(dx) = u_0(.)dx,$$

one has the locally uniform convergence in probability

$$\lim_{N\to\infty} \pi^N\left(\eta_{Nt}^N\right)(dx) = u(.,t)dx$$

Remark 1. Theorem 1 (resp. Theorem 2) is a special case of Theorem 4 (resp. Theorem 3). Indeed, it suffices to consider the "disorder" distribution Q that is the Dirac measure supported on the single homogeneous environment α_{hom} defined by $\alpha_{\text{hom}}(x) = 1$ for all $x \in \mathbb{Z}$ (see also Remark 2).

Note that the statement of Theorem 4 is stronger than hydrodynamic limit in the sense of Definition 2, because it states convergence of the empirical measure *process* rather then convergence at every fixed time. To define the *macroscopic flux* G^Q, we first define the *microscopic flux* as follows. The generator (4) can be decomposed as a sum of translates of a "seed" generator centered around the origin:

$$L_\alpha = \sum_{x\in\mathbb{Z}} L_\alpha^x \tag{17}$$

Note that such a decomposition is not unique. A natural choice of component L_α^x at $x \in \mathbb{Z}$ is given in the case of (4) by

$$L_\alpha^x f(\eta) = \alpha(x) \sum_{z\in\mathbb{Z}} p(z) \left[f\left(\eta^{x,x+z}\right) - f(\eta)\right]$$

We then define j to be either of the functions j_1, j_2 defined below:

$$j_1(\alpha,\eta) := L_\alpha\left[\sum_{x>0}\eta(x)\right], \quad j_2(\alpha,\eta) := L_\alpha^0\left[\sum_{x\in\mathbb{Z}} x\eta(x)\right] \tag{18}$$

The definition of j_1 is partly formal, because the function $\sum_{x>0}\eta(x)$ does not belong to the domain of the generator L_α. Nevertheless, the formal computation gives rise to a well-defined function j_1, because the rate b is a local function. Rigorously, one defines j_1 by difference, as the unique function such that

$$j_1 - \tau_x j_1(\alpha,\eta) = L_\alpha\left[\sum_{y=1}^{x}\eta(y)\right] \tag{19}$$

for every $x \in \mathbb{N}$. The action of the generator in (19) is now well defined, because we have a local function that belongs to its domain.

In the case of (4), we obtain the following microscopic flux functions:

$$\begin{aligned} j_1(\alpha,\eta) &= \sum_{(x,z)\in\mathbb{Z}^2, x\le 0<x+z} \alpha(x)b(\eta(x),\eta(x+z)) \\ &\quad - \sum_{(x,z)\in\mathbb{Z}^2, x+z\le 0<x} \alpha(x)b(\eta(x),\eta(x+z)) \\ j_2(\alpha,\eta) &= \alpha(0)\sum_{z\in\mathbb{Z}} zp(z)b(\eta(0),\eta(z)) \end{aligned}$$

Once a microscopic flux function j is defined, the macroscopic flux function is obtained by averaging with respect to a suitable family of measures, that we now introduce.

Invariant Measures. We define the Markovian *joint disorder-particle process* $(\alpha_t,\eta_t)_{t\ge 0}$ on $\mathbf{A}\times\mathbf{X}$ with generator given by, for any local function f on $\mathbf{A}\times\mathbf{X}$,

$$\mathfrak{L}f(\alpha,\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x)p(y-x)b(\eta(x),\eta(y))\left[f\left(\alpha,\eta^{x,y}\right) - f(\alpha,\eta)\right]$$

Given $\alpha_0 = \alpha$, this dynamics simply means that $\alpha_t = \alpha$ for all $t \ge 0$, while $(\eta_t)_{t\ge 0}$ is a Markov process with generator L_α given by (4). Note that $\mathfrak{L}$ is *translation invariant*, that is

$$\tau_x\mathfrak{L} = \mathfrak{L}\tau_x \tag{20}$$

where τ_x acts jointly on (α,η). This is equivalent to the commutation relation (15) for the quenched dynamics.

Let $\mathcal{I}_{\mathfrak{L}}$, $\mathcal{S}$ and $\mathcal{S}^{\mathbf{A}}$ denote the sets of probability measures that are respectively invariant for $\mathfrak{L}$, shift-invariant on $\mathbf{A}\times\mathbf{X}$ and shift-invariant on $\mathbf{A}$.

Proposition 2 ([9, Proposition 3.1]). *For every $Q \in \mathcal{S}^{\mathbf{A}}_e$, there exists a closed subset $\mathcal{R}^Q$ of $[0,K]$ containing 0 and K, depending on $p(.)$ and $b(.,.)$, such that*

$$\left(\mathcal{I}_{\mathfrak{L}}\cap\mathcal{S}\right)_e = \left\{\nu^{Q,\rho},\ Q\in\mathcal{S}^{\mathbf{A}}_e,\ \rho\in\mathcal{R}^Q\right\}$$

where index e denotes the set of extremal elements, and $(\nu^{Q,\rho}:\ \rho\in\mathcal{R}^Q)$ is a family of shift-invariant measures on $\mathbf{A}\times\mathbf{X}$, weakly continuous with respect to ρ, such that

$$\int \eta(0)\nu^{Q,\rho}(d\alpha,d\eta) = \rho$$

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:|x|\le l}\eta(x) = \rho, \quad \nu^{Q,\rho} - a.s. \tag{21}$$

$$\rho\le\rho' \Rightarrow \nu^{Q,\rho} \ll \nu^{Q,\rho'} \tag{22}$$

Here, $\ll$ denotes the conditional stochastic order defined in Lemma 1 below.

From the family of invariant measures in the above proposition for the joint disorder-particle process, one may deduce a family of invariant measures for the quenched particle process.

Corollary 1 ([9, Corollary 3.1]). *There exists a subset* $\widetilde{\mathbf{A}}^Q$ *of* $\mathbf{A}$ *with* Q*-probability* 1 *(depending on* $p(.)$ *and* $b(.,.)$*), such that the family of probability measures* $(\nu_\alpha^{Q,\rho} : \alpha \in \widetilde{\mathbf{A}}^Q, \rho \in \mathcal{R}^Q)$ *on* $\mathbf{X}$*, defined by* $\nu_\alpha^{Q,\rho}(.) := \nu^{Q,\rho}(.|\alpha)$ *satisfies the following properties, for every* $\rho \in \mathcal{R}^Q$*:*

(B1) For every $\alpha \in \widetilde{\mathbf{A}}^Q$*,* $\nu_\alpha^{Q,\rho}$ *is an invariant measure for* L_α*.*
(B2) For every $\alpha \in \widetilde{\mathbf{A}}^Q$*,* $\nu_\alpha^{Q,\rho}$*-a.s.,*

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:\,|x|\leq l}\eta(x)=\rho$$

(B3) The quantity

$$G_\alpha^Q(\rho) := \int j(\alpha,\eta)\nu_\alpha^{Q,\rho}(d\eta) =: G^Q(\rho), \quad \rho \in \mathcal{R}^Q \tag{23}$$

does not depend on $\alpha \in \widetilde{\mathbf{A}}^Q$*.*

Remark 2. As already observed in Remark 1 above, we can view the non-disordered model as a special “disordered” model by taking Q to be the Dirac measure on the homogeneous environment α_{hom} with constant value 1. Hence, Proposition 1 is a special case of Proposition 2. Note that we then have $\nu^\rho = \nu_\alpha^{Q,\rho}$ and $\nu^{Q,\rho} = \delta_{\alpha_{\text{hom}}} \otimes \nu^\rho$ for Q-a.e. $\alpha \in \mathbf{A}$.

We now come back to the macroscopic flux function $G^Q(\rho)$. We define it as (23) for $\rho \in \mathcal{R}^Q$ and we extend it by linear interpolation on the complement of $\mathcal{R}^Q$, which is a finite or countably infinite union of disjoint open intervals: that is, we set

$$G^Q(\rho) := \frac{\rho-\rho^-}{\rho^+-\rho^-}G(\rho^+) + \frac{\rho^+-\rho}{\rho^+-\rho^-}G(\rho^-), \quad \rho \notin \mathcal{R}^Q \tag{24}$$

where

$$\rho^- := \sup[0,\rho]\cap\mathcal{R}^Q, \quad \rho^+ := \inf[\rho,+\infty)\cap\mathcal{R}^Q \tag{25}$$

By definition of $\nu_\alpha^{Q,\rho}$ and statement *(B3)* of Corollary 1, we also have

$$G^Q(\rho) := \int j(\alpha,\eta)\nu^{Q,\rho}(d\alpha,d\eta), \quad \rho \in \mathcal{R}^Q \tag{26}$$

We point out that (26) yields the same macroscopic flux function, whether $j = j_1$ or $j = j_2$ defined in (18) is plugged into it. It does not depend on the choice of a particular decomposition (17) either. These invariance properties follow from translation invariance of $\nu^{Q,\rho}$ and translation invariance (20) of the joint dynamics. Definitions (18) of the microscopic flux, and (23)–(26) of the macroscopic flux are model-independent, and can thus be used for other models, such as the

k-exclusion process, or the models reviewed in Sect. 6. Of course, the invariant measures involved in (23)–(26) depend on the model and disorder.

The function G^Q can be shown to be Lipschitz continuous [9, Remark 3.3]. This is the minimum regularity required for the classical theory of entropy solutions, see Sect. 4. We cannot say more about G^Q in general, because the measures $\nu_\alpha^{Q,\rho}$ are most often not explicit.

Note that in the special case of the non-disordered model investigated in [7] (see Proposition 1, Theorem 1, Remarks 1 and 2 above), the microscopic flux functions do not depend on α, and (23) or (26) both reduce to

$$G(\rho) := \int j(\eta) d\nu^\rho(\eta), \quad \rho \in \mathcal{R}$$

Even in the absence of disorder, only a few models have explicit invariant measures, and thus an explicit flux function. In Sect. 6, we define a variant of the k-step exclusion process, that we call the exclusion process with overtaking. It is possible to tune the microscopic parameters of this model so as to obtain any prescribed polynomial flux function (constrained to vanish at density values 0 and 1 due to the exclusion rule).

In contrast, a most natural and seemingly simple generalization of the asymmetric exclusion process, the asymmetric K-exclusion process, for which $K \geq 2$ and $b(n,m) = \mathbf{1}_{\{n>0\}}\mathbf{1}_{\{m<K\}}$ in (4), does not have explicit invariant measures. Thus, nothing more than the Lipschitz property can be said about its flux function in general. In the special case of the totally asymmetric K-exclusion process, that is for $p(1) = 1$, the flux function is shown to be concave in [43], as a consequence of the variational approach used there to derive hydrodynamic limit. But this approach does not apply to the models we consider in the present paper.

An important open question is whether the set $\mathcal{R}^Q$ (or its analogue $\mathcal{R}$ in the absence of disorder) covers the whole range of possible densities, or if it contains gaps corresponding to phase transitions. The only partial answer to this question so far was given by the following result from [7] for the totally asymmetric K-exclusion process without disorder.

Theorem 5 ([7, Corollary 2.1]). *For the totally asymmetric K-exclusion process without disorder,* 0 *and K are limit points of $\mathcal{R}$, and $\mathcal{R}$ contains at least one point in* $[1/3, K-1/3]$.

We end this section with a skeleton of proof for Proposition 2. We combine the steps done to prove [7, Proposition 3.1] (without disorder) and [9, Proposition 3.1].

Proof of Proposition 2. The proof has two parts.
Part 1. It is an extension to the joint particle-disorder process of a classical scheme in a non-disordered setting due to [35], which is also the basis for similar results in [1,19,27,29]. It relies on couplings.

(a) We first need to couple measures, through the following lemma, analogous to Strassen's Theorem [46].

Lemma 1 **([9, Lemma 3.1]).** *For two probability measures μ^1, μ^2 on $\mathbf{A} \times \mathbf{X}$, the following properties (denoted by $\mu^1 \ll \mu^2$) are equivalent:*

(i) For every bounded measurable local function f on $\mathbf{A} \times \mathbf{X}$, such that $f(\alpha, .)$ is nondecreasing for all $\alpha \in \mathbf{A}$, we have $\int f\, d\mu^1 \leq \int f\, d\mu^2$.
(ii) The measures μ^1 and μ^2 have a common α-marginal Q, and $\mu^1(d\eta|\alpha) \leq \mu^2(d\eta|\alpha)$ for Q-a.e. $\alpha \in \mathbf{A}$.
(iii) There exists a coupling measure $\overline{\mu}(d\alpha, d\eta, d\xi)$ supported on $\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \eta \leq \xi\}$ under which $(\alpha, \eta) \sim \mu^1$ and $(\alpha, \xi) \sim \mu^2$.

(b) Then, for the dynamics, we denote by $\overline{\mathfrak{L}}$ the coupled generator for the joint process $(\alpha_t, \eta_t, \xi_t)_{t \geq 0}$ on $\mathbf{A} \times \mathbf{X}^2$ defined by

$$\overline{\mathfrak{L}} f(\alpha, \eta, \xi) = (\overline{L}_\alpha f(\alpha, .))(\eta, \xi)$$

for any local function f on $\mathbf{A} \times \mathbf{X}^2$, where $\overline{L}_\alpha$ was defined in (12). Given $\alpha_0 = \alpha$, this means that $\alpha_t = \alpha$ for all $t \geq 0$, while $(\eta_t, \xi_t)_{t \geq 0}$ is a Markov process with generator $\overline{L}_\alpha$. We denote by $\overline{\mathcal{S}}$ the set of probability measures on $\mathbf{A} \times \mathbf{X}^2$ that are invariant by space shift $\tau_x(\alpha, \eta, \xi) = (\tau_x \alpha, \tau_x \eta, \tau_x \xi)$. We prove successively (next lemma combines [9, Lemmas 3.2, 3.4 and Proposition 3.2]):

Lemma 2. *(i) Let $\mu', \mu'' \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e$ with a common α-marginal Q. Then there exists $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$ such that the respective marginal distributions of (α, η) and (α, ξ) under $\overline{\nu}$ are μ' and μ''.*
(ii) Let $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$. Then $\overline{\nu}\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \eta \leq \xi\}$ and $\overline{\nu}\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \xi \leq \eta\}$ belong to $\{0, 1\}$.
(iii) Every $\overline{\nu} \in \left(\mathcal{I}_{\overline{\mathfrak{L}}} \cap \overline{\mathcal{S}}\right)_e$ is supported on $\{(\alpha, \eta, \xi) \in \mathbf{A} \times \mathbf{X}^2 : \eta \leq \xi \text{ or } \xi \leq \eta\}$.

(c) This last point *(iii)* is the core of the proof of Proposition 2: Attractiveness assumption ensures that an initially ordered pair of coupled configurations remains ordered at later times. We say that there is a positive (resp. negative) discrepancy between two coupled configurations ξ, ζ at some site x if $\xi(x) > \zeta(x)$ (resp. $\xi(x) < \zeta(x)$). Irreducibility assumption (7) induces a stronger property: pairs of discrepancies of opposite signs between two coupled configurations eventually get killed, so that the two configurations become ordered.

Part 2. We define

$$\mathcal{R}^Q := \left\{ \int \eta(0) \nu(d\alpha, d\eta) : \nu \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e, \nu \text{ has } \alpha\text{-marginal } Q \right\}$$

Let $\nu^i \in (\mathcal{I}_{\mathfrak{L}} \cap \mathcal{S})_e$ with α-marginal Q and $\rho^i := \int \eta(0) \nu^i(d\alpha, d\eta) \in \mathcal{R}^Q$ for $i \in \{1, 2\}$. Assume $\rho^1 \leq \rho^2$. Using Lemma 1,*(iii)*, then Lemma 2, we obtain $\nu^1 \ll \nu^2$, that is (22). Existence (21) of an asymptotic particle density can be obtained by a proof analogous to [38, Lemma 14], where the space-time ergodic theorem is applied to the joint disorder-particle process. Then, closedness of $\mathcal{R}^Q$ is established as in [7, Proposition 3.1]: it uses (22), (21). Given the rest of the proposition, the weak continuity statement comes from a coupling argument, using (22) and Lemma 1. □

3.3 Required Properties of the Model

For the proofs of Theorem 3 and Proposition 2, we have not used the particular form of L_α in (4), but the following properties.

1) The set of environments is a probability space $(\mathbf{A}, \mathcal{F}_\mathbf{A}, Q)$, where $\mathbf{A}$ is a compact metric space and $\mathcal{F}_\mathbf{A}$ its Borel σ-field. On $\mathbf{A}$ we assume given a group of space shifts $(\tau_x : x \in \mathbb{Z})$, with respect to which Q is ergodic. For each $\alpha \in \mathbf{A}$, L_α is the generator of a Feller process on $\mathbf{X}$ that satisfies (15). The latter should be viewed as the assumption on "how the disorder enters the dynamics". It is equivalent to $\mathfrak{L}$ satisfying (20), that is being a translation-invariant generator on $\mathbf{A} \times \mathbf{X}$.
2) For L_α we can define a graphical construction on a space-time Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the complete monotonicity property (11).
3) Irreducibility assumption (7), combined with attractiveness assumption (*M2*), are responsible for Lemma 2,*(iii)*.

Indeed, given the generator (4) of the process, there is not a unique graphical construction. The strong convergence in Theorem 3 would hold for *any* graphical construction satisfying (11) *plus* the existence of a sequence of Poissonian events killing any remaining pair of discrepancies of opposite signs (see Part 1,(c)) of the proof of Proposition 2 for this last point). The latter property follows in the case of the misanthropes process from irreducibility assumption (7).

In Sect. 6 we shall therefore introduce a general framework to consider other models satisfying 1) and 2), with appropriate assumptions replacing (7) to imply Proposition 2. We refer to Lemma 11 for a statement and proof of Property (11) in the context of a general model, including the k-step exclusion process. The coupled process linked to this property can be tedious to write in the usual form of explicit coupling rates for more than two components, or for complex models. We shall see that it can be written in a simple model-independent way using the framework of Sect. 6.1.

4 Scalar Conservation Laws and Entropy Solutions

In Subsect. 4.1, we recall the definition and characterizations of entropy solutions to scalar conservation laws, which will appear as hydrodynamic limits of the above models. Then in Subsect. 4.2, we explain our variational formula for the entropy solution in the Riemann case, first when the flux function G is Lipschitz continuous, then when $G \in \mathcal{C}^2(\mathbb{R})$ has a single inflexion point, so that the entropy solution has a more explicit form. Finally, in Subsect. 4.3, we explain the approximation schemes to go from a Riemann initial profile to a general initial profile.

4.1 Definition and Properties of Entropy Solutions

This section is taken from [7, Section 2.2] and [8, Section 4.1]. For more details, we refer to the textbooks [28,44], or [18]. Equation (3) has no strong solutions in general: even starting from a smooth Cauchy datum $u(.,0) = u_0$, discontinuities (called shocks in this context) appear in finite time. Therefore it is necessary to consider weak solutions, but then uniqueness is lost for the Cauchy problem. To recover uniqueness, we need to define *entropy solutions.*

Let $\phi : [0,K] \cap \mathbb{R} \to \mathbb{R}$ be a convex function. In the context of hyperbolic systems, such a function is called an *entropy.* We define the associated *entropy flux* ψ on $[0,K]$ as

$$\psi(u) := \int_0^u \phi'(v)G'(v)dv$$

(ϕ,ψ) is called an *entropy-flux pair.* A Borel function $u : \mathbb{R} \times \mathbb{R}^{+*} \to [0,K]$ is called an *entropy solution* to (3) if and only if it is entropy-dissipative, *i.e.*

$$\partial_t \phi(u) + \partial_x \psi(u) \leq 0 \tag{27}$$

in the sense of distributions on $\mathbb{R} \times \mathbb{R}^{+*}$ for any entropy-flux pair (ϕ,ψ). Note that, by taking $\phi(u) = \pm u$ and hence $\psi(u) = \pm G(u)$, we see that an entropy solution is indeed a weak solution to (3). This definition can be motivated by the following points: *(i)* when G and ϕ are continuously differentiable, (3) implies equality in strong sense in (27) (this follows from the chain rule for differentiation); *(ii)* this no longer holds in general if u is only a weak solution to (3); *(iii)* the inequality (27) can be seen as a macroscopic version of the second law of thermodynamics that selects physically relevant solutions. Indeed, one should think of the *concave* function $h = -\phi$ as a thermodynamic entropy, and spatial integration of (27) shows that the total thermodynamic entropy may not decrease during the evolution (this is rigorously true for periodic boundary conditions, in which case the total entropy is well defined).

Kružkov proved the following fundamental existence and uniqueness result:

Theorem 6 ([34, Theorem 2 and Theorem 5]). *Let $u_0 : \mathbb{R} \to [0,K]$ be a Borel measurable initial datum. Then there exists a unique (up to a Lebesgue-null subset of $\mathbb{R} \times \mathbb{R}^{+*}$) entropy solution u to (3) subject to the initial condition*

$$\lim_{t \to 0^+} u(.,t) = u_0(.) \text{ in } L^1_{\text{loc}}(\mathbb{R})$$

This solution (has a representative in its $L^\infty(\mathbb{R} \times \mathbb{R}^{+*})$ equivalence class that) is continuous as a mapping $t \mapsto u(.,t)$ from $\mathbb{R}^{+*}$ to $L^1_{\text{loc}}(\mathbb{R})$.

We recall here that a sequence $(u_n, n \in \mathbb{N})$ of Borel measurable functions on $\mathbb{R}$ is said to converge to u in $L^1_{\text{loc}}(\mathbb{R})$ if and only if

$$\lim_{n \to \infty} \int_I |u_n(x) - u(x)|\, dx = 0$$

for every bounded interval $I \subset \mathbb{R}$.

Remark 3. Kružkov's theorems are stated for a continuously differentiable G. However the proof of the uniqueness result [34, Theorem 2] uses only Lipschitz continuity. In the Lipschitz-continuous case, existence could be derived from Kružkov's result by a flux approximation argument. However a different, self-contained (and constructive) proof of existence in this case can be found in [18, Chapter 6].

The following proposition is a collection of results on entropy solutions. We first recall the following definition. Let TV_I denote the variation of a function defined on some bounded closed interval $I=[a,b]\subset\mathbb{R}$, *i.e.*

$$\mathrm{TV}_I[u(.)] := \sup_{x_0=a<x_1\cdots<x_n=b}\sum_{i=0}^{n-1}|u(x_{i+1})-u(x_i)|$$

The total variation of u is defined by

$$\mathrm{TV}[u(.)] := \sup_{I\subset\mathbb{R}}\mathrm{TV}_I[u(.)]$$

Let us say that $u=u(.,.)$ defined on $\mathbb{R}\times\mathbb{R}^{+*}$ has locally bounded space variation if

$$\sup_{t\in J}\mathrm{TV}_I[u(.,t)]<+\infty$$

for every bounded closed space interval $I\subset\mathbb{R}$ and bounded time interval $J\subset\mathbb{R}^{+*}$.

For two measures $\alpha,\beta\in\mathcal{M}^+(\mathbb{R})$ with compact support, we define

$$\Delta(\alpha,\beta) := \sup_{x\in\mathbb{R}}|\alpha((-\infty,x])-\beta((-\infty,x])| \tag{28}$$

When α or β is of the form $u(.)dx$ for $u(.)\in L^\infty(\mathbb{R})$ with compact support, we simply write u in (28) instead of $u(.)dx$. For a sequence $(\mu_n)_{n\geq 0}$ of measures with uniformly bounded support, the following equivalence holds:

$$\mu_n\to\mu \text{ vaguely if and only if } \lim_{n\to\infty}\Delta(\mu_n,\mu)=0 \tag{29}$$

Proposition 3 ([8, Proposition 4.1]).

(i) Let $u(.,.)$ be the entropy solution to (3) with Cauchy datum $u_0\in L^\infty(\mathbb{R})$. Then the mapping $t\mapsto u_t=u(.,t)$ lies in $C^0([0,+\infty),L^1_{\mathrm{loc}}(\mathbb{R}))$.

(ii) If u_0 has constant value c, then for all $t>0$, u_t has constant value c.

(iii) If $u_0^i(.)$ has finite variation, that is $\mathrm{TV}u_0^i(.)<+\infty$, then so does $u^i(.,t)$ for every $t>0$, and $\mathrm{TV}u^i(.,t)\leq\mathrm{TV}u_0^i(.)$.

(iv) Finite propagation property: Assume $u^i(.,.)$ ($i\in\{1,2\}$) is the entropy solution to (3) with Cauchy data $u_0^i(.)$. Let

$$V=||G'||_\infty := \sup_\rho|G'(\rho)| \tag{30}$$

Then, for every $x<y$ *and* $0\le t<(y-x)/2V$,

$$\int_{x+Vt}^{y-Vt}\left[u^1(z,t)-u^2(z,t)\right]^{\pm}dz\le\int_x^y\left[u_0^1(z)-u_0^2(z)\right]^{\pm}dz$$

In particular, assume $u_0^1=u_0^2$ *(resp.* $u_0^1\le u_0^2$*) on* $[a,b]$ *for some* $a,b\in\mathbb{R}$ *such that* $a<b$*. Then, for all* $t\le(b-a)/(2V)$, $u_t^1=u_t^2$ *(resp.* $u_t^1\le u_t^2$*) on* $[a+Vt,b-Vt]$*.*

(v) If $\int_{\mathbb{R}}u_0^i(z)dz<+\infty$*, then*

$$\Delta(u^1(.,t),u^2(.,t))\le\Delta(u_0^1(.),u_0^2(.)) \tag{31}$$

Properties *(i)–(iv)* are standard. Property *(v)* can be deduced from the correspondence between entropy solutions of (3) and viscosity solutions of the Hamilton–Jacobi equation

$$\partial_t h(x,t)+G[\partial_x h(x,t)]=0 \tag{32}$$

Namely, h is a viscosity solution of (32) if and only if $u=\partial_x h$ is an entropy solution of (3). Then *(v)* follows from the monotonicity of the solution semigroup for (32). Properties *(iv)* and *(v)* have microscopic analogues (respectively Lemma 10 and Proposition 10) in the class of particle systems we consider, which play an important role in the proof of the hydrodynamic limit, as will be sketched in Sect. 5.

We next recall a possibly more familiar definition of entropy solutions based on shock admissibility conditions, but valid only for solutions with bounded variation. This point of view selects the relevant weak solutions by specifying what kind of discontinuities are permitted. First, in particular, the following two conditions are necessary and sufficient for a piecewise smooth function $u(x,t)$ to be a weak solution to Eq. (3) with initial condition (34) (see [14]):

1. $u(x,t)$ solves Eq. (3) at points of smoothness.
2. If $x(t)$ is a curve of discontinuity of the solution then the *Rankine–Hugoniot condition*

$$\frac{d}{dt}x(t)=S[u^+;u^-]:=\frac{G(u^-)-G(u^+)}{u^--u^+}$$

holds along $x(t)$.

Moreover, to ensure uniqueness, the following geometric condition, known as *Oleĭnik's entropy condition* (see *e.g.* [28] or [44]), is sufficient. A discontinuity (u^-,u^+), with $u^\pm:=u(x\pm 0,t)$, is called an entropy shock, if and only if:

The chord of the graph of G between u^- and u^+ lies:
below the graph if $u^-<u^+$, above the graph if $u^+<u^-$. (33)

In the above condition, "below" or "above" is meant in wide sense, *i.e.* does not exclude that the graph and chord coincide at some points between u^- and u^+.

In particular, when G is strictly convex (resp. concave), one recovers the fact that only (and all) decreasing (resp. increasing) jumps are admitted (as detailed in Subsect. 4.2 below). Note that, if the graph of G is linear on some nontrivial interval, condition (33) implies that any increasing or decreasing jump within this interval is an entropy shock.

Indeed, condition (33) can be used to select entropy solutions among weak solutions. The following result is a consequence of [48].

Proposition 4 ([7, Proposition 2.2]). *Let u be a weak solution to (3) with locally bounded space variation. Then u is an entropy solution to (3) if and only if, for a.e. $t > 0$, all discontinuities of $u(.,t)$ are entropy shocks.*

One can show that, if the Cauchy datum u_0 has locally bounded variation, the unique entropy solution given by Theorem 6 has locally bounded space variation. Hence Proposition 4 extends into an existence and uniqueness theorem within functions of locally bounded space variation, where entropy solutions may be defined as weak solutions satisfying (33), without reference to (27).

4.2 The Riemann Problem

This subsection is based first on [7, Section 4.1], then on [6, Section 2.1]. Of special importance among entropy solutions are the solutions of the Riemann problem, *i.e.* the Cauchy problem for particular initial data of the form

$$R_{\lambda,\rho}(x) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x\geq 0\}} \tag{34}$$

Indeed: *(i)* as developed in the sequel of this subsection, these solutions can be computed explicitly and have a variational representation; *(ii)* as will be seen in Subsect. 4.3, one can construct approximations to the solution of the general Cauchy problem by using only Riemann solutions. This has inspired our belief that one could derive general hydrodynamics from Riemann hydrodynamics.

In connection with Theorem 3 and Proposition 2, it will be important in the sequel to consider flux functions G with possible linear degeneracy on some density intervals. Therefore, in the sequel of this section, $\mathcal{R}$ will denote a closed subset of $[0, K] \cap \mathbb{R}$ such that G is affine on each of the countably many disjoint open intervals whose union is the complement of $\mathcal{R}$. Such a subset exists (for instance one can take $\mathcal{R} = [0, K] \cap \mathbb{R}$) and is not necessarily unique.

From now on we assume $\lambda < \rho$; adapting to the case $\lambda > \rho$ is straightforward, by replacing in the sequel lower with upper convex hulls, and minima/minimizers with maxima/maximizers. Consider G_c, the lower convex envelope of G on $[\lambda, \rho]$. There exists a nondecreasing function H_c (hence with left/right limits) such that G_c has left/right hand derivative denoted by $H_c(\alpha \pm 0)$ at every α. The function H_c is defined uniquely outside the at most countable set of non-differentiability points of G_c

$$\Theta = \{\alpha \in [\lambda, \rho] : H_c(\alpha - 0) < H_c(\alpha + 0)\}$$

As will appear below, the particular choice of H_c on Θ does not matter. Let $v_* = v_*(\lambda,\rho) := H_c(\lambda+0)$ and $v^* = v^*(\lambda,\rho) := H_c(\rho-0)$. Since H_c is nondecreasing, there is a nondecreasing function h_c on $[v_*, v^*]$ such that, for every $v \in [v_*, v^*]$,

$$\begin{aligned} \alpha < h_c(v) &\Rightarrow H_c(\alpha) \le v \\ \alpha > h_c(v) &\Rightarrow H_c(\alpha) \ge v \end{aligned} \tag{35}$$

Any such h_c satisfies

$$\begin{aligned} h_c(v-0) &= \inf\{\alpha \in \mathbb{R} :\ H_c(\alpha) \ge v\} = \sup\{\alpha \in \mathbb{R} :\ H_c(\alpha) < v\} \\ h_c(v+0) &= \inf\{\alpha \in \mathbb{R} :\ H_c(\alpha) > v\} = \sup\{\alpha \in \mathbb{R} :\ H_c(\alpha) \le v\} \end{aligned} \tag{36}$$

We have that, anywhere in (36), $H_c(\alpha)$ may be replaced with $H_c(\alpha \pm 0)$. The following properties can be derived from (35) and (36):

1. Given G, h_c is defined uniquely, and is continuous, outside the at most countable set

$$\Sigma_{low}(G) \ = \{v \in [v_*, v^*] : G_c \text{ is differentiable with derivative } v \text{ in a nonempty open subinterval of } [\lambda,\rho]\}$$

   By "defined uniquely" we mean that for such v's, there is a unique $h_c(v)$ satisfying (35), which does not depend on the choice of H_c on Θ.

2. Given G, $h_c(v\pm 0)$ is uniquely defined, *i.e.* independent of the choice of H_c on Θ, for any $v \in [v_*, v^*]$. For $v \in \Sigma_{low}(G)$, $(h_c(v-0), h_c(v+0))$ is the maximal open interval over which H_c has constant value v.

3. For every $\alpha \in \Theta$ and $v \in (H_c(\alpha-0), H_c(\alpha+0))$, $h_c(v)$ is uniquely defined and equal to α.

In the sequel we extend h_c outside $[v_*, v^*]$ in a natural way by setting

$$h_c(v) = \lambda \text{ for } v < v_*, \quad h_c(v) = \rho \text{ for } v > v^*$$

Next proposition extends [6, Proposition 2.1], where we assumed $G \in \mathcal{C}^2(\mathbb{R})$.

Proposition 5 ([7, Proposition 4.1]). *Let $\lambda,\rho \in [0,K]\cap\mathbb{R}$, $\lambda < \rho$. For $v \in \mathbb{R}$, we set*

$$\mathcal{G}_v(\lambda,\rho) := \inf\{G(r) - vr :\ r \in [\lambda,\rho] \cap \mathcal{R}\} \tag{37}$$

Then

(i) For every $v \in \mathbb{R}\setminus\Sigma_{low}(G)$, the minimum in (37) is achieved at the unique point $h_c(v)$, and $u(x,t) := h_c(x/t)$ is the weak entropy solution to (3) with Riemann initial condition (34), denoted by $R_{\lambda,\rho}(x,t)$.

(ii) If $\lambda \in \mathcal{R}$ and $\rho \in \mathcal{R}$, the previous minimum is unchanged if restricted to $[\lambda, \rho] \cap \mathcal{R}$. As a result, the Riemann entropy solution is a.e. $\mathcal{R}$-valued.
(iii) For every $v, w \in \mathbb{R}$,

$$\int_v^w R_{\lambda,\rho}(x,t)dx = t[\mathcal{G}_{v/t}(\lambda,\rho) - \mathcal{G}_{w/t}(\lambda,\rho)] \tag{38}$$

In the case when $G \in \mathcal{C}^2(\mathbb{R})$ is such that G'' vanishes only finitely many times, the expression of $u(x,t)$ is more explicit. Let us detail, as in [6, Section 2.1], following [14], the case where G has a single inflexion point $a \in (0, K)$ and $G(u)$ is strictly convex in $0 \leq u < a$ and strictly concave in $a < u \leq 1$. We denote $H = G'$, h_1 the inverse of H restricted to $(-\infty, a)$, and h_2 the inverse of H restricted to $(a, +\infty)$. We have

Lemma 3 ([14, Lemma 2.2, Lemma 2.4]). *Let $w < a$ be given, and define*

$$w^* := \sup\{u > w : \, S[w;u] > S[v;w], \, \forall v \in (w,u)\}$$

Suppose that $w^* < \infty$. Then

(a) $S[w;w^*] = H(w^*)$;
(b) w^* is the only zero of $S[u;w] - H(u)$, $u > w$.

If $\rho < \lambda < \rho^*$ ($\rho < a$), then $H(\lambda) > H(\rho)$: the characteristics starting from $x \leq 0$ have a speed (given by H) greater than the speed of those starting from $x > 0$. If the characteristics intersect along a curve $x(t)$, then Rankine–Hugoniot condition will be satisfied if

$$x'(t) = S[u^+;u^-] = \frac{G(\lambda) - G(\rho)}{\lambda - \rho} = S[\lambda;\rho].$$

The convexity of G implies that Oleĭnik's Condition is satisfied across $x(t)$. Therefore the unique entropy weak solution is the *shock*:

$$u(x,t) = \begin{cases} \lambda, & x \leq S[\lambda;\rho]t; \\ \rho, & x > S[\lambda;\rho]t; \end{cases}$$

If $\lambda < \rho < a$, the relevant part of the flux function is convex. The characteristics starting respectively from $x \leq 0$ and $x > 0$ never meet, and they never enter the space-time wedge between lines $x = H(\lambda)t$ and $x = H(\rho)t$. It is possible to define piecewise smooth weak solutions with a jump occurring in the wedge satisfying the Rankine–Hugoniot condition. But the convexity of G prevents such solutions from satisfying Oleĭnik's Condition. Thus the unique entropy weak solution is the *continuous solution with a rarefaction fan*:

$$u(x,t) = \begin{cases} \lambda, & x \leq H(\lambda)t; \\ h_1(x/t), & H(\lambda)t < x \leq H(\rho)t; \\ \rho, & H(\rho)t < x; \end{cases}$$

Let $\rho < \rho^* < \lambda$ ($\rho < a$): Lemma 3 applied to ρ suggests that a jump from ρ^* to ρ along the line $x = H(\rho^*)t$ will satisfy the Rankine–Hugoniot condition. Due to the definition of ρ^*, a solution with such a jump will also satisfy Oleĭnik's Condition, therefore it would be the unique entropic weak solution. Notice that since $H(\lambda) < H(\rho^*)$, no characteristics intersect along the line of discontinuity $x = H(\rho^*)t$. This case is called a *contact discontinuity* in [14]. The solution is defined by

$$u(x,t) = \begin{cases} \lambda, & x \le H(\lambda)t; \\ h_2(x/t), & H(\lambda)t < x \le H(\rho^*)t; \\ \rho, & H(\rho^*)t < x; \end{cases}$$

Corresponding cases on the concave side of G are treated similarly.

Let us illustrate this description on a reference example, *the totally asymmetric 2-step exclusion* (what follows is taken from [6, Section 4.1.1]):

Its flux function $G_2(u) = u + u^2 - 2u^3$ is strictly convex in $0 \le u < 1/6$ and strictly concave in $1/6 < u \le 1$. For $w < 1/6$, $w^* = (1-2w)/4$, and for $w > 1/6$, $w_* = (1-2w)/4$; $h_1(x) = (1/6)(1-\sqrt{7-6x})$ for $x \in (-\infty, 7/6)$, and $h_2(x) = (1/6)(1+\sqrt{7-6x})$ for $x \in (7/6, +\infty)$. We reproduce here [6, Figure 1], which shows the six possible behaviors of the (self-similar) solution $u(v,1)$, namely a rarefaction fan with either an increasing or a decreasing initial condition, a decreasing shock, an increasing shock, and a contact discontinuity with either an increasing or a decreasing initial condition. Cases (a) and (b) present respectively a rarefaction fan with increasing initial condition and a preserved decreasing shock. These situations as well as cases (c) and (f) cannot occur for simple exclusion. Observe also that $\rho \ge 1/2$ implies $\rho_* \le 0$, which leads only to cases (d),(e), and excludes case (f) (going back to a simple exclusion behavior).

4.3 From Riemann to Cauchy Problem

The beginning of this subsection is based on [7, Section 2.4]. We will briefly explain here the principle of approximation schemes based on Riemann solutions, the most important of which is probably Glimm's scheme, introduced in [26]. Consider as initial datum a piecewise constant profile with finitely many jumps. The key observation is that, for small enough times, this can be viewed as a succession of noninteracting Riemann problems. To formalize this, we recall part of [6, Lemma 3.4], which is a consequence of the finite propagation property for (3), see statement *(iv)* of Proposition 3. We denote by $R_{\lambda,\rho}(x,t)$ the entropy solution to the Riemann problem with initial datum (34).

Lemma 4 ([7, Lemma 2.1]). *Let* $x_0 = -\infty < x_1 < \cdots < x_n < x_{n+1} = +\infty$, *and* $\varepsilon := \min_k (x_{k+1} - x_k)$. *Consider the Cauchy datum*

$$u_0 := \sum_{k=0}^{n} r_k \mathbf{1}_{(x_k, x_{k+1})}$$

where $r_k \in [0, K]$. *Then for* $t < \varepsilon/(2V)$, *with* V *given by* (30), *the entropy solution* $u(.,t)$ *at time* t *coincides with* $R_{r_{k-1},r_k}(. - x_k, t)$ *on* $(x_{k-1} + Vt, x_{k+1} - Vt)$. *In particular,* $u(.,t)$ *has constant value* r_k *on* $(x_k + Vt, x_{k+1} - Vt)$.

Given some Cauchy datum u_0, we construct an approximate solution $\tilde{u}(.,.)$ for the corresponding entropy solution $u(.,.)$. To this end we define an approximation scheme based on a time discretization step $\Delta t > 0$ and a space discretization step $\Delta x > 0$. In the limit we let $\Delta x \to 0$ with the ratio $R := \Delta t/\Delta x$ kept constant, under the condition

$$R \leq 1/(2V) \tag{39}$$

known as the *Courant–Friedrichs–Lewy (CFL) condition*. Let $t_k := k\Delta t$ denote discretization times. We start with $k = 0$, setting $\tilde{u}_0^- := u_0$.

Step one (approximation step): Approximate $\tilde{u}_k^-$ with a piecewise constant profile $\tilde{u}_k^+$ whose step lengths are bounded below by Δx.

Step two (evolution step): For $t \in [t_k, t_{k+1})$, denote by $\tilde{u}_k(.,t)$ the entropy solution at time t with initial datum $\tilde{u}_k^+$ at time t_k. By (39) and Lemma 4, $\tilde{u}_k(.,t)$ can be computed solving only Riemann problems. Set $\tilde{u}_{k+1}^- = \tilde{u}_k(., t_{k+1})$.

Step three (iteration): increment k and go back to step one.

The approximate entropy solution is then defined by

$$\tilde{u}(.,t) := \sum_{k \in \mathbb{N}} \tilde{u}_k(.,t) \mathbf{1}_{[t_k, t_{k+1})}(t) \tag{40}$$

The efficiency of the scheme depends on how the approximation step is performed. In Glimm's scheme, the approximation $\tilde{u}_k^+$ is defined as

$$\tilde{u}_k^+ := \sum_{j \in k/2 + \mathbb{Z}} \tilde{u}_k^-((j + a_k/2)\Delta x) \mathbf{1}_{((j-1/2)\Delta x, (j+1/2)\Delta x)} \tag{41}$$

where $a_k \in (-1, 1)$. Then we have the following convergence result.

Theorem 7 ([7, Theorem 2.3]). *Let* u_0 *be a given measurable initial datum. Then every sequence* $\varepsilon_n \downarrow 0$ *as* $n \to \infty$ *has a subsequence* $\delta_n \downarrow 0$ *such that, for a.e. sequence* (a_k) *w.r.t. product uniform measure on* $(-1, 1)^{\mathbb{Z}^+}$, *the Glimm approximation defined by* (40) *and* (41) *converges to* u *in* $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^{+*})$ *as* $\Delta x = \delta_n \downarrow 0$.

When u_0 has locally bounded variation, the above result is a specialization to scalar conservation laws of a more general result for systems of conservation laws: see Theorems 5.2.1, 5.2.2, 5.4.1 and comments following Theorem 5.2.2 in [44]. In [7, Appendix B], we prove that it is enough to assume u_0 measurable.

Due to the nature of the approximation step (41), the proof of Theorem 7 does not proceed by direct estimation of the error between $\tilde{u}_k^\pm$ and $u(.,t_k)$, but indirectly, by showing that limits of the scheme satisfy (27).

We will now present a different Riemann-based approximation procedure, introduced first in [6, Lemma 3.6], and refined in [7,8]. This approximation allows direct control of the error by using the distance Δ defined in (28). Intuitively, errors accumulate during approximation steps, but might be amplified by the resolution steps. The key properties of our approximation are that the total error accumulated during the approximation step is negligible as $\varepsilon \to 0$, and the error is not amplified by the resolution step, because Δ does not increase along entropy solutions, see Proposition 3*(v)*.

Theorem 8 ([6, Theorem 3.1]). *Assume $(T_t)_{t\geq 0}$ is a semigroup on the set of bounded $\mathcal{R}$-valued functions, with the following properties:*

1) For any Riemann initial condition u_0, $t \mapsto u_t = T_t u_0$ is the entropy solution to (3) with Cauchy datum u_0.
2) (Finite speed of propagation). There is a constant v such that, for any $a, b \in \mathbb{R}$, any two initial conditions u_0 and u_1 coinciding on $[a;b]$, and any $t < (b-a)/(2V)$, $u_t = T_t u_0$ and $v_t = T_t v_0$ coincide on $[a+Vt; b-Vt]$.
3) (Time continuity). For every bounded initial condition u_0 with bounded support and every $t \geq 0$, $\lim_{\varepsilon\to 0+}\Delta(T_t u_0, T_{t+\varepsilon} u_0) = 0$.
4) (Stability). For any bounded initial conditions u_0 and v_0, with bounded support, $\Delta(T_t u_0, T_t v_0) \leq \Delta(u_0, v_0)$.

Then, for any bounded u_0, $t \mapsto T_t u_0$ is the entropy solution to (3) with Cauchy datum u_0.

A crucial point is that properties 1)–4) in the above Theorem 8 hold at particle level, where this will allow us to mimic the scheme. The proof of Theorem 8 relies on the following uniform approximation (in the sense of distance Δ) by step functions, which is also important at particle level.

Lemma 5 ([8, Lemma 4.2]). *Assume $u_0(.)$ is a.e. $\mathcal{R}$-valued, has bounded support and finite variation, and let $(x,t) \mapsto u(x,t)$ be the entropy solution to (3) with Cauchy datum $u_0(.)$. For every $\varepsilon > 0$, let $\mathcal{P}_\varepsilon$ be the set of piecewise constant $\mathcal{R}$-valued functions on $\mathbb{R}$ with compact support and step lengths at least ε, and set*

$$\delta_\varepsilon(t) := \varepsilon^{-1} \inf\{\Delta(u(.), u(., t)) : u(.) \in \mathcal{P}_\varepsilon\}$$

Then there is a sequence $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that δ_{ε_n} converges to 0 uniformly on any bounded subset of $\mathbb{R}^+$.

5 Proof of Hydrodynamics

In this section, based on [9, Section 4], we prove the hydrodynamic limit in the quenched disordered setting, that is Theorem 3, following the strategy introduced in [6,7] and significantly strengthened in [8] and [9]. First, we prove the hydrodynamic limit for $\mathcal{R}^Q$-valued Riemann initial conditions (the so-called Riemann problem), and then use a constructive scheme to mimic the proof of Theorem 8 at microscopic level.

5.1 Riemann Problem

Let $\lambda, \rho \in \mathcal{R}^Q$ with $\lambda < \rho$ (for $\lambda > \rho$ replace infimum with supremum below). We first need to derive hydrodynamics for the Riemann initial condition $R_{\lambda,\rho}$ defined in (34). Microscopic Riemann states with profile (34) can be constructed using the following lemma.

Lemma 6 ([9, Lemma 4.1]). *There exist random variables* α *and* $(\eta^\rho : \rho \in \mathcal{R}^Q)$ *on a probability space* $(\Omega_{\mathbf{A}}, \mathcal{F}_{\mathbf{A}}, \mathbb{P}_{\mathbf{A}})$ *such that*

$$(\alpha, \eta^\rho) \sim \nu^{Q,\rho}, \;\; \alpha \sim Q$$
$$\mathbb{P}_{\mathbf{A}} - a.s., \;\; \rho \mapsto \eta^\rho \text{ is nondecreasing}$$

Let $\overline{\nu}^{Q,\lambda,\rho}$ denote the distribution of $(\alpha, \eta^\lambda, \eta^\rho)$, and $\overline{\nu}_\alpha^{\lambda,\rho}$ the conditional distribution of $(\alpha, \eta^\lambda, \eta^\rho)$ given α. Recall the definition (13) of the space-time shift θ_{x_0,t_0} on Ω for $(x_0, t_0) \in \mathbb{Z} \times \mathbb{R}^+$. We now introduce an extended shift θ' on $\Omega' = \mathbf{A} \times \mathbf{X}^2 \times \Omega$. If $\omega' = (\alpha, \eta, \xi, \omega)$ denotes a generic element of Ω', we set

$$\theta'_{x,t}\omega' = (\tau_x \alpha, \tau_x \eta_t(\alpha, \eta, \omega), \tau_x \eta_t(\alpha, \xi, \omega), \theta_{x,t}\omega)$$

It is important to note that this shift incorporates disorder. Let $T : \mathbf{X}^2 \to \mathbf{X}$ be given by

$$T(\eta, \xi)(x) = \eta(x)\mathbf{1}_{\{x<0\}} + \xi(x)\mathbf{1}_{\{x\geq 0\}}$$

A strong (that is almost sure with respect to the Poisson space) form of hydrodynamic limit for Riemann data can now be stated as follows.

Proposition 6 ([9, Proposition 4.1]). *Set, for* $t \geq 0$,

$$\beta_t^N(\omega')(dx) := \pi^N(\eta_t(\alpha, T(\eta, \xi), \omega))(dx)$$

For all $t > 0$, $s_0 \geq 0$ *and* $x_0 \in \mathbb{R}$, *we have that, for* Q*-a.e.* $\alpha \in \mathbf{A}$,

$$\lim_{N\to\infty} \beta_{Nt}^N(\theta'_{\lfloor Nx_0 \rfloor, Ns_0}\omega')(dx) = R_{\lambda,\rho}(.,t)dx, \quad \overline{\nu}_\alpha^{\lambda,\rho} \otimes \mathbb{P}\text{-}a.s.$$

Proposition 6 will follow from a law of large numbers for currents. Let $x_. = (x_t, t \geq 0)$ be a $\mathbb{Z}$-valued *càdlàg* random path, with $|x_t - x_{t^-}| \leq 1$, independent of the Poisson measure ω. We define the particle current seen by an observer travelling along this path by

$$\varphi_t^{x_.}(\alpha, \eta_0, \omega) = \varphi_t^{x_.,+}(\alpha, \eta_0, \omega) - \varphi_t^{x_.,-}(\alpha, \eta_0, \omega) + \widetilde{\varphi}_t^{x_.}(\alpha, \eta_0, \omega) \tag{42}$$

where $\varphi_t^{x_.,\pm}(\alpha, \eta_0, \omega)$ count the number of rightward/leftward crossings of $x_.$ due to particle jumps, and $\widetilde{\varphi}_t^{x_.}(\alpha, \eta_0, \omega)$ is the current due to the self-motion of the observer. We shall write φ_t^v in the particular case $x_t = \lfloor vt \rfloor$. Set $\phi_t^v(\omega') := \varphi_t^v(\alpha, T(\eta, \xi), \omega)$. Note that for $(v, w) \in \mathbb{R}^2$,

$$\beta_{Nt}^N(\omega')([v, w]) = t(Nt)^{-1}(\phi_{Nt}^{v/t}(\omega') - \phi_{Nt}^{w/t}(\omega')) \tag{43}$$

We view (43) as a microscopic analogue of (38). Thus, Proposition 6 boils down to showing that each term of (43) converges to its counterpart in (38).

Proposition 7 **([9, Proposition 4.2]).** *For all* $t > 0$, $a \in \mathbb{R}^+, b \in \mathbb{R}$ *and* $v \in \mathbb{R}$,

$$\lim_{N\to\infty}(Nt)^{-1}\phi^v_{Nt}(\theta'_{\lfloor bN\rfloor,aN}\omega') = \mathcal{G}_v(\lambda,\rho) \qquad \overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}-a.s.$$

where $\mathcal{G}_v(\lambda,\rho)$ *is defined by* (37).

To prove Proposition 7, we introduce a probability space Ω^+, whose generic element is denoted by ω^+, on which is defined a Poisson process $(N_t(\omega^+))_{t\geq 0}$ with intensity $|v|$ ($v \in \mathbb{R}$). Denote by $\mathbb{P}^+$ the associated probability. Set

$$\begin{aligned} x^N_s(\omega^+) &:= (\mathrm{sgn}(v))\left[N_{aN+s}(\omega^+) - N_{aN}(\omega^+)\right] \\ \widetilde{\eta}^N_s(\alpha,\eta_0,\omega,\omega^+) &:= \tau_{x^N_s(\omega^+)}\eta_s(\alpha,\eta_0,\omega) \\ \widetilde{\alpha}^N_s(\alpha,\omega^+) &:= \tau_{x^N_s(\omega^+)}\alpha \end{aligned} \tag{44}$$

Thus $(\widetilde{\alpha}^N_s,\widetilde{\eta}^N_s)_{s\geq 0}$ is a Feller process with generator

$$L^v = \mathfrak{L} + S^v, \quad S^v f(\alpha,\zeta) = |v|\left[f(\tau_{\mathrm{sgn}(v)}\alpha,\tau_{\mathrm{sgn}(v)}\zeta) - f(\alpha,\zeta)\right]$$

for f local and $\alpha \in \mathbf{A}$, $\zeta \in \mathbf{X}$. Since any translation invariant measure on $\mathbf{A}\times\mathbf{X}$ is stationary for the pure shift generator S^v, we have $\mathcal{I}_{\mathfrak{L}}\cap\mathcal{S} = \mathcal{I}_{L^v}\cap\mathcal{S}$. Define the time and space-time empirical measures (where $\varepsilon > 0$) by

$$\begin{aligned} m_{tN}(\omega',\omega^+) &:= (Nt)^{-1}\int_0^{tN}\delta_{(\widetilde{\alpha}^N_s(\alpha,\omega^+),\widetilde{\eta}^N_s(\alpha,T(\eta,\xi),\omega,\omega^+))}ds \\ m_{tN,\varepsilon}(\omega',\omega^+) &:= |\mathbb{Z}\cap[-\varepsilon N,\varepsilon N]|^{-1}\sum_{x\in\mathbb{Z}:\,|x|\leq\varepsilon N}\tau_x m_{tN}(\omega',\omega^+) \end{aligned} \tag{45}$$

Notice that there is a disorder component we cannot omit in the empirical measure, although ultimately we are only interested in the behavior of the η-component. Let $\mathcal{M}^Q_{\lambda,\rho}$ denote the compact set of probability measures $\mu(d\alpha,d\eta) \in \mathcal{I}_{\mathfrak{L}}\cap\mathcal{S}$ such that μ has α-marginal Q, and $\nu^{Q,\lambda} \ll \mu \ll \nu^{Q,\rho}$. By Proposition 2,

$$\mathcal{M}^Q_{\lambda,\rho} = \left\{\nu(d\alpha,d\eta) = \int \nu^{Q,r}(d\alpha,d\eta)\gamma(dr) : \gamma \in \mathcal{P}([\lambda,\rho]\cap\mathcal{R}^Q)\right\}$$

The key ingredients for Proposition 7 are the following lemmas.

Lemma 7 **([9, Lemma 4.2]).** *The function* $\phi^v_t(\alpha,\eta,\xi,\omega)$ *is increasing in* η, *decreasing in* ξ.

Lemma 8 **([9, Lemma 4.3]).** *With* $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$*-probability one, every subsequential limit as* $N\to\infty$ *of* $m_{tN,\varepsilon}(\theta'_{\lfloor bN\rfloor,aN}\omega',\omega^+)$ *lies in* $\mathcal{M}^Q_{\lambda,\rho}$.

Lemma 7 is a consequence of the monotonicity property (11). Lemma 8 relies in addition on a space-time ergodic theorem and on a general uniform large deviation upper bound for space-time empirical measures of Markov processes. We state these two results before proving Proposition 7.

Proposition 8 ([8, Proposition 2.3]). *Let $(\eta_t)_{t\geq 0}$ be a Feller process on $\mathbf{X}$ with a translation invariant generator L, that is*

$$\tau_1 L \tau_{-1} = L$$

Assume further that

$$\mu \in (\mathcal{I}_L \cap \mathcal{S})_e$$

where $\mathcal{I}_L$ denotes the set of invariant measures for L. Then, for any local function f on $\mathbf{X}$, and any $a > 0$

$$\lim_{\ell\to\infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=0}^{\ell} \tau_i f(\eta_t) dt = \int f d\mu = \lim_{\ell\to\infty} \frac{1}{a\ell^2} \int_0^{a\ell} \sum_{i=-\ell}^{-1} \tau_i f(\eta_t) dt$$

a.s. with respect to the law of the process with initial distribution μ.

Lemma 9 ([8, Lemma 3.4]). *Let $\mathbf{P}_\nu^v$ denote the law of a Markov process $(\widetilde{\alpha}_., \widetilde{\xi}_.)$ with generator L^v and initial distribution ν. For $\varepsilon > 0$, let*

$$\pi_{t,\varepsilon} := |\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]|^{-1} \sum_{x\in\mathbb{Z}\cap[-\varepsilon t,\varepsilon t]} t^{-1} \int_0^t \delta_{(\tau_x \widetilde{\alpha}_s, \tau_x \widetilde{\xi}_s)} ds$$

Then, there exists a functional $\mathcal{D}_v$ which is nonnegative, l.s.c., and satisfies $\mathcal{D}_v^{-1}(0) = \mathcal{I}_{L^v}$, such that, for every closed subset F of $\mathcal{P}(\mathbf{A} \times \mathbf{X})$,

$$\limsup_{t\to\infty} t^{-1} \log \sup_{\nu\in\mathcal{P}(\mathbf{A}\times\mathbf{X})} \mathbf{P}_\nu^v \left(\pi_{t,\varepsilon}(\widetilde{\xi}_.) \in F\right) \leq -\inf_{\mu\in F} \mathcal{D}_v(\mu)$$

Proof of Proposition 7. We will show that

$$\liminf_{N\to\infty} (Nt)^{-1} \phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}(\omega') \geq \mathcal{G}_v(\lambda, \rho), \quad \overline{\nu}^{Q,\lambda,\rho} \otimes \mathbb{P}\text{-a.s.} \tag{46}$$

$$\limsup_{N\to\infty} (Nt)^{-1} \phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}(\omega') \leq \mathcal{G}_v(\lambda, \rho), \quad \overline{\nu}^{Q,\lambda,\rho} \otimes \mathbb{P}\text{-a.s.} \tag{47}$$

Step one: Proof of (46).
Setting $\varpi_{aN} = \varpi_{aN}(\omega') := T\left(\tau_{\lfloor bN \rfloor}\eta_{aN}(\alpha, \eta, \omega), \tau_{\lfloor bN \rfloor}\eta_{aN}(\alpha, \xi, \omega)\right)$, we have

$$(Nt)^{-1} \phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}(\omega') = (Nt)^{-1} \varphi_{tN}^v (\tau_{\lfloor bN \rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN \rfloor, aN}\omega)$$

Let, for every $(\alpha, \zeta, \omega, \omega^+) \in \mathbf{A} \times \mathbf{X} \times \Omega \times \Omega^+$ and $x_.^N(\omega^+)$ given by (44),

$$\psi_{tN}^{v,\varepsilon}(\alpha, \zeta, \omega, \omega^+) := |\mathbb{Z} \cap [-\varepsilon N, \varepsilon N]|^{-1} \sum_{y\in\mathbb{Z}:\,|y|\leq\varepsilon N} \varphi_{tN}^{x_.^N(\omega^+)+y}(\alpha, \zeta, \omega) \tag{48}$$

Note that $\lim_{N\to\infty}(Nt)^{-1} x_{tN}^N(\omega^+) = v$, $\mathbb{P}^+$-a.s., and that for two paths $y_., z_.$ (see (42)),

$$|\varphi_{tN}^{y_.}(\alpha, \eta_0, \omega) - \varphi_{tN}^{z_.}(\alpha, \eta_0, \omega)| \leq K\left(|y_{tN} - z_{tN}| + |y_0 - z_0|\right)$$

Hence the proof of (46) reduces to that of the same inequality where we replace $(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')$ by $(Nt)^{-1}\psi^{v,\varepsilon}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)$ and $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}$ by $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$. By definitions (18), (42) of flux and current, for any $\alpha\in\mathbf{A}$, $\zeta\in\mathbf{X}$,

$$\begin{aligned} M^{x,v}_{tN}(\alpha,\zeta,\omega,\omega^+) := \varphi^{x^N(\omega^+)+x}_{tN}(\alpha,\zeta,\omega) \\ -\int_0^{tN}\tau_x\big\{j(\widetilde{\alpha}^N_s(\alpha,\omega^+),\widetilde{\eta}^N_s(\alpha,\zeta,\omega,\omega^+)) \\ -v(\widetilde{\eta}^N_s(\alpha,\zeta,\omega,\omega^+))(\mathbf{1}_{\{v>0\}})\big\}ds \end{aligned}$$

is a mean 0 martingale under $\mathbb{P}\otimes\mathbb{P}^+$. Let

$$\begin{aligned} R^{\varepsilon,v}_{tN} &:= (Nt\,|\mathbb{Z}\cap[-\varepsilon N,\varepsilon N]|)^{-1}\sum_{x\in\mathbb{Z}:\,|x|\le\varepsilon N}M^{x,v}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)\\ &= (Nt)^{-1}\psi^{v,\varepsilon}_{tN}(\tau_{\lfloor bN\rfloor}\alpha,\varpi_{aN},\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)\\ &\quad -\int[j(\alpha,\eta)-v\eta(\mathbf{1}_{\{v>0\}})]m_{tN,\varepsilon}(\theta'_{\lfloor bN\rfloor,aN}\omega',\omega^+)(d\alpha,d\eta) \end{aligned} \tag{49}$$

where the last equality comes from (45), (48). The exponential martingale associated with $M^{x,v}_{tN}$ yields a Poissonian bound, uniform in (α,ζ), for the exponential moment of $M^{x,v}_{tN}$ with respect to $\mathbb{P}\otimes\mathbb{P}^+$. Since ϖ_{aN} is independent of $(\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)$ under $\overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$, the bound is also valid under this measure, and Borel–Cantelli's lemma implies $\lim_{N\to\infty}R^{\varepsilon,v}_{tN}=0$. From (49), Lemma 8 and Corollary 1, *(B2)* imply (46), as well as

$$\limsup_{N\to\infty}(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')\le\sup_{r\in[\lambda,\rho]\cap\mathcal{R}^Q}[G^Q(r)-vr],\quad \overline{\nu}^{Q,\lambda,\rho}\otimes\mathbb{P}\text{-a.s.} \tag{50}$$

Step two: Proof of (47). Let $r\in[\lambda,\rho]\cap\mathcal{R}^Q$. We define $\overline{\nu}^{Q,\lambda,r,\rho}$ as the distribution of $(\alpha,\eta^\lambda,\eta^r,\eta^\rho)$. With respect to this measure, by (46) and (50), we have the almost sure limit

$$\lim_{N\to\infty}(Nt)^{-1}\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\alpha,\eta^r,\eta^r,\omega)=G^Q(r)-vr$$

By Lemma 7,

$$\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')\le\phi^v_{tN}\circ\theta'_{\lfloor bN\rfloor,aN}(\alpha,\eta^r,\eta^r,\omega)$$

The result follows by continuity of G^Q and minimizing over r. □

5.2 Cauchy Problem

Using (31) and the fact that an arbitrary function can be approximated by a $\mathcal{R}^Q$-valued function with respect to the distance Δ defined by (28), the proof of Theorem 3 for general initial data u_0 can be reduced (see [7]) to the case of $\mathcal{R}^Q$-valued initial data by coupling and approximation arguments (see [8, Section 4.2.2]).

Proposition 9 ([9, Proposition 4.3]). *Assume (η_0^N) is a sequence of configurations such that: (i) there exists $C > 0$ such that for all $N \in \mathbb{N}$, η_0^N is supported on $\mathbb{Z} \cap [-CN, CN]$;*
(ii) $\pi^N(\eta_0^N) \to u_0(.)dx$ as $N \to \infty$, where u_0 has compact support, is a.e. $\mathcal{R}^Q$-valued and has finite space variation.
Let $u(.,t)$ denote the unique entropy solution to (16) with Cauchy datum $u_0(.)$. Then, $Q \otimes \mathbb{P}$-a.s. as $N \to \infty$,

$$\Delta^N(t) := \Delta(\pi^N(\eta_{Nt}^N(\alpha, \eta_0^N, \omega)), u(.,t)dx)$$

converges uniformly to 0 on $[0, T]$ for every $T > 0$.

Before proving this proposition, we state two crucial tools in their most complete form (see [38]), the macroscopic stability and the finite propagation property for the particle system. Macroscopic stability yields that the distance Δ defined in (28) is an "almost" nonincreasing functional for two coupled particle systems. It is thus a microscopic analogue of property (31) in Proposition 3. The finite propagation property is a microscopic analogue of Proposition 3, *iv)*. For the misanthropes process, it follows essentially from the finite mean assumption (*M4*).

Proposition 10 ([8, Proposition 4.2] with [38, Theorem 2]). *Assume $p(.)$ has a finite first moment and a positive mean. Then there exist constants $C > 0$ and $c > 0$, depending only on $b(.,.)$ and $p(.)$, such that the following holds. For every $N \in \mathbb{N}$, $(\eta_0, \xi_0) \in \mathbf{X}^2$ with $|\eta_0| + |\xi_0| := \sum_{x\in\mathbb{Z}}[\eta_0(x) + \xi_0(x)] < +\infty$, and every $\gamma > 0$, the event*

$$\forall t > 0 : \; \Delta(\pi^N(\eta_t(\alpha, \eta_0, \omega)), \pi^N(\eta_t(\alpha, \xi_0, \omega))) \le \Delta(\pi^N(\alpha, \eta_0), \pi^N(\alpha, \xi_0)) + \gamma \tag{51}$$

has $\mathbb{P}$-probability at least $1 - C(|\eta_0| + |\xi_0|)e^{-cN\gamma}$.

Lemma 10 ([38, Lemma 15]). *Assume $p(.)$ has a finite third moment. There exist a constant v, and a function $A(.)$ (satisfying $\sum_n A(n) < \infty$), depending only on $b(.,.)$ and $p(.)$, such that the following holds. For any $x, y \in \mathbb{Z}$, any $(\eta_0, \xi_0) \in \mathbf{X}^2$, and any $0 < t < (y - x)/(2v)$: if η_0 and ξ_0 coincide on the site interval $[x, y]$, then with $\mathbb{P}$-probability at least $1 - A(t)$, $\eta_s(\alpha, \eta_0, \omega)$ and $\eta_s(\alpha, \xi_0, \omega)$ coincide on the site interval $[x + vt, y - vt] \cap \mathbb{Z}$ for every $s \in [0, t]$.*

To prove Proposition 10, we work with a coupled process, and reduce the problem to analysing the evolution of labeled positive and negative discrepancies to control their coalescences. For this we order the discrepancies, we possibly relabel them according to their movements to favor coalescences, and define *windows*, which are space intervals on which coalescences are favored, in the same spirit as in [17]. The irreducibility assumption (that is (7) in the case of the misanthropes process) plays an essential role there. To take advantage of [17], we treat separately the movements of discrepancies corresponding to big jumps, which can be controlled thanks to the finiteness of the first moment (that is, assumption (*M4*)).

Proof of Proposition 9. By assumption *(ii)* of the proposition, $\lim_{N\to\infty}\Delta^N(0)=0$. Let $\varepsilon>0$, and $\varepsilon'=\varepsilon/(2V)$, for V given by (30). Set $t_k=k\varepsilon'$ for $k\le\kappa:=\lfloor T/\varepsilon'\rfloor$, $t_{\kappa+1}=T$. Since the number of steps is proportional to ε^{-1}, if we want to bound the total error, the main step is to prove

$$\limsup_{N\to\infty}\sup_{k=0,\dots,\mathcal{K}-1}\left[\Delta^N(t_{k+1})-\Delta^N(t_k)\right]\le 3\delta\varepsilon,\quad Q\otimes\mathbb{P}\text{-a.s.}\tag{52}$$

where $\delta:=\delta(\varepsilon)$ goes to 0 as ε goes to 0; the gaps between discrete times are filled by an estimate for the time modulus of continuity of $\Delta^N(t)$ (see [8, Lemma 4.5]).

Proof of (52). Since $u(.,t_k)$ has locally finite variation, by [8, Lemma 4.2], for all $\varepsilon>0$ we can find functions

$$v_k=\sum_{l=0}^{l_k}r_{k,l}\mathbf{1}_{[x_{k,l},x_{k,l+1})}$$

with $-\infty=x_{k,0}<x_{k,1}<\ldots<x_{k,l_k}<x_{k,l_k+1}=+\infty$, $r_{k,l}\in\mathcal{R}^Q$, $r_{k,0}=r_{k,l_k}=0$, such that $x_{k,l}-x_{k,l-1}\ge\varepsilon$, and

$$\Delta(u(.,t_k)dx,v_kdx)\le\delta\varepsilon\tag{53}$$

For $t_k\le t<t_{k+1}$, we denote by $v_k(.,t)$ the entropy solution to (16) at time t with Cauchy datum $v_k(.)$. The configuration $\xi^{N,k}$ defined on $(\Omega_{\mathbf{A}}\otimes\Omega,\mathcal{F}_{\mathbf{A}}\otimes\mathcal{F},\mathbb{P}_{\mathbf{A}}\otimes\mathbb{P})$ (see Lemma 6) by

$$\xi^{N,k}(\omega_{\mathbf{A}},\omega)(x):=\eta_{Nt_k}(\alpha(\omega_{\mathbf{A}}),\eta^{r_{k,l}}(\omega_{\mathbf{A}}),\omega)(x),\ \text{ if } \lfloor Nx_{k,l}\rfloor\le x<\lfloor Nx_{k,l+1}\rfloor$$

is a microscopic version of $v_k(.)$, since by Proposition 6 with $\lambda=\rho=r^{k,l}$,

$$\lim_{N\to\infty}\pi^N(\xi^{N,k}(\omega_{\mathbf{A}},\omega))(dx)=v_k(.)dx,\quad \mathbb{P}_{\mathbf{A}}\otimes\mathbb{P}\text{-a.s.}\tag{54}$$

We denote by $\xi_t^{N,k}(\omega_{\mathbf{A}},\omega)=\eta_t(\alpha(\omega_{\mathbf{A}}),\xi^{N,k}(\omega_{\mathbf{A}},\omega),\theta_{0,Nt_k}\omega)$ the evolved configuration starting from $\xi^{N,k}$. By triangle inequality,

$$\Delta^N(t_{k+1})-\Delta^N(t_k)\le\Delta\left[\pi^N(\eta^N_{Nt_{k+1}}),\pi^N(\xi^{N,k}_{N\varepsilon'})\right]-\Delta^N(t_k)\tag{55}$$

$$+\ \Delta\left[\pi^N(\xi^{N,k}_{N\varepsilon'}),v_k(.,\varepsilon')dx\right]\tag{56}$$

$$+\ \Delta(v_k(.,\varepsilon')dx,u(.,t_{k+1})dx)\tag{57}$$

To conclude, we rely on Properties (29), (51) and (31) of Δ: Since $\varepsilon'=\varepsilon/(2V)$, finite propagation property for (16) and for the particle system (see Proposition 3, *iv)* and Lemma 10) and Proposition 6 imply

$$\lim_{N\to\infty}\pi^N(\xi^{N,k}_{N\varepsilon'}(\omega_{\mathbf{A}},\omega))=v_k(.,\varepsilon')dx,\qquad \mathbb{P}_A\otimes\mathbb{P}\text{-a.s.}$$

Hence, the term (56) converges a.s. to 0 as $N \to \infty$. By Δ-stability for (16), the term (57) is bounded by $\Delta(v_k(.)dx, u(., t_k)dx) \le \delta\varepsilon$. We now consider the term (55). By macroscopic stability (Proposition 10), outside probability $e^{-CN\delta\varepsilon}$,

$$\Delta\left[\pi^N(\eta^N_{Nt_{k+1}}), \pi^N(\xi^{N,k}_{N\varepsilon'})\right] \le \Delta\left[\pi^N(\eta^N_{Nt_k}), \pi^N(\xi^{N,k})\right] + \delta\varepsilon \tag{58}$$

Thus the event (58) holds a.s. for N large enough. By triangle inequality,

$$\begin{aligned}
&\Delta\left[\pi^N(\eta^N_{Nt_k}), \pi^N(\xi^{N,k})\right] - \Delta^N(t_k)\\
&\le \Delta\left(u(., t_k)dx, v_k(.)dx\right) + \Delta\left[v_k(.)dx, \pi^N(\xi^{N,k})\right]
\end{aligned}$$

for which (53), (54) yield as $N \to \infty$ an upper bound $2\delta\varepsilon$, hence $3\delta\varepsilon$ for the term (55). □

6 Other Models Under a General Framework

As announced in Sect. 3, we first define in Subsect. 6.1, as in [5,9,47] a general framework (which encompasses all known examples), that we illustrate with our reference examples, the misanthropes process (with generator (4)), and the k-exclusion process (with generator (5)). Next in Subsect. 6.2, we study examples of more complex models thanks to this new framework.

6.1 Framework

This section is based on parts of [9, Sections 2, 5]. The interest of an abstract description is to summarize all details of the microscopic dynamics in a single mapping, hereafter denoted by $\mathcal{T}$. This mapping contains both the generator description of the dynamics and its graphical construction. Once a given model is written in this framework, all proofs can be done without any model-specific computations, only relying on the properties of $\mathcal{T}$.

Monotone Transformations. Given an environment $\alpha \in \mathbf{A}$, we are going to define a Markov generator whose associated dynamics can be generically understood as follows: we pick a location on the lattice and around this location, apply a random monotone transformation to the current configuration. Let $(\mathcal{V}, \mathcal{F}_{\mathcal{V}}, m)$ be a measure space, where m is a nonnegative finite measure. This space will be used to generate a monotone conservative transformation, that is a mapping $\mathcal{T} : \mathbf{X} \to \mathbf{X}$ such that:

(i) $\mathcal{T}$ is nondecreasing: that is, for every $\eta \in \mathbf{X}$ and $\xi \in \mathbf{X}$, $\eta \le \xi$ implies $\mathcal{T}\eta \le \mathcal{T}\xi$;
(ii) $\mathcal{T}$ acts on finitely many sites, that is, there exists a finite subset S of $\mathbb{Z}$ such that, for all $\eta \in \mathbf{X}$, $\mathcal{T}\eta$ only depends on the restriction of η to S, and coincides with η outside S;

(iii) $\mathcal{T}\eta$ is conservative, that is, for every $\eta \in \mathbf{X}$,

$$\sum_{x\in S}\mathcal{T}\eta(x)=\sum_{x\in S}\eta(x)$$

We denote by $\mathfrak{T}$ the set of monotone conservative transformations, endowed with the σ-field $\mathcal{F}_{\mathfrak{T}}$ generated by the evaluation mappings $\mathcal{T}\mapsto\mathcal{T}\eta$ for all $\eta\in\mathbf{X}$.

Definition of the Dynamics. In order to define the process, we specify a mapping

$$\mathbf{A}\times\mathcal{V}\to\mathfrak{T},\quad (\alpha,v)\mapsto\mathcal{T}^{\alpha,v}$$

such that for every $\alpha\in\mathbf{A}$ and $\eta\in\mathbf{X}$, the mapping $v\mapsto\mathcal{T}^{\alpha,v}\eta$ is measurable from $(\mathcal{V},\mathcal{F}_{\mathcal{V}},m)$ to $(\mathfrak{T},\mathcal{F}_{\mathfrak{T}})$. When m is a probability measure, this amounts to saying that for each $\alpha\in\mathbf{A}$, the mapping $v\mapsto\mathcal{T}^{\alpha,v}$ is a $\mathfrak{T}$-valued random variable. The transformation $\mathcal{T}^{\alpha,v}$ must be understood as applying a certain update rule around 0 to the current configuration, depending on the environment around 0. If $x\in\mathbb{Z}\setminus\{0\}$, we define

$$\mathcal{T}^{\alpha,x,v}:=\tau_{-x}\mathcal{T}^{\tau_x\alpha,v}\tau_x \tag{59}$$

This definition can be understood as applying the same update rule around site x, which involves simultaneous shifts of the initial environment and transformation.

We now define the Markov generator

$$L_\alpha f(\eta)=\sum_{x\in\mathbb{Z}}\int_{\mathcal{V}}\left[f\left(\mathcal{T}^{\alpha,x,v}\eta\right)-f(\eta)\right]m(dv) \tag{60}$$

As a result of (59), the generator (60) satisfies the commutation property (15).

Basic Examples. To illustrate the above framework, we come back to our reference examples of Sect. 2.2.

The misanthropes process. Let

$$\mathcal{V}:=\mathbb{Z}\times[0,1],\quad v=(z,u)\in\mathcal{V},\quad m(dv)=c^{-1}||b||_\infty p(dz)\lambda_{[0,1]}(du) \tag{61}$$

For $v=(z,u)\in\mathcal{V}$, $\mathcal{T}^{\alpha,v}$ is defined by

$$\mathcal{T}^{\alpha,v}\eta=\begin{cases}\eta^{0,z} & \text{if } u<\alpha(0)\dfrac{b(\eta(0),\eta(z))}{c^{-1}||b||_\infty}\\ \eta & \text{otherwise}\end{cases} \tag{62}$$

Once given $\mathcal{T}^{\alpha,v}$ in (62), we deduce $\mathcal{T}^{\alpha,x,v}$ from (59):

$$\mathcal{T}^{\alpha,x,v}\eta=\begin{cases}\eta^{x,x+z} & \text{if } u<\alpha(x)\dfrac{b(\eta(x),\eta(x+z))}{c^{-1}||b||_\infty}\\ \eta & \text{otherwise}\end{cases} \tag{63}$$

Though $\mathcal{T}^{\alpha,v}$ is the actual input of the model, from which $\mathcal{T}^{\alpha,x,v}$ follows, in the forthcoming examples, for the sake of readability, we will directly define $\mathcal{T}^{\alpha,x,v}$.

Monotonicity of the transformation $\mathcal{T}^{\alpha,x,z}$ given in (63) follows from assumption (*M2*). One can deduce from (60) and (63) that L_α is indeed given by (4).

The k-step exclusion process. Here we let $\mathcal{V}=\mathbb{Z}^k$ and m denote the distribution of the first k steps of a random walk on $\mathbb{Z}$ with increment distribution $p(.)$ absorbed at 0. We define, for $(x,v,\eta)\in\mathbb{Z}\times\mathcal{V}\times\mathbf{X}$, with $v=(z_1,\dots,z_k)$,

$$N(x,v,\eta)=\inf\{i\in\{1,\dots,k\}:\eta(x+z_i)=0\}$$

with the convention that $\inf\emptyset=+\infty$. We then set

$$\mathcal{T}^{\alpha,x,v}\eta=\begin{cases}\eta^{x,x+N(x,v,\eta)} & \text{if } N(x,v,\eta)<+\infty\\ \eta & \text{if } N(x,v,\eta)=+\infty\end{cases} \tag{64}$$

One can show that this transformation is monotone, either directly, or by application of Lemma 11, since the k-step exclusion is a particular k-step misanthropes process (see special case 2a below). Plugging (64) into (60) yields (5).

We now describe the so-called *graphical construction* of the system given by (60), that is its pathwise construction on a Poisson space. We consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ of locally finite point measures $\omega(dt,dx,dv)$ on $\mathbb{R}^+\times\mathbb{Z}\times\mathcal{V}$, where $\mathcal{F}$ is generated by the mappings $\omega\mapsto\omega(S)$ for Borel sets S of $\mathbb{R}^+\times\mathbb{Z}\times\mathcal{V}$, and $\mathbb{P}$ makes ω a Poisson process with intensity

$$M(dt,dx,dv)=\lambda_{\mathbb{R}^+}(dt)\lambda_{\mathbb{Z}}(dx)m(dv)$$

denoting by λ either the Lebesgue or the counting measure. We write $\mathbb{E}$ for expectation with respect to $\mathbb{P}$. There exists a unique mapping

$$(\alpha,\eta_0,t)\in\mathbf{A}\times\mathbf{X}\times\mathbb{R}^+\mapsto\eta_t=\eta_t(\alpha,\eta_0,\omega)\in\mathbf{X} \tag{65}$$

satisfying: *(a)* $t\mapsto\eta_t(\alpha,\eta_0,\omega)$ is right-continuous; *(b)* $\eta_0(\alpha,\eta_0,\omega)=\eta_0$; *(c)* the particle configuration is updated at points $(t,x,v)\in\omega$ (and only at such points; by $(t,x,v)\in\omega$ we mean $\omega\{(t,x,v)\}=1$) according to the rule

$$\eta_t(\alpha,\eta_0,\omega)=\mathcal{T}^{\alpha,x,v}\eta_{t^-}(\alpha,\eta_0,\omega) \tag{66}$$

The processes defined by (60) and (65)–(66) exist and are equal in law under general conditions given in [47], see also [11] for a summary of this construction).

Coupling and Monotonicity. The monotonicity of $\mathcal{T}^{\alpha,x,u}$ implies monotone dependence (11) with respect to the initial state. Thus, an arbitrary number of processes can be coupled via the graphical construction. This implies complete monotonicity and thus attractiveness. It is also possible to define the coupling of any number of processes using the transformation $\mathcal{T}$. For instance, in order to couple two processes, we define the coupled generator $\overline{L}_\alpha$ on $\mathbf{X}^2$ by

$$\overline{L}_\alpha f(\eta,\xi):=\sum_{x\in\mathbb{Z}}\int_{\mathcal{V}}\left[f\left(\mathcal{T}^{\alpha,x,v}\eta,\mathcal{T}^{\alpha,x,v}\xi\right)-f(\eta,\xi)\right]m(dv)$$

for any local function f on $\mathbf{X}^2$.

6.2 Examples

We refer the reader to [9, Section 5] for various examples of completely monotone models defined using this framework. We now review two of the models introduced in [9, Section 5], then present a new model containing all the other models in this paper, *the k-step misanthropes process.*

The Generalized Misanthropes Process. [9, Section 5.1]. Let $K \in \mathbb{N}$. Let $c \in (0,1)$, and $p(.)$ (resp. $P(.)$), be a probability distribution on $\mathbb{Z}$. Define $\mathbf{A}$ to be the set of functions $B : \mathbb{Z}^2 \times \{0, \dots, K\}^2 \to \mathbb{R}^+$ such that:

(GM1) For all $(x, z) \in \mathbb{Z}^2$, $B(x, z, ., .)$ satisfies assumptions (*M1*)–(*M3*);
(GM2) There exists a constant $C > 0$ and a probability measure $P(.)$ on $\mathbb{Z}$ such that $B(x, z, K, 1) \leq CP(z)$ for all $x \in \mathbb{Z}$.

Assumption *(GM2)* is a natural sufficient assumption for the existence of the process and graphical construction below. The shift operator τ_y on $\mathbf{A}$ is defined by $(\tau_y B)(x, z, n, m) = B(x + y, z, n, m)$ We generalize (4) by setting

$$L_B f(\eta) = \sum_{x,y \in \mathbb{Z}} B(x, y - x, \eta(x), \eta(y)) \left[f\left(\eta^{x,y}\right) - f(\eta) \right]$$

Thus, the environment at site x is given here by the jump rate function $B(x, ., .)$ with which jump rates from site x are computed.

For $v = (z, u)$, set $m(dv) = CP(dz)\lambda_{[0,1]}(du)$ in (61), and replace (62) with

$$\mathcal{T}^{B,x,v}\eta = \begin{cases} \eta^{x,x+z} & \text{if } u < \dfrac{B(x, z, \eta(x), \eta(x + z))}{CP(z)} \\ \eta & \text{otherwise} \end{cases}$$

A natural irreducibility assumption generalizing (7) is the existence of a constant $c > 0$ and a probability measure $p(.)$ on $\mathbb{Z}$ satisfying (7), such that

$$\forall z \in \mathbb{Z}, \quad \inf_{x \in \mathbb{Z}} b(x, z, 1, K - 1) \geq cp(z)$$

The basic model (4) is recovered for $B(x, z, n, m) = \alpha(x)p(z)b(n, m)$. Another natural example is the misanthropes process with bond disorder. Here $\mathbf{A} = [c, 1/c]^{\mathbb{Z}^2}$ with the space shift defined by $\tau_z \alpha = \alpha(. + z, . + z)$. We set $B(x, z, n, m) = \alpha(x, x + z)b(n, m)$, where $\alpha \in \mathbf{A}$. Assumption *(GM2)* is now equivalent to existence of a constant $C > 0$ and a probability measure $P(.)$ on $\mathbb{Z}$ such that $\alpha(x, y) \leq CP(y - x)$.

The microscopic flux function j_2 in (18) is given here by

$$j_2(\alpha, \eta) = \sum_{z \in \mathbb{Z}} zB(0, z, \eta(0), \eta(z))$$

that is well defined under the assumption that $P(.)$ has a finite first moment.

Asymmetric Exclusion Process with Overtaking. This example is a particular case of the generalized k-step K-exclusion studied in [9, Section 5.2, Example 5.4], see also the traffic flow model in [9, Section 5.3]. The former model is itself a special case of the k-step misanthropes process defined below.

Let $K = 1$, $k \in \mathbb{N}$, and $\mathfrak{K}$ denote the set of $(2k)$-tuples $(\beta^j)_{j\in\{-k,\dots,k\}\setminus\{0\}}$ such that $\beta^{j+1} \le \beta^j$ for every $j = 1, \dots, k-1$, $\beta^{j-1} \le \beta^j$ for every $j = -1, \dots, -k+1$, and $\beta^1 + \beta^{-1} > 0$. We define $\mathbf{A} = \mathfrak{K}^{\mathbb{Z}}$. An element of $\mathbf{A}$ is denoted by $\beta = (\beta_x^j)_{j\in\{-k,\dots,k\},x\in\mathbb{Z}}$. The dynamics of this model is defined informally as follows. A site $x \in \mathbb{Z}$ is chosen as the initial site, then a jump direction (right or left) is chosen, and in this direction, the particle jumps to the first available site if it is no more than k sites ahead. The jump occurs at rate β_x^j if the first available site is $x + j$. Let $\mathcal{V} = [0, 1] \times \{-1, 1\}$ and $m = \delta_1 + \delta_{-1}$. For $x \in \mathbb{Z}$ and $v \in \{-1, 1\}$, we set

$$N(x, v, \eta) := \inf\{i \in \{1, \dots, k\} : \eta(x + iv) = 0\}$$

with the usual convention $\inf\emptyset = +\infty$. The corresponding monotone transformation is defined for $(u, v) \in \mathcal{V}$ by

$$\mathcal{T}^{\beta,x,v}\eta = \begin{cases} \eta^{x,x+N(x,v,\eta)v} & \text{if} \quad N(x,v,\eta) < +\infty \text{ and } u \le \beta_x^{vN(x,v,\eta)} \\ \eta & \text{otherwise} \end{cases} \tag{67}$$

Monotonicity of the transformation $\mathcal{T}^{\alpha,x,z}$ is given by [9, Lemma 5.1], and is also a particular case of Lemma 11 below, which states the same property for the k-step misanthropes process. It follows from (60) and (67) that the generator of this process is given for $\beta \in \mathbf{A}$ by

$$\begin{aligned} L_\beta f(\eta) &= \sum_{x\in\mathbb{Z}} \eta(x) \sum_{j=1}^{k} \Bigg\{ \beta_x^j [1 - \eta(x+j)] \prod_{i=1}^{i=j-1} \eta(x+i) \\ &\qquad + \beta_x^{-j} [1 - \eta(x-j)] \prod_{i=1}^{i=j-1} \eta(x-i) \Bigg\} \end{aligned} \tag{68}$$

A sufficient irreducibility property replacing (7) is

$$\inf_{x\in\mathbb{Z}} \left(\beta_x^1 + \beta_x^{-1}\right) > 0$$

The microscopic flux function j_2 in (18) is given here by

$$\begin{aligned} j_2(\beta, \eta) &= \eta(0) \sum_{j=1}^{k} j\beta_0^j [1 - \eta(j)] \prod_{i=1}^{j-1} \eta(i) \\ &\qquad - \eta(0) \sum_{j=1}^{k} j\beta_0^{-j} [1 - \eta(j)] \prod_{i=1}^{j-1} \eta(i) \end{aligned} \tag{69}$$

with the convention that an empty product is equal to 1. For $\rho \in [0, 1]$, let $\mathcal{B}_\rho$ denote the Bernoulli distribution on $\{0, 1\}$. In the absence of disorder, that is when β^i_x does not depend on x, the measure ν_ρ defined by

$$\nu_\rho(d\eta) = \bigotimes_{x\in\mathbb{Z}} \mathcal{B}_\rho[d\eta(x)]$$

is invariant for this process. It follows from (69) that the macroscopic flux function for the model without disorder is given by

$$G(u) = (1-u)\sum_{j=1}^{k} j[\beta^j - \beta^{-j}]u^j$$

The k-step Misanthropes Process. In the sequel, an element of $\mathbb{Z}^k$ is denoted by $\underline{z} = (z_1, \ldots, z_k)$. Let $K \geq 1$, $k \geq 1$, $c \in (0,1)$.

Define $\mathcal{D}_0$ to be the set of functions $b : \{0,\ldots,K\}^2 \to \mathbb{R}^+$ such that $b(0,.) = b(.,K) = 0$, $b(n,m) > 0$ for $n > 0$ and $m < K$, and b is nondecreasing (resp. nonincreasing) w.r.t. its first (resp. second) argument. Let $\mathcal{D}$ denote the set of functions $b = (b^1,\ldots,b^k)$ from $\mathbb{Z}^k \times \{0,\ldots,K\}^2 \to (\mathbb{R}^+)^k$ such that $b^j(\underline{z},.,.) \in \mathcal{D}_0$ for each $j = 1,\ldots,k$, and

$$\forall j = 2,\ldots,k, \quad b^j(.,K,0) \leq b^{j-1}(.,1,K-1) \tag{70}$$

Let q be a probability distribution on $\mathbb{Z}^k$, and $b \in \mathcal{D}$. We define the (q,b) k-step misanthrope process as follows. A particle at x (if some) picks a q-distributed random vector $\underline{Z} = (Z_1,\ldots,Z_k)$, and jumps to the first site $x+Z_i$ ($i \in \{1,\ldots,k\}$) with strictly less than K particles along the path $(x+Z_1,\ldots,x+Z_k)$, if such a site exists, with rate $b^i(\underline{Z},\eta(x),\eta(x+Z_i))$. Otherwise, it stays at x.

Next, disorder is introduced: the environment is a field $\alpha = ((q_x,b_x) : x \in \mathbb{Z}) \in \mathbf{A} := (\mathcal{P}(\mathbb{Z}^k)\times\mathcal{D})^{\mathbb{Z}}$. For a given realization of the environment, the distribution of the path $\underline{Z}$ picked by a particle at x is q_x, and the rate at which it jumps to $x+Z_i$ is $b^i_x(\underline{Z},\eta(x),\eta(x+Z_i))$. The corresponding generator is given by

$$L_\alpha f(\eta) = \sum_{i=1}^{k}\sum_{x,y\in\mathbb{Z}} c^i_\alpha(x,y,\eta)\left[f(\eta^{x,y}) - f(\eta)\right] \tag{71}$$

for a local function f on $\mathbf{X}$, where (with the convention that an empty product is equal to 1)

$$c^i_\alpha(x,y,\eta) = \int \left[b^i_x(\underline{z},\eta(x),\eta(y))\mathbf{1}_{\{x+z_i=y\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{\eta(x+z_j)=K\}} \right] dq_x(\underline{z})$$

The distribution Q of the environment on $\mathbf{A}$ is assumed ergodic with respect to the space shift τ_y, where $\tau_y\alpha = ((q_{x+y},b_{x+y}) : x \in \mathbb{Z})$.

A sufficient condition for the existence of the process and graphical construction below is the existence of a probability measure $P(.)$ on $\mathbb{Z}$ and a constant $C > 0$ such that

$$\sup_{i=1,\ldots,k}\sup_{x\in\mathbb{Z}} q^i_x(.) \leq C^{-1}P(.) \tag{72}$$

where q_x^i denotes the i-th marginal of q_x. On the other hand, a natural irreducibility assumption sufficient for Proposition 2 and Theorem 3, is the existence of a constant $c > 0$, and a probability measure $p(.)$ on $\mathbb{Z}$ satisfying (72), such that

$$\inf_{x\in\mathbb{Z}} q_x^1(.) \geq cp(.)$$

To define a graphical construction, we set, for $(x, \underline{z}, \eta) \in \mathbb{Z}\times\mathbb{Z}^k \times \mathbf{X}$, $b \in \mathcal{D}_0$ and $u \in [0, 1]$,

$$\begin{aligned}
N(x,\underline{z},\eta) &= \inf\{i \in \{1,\dots,k\} : \eta(x+z_i) < K\} \ \text{ with} \inf\emptyset = +\infty \\
Y(x,\underline{z},\eta) &= \begin{cases} x + z_{N(x,\underline{z},\eta)} & \text{if } N(x,\underline{z},\eta) < +\infty \\ x & \text{if } N(x,\underline{z},\eta) = +\infty \end{cases} \\
\mathcal{T}_0^{x,\underline{z},b,u}\eta &= \begin{cases} \eta^{x,Y(x,\underline{z},\eta)} & \text{if } u < b^{N(x,\underline{z},\eta)}(\underline{z},\eta(x),\eta(Y(x,\underline{z},\eta))) \\ \eta & \text{otherwise} \end{cases}
\end{aligned} \tag{73}$$

Let $\mathcal{V} = [0, 1] \times [0, 1]$, $m = \lambda_{[0,1]} \otimes \lambda_{[0,1]}$. For each probability distribution q on $\mathbb{Z}^k$, there exists a mapping $F_q : [0, 1] \to \mathbb{Z}^k$ such that $F_q(V_1)$ has distribution q if V_1 is uniformly distributed on $[0, 1]$. Then the transformation $\mathcal{T}$ in (66) is defined by (with $v = (v_1, v_2)$ and $\alpha = ((q_x, \beta_x) : x \in \mathbb{Z})$)

$$\mathcal{T}^{\alpha,x,v}\eta = \mathcal{T}_0^{x,F_{q_x}(v_1),b_x(F_{q_x}(v_1),.,.),v_2}\eta$$

The definition of j_2 in (18), applied to the generator (71), yields

$$j_2(\alpha,\eta) = \sum_{z\in\mathbb{Z}} zc_\alpha(0,z,\eta)$$

where

$$c_\alpha(x,y,\eta) := \sum_{i=1}^{k} c_\alpha^i(x,y,\eta)$$

Special Cases

1. The generalized misanthropes process is recovered for $k = 1$, because then in (71) we have $c_\alpha^1(x, y, \eta) = q_x^1(y - x)b_x^1(y - x, \eta(x), \eta(y))$.
2. A *generalized disordered* k-step exclusion process is obtained if $K = 1$ and $b_x^j(\underline{z}, n, m) = \beta_x^j(\underline{z})n(1 - m)$. In this process, if site x is the initial location of an attempted jump, and a particle is indeed present at x, a random path of length k with distribution q_x is picked, and the particle tries to find an empty location along this path. If it finds none, then it stays at x. Previous versions of the k-step exclusion process are recovered if one makes special choices for the distribution q_x:
   2a. The usual k-step exclusion process with site disorder, whose generator was given by (5), corresponds to the case where q_x is the distribution of the first k steps of a random walk with kernel $p(.)$ absorbed at 0, and $\beta_x^j = \alpha(x)$.

2b. The exclusion process with overtaking, whose generator was given by (68), corresponds to the case where the random path is chosen as follows: first, one picks with equal probability a jumping direction (left or right); next, one moves in this direction by successive deterministic jumps of size 1.

3. For $K \geq 2$, the generalized k-step K-exclusion process ([9, Subsection 5.2]) corresponds to $b_x^j(\underline{z}, n, m) = \beta_x^j(\underline{z})\mathbf{1}_{\{n>0\}}\mathbf{1}_{\{m<K\}}$.

Returning to the general case, condition (70) is the relevant extension of the condition $\beta_x^j(\underline{z}) \leq \beta_x^{j-1}(\underline{z})$ in the exclusion process with overtaking. If $K \geq 2$, it means that *any* possible j-step jump has rate larger or equal than *any* $(j-1)$-step jump.

The monotonicity property (11) of the graphical construction, and thus the complete monotonicity of the process, is a consequence of the following lemma.

Lemma 11. *For every $(x, \underline{z}, u) \in \mathbb{Z} \times \mathbb{Z}^k \times [0, 1]$ and $b \in \mathcal{D}_0$, $\mathcal{T}_0^{x,\underline{z},b,u}$ is an increasing mapping from $\mathbf{X}$ to $\mathbf{X}$.*

Proof of Lemma 11. Let $(\eta, \xi) \in \mathbf{X}^2$ with $\eta \leq \xi$. To prove that $\mathcal{T}_0^{x,\underline{z},b,u}\eta \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi$, since η and ξ can only possibly change at sites x, $y := Y(x, \underline{z}, \eta)$ and $y' := Y(x, \underline{z}, \xi)$, it is sufficient to verify the inequality at these sites.

If $\xi(x) = 0$, then by (73), η and ξ are both unchanged by $\mathcal{T}_0^{x,\underline{z},b,u}$. If $\eta(x) = 0 < \xi(x)$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y') \geq \xi(y') \geq \eta(y') = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y')$.

Now assume $\eta(x) > 0$. Then $\eta \leq \xi$ implies $N(x, \underline{z}, \eta) \leq N(x, \underline{z}, \xi)$. If $N(x, \underline{z}, \eta) = +\infty$, η and ξ are unchanged. If $N(x, \underline{z}, \eta) < N(x, \underline{z}, \xi) = +\infty$, then $\mathcal{T}_0^{x,\underline{z},b,u}\eta = \eta^{x,y}$, and $\xi(y) = K$. Thus, $\mathcal{T}_0^{x,\underline{z},b,u}\eta(x) = \eta(x) - 1 \leq \xi(x) = \mathcal{T}_0^{x,\underline{z},b,u}\xi(x)$, and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.

In the sequel, we assume $N(x, \underline{z}, \eta)$ and $N(x, \underline{z}, \xi)$ both finite. Let

$$\beta := b^{N(x,\underline{z},\eta)}(\eta(x), \eta(y))$$

and

$$\beta' := b^{N(x,\underline{z},\xi)}(\xi(x), \xi(y')).$$

(1) Assume $N(x, \underline{z}, \eta) = N(x, \underline{z}, \xi) < +\infty$, then $y = y'$. If $u \geq \max(\beta, \beta')$, both η and ξ are unchanged. If $u < \min(\beta, \beta')$, $\mathcal{T}_0^{x,\underline{z},b,u}\eta = \eta^{x,y}$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi = \xi^{x,y}$, whence the conclusion. We are left to examine different cases where $\min(\beta, \beta') \leq u < \max(\beta, \beta')$.
   (a) If $\eta(x) = \xi(x)$, then $\beta' \leq \beta$, and $\beta' \leq u < \beta$ implies $\eta(y) < \xi(y)$. In this case, $\mathcal{T}_0^{x,\underline{z},b,u}\xi(x) = \xi(x) \geq \eta(x) > \mathcal{T}_0^{x,\underline{z},b,u}\eta(x)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.
   (b) If $\eta(x) < \xi(x)$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(x) \geq \xi(x) - 1 \geq \eta(x) \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(x)$. If $\beta \leq u < \beta'$, then $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) + 1 > \eta(y) = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$. If $\beta' \leq u < \beta$, then $\eta(y) < \xi(y)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) \geq \eta(y) + 1 = \mathcal{T}_0^{x,\underline{z},b,u}\eta(y)$.

(2) Assume $N(x, \underline{z}, \eta) < N(x, \underline{z}, \xi) < +\infty$, hence $\beta \geq \beta'$ by (70), and $\eta(y) < \xi(y) = K$. If $u \geq \beta$, η and ξ are unchanged. If $u < \beta'$, then $\mathcal{T}_0^{x,\underline{z},b,u}\eta(y) = \eta(y) + 1 \leq \xi(y) = \mathcal{T}_0^{x,\underline{z},b,u}\xi(y)$, and $\mathcal{T}_0^{x,\underline{z},b,u}\xi(y') = \xi(y') + 1 \geq \mathcal{T}_0^{x,\underline{z},b,u}\eta(y') = \eta(y')$. If $\beta' \leq u < \beta$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(x) = \eta(x) - 1 \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi(x)$ and $\mathcal{T}_0^{x,\underline{z},b,u}\eta(y) = \eta(y) + 1 \leq \mathcal{T}_0^{x,\underline{z},b,u}\xi(y) = \xi(y) = K$. $\square$

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Scheduling of Non-Colliding Random Walks

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To Chuck in celebration of his 70th birthday

Abstract. On the complete graph $\mathcal{K}_M$ with $M \geq 3$ vertices consider two independent discrete time random walks $\mathbb{X}$ and $\mathbb{Y}$, choosing their steps uniformly at random. A pair of trajectories $\mathbb{X} = \{X_1, X_2, \dots\}$ and $\mathbb{Y} = \{Y_1, Y_2, \dots\}$ is called *non-colliding*, if by delaying their jump times one can keep both walks at distinct vertices forever. It was conjectured by P. Winkler that for large enough M the set of pairs of non-colliding trajectories $\{\mathbb{X}, \mathbb{Y}\}$ has positive measure. N. Alon translated this problem to the language of coordinate percolation, a class of dependent percolation models, which in most situations is not tractable by methods of Bernoulli percolation. In this representation Winkler's conjecture is equivalent to the existence of an infinite open cluster for large enough M. In this paper we establish the conjecture building upon the renormalization techniques developed in [4].

Keywords: Dependent percolation · Renormalization

1 Introduction

Bernoulli percolation has been a paradigm model for spatial randomness for last half a century. The deep and rich understanding that emerged is a celebrated success story of contemporary probability. In the mean time several natural questions arising from mathematical physics and theoretical computer science has necessitated the study of models containing more complicated dependent structures, which are not amenable to the tools of Bernoulli percolation. Among them we could mention classical gas of interacting Brownian paths [19], loop

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soups [18] and random interlacements [1,20]. A particular subclass of models that has received attention is a class of "coordinate percolation" models, which were introduced, motivated by problems of statistical physics, in late eighties by B. Tóth under the name "corner percolation", later studied in [17], and in early nineties in theoretical computer science by P. Winkler, later studied in several its variants in [5,6,15,21]. Problems of embedding one random sequence into another can also be cast into this framework [4,9,11–14], which in turn is intimately related to quasi-isometries of random objects [4,16].

In this work we focus on one particular model in this class, introduced by Winkler, which in its original formulation relates to clairvoyant scheduling of two independent random walks on a complete graph. More precisely, on the complete graph $\mathcal{K}_M$ with $M \geq 3$ vertices consider two independent discrete time random walks $\mathbb{X}$ and $\mathbb{Y}$ which move by choosing steps uniformly at random. Two trajectories (realizations) $\mathbb{X} = \{X_1, X_2, \dots\}$ and $\mathbb{Y} = \{Y_1, Y_2, \dots\}$ are called *non-colliding*, if, knowing all steps of $\mathbb{X}$ and $\mathbb{Y}$, one can keep both walks on distinct vertices forever by delaying their jump-times appropriately. The question of interest here is whether the set of non-colliding pairs of trajectories have positive probability. For $M = 3$ the measure of non-colliding pairs is zero (see Corollary 3.4 [21]). It was conjectured by Winkler [6] that for large enough M, in particular it is believed for $M \geq 4$ based on simulations, the set of non-colliding trajectories $\{\mathbb{X}, \mathbb{Y}\}$ has positive measure. The question became prominent as the *clairvoyant demon problem*.

N. Alon translated this problem into the language of coordinate percolation. Namely, let $\mathbb{X} = (X_1, X_2, \dots)$ and $\mathbb{Y} = (Y_1, Y_2, \dots)$ be two i.i.d. sequences with

$$\mathbb{P}(X_i = k) = \mathbb{P}(Y_j = k) = \frac{1}{M} \text{ for } \ k = 1, 2, \dots, M \text{ and for } i, j = 1, 2, \dots.$$

Define an oriented percolation process on $\mathbb{Z}_+ \times \mathbb{Z}_+$: the vertex $(i_1, i_2) \in \mathbb{Z}^2_{>0}$ will be called "closed" if $X_{i_1} = Y_{i_2}$. Otherwise it is called "open". It is curious to notice that this percolation process (for M=2) was introduced much earlier by Diaconis and Freedman [7] in the completely different context of studying visually distinguishable random patterns in connection with Julesez's conjecture. It is easy to observe that a pair of trajectories $\{\mathbb{X}, \mathbb{Y}\}$ is non-colliding if and only if there is an open oriented infinite path starting at the vertex $(1, 1)$. The issue of settling Winkler's conjecture then translates to proving that for M sufficiently large, there is percolation with positive probability, which is our main result in this paper. For $\mathbb{X}$ and $\mathbb{Y}$ as above, we say $\mathbb{X} \longleftrightarrow \mathbb{Y}$ if there exists an infinite open oriented path starting from $(1, 1)$.

Theorem 1. *For all M sufficiently large, $\mathbb{P}(\mathbb{X} \longleftrightarrow \mathbb{Y}) > 0$, thus clairvoyant scheduling is possible.*

1.1 Related Works

This scheduling problem first appeared in the context of distributed computing [6] where it is shown that two independent random walks on a finite connected non-bipartite graph will collide in a polynomial time even if a scheduler tries to keep them apart, unless the scheduler is clairvoyant. In a recent work [2], instead of independent random walks, by allowing coupled random walks, it was shown that a large number of random walks can be made to avoid one another forever. In the context of clairvoyant scheduling of two independent walks, the non-oriented version of the oriented percolation process described above was studied independently in [21] and [3] where they establish that in the non-oriented model there is percolation with positive probability if and only if $M \geq 4$. In [8] it was established that, if there is percolation, the chance that the cluster dies out after reaching distance n must decay polynomially in n, which showed that, unlike the non-oriented models, this model was fundamentally different from Bernoulli percolation, where such decay is exponential.

In [4] a multi-scale structure was developed to tackle random embedding problems which can be recast in co-ordinate percolation framework. As a corollary of a general embedding theorem, it was proved there that an i.i.d. Bernoulli sequence can almost surely be embedded into another in a Lipschitz manner provided that the Lipschitz constant is sufficiently large. It also led to a proof of rough isometry of two one-dimensional Poisson processes as well as a new proof of Winkler's compatible sequence problem. In this work we build upon the methods of [4], using a similar multi-scale structure, but with crucial adaptations. An earlier proof of Theorem 1 appeared in [10] with a very difficult multi-scale argument. Our proof is different and we believe gives a clearer inductive structure. We also believe that our proof can be adapted to deal with this problem on several other graphs, as well as in the case where there are multiple random walks.

1.2 Outline of the Proof

Our proof relies on multi-scale analysis. The key idea is to divide the original sequences into blocks of doubly exponentially growing length scales $L_j = L_0^{\alpha^j}$, for $j \geq 1$, and at each of these levels j we have a definition of a "good" block. The multi-scale structure that we construct has a number of parameters, $\alpha, \beta, \delta, m, k_0, R$ and L_0 which must satisfy a number of relations described in the next subsection. Single characters in the original sequences $\mathbb{X}$ and $\mathbb{Y}$ constitute the level 0 blocks.

Suppose that we have constructed the blocks up to level j denoting the sequence of blocks of level j as $(X_1^{(j)}, X_2^{(j)} \ldots)$. In Sect. 2 we give a construction of $(j+1)$-level blocks out of j-level sub-blocks in such way that the blocks are independent and, apart from the first block, identically distributed. Construction of blocks at level 1 has slight difference from the general construction.

At each level we have a definition which distinguishes some of the blocks as good. This is designed in such a manner that at each level, if we look at the rectangle in the lattice determined by a good block X and a random block Y, then, with high probability, it will have many open paths with varying slopes through it. For a precise definition see Definitions 4 and 5. Having these paths with different slopes will help achieve improving estimates of the probability of the event of having a path from the bottom left corner to the top right corner of the lattice rectangle determined by random blocks X and Y, denoted by $[X \overset{c,c}{\longleftrightarrow} Y]$, at higher levels.

The proof then involves a series of recursive estimates at each level, given in Sect. 3. We require that at level j the probability of a block being good is at least $1 - L_j^{-\delta}$, so that the vast majority of blocks are good. Furthermore, we obtain tail bounds on $\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X)$ by showing that for $0 < p \leq 3/4 + 2^{-(j+3)}$,

$$\mathbb{P}\left(\mathbb{P}\left(X \overset{c,c}{\longleftrightarrow} Y \mid X\right) \leq p\right) \leq p^{m+2^{-j}} L_j^{-\beta},$$

where β and m are parameters mentioned at the beginning of this section. We show the similar bound for $\mathbb{Y}$-blocks as well. We also ask that the length of blocks satisfy an exponential tail estimate. The full inductive step is given in Theorem 2. Proving this constitutes the main work of the paper.

We use the key quantitative estimate provided by Lemma 14 which is taken from [4] (see Lemma 7.3, [4]), which bounds the probability of a block having: (a) an excessive length, (b) too many bad sub-blocks, (c) a particularly difficult collection of sub-blocks, where we quantify the difficulty of a collection of bad sub-blocks $\{X_i\}_{i=1}^k$ by the value of $\prod_{i=1}^k \mathbb{P}[X_i \overset{c,c}{\longleftrightarrow} Y | X]$, where Y is a random block at the same level. In order to achieve the improvement on the tail bounds of $\mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y \mid X)$ at each level, we take advantage of the flexibility in trying a large number of potential positions to cross the rectangular strips determined by each member of a small collection of bad sub-blocks, obtained by using the recursive estimates on probabilities of existence of paths of varying slopes through rectangles determined by collections of good sub-blocks.

To this effect we also borrow the notion of generalised mappings developed in [4] to describe such potential mappings. Our analysis is split into 5 different cases. To push through the estimate of the probability of having many open paths of varying slopes at a higher level, we make some finer geometric constructions. To complete the proof we note that $X_1^{(j)}$ and $Y_1^{(j)}$ are good for all j with positive probability. Using the definition of good blocks and a compactness argument we conclude the existence of an infinite open path with positive probability.

1.3 Parameters

Our proof involves a collection of parameters $\alpha, \beta, \delta, k_0, m$ and R which must satisfy a system of constraints. The required constraints are

$$\alpha > 6,\ \delta > 2\alpha \vee 48,\ \beta > \alpha(\delta + 1),\ m > 9\alpha\beta,\ k_0 > 36\alpha\beta,\ R > 6(m + 1).$$

To fix on a choice we will set

$$\alpha = 10,\ \delta = 50,\ \beta = 600,\ m = 60000,\ k_0 = 300000,\ R = 400000. \tag{1}$$

Given these choices we then take L_0 to be a sufficiently large integer. We did not make a serious attempt to optimize the parameters or constraints, sometimes for the sake of clarity of exposition.

Organization of the Paper

Rest of this paper is organised as follows. In Sect. 2 we describe our block constructions and formally define good blocks. In Sect. 3 we state the main recursive theorem and show that it implies Theorem 1. In Sect. 4 we construct a collection of paths across a block which we shall use to improve the recursive estimates from one scale to the next. In Sect. 6 we prove the main recursive tail estimate for the corner to corner connection probabilities. Section 7 and Sect. 8 are devoted to proving estimates for corner to side and side to side connection probabilities respectively. In Sect. 9 we show that good blocks have the required inductive properties thus completing the induction.

2 The Multi-scale Structure

Our strategy for the proof of Theorem 1 is to partition the sequences $\mathbb{X}$ and $\mathbb{Y}$ into blocks at each level $j \geq 1$. For each $j \geq 1$, we write $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \ldots)$ where we call each $X_i^{(j)}$ a level j $\mathbb{X}$-block, similarly we write $\mathbb{Y} = (Y_1^{(j)}, Y_2^{(j)}, \ldots)$. Most of the time we would clearly state that something is a level j block and drop the superscript j. Each of the $\mathbb{X}$-block (resp. $\mathbb{Y}$-block) at level $(j+1)$ is a concatenation of a number of level j $\mathbb{X}$-blocks, where the level 0 blocks are just the elements of the original sequence.

2.1 Recursive Construction of Blocks

Level 1 blocks are constructed inductively as follows:

Suppose the first k blocks $X_1^{(1)}, \ldots, X_k^{(1)}$ at level 1 have already been constructed and suppose that the rightmost element of $X_k^{(1)}$ is $X_{n_k}^{(0)}$. Then $X_{n_k+1}^{(1)}$ consists of the elements $X_{n_k+1}^{(0)}, X_{n_k+2}^{(0)}, \ldots, X_{n_k+l}^{(0)}$ where

$$l = \min\left\{t \geq L_1 : X_{n_k+t}^{(0)} = 1 \bmod 4 \text{ and } X_{n_k+t+1}^{(0)} = 0 \bmod 4\right\}. \tag{2}$$

The same definition holds for $k = 0$, assuming $n_0 = -1$. Recall that $L_1 = L_0^{\alpha}$.

Similarly, suppose the first k $\mathbb{Y}$-blocks at level 1 are $Y_1^{(1)}, \ldots, Y_k^{(1)}$ and also suppose that the rightmost element of $Y_k^{(1)}$ is $Y_{n_k}^{(0)}$. Then $Y_{k+1}^{(1)}$ consists of the elements $Y_{n_k+1}^{(0)}, Y_{n_k+2}^{0)}, \ldots, Y_{n_k+l}^{(0)}$ where

$$l = \min\left\{t \geq L_1 : Y_{n_k+t}^{(0)} = 3 \bmod 4 \text{ and } Y_{n_k+t+1}^{(0)} = 2 \bmod 4\right\}. \tag{3}$$

We shall denote the length of an $\mathbb{X}$-block X (resp. a $\mathbb{Y}$-block Y) at level 1 by $L_X = L_1 + T_X^{(1)}$ (resp. $L_Y = L_1 + T_Y^{(1)}$). Notice that this construction, along with Assumption 1, ensures that the blocks at level one are independent and identically distributed.

At each level $j \geq 1$, we also have a recursive definition of "*good*" blocks (see Definition 7). Let $G_j^{\mathbb{X}}$ and $G_j^{\mathbb{Y}}$ denote the set of good $\mathbb{X}$-blocks and good

$\mathbb{Y}$-blocks at j-th level respectively. Now we are ready to describe the recursive construction of the blocks $X_i^{(j)}$ and $Y_i^{(j)}$ for $j \geq 2$.

The construction of blocks at level $j \geq 2$ is similar for both $\mathbb{X}$ and $\mathbb{Y}$ and we only describe the procedure to form the blocks for the sequence $\mathbb{X}$. Let us suppose we have already constructed the blocks of partition up to level j for some $j \geq 1$ and we have $X = (X_1^{(j)}, X_2^{(j)}, \ldots)$. Also assume we have defined the "good" blocks at level j, i.e., we know $G_j^{\mathbb{X}}$. We describe how to partition $\mathbb{X}$ into level $(j+1)$ blocks: $\mathbb{X} = (X_1^{(j+1)}, X_2^{(j+1)}, \ldots)$.

Suppose the first k blocks $X_1^{(j+1)}, \ldots, X_k^{(j+1)}$ at level $(j+1)$ has already been constructed and suppose that the rightmost level j-subblock of $X_k^{(j+1)}$ is $X_m^{(j)}$. Then $X_{k+1}^{(j+1)}$ consists of the sub-blocks $X_{m+1}^{(j)}, X_{m+2}^{(j)}, \ldots, X_{m+l+L_j^3}^{(j)}$ where $l > L_j^3 + L_j^{\alpha-1}$ is selected in the following manner. Let $W_{k+1,j+1}$ be a geometric random variable having $\mathrm{Geom}(L_j^{-4})$ distribution and independent of everything else. Then

$$l = \min\left\{s \geq L_j^3 + L_j^{\alpha-1} + W_{k+1,j+1} : X_{m+s+i} \in G_j^{\mathbb{X}} \text{ for } 1 \leq i \leq 2L_j^3\right\}.$$

That such an l is finite with probability 1 will follow from our recursive estimates. The case $k = 0$ is dealt with as before.

Put simply, our block construction mechanism at level $(j+1)$ is as follows: *Starting from the right boundary of the previous block, we include L_j^3 many sub-blocks, then further $L_j^{\alpha-1}$ many sub-blocks, then a* $\mathrm{Geom}(L_j^{-4})$ *many sub-blocks. Then we wait for the first occurrence of a run of $2L_j^3$ many consecutive good sub-blocks, and end our block at the midpoint of this run.*

We now record two simple but useful properties of the blocks thus constructed in the following observation. Once again a similar statement holds for $\mathbb{Y}$-blocks.

Observation 1. Let $\mathbb{X} = (X_1^{(j+1)}, X_2^{(j+1)}, \ldots) = (X_1^{(j)}, X_2^{(j)}, \ldots)$ denote the partition of $\mathbb{X}$ into blocks at levels $(j+1)$ and j respectively. Then the following hold.

1. Let $X_i^{(j+1)} = (X_{i_1}^{(j)}, X_{i_1+1}^{(j)}, \ldots X_{i_1+l}^{(j)})$. For $i \geq 1$, $X_{i_1+l+1-k}^{(j)} \in G_j^{\mathbb{X}}$ for each k, $1 \leq k \leq L_j^3$. Further, if $i > 1$, then $X_{i_1+k-1}^{(j)} \in G_j^{\mathbb{X}}$ for each k, $1 \leq k \leq L_j^3$. That is, all blocks at level $(j+1)$, except possibly the leftmost one, $X_1^{(j+1)}$, are guaranteed to have at least L_j^3 "good" level j sub-blocks at either end. Even $X_1^{(j+1)}$ ends in L_j^3 many good sub-blocks.
2. The blocks $X_1^{(j+1)}, X_2^{(j+1)}, \ldots$ are independently distributed. In fact, $X_2^{(j+1)}$, $X_3^{(j+1)}, \ldots$ are independently and identically distributed according to some law, say $\mu_{j+1}^{\mathbb{X}}$. Furthermore, conditional on the event $\{X_i^{(k)} \in G_k^{\mathbb{X}}$ for $i = 1, 2, \ldots, L_k^3,$ for all $k \leq j\}$, the $(j+1)$-th level blocks $X_1^{(j+1)}, X_2^{(j+1)}, \ldots$ are independently and identically distributed according to the law $\mu_{j+1}^{\mathbb{X}}$.

From now on whenever we say "a (random) $\mathbb{X}$-block at level j", we would imply that it has law $\mu_j^{\mathbb{X}}$, unless explicitly stated otherwise. Similarly let us denote the corresponding law of "a (random) $\mathbb{Y}$-block at level j" by $\mu_j^{\mathbb{Y}}$.

Also, for $j > 0$, let $\mu_{j,G}^{\mathbb{X}}$ denote the conditional law of an $\mathbb{X}$ block at level j, given that it is in $G_j^{\mathbb{X}}$. We define $\mu_{j,G}^{\mathbb{Y}}$ similarly.

We observe that we can construct a block with law $\mu_{j+1}^{\mathbb{X}}$ (resp. $\mu_{j+1}^{\mathbb{Y}}$) in the following alternative manner without referring to the sequence $\mathbb{X}$ (resp. $\mathbb{Y}$):

Observation 2. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent level j $\mathbb{X}$-blocks such that $X_i \sim \mu_{j,G}^{\mathbb{X}}$ for $1 \le i \le L_j^3$ and $X_i \sim \mu_j^{\mathbb{X}}$ for $i > L_j^3$. Now let W be a $Geom(L_j^{-4})$ variable independent of everything else. Define as before

$$l = \min\left\{i \ge L_j^3 + L_j^{\alpha-1} + W : X_{i+k} \in G_j^{\mathbb{X}} \text{ for } 1 \le k \le 2L_j^3\right\}.$$

Then $X = (X_1, X_2, \ldots, X_{l+L_j^3})$ has law $\mu_{j+1}^{\mathbb{X}}$.

Whenever we have a sequence $X_1, X_2, ...$ satisfying the condition in the observation above, we shall call X the (random) level $(j+1)$ block constructed from $X_1, X_2, ...$ and we shall denote the corresponding geometric variable by W_X and set $T_X = l - L_j^3 - L_j^{\alpha-1}$.

We still need to define good blocks, to complete the structure, we now move towards that direction.

2.2 Corner to Corner, Corner to Side and Side to Side Mapping Probabilities

Now we make some definitions that we are going to use throughout our proof. Let

$$X = \left(X_{s+1}^{(j)}, X_{s+2}^{(j)}, \ldots, X_{s+l_X}^{(j)}\right) = \left(X_{a_1}^{(0)}, \ldots, X_{a_2}^{(0)}\right)$$

be a level $(j+1)$ $\mathbb{X}$-block ($j \ge 1$) where $X_i^{(j)}$'s and $X_i^{(0)}$ are the level j sub-blocks and the level 0 sub-blocks constituting it respectively. Similarly let $Y = (Y_{s'+1}^{(j)}, Y_{s'+2}^{(j)}, ..., Y_{s+l_Y}^{(j)}) = (Y_{b_1}^{(0)}, \ldots, Y_{b_2}^{(0)})$ is a level $(j+1)$ $\mathbb{Y}$-block. Let us consider the lattice rectangle $[a_1, a_2] \times [b_1, b_2] \cap \mathbb{Z}^2$, and denote it by $X \times Y$. It follows from (2) and (3) that sites at all the four corners of this rectangle are open.

Definition 1 (Corner to Corner Path). *We say that there is a corner to corner path in $X \times Y$, denoted by*

$$X \stackrel{c,c}{\longleftrightarrow} Y,$$

if there is an open oriented path in $X \times Y$ from (a_1, b_1) to (a_2, b_2).

A site (x, b_2) and respectively a site (a_2, y), on the top, respectively on the right side of $X \times Y$, is called "reachable from bottom left site" if there is an open oriented path in $X \times Y$ from (a_1, b_1) to that site.

Further, the intervals $[a_1, a_2]$ and $[b_1, b_2]$ will be partitioned into "chunks"

$$\{C_r^X\}_{r\geq 1} \text{ and } \{C_r^Y\}_{r\geq 1}$$

respectively in the following manner. Let for any $\mathbb{X}$-block $\tilde{X}$ at any level $j \geq 1$,

$$\mathcal{I}(\tilde{X}) = \left\{a \in \mathbb{N} : \tilde{X} \text{ contains the level 0 block } X_a^{(0)}\right\}.$$

Let $X = (X_{s+1}^{(j)}, X_{s+2}^{(j)}, \ldots, X_{s+l_X}^{(j)})$, and $n_X := \lfloor l_X/L_j^4 \rfloor$. Similarly we define $n'_Y := \lfloor l_Y/L_j^4 \rfloor$.

Definition 2 (Chunks). *The discrete segment $C_k^X \subset \mathcal{I}(X)$ defined as*

$$C_k^X := \begin{cases} \cup_{t=(k-1)L_j^4+1}^{kL_j^4} \mathcal{I}(X_{s+t}^{(j)}), & k = 1, \ldots, n_X - 1; \\ \cup_{t=k_X L_j^4+1}^{l_X} \mathcal{I}(X_{s+t}^{(j)}), & k = n_X; \end{cases}$$

is called the k^{th} chunk of X.

By C^X and C^Y we denote the set of all chunks $\{C_k^X\}_{k=1}^{n_X}$ and $\{C_k^Y\}_{k=1}^{n'_Y}$ of X and Y respectively. In what follows the letters $\mathcal{T}, \mathcal{B}, \mathcal{L}, \mathcal{R}$ will stand for "top", "bottom", "left", and "right", respectively. Define:

$$C_{\mathcal{B}}^X = C^X \times \{1\}, \quad C_{\mathcal{T}}^X = C^X \times \{n'_Y\},$$
$$C_{\mathcal{L}}^Y = \{1\} \times C^Y, \quad C_{\mathcal{R}}^Y = \{n_X\} \times C^Y.$$

Definition 3 (Entry/Exit Chunk, Slope Conditions). *A pair $(C_k^X, 1) \in C_{\mathcal{B}}^X$, $k \in [L_j, n_X - L_j]$ is called an entry chunk (from the bottom) if it satisfies the slope condition*

$$\frac{1 - 2^{-(j+4)}}{R} \leq \frac{n'_Y - 1}{n_X - k} \leq R\left(1 + 2^{-(j+4)}\right).$$

Similarly, $(1, C_k^Y) \in C_{\mathcal{L}}^Y$, $k \in [L_j, n'_Y - L_j]$, is called an entry chunk (from the left) if it satisfies the slope condition

$$\frac{1 - 2^{-(j+4)}}{R} \leq \frac{n'_Y - k}{n_X - 1} \leq R\left(1 + 2^{-(j+4)}\right).$$

The set of all entry chunks is denoted by $\mathcal{E}_{in}(X, Y) \subseteq (C_{\mathcal{B}}^X \cup C_{\mathcal{L}}^Y)$. The set of all exit chunks $\mathcal{E}_{out}(X, Y)$ is defined in a similar fashion.

We call $(e_1, e_2) \in (C_B^X \cup C_L^Y) \times (C_T^X \cup C_R^Y)$ is an "entry-exit pair of chunks" if the following conditions are satisfied. Without loss of generality assume $e_1 = (C_k^X, 1) \in C_B^X$ and $e_2 = (n'_X, C_{k'}^Y) \in C_R^Y$. Then (e_1, e_2) is called an "entry-exit pair" if $k \in [L_j, n_X - L_j]$, $k' \in [L_j, n'_Y - L_j]$ and they satisfy the slope condition

$$\frac{1 - 2^{-(j+4)}}{R} \leq \frac{k' - 1}{n_X - k} \leq R\left(1 + 2^{-(j+4)}\right).$$

Let us denote the set of all "entry-exit pair of chunks" by $\mathcal{E}(X, Y)$.

Definition 4 (Corner to Side and Side to Corner Path). *We say that there is a corner to side path in* $X \times Y$*, denoted by*

$$X \overset{c,s}{\longleftrightarrow} Y$$

if for each $(C_k^X, n_X), (n'_Y, C_{k'}^Y) \in \mathcal{E}_2(X,Y)$

$$\#\Big\{a \in C_k^X : (a, b_2) \textit{ is reachable } \textit{from } (a_1, b_1) \textit{ in } X \times Y\Big\} \geq \left(\frac{3}{4} + 2^{-(j+5)}\right) |C_k^X|,$$

$$\#\Big\{b \in C_k^Y : (a_2, b) \textit{ is reachable } \textit{from } (a_1, b_1) \textit{ in } X \times Y\Big\} \geq \left(\frac{3}{4} + 2^{-(j+5)}\right) |C_k^Y|.$$

Side to corner paths in $X \times Y$, denoted $X \overset{s,c}{\longleftrightarrow} Y$ is defined in the same way except that in this case we want paths from the bottom or left side of the rectangle $X \times Y$ to its top right corner and use $\mathcal{E}_1(X,Y)$ instead of $\mathcal{E}_2(X,Y)$.

Condition S: Let $(e_1, e_2) \in \mathcal{E}(X,Y)$. Without loss of generality we assume $e_1 = (C_{k_1}^X, 1) \in C_B^X$ and $e_2 = (n_X, C_{k_2}^Y) \in C_R^Y$. (e_1, e_2) is said to satisfy condition S if there exists $A \subseteq C_{k_1}^X$ with $|A| \geq \left(3/4 + 2^{-(j+5)}\right)|C_{k_1}^X|$ and $B \subseteq C_{k_2}^Y$ with $|B| \geq \left(3/4 + 2^{-(j+5)}\right)|C_{k_2}^Y|$ such that for all $a \in A$ and for all $b \in B$ there exist an open path in $X \times Y$ from (a, b_1) to (a_2, b). Condition S is defined similarly for the other cases.

Definition 5 (Side to Side Path). *We say that there is a side to side path in* $X \times Y$*, denoted by*

$$X \overset{s,s}{\longleftrightarrow} Y$$

if each $(e_1, e_2) \in \mathcal{E}(X,Y)$ *satisfies condition* S*.*

It will be convenient for us to define corner to corner, corner to side, and side to side paths not only in rectangles determined by one $\mathbb{X}$-block and one $\mathbb{Y}$-block. Consider a $j+1$-level $\mathbb{X}$-block $X = (X_1, X_2, \ldots, X_n)$ and a $j+1$-level $\mathbb{Y}$ block $Y = (Y_1, \ldots, Y_{n'})$ where X_i, Y_i are j level subblocks constituting it. Let $\tilde{X}$ (resp. $\tilde{Y}$) denote a sequence of consecutive sub-blocks of X (resp. Y), e.g., $\tilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ for $1 \leq t_1 \leq t_2 \leq n$. Call $\tilde{X}$ to be a *segment* of X. Let $\tilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ be a segment of X and let $\tilde{Y} = (Y_{t'_1}, Y_{t'_1+1}, \ldots, Y_{t'_2})$ be a segment of Y. Let $\tilde{X} \times \tilde{Y}$ denote the rectangle in $\mathbb{Z}^2$ determined by $\tilde{X}$ and $\tilde{Y}$. Also let $X_{t_1} = (X_{a_1}^{(0)}, \ldots, X_{a_2}^{(0)})$, $X_{t_2} = (X_{a_3}^{(0)}, \ldots, X_{a_4}^{(0)})$, $Y_{t'_1} = (Y_{b_1}^{(0)}, \ldots, Y_{a_2}^{(0)})$, $Y_{t'_2} = (Y_{b_3}^{(0)}, \ldots, Y_{b_4}^{(0)})$.

- We denote by $\tilde{X} \overset{c,c}{\longleftrightarrow} \tilde{Y}$, the event that there exists an open oriented path from the bottom left corner to the top right corner of $\tilde{X} \times \tilde{Y}$.

- Let $\tilde{X} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}$ denote the event that

$$\Big\{\#\{b \in [b_3, b_4] : (a_2, b) \text{ is reachable from } (a_1, b_1)\} \geq \left(\frac{3}{4} + 2^{-(j+7/2)}\right)(b_4 - b_3)\Big\}$$

and

$$\Big\{\#\{a \in [a_3, a_4] : (a, b_4) \text{ is reachable from } (a_1, b_1)\} \geq \left(\frac{3}{4} + 2^{-(j+7/2)}\right)(a_4 - a_3)\Big\}.$$

$\tilde{X} \overset{s,c,*}{\longleftrightarrow} \tilde{Y}$ is defined in a similar manner.

- We set $\tilde{X} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}$ to be the following event. There exists $A \subseteq [a_1, a_2]$ with $|A| \geq (3/4 + 2^{-(j+7/2)})(a_2 - a_1)$, $A' \subseteq [a_3, a_4]$ with $|A'| \geq (3/4 + 2^{-(j+7/2)})(a_4 - a_3)$, $B \subseteq [b_1, b_2]$ with $|B| \geq (3/4 + 2^{-(j+7/2)})(b_2 - b_1)$ and $B' \subseteq [b_3, b_4]$ with $|B| \geq (3/4 + 2^{-(j+7/2)})(b_4 - b_3)$ such that for all $a \in A, a' \in A'$, $b \in B$, $b' \in B'$ we have that (a_4, b') and $(a', b_4))$ are reachable from (a, b_1) and (a_1, b).

Definition 6 (Corner to Corner Connection probability). *For $j \geq 1$, let X be an $\mathbb{X}$-block at level j and let Y be a $\mathbb{Y}$-block at level j. We define the corner to corner connection probability of X to be $S_j^{\mathbb{X}}(X) = \mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|X)$. Similarly we define $S_j^{\mathbb{Y}}(Y) = \mathbb{P}(X \overset{c,c}{\longleftrightarrow} Y|Y)$.*

As noted above the law of Y is $\mu_j^{\mathbb{Y}}$ in the definition of $S_j^{\mathbb{X}}$ and the law of X is $\mu_j^{\mathbb{X}}$ in the definition of $S_j^{\mathbb{Y}}$.

2.3 Good Blocks

To complete the description, we need to give the definition of "good" blocks at level j for each $j \geq 1$ which we have alluded to above. With the definitions from the preceding section, we are now ready to give the recursive definition of a "good" block as follows. As usual we only give the definition for $\mathbb{X}$-blocks, the definition for $\mathbb{Y}$ is similar.

Let $X^{(j+1)} = (X_1^{(j)}, X_2^{(j)}, \ldots, X_n^{(j)})$ be an $\mathbb{X}$ block at level $(j+1)$. Notice that we can form blocks at level $(j+1)$ since we have assumed that we already know $G_j^{\mathbb{X}}$.

Definition 7 (Good Blocks). *We say $X^{(j+1)}$ is a good block at level $(j+1)$ (denoted $X^{(j+1)} \in G_{j+1}^{\mathbb{X}}$) if the following conditions hold.*

(i) *It starts with* L_j^3 *good sub-blocks, i.e.,* $X_i^{(j)} \in G_j^{\mathbb{X}}$ *for* $1 \leq i \leq L_j^3$. *(This is required only for* $j > 0$*, as there are no good blocks at level* 0 *this does not apply for the case* $j = 0$*).*
(ii) $\mathbb{P}\left(X \stackrel{s,s}{\longleftrightarrow} Y|X\right) \geq 1 - L_{j+1}^{-2\beta}$
(iii) $\mathbb{P}\left(X \stackrel{c,s}{\longleftrightarrow} Y|X\right) \geq 9/10 + 2^{-(j+4)}$ *and* $\mathbb{P}\left(X \stackrel{s,c}{\longleftrightarrow} Y|X\right) \geq 9/10 + 2^{-(j+4)}$.
(iv) $S_j^{\mathbb{X}}(X) \geq 3/4 + 2^{-(j+4)}$.
(v) *The length of the block satisfies* $n \leq L_j^{\alpha-1} + L_j^5$.

3 Recursive Estimates

Our proof of the theorem depends on a collection of recursive estimates, all of which are proved together by induction. In this section we list these estimates for easy reference. The proof of these estimates are provided in the next few sections. We recall that for all $j > 0$, $L_j = L_{j-1}^{\alpha} = L_0^{\alpha^j}$.

Tail Estimate

I. Let $j \geq 1$. Let X be a $\mathbb{X}$-block at level j and let $m_j = m + 2^{-j}$. Then

$$\mathbb{P}\left(S_j^{\mathbb{X}}(X) \leq p\right) \leq p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \leq \frac{3}{4} + 2^{-(j+3)}. \tag{4}$$

Let Y be a $\mathbb{Y}$-block at level j. Then

$$\mathbb{P}\left(S_j^{\mathbb{Y}}(Y) \leq p\right) \leq p^{m_j} L_j^{-\beta} \quad \text{for} \quad p \leq \frac{3}{4} + 2^{-(j+3)}. \tag{5}$$

Length Estimate

II. For X an $\mathbb{X}$-block at level $j \geq 1$,

$$\mathbb{E}\left[\exp(L_{j-1}^{-6}(|X| - (2 - 2^{-j})L_j))\right] \leq 1. \tag{6}$$

Similarly for Y, a $\mathbb{Y}$-block at level j, we have

$$\mathbb{E}\left[\exp(L_{j-1}^{-6}(|Y| - (2 - 2^{-j})L_j))\right] \leq 1. \tag{7}$$

Probability of Good Blocks

III. Most blocks are “good”.

$$\mathbb{P}\left(X \in G_j^{\mathbb{X}}\right) \geq 1 - L_j^{-\delta}. \tag{8}$$

$$\mathbb{P}\left(Y \in G_j^{\mathbb{Y}}\right) \geq 1 - L_j^{-\delta}. \tag{9}$$

3.1 Consequences of the Estimates

For now let us assume that the estimates $I - III$ hold at some level j. Then we have the following consequences (we only state the results for $\mathbb{X}$, but similar results hold for $\mathbb{Y}$ as well).

Lemma 1. *Let us suppose* (4) *and* (8) *hold at some level* j*. Then for all* $X \in G_j^{\mathbb{X}}$ *we have the following.*

(i)

$$\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X\right] \geq \frac{3}{4} + 2^{-(j+7/2)}. \tag{10}$$

(ii)

$$\mathbb{P}\left[X \overset{c,s}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X\right] \geq \frac{9}{10} + 2^{-(j+7/2)},$$

$$\mathbb{P}\left[X \overset{s,c}{\longleftrightarrow} Y \mid Y \in G_j^{Y}, X\right] \geq \frac{9}{10} + 2^{-(j+7/2)}.$$

(iii)

$$\mathbb{P}\left[X \overset{s,s}{\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X\right] \geq 1 - L_j^{-\beta}. \tag{11}$$

Proof. We only prove (11), other two are similar. We have

$$\mathbb{P}\left[X \overset{s,s}{\not\longleftrightarrow} Y \mid Y \in G_j^{\mathbb{Y}}, X\right] \leq \frac{\mathbb{P}\left[X \overset{s,s}{\not\longleftrightarrow} Y \mid X\right]}{\mathbb{P}[Y \in G_j^{\mathbb{Y}}]} \leq L_j^{-2\beta}(1 - L_j^{-\delta})^{-1} \leq L_j^{-\beta}$$

which implies (11). □

3.2 The Main Recursive Theorem

We can now state the main recursive theorem.

Theorem 2 (Recursive Theorem). *There exist positive constants* α*,* β*,* δ*,* m*,* k_0 *and* R *such that for all large enough* L_0 *the following holds. If the recursive estimates (4), (5), (6), (7), (8), (9) and hold at level* j *for some* $j \geq 1$ *then all the estimates hold at level* $(j+1)$ *as well.*

We will choose the parameters as in Eq. (1). Before giving a proof of Theorem 2 we show how using this theorem we can prove the general theorem. To use the recursive theorem we first need to show that the estimates I, II and III hold at the base level $j = 1$. Because of the obvious symmetry between $\mathbb{X}$ and $\mathbb{Y}$ we need only show that (4), (6) and (8) hold for $j = 1$ if M is sufficiently large.

3.3 Proving the Recursive Estimates at Level 1

Without loss of generality we shall assume that M is a multiple of 4. Let

$$X = \left(X_1^{(0)}, X_{(2)}^{(0)}, \ldots, X_{(L_1 + T_X^{(1)})}^{(0)} \right) \sim \mu_1^{\mathbb{X}}$$

be an $\mathbb{X}$-block at level 1. Let

$$Y = \left(Y_1^{(0)}, Y_{(2)}^{(0)}, \ldots, Y_{(L_1 + T_Y^{(1)})}^{(0)} \right) \sim \mu_1^{\mathbb{Y}}.$$

We first have the following lemma which proves the length estimate in the base case.

Lemma 2. *Let X be an $\mathbb{X}$ block at level 1 as above. Then we have for all $l \geq 1$,*

$$\mathbb{P}\left(T_X^{(1)} \geq l\right) \leq \left(\frac{15}{16}\right)^{(l-1)/2}. \tag{12}$$

Further we have,

$$\mathbb{E}\left[\exp\left(L_0^{-6}\left(|X| - \frac{3}{2}L_1\right)\right)\right] \leq 1. \tag{13}$$

Proof. It follows from the construction of blocks at level 1 that $T_X^{(1)} \preceq 2V$ where V has a Geom(1/16) distribution, (12) follows immediately from this. To prove (13) we notice the following two facts.

$$\begin{aligned}\mathbb{P}\left[\exp(L_0^{-6}(|X| - 3/2L_1)) \geq \frac{1}{2}\right] \leq & \mathbb{P}\left[|X| \geq \frac{3}{2}L_1 - L_0^6 \log 2\right] \\ \leq & \mathbb{P}\left[|X| \geq 5/4L_1\right] \leq (15/16)^{L_1/10} \leq 1/4\end{aligned}$$

for L_0 large enough using (12).

Also, for all $x \geq 0$ using (12),

$$\mathbb{P}\left[\frac{|X| - 3/2L_1}{L_0^6} \geq x\right] \leq \left(\frac{15}{16}\right)^{xL_0^6/2 + L_1/4} \leq \frac{1}{10}\exp(-3x).$$

Now it follows from above that

$$\begin{aligned}\mathbb{E}[\exp(L_0^{-6}(|X| - 3/2L_1))] &= \int_0^\infty \mathbb{P}\left[\exp(L_0^{-6}(|X| - 3/2L_1)) \geq y\right] \, \mathrm{d}y \\ &= \int_0^{\frac{1}{2}} \mathbb{P}\left[\exp(L_0^{-6}(|X| - 3/2L_1)) \geq y\right] \, \mathrm{d}y \\ &\quad + \int_{\frac{1}{2}}^1 \mathbb{P}\left[\exp(L_0^{-6}(|X| - 3/2L_1)) \geq y\right] \, \mathrm{d}y \\ &\quad + \int_1^\infty \mathbb{P}\left[\exp(L_0^{-6}(|X| - 3/2L_1)) \geq y\right] \, \mathrm{d}y \\ &\leq \frac{1}{2} + \frac{1}{8} + \frac{1}{10}\int_0^\infty \mathbb{P}\left[(L_0^{-6}(|X| - 3/2L_1)) \geq z\right] \mathrm{e}^z \, \mathrm{d}z \\ &\leq \frac{1}{2} + \frac{1}{8} + \frac{1}{10} \leq 1.\end{aligned}$$

This completes the proof. □

Next we have the following theorem for the two remaining estimates.

Theorem 3. *For all sufficiently large L_0, if M (depending on L_0) is sufficiently large, then*

$$\mathbb{P}\left(S_j^{\mathbb{X}}(X) \leq p\right) \leq p^{m+2^{-1}} L_1^{-\beta} \quad \text{for} \quad p \leq \frac{3}{4} + 2^{-4}, \tag{14}$$

and

$$\mathbb{P}\left(X \in G_j^{\mathbb{X}}\right) \geq 1 - L_1^{-\delta}. \tag{15}$$

Before starting with the proof of Theorem 3, we define $\mathcal{A}_{X,1}^{(1)}$ to be the set of level 1 $\mathbb{X}$-blocks defined by

$$\mathcal{A}_{X,1}^{(1)} := \left\{X : T_X^{(1)} \leq 100 m L_1\right\}.$$

It follows from Lemma 2 that for L_0 sufficiently large

$$\mathbb{P}\left(X \in \mathcal{A}_{X,1}^{(1)}\right) \geq 1 - L_1^{-3\beta}. \tag{16}$$

We have the following two lemmas which will be used to prove Theorem 3.

Lemma 3. *For M sufficiently large, the following inequalities hold for each $X \in \mathcal{A}_{X,1}^{(1)}$.*

(i)

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] \geq \frac{3}{4} + 2^{-4}. \tag{17}$$

(ii)

$$\mathbb{P}[X \overset{c,s}{\longleftrightarrow} Y \mid X] \geq \frac{9}{10} + 2^{-4} \text{ and } \mathbb{P}[X \overset{s,c}{\longleftrightarrow} Y \mid X] \geq \frac{9}{10} + 2^{-4}.$$

(iii)

$$\mathbb{P}[X \overset{s,s}{\longleftrightarrow} Y \mid X] \geq 1 - L_1^{-2\beta}.$$

Proof. Let Y be a level 1 block constructed out of the sequence $Y_1^{(0)}, \ldots$. Let $\mathcal{C}(X)$ be the event

$$\left\{Y_i^{(0)} \neq X_{i'}^{(0)} \forall\, i, i', i \in [(10m+1)L_1], i' \in [L_1 + T_X^{(1)}]\right\}.$$

Let $\mathcal{E}$ denote the event

$$\left\{Y \in \mathcal{A}_{Y,1}^{(1)}\right\}.$$

Using the definition of the sequence $Y_1^{(0)}, \dots$ and the $\mathbb{Y}$-version of (16) we get that

$$\mathbb{P}[\mathcal{C}(X) \cap \mathcal{E} \mid X] \geq \left(1 - \frac{4(100m+1)L_1}{M}\right)^{(100m+1)L_1} - L_1^{-3\beta}$$
$$\geq \max\left\{1 - L_1^{-2\beta}, \frac{9}{10} + 2^{-4}\right\}$$

for M large enough.

Since $X \xleftrightarrow{s,s} Y$, $X \xleftrightarrow{s,c} Y$, $X \xleftrightarrow{c,s} Y$, $X \xleftrightarrow{c,c} Y$ each hold if $\mathcal{C}(X)$ and $\mathcal{E}$ both hold, the lemma follows immediately. □

Lemma 4. *If M is sufficiently large then*

$$\mathbb{P}\left(\mathbb{P}(X \xleftrightarrow{c,c} Y \mid X) \leq p\right) \leq p^{m+\frac{1}{2}} L_1^{-\beta} \quad \textit{for} \quad p \leq \frac{3}{4} + 2^{-4}.$$

Proof. Since L_1 is sufficiently large, (17) implies that it suffices to consider the case $p < 1/500$ and $X \notin \mathcal{A}_{X,1}^{(1)}$. We prove that for $p < 1/500$

$$\mathbb{P}\left[\mathbb{P}(X \xleftrightarrow{c,c} Y \mid X) \leq p, X \notin \mathcal{A}_{X,1}^{(1)}\right] \leq p^{m+2^{-1}} L_1^{-\beta}.$$

Let $\mathcal{E}(X)$ denote the event

$$\left\{T_Y^{(1)} = \left\lfloor \frac{1}{50m} T_X^{(1)} \right\rfloor, Y_i^{(0)} \neq 2 \bmod 4, \forall i \in \left[L_1 + 1, L_1 + T_Y^{(1)}\right]\right\}$$

It follows from definition that

$$\mathbb{P}[\mathcal{E}(X) \mid X] \geq \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{T_X^{(1)}/(50m)}.$$

Now let D_k denote the event that

$$D_k = \left\{Y_k^{(0)} \neq X_{i'}^{(0)} \forall\ i' \in \left[50km, 50(k+2)m \wedge T_Y^{(1)}\right]\right\}.$$

Let

$$D = \bigcap_{k=1}^{L_1 + T_Y^{(1)}} D_k.$$

It follows that

$$\mathbb{P}[D_k \mid X, \mathcal{E}(X)] \geq \left(1 - \frac{400m}{M}\right).$$

Since D_k are independent conditional on X and $\mathcal{E}(X)$

$$\mathbb{P}[D \mid X, \mathcal{E}(X)] \geq \left(1 - \frac{400m}{M}\right)^{L_1 + T_X^{(1)}/50m}.$$

It follows that

$$\mathbb{P}[X \overset{c,c}{\longleftrightarrow} Y \mid X] \geq \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{\frac{T_X^{(1)}}{50m}} \left(1 - \frac{200m}{M}\right)^{L_1 + T_X^{(1)}/50m}$$
$$\geq \frac{1}{20}\left(\frac{7}{10}\right)^{T_X^{(1)}/(50m)}$$

for M sufficiently large.

It follows that

$$\mathbb{P}\left[\mathbb{P}\left(X \overset{c,c}{\longleftrightarrow} Y \mid X\right) \leq p, X \notin \mathcal{A}_{X,1}^{(1)}\right] \leq \mathbb{P}\left[T_X^{(1)} \geq \left(50m\frac{\log 20p}{\log \frac{7}{10}}\right) \vee 100mL_1\right]$$
$$\leq \left(\frac{15}{16}\right)^{20m(\log 20p)/(\log \frac{7}{10})} \wedge \left(\frac{15}{16}\right)^{40mL_1}$$
$$\leq (20p)^{2m} \wedge \left(\frac{15}{16}\right)^{40mL_1} \leq p^{m+2^{-1}} L_1^{-\beta}$$

since $(15/16)^{10} < 7/10$ and L_0 is sufficiently large and $m > 100$. □

Proof of Theorem 3. We have established (14) in Lemma 4. That (15) holds follows from Lemma 3 and (16) noting $\beta > \delta$. □

Now we prove Theorem 1 using Theorem 2.

Proof of Theorem 1. Let $\mathbb{X} = (X_1, X_2, \ldots)$, $\mathbb{Y} = (Y_1, Y_2, \ldots)$ be as in the statement of the theorem. Let for $j \geq 1$, $\mathbb{X} = (X_1^{(j)}, X_2^{(j)}, \ldots)$ denote the partition of $\mathbb{X}$ into level j blocks as described above. Similarly let $\mathbb{Y} = (Y_1^{(j)}, Y_2^{(j)}, \ldots)$ denote the partition of $\mathbb{Y}$ into level j blocks. Let β, δ, m, R be as in Theorem 2. It follows form Theorem 3 that for all sufficiently large L_0, estimates I and II hold for $j = 1$ for all sufficiently large M. Hence the Theorem 2 implies that if L_0 is sufficiently large then I and II hold for all $j \geq 1$ for M sufficiently large.

Let $\mathcal{T}_j^{\mathbb{X}} = \{X_k^{(j)} \in G_j^{\mathbb{X}}, 1 \leq k \leq L_j^3\}$ be the event that the first L_j^3 blocks at level j are good. Notice that on the event $\cap_{k=1}^{j-1} \mathcal{T}_k^{\mathbb{X}}$, $X_1^{(j)}$ has distribution $\mu_j^{\mathbb{X}}$ by Observation 1 and so $\{X_i^{(j)}\}_{i \geq 1}$ is i.i.d. with distribution $\mu_j^{\mathbb{X}}$. Hence it follows from Eq. (8) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{X}} | \cap_{k=1}^{j-1} \mathcal{T}_k^{\mathbb{X}}) \geq (1 - L_j^{-\delta})^{L_j^3}$. Similarly defining $\mathcal{T}_j^{\mathbb{Y}} = \{Y_k^{(j)} \in G_j^{\mathbb{Y}}, 1 \leq k \leq L_j^3\}$ we get using (9) that $\mathbb{P}(\mathcal{T}_j^{\mathbb{Y}} | \cap_{k=0}^{j-1} \mathcal{T}_k^{\mathbb{Y}}) \geq (1 - L_j^{-\delta})^{L_j^3}$.

Let $\mathcal{A} = \cap_{j \geq 0}(\mathcal{T}_j^{\mathbb{X}} \cap \mathcal{T}_j^{\mathbb{Y}})$. It follows from above that $\mathbb{P}(\mathcal{A}) > 0$ since $\delta > 3$

Let $\mathcal{A}_{j+1} = \cap_{k \leq j}(\mathcal{T}_k^{\mathbb{X}} \cap \mathcal{T}_k^{\mathbb{Y}})$. It follows from (10) and (8) that

$$P\left[X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)} \mid \mathcal{A}_{j+1}\right] \geq \frac{3}{4} + 2^{-(j+9/2)} - 2L_{j+1}^{-\delta} \geq \frac{3}{4}.$$

Let $\mathcal{B}_{j+1}$ denote the event

$$\mathcal{B}_{j+1} = \{\exists \text{ an open path from } (0,0) \to (m,n) \text{ for some } m,n \geq L_{j+1}\}.$$

Then $\mathcal{B}_{j+1} \downarrow$ and $\mathcal{B}_{j+1} \supseteq \left\{X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)}\right\}$. It follows that

$$\mathbb{P}[\cap \mathcal{B}_{j+1}] \geq \liminf P\left[X_1^{(j+1)} \overset{c,c}{\longleftrightarrow} Y_1^{(j+1)}\right] \geq \frac{3}{4}\mathbb{P}[\mathcal{A}] > 0.$$

A standard compactness argument shows that $\cap\mathcal{B}_{j+1} \subseteq \{\mathbb{X} \leftrightarrow \mathbb{Y}\}$ and hence $\mathbb{P}[\mathbb{X} \leftrightarrow \mathbb{Y}] > 0$, which completes the proof of the theorem. □

The remainder of the paper is devoted to the proof of the estimates in the induction. Throughout these sections we assume that the estimates $I - III$ hold for some level $j \geq 1$ and then prove the estimates at level $j+1$. Combined they complete the proof of Theorem 2.

From now on, in every Theorem, Proposition and Lemma we state, we would implicitly assume the hypothesis that all the recursive estimates hold upto level j, the parameters satisfy the constraints described in Sect. 1.3 and L_0 is sufficiently large.

4 Geometric Constructions

We shall join paths across blocks at a lower level to form paths across blocks at a higher level. The general strategy will be as follows. Suppose we want to construct a path across $X \times Y$ where X, Y are level $j+1$ blocks. Using the recursive estimates at level j we know we are likely to find many paths across $X_i \times Y$ where X_i is a good sub-block of X. So we need to take special care to ensure that we can find open paths crossing bad-subblocks of X (or Y). To show the existence of such paths, we need some geometric constructions, which we shall describe in this section. We start with the following definition.

Definition 8 (Admissible Assignments). *Let $I_1 = [a+1, a+t] \cap \mathbb{Z}$ and $I_2 = [b+1, b+t'] \cap \mathbb{Z}$ be two intervals of consecutive positive integers. Let $I_1^* = [a + L_j^3 + 1, a + t - L_j^3] \cap \mathbb{Z}$ and $I_2^* = [b + L_j^3 + 1, b + t' - L_j^3] \cap \mathbb{Z}$. Also let $B \subseteq I_1^*$ and $B' \subseteq I_2^*$ be given. We call $\Upsilon(I_1, I_2, B, B') = (H, H', \tau)$ to be an admissible assignment at level j of (I_1, I_2) w.r.t. (B, B') if the following conditions hold.*

(i) $B \subseteq H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq I_1$ *and* $B' \subseteq H' = \{b_1 < b_2 < \cdots < b_\ell\} \subseteq I_2^*$ *with* $\ell = |B| + |B'|$.

(ii) $\tau(a_i) = b_i$ *and* $\tau(B) \cap B' = \emptyset$.

(iii) Set $a_0 = a, a_{\ell+1} = a + t + 1$; $b_0 = b, b_{\ell+1} = b + t' + 1$. *Then we have for all* $i \geq 0$

$$\frac{1 - 2^{-(j+7/2)}}{R} \leq \frac{b_{i+1} - b_i - 1}{a_{i+1} - a_i - 1} \leq R\left(1 + 2^{-(j+7/2)}\right).$$

The following proposition concerning the existence of admissible assignment follows from the results in Sect. 6 of [4]. We omit the proof.

Proposition 1. *Assume the set-up in Definition 8. We have the following.*

(i) Suppose we have

$$\frac{1-2^{-(j+4)}}{R} \le \frac{t'}{t} \le R\left(1+2^{-(j+4)}\right).$$

Also suppose $|B|, |B'| \le 3k_0$. *Then there exist* L_j^2 *level* j *admissible assignments* (H_i, H_i', τ_i) *of* (I_1, I_2) *w.r.t.* (B, B') *such that for all* $x \in B$, $\tau_i(x) = \tau_1(x) + i - 1$ *and for all* $y \in B'$, $\tau_i^{-1}(y) = \tau_1^{-1}(y) - i + 1$.

(ii) Suppose

$$\frac{3}{2R} \le \frac{t'}{t} \le \frac{2R}{3}$$

and $|B| \le (t - 2L_j^3)/(10R_j^+)$. *Then there exists an admissible assignment* (H, H', τ) *at level* j *of* (I_1, I_2) *w.r.t.* $(B, \emptyset)$.

Constructing suitable admissible assignments will let us construct different types of open paths in different rectangles. To demonstrate this we first define the following somewhat abstract set-up.

4.1 Admissible Connections

Assume the set-up in Definition 8. Consider the lattice $A = I_1 \times I_2$. Let $\mathcal{B} = (B_{i_1,i_2})_{(i_1,i_2)\in A}$ be a collection of finite rectangles where $B_{i_1,i_2} = [n_{i_1}] \times [n'_{i_2}]$. Let $A \otimes \mathcal{B}$ denote the bi-indexed collection

$$\{((a_1, b_1), (a_2, b_2)) : (a_1, a_2) \in A, (b_1, b_2) \in B_{a_1,a_2}\}.$$

We think of $A \otimes \mathcal{B}$ as a $\sum_{i_1} n_{i_1} \times \sum_{i_2} n'_{i_2}$ rectangle which is further divided into rectangles indexed by $(i_1, i_2) \in A$ in the obvious manner.

Definition 9 (Route). *A **route** P at level j in $A \otimes \mathcal{B}$ is a sequence of points*

$$\left\{\left((v_i, b^{1,v_i}), (v_i, b^{2,v_i})\right)\right\}_{i\in[\ell]}$$

in $A \otimes \mathcal{B}$ satisfying the following conditions.

(i) $V(P) = \{v_1, v_2, \ldots, v_\ell\}$ *is an oriented path from* $(a+1, b+1)$ *to* $(a+t, b+t')$ *in* A.

(ii) Let $v_i = (v_i^1, v_i^2)$. *For each* i,

$$b^{1,v_i} \in [L_{j-1}, n_{v_i^1} - L_{j-1}] \times \{1\} \cup \{1\} \times [L_{j-1}, n'_{v_i^2} - L_{j-1}]$$

and

$$b^{2,v_i} \in [L_{j-1}, n_{v_i^1} - L_{j-1}] \times \{n'_{v_i^2}\} \cup \{n_{v_i^1}\} \times [L_{j-1}, n'_{v_i^2} - L_{j-1}]$$

except that $b^{1,v_1} = (1, 1)$ *and* $b^{2,v_\ell} = (n_{v_\ell^1}, n'_{v_{ell}^2})$ *are also allowed.*

(iii) *For each i (we drop the superscript v_i), let $b^1 = (b^1_1, b^1_2)$ and $b^2 = (b^2_1, b^2_2)$. Then for each i, we have*

$$\frac{1 - 2^{-(j+3)}}{R} \leq \frac{b^2_2 - b^1_2}{b^2_1 - b^1_1} \leq R(1 + 2^{-(j+3)}).$$

(iv) *b^{2,v_i} and $b^{1,v_{i+1}}$ agree in one co-ordinate.*

*A route P defined as above is called a route in $A \otimes B$ from (v_1, b^{1,v_1}) to (v_ℓ, b^{2,v_ℓ}). We call P a **corner to corner route** if $b^{1,v_1} = (1,1)$ and $b^{2,v_\ell} = (n_{v^1_\ell}, n'_{v^2_\ell})$. For $k \in I_2$, the k-section of the route P is defined to be the set of $k' \in I_1$ such that $(k', k) \in V(P)$.*

Now gluing together these routes one can construct corner to corner (resp. corner to side or side to side) paths under certain circumstances. We make the following definition to that end.

Definition 10 (Admissible Connections). *Consider the above set-up. Let*

$$S_{in} = [L_{j-1}, n_{a+1} - L_{j-1}] \times \{1\} \cup \{1\} \times [L_{j-1}, n'_{b+1} - L_{j-1}]$$

and

$$S_{out} = [L_{j-1}, n_{a+t} - L_{j-1}] \times \{n'_{b+t'}\} \cup \{n_{a+t}\} \times [L_{j-1}, n'_{b+t} - L_{j-1}].$$

Suppose for each $b \in S_{out}$ there exists a level j route P^b in $A \otimes \mathcal{B}$ from $(1,1)$ to b. The collection $\mathcal{P} = \{P^b\}$ is called a corner to side admissible connection in $A \otimes \mathcal{B}$. A side to corner admissible connection is defined in a similar manner. Now suppose for each $b \in S_{in}$, $b' \in S_{out}$ there exists a level j route $P^{b,b'}$ in $A \otimes \mathcal{B}$ from b to b'. The collection $\mathcal{P} = \{P^{b,b'}\}$ in this case is called a side to side admissible connection in $A \otimes \mathcal{B}$. We also define $V(\mathcal{P}) = \cup_{P \in \mathcal{P}} V(P)$.

The usefulness of having these abstract definitions is demonstrated by the next few lemmata. These follow directly from definition and hence we shall omit the proofs.

Now let $X = (X_1, X_2, ..., X_t)$ be an $\mathbb{X}$-blocks at level $j+1$ with X_i being the j-level subblocks constituting it. Let X_i consisting of n_i many chunks of $(j-1)$-level subblocks. Similarly let $Y = (Y_1, Y_2, ..., Y_{t'})$ be a $\mathbb{Y}$-block at level $j+1$ with j-level subblocks Y_i consisting of n'_i many chunks of $(j-1)$ level subblocks. Then we have the following Lemmata. Set $A = [t] \times [t']$. Define $\mathcal{B} = \{B_{i,j}\}$ where $B_{i_1,i_2} = [n'_{i_1}] \times [n'_{i_2}]$.

Lemma 5. *Consider the set-up described above. Let $H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq [t]$ and $H' = \{b_1 < b_2 < \cdots < b_\ell\}$. Set $\tilde{X}_{(s)} = (X_{a_s+1}, \ldots, X_{a_{s+1}-1})$ and $\tilde{Y}_{(s)} = (Y_{b_s+1}, \ldots, Y_{b_{s+1}-1})$. Suppose further that for each s, $\tilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \tilde{Y}_{(s)}$ and $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. Then we have $X \overset{c,c}{\longleftrightarrow} Y$ (Fig. 1).*

The next lemma gives sufficient conditions under which we have $\tilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \tilde{Y}_{(s)}$.

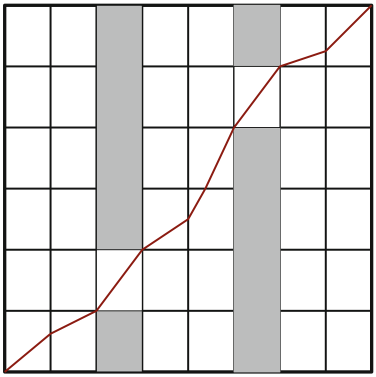

Fig. 1. Corner to corner paths

Lemma 6. *In the above set-up, let* $I_1^s = [a_s+1, a_{s+1}-1]$, $I_2^s = [a_s+1, a_{s+1}-1]$. *Set* $A^s = I_1^s \times I_2^s$ *and let* $\mathcal{B}^s$ *be the restriction of* $\mathcal{B}$ *to* A^s. *Suppose there exists a corner to corner route* P *in* $A^s \otimes B^s$ *such that* $X_{a_s+1} \overset{c,s}{\longleftrightarrow} Y_{b_s+1}$, $X_{a_{s+1}-1} \overset{s,c}{\longleftrightarrow} Y_{b_{s+1}-1}$ *and for all other* $(v_1, v_2) \in V(P)$ $X_{v_1} \overset{s,s}{\longleftrightarrow} Y_{v_2}$. *Then* $\tilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \tilde{Y}_{(s)}$.

The above lemmata are immediate from definition. Now we turn to corner to side, side to corner and side to side connections. We have the following lemma.

Lemma 7. *Consider the set-up as above. Suppose* X *and* Y *contain* n_X *and* n_Y *many chunks respectively. Further suppose that none of the subblock* X_i *or* Y_i *contain more than* $3L_j$ *level* 0 *subblocks.*

(i) Suppose for every exit chunk in $\mathcal{E}_{out}(X,Y)$ *the following holds. For concreteness consider the chunk* (k, n_Y). *Let* T_k *denote the set of all* i *such that* X_i *is contained in* C_k^X. *There exists* $T_k^* \subseteq T_k$ *with* $|T_k^*| \geq (1-10k_0L_j^{-1})|T_k|$ *such that for all* $r \in T_k^*$ *and* $\tilde{X} = (X_1, \ldots, X_r)$ *we have* $\tilde{X} \overset{c,s,*}{\longleftrightarrow} Y$.

Then we have $X \overset{c,s}{\longleftrightarrow} Y$.

(ii) A similar statement holds for $X \overset{s,c}{\longleftrightarrow} Y$.

(iii) Suppose for every pair of entry-exit chunks in $\mathcal{E}(X,Y)$ *the following holds. For concreteness consider the pair of entry-exit chunks* $((k_1,1),(n_X,k_2))$. *Let* T_{k_1} *(resp.* T'_{k_2}*) denote the set of all* i *such that* X_i *(resp.* Y_i*) is contained in* $C_{k_1}^X$ *(resp.* $C_{k_2}^Y$*). There exists* $T_{k_1,*} \subseteq T_{k_1}$, $T'_{k_2,*} \subseteq T'_{k_2}$ *with* $|T_{k_1,*}| \geq (1-10k_0L_j^{-1})|T_{k_1}|$, $|T'_{k_2,*}| \geq (1-10k_0L_j^{-1})|T'_{k_2}|$ *such that for all* $r \in T_{k_1,*}$, $r' \in T'_{k_2,*}$ *and* $\tilde{X} = (X_r, \ldots, X_t)$, $\tilde{Y} = (Y_1, \ldots, Y_{r'})$ *we have* $\tilde{X} \overset{s,s,*}{\longleftrightarrow} Y$. *Then we have* $X \overset{s,s}{\longleftrightarrow} Y$.

Proof. Parts (i) and (ii) are straightforward from definitions. Part (iii) follows from definitions by noting the following consequence of planarity. Suppose there are open oriented paths in $\mathbb{Z}^2$ from $v_1 = (x_1, y_1)$ to $v_2 = (x_2, y_2)$ and also from

$v_3 = (x_3, y_1)$ to $v_4(x_2, y_3)$ such that $x_1 < x_3 < x_2$ and $y_1 < y_2 < y_3$. Then these paths must intersect and hence there are open paths from v_1 to v_4 and also from v_2 to v_3. The condition on the length of sub-blocks is used to ensure that none of the subblocks in $T_{k_1} \setminus T_{k_1,*}$ are extremely long. □

The next lemma gives sufficient conditions for $\tilde{X} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}$ and $\tilde{X} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}$ in the set-up of the above lemma. This lemma also easily follows from definitions.

Lemma 8. *Assume the set-up of Lemma 7. Let* $\tilde{X} = (X_{t_1}, X_{t_1+1}, \ldots, X_{t_2})$ *and* $\tilde{Y} = (Y_{t'_1}, \ldots, Y_{t'_2})$. *Let* $H = \{a_1 < a_2 < \cdots < a_\ell\} \subseteq [t_1, t_2]$ *and* $H' = \{b_1 < b_2 < \cdots < b_\ell\} \subseteq [t'_1, t'_2]$. *Set* $\tilde{X}_{(s)} = (X_{a_s+1}, \ldots, X_{a_{s+1}-1})$ *and* $\tilde{Y}_{(s)} = (Y_{b_s+1}, \ldots, Y_{b_{s+1}-1})$ *(a_0, b_0 etc. are defined in the natural way).*

(i) Suppose that for each $s < \ell$, $\tilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \tilde{Y}_{(s)}$ and $\tilde{X}_{(\ell)} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}_{(\ell)}$. Also suppose for each s, $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. Then we have $\tilde{X} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}$ (Fig. 2).

(ii) A similar statement holds for $\tilde{X} \overset{s,c,*}{\longleftrightarrow} \tilde{Y}$.

(iii) Suppose that for each $s \in [\ell - 1]$, $\tilde{X}_{(s)} \overset{c,c}{\longleftrightarrow} \tilde{Y}_{(s)}$, $\tilde{X}_{(0)} \overset{s,c,*}{\longleftrightarrow} \tilde{Y}_{(0)}$ $\tilde{X}_{(\ell)} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}_{(\ell)}$. Also suppose for each s, $X_{a_s} \overset{c,c}{\longleftrightarrow} Y_{b_s}$. Then we have $\tilde{X} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}$ (Fig. 3).

Now we give sufficient conditions for $\tilde{X} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}$ and $\tilde{X} \overset{s,c,*}{\longleftrightarrow} \tilde{Y}$ in terms of routes.

Lemma 9. *In the above set-up, further suppose that none of the level $(j-1)$ sub-blocks of X_{t_1}, X_{t_2}, $Y_{t'_1}$, $Y_{t'_2}$ contain more than $3L_{j-1}$ level 0 sub-blocks. Set $I_1^s = [a_s + 1, a_{s+1} - 1]$, $I_2^s = [a_s + 1, a_{s+1} - 1]$. Set $A^s = I_1^s \times I_2^s$ and let $\mathcal{B}^s$ be the restriction of $\mathcal{B}$ to A^s. Suppose there exists a corner to side admissible connection $\mathcal{P}$ in $A^s \otimes B^s$ such that $X_{a_s+1} \overset{c,s}{\longleftrightarrow} Y_{b_s+1}$ and for all other $(v_1, v_2) \in V(\mathcal{P})$ $X_{v_1} \overset{s,s}{\longleftrightarrow} Y_{v_2}$. Then $\tilde{X}_{(s)} \overset{c,s,*}{\longleftrightarrow} \tilde{Y}_{(s)}$. Similar statements hold for $\tilde{X}_{(s)} \overset{s,c,*}{\longleftrightarrow} \tilde{Y}_{(s)}$ and $\tilde{X}_{(s)} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}_{(s)}$.*

Proof. Proof is immediate from definition of admissible connections and the inductive hypotheses (this is where we need the assumption on the lengths of $j-1$ level subblocks). For $\tilde{X}_{(s)} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}_{(s)}$, we again need to use planarity as before. □

Now we connect it up with the notion of admissible assignments defined earlier in this section. Consider the set-up in Lemma 5. Let $B_1 \subseteq I_1 = [t]$, $B_2 \subseteq I_2 = [t']$, let $B_1^* \supseteq B_1$ (resp. $B_2^* \supseteq B_2$) be the set containing elements of B_1 (resp. B_2) and its neighbours. Let Υ be a level j admissible assignment of (I_1, I_2) w.r.t. (B_1^*, B_2^*) with associated τ. Suppose $H = \tau^{-1}(B_2) \cup B_1$ and $H' = B_2^* \cup \tau(B_2)$. We have the following lemmata.

Lemma 10. *Consider $(\tilde{X}_{(s)}, \tilde{Y}_{(s)})$ in the above set-up. There exists a corner to corner route P in $A^s \otimes \mathcal{B}^s$. Further for each $k \in I_2^s$, there exist sets $H_k^\tau \subseteq I_1^s$ with $|H_k^\tau| \leq L_j$ such that the k-section of the route P is contained in H_k^τ for all k.*

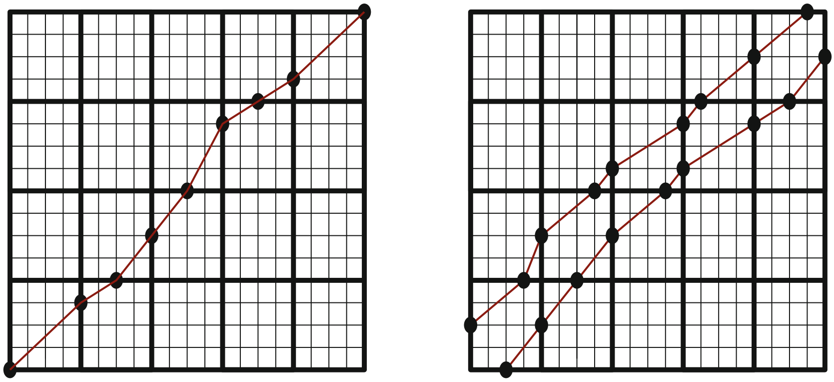

Fig. 2. Corner to corner and side to side routes

In the special case where $t = t'$ *and* $\tau(i) = i$ *for all* i*, one case take* $H_k^\tau = \{k-1, k, k+1\}$*. Further Let* $A' \subseteq A^s$ *with* $|A'| \leq k_0$*. Suppose Further that for all* $v = (v_1, v_2) \in A'$ *and for* $i \in \{s, s+1\}$ *we have* $||v - (a_i, b_i)||_\infty \geq k_0 R^3 10^{j+8}$*. Then we can take* $V(P) \cap A' = \emptyset$*.*

Proof. This lemma is a consequence of Lemma 12 below. □

Lemma 11. *In the above set-up, consider* $(\tilde{X}_{(s)}, \tilde{Y}_{(s)})$*. Assume for each* $i \in [a_s+1, a_{s+1}-1]$*,* $i' \in [b_s+1, b_{s+1}-1]$ *we have* $L_{j-1}^{\alpha-5} \leq n_i, n'_{i'} \leq L_{j-1}^{\alpha-5} + L_{j-1}$*. Let* $A' \subseteq A^s$ *with* $|A'| \leq k_0$*. Suppose further that for all* $v = (v_1, v_2) \in A'$ *and for* $i \in s, s+1$ *we have* $||v - (a_i, b_i)||_\infty \geq k_0 R^3 10^{j+8}$*. Assume also* $a_{s+1} - a_s, b_{s+1} - b_s \geq 5^{j+6} R$*. Then there exists a corner to side (resp. side to corner, side to side) admissible connection* $\mathcal{P}$ *in* $A^s \otimes \mathcal{B}^s$ *such that* $V(P) \cap A' = \emptyset$*.*

Proof. This lemma also follows from Lemma 12 below. □

Lemma 12. *Let* $A \otimes \mathcal{B}$ *be as in Definition 9. Assume that*

$$\frac{1 - 2^{-(j+7/2)}}{R} \leq \frac{t'}{t} \leq R(1 + 2^{-(j+7/2)}),$$

and $L_{j-1}^{\alpha-5} + L_{j-1} \geq n_i, n'_{i'} \geq L_{j-1}^{\alpha-5}$*. Then the following holds.*

(i) There exists a corner to corner route P *in* $A \otimes \mathcal{B}$ *where* $V(P) \subseteq R(A)$ *where*

$$R(A) = \{v = (v_1, v_2) \in A : |v - (a + xt, b + xt')|_1 \leq 50 \text{ for some } x \in [0, 1]\}.$$

(ii) Further, if $t, t' \geq 5^{j+6} R$*, then there exists a corner to side (resp. side to corner, side to side) admissible connection* $\mathcal{P}$ *with* $V(\mathcal{P}) \subseteq R(A)$*.*

(iii) Let A' *be a given subset of* A *with* $|A'| \leq k_0$ *such that* $A' \bigcap ([k_0 R^3 10^{j+8}] \times [k_0 R^3 10^{j+8}] \cup ([n - k_0 R^3 10^{j+8}, n] \times [n' - k_0 R^3 10^{j+8}, n']) = \emptyset$*. Then there is a corner to corner route* P *in* $A \otimes \mathcal{B}$ *such that* $V(P) \cap A' = \emptyset$*. Further, if* $t, t' \geq 5^{j+6} R$*, then there exists a corner to side (resp. side to corner, side to side) admissible connection* $\mathcal{P}$ *with* $V(\mathcal{P}) \cap A' = \emptyset$*.*

Proof. Without loss of generality, for this proof we shall assume $a = b = 0$. We prove (i) first. Let $y_i = \lfloor it'/t \rfloor + 1$ for $i \in [t]$ and let $x_i = \lceil it/t' \rceil$ for $i \in [t']$. Define $\tilde{y_i} = (it'/t - y_i + 1)$ and $\tilde{x_i} = (it/t' - x_i + 1)$.

Define $y_i^* = \lfloor \tilde{y}_i n'_{y_i} \rfloor + 1$ and $x_i^* = \lceil \tilde{x}_i n_{x_i} \rceil$. Observe that it follows from the definitions that $y_i^* \in [n'_{y_i}]$ and $x_i^* \in [n_{x_i}]$. Now define $y_i^{**} = y_i^*$ if $y_i^* \in [L_{j-1}, n'_{y_i} - L_{j-1}]$. If $y_i^* \in [L_{j-1}]$ define $y_i^{**} = L_{j-1}$, if $y_i^* \in [n'_{y_i} - L_{j-1}, n'_{y_i}]$ define $y_i^{**} = n'_{y_i} - L_{j-1}$. Similarly define $x_i^{**} = x_i^*$ if $x_i^* \in [L_{j-1}, n_{x_i} - L_{j-1}]$. If $x_i^* \in [L_{j-1}]$ define $x_i^{**} = L_{j-1}$, if $x_i^* \in [n_{x_i} - L_{j-1}, n_{x_i}]$ define $x_i^{**} = n_{x_i} - L_{j-1}$. Now for $i \in [t-1], i' \in [t'-1]$ consider points $((i, n_i), (y_i, y_i^{**}))$, $((i+1, 1), (y_i, y_i^{**}))$, $((x_{i'}, x_{i'}^{**}), (i', n'_{i'}))$, $((x_{i'}, x_{i'}^{**}), (i'+1, 1))$ alongwith the two corner points. We construct a corner to corner route using these points.

Let us define $V(P) = \{(i, y_i), (x_{i'}, i') : i \in [t-1], i' \in [t'-1]\} \cup \{(t, t')\}$. We notice that either $y_1 = 1$ or $x_1 = 1$. It is easy to see that the vertices in $V(P)$ defines an oriented path from $(1,1)$ to (t, t') in A. Denote the path by $(v^1, v^2, \ldots v^{t+t'-1})$. For $v = v^r, r \in [2, t+t'-2]$, we define points b^{1,v^r} and b^{2,v^r} as follows. Without loss of generality assume $v = v^r = (i, y_i)$. Then either $v^{r-1} = (i-1, y_i) = (i-1, y_{i-1})$ or $v^{r-1} = (i, y_i - 1) = (x_{y_i - 1}, y_i - 1)$. If $v^{r-1} = (i-1, y_{i-1})$, then define $\{(b_1^{1,v}, b_2^{1,v}), (b_1^{2,v}, b_2^{2,v})\}$ by $b_1^{1,v} = 1$, $b_2^{1,v} = y_{i-1}^{**}$, $b_1^{2,v} = n_i$, $b_2^{2,v} = y_i^{**}$. If $v^{t-1} = (x_{y_i - 1}, y_i - 1)$ then define $K^v = \{(b_1^{1,v}, b_2^{1,v}), (b_1^{2,v}, b_2^{2,v})\}$ by $b_1^{1,v} = x_{y_i - 1}^{**}$, $b_2^{1,v} = 1$, $b_1^{2,v} = n_i$, $b_2^{2,v} = y_i^{**}$. To prove that this is indeed a route we only need to check the slope condition in Definition 9 in both the cases. We do that only for the latter case and the former one can be treated similarly.

Notice that from the definition it follows that the slope between the points (in $\mathbb{R}^2$) $(\tilde{x}_{y_i - 1}, 0)$ and $(1, \tilde{y}_i)$ is t'/t. We need to show that

$$\frac{1 - 2^{-(j+3)}}{R} \leq \frac{b_2^2 - b_2^1}{b_1^2 - b_1^1} = \frac{y_i^{**} - 1}{n_i - x_{y_i - 1}^{**}} \leq R\left(1 + 2^{-(j+3)}\right)$$

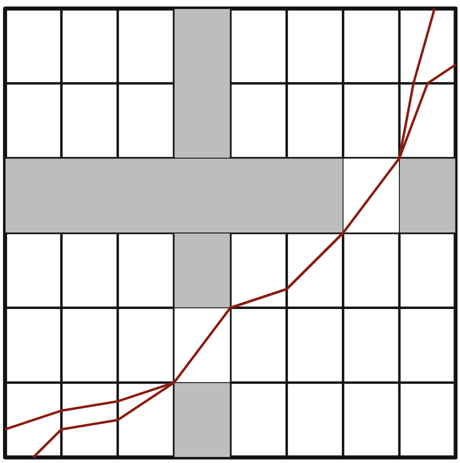

Fig. 3. Side to side admissible connections

where once more we have dropped the superscript v for convenience. Now if $x^*_{y_i-1} \in [n_i - L_{j-1}, n_i]$ and $y^*_i \in [L_{j-1}]$ then from definition it follows that $(b^2_2 - b^1_2)/(b^2_1 - b^1_1) = 1$ and hence the slope condition holds. Next let us suppose $y^*_i \in [L_{j-1}]$ but $x^*_{y_i-1} \notin [n_i - L_{j-1}, n_i]$. Then clearly, $(y^{**}_i - 1)/(n_i - x^{**}_{y_i-1}) \leq 1$. Also notice that in this case $x^*_{y_i-1} > L_{j-1}$ and $y^{**}_i - 1 \geq n'_{y_i}\tilde{y}_i(1 - L^{-1}_{j-1})$. It follows that

$$1 - \frac{x^{**}_{y_i-1}}{n_i} = 1 - \frac{x^*_{y_i-1}}{n_i} \leq 1 - \tilde{x}_{y_i-1} + \frac{1}{n_i} \leq (1 - \tilde{x}_{y_i-1})\left(1 + L^{-1}_{j-1}\right).$$

Hence

$$\begin{aligned}\frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} &\geq \frac{n'_{y_i}}{n_i}\frac{\tilde{y}_i}{1 - \tilde{x}_{y_i-1}}\frac{1 - L^{-1}_{j-1}}{1 + L^{-1}_{j-1}} \\ &\geq \frac{t'}{t}\frac{L^{\alpha-5}_{j-1}(1 - L^{-1}_{j-1})}{(L^{\alpha-5}_{j-1} + L_{j-1})(1 + L^{-1}_{j-1})} \geq \frac{1 - 2^{-(j+3)}}{R}\end{aligned}$$

for L_0 sufficiently large. The case where $y^*_i \notin [L_{j-1}]$ but $x^*_{y_i-1} \in [n_i - L_{j-1}, n_i]$ can be treated similarly.

Next we treat the case where $x^*_{y_i-1} \in [L_{j-1} + 1, n_i - L_{j-1} - 1]$ and $y^*_i \in [L_{j-1} + 1, n'_{y_i} - L_{j-1} - 1]$. Here we have similarly as before

$$\left(1 - L^{-1}_{j-1}\right)(1 - \tilde{x}_{y_i-1}) \leq 1 - \frac{x^{**}_{y_i-1}}{n_i} \leq (1 - \tilde{x}_{y_i-1})\left(1 + L^{-1}_{j-1}\right)$$

and

$$\tilde{y}_i\left(1 + 2L^{-1}_{j-1}\right) \geq \frac{y^{**}_i - 1}{n'_{y_i}} \geq \tilde{y}_i\left(1 - 2L^{-1}_{j-1}\right).$$

It follows as before that

$$\frac{(1 + 2Lj - 1^{-1})}{1 - L^{-1}_{j-1}}\frac{n'_{y_i}}{n_i}\frac{n'}{n} \geq \frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} \geq \frac{n'_{y_i}}{n_i}\frac{t'}{t}\frac{(1 + 2Lj - 1^{-1})}{1 - L^{-1}_{j-1}}$$

and hence

$$R\left(1 + 2^{-(j+3)}\right)\frac{y^{**}_i - 1}{n_i - x^{**}_{y_i-1}} \geq \frac{1 - 2^{-(j+3)}}{R}$$

for L_0 sufficiently large.

Other cases can be treated in similar vein and we only provide details in the case where $y^*_i \in [n'_{y_i} - L_{j-1}, n'_{y_i}]$ and $x^*_{y_i-1} \in [L_{j-1}]$. In this case we have that

$$\tilde{y}_i\left(1 - \frac{2L_{j-1}}{n'_{y_i}}\right) \leq \frac{y^{**}_i - 1}{n'_{y_i}} \leq \tilde{y}_i.$$

We also have that

$$(1 - \tilde{x}_{y_i-1})\left(1 - \frac{L_{j-1}}{n_i}\right)1 - \frac{x^{**}_{y_i-1}}{n_i} \leq 1 - \tilde{x}_{y_i-1}.$$

Combining these two relations we get as before that

$$R\left(1+2^{-(j+3)}\right)\frac{y_i^{**}-1}{n_i-x_{y_i-1}^{**}} \geq \frac{1-2^{-(j+3)}}{R}$$

for L_0 sufficiently large.

Thus we have constructed a corner to corner route in $A \otimes \mathcal{B}$. From the definitions it follows easily that for P as above $V(P) \subseteq R(A)$ and hence proof of (i) is complete.

Proof of (ii) is similar. Say, for the side to corner admissible connection, for a given $b \in S_{in}$, in stead of starting with the line $y = (t'/t)x$, we start with the line passing through $(b_1/n_1, 0)$ and (t, t'), and define $\tilde{x}_i$, $\tilde{y}_i$ to be the intersection of this line with the lines $y = i$ and $x = i$ respectively. Rest of the proof is almost identical, we use the fact $t, t' > 5^{j+6}R$ to prove that the slope of this new line is still sufficiently close to t'/t.

For part (iii), instead of a straight line we start with a number of piecewise linear functions which approximate $V(P)$. By taking a large number of such choices, it follows that for one of the cases $V(P)$ must be disjoint with the given set A', we omit the details. □

Finally we show that if we try a large number of admissible assignments, at least one of them must obey the hypothesis in Lemmas 10 and 11 regarding A'

Lemma 13. *Assume the set-up in Proposition 1. Let $\Upsilon_h, h \in [L_j^2]$ be the family of admissible assignments of (I_1, I_2) w.r.t. (B, B') described in Proposition 1(i). Fix any arbitrary $\mathcal{T} \subset [L_j^2]$ with $|\mathcal{T}| = R^6 k_0^5 10^{2j+20}$. Then for every $S \subset I_1 \times I_2$ with $|S| = k_0$, there exist $h_0 \in \mathcal{T}$ such that*

$$\min_{x \in B_X, y \in B_Y, s \in S} \left\{ |(x, \tau_{h_0}(x)) - s|, |(\tau_{h_0}^{-1}(y), y)| - s| \right\} \geq 2k_0 R^3 10^{j+8}.$$

Proof. Call $(x, y) \in I_1 \times I_2$ *forbidden* if there exist $s \in S$ such that $|(x, y) - s| \leq 2k_0 R^3 10^{j+8}$. For each $s \in S$, let $B_s \subset I_1 \times I_2$ denote the set of vertices which are *forbidden* because of s, i.e., $B_s = \{(x, y) : |(x, y) - s| \leq 2k_0 R^3 10^{j+8}\}$. Clearly $|B_s| \leq 10^{2j+18} k_0^2 R^6$. So the total number of forbidden vertices is $\leq 10^{2j+18} k_0^3 R^6$. Since $|B|, |B'| \leq k_0$, there exists $\mathcal{H} \subset \mathcal{T}$ with $|\mathcal{H}| = 10^{2j+19} R^6 k_0^4$ such that for all $x, x' \in B$, $x \neq x'$, $y, y' \in B'$, $y \neq y'$, $h_1, h_2 \in \mathcal{H}$, we have $\tau_{h_1}(x) \neq \tau_{h_2}(x')$ and $\tau_{h_1}^{-1}(y) \neq \tau_{h_2}^{-1}(y')$. Now for each $x \in B$ (resp. $y \in B'$), $(x, \tau_h(x))$ (resp. $(\tau_h^{-1}(y), y)$) can be *forbidden* for at most $10^{2j+18} k_0^3 R^6$ many different $h \in \mathcal{H}$. Hence,

$$\begin{aligned} &\# \bigcup_{x \in B, y \in B'} \left\{ h \in \mathcal{H} : (x, \tau_h(x)) \text{ or } (\tau_h^{-1}(y), y) \text{ is forbidden} \right\} \\ &\leq 2 \times 10^{2j+18} R^6 k_0^4 < |\mathcal{H}|. \end{aligned}$$

It follows that there exist $h_0 \in \mathcal{H}$ which satisfies the condition in the statement of the lemma. □

5 Length Estimate

We shall quote the following theorem directly from [4].

Theorem 4 (Theorem 8.1, [4]). *Let X be an $\mathbb{X}$ block at level $(j+1)$ we have that*

$$\mathbb{E}\left[\exp(L_j^{-6}(|X| - (2-2^{-(j+1)})L_{j+1}))\right] \le 1.$$

and hence for $x \ge 0$,

$$\mathbb{P}(|X| > ((2-2^{-(j+1)})L_{j+1} + xL_j^6)) \le \mathrm{e}^{-x}.$$

The proof is exactly the same as in [4].

6 Corner to Corner Estimate

In this section we prove the recursive tail estimate for the corner to corner connection probabilities.

Theorem 5. *Assume that the inductive hypothesis holds up to level j. Let X and Y be random $(j+1)$-level blocks according to $\mu^{\mathbb{X}}_{j+1}$ and $\mu^{\mathbb{Y}}_{j+1}$. Then*

$$\mathbb{P}\left(\mathbb{P}\left(X \overset{c,c}{\longleftrightarrow} Y|X\right) \le p\right) \le p^{m_{j+1}} L_{j+1}^{-\beta},$$
$$\mathbb{P}\left(\mathbb{P}\left(X \overset{c,c}{\longleftrightarrow} Y|Y\right) \le p\right) \le p^{m_{j+1}} L_{j+1}^{-\beta}$$

for $p \le 3/4 + 2^{-(j+4)}$ and $m_{j+1} = m + 2^{-(j+1)}$.

Due to the obvious symmetry between our X and Y bounds and for brevity all our bounds will be stated in terms of X and $S^{\mathbb{X}}_{j+1}$ but will similarly hold for Y and $S^{\mathbb{Y}}_{j+1}$. For the rest of this section we drop the superscript $\mathbb{X}$ and denote $S^{\mathbb{X}}_{j+1}$ (resp. $S^{\mathbb{X}}_j$) simply by S_{j+1} (resp. S_j).

The block X is constructed from an i.i.d. sequence of j-level blocks $X_1, X_2, \ldots$ conditioned on the event $X_i \in G^{\mathbb{X}}_j$ for $1 \le i \le L_j^3$ as described in Sect. 2. The construction also involves a random variable $W_X \sim \mathrm{Geom}(L_j^{-4})$ and let T_X denote the number of extra sub-blocks of X, that is the length of X is $L_j^{\alpha-1} + 2L_j^3 + T_X$. Let K_X denote the number of bad sub-blocks of X. Let us also denote the position of bad subblock of X and their neighbours by $\{\ell_1 < \ell_2 < \cdots < \ell_{K'_X}\}$, where K'_X denotes the number of such blocks. Trivially, $K'_X \le 3K_X$. We define $Y, \ldots, W_Y, T_Y$ and K_Y similarly. The proof of Theorem 5 is divided into 5 cases depending on the number of bad sub-blocks, the total number of sub-blocks of X and how "bad" the sub-blocks are.

We note here that the proof of Theorem 5 follows along the same general line of argument as the proof of Theorem 7.1 in [4], with significant adaptations resulting from the specifics of the model and especially the difference in the definition of good blocks. As such this section is similar to Sect. 7 in [4].

The following key lemma provides a bound for the probability of blocks having large length, number of bad sub-blocks or small $\prod_{i \in B_X} S_j(X_i)$.

Lemma 14. *For all $t', k', x \geq 0$ we have that*

$$\mathbb{P}\left[T_X \geq t', K_X \geq k', -\log \prod_{i \in B_X} S_j(X_i) > x\right]$$
$$\leq 2L_j^{-\delta k'/4} \exp\left(-x m_{j+1} - \frac{1}{2} t' L_j^{-4}\right).$$

The proof of this Lemma is same as the proof of Lemma 7.3 in [4] and we omit the details.

We now proceed with the 5 cases we need to consider.

6.1 Case 1

The first case is the scenario where the blocks are of typical length, have few bad sub-blocks whose corner to corner connection probabilities are not too small. This case holds with high probability.

We define the event $\mathcal{A}^{(1)}_{X,j+1}$ to be the set of $(j+1)$ level blocks such that

$$\mathcal{A}^{(1)}_{X,j+1} := \left\{X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i \in B_X} S_j(X_i) > L_j^{-1/3}\right\}.$$

The following Lemma is an easy corollary of Lemma 14 and the choices of parameters, we omit the proof.

Lemma 15. *The probability that $X \in \mathcal{A}^{(1)}_{X,j+1}$ is bounded below by*

$$\mathbb{P}\left[X \notin \mathcal{A}^{(1)}_{X,j+1}\right] \leq L_{j+1}^{-3\beta}.$$

Lemma 16. *We have that for all $X \in \mathcal{A}^{(1)}_{X,j+1}$,*

$$\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}^{(1)}_{Y,j+1}, X\right] \geq \frac{3}{4} + 2^{-(j+3)},$$

Proof. Suppose that $X \in \mathcal{A}^{(1)}_{X,j+1}$ with length $L_j^{\alpha-1} + 2L_j^3 + T_X$. Let B_X denote the location of bad subblocks of X. let K'_X be the number of bad sub-blocks and their neighbours and let set of their locations be $B^* = \{\ell_1 < \cdots < \ell_{K'_X}\}$. Notice that $K'_X \leq 3k_0$. We condition on $Y \in \mathcal{A}^{(1)}_{Y,j+1}$ having no bad subblocks. Denote this conditioning by

$$\mathcal{F} = \left\{Y \in \mathcal{A}^{(1)}_{Y,j+1}, T_Y, K_Y = 0\right\}.$$

Let $I_1 = [L_j^{\alpha-1} + 2L_j^3 + T_X]$ and $I_2 = [L_j^{\alpha-1} + 2L_j^3 + T_Y]$. By Proposition 1(i), we can find L_j^2 admissible assignments Υ_h at level j w.r.t. $(B^*, \emptyset)$, with associated τ_h for $1 \leq h \leq L_j^2$, such that $\tau_h(\ell_i) = \tau_1(\ell_i) + h - 1$ and in particular each block ℓ_i is mapped to L_j^2 distinct sub-blocks. Hence we get $\mathcal{H} \subset [L_j^2]$ of size $L_j < \lfloor L_j^2/9k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$, that is that all the positions bad blocks and their neighbours are mapped to are distinct.

Our construction ensures that all $Y_{\tau_h(\ell_i)}$ are uniformly chosen good j-blocks conditional on $\mathcal{F}$ and since $S_j(X_{\ell_i}) \geq L_j^{-1/3}$ we have that if $X_{\ell_i} \notin G_j^{\mathbb{X}}$,

$$\mathbb{P}\left[X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \mid \mathcal{F}\right] \geq S_j(X_{\ell_i}) - \mathbb{P}\left[Y_{\tau_h(\ell_i)} \notin G_j^{\mathbb{X}}\right] \geq \frac{1}{2} S_j(X_{\ell_i}). \tag{18}$$

Also if $X_{\ell_i} \in G_j^{\mathbb{X}}$ then from the recursive estimates it follows that

$$\mathbb{P}\left[X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \mid \mathcal{F}\right] \geq \frac{3}{4};$$

$$\mathbb{P}\left[X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \mid \mathcal{F}\right] \geq \frac{9}{10};$$

$$\mathbb{P}\left[X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \mid \mathcal{F}\right] \geq \frac{9}{10}.$$

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor $X_{\ell_i + 1}$ is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)}\right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)}\right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)}\right\}.$$

Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}.$$

Further, $\mathcal{S}$ denote the event

$$\mathcal{S} = \left\{X_k \overset{s,s}{\longleftrightarrow} Y_{k'} \forall k \in [L_j^{\alpha-1} + 2L_j^3 + T_X] \setminus B_X, \forall k' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y]\right\}.$$

Also let

$$\mathcal{C}_1 = \left\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\right\} \text{ and } \mathcal{C}_2 = \left\{X_{L_j^{\alpha-1}+2L_j^3+T_X} \overset{s,c}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y}\right\}.$$

By Lemmas 5, 6 and 10 if $\cup_{h\in\mathcal{H}} \mathcal{D}_h, \mathcal{S}, \mathcal{C}_1, \mathcal{C}_2$ all hold then $X \overset{c,c}{\longleftrightarrow} Y$. Conditional on $\mathcal{F}$, for $h \in \mathcal{H}$, the $\mathcal{D}_h$, $\mathcal{C}_1$, $\mathcal{C}_2$ are independent and by (18) and the recursive estimates,

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] \geq 2^{-5k_0} 3^{2k_0} L_j^{-1/3}.$$

Hence

$$\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h \mid \mathcal{F}] \geq 1 - \left(1 - 2^{-5k_0}3^{2k_0}L_j^{-1/3}\right)^{L_j} \geq 1 - L_{j+1}^{-3\beta}.$$

It follows from the recursive estimates that

$$\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \mathcal{C}_1, \mathcal{C}_2 \mid \mathcal{F}] \geq \left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right)$$

Also a union bound using the recursive estimates at level j gives

$$\mathbb{P}[\neg\mathcal{S} \mid \mathcal{F}] \leq \left(1 + \frac{R}{2}\right)^2 L_j^{2\alpha-2}L_j^{-2\beta} \leq L_j^{-\beta}.$$

It follows that

$$\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid \mathcal{F}\right] \geq \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \mathcal{C}_1, \mathcal{C}_2, \mathcal{S}] \geq \left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}.$$

Hence

$$\begin{aligned}
&\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X, T_Y\right] \\
\geq&\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid \mathcal{F}\right]\mathbb{P}\left[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y\right] \\
\geq&\left(0.81\left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}\right)\mathbb{P}\left[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, T_Y\right].
\end{aligned}$$

Removing the conditioning on T_Y we get

$$\begin{aligned}
&\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X\right] \\
\geq&\left(\left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}\right)\mathbb{P}\left[K_Y = 0 \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}\right] \\
\geq&\left(\left(\frac{9}{10}\right)^2 \left(1 - L_{j+1}^{-3\beta}\right) - L_j^{-\beta}\right)\left(1 - L_{j+1}^{-3\beta} - 2L_j^{-\delta/4}\right) \\
\geq&\frac{3}{4} + 2^{-(j+1)}
\end{aligned}$$

for large enough L_0, where the penultimate inequality follows from Lemmas 14 and 15. This completes the lemma. □

Lemma 17. *When* $1/2 \leq p \leq 3/4 + 2^{-(j+4)}$

$$\mathbb{P}(S_{j+1}(X) \leq p) \leq p^{m_{j+1}}L_{j+1}^{-\beta}$$

Proof. By Lemmas 15 and 16 we have that for all $X \in \mathcal{A}^{(1)}_{X,j+1}$

$$\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X\right] \geq \mathbb{P}\left[Y \in \mathcal{A}^{(1)}_{Y,j+1}\right] \mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X, Y \in \mathcal{A}^{(1)}_{Y,j+1}\right]$$
$$\geq \frac{3}{4} + 2^{-(j+4)}.$$

Hence if $1/2 \leq p \leq 3/4 + 2^{-(j+4)}$

$$\mathbb{P}\left(\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X\right] \leq p\right) \leq \mathbb{P}\left[X \notin \mathcal{A}^{(1)}_{X,j+1}\right]$$
$$\leq L_{j+1}^{-3\beta} \leq 2^{-m_{j+1}} L_{j+1}^{-\beta} \leq p^{m_{j+1}} L_{j+1}^{-\beta}.$$

□

6.2 Case 2

The next case involves blocks which are not too long and do not contain too many bad sub-blocks but whose bad sub-blocks may be very bad in the since that corner to corner connection probabilities of those might be really small. We define the class of blocks $\mathcal{A}^{(2)}_{X,j+1}$ as

$$\mathcal{A}^{(2)}_{X,j+1} := \left\{X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, K_X \leq k_0, \prod_{i \in B_X} S_j(X_i) \leq L_j^{-1/3}\right\}.$$

Lemma 18. *For $X \in \mathcal{A}^{(2)}_{X,j+1}$,*

$$S_{j+1}(X) \geq \min\left\{\frac{1}{2}, \frac{1}{10}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i)\right\}$$

Proof. Suppose that $X \in \mathcal{A}^{(2)}_{X,j+1}$. Let $\mathcal{E}$ denote the event

$$\mathcal{E} = \left\{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\right\}.$$

Then by definition of W_Y, $\mathbb{P}[W_Y \leq L_j^{\alpha-1}] \geq 1 - (1 - L_j^{-4})^{L_j^{\alpha-1}} \geq 9/10$ while by the definition of the block boundaries the event $T_Y = W_Y$ is equivalent to their being no bad sub-blocks amongst $Y_{L_j^3+L_j^{\alpha-1}+W_Y+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+W_Y+2L_j^3}$, that is that we don't need to extend the block because of bad sub-blocks. Hence $\mathbb{P}[T_Y = W_Y] \geq (1 - L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{19}$$

By our block construction procedure, on the event $T_Y = W_Y$ we have that the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-level blocks.

Define I_1, I_2, B_X and B^* as in the proof of Lemma 16. Also set $[L_j^{\alpha-1} + 2L_j^3 + T_X] \setminus B_X = G_X$. Using Proposition 1 again we can find L_j^2 level j admissible assignments Υ_h of (I_1, I_2) w.r.t. $(B^*, \emptyset)$ for $1 \le h \le L_j^2$ with associated τ_h. As in Lemma 16 we can construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = L_j < \lfloor L_j^2/9k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$, that is that all the positions bad blocks are assigned to are distinct. We will estimate the probability that one of these assignments work.

In trying out these L_j different assignments there is a subtle conditioning issue since conditioned on an assignment not working (e.g., the event $X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)}$ failing) the distribution of $Y_{\tau_h(l_i)}$ might change. As such we condition on an event $\mathcal{D}_h \cup \mathcal{G}_h$ which holds with high probability.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \forall k \in G_X \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i}$ denote the event

$$\mathcal{D}_{h,i} = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \forall k \in G_X \right\}.$$

Let $\mathcal{D}_h$ denote the event

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}.$$

Further, let

$$\mathcal{G}_h = \left\{ Y_{\tau_h(\ell_i)} \in G_j^{\mathbb{Y}} \text{ and } Y_{\tau_h(\ell_i)} \overset{s,s}{\longleftrightarrow} X_k \text{ for } 1 \le i \le K'_X, k \in G_X \right\}.$$

Then it follows from the recursive estimates and since $\beta > \alpha + \delta + 1$ that

$$\mathbb{P}[\mathcal{D}_h \cup \mathcal{G}_h \mid X, \mathcal{E}] \ge \mathbb{P}[\mathcal{G}_h \mid X, \mathcal{E}] \ge 1 - 10k_0 L_j^{-\delta}.$$

and since they are conditionally independent given X and $\mathcal{E}$,

$$\mathbb{P}[\cap_{h \in \mathcal{H}} (\mathcal{D}_h \cup \mathcal{G}_h) \mid X, \mathcal{E}] \ge (1 - 10k_0 L_j^{-\delta})^{L_j} \ge 9/10. \tag{20}$$

Now

$$\mathbb{P}[\mathcal{D}_h \mid X, \mathcal{E}, (\mathcal{D}_h \cup \mathcal{G}_h)] \ge \mathbb{P}[\mathcal{D}_h \mid X, \mathcal{E}] \ge \left(\frac{3}{4}\right)^{2k_0} \prod_{i \in B_X} S_j(X_i)$$

and hence

$$\begin{aligned} \mathbb{P}[\cup_{h \in \mathcal{H}} \mathcal{D}_h \mid X, \mathcal{E}, \cap_{h \in \mathcal{H}} (\mathcal{D}_h \cup \mathcal{G}_h)] &\ge 1 - \left(1 - \left(\frac{3}{4}\right)^{2k_0} \prod_{i \in B_X} S_j(X_i)\right)^{L_j} \\ &\ge \frac{9}{10} \wedge \frac{1}{4} \left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i \in B_X} S_j(X_i) \end{aligned} \tag{21}$$

since $1 - \mathrm{e}^{-x} \geq x/4 \wedge 9/10$ for $x \geq 0$. Furthermore, if

$$\mathcal{M} = \{\exists h_1 \neq h_2 \in \mathcal{H} : \mathcal{D}_{h_1} \setminus \mathcal{G}_{h_1}, \mathcal{D}_{h_2} \setminus \mathcal{G}_{h_2}\},$$

then

$$\begin{aligned}\mathbb{P}[\mathcal{M} \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] &\leq \binom{L_j}{2}\mathbb{P}[\mathcal{D}_h \setminus \mathcal{G}_h \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)]^2 \\ &\leq \binom{L_j}{2}\left(2\left(\frac{3}{4}\right)^{2k_0} \prod_{i\in B_X} S_j(X_i) \wedge 2L_j^{-\delta}\right)^2 \\ &\leq L_j^{-(\delta-2)}\left(\frac{3}{4}\right)^{2k_0} \prod_{i\in B_X} S_j(X_i). \end{aligned} \tag{22}$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1}+2L_j^3+T_Y}$ denote the events

$$\mathcal{J}_I = \left\{X_1 \xleftrightarrow{c,s} Y_1 \text{ and } X_k \xleftrightarrow{s,s} Y_1 \text{ for all } k \in G_X\right\};$$

$$\begin{aligned}\mathcal{J}_F \ = \Big\{X_{L_j^{\alpha-1}+2L_j^3+T_Y} &\xleftrightarrow{s,c} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \\ &\text{and } X_k \xleftrightarrow{s,s} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in G_X\Big\}.\end{aligned}$$

For $k \in \{2, \ldots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1, \} \setminus \cup_{h\in\mathcal{H},1\leq i\leq K'_X}\{\tau_h(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \left\{Y_k \in G_j^{\mathbb{Y}}, X_{k'} \xleftrightarrow{s,s} Y_k \text{ for all } k' \in G_X\right\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k\in[L_j^{\alpha-1}+2L_j^3+T_Y]\setminus\cup_{h\in\mathcal{H},1\leq i\leq K'_X}\{\tau_h(\ell_i)\}} \mathcal{J}_k.$$

Then it follows from the recursive estimates and the fact that $\mathcal{J}_k$ are conditionally independent that

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \geq \left(\frac{9}{10}\right)^2 \left(1 - RL_j^{\alpha-1-\beta}\right)^{2L_j^{\alpha-1}} \geq 3/4. \tag{23}$$

If $\mathcal{J}, \cup_{h\in\mathcal{H}}\mathcal{D}_h$ and $\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)$ all hold and $\mathcal{M}$ does not hold then we can find at least one $h \in \mathcal{H}$ such that $\mathcal{D}_h$ holds and $\mathcal{G}_{h'}$ holds for all $h' \in \mathcal{H} \setminus \{h\}$. Then by Lemma 10 as before we have that $X \xleftrightarrow{c,c} Y$. Hence by (20), (21), (22), and (23) and the fact that $\mathcal{J}$ is conditionally independent of the other events that

$$\begin{aligned}
&\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid X, \mathcal{E}\right] \\
&\geq \mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h), \mathcal{J},\ \neg\mathcal{M} \mid X, \mathcal{E}] \\
&= \mathbb{P}[\mathcal{J} \mid X, \mathcal{E}]\mathbb{P}[\cup_{h\in\mathcal{H}}\mathcal{D}_h,\ \neg\mathcal{M} \mid X, \mathcal{E}, \cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h)] \\
&\qquad \times \mathbb{P}[\cap_{h\in\mathcal{H}}(\mathcal{D}_h \cup \mathcal{G}_h) \mid X, \mathcal{E}] \\
&\geq \frac{27}{40}\left[\frac{9}{10} \wedge \frac{1}{4}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i\in B_X} S_j(X_i) - L_j^{-(\delta-2)} \prod_{i\in B_X} S_j(X_i)\right] \\
&\geq \frac{3}{5} \wedge \frac{1}{5}L_j \left(\frac{3}{4}\right)^{2k_0} \prod_{i\in B_X} S_j(X_i).
\end{aligned}$$

Combining with (19) we have that

$$\mathbb{P}[X \hookrightarrow Y \mid X] \geq \frac{1}{2} \wedge \frac{1}{10}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i\in B_X} S_j(X_i),$$

which completes the proof. □

Lemma 19. *When* $0 < p < 1/2$,

$$\mathbb{P}\left(X \in \mathcal{A}^{(2)}_{X,j+1}, S_{j+1}(X) \leq p\right) \leq \frac{1}{5}p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}
\mathbb{P}\left(X \in \mathcal{A}^{(2)}_{X,j+1}, S_{j+1}(X) \leq p\right) &\leq \mathbb{P}\left[\frac{1}{10}\left(\frac{3}{4}\right)^{2k_0} L_j \prod_{i\in B_X} S_j(X_i) \leq p\right] \\
&\leq 2\left(\frac{10p}{L_j}\left(\frac{4}{3}\right)^{2k_0}\right)^{m_{j+1}} \leq \frac{1}{5}p^{m_{j+1}} L_{j+1}^{-\beta} \quad (24)
\end{aligned}$$

where the first inequality holds by Lemma 18, the second by Lemma 14 and the third holds for large enough L_0 since $m_{j+1} > m > \alpha\beta$. □

6.3 Case 3

The third case allows for a greater number of bad sub-blocks. The class of blocks $\mathcal{A}^{(3)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(3)}_{X,j+1} := \left\{X : T_X \leq \frac{RL_j^{\alpha-1}}{2}, k_0 \leq K_X \leq \frac{L_j^{\alpha-1} + T_X}{10R_j^{+}}\right\}.$$

Lemma 20. *For* $X \in \mathcal{A}^{(3)}_{X,j+1}$,

$$S_{j+1}(X) \geq \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i)$$

Proof. For this proof we only need to consider a single admissible assignment Υ. Suppose that $X \in \mathcal{A}^{(3)}_{X,j+1}$. Again let $\mathcal{E}$ denote the event

$$\mathcal{E} = \{W_Y \leq L_j^{\alpha-1}, T_Y = W_Y\}.$$

Similarly to (19) we have that,

$$\mathbb{P}[\mathcal{E}] \geq 8/10. \tag{25}$$

As before we have, on the event $T_Y = W_Y$, the blocks $Y_{L_j^3+1}, \ldots, Y_{L_j^3+L_j^{\alpha-1}+T_Y}$ are uniform j-blocks since the block division did not evaluate whether they are good or bad.

Set I_1, I_2, B_X, G_X and B^* as in the proof of Lemma 18. By Proposition 1 we can find a level j admissible assignment Υ of (I_1, I_2) w.r.t. (B^*, ϕ) with associated τ so that for all i, $L_j^3 + 1 \leq \tau_h(\ell_i) \leq L_j^3 + L_j^{\alpha-1} + T_Y$. We estimate the probability that this assignment works.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither X_{ℓ_i-1} nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau(\ell_i)}\right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in G_X\right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} \} Y_{\tau(\ell_i)} \forall k \in G_X\right\}.$$

Let $\mathcal{D}$ denote the event

$$\mathcal{D} = \bigcap_{i=1}^{K'_X} \mathcal{D}_i.$$

By definition and the recursive estimates,

$$\mathbb{P}[\mathcal{D} \mid X, \mathcal{E}] \geq \left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i) \tag{26}$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1}+2L_j^3+T_Y}$ denote the events

$$\mathcal{J}_I = \left\{X_1 \overset{c,s}{\longleftrightarrow} Y_1 \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_1 \text{ for all } k \in G_X\right\};$$

$$\mathcal{J}_F = \Big\{X_{L_j^{\alpha-1}+2L_j^3+T_Y} \overset{s,c}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in G_X\Big\}.$$

For $k \in \{2, \ldots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1,\} \setminus \cup_{1\le i\le K'_X}\{\tau(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \Big\{Y_k \in G_j^{\mathbb{Y}}, X_{k'} \overset{s,s}{\longleftrightarrow} Y_k \text{ for all } k' \in G_X\Big\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k\in[L_j^{\alpha-1}+2L_j^3+T_Y]\setminus\cup_{1\le i\le K'_X}\{\tau(\ell_i)\}} \mathcal{J}_k.$$

From the recursive estimates

$$\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \ge \frac{3}{4}. \tag{27}$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Lemma 10 we have that $X \overset{c,c}{\longleftrightarrow} Y$. Hence by (26) and (27) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X, \mathcal{E}\right] &\ge \mathbb{P}[\mathcal{D}, \mathcal{J} \mid X, \mathcal{E}] \\ &= \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\ge \frac{3}{4}\left(\frac{3}{4}\right)^{2K_X} \prod_{i\in B_X} S_j(X_i).\end{aligned}$$

Combining with (25) we have that

$$\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X\right] \ge \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i\in B_X} S_j(X_i),$$

which completes the proof. □

Lemma 21. *When* $0 < p \le 1/2$,

$$\mathbb{P}\left(X \in \mathcal{A}_{X,j+1}^{(3)}, S_{j+1}(X) \le p\right) \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. We have that

$$\begin{aligned}\mathbb{P}\left(X \in \mathcal{A}_{X,j+1}^{(3)}, S_{j+1}(X) \le p\right) &\le \mathbb{P}\left[K_X > k_0, \frac{1}{2}\left(\frac{3}{4}\right)^{2K_X} \prod_{i\in B_X} S_j(X_i) \le p\right] \\ &\le \sum_{k=k_0}^{\infty} \mathbb{P}\left[K_X = k, \prod_{i\in B_X} S_j(X_i) \le 2p\left(\frac{4}{3}\right)^{2k}\right] \\ &\le 2\sum_{k=k_0}^{\infty} \left(2p\left(\frac{4}{3}\right)^{2k}\right)^{m_{j+1}} L_j^{-\delta k/4} \\ &\le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}\end{aligned}$$

where the first inequality holds by Lemma 20, the third follows from Lemma 14 and the last one holds for large enough L_0 since $\delta k_0 > 4\alpha\beta$. □

6.4 Case 4

In Case 4 we allow blocks of long length but not too many bad sub-blocks. The class of blocks $\mathcal{A}^{(4)}_{X,j+1}$ is defined as

$$\mathcal{A}^{(4)}_{X,j+1} := \left\{ X : T_X > \frac{RL_j^{\alpha-1}}{2}, K_X \leq \frac{L_j^{\alpha-1} + T_X}{10R_j^+} \right\}.$$

Lemma 22. *For $X \in \mathcal{A}^{(4)}_{X,j+1}$,*

$$S_{j+1}(X) \geq \left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i) \exp\left(-\frac{3T_X L_j^{-4}}{R}\right)$$

Proof. In this proof we allow the length of Y to grow at a slower rate than that of X. Suppose that $X \in \mathcal{A}^{(4)}_{X,j+1}$ and let $\mathcal{E}(X)$ denote the event

$$\mathcal{E}(X) = \{W_Y = \lfloor 2T_X/R \rfloor, T_Y = W_Y\}.$$

Then by definition $\mathbb{P}[W_Y = \lfloor 2T_X/R \rfloor] = L_j^{-4}(1 - L_j^{-4})^{\lfloor 2T_X/R \rfloor}$. Similarly to Lemma 18, $\mathbb{P}[T_Y = W_Y \mid W_Y] \geq (1 - L_j^{-\delta})^{2L_j^3} \geq 9/10$. Combining these we have that

$$\mathbb{P}[\mathcal{E}(X)] \geq \frac{9}{10} L_j^{-4}(1 - L_j^{-4})^{\lfloor 2T_X/R \rfloor}. \tag{28}$$

Set I_1, I_2, B_X, B^* as before. By Proposition 1 we can find an admissible assignment at level j, Υ of (I_1, I_2) w.r.t. $(B^*, \emptyset)$ with associated τ so that for all i, $L_j^3 + 1 \leq \tau(\ell_i) \leq L_j^3 + L_j^{\alpha-1} + T_Y$. We again estimate the probability that this assignment works.

We need to modify the definition of $\mathcal{D}$ and $\mathcal{J}$ in this case since the length of X could be arbitrarily large. For $k \in [L_j^{\alpha-1} + 2L_j^3 + T_Y] \setminus \tau(B_X)$, let $H_k^\tau \subseteq [L_j^{\alpha-1} + 2L_j^3 + T_Y] \setminus B_X$ be the sets given by Lemma 10 such that $|H_k^\tau| \leq L_j$ and there exists a τ-compatible admissible route with k-sections contained in H_k^τ for all k. We define $\mathcal{D}$ and $\mathcal{J}$ in this case as follows.

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither $X_{\ell_i - 1}$ nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in H^\tau_{\tau(l_i)} \right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_i$ denote the event

$$\mathcal{D}_i = \left\{ Y_{\tau(\ell_i)} \in G_j^{\mathbb{Y}}, X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau(\ell_i)} \text{ and } X_k \overset{s,s}{\longleftrightarrow} Y_{\tau(\ell_i)} \forall k \in H^\tau_{\tau(l_i)} \right\}.$$

Let $\mathcal{D}$ denote the event

$$\mathcal{D} = \bigcap_{i=1}^{K'_X} \mathcal{D}_i.$$

Let $\mathcal{J}_I = \mathcal{J}_1$ and $\mathcal{J}_F = \mathcal{J}_{L_j^{\alpha-1}+2L_j^3+T_Y}$ denote the events

$$\mathcal{J}_I = \left\{ X_1 \xleftrightarrow{c,s} Y_1 \text{ and } X_k \xleftrightarrow{s,s} Y_1 \text{ for all } k \in H_1^\tau \right\};$$

$$\mathcal{J}_F = \Big\{ X_{L_j^{\alpha-1}+2L_j^3+T_Y} \xleftrightarrow{s,c} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \text{ and } X_k \xleftrightarrow{s,s} Y_{L_j^{\alpha-1}+2L_j^3+T_Y} \forall k \in H^\tau_{L_j^{\alpha-1}+2L_j^3+T_Y} \Big\}.$$

For $k \in \{2, \dots L_j^{\alpha-1} + 2L_j^3 + T_Y - 1, \} \setminus \cup_{1\le i\le K'_X} \{\tau(\ell_i)\}$, let $\mathcal{J}_k$ denote the event

$$\mathcal{J}_k = \left\{ Y_k \in G_j^{\mathbb{Y}}, X_{k'} \xleftrightarrow{s,s} Y_k \text{ for all } k' \in H_k^\tau \right\}.$$

Finally let

$$\mathcal{J} = \bigcap_{k\in[L_j^{\alpha-1}+2L_j^3+T_Y]\setminus\cup_{1\le i\le K'_X}\{\tau(\ell_i)\}} \mathcal{J}_k.$$

If $\mathcal{D}$ and $\mathcal{J}$ hold then by Lemma 10 we have that $X \xleftrightarrow{c,c} Y$. It is easy to see that, in this case (26) holds. Also we have for large enough L_0,

$$\begin{aligned}\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] &\ge \frac{3}{4}\left(1 - 2L_j^{-\delta}\right)^{L_j^{\alpha-1}+\lfloor 2T_X/R\rfloor+2L_j^3} \\ &\ge \frac{1}{4}\exp\left(-2L_j^{-\delta}(L_j^{\alpha-1} + \lfloor 2T_X/R\rfloor + 2L_j^3)\right). \end{aligned} \tag{29}$$

Hence by (26) and (29) and the fact that $\mathcal{D}$ and $\mathcal{J}$ are conditionally independent we have that,

$$\begin{aligned}&\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid X, \mathcal{E}\right] \\ &\ge \mathbb{P}[\mathcal{D} \mid X, \mathcal{E}]\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}] \\ &\ge \frac{1}{4}\exp\left(-L_j^{-\delta}\left(L_j^{\alpha-1} + \lfloor 2T_X/R\rfloor + 2L_j^3\right)\right)\left(\frac{3}{4}\right)^{2K_X}\prod_{i\in B_X} S(X_i).\end{aligned}$$

Combining with (28) we have that

$$\mathbb{P}\left[X \xleftrightarrow{c,c} Y \mid X\right] \ge \exp\left(-3T_X L_j^{-4}/R\right)\left(\frac{3}{4}\right)^{2K_X}\prod_{i\in B_X} S_j(X_i),$$

since $T_X L_j^{-4} = \Omega(L_j^{\alpha-6})$ and $\delta > 5$ which completes the proof. □

Lemma 23. *When* $0 < p \le 1/2$,

$$\mathbb{P}\left(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \le p\right) \le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. Set $t_0 = RL_j^{\alpha-1}/2 + 1$ and for $k \ge k_0$, set

$$S(k) = \left(\frac{3}{4}\right)^{2k} \prod_{i \in B_X} S_j(X_i).$$

We have that

$$\begin{aligned}
&\mathbb{P}\left(X \in \mathcal{A}^{(4)}_{X,j+1}, S_{j+1}(X) \le p\right) \\
&\le \sum_{t=t_0}^{\infty} \sum_{k=k_0}^{\infty} \mathbb{P}\left[T_X = t, K_X = k, S(k) \exp\left(-\frac{3tL_j^{-4}}{R}\right) \le p\right] \\
&\le \sum_{t=t_0}^{\infty} \sum_{k=k_0}^{\infty} 2\left(\frac{4^{2k} p}{3^{2k}}\right)^{m_{j+1}} \exp\left(\frac{3m_{j+1} t L_j^{-4}}{R} - \frac{tL_j^{-4}}{2}\right) L_j^{-\delta k/4} \\
&\le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}
\end{aligned}$$

where the first inequality holds by Lemma 22, the second by Lemma 14 and the third holds for large enough L_0 since $3m_{j+1}/R < 1/2$ and so for large enough L_0, $(4/3)^{2(m+1)} L_j^{-\delta/4} \le 1/2$ and

$$\sum_{t=RL_j^{\alpha-1}/2+1}^{\infty} \exp\left(-tL_j^{-4}\left(\frac{1}{2} - \frac{3m_{j+1}}{R}\right)\right) < \frac{1}{10} L_{j+1}^{-\beta}.$$

□

6.5 Case 5

It remains to deal with the case involving blocks with a large density of bad sub-blocks. Define the class of blocks $\mathcal{A}^{(5)}_{X,j+1}$ is as

$$\mathcal{A}^{(5)}_{X,j+1} := \left\{X : K_X > \frac{L_j^{\alpha-1} + T_X}{10R_j^+}\right\}.$$

Lemma 24. *For* $X \in \mathcal{A}^{(5)}_{X,j+1}$,

$$S_{j+1}(X) \ge \exp\left(-2T_X L_j^{-4}\right) \left(\frac{3}{4}\right)^{2K_X} \prod_{i \in B_X} S_j(X_i)$$

Proof. The proof is a minor modification of the proof of Lemma 22. We take $\mathcal{E}(X)$ to denote the event

$$\mathcal{E}(X) = \{W_Y = T_X, T_Y = W_Y\}.$$

and get a bound of

$$\mathbb{P}[\mathcal{E}(X)] \geq \frac{9}{10} L_j^{-4} \left(1 - L_j^{-4}\right)^{T_X}.$$

We consider the admissible assignment Υ given by $\tau(i) = i$ for $i \in B^*$. It follows from Lemma 10 that in this case we can define $H_k^\tau = k-1, k, k+1$. We define $\mathcal{D}$ and $\mathcal{J}$ as before. The new bound for $\mathcal{J}$ becomes

$$\begin{aligned}\mathbb{P}[\mathcal{J} \mid X, \mathcal{E}(X)] &\geq \frac{3}{4}\left(1 - 2L_j^{-\delta}\right)^{L_j^{\alpha-1} + T_X + 2L_j^3} \\ &\geq \frac{1}{4}\exp\left(-2L_j^{-\delta}\left(L_j^{\alpha-1} + T_X + 2L_j^3\right)\right).\end{aligned}$$

We get the result proceeding as in the proof of Lemma 22. □

Lemma 25. *When* $0 < p \leq 1/2$*,*

$$\mathbb{P}\left(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \leq p\right) \leq \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}$$

Proof. First note that since $\alpha > 4$,

$$L_j^{-\delta/(50R_j^+)} = L_0^{-(\delta\alpha^j)/(50R_j^+)} \to 0$$

as $j \to \infty$. Hence for large enough L_0,

$$\sum_{t=0}^{\infty}\left(\exp\left(2m_{j+1}L_j^{-4}\right) L_j^{-\delta/(50R_j^+)}\right)^t < 2. \tag{30}$$

Set

$$k_* = \frac{L_j^{\alpha-1} + t}{10R_j^+}$$

and for $k \geq k_*$ set

$$S(k) = \left(\frac{3}{4}\right)^{2k} \prod_{i \in B_X} S_j(X_i).$$

We have that

$$\begin{aligned}
&\mathbb{P}\left(X \in \mathcal{A}^{(5)}_{X,j+1}, S_{j+1}(X) \le p\right)\\
&\le \sum_{t=0}^{\infty}\sum_{k=k_*}^{\infty} \mathbb{P}\left[T_X = t, K_X = k, S(k)\exp(-2tL_j^{-4}) \le p\right]\\
&\le p^{m_{j+1}} \sum_{t=0}^{\infty}\sum_{k=k_*}^{\infty} 2\left(\exp(2m_{j+1}tL_j^{-4})\right)\left(\left(\frac{16}{9}\right)^{m_j+1} L_j^{-\delta/4}\right)^k\\
&\le p^{m_{j+1}} \sum_{t=0}^{\infty} 4\left(\exp(2m_{j+1}tL_j^{-4})\right) L_j^{-(L_j^{\alpha-1}+t)/(50R_j^+)}\\
&\le \frac{1}{5} p^{m_{j+1}} L_{j+1}^{-\beta}
\end{aligned}$$

where the first inequality holds be by Lemma 24, the second by Lemma 14 and the third follows since L_0 is sufficiently large and the last one by (30) and the fact that

$$L_j^{-(\delta L_j^{\alpha-1})/(50R_j^+)} \le \frac{1}{40} L_{j+1}^{-\beta},$$

for large enough L_0. □

6.6 Theorem 5

Putting together all the five cases we now prove Theorem 5.

Proof of Theorem 5. The case of $1/2 \le p \le 1 - L_{j+1}^{-1}$ is established in Lemma 17. By Lemma 16 we have that $S_{j+1}(X) \ge 1/2$ for all $X \in \mathcal{A}^{(1)}_{X,j+1}$. Hence we need only consider $0 < p < 1/2$ and cases 2 to 5. By Lemmas 19, 21, 23 and 25 then

$$\mathbb{P}(S_{j+1}(X) \le p) \le \sum_{l=2}^{5} \mathbb{P}\left(X \in \mathcal{A}^{(l)}_{X,j+1}, S_{j+1}(X) \le p\right) \le p^{m_{j+1}} L_{j+1}^{-\beta}.$$

The bound for $S^{\mathbb{Y}}_{j+1}$ follows similarly. □

7 Side to Corner and Corner to Side Estimates

The aim of this section is to show that for a large class of $\mathbb{X}$-blocks (resp. $\mathbb{Y}$-blocks), $\mathbb{P}(X \overset{c,s}{\longleftrightarrow} Y \mid X)$ and $\mathbb{P}(X \overset{s,c}{\longleftrightarrow} Y \mid X)$ (resp. $\mathbb{P}(X \overset{c,s}{\longleftrightarrow} Y \mid Y)$ and $\mathbb{P}(X \overset{s,c}{\longleftrightarrow} Y \mid Y)$) is large. We shall state and prove the result only for $\mathbb{X}$-blocks.

Here we need to consider a different class of blocks where the blocks have few bad sub-blocks whose corner to corner connection probabilities are not too small, where the excess number of subblocks is of smaller order than the typical length and none of the subblocks, and their chunks contain too many level 0 blocks. This case holds with high probability. Let X be a level $(j+1)$ X-block constructed out of the independent sequence of j level blocks $X_1, X_2, \ldots$ where the first L_j^3 ones are conditioned to be good.

For $i = 1, 2, \ldots, L_j^{\alpha-1} + 2L_j^3 + T_X$, let $\mathcal{G}_i$ denote the event that all level $j-1$ subblocks contained in X_i contains at most $3L_{j-1}$ level 0 blocks, and X_i contains at most $3L_j$ level 0 blocks. Let $\mathcal{G}_X$ denote the event that for all good blocks X_i contained in X, $\mathcal{G}_i$ holds. We define $\mathcal{A}_{X,j+1}^{(*)}$ to be the set of $(j+1)$ level blocks such that

$$\mathcal{A}_{X,j+1}^{(*)} := \left\{ X : T_X \le L_j^5 - 2L_j^3, K_X \le k_0, \prod_{i \in B_X} S_j(X_i) > L_j^{-1/3}, \mathcal{G}_X \right\}.$$

It follows from Theorem 4 that $\mathbb{P}[\mathcal{G}_X^c]$ is exponentially small in L_{j-1} and hence we shall be able to safely ignore this conditioning while calculating probability estimates since L_0 is sufficiently large.

Similarly to Lemma 15 it can be proved that

$$\mathbb{P}\left[X \in \mathcal{A}_{X,j+1}^{(*)}\right] \ge 1 - L_{j+1}^{-3\beta}. \tag{31}$$

We have the following proposition.

Proposition 2. *We have that for all $X \in \mathcal{A}_{X,j+1}^{(*)}$,*

$$\mathbb{P}\left[X \overset{c,s}{\longleftrightarrow} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(*)}, X\right] \ge \frac{9}{10} + 2^{-(j+15/4)},$$
$$\mathbb{P}\left[X \overset{s,c}{\longleftrightarrow} Y \mid Y \in \mathcal{A}_{Y,j+1}^{(1)}, X\right] \ge \frac{9}{10} + 2^{-(j+15/4)}.$$

We shall only prove the corner to side estimate, the other one follows by symmetry. Suppose that $X \in \mathcal{A}_{X,j+1}^{(*)}$ with length $L_j^{\alpha-1} + 2L_j^3 + T_X$, define B_X, B^*, K'_X, T_Y and K_Y as in the proof of Lemma 16. We condition on $Y \in \mathcal{A}_{Y,j+1}^{(*)}$ having no bad subblocks. Denote this conditioning by

$$\mathcal{F} = \left\{Y \in \mathcal{A}_{Y,j+1}^{(*)}, T_Y, K_Y = 0\right\}.$$

Let n_X and n_Y denote the number of chunks in X and Y respectively. We first prove the following lemma.

Lemma 26. *Consider an exit chunk (k, n_Y) (resp. (n_X, k)) in $\mathcal{E}_{out}(X, Y)$. Fix $t \in [L_j^{\alpha-1} + 2L_j^3 + T_X]$ contained in C_k^X such that $[t, t - L_j^3] \cap B_X = \emptyset$ (resp. fix $t' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y]$ contained in C_k^Y). Consider $\tilde{X} = (X_1, \ldots, X_t)$ (or $\tilde{Y} = (Y_1, \ldots, Y_{t'})$). Then there exists an event $\mathcal{S}_t$ with $\mathbb{P}[S_t \mid \mathcal{F}] \ge 1 - L_j^{-\alpha}$ and on S_t, $\mathcal{F}$ and $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$ we have $\tilde{X} \overset{c,s,*}{\longleftrightarrow} Y$ (resp. $\mathcal{S}_{t'}$ with $\mathbb{P}[S_{t'} \mid \mathcal{F}] \ge 1 - L_j^{-\alpha}$ and on $S_{t'}$, $\mathcal{F}$ and $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$ we have $X \overset{c,s,*}{\longleftrightarrow} \tilde{Y}$).*

Proof. We shall only prove the first case, the other case follows by symmetry. Set $I_1 = [t]$, $I_2 = [L_j^{\alpha-1} + 2L_j^3 + T_Y]$. Also define $B_{\tilde{X}}$ and B^* as in the proof of Lemma 16. The slope condition in the definition of $\mathcal{E}_{out}(X,Y)$, and the fact that B_X is disjoint with $[t - L_j^3, t]$ implies that by Proposition 1 we can find L_j^2 admissible generalized mappings Υ_h of (I_1, I_2) with respect to $(B^*, \emptyset)$ with associated τ_h for $1 \le h \le L_j^2$ as in the proof of Lemma 16. As in there, we construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = L_j < \lfloor L_j^2/3k_0 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$.

For $h \in \mathcal{H}, i \in B^*$, define the events $\mathcal{D}_{h,i}$ similarly as in the proof of Lemma 16. Set

$$\mathcal{D}_h = \bigcap_{i=1}^{K'_X} \mathcal{D}_{h,i}^k \text{ and } \mathcal{D} = \bigcup_{h \in \mathcal{H}} \mathcal{D}_h^k.$$

Further, $\mathcal{S}$ denote the event

$$\mathcal{S} = \left\{ X_k \overset{s,s}{\longleftrightarrow} Y_{k'} \forall k \in [t] \setminus \{\ell_1, \ldots, \ell_{K'_X}\}, \forall k' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y] \right\}.$$

Same arguments as in the proof of yields

$$\mathbb{P}[\mathcal{D} \mid \mathcal{F}] \ge 1 - L_{j+1}^{-3\beta}$$

and

$$\mathbb{P}[\neg\mathcal{S} \mid \mathcal{F}] \le 4L_j^{2\alpha-2} L_j^{-2\beta} \le L_j^{-\beta}.$$

Now it follows from Lemmas 8 and 11, that on $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\}$, $\mathcal{S}, \mathcal{D}$ and $\mathcal{F}$, we have $\tilde{X} \overset{c,s,*}{\longleftrightarrow} Y$. The proof of the Lemma is completed by setting $\mathcal{S}_t = \mathcal{S} \cap \mathcal{D}$.

Now we are ready to prove Proposition 2.

Proof of Proposition 2. Fix an exit chunk (k, n_Y) or (n_X, k') in $\mathcal{E}_{out}(X,Y)$. In the former case set T_k to be the set of all blocks X_t contained in C_k^X such that $[t, t - L_j^3] \cap B_X = \emptyset$, in the later case set $T'_{k'}$ to be the set of all blocks $Y_{t'}$ contained in $C_{k'}^Y$. Notice that the number of blocks contained in T_k is at least $(1 - 2k_0 L_j^{-1})$ fraction of the total number of blocks contained in C_k^X. For $t \in T_k$ (resp. $t' \in T'_{k'}$), let S_t (resp. $S_{t'}$) be the event given by Lemma 26 Hence it follows from Lemma 7(i), that on $\{X_1 \overset{c,s}{\longleftrightarrow} Y_1\} \bigcap \cap_{k,T_k} \mathcal{S}_t \bigcap \cap_{k',T_{k'}} \mathcal{S}_{t'}$, we have $X \overset{c,s}{\longleftrightarrow} Y$. Taking a union bound and using Lemma 26 and also using the recursive lower bound on $\mathbb{P}[X_1 \overset{c,s}{\longleftrightarrow} Y_1]$ yields,

$$\mathbb{P}\left[X \overset{c,s}{\longleftrightarrow} Y \mid \mathcal{F}, X\right] \ge \frac{9}{10} + 2^{-(j+31/8)}.$$

The proof can now be completed by removing the conditioning on T_Y and proceeding as in Lemma 16.

8 Side to Side Estimate

In this section we estimate the probability of having a side to side path in $X \times Y$. We work in the set up of previous section. We have the following theorem.

Proposition 3. *We have that*

$$\mathbb{P}\left[X \overset{s,s}{\longleftrightarrow} Y \mid X \in \mathcal{A}^{(*)}_{X,j+1}, Y \in \mathcal{A}^{(*)}_{Y,j+1}\right] \geq 1 - L_{j+1}^{-3\beta}. \tag{32}$$

Suppose that $X \in \mathcal{A}^{(*)}_{X,j+1}, Y \in \mathcal{A}^{(*)}_{Y,j+1}$. Let T_X, T_Y, B_X, B_Y, G_X, G_Y be as before. Let $B_1^* = \{\ell_1 < \cdots < \ell_{K'_X}\}$ and $B_2^* = \{\ell'_1 < \cdots < \ell'_{K'_Y}\}$ denote the locations of bad blocks and their neighbours in X and Y respectively. Let us condition on the block lengths T_X, T_Y, B_1^*, B_2^* and the bad-sub-blocks and their neighbours themselves. Denote this conditioning by

$$\mathcal{F} = \Big\{X \in \mathcal{A}^{(1)}_{X,j+1}, Y \in \mathcal{A}^{(1)}_{Y,j+1}, T_X, T_Y, K'_X, K'_Y, \ell_1, \ldots, \ell_{K'_X}, \ell'_1, \ldots, \ell'_{K'_Y}, \\ X_{\ell_1}, \ldots, X_{\ell_{K'_X}}, Y_{\ell'_1}, \ldots, Y_{\ell'_{K'_Y}}\Big\}.$$

Let

$$B_{X,Y} = \left\{(k,k') \in G_X \times G_Y : X_k \overset{s,s}{\not\longleftrightarrow} Y_{k'}\right\}$$

and $N_{X,Y} = |B_{X,Y}|$. Let $\mathcal{S}$ denote the event $\{N_{X,Y} \leq k_0\}$. We first prove the following lemma.

Lemma 27. *Let n_X and n_Y denote the number of chunks in X and Y respectively. Fix an entry exit pair of chunks. For concreteness, take $((k,1),(n_X,k')) \in \mathcal{E}(X,Y)$. Fix $t \in [L_j^{\alpha-1} + 2L_j^3 + T_X]$ and $t' \in [L_j^{\alpha-1} + 2L_j^3 + T_Y]$ such that X_t is contained in C_k^X, $Y_{t'}$ contained in $C_{k'}^Y$ also such that $[t, t+L_j^3] \cap B_X = \emptyset$. Also let $A_{t,t'}$ denote the event that $[t, t+L_j^3] \times [1, L_j^3] \cup [L_j^{\alpha-1} + T_X + L_j^3, L_j^{\alpha-1} + T_X + 2L_j^3] \times [t' - L_j^3, t']$ is disjoint with $B_{X,Y}$. Set $\tilde{X} = (X_t, X_{t+1}, \ldots, X_{L_j^{\alpha-1} + T_X + 2L_j^3})$ and $\tilde{Y} = (Y_1, Y_2, \ldots, Y_{t'})$, call such a pair $(\tilde{X}, \tilde{Y})$ to be a* proper section *of (X,Y). Then there exists an event $S_{t,t'}$ with $\mathbb{P}[S_{t,t'} \mid \mathcal{F}] \geq 1 - L_{j+1}^{-4\beta}$ and such that on $S \cap S_{t,t'} \cap A_{t,t'}$, we have $\tilde{X} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}$.*

Proof. Set $I_1 = [t, L_j^{\alpha-1} + T_X + 2L_j^3] \cap \mathbb{Z}$, $I_2 = [1, t'] \cap \mathbb{Z}$. By Proposition 1 we can find L_j^2 admissible assignments mappings Υ_h with associated τ_h of (I_1, I_2) w.r.t. $(B_1^* \cap I_1, B_2^* \cap I_2)$ such that we have $\tau_h(\ell_i) = \tau_1(\ell_i) + h - 1$ and $\tau_h^{-1}(\ell'_i) = \tau_1^{-1}(\ell'_i) - h + 1$. As before we can construct a subset $\mathcal{H} \subset [L_j^2]$ with $|\mathcal{H}| = 10k_0 L_j < \lfloor L_j^2/36k_0^2 \rfloor$ so that for all $i_1 \neq i_2$ and $h_1, h_2 \in \mathcal{H}$ we have that $\tau_{h_1}(\ell_{i_1}) \neq \tau_{h_2}(\ell_{i_2})$ and $\tau_{h_1}^{-1}(\ell'_{i_1}) \neq \tau_{h_2}^{-1}(\ell'_{i_2})$, that is that all the positions bad blocks and their neighbours are assigned to are distinct.

Hence we have for all $h \in \mathcal{H}$

$$\mathbb{P}\left[X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h(\ell_i)} \mid \mathcal{F}\right] \geq \frac{1}{2} S_j(X_{\ell_i}); \tag{33}$$

$$\mathbb{P}\left[X_{\tau_h^{-1}(\ell_i')} \overset{c,c}{\longleftrightarrow} Y_{\ell_i'} \mid \mathcal{F}\right] \geq \frac{1}{2} S_j(Y_{\ell_i'}). \tag{34}$$

If $X_{\ell_i} \notin G_j^{\mathbb{X}}$, or, if neither X_{ℓ_i-1} nor X_{ℓ_i+1} is $\in G_j^{\mathbb{X}}$, let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X} = \left\{X_{\ell_i} \overset{c,c}{\longleftrightarrow} Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

If $X_{\ell_i}, X_{\ell_i+1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X} = \left\{X_{\ell_i} \overset{c,s}{\longleftrightarrow} Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

If $X_{\ell_i}, X_{\ell_i-1} \in G_j^{\mathbb{X}}$ then let $\mathcal{D}_{h,i,X}$ denote the event

$$\mathcal{D}_{h,i,X} = \left\{X_{\ell_i} \overset{s,c}{\longleftrightarrow} Y_{\tau_h^{k,k'}(\ell_i)}\right\}.$$

Let $\mathcal{D}_{h,X}$ denote the event

$$\mathcal{D}_{h,X} = \bigcap_{i=1}^{K_X'} \mathcal{D}_{h,i,X}$$

Let us define the event $\mathcal{D}_{h,Y}$ similarly and let

$$\mathcal{D}_h = \mathcal{D}_{h,X} \cap \mathcal{D}_{h,Y}$$

Finally, let

$$\mathcal{D} = \left\{\sum_{h\in\mathcal{H}} \mathbf{1}_{\mathcal{D}_h} \geq R^6 k_0^5 10^{2j+20}\right\}.$$

Conditional on $\mathcal{F}$, for $h \in \mathcal{H}$, the $\mathcal{D}_h$ are independent and by (33), (34) and the recursive estimates,

$$\mathbb{P}[\mathcal{D}_h \mid \mathcal{F}] \geq 2^{-10k_0} 3^{4k_0} L_j^{-2/3}.$$

Hence using a large deviation estimate for binomial tail probabilities we get,

$$\mathbb{P}\left[\mathcal{D} \mid \mathcal{F}\right] \geq \mathbb{P}\left[\mathrm{Bin}(10k_0 L_j, 2^{-10k_0} 3^{4k_0} L_j^{-2/3}) \geq R^6 k_0^5 10^{2j+20}\right] \geq 1 - L_{j+1}^{-4\beta}$$

for L_0 sufficiently large. Now it follows from Lemma 13 and Lemma 11 that if $\mathcal{D}$, $\mathcal{S}$, and $A_{t,t'}$ all holds than $\tilde{X} \overset{s,s,*}{\longleftrightarrow} \tilde{Y}$. This completes the proof of the lemma. □

Before proving Proposition 3, we need the following lemma bounding the probability of $\mathcal{S}$.

Lemma 28. *We have*

$$\mathbb{P}[\neg S \mid \mathcal{F}] \leq \frac{1}{3} L_{j+1}^{-3\beta}.$$

Proof. Let for $k' \in G_Y$,

$$V_{k'}^Y = I\left[\left\{\#\left\{k \in G_X : X_k \overset{s,s}{\not\longrightarrow} Y_{k'}\right\} \geq 1\right\}\right].$$

It follows from taking a union bound and using the recursive estimates that

$$\mathbb{P}\left[V_{k'}^Y = 1 \mid \mathcal{F}, X\right] \leq 2L_j^{\alpha-1-\beta}.$$

Since $V_{k'}^Y$ are conditionally independent given X and $\mathcal{F}$, a stochastic domination argument yields

$$\mathbb{P}\left[\sum_{k'} V_{k'}^Y \geq k_0^{1/2} \mid X, \mathcal{F}\right] \leq \mathbb{P}\left[\text{Bin}\left(2L_j^{\alpha-1}, 2L_j^{\alpha-1-\beta}\right) \geq k_0^{1/2}\right].$$

Using a Chernoff bound and setting $\lambda = k_0^{1/2} L_j^{-2\alpha+2+\beta}/4$ (note $\lambda > 1$ as $\beta > 2\alpha$ and L_0 is large enough) we get

$$\begin{aligned}\mathbb{P}\left[\sum_{k'} V_{k'}^Y \geq k_0^{1/2} \mid \mathcal{F}, X\right] &\leq \exp\left(4L_j^{2\alpha-2-\beta}(\lambda - 1 - \lambda\log\lambda)\right)\\ &\leq \exp\left(-2L_j^{2\alpha-2-\beta}\lambda\log\lambda\right)\\ &\leq \left(\frac{1}{4}k_0^{1/2}L_j^{-2\alpha+2+\beta}\right)^{k_0^{1/2}/2} \leq \frac{1}{6}L_{j+1}^{-3\beta}\end{aligned}$$

for L_0 large enough since $k_0^{1/2}(\beta + 2 - 2\alpha) > 6\alpha\beta$.

Removing the conditioning on X we get,

$$\mathbb{P}\left[\sum_{k'} V_k^Y \geq k_0^{1/2} \mid \mathcal{F}\right] \leq \frac{1}{6}L_{j+1}^{-3\beta}.$$

Defining V_k^X's similarly we get

$$\mathbb{P}\left[\sum_{k} V_k^X \geq k_0^{1/2} \mid \mathcal{F}\right] \leq \frac{1}{6}L_{j+1}^{-3\beta}.$$

Since on $\mathcal{F}$,

$$\neg\mathcal{S} \subseteq \left\{\sum_k V_k^X \geq k_0^{1/2}\right\} \cup \left\{\sum_k V_k^X \geq k_0^{1/2}\right\},$$

the lemma follows. □

Now we are ready to prove Proposition 3.

Proof of Proposition 3. Consider the set-up of Lemma 27. Let T_k (resp. $T'_{k'}$) denote the set of indices t (resp. t') such that X_t is contained in C_k^X (resp. $Y_{t'}$ is contained in $C_{k'}^Y$). It is easy to see that there exists $T_{k,*} \subset T_k$ (resp. $T'_{k',*} \subset T'_{k'}$) with $|T_{k,*}| \geq (1-10k_0L_j^{-1})|T_k|$ (resp. $|T'_{k',*}| \geq (1-10k_0L_j^{-1})|T'_{k'}|$) such that for all $t \in T_{k,*}$ and for all $t' \in T'_{k',*}$, $\tilde{X}$ and $\tilde{Y}$ defined as in Lemma 27 satisfies that $(\tilde{X}, \tilde{Y})$ is a *proper section* of (X,Y) and $A_{t,t'}$ holds.

It follows now by taking a union bound over all $t \in T_k$, $t' \in T'_{k'}$, and all pairs of entry exit chunks in $\mathcal{E}(X,Y)$ and using Lemma 7 that

$$\mathbb{P}\left[X \overset{s,s}{\longleftrightarrow} Y \mid \mathcal{F}\right] \geq 1 - \frac{1}{3}L_{j+1}^{-3\beta} - 4L_j^{2\alpha}L_{j+1}^{-3\beta} \geq 1 - L_{j+1}^{-3\beta}$$

for L_0 sufficiently large since $\beta > 2\alpha$. Now removing the conditioning we get (32). □

9 Good Blocks

Now we are ready to prove that a block is good with high probability.

Theorem 6. *Let X be a $\mathbb{X}$-block at level $(j+1)$. Then $\mathbb{P}(X \in G_{j+1}^{\mathbb{X}}) \geq 1 - L_{j+1}^{-\delta}$. Similarly for $\mathbb{Y}$-block Y at level $(j+1)$, $\mathbb{P}(Y \in G_{j+1}^{\mathbb{Y}}) \geq 1 - L_{j+1}^{-\delta}$.*

Proof. To avoid repetition, we only prove the theorem for $\mathbb{X}$-blocks. Let X be a $\mathbb{X}$-block at level $(j+1)$ with length $L_j^{\alpha-1}$.

Let the events $A_i, i = 1, \ldots 5$ be defined as follows.

$$A_1 = \left\{T_X \leq L_j^5 - 2L_j^3\right\}.$$

$$A_2 = \left\{\mathbb{P}\left[X \overset{c,c}{\longleftrightarrow} Y \mid X\right] \geq \frac{3}{4} + 2^{-(j+4)}\right\}.$$

$$A_3 = \left\{\mathbb{P}\left[X \overset{c,s}{\longleftrightarrow} Y \mid X\right] \geq \frac{9}{10} + 2^{-(j+4)}\right\}.$$

$$A_4 = \left\{\mathbb{P}\left[X \overset{s,c}{\longleftrightarrow} Y \mid X\right] \geq \frac{9}{10} + 2^{-(j+4)}\right\}.$$

$$A_5 = \left\{\mathbb{P}\left[X \overset{s,s}{\longleftrightarrow} Y \mid X\right] \geq 1 - L_j^{2\beta}\right\}.$$

From Lemma 14 it follows that

$$\mathbb{P}[A_1^c] \leq L_{j+1}^{-3\beta}.$$

From Lemmas 15 and 16 it follows that

$$\mathbb{P}[A_2^c] \leq L_{j+1}^{-3\beta}.$$

From (31) and Proposition 2 it follows that

$$\mathbb{P}[A_3^c] \le L_{j+1}^{-3\beta},\ \mathbb{P}[A_4^c] \le L_{j+1}^{-3\beta}.$$

Using Markov's inequality, it follows from Proposition 3

$$\begin{aligned}
\mathbb{P}[A_5^c] &= \mathbb{P}\left[\mathbb{P}\left[X \overset{s,s}{\not\longrightarrow} Y \mid X\right] \ge L_{j+1}^{-2\beta}\right] \\
&\le \mathbb{P}\left[X \overset{s,s}{\not\longrightarrow} Y\right] L_{j+1}^{2\beta} \\
&\le \Big(\mathbb{P}\left[X \overset{s,s}{\not\longrightarrow} Y, X \in \mathcal{A}_{X,j+1}^{(*)}, Y \in \mathcal{A}_{Y,j+1}^{(*)}\right] \\
&\qquad + \mathbb{P}\left[X \notin \mathcal{A}_{X,j+1}^{(*)}\right] + \mathbb{P}\left[Y \notin \mathcal{A}_{Y,j+1}^{(*)}\right]\Big) L_{j+1}^{2\beta} \\
&\le 3L_{j+1}^{-\beta}.
\end{aligned}$$

Putting all these together we get

$$\mathbb{P}\left[X \in G_{j+1}^{\mathbb{X}}\right] \ge \mathbb{P}[\cap_{i=1}^{5} A_i] \ge 1 - L_{j+1}^{-\delta}$$

for L_0 large enough since $\beta > \delta$. □

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Self-Avoiding Walks on the UIPQ

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We dedicate this work to Chuck Newman on the occasion of his 70th birthday

Abstract. We study an annealed model of Uniform Infinite Planar Quadrangulation (UIPQ) with an infinite two-sided self-avoiding walk (SAW), which can also be described as the result of glueing together two independent uniform infinite quadrangulations of the half-plane (UIHPQs). We prove a lower bound on the displacement of the SAW which, combined with the estimates of [15], shows that the self-avoiding walk is diffusive. As a byproduct this implies that the volume growth exponent of the lattice in question is 4 (as is the case for the standard UIPQ); nevertheless, using our previous work [9] we show its law to be singular with respect to that of the standard UIPQ, that is – in the language of statistical physics – the fact that disorder holds.

Keywords: Uniform Infinite Planar Quadrangulation · Random planar maps · Self-avoiding walk · Peeling process

1 Introduction

Much of the recent mathematical work on the geometry of random planar maps is focused on the "pure gravity" case where the random lattice is not affected by "matter": in probabilistic terms, this corresponds to choosing a map uniformly at random within a certain class, e.g. triangulations, quadrangulations, p-angulations... In this work we study the geometry of random planar quadrangulations weighted by the number of their self-avoiding walks (SAWs for short); that is, we study the model of annealed SAWs on random quadrangulations. We start by presenting our main objects of interest:

Surgeries. The Uniform Infinite Quadrangulation of the Plane (UIPQ), denoted by $\mathcal{Q}_\infty$, is the local limit of uniform random quadrangulations whose size is sent to infinity. This object has been defined by Krikun [21] following the earlier work of Angel and Schramm [6] in the case of triangulations; since then the UIPQ has attracted a lot of attention, see [2,3,12,14,17] and references therein.

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One can also define a related object (see [1,15]) called the Uniform Infinite Quadrangulation of the Half-Plane with a Simple Boundary, or simple boundary UIHPQ, denoted here[1] by $\mathcal{H}_\infty$, which is – as the name suggests – a random quadrangulation with an infinite simple boundary.

The simple boundary UIHPQ can be obtained as the local limit of uniform quadrangulations with n faces and a simple boundary of length $2p$ by first letting $n \to \infty$ and then $p \to \infty$ (see Sect. 2 for more details). From it, we shall construct two additional objects by means of "surgery" operations. First, we define a random infinite quadrangulation of the plane by folding the infinite simple boundary of $\mathcal{H}_\infty$ onto itself as in Fig. 1. The resulting map $\mathcal{Q}_\infty^{\rightarrow}$ is naturally endowed with an infinite one-ended self-avoiding path $(\mathsf{P}_i^{\rightarrow})_{i\geq 0}$ which is the image of the boundary of $\mathcal{H}_\infty$.

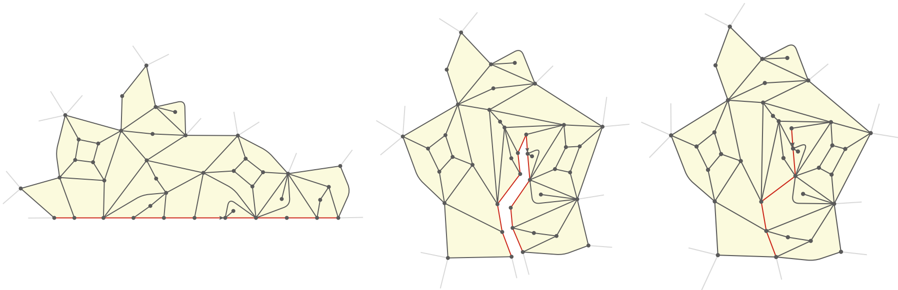


Fig. 1. A simple boundary UIHPQ and the resulting quadrangulation with a self-avoiding path obtained by folding the boundary onto itself.

We also perform a variant of the former construction. Consider two independent copies $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$ of the simple boundary UIHPQ and form a quadrangulation of the plane $\mathcal{Q}_\infty^{\leftrightarrow}$ by glueing together $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$ along their boundaries (identifying the root edges with opposite orientations). This rooted infinite quadrangulation also comes with a distinguished bi-infinite self-avoiding path $(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}}$ resulting from the identified boundaries.

These will be the main objects of study within this work. We will show in Sect. 2 that $(\mathcal{Q}_\infty^{\rightarrow}, \mathsf{P}^{\rightarrow})$ and $(\mathcal{Q}_\infty^{\leftrightarrow}, \mathsf{P}^{\leftrightarrow})$ are the natural models of annealed self-avoiding walks (respectively one-sided and two-sided) on the UIPQ, which means that they can be obtained as local limits of random objects uniformly sampled among quadrangulations endowed with a self-avoiding path.

Results. According to the physics literature [16], the three infinite random quadrangulations of the plane $\mathcal{Q}_\infty, \mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ should be described by the

[1] Remark that this notation is not coherent with that of [9], where we denoted by $\mathcal{H}_\infty$ an object with a boundary that is not necessarily simple, and by $\widetilde{\mathcal{H}}_\infty$ the one that is central to this paper, obtained from $\mathcal{H}_\infty$ by a pruning procedure. Since the general boundary UIHPQ will make no appearance in this paper, we shall drop the tilde with no fear of confusion.

same conformal field theory with central charge $c = 0$; that is to say, roughly speaking, the large scale properties of $\mathcal{Q}_\infty$, $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ should be close to each other. We confirm this prediction by showing that these random lattices share the same volume growth exponent of 4 (or "Hausdorff dimension", as it is commonly referred to by the physicists), a fact that is well-known in the case of the UIPQ, see [10,24]. The key is to first show that the self-avoiding walks on $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ are diffusive:

Theorem 1 (Diffusivity of the SAWs). *If* $(\mathsf{P}_i^{\rightarrow})_{i\geq 0}$ *and* $(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}}$ *are the edges visited by the self-avoiding walks on* $\mathcal{Q}_\infty^{\rightarrow}$ *and* $\mathcal{Q}_\infty^{\leftrightarrow}$ *then we have*

$$\mathrm{d_{gr}}(\mathsf{P}_0^{\rightarrow}, \mathsf{P}_n^{\rightarrow}) \approx \sqrt{n} \text{ and } \mathrm{d_{gr}}(\mathsf{P}_0^{\leftrightarrow}, \mathsf{P}_n^{\leftrightarrow}) \approx \sqrt{n}.$$

Notation: Here and later, for a random process $(X_n)_{n\geq 0}$ with values in $\mathbb{R}_+$ and a function $f : \mathbb{Z}_+ \to \mathbb{R}_+$, we write $X_n \preceq f(n)$ if

$$\lim_{a\to\infty} \limsup_{n\to\infty} \mathbb{P}(X_n > af(n)) = 0$$

and similarly for $X_n \succeq f(n)$ with the reversed inequality and a tending to 0. We write $X_n \approx f(n)$ if we have both $X_n \preceq f(n)$ and $X_n \succeq f(n)$.

The ball of radius r in a planar quadrangulation $\mathfrak{q}$ is the map $\mathfrak{B}_r(\mathfrak{q})$ obtained by keeping only those (internal) faces of $\mathfrak{q}$ that have at least one vertex at graph distance smaller than or equal to r from the origin of the map (as usual all our maps are *rooted*, that is given with one distinguished oriented edge whose tail vertex is the origin of the map) and keeping the root edge. The notation $\#\mathfrak{B}_r(\mathfrak{q})$ stands for the number of vertices in $\mathfrak{B}_r(\mathfrak{q})$.

Corollary 1 (Volume growth). *We have*

$$\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow}) \approx r^4$$

as well as

$$\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\leftrightarrow}) \approx r^4.$$

Since graph distances in $\mathcal{Q}_\infty^{\rightarrow}$ and in $\mathcal{Q}_\infty^{\leftrightarrow}$ are trivially bounded above by distances between corresponding vertices in $\mathcal{H}_\infty$ (and $\mathcal{H}'_\infty$) the lower bound for the volume growth in $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ follows from known results on the geometry of the UIHPQ. The nontrivial part of the statement is the upper bound, for whose proof we employ a lower bound on the displacement of the self-avoiding paths $\mathsf{P}^{\rightarrow}$ and $\mathsf{P}^{\leftrightarrow}$ (the upper bound also follows from known results on the UIHPQ [9]).

Although $\mathcal{Q}_\infty^{\leftrightarrow}$ and $\mathcal{Q}_\infty$ share the same volume growth exponent, we show that their laws are very different:

Theorem 2 (Glueing two half-planes does not produce a plane). *The two random variables* $\mathcal{Q}_\infty$ *and* $\mathcal{Q}_\infty^{\leftrightarrow}$ *are singular with respect to each other.*

In the language of statistical mechanics, the former result shows that *disorder* holds on the UIPQ, meaning that the (quenched) number of SAWs on $\mathcal{Q}_\infty$ is typically much less than its expectation (see Corollary 2). However, as we shall see, the proof of Theorem 2 does not involve enumerating self-avoiding walks, and is instead based on a volume argument. Unfortunately, this argument does not yield a proof of the similar result for $\mathcal{Q}_\infty^{\rightarrow}$ instead of $\mathcal{Q}_\infty^{\leftrightarrow}$, see Conjecture 1.

TECHNIQUES. In order to understand the geometry of the quadrangulations obtained from surgical operations involving the (simple boundary) UIHPQ, it is first necessary to deeply understand the geometry of the UIHPQ itself. To this end we devoted the paper [9], in which we first considered a general boundary UIHPQ, also obtained as a local limit of uniform random quadrangulations with a boundary (on which no simplicity constraint is imposed, see [15]), whose study is simpler thanks to its construction "à la Schaeffer" from a random infinite labelled tree. The results of [9] and [15] are nonetheless easily transferred to our context and yield the upper bound in Theorem 1 and the lower bound in Corollary 1.

In order to prove the diffusive lower bound for the self-avoiding walks, the main idea is to construct disjoint paths in the UIHPQ whose endpoints lie on the boundary and are symmetric around the origin, so that after the folding of the boundary these paths become disjoint nested loops separating the origin from infinity in $\mathcal{Q}_\infty^{\rightarrow}$, see Fig. 9 (a similar geometric construction is made in the case of $\mathcal{Q}_\infty^{\leftrightarrow}$). We build these paths inductively as close to each other as possible using the technique of peeling (see [1]) on the UIHPQ. We prove that we can construct $\approx n$ such paths on a piece of boundary of length $\approx n^2$ around the origin. Through the "folding" operation, this will yield the diffusivity of the self-avoiding walk $\mathsf{P}^{\rightarrow}$ (resp. $\mathsf{P}^{\leftrightarrow}$). Corollary 1 then follows easily by using rough bounds on the volume of balls in the UIHPQ.

The main ingredient in the proof of Theorem 2 is a series of *precise* estimates of the volume growth in the UIPQ and in its half-plane analogue. Indeed, the work of Le Gall & Ménard on the UIPQ [24,25] (see also [7, Chapitre 4]) as well as our previous work on the UIHPQ [9] show[2] that

$$\mathbb{E}[\#\mathfrak{B}_r(\mathcal{Q}_\infty)] \sim \frac{3}{28}r^4 \quad \text{and} \quad \mathbb{E}[\#\mathfrak{B}_r(\mathcal{H}_\infty)] \sim \frac{1}{12}r^4, \text{ as } \quad r \to \infty. \tag{1}$$

As the reader will see, the constants in the above display are crucial for our purpose: since the surgeries used to create $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ from $\mathcal{H}_\infty$ can only decrease distances we deduce that

$$\mathbb{E}[\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow})] \geq r^4/12 \quad \text{and} \quad \mathbb{E}[\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\leftrightarrow})] \geq 2 \times \frac{1}{12}r^4 = \frac{1}{6}r^4, \quad \text{as } r \to \infty.$$

Finally, the fact that $1/6 > 3/28$ implies that balls (of large radius) around the origin in $\mathcal{Q}_\infty^{\leftrightarrow}$ are typically larger than those in $\mathcal{Q}_\infty$. This fact applied to different

[2] Actually the works [9,24,25] show a convergence in distribution and one needs to prove uniform integrability to be able to pass to the expectation. We do not give the details since the actual proof bypasses this technical issue.

scales (so that the corresponding balls are roughly independent) is the core of the proof of Theorem 2. However, since $1/12 < 3/28$, this strategy does not work directly to prove that the laws of $\mathcal{Q}_\infty$ and of $\mathcal{Q}_\infty^{\rightarrow}$ are singular with respect to each other.

Remark 1. During the final stages of this work we became aware of the recent progresses of Gwynne and Miller [18–20], who study the scaling limits in the Gromov–Hausdorff sense of the objects considered in this paper. In particular in [18] they prove, roughly speaking, that the glueing of random planar quadrangulations along their boundaries defines a proper glueing operation in the continuous setting after taking the scaling limit (i.e. the image of the boundaries is a simple curve and the quotient metric does not collapse along the boundary). To do so, they use a peeling procedure which is equivalent to the one we study in Sect. 3. However, the estimates provided in [18] are much more precise than those required and proved in this paper (their work thus greatly improves upon our Sect. 3). Using the powerful theory developed by Miller & Sheffield, the work [18] combined with [19] yields an impressive description of the Brownian surfaces glued along their boundaries in terms of $\sqrt{8/3}$-Liouville Quantum Gravity surfaces. In particular according to [18], the scaling limit of $\mathcal{Q}_\infty^{\leftrightarrow}$ is a weight 4-quantum cone, the scaling limit of $\mathcal{Q}_\infty^{\rightarrow}$ is a weight 2-quantum cone, whereas the Brownian plane (scaling limit of $\mathcal{Q}_\infty$ itself) is a weight 4/3-quantum cone. This difference of laws in the scaling limit could probably be used instead of (1) as the main input to prove Theorem 2 and could probably yield a proof of our Conjecture 1.

2 Annealed Self-Avoiding Walks on Quadrangulations

We start by recalling notation and classical convergence results about random quadrangulations with a boundary. The curious reader may consult [2, 9, 15] for details.

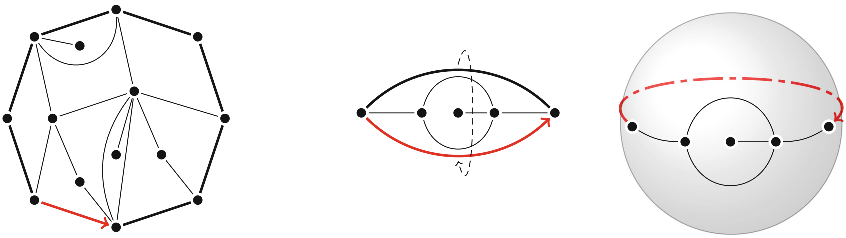
2.1 Quadrangulations with a Boundary

Recall that all the maps we consider here are planar and *rooted*, that is endowed with one distinguished oriented edge whose tail vertex is called the origin of the map.

A *quadrangulation with a boundary* $\mathfrak{q}$ is a planar map all of whose faces have degree four, with the possible exception of the face lying directly *to the right* of the root edge (also called the *external face*, or *outerface*). The external face of $\mathfrak{q}$, whose boundary is called by extension the boundary of $\mathfrak{q}$, necessarily has even degree (since $\mathfrak{q}$ is bipartite); we refer to this degree as the *perimeter* of $\mathfrak{q}$, while the *size* of $\mathfrak{q}$ is the number of its faces minus 1 (so the external face is excluded). We say that $\mathfrak{q}$ has a *simple* boundary if its boundary has no pinch point, that is, if it is a cycle with no self-intersection (see Fig. 2(a)).

We denote[3] by $\mathsf{Q}_{n,p}$ the set of all rooted quadrangulations with a simple boundary having size n and perimeter $2p$ (and by $\#\mathsf{Q}_{n,p}$ its cardinal). Within this paper, *all quadrangulations with a boundary will be implicitly required to have a simple boundary, unless otherwise stated.*

By convention, the set $\mathsf{Q}_{0,0}$ contains a unique "vertex" map; more importantly, $\mathsf{Q}_{0,1}$ is the set containing the unique map with one oriented edge (which has a simple boundary and no inner face). We remark that any quadrangulation with a boundary of perimeter 2 can be seen as a rooted quadrangulation of the sphere (i.e. without a boundary) by contracting the external face of degree two (see Fig. 2(b)); thus the set $\mathsf{Q}_{n,1}$ can be identified with the set of all (rooted) quadrangulations of the sphere with n faces, which we denote by Q_n.



(a) (b)

Fig. 2. (a) A quadrangulation in $\mathsf{Q}_{9,4}$. (b) The two boundary edges of the above quadrangulation from $\mathsf{Q}_{3,1}$ are "glued together" to obtain a rooted quadrangulation of the sphere with three faces (i.e. an element of Q_3) on the right.

From [8, Eq. (2.11)], we report estimates for the cardinals of the sets $\mathsf{Q}_{n,p}$, where $n \geq 0, p \geq 1$:

$$\#\mathsf{Q}_{n,p} = 3^{-p}\frac{(3p)!}{p!(2p-1)!}3^n\frac{(2n+p-1)!}{(n-p+1)!(n+2p)!}, \underset{n\to\infty}{\sim} C_p 12^n n^{-5/2}, \quad (2)$$

$$C_p = \frac{1}{2\sqrt{\pi}}\frac{(3p)!}{p!(2p-1)!}\left(\frac{2}{3}\right)^p \underset{p\to\infty}{\sim} \frac{\sqrt{3p}}{2\pi}\left(\frac{9}{2}\right)^p. \quad (3)$$

The sum of the series $\sum_{n\geq 0}\#\mathsf{Q}_{n,p}12^{-n}$ (which is finite) is classically denoted by $Z(p)$ and can be explicitly computed: we have $Z(1)=\frac{4}{3}$ and for $p \geq 2$,

$$Z(p) = 2\left(\frac{2}{3}\right)^p\frac{(3p-3)!}{p!(2p-1)!} \underset{p\to\infty}{\sim} \frac{2}{9\sqrt{3\pi}}p^{-5/2}\left(\frac{9}{2}\right)^p. \quad (4)$$

One can define a Boltzmann quadrangulation of the $2p$-gon as a random variable with values in $\bigcup_{n\geq 0}\mathsf{Q}_{n,p}$, distributed according to the measure that assigns a weight $12^{-n}Z(p)^{-1}$ to each map in $\mathsf{Q}_{n,p}$.

[3] Notice that this is in contrast with the notation of [9], where a distinction needed to be made between quadrangulations with a general boundary and ones whose boundary was required to be simple, which we usually signalled with a "tilde" over the relevant symbol.

In what follows, for all $n \geq 0$ and $p \geq 1$, we shall denote by $\mathcal{Q}_{n,p}$ a random variable uniformly distributed over $\mathsf{Q}_{n,p}$. When $p = 1$ we also denote $\mathcal{Q}_n := \mathcal{Q}_{n,1}$ a uniform quadrangulation with n faces.

2.2 Uniform Infinite (Half-)Planar Quadrangulations

Recall that if $\mathfrak{q}, \mathfrak{q}'$ are two rooted (planar) quadrangulations (with or without a boundary), the local distance between the two is

$$\mathrm{d_{loc}}(\mathfrak{q}, \mathfrak{q}') = \left(1 + \sup\{r \geq 0 : \mathfrak{B}_r(\mathfrak{q}) = \mathfrak{B}_r(\mathfrak{q}')\}\right)^{-1}, \tag{5}$$

where $\mathfrak{B}_r(\mathfrak{q})$ is obtained by erasing from $\mathfrak{q}$ everything but those inner faces that have at least one vertex at distance smaller than or equal to r from the origin (thus the outerface is not automatically preserved if $\mathfrak{q}$ has a boundary). The set of all finite quadrangulations with a boundary is not complete for this metric: we shall work in its completion, obtained by adding locally finite infinite quadrangulations with a finite or infinite simple boundary, see [14] for details. Recall that $\mathcal{Q}_{n,p}$ is uniformly distributed over $\mathsf{Q}_{n,p}$. The following convergences in distribution for $\mathrm{d_{loc}}$ are by now well known:

$$\mathcal{Q}_{n,p} \quad \xrightarrow[n\to\infty]{(d)} \quad \mathcal{Q}_{\infty,p} \quad \xrightarrow[p\to\infty]{(d)} \quad \mathcal{H}_\infty. \tag{6}$$

The first convergence in the special case $p = 1$ constitutes the definition of the UIPQ by Krikun [21]; the second one is found in [15] (see also the pioneering work [1] concerning the triangulation case). The object $\mathcal{Q}_{\infty,p}$ is the so-called UIPQ of the $2p$-gon and $\mathcal{H}_\infty$ is the simple boundary UIHPQ.

It is worthwhile to note (such a fact will be useful later) that $\mathcal{H}_\infty$ enjoys a property of invariance under rerooting: if we shift the root edge by one along the boundary (to the left or right), the random map thus obtained still has the law of a (simple boundary) UIHPQ.

2.3 Zipper

Let us now give a precise definition of a self-avoiding path:

Definition 1. *Let $\mathfrak{q}$ be a (finite or infinite) planar quadrangulation, and let $b \in \{0, 1, \ldots\} \cup \{\infty\}$, $f \in \{1, 2, \ldots\} \cup \{\infty\}$ ("b" stands for backward and "f" for forward). A (b, f)-SAW on $\mathfrak{q}$ is a sequence*

$$\mathbf{w} = (\vec{e}_i)_{-b \leq i < f}$$

of successive oriented edges of the map, where $\vec{e}_i$ has tail vertex x_i and target vertex x_{i+1}, so that the target of $\vec{e}_i$ coincides with the tail of $\vec{e}_{i+1}$, the oriented edge $\vec{e}_0$ is the root of $\mathfrak{q}$ (thus x_0 the origin) and the vertices in the sequence $(x_i)_{-b \leq i \leq f}$ are distinct, see Fig. 3.

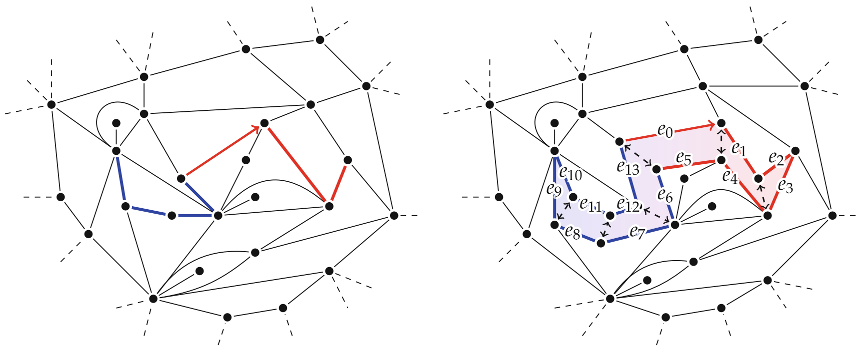


Fig. 3. Left: A $(4, 3)$-SAW in a rooted quadrangulation. Right: The function $Z^{4,3}$ is applied to a quadrangulation with a boundary of length 14 (whose outerface is drawn as finite to aid visualisation) to obtain the quadrangulation with a distinguished $(4, 3)$-SAW depicted on the Left.

We shall call $\mathsf{Q}_n^{b,f}$, the set of all pairs $(\mathfrak{q}, P)$, where $\mathfrak{q} \in \mathsf{Q}_n$ and P is a (b, f)-SAW on $\mathfrak{q}$ (so that the set $\mathsf{Q}_n^{0,1}$ is automatically identified with the set Q_n).

Fix $p \geq 1$. There is an obvious bijective correspondence between, on the one hand, the set $\mathsf{Q}_n^{0,p}$ of quadrangulations of size n with a $(0, p)$-SAW and, on the other hand, the set $\mathsf{Q}_{n,p}$ of quadrangulations with a simple boundary of perimeter $2p$ and size n; such a correspondence is an immediate generalisation of the one between $\mathsf{Q}_{n,1}$ and Q_n mentioned in Sect. 2.1 (Fig. 2(b)): simply let the self-avoiding walk act as a "zipper", eliminating the external face by pairwise identifying its edges.

In fact, we may generalize this construction further: for $b \geq 0, f \geq 1$ such that $b + f = p$ one can build a bijection $Z^{b,f}$ between the set of all finite quadrangulations with a simple boundary of length $2p$ and the set of all finite quadrangulations of the sphere endowed with a (b, f)-SAW. Such a mapping works as follows: write $\vec{e}_0, \ldots, \vec{e}_{2p-1}$ for the $2p$ edges of the boundary of a quadrangulation $\mathfrak{q} \in \mathsf{Q}_{n,p}$, taken in clockwise order and in such a way that $\vec{e}_0$ is the root edge, each edge oriented clockwise with respect to the outerface. We set $Z^{b,f}(\mathfrak{q})$ to be the quadrangulation of the sphere obtained by identifying $\vec{e}_i$ with $-\vec{e}_{2f-1-i}$ (where indices are to be read modulo $2p$ and the minus sign represents a change in orientation), endowed with the distinguished self-avoiding path of length p that is the image of the original cycle $\vec{e}_0, \ldots \vec{e}_{2p-1}$ and rooted at the image of $\vec{e}_0$, see Fig. 3.

Since the above mappings are bijections, if $\mathcal{Q}_{n,p}$ is uniformly distributed over $\mathsf{Q}_{n,p}$ then for any fixed quadrangulation of the sphere $\mathfrak{q}$ with n faces we have

$$\mathbb{P}\big(Z^{b,f}(\mathcal{Q}_{n,p}) = (\mathfrak{q}, \mathbf{w}) \text{ for some } (b, f)\text{-SAW } \mathbf{w}\big) = \frac{\#\mathsf{SAW}^{b,f}(\mathfrak{q})}{\#\mathsf{Q}_{n,b+f}}, \tag{7}$$

where $\#\mathsf{SAW}^{b,f}(\mathfrak{q})$ is the number of (b,f)-SAWs on $\mathfrak{q}$. In other words, the underlying quadrangulation of $Z^{b,f}(\mathcal{Q}_{n,p})$ is *not* uniformly distributed, but biased by its number of (b,f)-SAWs.

2.4 Annealed Infinite Self-Avoiding Walks on the UIPQ

One can extend the definition of the local distance to maps endowed with a distinguished SAW as a variant of (5), by providing an appropriate notion of a ball: if $(\mathfrak{q},(\vec{e}_i)_{-b-1<i<f})$ is a quadrangulation with a distinguished SAW of type (b,f), for each $r\geq 2$ we set

$$\mathfrak{B}_r(\mathfrak{q},(\vec{e}_i)_{-b-1<i<f})=\Big(\mathfrak{B}_r(\mathfrak{q}),(\vec{e}_i)_{-(b\wedge(r-1))-1<i<(f\wedge(r-1))}\Big).$$

For any fixed b,f, it is clear that the zipper map $Z^{b,f}$ is continuous for the local topology, hence one may deduce from (6) that for any $b\geq 0$, $f\geq 1$ such that $b+f=p$ one has $Z^{b,f}(\mathcal{Q}_{n,p})\to Z^{b,f}(\mathcal{Q}_{\infty,p})$ in distribution as $n\to\infty$. We are now interested in letting b and f tend to ∞.

Proposition 1 (Annealed UIPQs with SAW). *We have the following convergences in distribution for the local topology on quadrangulations endowed with a self-avoiding walk:*

$$Z^{0,p}(\mathcal{Q}_{\infty,p}) \xrightarrow[p\to\infty]{(d)} (\mathcal{Q}_\infty^{\rightarrow},(\mathsf{P}_i^{\rightarrow})_{i\geq 0}), \tag{8}$$

$$Z^{p,p'}(\mathcal{Q}_{\infty,p+p'}) \xrightarrow[p,p'\to\infty]{(d)} (\mathcal{Q}_\infty^{\leftrightarrow},(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}}), \tag{9}$$

where $(\mathcal{Q}_\infty^{\rightarrow},(\mathsf{P}_i^{\rightarrow})_{i\geq 0})$ can be obtained as $Z^{0,\infty}(\mathcal{H}_\infty)$ by "zipping up" the boundary of a UIHPQ, whereas $(\mathcal{Q}_\infty^{\leftrightarrow},(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}})$ is the result of the glueing of two independent UIHPQs along their boundaries (so that their root edges are identified with opposite orientations).

Proof. Consider the first convergence (8); we claim that it is a consequence of the second convergence in (6). To see this, notice first that we can extend the definition of the zipper map and consider $Z^{b,f}$ when one out of b,f is finite, the other infinite; such a correspondence maps an infinite quadrangulation with an infinite boundary to an infinite quadrangulation endowed with a (b,f)-SAW, as depicted in Fig. 4.

Lemma 1. *Let $\mathfrak{q}_p\to\mathfrak{q}_\infty$ be a sequence of quadrangulations with a boundary, $\mathfrak{q}_p$ having perimeter $2p$, which converges for $\mathrm{d_{loc}}$ towards an infinite quadrangulation with an infinite boundary; then we have*

$$Z^{0,p}(\mathfrak{q}_p) \xrightarrow[p\to\infty]{(\mathrm{d_{loc}})} Z^{0,\infty}(\mathfrak{q}_\infty).$$

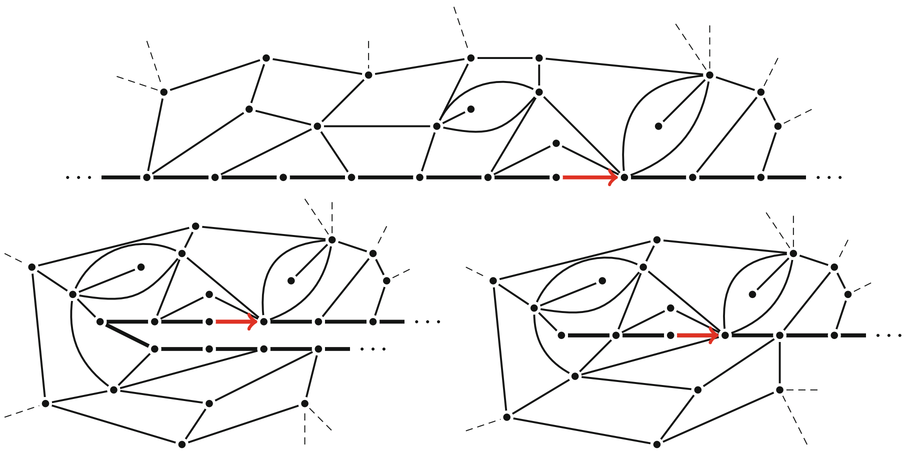


Fig. 4. The map $Z^{2,\infty}$ applied to a quadrangulation with an infinite boundary.

Proof. Fix $r \geq 1$. Although $\mathfrak{B}_r(Z^{0,\infty}(\mathfrak{q}_\infty))$ may not be a measurable function of $\mathfrak{B}_r(\mathfrak{q}_\infty)$ (because graph distances may be decreased by applying $Z^{0,\infty}$), it is easy to see that one can find $r' \geq r$ (depending on $\mathfrak{q}_\infty$) such that if $\mathfrak{B}_{r'}(\mathfrak{q}_\infty) = \mathfrak{B}_{r'}(\mathfrak{q}_p)$ then we have $\mathfrak{B}_r(Z^{0,p}(\mathfrak{q}_p)) = \mathfrak{B}_r(Z^{0,\infty}(\mathfrak{q}_\infty))$. This proves the lemma. □

Coming back to the proof of the theorem, by (6) and the Skorokhod embedding theorem one can suppose that $\mathcal{Q}_{\infty,p} \to \mathcal{H}_\infty$ almost surely. It thus follows from the above lemma that $Z^{0,p}(\mathcal{Q}_{\infty,p}) \to Z^{0,\infty}(\mathcal{H}_\infty)$ almost surely as $p \to \infty$. This proves the desired convergence in distribution.

We now move on to the second convergence (9), which is not this time a simple consequence of (6), as one cannot define $Z^{\infty,\infty}(\mathcal{H}_\infty)$. The idea is that the two parts of $\mathcal{Q}_{\infty,p+p'}$ which are facing together near the root edge in $Z^{p,p'}(\mathcal{Q}_{\infty,p+p'})$ are distant from each other when $p, p' \to \infty$ and become asymptotically independent. Here is the proper lemma from which the second convergence (9) immediately follows:

Lemma 2. *For $k \in \{0, \ldots, 2p\}$ denote by $\mathcal{Q}^{(k)}_{\infty,p}$ the random infinite quadrangulation with a boundary of perimeter $2p$ obtained by re-rooting $\mathcal{Q}_{\infty,p}$ at the k-th edge along the boundary of its external face. Then we have*

$$(\mathcal{Q}_{\infty,p}, \mathcal{Q}^{(k)}_{\infty,p}) \xrightarrow[\substack{k\to\infty \\ (2p-k)\to\infty}]{} (\mathcal{H}_\infty, \mathcal{H}'_\infty),$$

where $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$ are two independent copies of the UIHPQ.

Proof. Notice first that by invariance under re-rooting $\mathcal{Q}_{\infty,p}$ and $\mathcal{Q}^{(k)}_{\infty,p}$ have the same law and both converge in law towards $\mathcal{H}_\infty$ by (6). The only nontrivial point is the asymptotic independence. Let $r \geq 1$; we will show that the r-neighborhoods around the root edge and the k-th edge along the boundary of

$\mathcal{Q}_{\infty,p}$ become independent as $k \to \infty$ and $2p - k \to \infty$. We write $\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p})$ for the hull of the ball of radius r inside $\mathcal{Q}_{\infty,p}$: this is the submap obtained by filling in all the finite holes that $\mathfrak{B}_r(\mathcal{Q}_{\infty,p})$ together with the boundary of $\mathcal{Q}_{\infty,p}$ may create (recall that $\mathcal{Q}_{\infty,p}$ only has one end), see Fig. 5. Hence $\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p})$ is a finite quadrangulation with a simple boundary made up of two joined paths: one belonging to the boundary of $\mathcal{Q}_{\infty,p}$ (the outer boundary) and the other (the inner boundary) on which one needs to glue an infinite quadrangulation with a boundary in order to recover $\mathcal{Q}_{\infty,p}$ (see [13, Section 4.1] for a similar definition in the context of triangulations).

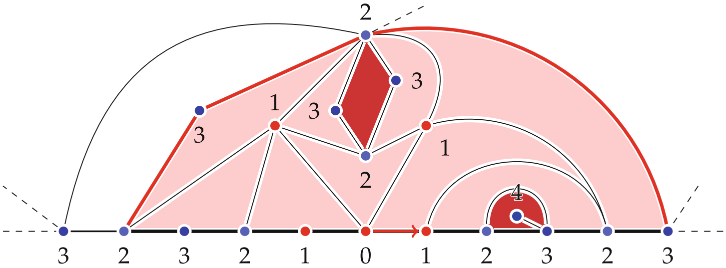


Fig. 5. The hull of the ball of radius 1 inside an infinite quadrangulation of the half-plane. The outer boundary is the thick black line and the inner boundary is in red.

The spatial Markov property of the UIPQ of the $2p$-gon (see [2,11,15]) shows that conditionally on $\{\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p}) = \mathfrak{a}\}$ the law of the remaining part of $\mathcal{Q}_{\infty,p}$ is that of of a UIPQ of the $(2p+\ell_{\text{in}}-\ell_{\text{out}})$-gon where ℓ_{in} and ℓ_{out} are respectively the length of the inner and outer boundary of $\mathfrak{a}$. Now if $k \to \infty$ and $2p - k \to \infty$, by local finiteness, it is very unlikely that $\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p})$ intersects $\mathfrak{B}^\bullet_r(\mathcal{Q}^{(k)}_{\infty,p})$ and, conditionally on the event that they are disjoint, by invariance under re-rooting the law of $\mathfrak{B}^\bullet_r(\mathcal{Q}^{(k)}_{\infty,p})$ is the same as that of $\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p+\ell_{\text{in}}-\ell_{\text{out}}})$, which converges to the law of $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ by (6). In particular this shows that $\mathfrak{B}^\bullet_r(\mathcal{Q}^{(k)}_{\infty,p})$ is asymptotically independent of $\{\mathfrak{B}^\bullet_r(\mathcal{Q}_{\infty,p}) = \mathfrak{a}\}$ as $k \to \infty$ with $2p - k \to \infty$. This yields our claim. □

2.5 Annealed and Quenched Connective Constants

In a lattice, the connective constant is generally defined as the exponential growth rate (when it exists) of the number of $(0,n)$-SAWs. In our context, given an infinite quadrangulation $\mathfrak{q}$, we call the connective constant of $\mathfrak{q}$ the quantity

$$\mu(\mathfrak{q}) = \limsup_{n\to\infty} \left(\#\mathsf{SAW}^{0,n}(\mathfrak{q})\right)^{1/n}.$$

Proposition 2 (Existence of the quenched connective constant). *The connective constant $\mu(\mathcal{Q}_\infty)$ of the UIPQ is almost surely constant.*

Proof. It follows from Lemma 2.1 of [22] that the value of the connective constant on an infinite connected locally finite graph does not depend on the starting point of the self-avoiding walks (actually Lacoin assumes a uniform bound on the degrees but local finiteness is sufficient for the proof). In particular, the value of the connective constant on the UIPQ is invariant by changing the root edge. By ergodicity of the UIPQ (see [2, Theorem 7.2] for the case of triangulations, which is easily adapted to our quadrangular case) any random variable which is invariant under changing the root edge must be almost surely constant (see [4, Theorem 3.1]). Hence $\mu(\mathcal{Q}_\infty)$ is almost surely constant. □

Although the value of the almost sure connective constant of the UIPQ (sometimes called the quenched connective constant) remains a mystery, we can precisely compute the average number of self-avoiding walks of any given type in the UIPQ.

Proposition 3 (Annealed connective constant). *With the same notation as (3), for any $b \geq 0$ and $f \geq 1$ we have*

$$\mathbb{E}[\#\mathsf{SAW}^{b,f}(\mathcal{Q}_\infty)] = \frac{C_{b+f}}{C_1} = \left(\frac{9}{2}\right)^{b+f+o(b+f)}.$$

Hence we could say that the "annealed" connective constant of the UIPQ is $9/2$.

Proof. Let $b \geq 0$ and $f \geq 1$. Having fixed n, if $\mathcal{Q}_n$ is a uniform quadrangulation of the sphere with n faces, by the bijection between quadrangulations endowed with a (b,f)-SAW and quadrangulations of the $2(b+f)$-gon we have (thanks to (7) and (3))

$$\mathbb{E}[\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)] = \frac{\#\mathsf{Q}_{n,b+f}}{\#\mathsf{Q}_{n,1}} \xrightarrow[n\to\infty]{} \frac{C_{b+f}}{C_1}.$$

On the other hand, the convergence of uniform quadrangulations towards the UIPQ implies that for any fixed b, f the random variables $\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)$ converge in law towards $\#\mathsf{SAW}^{b,f}(\mathcal{Q}_\infty)$ as $n \to \infty$. The statement of the proposition thus follows once we prove that $(\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n))_{n\geq 0}$ is uniformly integrable. In other words, for any $\varepsilon > 0$ we want to find $A > 0$ such that

$$\mathbb{E}[\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)\mathbf{1}_{\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)>A}] \leq \varepsilon$$

for all $n \geq 0$. If we denote $Z^{b,f}(\mathcal{Q}_{n,b+f}) = (\mathcal{Q}_n^{b,f}, \mathbf{w}_n^{b,f})$ we re-express the last quantity using the fact (7) that the density of $\mathcal{Q}_n^{b,f}$ with respect to $\mathcal{Q}_n$ is proportional to $\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)$:

$$\mathbb{E}\big[\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)\mathbf{1}_{\#\mathsf{SAW}^{b,f}(\mathcal{Q}_n)>A}\big] \underset{(7)}{=} \frac{\#\mathsf{Q}_{n,1}}{\#\mathsf{Q}_{n,(b+f)}}\mathbb{E}\big[\mathbf{1}_{\#\mathsf{SAW}^{b,f}\left(\mathcal{Q}_n^{b,f}\right)>A}\big]. \qquad (10)$$

Since we know that $\mathcal{Q}_n^{b,f}$ converges locally in distribution (see the discussion above Proposition 1), it follows that $\big(\#\mathsf{SAW}^{b,f}\big(\mathcal{Q}_n^{b,f}\big)\big)_{n\geq 1}$ converges in distribution as well and in particular is tight. Using this fact and the asymptotics (2) and (3) we can find A large enough so that the right-hand side of (10) is less than ε uniformly in $n \geq 0$ as desired. □

Open question 1 (Coincidence of the quenched and annealed connective constants). Combining the last two results we have $\mu(\mathcal{Q}_\infty) \leq 9/2$. Do we actually have a strict inequality?

Order and Disorder. The question of the coincidence of the quenched and annealed connective constants is usually referred to as weak/strong disorder in the statistical physics literature, see e.g. [22,23]. However, our context is different from the standard one where an underlying probability measure is tilted via a martingale biasing, so we shall use this section to clarify what we mean here by disorder.

For simplicity we restrict ourselves to the case of a two-sided SAW in order to connect this section with Theorem 2. To simplify notation a little, we shall write $\mathcal{Q}_\infty^{p,\leftrightarrow}$ for the underlying rooted quadrangulation of $Z^{p,p}(\mathcal{Q}_{\infty,2p})$. Since the random variable giving the number of self-avoiding paths of a given type is continuous for the local topology, we can combine Proposition 1 with (7) to deduce that the Radon–Nikodym derivative of $\mathcal{Q}_\infty^{p,\leftrightarrow}$ with respect to $\mathcal{Q}_\infty$ is given by

$$\frac{\mathrm{d}\mathcal{Q}_\infty^{p,\leftrightarrow}}{\mathrm{d}\mathcal{Q}_\infty} = \frac{C_1}{C_{2p}} \#\mathsf{SAW}^{p,p}(\mathcal{Q}_\infty).$$

Hence $\mathcal{Q}_\infty^{p,\leftrightarrow}$ is only a "mild" modification of $\mathcal{Q}_\infty$, since the laws of the two random quadrangulations are equivalent. However, the distortion effect might become dramatic as $p \to \infty$. Borrowing terminology from statistical physics, we will say that *disorder* holds if the law of $\mathcal{Q}_\infty^{\leftrightarrow} = \lim_p \mathcal{Q}_\infty^{p,\leftrightarrow}$ is singular with respect to that of $\mathcal{Q}_\infty$. This is exactly the content of Theorem 2 which we will prove below. First, however, let us state a direct corollary:

Corollary 2. *We have* $\frac{C_1}{C_{2p}} \#\mathsf{SAW}^{p,p}(\mathcal{Q}_\infty) \to 0$ *in probability as* $p \to \infty$.

In other words, as $p \to \infty$ the typical number of (p,p)-SAWs on the UIPQ becomes much less than its expectation. Notice that even when disorder holds, the quenched and the annealed connective constants may very well be equal.

Proof. We prove the result in greater generality. Let (E, d) be a Polish space and $\mu, \nu, (\mu_n)_{n\geq 0}$ be probability measures on E such that $\mu_n \to \mu$ in distribution as $n \to \infty$ and such that μ_n is absolutely continuous with respect to ν, with density f_n (here μ_p is the law of $\mathcal{Q}_\infty^{p,\leftrightarrow}$, μ the law of $\mathcal{Q}_\infty^{\leftrightarrow}$ and ν that of the UIPQ). Assuming ν and μ are singular with respect to each other, the goal is to prove that

$$f_n \to 0, \qquad \text{in probability for } \nu.$$

Notice that, in general, there is no equivalence between the fact that ν and μ are singular and the fact that $f_n \to 0$ in ν-probability. We pick a measurable subset A such that $\mu(A) = 0$ and $\nu(A) = 1$. By regularity we can find a closed subset $F \subseteq A$ such that $\nu(F) \geq 1 - \varepsilon$ and a fortiori $\mu(F) = 0$. By the Portmanteau theorem we thus have

$$0 = \mu(F) \geq \limsup_{n\to\infty} \mu_n(F) = \limsup_{n\to\infty} \int \mathrm{d}\nu \, f_n \mathbf{1}_F.$$

It follows that $\nu(\{f_n \geq \varepsilon\}) \leq \nu(\{f_n \geq \varepsilon\} \cap F) + \varepsilon \leq \varepsilon^{-1} \int \mathrm{d}\nu\, f_n \mathbf{1}_F + \varepsilon$, which is eventually less than 2ε by the above display. We have thus proved that $f_n \to 0$ in ν-probability as desired. □

3 Displacement of the Distinguished SAW in $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$

This section is devoted to proving Theorem 1. Recall the notation $(\mathsf{P}_i^{\rightarrow})_{i\geq 0}$ and $(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}}$ for the distinguished self-avoiding walks on $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ respectively. From our previous paper [9], the following is easily inferred:

$$\mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\rightarrow}}(\mathsf{P}_0^{\rightarrow}, \mathsf{P}_n^{\rightarrow}) \preceq \sqrt{n} \text{ and } \mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\leftrightarrow}}(\mathsf{P}_0^{\leftrightarrow}, \mathsf{P}_{\pm n}^{\leftrightarrow}) \preceq \sqrt{n}, \qquad n \geq 0, \tag{11}$$

where the notation $\mathrm{d}_{\mathrm{gr}}^{\mathfrak{q}}(\vec{u}, \vec{v})$ stands for the minimum graph distance between an endpoint of $\vec{u}$ and an endpoint of $\vec{v}$ in the quadrangulation $\mathfrak{q}$. This is quite immediate from [9, Proposition 6.1]: if we write $(x_i)_{i\in\mathbb{Z}}$ for the boundary vertices of a UIHPQ $\mathcal{H}_\infty$ (labelling them in the natural way, so that $x_0 \to x_1$ is the root edge) then the construction of $\mathcal{Q}_\infty^{\rightarrow}$ from $\mathcal{H}_\infty$ may only decrease distances, so that

$$\mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\rightarrow}}(\mathsf{P}_0^{\rightarrow}, \mathsf{P}_n^{\rightarrow}) \leq \mathrm{d}_{\mathrm{gr}}^{\mathcal{H}_\infty}(x_0, x_n).$$

Thus $\mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\rightarrow}}(\mathsf{P}_0^{\rightarrow}, \mathsf{P}_n^{\rightarrow}) \preceq \sqrt{n}$, and the case of $\mathcal{Q}_\infty^{\leftrightarrow}$ is analogous.

To establish Theorem 1 what we wish to obtain is a corresponding lower bound

$$\mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\rightarrow}}(\mathsf{P}_0^{\rightarrow}, \mathsf{P}_n^{\rightarrow}) \succeq \sqrt{n} \quad \text{ and } \quad \mathrm{d}_{\mathrm{gr}}^{\mathcal{Q}_\infty^{\leftrightarrow}}(\mathsf{P}_0^{\leftrightarrow}, \mathsf{P}_{\pm n}^{\leftrightarrow}) \succeq \sqrt{n}, \qquad n \geq 0. \tag{12}$$

We shall establish such a bound by constructing a sequence of nested "fences" $(P_i')_{i\geq 1}$ inside $\mathcal{H}_\infty$, that is a sequence of disjoint paths whose endpoints lie on the boundary of $\mathcal{H}_\infty$ and are of the form $x_{-r(i)}$ and $x_{r(i)}$ for an increasing integer sequence $r(i)$. After the folding of $\mathcal{H}_\infty$ to form $\mathcal{Q}_\infty^{\rightarrow}$ these paths will create nested loops so that $k \geq r(i)$ implies $\mathrm{d}_{\mathrm{gr}}(x_0, x_k) \geq i$, see Fig. 9.

The construction of these fences will be achieved by a (deterministic) algorithm which, when applied to the UIHPQ, will yield a random sequence $(P_i')_{i\geq 1}$ for which one wishes to control the "growth" on the boundary $(r(i))_{i\geq 1}$. This will be possible thanks to the spatial Markov property of the simple boundary UIHPQ.

3.1 Building One Fence

Suppose you are given a (deterministic) one-ended quadrangulation $\mathfrak{h}$ with an infinite simple boundary on the sequence of (successive) vertices $(v_i)_{i\in\mathbb{Z}}$ so that $v_0 \to v_1$ is the root edge, and a positive integer k.

We shall build a 'fence' P that avoids vertices $v_1, \ldots, v_k$ via a peeling process. We first set e_0 to be the root edge and then iteratively perform the following loop, starting with $i = 0$:

– reveal the face f_i lying left of e_i;
– consider the *rightmost* vertex v_{ρ_i} of f_i lying on the boundary of $\mathfrak{h}$; set e_{i+1} to be the *rightmost* edge in $\mathfrak{h}$ which has v_{ρ_i} as an endpoint and belongs to f_i (oriented towards v_{ρ_i});
– if $\rho_i > k$, then `STOP`; otherwise restart the loop, increasing i by 1.

Notice that each revealed face does have a vertex on the boundary, thus the operations required are well defined, and that the sequence $(\rho_i)_{i\geq 0}$ is weakly increasing, so that (thanks to local finiteness of $\mathfrak{h}$) the algorithm does eventually terminate (see Fig. 6).

Once the end condition is met (at – say – iteration T, where iterations are numbered from 0), we have a final (connected) set of revealed faces $F = \{f_0, \dots, f_T\}$. We consider then the hull $F^\bullet$ of F obtained by "filling in" any finite holes between F and the boundary of $\mathfrak{h}$, and set our fence P to be the inner boundary of $F^\bullet$ (i.e. the part of the boundary of $F^\bullet$ which is not in common with the boundary of $\mathfrak{h}$). It is easy to show that P is indeed a simple path, that one of its endpoints is v_{r+k} for some $r \geq 1$ while the other is some $v_{1-\ell}$ with $\ell \geq 1$, and that it has no vertices on the boundary of $\mathfrak{h}$ except for its endpoints, so that it does not intersect $\{v_1, \dots, v_k\}$.

Definition 2. *We call the quantities ℓ and r respectively the* left *and* right overshoots *of the construction.*

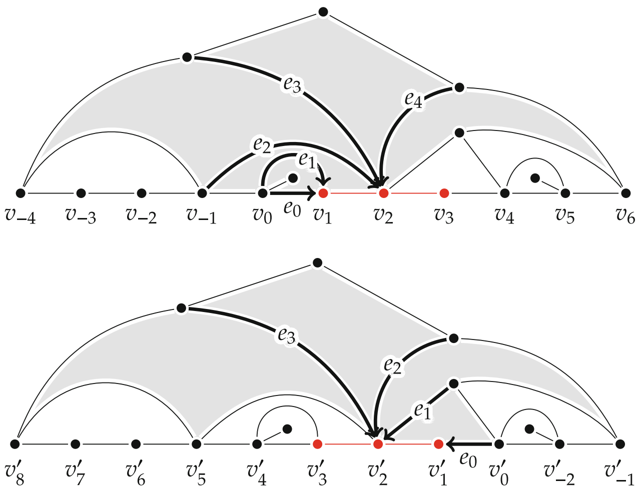


Fig. 6. Above, the algorithm run with $k = 3$ reveals 5 faces; the left overshoot is $|-4|+1 = 5$ and the right overshoot is $6-3 = 3$. Below, the flipped algorithm reveals only 3 faces, but produces the same "fence", with a left overshoot of $|-2|+1 = 3$ and a right overshoot of $8-3 = 5$.

Remark 2. We make here an alternative direct definition of $F^\bullet$ which will be useful later. We claim that $F^\bullet$ is also the hull of the set of faces of the quadrangulation having (at least) a vertex in the set $\{v_1, \ldots, v_k\}$. One inclusion is clear, since all faces with a vertex in $\{v_1, \ldots, v_k\}$ must lie below P, which separates $\{v_1, \ldots, v_k\}$ from infinity; the other is also clear, since all faces revealed during the construction of P have a vertex in the set $\{v_1, ..., v_k\}$.

This alternative construction of P highlights the inherent symmetry in the roles of the left and right overshoots. If we flip the quadrangulation $\mathfrak{h}$ (exchanging left and right) and relabel its boundary vertices as $(v'_i)_{i\in\mathbb{Z}}$ so that v'_i is v_{k+1-i}, then perform the algorithm to build a fence as above (starting with $e'_0 = (v'_0, v'_1)$, which corresponds to (v_{k+1}, v_k)), then the left overshoot of this new fence is the right overshoot of P and vice versa (see Fig. 6).

3.2 One Fence in the Simple Boundary UIHPQ

Given $k \geq 0$ and a copy of the simple boundary UIHPQ $\mathcal{H}_\infty$, whose boundary vertices we call $(x_i)_{i\in\mathbb{Z}}$, we can use the above algorithm to discover $F^\bullet$ and build a fence P enclosing the vertices $x_1, \ldots, x_k$. The spatial Markov property of the UIHPQ already used in the proof of Lemma 2 then show that conditionally on $F^\bullet$ the remaining part $\mathcal{H}_\infty \backslash F^\bullet$ (rooted for example at the rightmost edge on the boundary of $\mathcal{H}_\infty$ which lies to the left of $F^\bullet$) has the law of a UIHPQ. The easiest way to see this is to say that $F^\bullet$ has been discovered by the mean of a peeling process (we shall discuss this later in more detail).

In this setting we denote by $X^{+,(k)}$ and $X^{-,(k)}$ respectively the right and left overshoots in the construction of P. Obviously the law of these overshoots depends on k, but the first observation we can make is that Remark 2 (combined with the fact that the law of the UIHPQ is invariant under "flipping") implies that for all $k \geq 1$

$$X^{+,(k)} = X^{-,(k)} \qquad \text{in distribution.} \tag{13}$$

This being said, we will stochastically bound the variable $X^{-,(k)}$ by a random variable which is independent of k. To do so, we may consider the peeling process on the UIHPQ which consists in running the loop of Sect. 3.1 indefinitely (roughly speaking by setting $k = \infty$), and set X to be the (random) index of the leftmost boundary vertex of a face that is eventually revealed by it; it can be shown that X is almost surely well defined (i.e. finite), and naturally we have $X^{-,(k)} \leq |X| + 1$ stochastically for all $k \geq 1$.

The explicit construction of the peeling process makes it possible to show the following:

Lemma 3. *For the peeling process based on Sect. 3.1 applied to $\mathcal{H}_\infty$ with $k = \infty$, the (random) index X of the leftmost boundary vertex belonging to a face that is eventually revealed is such that*

$$\sup_{n\geq 0} \sqrt{n}\mathbb{P}(X \leq -n) < \infty.$$

For the sake of completeness, we shall first devote a subsection to an explicit description of the peeling process (based on [3], to which we refer the reader for details), and then use it to give a proof of the above lemma. Before doing so, we deduce the technical corollary that we will use:

Corollary 3. *There exists a (law of a) random variable $O \geq 1$ whose tail satisfies $\mathbb{P}(O \geq n) \leq Cn^{-1/2}$ for some $C > 0$ and such that for any $k \geq 1$ we can couple O and the overshoots $(X^{-,(k)}, X^{+,(k)})$ so that*

$$\max(X^{-,(k)}, X^{+,(k)}) \leq O.$$

Proof. It suffices to estimate the tail of $\max(X^{-,(k)}, X^{+,(k)})$. For $n \geq 1$ we have

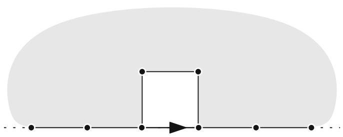
$$\begin{aligned}\mathbb{P}(\max(X^{-,(k)}, X^{+,(k)}) \geq n) &\leq \mathbb{P}(X^{+,(k)} \geq n/2) + \mathbb{P}(X^{-,(k)} \geq n/2)\\ &\underset{(13)}{=} 2\mathbb{P}(X^{-,(k)} \geq n/2)\\ &\leq 2\mathbb{P}(-X \geq n/2 - 1) \underset{\text{Lem. 3}}{\leq} Cn^{-1/2}.\end{aligned}$$

The statement of the corollary then follows from standard coupling arguments. □

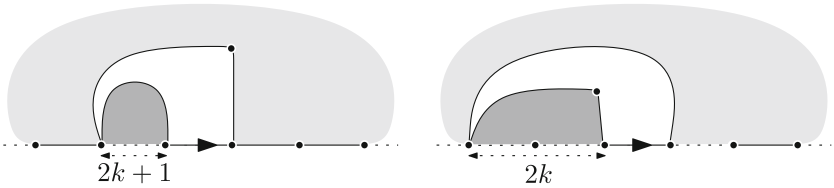
3.3 The Peeling Process

Let $\mathcal{H}_\infty$ be a simple boundary UIHPQ and let us reveal the quadrangular face that contains the root edge (an operation which we may call *peeling* the root edge). The revealed face can separate the map into one, two or three regions, only one of which is infinite according to the cases listed below. Conditionally on each of these cases, such regions are independent from each other, the infinite one (in light grey in the following figures) always being a copy of a UIHPQ while the finite ones (in dark grey) are Boltzmann quadrangulations of appropriate perimeter. We shall call edges of the revealed face that belong to the infinite region *exposed* edges; edges of the boundary of $\mathcal{H}_\infty$ that belong to the finite regions will be said to have been *swallowed*, and we will distinguish edges that are swallowed *to the left* and *to the right* according to whether they lie to the left or right of the root edge being peeled (see Fig. 8). We distinguish the following cases:

C Firstly, the revealed face may have exactly two vertices on the boundary of $\mathcal{H}_\infty$. In this case we say that the form of the quadrangle revealed is C (for center).



$\mathsf{R}_\ell, \mathsf{L}_\ell$ The revealed face can also have three of its vertices lying on the boundary of $\mathcal{H}_\infty$ and one in the interior, thus separating the map into a region with a finite boundary and one with an infinite boundary. We have two subcases, depending on whether the third vertex lies on the left (case L_ℓ) or on the right (case R_ℓ) of the root edge. Suppose for example that the third vertex lies on the left of the root edge; the fourth vertex of the quadrangle may lie on the boundary of the finite region or of the infinite region. Since all quadrangulations are bipartite, this is determined by the parity of the number ℓ of *swallowed* edges (see the figure below for the case of L_ℓ).



$\mathsf{L}_{\ell_1,\ell_2}$, $\mathsf{C}_{\ell_1,\ell_2}$, $\mathsf{R}_{\ell_1,\ell_2}$ The last case to consider is when the revealed quadrangle has all of its four vertices on the boundary. In this case the revealed face separates from infinity two segments of length ℓ_1 and ℓ_2 along the boundary, as depicted in the figure below. This could happen in three ways, as 0, 1, or 2 vertices could lie to the right of the root edge (see Fig. 7) and the corresponding subcases are denoted by $\mathsf{L}_{\ell_1,\ell_2}$, $\mathsf{C}_{\ell_1,\ell_2}$ and $\mathsf{R}_{\ell_1,\ell_2}$. Notice that the numbers $\ell_1 = 2k_1 + 1$ and $\ell_2 = 2k_2 + 1$ must both be odd.

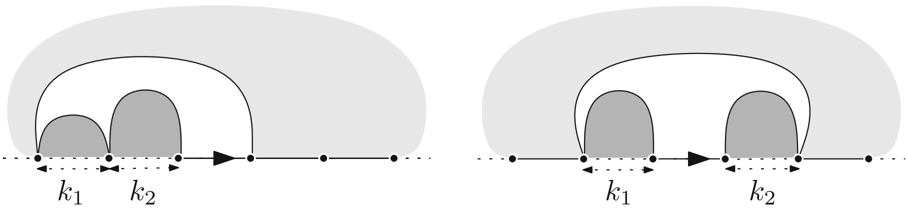


Fig. 7. Cases $\mathsf{L}_{\ell_1,\ell_2}$ and $\mathsf{C}_{\ell_1,\ell_2}$.

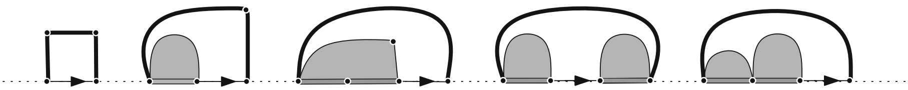


Fig. 8. The exposed edges are in fat black lines and the swallowed ones are in fat gray lines (the remaining cases are symmetric).

The exact probabilities of these events can be computed explicitly (see [3]) but we will only use the fact [3, Section 2.3.2] that, if $\mathcal{E}$ and $\mathcal{S}$ denote the number of edges respectively exposed and swallowed by a peeling step,

$$\mathbb{E}[\mathcal{E}] = 2, \quad \mathbb{E}[\mathcal{S}] = 1, \quad \text{and} \quad \mathbb{P}(\mathcal{S} = k) \sim Ck^{-5/2} \tag{14}$$

for some constant $C > 0$ as $k \to \infty$.

By iterating the one-step peeling described above one can define a growth algorithm that discovers (a subset of) the simple boundary UIHPQ step by step.

A *peeling process* is a randomized algorithm that explores $\mathcal{H}_\infty$ by revealing at each step the face in the unexplored part adjacent to a given edge, together with any finite regions that it encloses; in order to choose the next edge to peel, one can use the submap of $\mathcal{H}_\infty$ that has already been revealed and possibly another source of randomness as long as the choice remains independent of the unknown region (see [3, Section 2.3.3] for details). Under these assumptions the one-step peeling transitions and the invariance of $\mathcal{H}_\infty$ under re-rooting along the boundary show that the peeling steps are i.i.d., see [3, Proposition 4].

Notice that this is definitely the case with the algorithm described in Sect. 3.1, which in fact consists of the one-step peeling described above, with the chosen edge at each step being simply the rightmost exposed edge of the most recently revealed face (until the algorithm stops).

Furthermore, consider the number of *exposed*, *left swallowed* and *right swallowed* edges at each peeling step; each of these quantities is a random variable whose law can be computed explicitly thanks to the probabilities of the various events listed above, by referring to Table 1.

3.4 Overshoot Estimates

We now can proceed with the proof of Lemma 3.

Table 1.

Case	Exposed	Left swallowed	Right swallowed	ΔY
C	3	0	0	2
L_{2k}	1	$2k$	0	$-2k$
L_{2k+1}	2	$2k+1$	0	$-2k$
R_{2k}	1	0	$2k$	0
R_{2k+1}	2	0	$2k+1$	1
L_{k_1,k_2}	1	k_1+k_2	0	$-k_1-k_2$
C_{k_1,k_2}	1	k_1	k_2	$-k_1$
R_{k_1,k_2}	1	0	k_1+k_2	0

Proof of Lemma 3. Consider the peeling process on $\mathcal{H}_\infty$ as defined in Sect. 3.1, run indefinitely (with $k=\infty$). Steps are numbered from 0, and step i reveals a face f_i and outputs an oriented edge e_{i+1} incident to f_i, with an endpoint x_{ρ_i} on the boundary $(x_i)_{i\in\mathbb{Z}}$ of $\mathcal{H}_\infty$; the edge e_{i+1} is then peeled at step $i+1$; we denote by $\mathcal{H}_\infty(i)$ the map obtained from $\mathcal{H}_\infty$ after removing the hull of the faces discovered up to time i. We already know that $\mathcal{H}_\infty(i)$ – appropriately rooted – is distributed as a UIHPQ. We consider the section γ_i of the boundary of $\mathcal{H}_\infty(i)$ lying left of e_{i+1}. This is of course made of infinitely many edges, but γ_i and γ_0

differ by only finitely many edges. This makes possible to define for each $i \geq 0$ a quantity Y_i which represents the algebraic variation of the "length" of γ_i with respect to γ_0. The formal definition of Y_i is given from $Y_0 = 0$ via its variation $\Delta Y_i = Y_{i+1} - Y_i$ equal to the number of exposed edges minus the number of swallowed edges on the left of the current point minus 1 (see the above table). Clearly the definition of (Y) and the properties of the peeling process entail that (Y) is a random walk with i.i.d. increments whose law can be explicitly computed. In particular, $\mathbb{E}[\Delta Y] = \mathbb{E}[\mathcal{E}] - 1 - \mathbb{E}[\mathcal{S}]/2 = \frac{1}{2}$ with the notation of (14). Since (Y) has a positive drift we can define the overall infimum

$$\inf_{i \geq 0} Y_i > -\infty.$$

An easy geometric argument then shows that the variable X we are after is just $\inf_{i \geq 0} Y_i$. The tail of the overall infimum of the transient random walk (Y) can be estimated from the tail of ΔY (which is given by (14)) using [27, Theorem 2]. It follows that for some $C' > 0$

$$\mathbb{P}(X \leq -n) = \mathbb{P}\left(\inf_{i \geq 0} Y_i \leq -n\right) \leq C' n^{-1/2}, \quad \text{for all } n \geq 1.$$

3.5 Building the Final Fences in $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$

Now, in order to conclude our proof of Theorem 1, we need a little tweaking of the fence-building algorithm, so as to have fence endpoints coincide in the case of $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$.

Consider first $\mathcal{Q}_\infty^{\rightarrow}$, built from $\mathcal{H}_\infty$ by glueing the boundary onto itself as described in the Introduction. Start by setting $k = 1$, reroot $\mathcal{H}_\infty$ at $e_0 = (x_{-1}, x_0)$, and build a fence P_1 avoiding x_0 whose endpoints are $x_{-\ell_1}$ and x_{r_1} as described in Sect. 3.1; let $X_1^+ = r_1$ and $X_1^- = \ell_1$ be its right and left overshoots respectively. Before building P_2 we set P_1' to be P_1 with the addition of the portion of the boundary between x_{-r_1} and $x_{-\ell_1}$ if $r_1 > \ell_1$, or between x_{r_1} and x_{ℓ_1} if $r_1 < \ell_1$. Hence the path P_1' connects symmetric vertices on the boundary. We then consider the map $\mathcal{H}_\infty(1)$ obtained by erasing the region of $\mathcal{H}_\infty$ lying below P_1' (or, equivalently, below P_1), rooted at the first boundary edge on the left of P_1'. The map $\mathcal{H}_\infty(1)$ is a copy of the UIHPQ independent of the part erased. We then build P_2 by running the algorithm from Sect. 3.1 inside $\mathcal{H}_\infty(1)$, setting k to be the number of edges of P_1' plus 1. We then extend P_2 into P_2' as above to make it connect mirror vertices. Iterating the process we build disjoint paths P_i' connecting mirror vertices $x_{-r(i)}$ to $x_{r(i)}$.

Consider now the sequence $(P_i')_{i \geq 1}$ as seen in $\mathcal{Q}_\infty^{\rightarrow}$ (Fig. 9): the fences now form nested disjoint loops. By planarity if $j \geq r(i)$ then the j-th point on the distinguished self-avoiding walk $\mathsf{P}^{\rightarrow}$ is at distance at least i from the origin in $\mathcal{Q}_\infty^{\rightarrow}$. Our lower bound (12) (LHS) is then implied if we can show that

$$r(n) \preceq n^2. \tag{15}$$

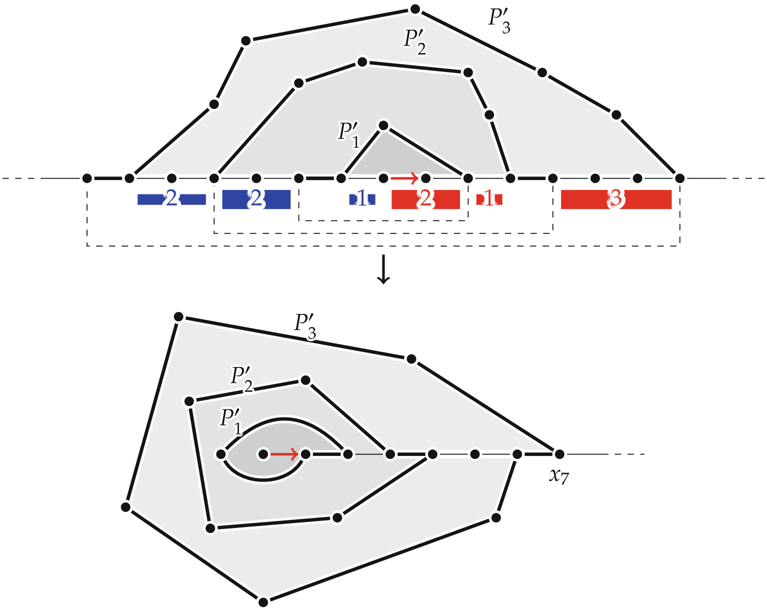


Fig. 9. The fences $(P_i)_{i\geq 1}$ are the subpaths of the fences $(P'_i)_{i\geq 1}$ above obtained by disregarding boundary edges; we have $X_1^+ = 2$, $X_2^+ = 1$, $X_3^+ = 3$, $X_1^- = 1$, $X_2^- = 2$, $X_3^- = 2$. The overshoots of P'_i are both equal to $\max\{X_i^+, X_i^-\}$ (hence 2, 2, 3 for $i = 1, 2, 3$), and $r(i)$ (see the folded version) is $X_1 + \ldots + X_i$ (e.g. we have $r(3) = 7$).

If we denote by X_i^- and X_i^+ the left and right overshoots of P_i, then

$$r(n) = \sum_{j=1}^{n} \max(X_j^+, X_j^-).$$

Notice that the overshoots $X_1^+, X_1^-, \ldots, X_i^+, X_i^-$ are *not* independent nor identically distributed; however, using Corollary 3 we can couple those overshoots with a sequence of i.i.d. random variables $(O_i)_{i\geq 1}$ having the law prescribed in Corollary 3 so that $\max(X_j^+, X_j^-) \leq O_j$ for all $j \geq 1$. Hence in this coupling we have $r(n) \leq O_1 + \cdots + O_n$. Standard estimates for i.i.d. variables with heavy tails then show that $O_1 + \cdots + O_n \preceq n^2$ which proves our goal (15) which – combined with Corollary 11 – proves the first part of Theorem 1.

The case of $\mathcal{Q}_\infty^{\leftrightarrow}$ is essentially the same. Supposing $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$ are the two independent "halves" sharing the distinguished path $(\mathsf{P}_i^{\leftrightarrow})_{i\in\mathbb{Z}}$ on the vertices $(x_i)_{i\in\mathbb{Z}}$, one proceeds to build a fence P_1 within $\mathcal{H}_\infty$ that avoids x_0, then a fence Q_1 in $\mathcal{H}'_\infty$ that does the same; one then builds P'_1 and Q'_1 by adding boundary edges to P_1 and Q_1 so that the endpoints of P'_1 coincide with those of Q'_1 (see Fig. 10), and thus the right overshoot of P'_1 is the maximum between the right overshoot of P_1 and the left overshoot of Q_1. Iterating such a construction and applying the very same estimates as for the case of $\mathcal{Q}_\infty^{\rightarrow}$ finally shows Theorem 1.

4 Volume Estimates and Singularity

4.1 Proof of Corollary 1

Proof of Corollary 1. We treat the case of $\mathcal{Q}_\infty^{\rightarrow}$, the argument being similar in the case of $\mathcal{Q}_\infty^{\leftrightarrow}$. We assume that $\mathcal{Q}_\infty^{\rightarrow}$ has been constructed from $\mathcal{H}_\infty$ by folding its boundary onto itself. Since the surgery operation performed to create $\mathcal{Q}_\infty^{\rightarrow}$ from $\mathcal{H}_\infty$ can only decrease distances we have $\mathfrak{B}_r(\mathcal{H}_\infty) \subseteq \mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow})$ and it follows from the estimates on the volume growth in the UIHPQ [9, Proposition 6.2] that

$$\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow}) \succeq r^4.$$

Let us now turn to the upper bound. Consider the portion of the folded boundary of $\mathcal{H}_\infty$ that is inside $\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow})$. By Theorem 1 the length L_r of this portion is $\preceq r^2$. Hence it is immediate that $\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow})$ is contained in the set of all faces having one vertex at distance at most r from the boundary $[\![-L_r, L_r]\!]$ in $\mathcal{H}_\infty$. Let us denote by M_r the maximal distance in $\mathcal{H}_\infty$ to the origin of this piece of boundary. By the above argument we have

$$\mathfrak{B}_r(\mathcal{Q}_\infty^{\rightarrow}) \subseteq \mathfrak{B}_{M_r+r}(\mathcal{H}_\infty).$$

But applying the distance estimates along the boundary inside the UIHPQ ([9, Proposition 6.1]) one deduces that $M_r \approx \sqrt{L_r} \approx r$ and using volume estimates once more we get $\#\mathfrak{B}_{M_r+r}(\mathcal{H}_\infty) \approx r^4$, which completes the proof of the upper bound and hence of Corollary 1. □

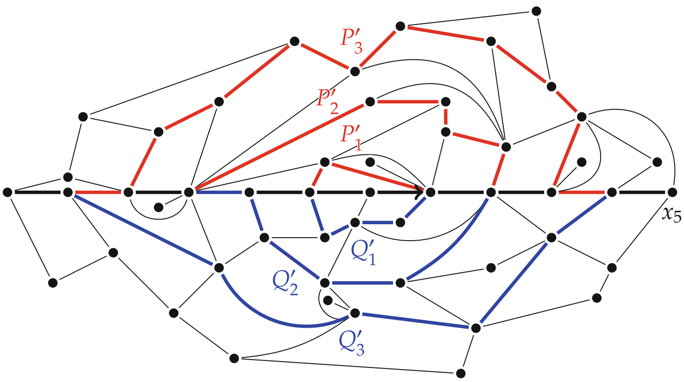


Fig. 10. Illustration of the construction of the fences in the case of $\mathcal{Q}_\infty^{\leftrightarrow}$.

4.2 Proof of Theorem 2

We first recall the scaling limit results that we will need. The work of Le Gall & Ménard on the UIPQ [24,25] as well as our previous work on the UIHPQ [9] show that there are two random variables $\mathcal{V}_p$ and $\mathcal{V}_h$ such that

$$r^{-4}\#\mathfrak{B}_r(\mathcal{Q}_\infty) \to \mathcal{V}_p \quad \text{and} \quad r^{-4}\#\mathfrak{B}_r(\mathcal{H}_\infty) \to \mathcal{V}_h,$$

in distribution as $r \to \infty$. The expectations of such continuous random variables have been computed and in particular $\mathbb{E}[\mathcal{V}_h] = \frac{7}{9}\mathbb{E}[\mathcal{V}_p]$. We will only use a trivial consequence of these calculations: if $\mathcal{V}_h'$ is an (independent) copy of $\mathcal{V}_h$ since $\mathbb{E}[\mathcal{V}_h + \mathcal{V}_h'] > \mathbb{E}[\mathcal{V}_p]$ one can find $\alpha > 0$ such that

$$\mathbb{P}(\mathcal{V}_h + \mathcal{V}_h' > \alpha) > \mathbb{P}(\mathcal{V}_p > \alpha), \tag{16}$$

we can and will furthermore assume that α is not an atom for the law of $\mathcal{V}_p$ nor for that of $\mathcal{V}_h + \mathcal{V}_h'$ (this is possible since there are at most a countable number of atoms for each law).

By construction, the volume of the ball of radius r inside $\mathcal{Q}_\infty^{\leftrightarrow}$ is at least $\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty')$, where $\mathcal{H}_\infty$ and $\mathcal{H}_\infty'$ are the two independent copies of the UIHPQ with a simple boundary used to construct $\mathcal{Q}_\infty^{\leftrightarrow}$, and $\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)$ is the set of inner vertices (not on the boundary) which are at distance less than r from the origin of the map (since we shall use estimates for $r^{-4}\#\mathfrak{B}_r(\mathcal{H}_\infty)$, the number of vertices on the boundary which is of order r^2 will turn out to be completely irrelevant, see [9, Section 6]). In particular from the scaling limit results recalled in the beginning of this section we deduce that for the α chosen in (16) we have

$$\begin{aligned}
\lim_{r\to\infty} \mathbb{P}(r^{-4}\#\mathfrak{B}_r(\mathcal{Q}_\infty) > \alpha) &= \mathbb{P}(\mathcal{V}_p > \alpha),\\
\liminf_{r\to\infty} \mathbb{P}(r^{-4}\#\mathfrak{B}_r(\mathcal{Q}_\infty^{\leftrightarrow}) > \alpha) &\geq \liminf_{r\to\infty} \mathbb{P}(r^{-4}(\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty')) > \alpha)\\
&= \mathbb{P}(\mathcal{V}_h + \mathcal{V}_h' > \alpha).
\end{aligned}$$

Given a quadrangulation of the plane $\mathfrak{q}$, set $\mathcal{X}_r(\mathfrak{q}) = 1$ if $\#\mathfrak{B}_r(\mathfrak{q}) > \alpha r^4$ and $\mathcal{X}_r(q) = 0$ otherwise. Similarly if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two quadrangulations of the half-plane then we set $\mathcal{Y}_r(\mathfrak{h}_1, \mathfrak{h}_2) = 1$ if

$$\#\widetilde{\mathfrak{B}}_r(\mathfrak{h}_1) + \#\widetilde{\mathfrak{B}}_r(\mathfrak{h}_2) > \alpha r^4.$$

Clearly with this notation we have $\mathcal{X}_r(\mathcal{Q}_\infty^{\leftrightarrow}) \geq \mathcal{Y}_r(\mathcal{H}_\infty, \mathcal{H}_\infty')$. The singularity result follows from an evaluation of the random variables $\mathcal{X}_r(\mathcal{Q}_\infty^{\leftrightarrow})$ and $\mathcal{X}_r(\mathcal{Q}_\infty)$ at different scales $0 \ll r_1 \ll r_2 \ll r_3 \ll \cdots$ chosen in such a way that $\mathcal{X}_{r_i}(\mathcal{Q}_\infty)$ is roughly independent of $\mathcal{X}_{r_j}(\mathcal{Q}_\infty)$ for $i \neq j$, so that one may invoke a law of large numbers. To make this precise we first state an independence lemma which we prove at the end of the paper.

Lemma 4 (Independance of scales). *For any $r \geq 0$ and any $\varepsilon > 0$ there exists $R \geq r$ such that for any $s \geq R$ we have*

$$\mathrm{Cov}\big(\mathcal{Y}_r(\mathcal{H}_\infty, \mathcal{H}_\infty'); \mathcal{Y}_s(\mathcal{H}_\infty, \mathcal{H}_\infty')\big) \leq \varepsilon,$$

$$\mathrm{Cov}\big(\mathcal{X}_r(\mathcal{Q}_\infty); \mathcal{X}_s(\mathcal{Q}_\infty)\big) \leq \varepsilon.$$

Using the above lemma we build a sequence $0 \ll r_1 \ll r_2 \ll \cdots$ such that for all $i < j$ one has $\mathrm{Cov}(\mathcal{X}_{r_i}(\mathcal{Q}_\infty), \mathcal{X}_{r_j}(\mathcal{Q}_\infty)) \le 2^{-j}$ as well as

$$\mathrm{Cov}\big(\mathcal{Y}_{r_i}(\mathcal{H}_\infty, \mathcal{H}'_\infty); \mathcal{Y}_{r_j}(\mathcal{H}_\infty, \mathcal{H}'_\infty)\big) \le 2^{-j}.$$

Hence the hypotheses for the strong law of large numbers for weakly correlated variables are satisfied, see e.g. [26], and this implies that

$$\frac{1}{k}\sum_{i=1}^{k} \mathcal{X}_{r_i}(\mathcal{Q}_\infty) \xrightarrow[k\to\infty]{a.s.} \mathbb{P}(\mathcal{V}_p > \alpha)$$

and

$$\frac{1}{k}\sum_{i=1}^{k} \mathcal{Y}_{r_i}(\mathcal{H}_\infty, \mathcal{H}'_\infty) \xrightarrow[k\to\infty]{a.s.} \mathbb{P}(\mathcal{V}_h + \mathcal{V}'_h > \alpha).$$

Since we have $\mathcal{X}_r(\mathcal{Q}_\infty^{\leftrightarrow}) \ge \mathcal{Y}_r(\mathcal{H}_\infty, \mathcal{H}'_\infty)$ this entails thanks to (16)

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \mathcal{X}_{r_i}(\mathcal{Q}_\infty^{\leftrightarrow}) \ge \mathbb{P}(\mathcal{V}_h + \mathcal{V}'_h > \alpha) > \mathbb{P}(\mathcal{V}_p > \alpha).$$

In other words, the event $\{\liminf_{k\to\infty} k^{-1}\sum_{i=1}^{k} \mathcal{X}_{r_i}(\mathfrak{q}) = \mathbb{P}(\mathcal{V}_p > \alpha)\}$ has probability 1 under the law of the UIPQ and probability 0 under the law of $\mathcal{Q}_\infty^{\leftrightarrow}$, finally establishing singularity of the two distributions, as expected.

4.3 Lemma 4: Decoupling the Scales

Proof of Lemma 4. We begin with the first statement concerning the half-planes. Let $\mathcal{H}_\infty$ and $\mathcal{H}'_\infty$ be two copies of the UIHPQ and let $r \ge 0$. As recalled above, in [9, Section 6] we established scaling limits for the volume process $(\#\mathfrak{B}_r(\mathcal{H}_\infty))_{r\ge 0}$ in the UIHPQ. It follows in particular from the almost sure continuity of the scaling limit process at time $t = 1$ and the fact that the boundary effects are negligible that for any function $o(r)$ negligible with respect to r we have

$$\frac{\#\mathfrak{B}_{r+o(r)}(\mathcal{H}_\infty)}{\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)} \xrightarrow[r\to\infty]{(\mathbb{P})} 1, \tag{17}$$

where we recall that $\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)$ are the inner vertices of $\mathfrak{B}_r(\mathcal{H}_\infty)$. Recall also that we denoted by $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ the hull of the ball of radius r inside $\mathcal{H}_\infty$, and that by the spatial Markov property of the UIHPQ the map $\mathcal{H}_\infty[r] := \mathcal{H}_\infty \backslash \mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ rooted, say, at the first edge on the boundary of $\mathcal{H}_\infty$ on the left of $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$, is distributed as a UIHPQ and independent of $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ (and also of $\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)$). An easy geometric argument shows that there exist two random constants $A_r, B_r \ge 0$ depending on $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ such that we have for all $s \ge r$

$$\#\widetilde{\mathfrak{B}}_{s-A_r}(\mathcal{H}_\infty[r]) - B_r \le \#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty) \le \#\widetilde{\mathfrak{B}}_{s+A_r}(\mathcal{H}_\infty[r]) + B_r.$$

Hence, using (17) we deduce that for any $r \geq 1$

$$\frac{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty)}{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r])} \xrightarrow[s\to\infty]{(\mathbb{P})} 1,$$

and similarly when considering the other copy $\mathcal{H}'_\infty$ of the UIHPQ. The point being that now $\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r])$ is independent of $\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)$. We use these convergences together with the fact that α is not an atom of the law of $\mathcal{V}_h + \mathcal{V}'_h$ to deduce that

$$\mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty)+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty)>\alpha s^4} - \mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r])+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty[r])>\alpha s^4} \xrightarrow[s\to\infty]{(\mathbb{P})} 0.$$

When developing the covariance $\mathrm{Cov}(\mathcal{Y}_r(\mathcal{H}_\infty, \mathcal{H}'_\infty); \mathcal{Y}_s(\mathcal{H}_\infty, \mathcal{H}'_\infty))$ we can then replace $\mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty)+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty)>\alpha s^4}$ by $\mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r])+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty[r])>\alpha s^4}$ with asymptotically no harm: For r fixed as $s \to \infty$ we have

$$\begin{aligned}
&\mathrm{Cov}\big(\mathcal{Y}_r(\mathcal{H}_\infty, \mathcal{H}'_\infty); \mathcal{Y}_s(\mathcal{H}_\infty, \mathcal{H}'_\infty)\big)\\
&\quad + \mathbb{E}\left[\mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty)+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty)>\alpha s^4}\mathbf{1}_{\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)+\#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty)>\alpha r^4}\right]\\
&\quad - \mathbb{P}(\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty) > \alpha s^4)\mathbb{P}(\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty) > \alpha r^4)\\
&= o(1)\\
&\quad + \mathbb{E}\left[\mathbf{1}_{\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r])+\#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty[r])>\alpha s^4}\mathbf{1}_{\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty)+\#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty)>\alpha r^4}\right]\\
&\quad - \mathbb{P}(\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty) > \alpha s^4)\mathbb{P}(\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty) > \alpha r^4)\\
&= o(1)\\
&\quad + \mathbb{P}(\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty[r]) + \#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty[r]) > \alpha s^4)\mathbb{P}(\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty) > \alpha r^4)\\
&\quad - \mathbb{P}(\#\widetilde{\mathfrak{B}}_s(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_s(\mathcal{H}'_\infty) > \alpha s^4)\mathbb{P}(\#\widetilde{\mathfrak{B}}_r(\mathcal{H}_\infty) + \#\widetilde{\mathfrak{B}}_r(\mathcal{H}'_\infty) > \alpha r^4),
\end{aligned}$$

where in the last line we used the independence of $\mathcal{H}_\infty[r]$ and $\mathfrak{B}^\bullet_r(\mathcal{H}_\infty)$ (and similarly for $\mathcal{H}'_\infty$). Since $\mathcal{H}_\infty[r]$ has the law of a UIHPQ the product of probabilities cancels out and the covariance indeed tends to 0 as $s \to \infty$ as desired.

For the case of the UIPQ things are a little more complicated since the spatial Markov property involves the perimeter of the discovered region. For $r \geq 1$ we also consider the hull $\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$ of the ball of radius r inside the UIPQ. Unfortunately $\mathcal{Q}_\infty\backslash\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$ (appropriately rooted) is not independent of $\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$: conditionally on $\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$ the map $\mathcal{Q}_\infty\backslash\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$ has the law of a UIPQ of the 2ℓ-gon where 2ℓ is the perimeter of the unique hole of $\mathfrak{B}^\bullet_r(\mathcal{Q}_\infty)$. Yet, it can be shown (for example using the techniques of [11]) that for any $\ell \geq 1$ if $\mathcal{Q}_{\infty,\ell}$ is a UIPQ of the 2ℓ-gon then we still have

$$r^{-4}\#\mathfrak{B}_r(\mathcal{Q}_{\infty,\ell}) \xrightarrow[r\to\infty]{(d)} \mathcal{V}_p.$$

This is in fact sufficient in order to adapt the above proof to the case of the UIPQ. At this point in the paper, we leave the details to the courageous reader.

5 Open Problems

In this section we discuss several open problems related to the topic of this paper, and indicate some possible directions for further research. First we state a natural conjecture motivated by Theorem 2 (see [5, Section 5] for a related conjecture):

Conjecture 1. The laws of $\mathcal{Q}_\infty$, $\mathcal{Q}_\infty^{\rightarrow}$ and of $\mathcal{Q}_\infty^{\leftrightarrow}$ are singular with respect to each other.

As stated in the Introduction, a possible way towards a proof of this conjecture would be to use the recent work [18,19] as a replacement of the input (1) and to adapt the proof of the last section. Recall also the open question about coincidence of quenched and annealed connective constants:

Open question 1 (Coincidence of the quenched and annealed connective constants). The quenched connective constant $\mu(\mathcal{Q}_\infty)$ of the UIPQ is less than the "annealed" connective constant which is equal to 9/2. Do we actually have equality?

In light of the works devoted to random walks on random planar maps (and in particular the fact that the UIPQ is recurrent [17]) the following question is also natural:

Open question 2. Are the random lattices $\mathcal{Q}_\infty^{\rightarrow}$ and $\mathcal{Q}_\infty^{\leftrightarrow}$ almost surely recurrent?

Both open questions would have a positive answer if the geometry of $\mathcal{Q}_\infty^{\rightarrow}$ (resp. that of $\mathcal{Q}_\infty^{\leftrightarrow}$) at a given scale were comparable (in a strong "local" sense) to that of the UIPQ. A starting point for this strong comparison would be to show that for any $r \geq 1$ the following two random variables

$$\mathfrak{B}_{2r}(Q_\infty)\backslash\mathfrak{B}_r^\bullet(Q_\infty) \quad \text{and} \quad \mathfrak{B}_{2r}(Q_\infty^{\rightarrow})\backslash\mathfrak{B}_r^\bullet(Q_\infty^{\rightarrow}),$$

are contiguous (i.e. every graph property that holds with high probability for the first random variable also holds for the second one and vice-versa). We do not, however, intend to conjecture this is true.

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A Max-Type Recursive Model: Some Properties and Open Questions

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In honor of Professor Charles M. Newman on the occasion of his 70th birthday

Abstract. We consider a simple max-type recursive model which was introduced in the study of depinning transition in presence of strong disorder, by Derrida and Retaux [5]. Our interest is focused on the critical regime, for which we study the extinction probability, the first moment and the moment generating function. Several stronger assertions are stated as conjectures.

Keywords: Max-type recursive model · Critical regime · Free energy · Survival probability

1 Introduction

Fix an integer $m \geq 2$. Let $X_0 \geq 0$ be a non-negative random variable (but very soon, taking values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$). To avoid trivial discussions, we assume that X_0 is not almost surely a constant. Consider the following recurrence relation:

$$X_{n+1} = (X_n^{(1)} + \cdots + X_n^{(m)} - 1)^+, \; n \geq 0, \tag{1}$$

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where $X_n^{(1)}, \ldots, X_n^{(m)}$ are independent copies of X_n, and for all $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$ is the positive part of x.

The model with recursion defined in (1) was introduced by Derrida and Retaux [5] as a simple hierarchical renormalization model to understand depinning transition of a line in presence of strong disorder. The study of depinning transition has an important literature both in mathematics and in physics. Of particular interest are problems about the relevance of the disorder, and if it is, the precise description of the transition. Similar problems are raised when the line has hierarchical constraints. We refer to Derrida, Hakim and Vannimenus [4], Giacomin, Lacoin and Toninelli [6], Lacoin [9], Berger and Toninelli [2], Derrida and Retaux [5], and Hu and Shi [8] for more details and references. Let us mention that the recursion (1) appears as a special case in the survey paper of Aldous and Bandyopadhyay [1], in a spin glass toy-model in Collet et al. [3], and is also connected to a parking scheme recently studied by Goldschmidt and Przykucki [7].

Since $x - 1 \leq (x-1)^+ \leq x$ for all $x \geq 0$, we have, by (1),

$$m\,\mathbb{E}(X_n) - 1 \leq \mathbb{E}(X_{n+1}) \leq m\,\mathbb{E}(X_n),$$

so the *free energy*

$$F_\infty := \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{m^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - \frac{1}{m-1}}{m^n}, \tag{2}$$

is well-defined.

In a sense, the free energy is positive for "large" random variables and vanishes for "small" ones. The two regimes are separated by a "surface" (called "critical manifold" in [5]) in the space of distributions that exhibits a critical behavior. Many interesting questions are related to the behavior of X_n at the critical regime or near it.

As an example, let us recall a key conjecture, due to Derrida and Retaux [5], which handles the following parametric setting.

For any random variable X, we write P_X for its law. Assume

$$P_{X_0} = (1-p)\,\delta_0 + p\,P_{Y_0},$$

where δ_0 is the Dirac measure at 0, Y_0 is a positive random variable, and $p \in [0, 1]$ a parameter. Since $F_\infty =: F_\infty(p)$ is non-decreasing in p, there exists $p_c \in [0, 1]$ such that $F_\infty > 0$ for $p > p_c$ and that $F_\infty(p) = 0$ for $p < p_c$. A conjecture of Derrida and Retaux [5] says that if $p_c > 0$ (and possibly under some additional integrability conditions on Y_0), then

$$F_\infty(p) = \exp\Big(- \frac{K + o(1)}{(p-p_c)^{1/2}} \Big), \ p \downarrow p_c\,, \tag{3}$$

for some constant $K \in (0, \infty)$.

When $p_c = 0$, it is possible to have other exponents than $\frac{1}{2}$ in (3), see [8], which also contains several open problems in the regime $p \downarrow p_c$.

We have not been able to prove or disprove the conjecture. In this paper, we are interested in the critical regime, i.e., $p = p_c$ in the Derrida–Retaux conjecture setting. However, we do not formulate the model in a parametric way. When X_0 is integer valued, the critical regime is characterized by the following theorem.

Theorem 1 (Collet et al.[3]). *If X_0 takes values in $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, the critical regime is given by*

$$(m-1)\mathbb{E}(X_0\, m^{X_0}) = \mathbb{E}(m^{X_0}) < \infty\,;$$

more precisely, $F_\infty > 0$ if either $\mathbb{E}(m^{X_0}) < (m-1)\mathbb{E}(X_0\, m^{X_0}) < \infty$, or $\mathbb{E}(X_0\, m^{X_0}) = \infty$ (a fortiori, if $\mathbb{E}(m^{X_0}) = \infty$), and $F_\infty = 0$ otherwise.

We assume **from now on** that X_0 is $\mathbb{Z}_+$-valued, and we work in the **critical regime**, i.e., assuming

$$(m-1)\mathbb{E}(X_0\, m^{X_0}) = \mathbb{E}(m^{X_0}) < \infty. \tag{4}$$

A natural question is whether $\mathbb{E}(X_n) \to 0$ in the critical regime. The answer is positive.

Theorem 2. *Assume (4). Then $\lim_{n\to\infty} \mathbb{E}(m^{X_n}) = 1$. A fortiori,*

$$\lim_{n\to\infty} \mathbb{E}(X_n) = 0.$$

It is natural to study the asymptotic behavior of X_n quantitatively. Although we have not succeeded in making many of our arguments rigorous, we are led by a general asymptotic picture described by the following two conjectures. The first of them (Conjecture 1) describes how $\mathbb{P}(X_n \neq 0)$ tends to 0, while the second one (Conjecture 2) describes the conditional asymptotic behavior of X_n provided that $X_n \neq 0$.

We use the notation $a_n \sim b_n$, $n \to \infty$, to denote $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Conjecture 1. Assume (4). Then we have

$$\mathbb{P}(X_n \neq 0) \sim \frac{4}{(m-1)^2}\, \frac{1}{n^2},\ n \to \infty.$$

When $m = 2$, Conjecture 1 was already given by Collet et al. [3] (see their equation (AIII.10)) and by Derrida and Retaux [5].

Our next conjecture concerns weak convergence of X_n given $X_n > 0$.

Conjecture 2. Assume (4). Then, conditionally on $X_n \neq 0$, the random variable X_n converges weakly to a limit Y_∞ with geometric law: $\mathbb{P}(Y_\infty = k) = \frac{m-1}{m^k}$ for integers $k \geq 1$.

The two conjectures above immediately lead to more specific quantitative assertions. In view of Theorem 2, it is natural to study how $\mathbb{E}(X_n)$ goes to zero in the critical regime.

Conjecture 3. Assume (4). Then we have

$$\mathbb{E}(X_n) \sim \frac{4m}{(m-1)^3}\,\frac{1}{n^2},\ n \to \infty\,,$$

and more generally, for all real numbers $r \in (0,\,\infty)$,

$$\mathbb{E}(X_n^r) \sim \frac{c(r)}{n^2},\ n \to \infty\,,$$

where $c(r) = c(r,\,m) := \frac{4}{m-1}\sum_{k=1}^{\infty}\frac{k^r}{m^k}$.

In view of Conjectures 1 and 2, we may also guess how fast the moment generating function converges.

Conjecture 4. Assume (4). Then

(a) We have, for $n \to \infty$ and $s \in (0,\,m),\ s \neq 1$,

$$\mathbb{E}(s^{X_n}) - 1 \sim \frac{4m}{(m-1)^2}\,\frac{s-1}{m-s}\,\frac{1}{n^2}\,. \tag{5}$$

(b) We have, for $n \to \infty$,

$$\mathbb{E}(m^{X_n}) - 1 \sim \frac{2}{m-1}\,\frac{1}{n}\,. \tag{6}$$

Possibly some additional integrability conditions, such as $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$ in Theorem 3 below, are necessary for our conjectures to hold.

The following weaker version of (6) can be rigorously proved.

Theorem 3. *Assume* (4)*. If* $\Lambda_0 := \mathbb{E}(X_0^3\, m^{X_0}) < \infty$*, then there exist constants* $c_2 \geq c_1 > 0$*, depending only on* m*, such that*

$$c_1\,\frac{[\mathbb{P}(X_0 = 0)]^{1/2}}{\Lambda_0^{1/2}}\,\frac{1}{n} \leq \mathbb{E}(m^{X_n}) - 1 \leq c_2\,\frac{\Lambda_0^{1/2}}{[\mathbb{P}(X_0 = 0)]^{1/2}}\,\frac{1}{n},\ n \geq 1\,.$$

Remark 1. (i) The assumption (4) guarantees that $\mathbb{P}(X_0 = 0) > 0$.
(ii) It will be seen (in Lemma 3) that in case $m \geq 3$, we have $\mathbb{P}(X_0 = 0) \geq \frac{m-2}{m-1}$.

The content of the rest of the paper is as follows:

- Section 2: A few key facts about the moment generating functions of X_n and their interrelations, with some technical proofs postponed to Sect. 7;
- Section 3: Proof of Theorem 1;
- Section 4: Proof of Theorem 2;
- Section 5: Heuristics for Conjectures 1 and 4b;
- Section 6: Heuristics for Conjectures 2, 3, and 4a;
- Section 8: Proof of Theorem 3;
- Section 9: Proofs of weaker versions of some of our conjectures.

Throughout the paper, $c_i = c_i(m)$, for $1 \leq i \leq 17$, denote finite and positive constants whose values depend only on m.

2 Moment Generating Functions and Their Evolution

2.1 Derivatives and Evolution

All the techniques used in this article are based on the evaluation of the moment generating functions and on their evolution during the recursive process (1). Write, for $n \geq 0$,

$$G_n(s) := \mathbb{E}(s^{X_n}),$$

the moment generating function of X_n.

In terms of generating functions, the recursion (1) writes as

$$G_{n+1}(s) = \frac{1}{s}\, G_n(s)^m + (1 - \frac{1}{s}) G_n(0)^m \,. \tag{7}$$

Moreover, if $G'_n(s)$ is well defined, then so is $G'_{n+1}(s)$ and differentiation yields

$$G'_{n+1}(s) = \frac{m}{s}\, G'_n(s)\, G_n(s)^{m-1} - \frac{1}{s^2}\, G_n(s)^m + \frac{1}{s^2}\, G_n(0)^m \,. \tag{8}$$

Eliminating $G_n(0)$ from the two identities, it follows that

$$(s-1)s\, G'_{n+1}(s) - G_{n+1}(s) = [m(s-1)\, G'_n(s) - G_n(s)]\, G_n(s)^{m-1} \,. \tag{9}$$

Taking $s = m$ yields a formula particularly convenient for iterations, namely,

$$(m-1)m\, G'_{n+1}(m) - G_{n+1}(m) = [(m-1)m\, G'_n(m) - G_n(m)]\, G_n(m)^{m-1} \,. \tag{10}$$

Further differentiation of (8) yields (if the involved derivatives are well defined for G_0)

$$\begin{aligned} G''_{n+1}(s) = &-\frac{2m}{s^2}\, G'_n(s)\, G_n(s)^{m-1} + \frac{m}{s}\, G''_n(s)\, G_n(s)^{m-1} \\ &+ \frac{m(m-1)}{s}\, G'_n(s)^2\, G_n(s)^{m-2} + \frac{2}{s^3}\, G_n(s)^m - \frac{2}{s^3}\, G_n(0)^m \end{aligned} \tag{11}$$

and

$$\begin{aligned} G'''_{n+1}(s) = &\frac{6m}{s^3}\, G'_n(s)\, G_n(s)^{m-1} - \frac{3m}{s^2}\, G''_n(s)\, G_n(s)^{m-1} \\ &- \frac{3m(m-1)}{s^2}\, G'_n(s)^2\, G_n(s)^{m-2} - \frac{6}{s^4}\, G_n(s)^m \\ &+ \frac{6}{s^4}\, G_n(0)^m + \frac{m}{s}\, G'''_n(s) G_n(s)^{m-1} \\ &+ \frac{3m(m-1)}{s}\, G'_n(s) G''_n(s) G_n(s)^{m-2} \\ &+ \frac{m(m-1)(m-2)}{s}\, G'_n(s)^3 G_n(s)^{m-3} \,. \end{aligned} \tag{12}$$

Notice that assumption (4) can be rewritten in the language of generating functions as

$$(m-1)m\, G'_0(m) = G_0(m).$$

It follows immediately from (10) that we have in this case for all $n \geq 0$,

$$(m-1)m\, G_n'(m) = G_n(m). \tag{13}$$

Assuming $\mathbb{E}(X_0^2\, m^{X_0}) < \infty$ and plugging (13) into (11) with $s = m$, we obtain, for all $n \geq 0$,

$$G_{n+1}''(m) = G_n''(m)\, G_n(m)^{m-1} + \frac{m-2}{m^3(m-1)}\, G_n(m)^m - \frac{2}{m^3}\, G_n(0)^m, \tag{14}$$

which, in combination with (7), yields

$$G_{n+1}''(m) + \frac{2}{m^2(m-1)}\, G_{n+1}(m) = \left[G_n''(m) + \frac{1}{m^2(m-1)}\, G_n(m)\right] G_n(m)^{m-1}. \tag{15}$$

This is still not a perfectly iterative relation because of the difference of the numerators on both sides.

However, let us continue the same operation with the third derivative. Assuming that $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$ and plugging (13) into (12) with $s = m$, we obtain for all $n \geq 0$,

$$G_{n+1}'''(m) = \frac{-2m^2 + 7m - 6}{m^4(m-1)^2}\, G_n(m)^m + \frac{6}{m^4}\, G_n(0)^m + G_n'''(m) G_n(m)^{m-1}. \tag{16}$$

By excluding $G_n(0)$ from (16) and (14), it follows that

$$\begin{aligned} &mG_{n+1}'''(m) + 3G_{n+1}''(m) \\ &= \big[mG_n'''(m) + 3G_n''(m)\big] G_n(m)^{m-1} + \frac{m-2}{m^2(m-1)^2}\, G_n(m)^m. \end{aligned} \tag{17}$$

Now we may get a completely iterative aggregate

$$\begin{aligned} \mathcal{D}_n(m) &:= m(m-1)G_n'''(m) + (4m-5)G_n''(m) + \frac{2(m-2)}{m^2(m-1)}\, G_n(m) \\ &= (m-1)\big[mG_n'''(m) + 3G_n''(m)\big] \\ &\quad + (m-2)\left[G_n''(m) + \frac{2}{m^2(m-1)}\, G_n(m)\right] \end{aligned} \tag{18}$$

because combining (15) and (17) yields, for all $n \geq 0$,

$$\mathcal{D}_{n+1}(m) = \mathcal{D}_n(m)\, G_n(m)^{m-1}. \tag{19}$$

This is another perfectly iterative relation along with (10). Recall that it is proved under (4) and assuming that $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$.

The approach based on analysis of the moment generating function is already adopted by Collet et al. [3] and Derrida and Retaux [5], where the main attention is focused on the case $m = 2$. As seen from (18), the case $m = 2$ only involves the first half of $\mathcal{D}_n(m)$. Our work reveals the importance of the second half; together with the first half, they serve as a useful tool in the study of the moment generating function, as we will see in Sect. 7.2.

2.2 Products

The main technical properties of generating functions are contained in the following two complementary propositions.

Proposition 1. *Assume* (4). *There exists a constant* $c_3 \in (0, \infty)$ *such that*

$$\prod_{i=0}^{n} G_i(m)^{m-1} \leq \frac{c_3}{\mathbb{P}(X_0 = 0)} n^2, \; n \geq 1.$$

Proposition 2. *Assume* (4). *If* $\Lambda_0 := \mathbb{E}(X_0^3 \, m^{X_0}) < \infty$, *then there exists a positive constant* c_4, *depending only on* m, *such that*

$$\prod_{i=0}^{n} G_i(m)^{m-1} \geq \frac{c_4}{\Lambda_0} n^2, \; n \geq 1.$$

The proofs of these propositions, rather technical, are postponed to Sect. 7.

3 Proof of Theorem 1

The proof of Theorem 1 essentially follows Collet et al. [3]. It is presented here, not only for the sake of self-containedness, but also for some simplification, which we consider as interesting, in both the upper and the lower bounds.

Assume for a while that $\mathbb{E}(X_0 \, m^{X_0}) < \infty$, which means, in the language of generating functions, that $G_0'(s)$ is well defined for all $0 \leq s \leq m$. Then, as we know from (8), the derivative $G_n'(s)$ is well defined for all $n \geq 0$ and $0 \leq s \leq m$.

Proof of Theorem 1: *Upper bound.* Suppose $(m-1)\mathbb{E}(X_0 \, m^{X_0}) \leq \mathbb{E}(m^{X_0})$ and $\mathbb{E}(X_0 \, m^{X_0}) < \infty$. Let us prove that $F_\infty = 0$. In the language of generating functions our assumption simply means $(m-1)m \, G_0'(m) - G_0(m) \leq 0$ and $G_0'(m) < \infty$. Then iterative identity (10) yields that we have the same relations $(m-1)m \, G_n'(m) - G_n(m) \leq 0$ and $G_n'(m) < \infty$ for all $n \geq 0$. Back to the moments' language, we obtain $\mathbb{E}(X_n \, m^{X_n}) < \infty$ and $\mathbb{E}(m^{X_n}) \geq (m-1)\mathbb{E}(X_n \, m^{X_n})$. The latter expression, by the FKG inequality, is greater than or equal to $(m-1)\mathbb{E}(X_n) \, \mathbb{E}(m^{X_n})$. Therefore, $\mathbb{E}(X_n) \leq \frac{1}{m-1}$, for all $n \geq 0$. By definition, we get $F_\infty = 0$. □

We now turn to the lower bound for the free energy.

Lemma 1. *If* $\mathbb{E}(m^{X_0}) < (m-1)\mathbb{E}(X_0 \, m^{X_0}) < \infty$, *then there exists* $s \in (1, m)$ *such that*

$$(s-1)\mathbb{E}(X_n \, s^{X_n}) - \mathbb{E}(s^{X_n}) \to \infty, \; n \to \infty.$$

Proof. Taking $s \in (1, m)$ sufficiently close to m we may assure that

$$(s-1)\mathbb{E}(X_0\, s^{X_0}) - \mathbb{E}(s^{X_0}) > 0.$$

In the language of generating functions, this means

$$(s-1)sG_0'(s) - G_0(s) > 0.$$

By (9),

$$\begin{aligned}(s-1)sG_{n+1}'(s) - G_{n+1}(s) &= \frac{m}{s}\Big[s(s-1)G_n'(s) - \frac{s}{m}G_n(s)\Big]G_n(s)^{m-1}\\ &\geq \frac{m}{s}[s(s-1)G_n'(s) - G_n(s)]G_n(s)^{m-1}\\ &\geq \frac{m}{s}[s(s-1)G_n'(s) - G_n(s)],\end{aligned}$$

where at the last step we used $G_n(s) \geq 1$ for all $s > 1$. By induction, we obtain

$$(s-1)sG_n'(s) - G_n(s) \geq \Big(\frac{m}{s}\Big)^n[(s-1)sG_0'(s) - G_0(s)] \to \infty$$

which is precisely equivalent to the claim of our lemma. □

Lemma 2. *If* $F_\infty = 0$, *then* $\sup_{n\geq 0}\mathbb{E}(X_n\, s^{X_n}) < \infty$ *for all* $s \in (0, m)$.

Proof. Let $k \geq 1$ and $n \geq 0$. Clearly,

$$X_{n+k} \geq \sum_{i=1}^{m^k} \mathbf{1}_{\{X_n^{(i)} \geq k+1\}},$$

where, as before, $X_n^{(i)}$, $i \geq 1$, are independent copies of X_n. It follows that

$$\mathbb{E}(X_{n+k}) \geq m^k\, \mathbb{P}(X_n \geq k+1)\,.$$

Suppose $F_\infty = 0$. Then by (2), $\mathbb{E}(X_{n+k}) \leq \frac{1}{m-1}$ for all $n \geq 0$ and $k \geq 1$; hence

$$\mathbb{P}(X_n \geq k+1) \leq \frac{1}{(m-1)m^k}$$

for all $n \geq 0$ and $k \geq 1$, implying the assertion of our lemma. □

Proof of Theorem 1: Lower bound. Assume first

$$\mathbb{E}(m^{X_0}) < (m-1)\mathbb{E}(X_0\, m^{X_0}) < \infty.$$

Let us prove that $F_\infty > 0$.

By Lemma 1, there exists $s \in (1, m)$ such that

$$\mathbb{E}(X_n\, s^{X_n}) \to \infty,\ n \to \infty. \tag{20}$$

If $F_\infty = 0$, then by Lemma 2 we would have $\sup_{n\ge 0} \mathbb{E}(X_n\, s^{X_n}) < \infty$, contradicting (20).

Consider now the remaining case $\mathbb{E}(X_0\, m^{X_0}) = \infty$. For any $k \ge 1$ let $\widetilde{X}_{0,k} := \min\{X_0,\, k\}$ be the trimmed version of X_0. By choosing k sufficiently large, one obtains

$$(m-1)\mathbb{E}(\widetilde{X}_{0,k}\, m^{\widetilde{X}_{0,k}}) > \mathbb{E}(m^{\widetilde{X}_{0,k}}).$$

The just proved part of our theorem asserts that the free energy associated with $\widetilde{X}_{0,k}$ is positive, and a fortiori $F_\infty > 0$ in this case. □

4 Proof of Theorem 2

We prove Theorem 2 in this section by means of Proposition 1. Write as before $G_n(s) := \mathbb{E}(s^{X_n})$.

Lemma 3. *Assume* (4).

(i) For any $n \ge 0$, $s \mapsto \frac{G_n(s)^{m-1}}{s}$ is non-increasing on $[1,\, m]$. In particular, we have $\frac{G_n(s)^{m-1}}{s} \ge \frac{G_n(m)^{m-1}}{m}$ for $n \ge 0$ and $s \in [1,\, m]$.
(ii) We have $\sup_{n\ge 0} G_n(m) \le m^{1/(m-1)}$.
(iii) We have $\inf_{n\ge 0} \mathbb{P}(X_n = 0) \ge \frac{m-2}{m-1}$.

Proof. (i) Since $s \mapsto \frac{s\, G_n'(s)}{G_n(s)}$ is non-decreasing on $[1,\, \infty)$ (this is a general property of moment generating functions, and has nothing to do with assumption (4)), we have, for $s \in [1,\, m]$,

$$(m-1)\frac{s\, G_n'(s)}{G_n(s)} - 1 \le (m-1)\frac{m\, G_n'(m)}{G_n(m)} - 1 = 0$$

by (13). This implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{G_n(s)^{m-1}}{s}\right) = \left[(m-1)\frac{s\, G_n'(s)}{G_n(s)} - 1\right]\frac{G_n(s)^{m-1}}{s^2} \le 0$$

for $s \in [1,\, m]$; hence $s \mapsto \frac{G_n(s)^{m-1}}{s}$ is non-increasing on $[1,\, m]$.

(ii) By (i), $\frac{G_n(m)^{m-1}}{m} \le G_n(1)^{m-1} = 1$, so $G_n(m)^{m-1} \le m$.

(iii) From (4), we get

$$\begin{aligned}\mathbb{P}(X_n = 0) &= \mathbb{E}(m^{X_n}) - \mathbb{E}(m^{X_n}\, \mathbf{1}_{\{X_n \ge 1\}})\\ &= (m-1)\mathbb{E}(X_n\, m^{X_n}\, \mathbf{1}_{\{X_n\ge 1\}}) - \mathbb{E}(m^{X_n}\, \mathbf{1}_{\{X_n\ge 1\}})\\ &\ge (m-2)\mathbb{E}(m^{X_n}\, \mathbf{1}_{\{X_n\ge 1\}}).\end{aligned}$$

This implies $(m-1)\mathbb{P}(X_n = 0) \ge (m-2)\mathbb{E}(m^{X_n}) \ge m-2$, as desired. □

Proof of Theorem 2. Assume (4). Let, for $n \ge 0$,

$$\varepsilon_n := G_n(m) - 1 > 0\,.$$

The proof of Theorem 2 consists of showing that $\varepsilon_n \to 0$.

By (7), $G_{n+1}(s) \le \frac{1}{s}\, G_n(s)^m + (1 - \frac{1}{s})$. In particular, taking $s = m$ gives that

$$\varepsilon_{n+1} \le \frac{(1+\varepsilon_n)^m - 1}{m}, \; n \ge 0\,. \tag{21}$$

We will now use the following elementary inequality

$$\frac{m}{\log(1+my)} < \frac{1}{\log(1+y)} + \frac{m-1}{2}\,, \; y > 0\,. \tag{22}$$

Indeed, the function $h(x) := \frac{x}{\log(1+x)}$ satisfies $h'(x) \le \frac{1}{2}$ for all $x > 0$. Therefore,

$$\frac{m}{\log(1+my)} - \frac{1}{\log(1+y)} = y^{-1}(h(my) - h(y)) \le \frac{m-1}{2}\,.$$

Let now $n > k \ge 0$. Since $\varepsilon_{n-k} \le \frac{(1+\varepsilon_{n-k-1})^m - 1}{m}$ (see (21)), we have

$$\frac{1}{\log(1+\varepsilon_{n-k-1})} \le \frac{m}{\log(1+m\varepsilon_{n-k})} < \frac{1}{\log(1+\varepsilon_{n-k})} + \frac{m-1}{2}\,,$$

the last inequality being a consequence of (22). Iterating the inequality yields that for all integers $0 \le k \le n$,

$$\frac{1}{\log(1+\varepsilon_{n-k})} \le \frac{1}{\log(1+\varepsilon_n)} + \frac{m-1}{2}\,k \le \frac{1}{\varepsilon_n} + 1 + \frac{m-1}{2}\,k\,,$$

the second inequality following from the inequality $\frac{1}{\log(1+x)} < \frac{1}{x} + 1$ (for $x > 0$). As a consequence, for integers $n > j \ge 0$,

$$\sum_{k=0}^{n-j-1} \log(1+\varepsilon_{n-k}) \ge \sum_{k=0}^{n-j-1} \frac{1}{\frac{1}{\varepsilon_n} + 1 + \frac{m-1}{2}\,k} \ge \int_0^{n-j} \frac{\mathrm{d}x}{\frac{1}{\varepsilon_n} + 1 + \frac{m-1}{2}\,x}\,.$$

The sum on the left-hand side is $\sum_{i=j+1}^{n} \log(1+\varepsilon_i)$, whereas the integral on the right-hand side is equal to

$$\frac{2}{m-1} \log\Big(\frac{\frac{1}{\varepsilon_n} + 1 + \frac{m-1}{2}\,(n-j)}{\frac{1}{\varepsilon_n} + 1}\Big) \ge \frac{2}{m-1} \log(c_5\,(n-j)\varepsilon_n),$$

for some constant $c_5 > 0$ whose value depends only on m (recalling from Lemma 3 (ii) that $\varepsilon_n \le m^{1/(m-1)} - 1$). This yields that for $n > j \ge 0$,

$$\prod_{i=j+1}^{n} (1+\varepsilon_i)^{(m-1)/2} \ge c_5\,(n-j)\varepsilon_n\,. \tag{23}$$

Replacing the pair (n, j) by $(j, 0)$, we also have

$$\prod_{i=1}^{j}(1+\varepsilon_i)^{(m-1)/2} \ge c_5\, j\varepsilon_j$$

for $j \ge 1$. Therefore,

$$c_5^2\, j(n-j)\,\varepsilon_j\,\varepsilon_n \le \prod_{i=1}^{n}(1+\varepsilon_i)^{(m-1)/2},$$

which is bounded by $\frac{c_3^{1/2}}{[\mathbb{P}(X_0=0)]^{1/2}}\, n$ (see Proposition 1). Consequently, with $c_6 := \frac{c_3^{1/2}}{c_5^2}$, we have $\varepsilon_j\,\varepsilon_n \le \frac{c_6}{[\mathbb{P}(X_0=0)]^{1/2}}\,\frac{n}{j(n-j)}$ for $n > j \ge 1$. In particular,

$$\varepsilon_j \sup_{n\ge 2j}\varepsilon_n \le \frac{2c_6}{[\mathbb{P}(X_0=0)]^{1/2}}\,\frac{1}{j}\,,\ j\ge 1\,. \tag{24}$$

This yields

$$(\limsup_{j\to\infty}\varepsilon_j)^2 = \limsup_{j\to\infty}\varepsilon_j \cdot \lim_{j\to\infty}\sup_{n\ge 2j}\varepsilon_n = \limsup_{j\to\infty}\Big(\varepsilon_j \sup_{n\ge 2j}\varepsilon_n\Big) = 0,$$

i.e., $\varepsilon_j \to 0$. □

5 Around Conjectures 1 and 4b

Let

$$\varepsilon_n := G_n(m) - 1.$$

By Theorem 2, $\varepsilon_n \to 0$, $n\to\infty$. Propositions 1 and 2 together say that $\prod_{i=0}^{n}(1+\varepsilon_i)$ is of order of magnitude $n^{2/(m-1)}$ when n is large. So if $n \mapsto \varepsilon_n$ were sufficiently regular, we would have

$$\varepsilon_n \sim \frac{2}{m-1}\,\frac{1}{n}\,,\ n\to\infty\,, \tag{25}$$

This is (6) in Conjecture 4b.

From the relation (7) we obtain, with $s = m$,

$$1+\varepsilon_{n+1} = \frac{1}{m}\,(1+\varepsilon_n)^m + (1-\frac{1}{m})\,G_n(0)^m.$$

Since $G_n(0) = 1 - \mathbb{P}(X_n \ne 0)$, whereas $\varepsilon_n \to 0$, this implies that

$$\mathbb{P}(X_n \ne 0) \sim \frac{\varepsilon_n - \varepsilon_{n+1}}{m-1} + \frac{1}{2}\,\varepsilon_n^2,\ n\to\infty. \tag{26}$$

Let us look back at (25). If we were able to show that $n^2(\varepsilon_n - \frac{2}{m-1}\,\frac{1}{n})$ admits a finite limit when $n\to\infty$, (26) would give an affirmative answer to Conjecture 1.

6 About Conjectures 2, 3 and 4a

Let us first look at Conjecture 4a. Theorem 2 yields $\mathbb{P}(X_n \neq 0) \to 0$. Therefore,

$$G_n(0)^m = [1 - \mathbb{P}(X_n \neq 0)]^m = 1 - m\,\mathbb{P}(X_n \neq 0) + o(\mathbb{P}(X_n \neq 0)), \quad n \to \infty.$$

Using this fact and the identity (7), we obtain

$$\begin{aligned}\frac{G_{n+1}(s) - 1}{\mathbb{P}(X_n \neq 0)} &= \frac{s^{-1}(G_n(s)^m - 1) + (1 - s^{-1})(G_n(0)^m - 1)}{\mathbb{P}(X_n \neq 0)} \\ &= \frac{s^{-1}(G_n(s)^m - 1)}{\mathbb{P}(X_n \neq 0)} - m(1 - s^{-1}) + o(1),\ n \to \infty\,.\end{aligned}$$

Suppose we were able to prove that $\mathbb{P}(X_{n+1} \neq 0) \sim \mathbb{P}(X_n \neq 0)$ (which would be a consequence of Conjecture 1), and that

$$\frac{G_n(s) - 1}{\mathbb{P}(X_n \neq 0)} \to H(s), \tag{27}$$

for $s \in (0, m)$, with some real-valued measurable function $H(\,\cdot\,)$. Then

$$H(s) = s^{-1}\, m\, H(s) - m(1 - s^{-1}),$$

which would lead to

$$H(s) = \frac{m(s-1)}{m-s},$$

for $s \in (0,\, m)$. This, in view of Conjecture 1, is what have led us to Conjecture 4a.

To see why Conjecture 2 would be true, let us note from (27) and $H(s) = \frac{m(s-1)}{m-s}$ that

$$\mathbb{E}(s^{X_n} \,|\, X_n \neq 0) \to \frac{(m-1)s}{m-s},\ s \in (0,\, m)\,. \tag{28}$$

On the right-hand side, $\frac{(m-1)s}{m-s} = \mathbb{E}(s^{Y_\infty})$ if Y_∞ is such that $\mathbb{P}(Y_\infty = k) = \frac{m-1}{m^k}$ for integers $k \geq 1$. In words, Y_∞ has a geometric distribution with parameter $\frac{1}{m}$. This would give a proof of Conjecture 2.

Conjecture 3 would be an immediate consequence of Conjecture 1 and (28).

We end this section with an elementary argument showing that our prediction for $\mathbb{E}(X_n)$ in Conjecture 3 would be a consequence of Conjecture 1 (without using the conjectured convergence in (28)).

Lemma 4. *Assume $\mathbb{E}(X_0) < \infty$. For all $n \geq 0$, we have $\mathbb{E}(X_n) < \infty$, and*

$$\mathbb{P}(X_n = 0) = [1 - m\,\mathbb{E}(X_n) + \mathbb{E}(X_{n+1})]^{1/m}.$$

Proof. Taking $s = 1$ in (8) gives that for $n \geq 0$, $\mathbb{E}(X_n) < \infty$ and

$$\mathbb{E}(X_{n+1}) = m\,\mathbb{E}(X_n) - 1 + G_n(0)^m\,,$$

which implies the lemma by noting that $G_n(0) = \mathbb{P}(X_n = 0)$. $\square$

If Conjecture 1 were true, then it would follow from Lemma 4 (and Theorem 2 which guarantees $\mathbb{E}(X_n) \to 0$) that

$$\mathbb{E}(X_n) - \frac{\mathbb{E}(X_{n+1})}{m} \sim \frac{4}{(m-1)^2}\frac{1}{n^2}, \; n \to \infty .$$

Applying Lemma 5 below to $a_n := \mathbb{E}(X_n)$, $\lambda := \frac{1}{m}$ and $b := 2$, this would yield the prediction for $\mathbb{E}(X_n)$ in Conjecture 3.

Lemma 5. *Let* $0 < \lambda < 1$ *and* $b > 0$. *Let* $(a_n, n \geq 1)$ *be a bounded sequence such that*

$$\lim_{n\to\infty} n^b(a_n - \lambda a_{n+1}) = x ,$$

for some real x. *Then*

$$\lim_{n\to\infty} n^b a_n = \frac{x}{1-\lambda} .$$

Proof. Let $p_n := a_n - \lambda a_{n+1}$. Then $\lim_{n\to\infty} n^b p_n = x$ and

$$M := \sup_{j\geq 1}(j^b|p_j|) < \infty.$$

In particular, the sequence (p_n) is bounded.

By iterating the identity $a_n = p_n + \lambda a_{n+1}$, we get a series representation

$$a_n = \sum_{k=0}^{\infty} p_{n+k}\lambda^k,$$

where the series converges because (p_n) is bounded. We may apply the dominated convergence theorem to the identity

$$n^b a_n = \sum_{k=0}^{\infty} n^b p_{n+k}\lambda^k,$$

because for every $k \geq 0$ we have $n^b p_{n+k} \to x$ by assumption, and the dominating summable majorant $(M_k, k \geq 0)$ is given by $M_k := M\lambda^k$. We arrive at

$$\lim_{n\to\infty} n^b a_n = x\sum_{k=0}^{\infty}\lambda^k = \frac{x}{1-\lambda},$$

as required. □

7 Proofs of Propositions 1 and 2

7.1 Proof of Proposition 1

Let us define, for $n \geq 0$ and $s \in [0, m)$,

$$\Delta_n(s) := [G_n(s) - s(s-1)G_n'(s)] - \frac{(m-1)(m-s)}{m}[2sG_n'(s) + s^2 G_n''(s)] .$$

Then by (9), (8) and (11),

$$\Delta_{n+1}(s) = \frac{m}{s}\,\Delta_n(s)\,G_n(s)^{m-1} - \frac{m-s}{s}\,[(m-1)sG_n'(s) - G_n(s)]^2 G_n(s)^{m-2}. \tag{29}$$

Lemma 6. *Assume* (4). *Then* $\Delta_n(s) \in [0,\,1]$ *for* $n \geq 0$ *and* $s \in [0,\,m)$.

Proof. By the definition of Δ_n, for $s \in [0,\,m)$,

$$\Delta_n'(s) = -\frac{m-s}{m}\,[2(m-2)G_n'(s) + (4m-5)sG_n''(s) + (m-1)s^2 G_n'''(s)],$$

which is non-positive. Hence Δ_n is non-increasing on $(0,\,m)$. Since $\Delta_n(0) = G_n(0) = \mathbb{P}(X_n = 0) \leq 1$, whereas under assumption (4), it is easily checked that $\lim_{s\to m-} \Delta_n(s) = 0$, the lemma follows. □

Lemma 7. *Assume* (4). *For all* $n \geq 0$,

$$[(m-1)sG_n'(s) - G_n(s)]^2 \leq 2G_n(0)\Delta_n(s),\ \ s \in [0,\,m).$$

Proof. By using (13) and writing $x := \frac{s}{m} \in [0,\,1)$ for brevity, we have

$$G_n(s) = (m-1)mG_n'(m) - G_n(m) + G_n(s) = \sum_{k=1}^{\infty} m^k (km-k-1+x^k)\,\mathbb{P}(X_n = k)\,.$$

[In particular, $G_n(0) = \sum_{k=1}^{\infty} m^k (km-k-1)\,\mathbb{P}(X_n = k)$.] Furthermore,

$$\begin{aligned} G_n(s) - (m-1)sG_n'(s) &= \sum_{k=1}^{\infty} m^k [km-k-1+x^k-(m-1)kx^k]\,\mathbb{P}(X_n = k) \\ &= \sum_{k=1}^{\infty} m^k (km-k-1)(1-x^k)\,\mathbb{P}(X_n = k)\,. \end{aligned}$$

Consequently,

$$\Delta_n(s) = \sum_{k=1}^{\infty} m^k (km-k-1)(1-(k+1)x^k + kx^{k+1})\,\mathbb{P}(X_n = k)\,. \tag{30}$$

By the Cauchy–Schwarz inequality,

$$\left(\sum_{k=1}^{\infty} a_k q_k\right)^2 \leq \left(\sum_{k=1}^{\infty} q_k\right)\left(\sum_{k=1}^{\infty} a_k^2 q_k\right),$$

with $a_k := 1 - x^k$ and $q_k := m^k(km-k-1)\mathbb{P}(X_n = k)$, the proof of the lemma is reduced to showing the following: for $x \in [0,1]$ and $k \geq 1$,

$$(1-x^k)^2 \leq 2(1-(k+1)x^k + kx^{k+1}). \tag{31}$$

This can be rewritten as $2k\,x^k(1-x) \le 1-x^{2k}$, which is proved by

$$\frac{1-x^{2k}}{1-x} = 1 + x + \ldots + x^{2k-1} \ge 2kx^k$$

(using that $2x^k \le 2x^{k-1/2} \le x^{k+\ell} + x^{k-\ell-1}$ for $0 \le \ell < k$). □

Proof of Proposition 1. By (29) and Lemma 7, for $s \in [0,\, m)$,

$$\begin{aligned}\Delta_{n+1}(s) &\ge \frac{m}{s}\,\Delta_n(s)\,G_n(s)^{m-1} - \frac{m-s}{s}\,2G_n(0)\Delta_n(s)\,G_n(s)^{m-2}\\ &\ge \frac{m}{s}\,\Delta_n(s)\,G_n(s)^{m-1} - \frac{2(m-s)}{s}\,\Delta_n(s)\,G_n(s)^{m-1}\\ &= \frac{2s-m}{s}\,\Delta_n(s)\,G_n(s)^{m-1}.\end{aligned}$$

Hence for any $s \in (\frac{m}{2},\, m)$,

$$\Delta_{n+1}(s) \ge \Delta_0(s)(2s-m)^{n+1}\prod_{i=0}^{n}\frac{G_i(s)^{m-1}}{s} \ge \Delta_0(s)(2s-m)^{n+1}\prod_{i=0}^{n}\frac{G_i(m)^{m-1}}{m},$$

where we used Lemma 3 (i) in the last inequality. Therefore, for $s \in (\frac{m}{2},\, m)$,

$$\prod_{i=0}^{n} G_i(m)^{m-1} \le \Big(\frac{m}{2s-m}\Big)^{n+1}\frac{\Delta_{n+1}(s)}{\Delta_0(s)} \le \Big(\frac{m}{2s-m}\Big)^{n+1}\frac{1}{\Delta_0(s)},$$

the second inequality being a consequence of Lemma 6. Taking $s = m - \frac{m}{n+2}$ gives that with $c_7 := \sup_{n\ge1}(1+\frac{2}{n})^{n+1} \in (1,\, \infty)$ and all $n \ge 1$,

$$\prod_{i=0}^{n} G_i(m)^{m-1} \le \frac{c_7}{\Delta_0(m-\frac{m}{n+2})}\,.$$

By (30) and (31), with $x := 1 - \frac{1}{n+2}$,

$$\begin{aligned}\Delta_0(m-\frac{m}{n+2}) &\ge \frac{1}{2}\sum_{k=1}^{\infty} m^k(km-k-1)(1-x^k)^2\,\mathbb{P}(X_0=k)\\ &\ge \frac{1}{2}\,(1-x)^2\sum_{k=1}^{\infty} m^k(km-k-1)\,\mathbb{P}(X_0=k)\,.\end{aligned}$$

On the right-hand side, $(1-x)^2 = \frac{1}{(n+2)^2}$, whereas

$$\begin{aligned}\sum_{k=1}^{\infty} m^k(km-k-1)\,\mathbb{P}(X_0=k) &= \mathbb{E}\{[(m-1)X_0-1]m^{X_0}\,\mathbf{1}_{\{X_0\ge1\}}\}\\ &= \mathbb{E}\{[(m-1)X_0-1]m^{X_0}\} + \mathbb{P}(X_0=0),\end{aligned}$$

which is $\mathbb{P}(X_0=0)$ because under (4), we have $\mathbb{E}\{[(m-1)X_0-1]m^{X_0}\} = 0$. Consequently,

$$\Delta_0(m-\frac{m}{n+2}) \ge \frac{\mathbb{P}(X_0=0)}{2(n+2)^2}\,,$$

for all $n \ge 1$. This yields the proposition. □

7.2 Proof of Proposition 2

Now we start to prepare the proof of Proposition 2. In the case $m = 2$, it is proved in Appendix III of [3].

Lemma 8. *Assume* (4) *and let* $n \ge 0$. *If* $G_n'''(m) < \infty$, *then*

$$G_n''(m) \le c_8 \max\{G_n'''(m)^{1/2}, 1\}, \tag{32}$$

where $c_8 = c_8(m) \in (0, \infty)$ *is a constant whose value depends only on* m.

Proof. According to Lemma 3 (ii), $G_n(m) \le m^{1/(m-1)}$. By plugging this into (4), we have $G_n'(m) \le \frac{m^{1/(m-1)}}{m(m-1)} =: c_9$. The function $G_n'(\cdot)$ being convex, we have

$$G_n''(s) \le \frac{G_n'(m) - G_n'(s)}{m-s} \le \frac{G_n'(m)}{m-s} \le \frac{c_9}{m-s}, \; s \in [0, m).$$

By the convexity of $G_n''(\cdot)$, this implies that, for $s \in [0, m)$,

$$G_n''(m) \le G_n''(s) + (m-s)\, G_n'''(m) \le \frac{c_9}{m-s} + (m-s)\, G_n'''(m).$$

The lemma follows by taking $s := m - \frac{1}{\max\{G_n'''(m)^{1/2},1\}}$. □

Lemma 9. *Assume* (4) *and let* $n \ge 0$. *If* $G_n'''(m) < \infty$, *then*

$$G_n(m) - 1 \ge \frac{c_{10}}{\max\{G_n'''(m)^{1/2}, 1\}}, \tag{33}$$

where $c_{10} = c_{10}(m) \in (0, \infty)$ *is a constant that does not depend on* n.

Proof. If $m = 2$ and $\mathbb{P}(X_n = 0) \le \frac{1}{2}$, then $G_n(2) - 1 \ge \mathbb{P}(X_n \ge 1) \ge \frac{1}{2}$. If $m = 2$ and $\mathbb{P}(X_n = 0) > \frac{1}{2}$, the equation (AIII.5) of [3] says that $G_n''(2)(G_n(2) - 1) > \frac{1}{4}\,[\mathbb{P}(X_n = 0)]^2 > \frac{1}{16}$. So the lemma (in case $m = 2$) follows from Lemma 8.

In the rest of the proof, we assume $m \ge 3$.

Write (4) as

$$\sum_{k=0}^{\infty} ((m-1)k - 1) m^k\, \mathbb{P}(X_n = k) = 0.$$

It follows that

$$\begin{aligned} \mathbb{P}(X_n = 0) - m(m-2)\, \mathbb{P}(X_n = 1) &= \sum_{k=2}^{\infty} ((m-1)k - 1) m^k\, \mathbb{P}(X_n = k) \\ &\le (m-1) \sum_{k=2}^{\infty} k m^k\, \mathbb{P}(X_n = k). \end{aligned}$$

Writing

$$G_n''(m) = \frac{1}{m^2} \sum_{k=2}^{\infty} k(k-1) m^k\, \mathbb{P}(X_n = k) \ge \frac{1}{2m^2} \sum_{k=2}^{\infty} k^2\, m^k\, \mathbb{P}(X_n = k),$$

$$G_n(m) - 1 = \sum_{k=1}^{\infty} (m^k - 1)\, \mathbb{P}(X_n = k) \ge (1 - \frac{1}{m}) \sum_{k=1}^{\infty} m^k\, \mathbb{P}(X_n = k),$$

it follows from the Cauchy–Schwarz inequality that

$$\mathbb{P}(X_n = 0) - m(m-2)\,\mathbb{P}(X_n = 1) \le \Big(2m^3(m-1)\,G_n''(m)[G_n(m)-1]\Big)^{1/2}.$$

Write $\varepsilon_n := G_n(m) - 1$ as before. We have $G_n(m) \ge 1 + (m-1)\mathbb{P}(X_n = 1)$, i.e.,

$$m(m-2)\,\mathbb{P}(X_n = 1) \le \frac{m(m-2)\,\varepsilon_n}{m-1}.$$

On the other hand, $\mathbb{P}(X_n = 0) \ge \frac{m-2}{m-1}$ by Lemma 3 (iii). Therefore,

$$\frac{m-2}{m-1} - \frac{m(m-2)\,\varepsilon_n}{m-1} \le (2m^3(m-1))^{1/2}\,G_n''(m)^{1/2}\,\varepsilon_n^{1/2},$$

which yields

$$G_n(m) - 1 = \varepsilon_n \ge \frac{c_{11}}{\max\{G_n''(m), 1\}}, \tag{34}$$

for some constant $c_{11} = c_{11}(m) \in (0, \infty)$ and all n. The lemma (in case $m \ge 3$) follows from Lemma 8. □

Proof of Proposition 2. Write $b_j = \prod_{i=0}^{j} G_i(m)^{m-1}$. By (19),

$$\mathcal{D}_n(m) = \mathcal{D}_0(m) b_{n-1}\,.$$

By Definition (18), we have $\mathcal{D}_n(m) \ge m(m-1)\,G_n'''(m)$ and $\frac{\mathcal{D}_0(m)}{m(m-1)} \le c_{12}\Lambda_0$ for some $c_{12} = c_{12}(m) \in (0, \infty)$, where $\Lambda_0 := \mathbb{E}(X_0^3 m^{X_0})$. Therefore,

$$G_n'''(m) \le c_{12}\,\Lambda_0\,b_{n-1}\,. \tag{35}$$

We know that $b_{n-1} \ge 1$, and that $\Lambda_0 \ge \mathbb{E}(X_0 m^{X_0}) = \frac{1}{m-1}\mathbb{E}(m^{X_0}) \ge \frac{1}{m-1}$. So, with $c_{13} := \max\{c_{12}, m-1\}$, we have

$$\max\{G_n'''(m),\, 1\} \le c_{13}\,\Lambda_0\,b_{n-1}\,.$$

On the other hand, by Lemma 9, $\max\{G_n'''(m),\, 1\} \ge (\frac{c_{10}}{G_n(m)-1})^2$. Therefore, with $c_{14} := \frac{c_{10}}{c_{13}^{1/2}}$, we have $G_n(m) \ge 1 + \frac{c_{14}}{\Lambda_0^{1/2} b_{n-1}^{1/2}}$, i.e.,

$$\frac{b_n^{1/(m-1)}}{b_{n-1}^{1/(m-1)}} \ge 1 + \frac{c_{14}}{\Lambda_0^{1/2} b_{n-1}^{1/2}}\,.$$

Since $\frac{c_{14}}{\Lambda_0^{1/2} b_{n-1}^{1/2}} \le \frac{c_{14}}{\Lambda_0^{1/2}} \le (m-1)^{1/2} c_{14}$, this yields that for some $c_{15} = c_{15}(m) \in (0, \infty)$ depending only on m,

$$\frac{b_n^{1/2}}{b_{n-1}^{1/2}} \ge 1 + \frac{c_{15}}{\Lambda_0^{1/2} b_{n-1}^{1/2}}\,.$$

Hence $b_n^{1/2} - b_{n-1}^{1/2} \ge \frac{c_{15}}{\Lambda_0^{1/2}}$. Thus $b_n^{1/2} \ge \frac{c_{15}}{\Lambda_0^{1/2}} n$, as desired.

□

8 Proof of Theorem 3

Proof. Assume (4). Write as before $G_n(m) := \mathbb{E}(m^{X_n})$ and $\varepsilon_n := G_n(m) - 1$. Recall from (23) that for integers $n > j \geq 0$,

$$\prod_{i=j+1}^{n} (1+\varepsilon_i)^{(m-1)/2} \geq c_5\,(n-j)\varepsilon_n.$$

Taking $j := \lfloor \frac{n}{2} \rfloor$ gives that for $n \geq 1$,

$$\prod_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (1+\varepsilon_i)^{(m-1)/2} \geq \frac{c_5}{2}\, n\varepsilon_n\,.$$

By Proposition 1,

$$\prod_{i=0}^{n} (1+\varepsilon_i)^{(m-1)/2} \leq \frac{c_3^{1/2}}{\mathbb{P}(X_0=0)^{1/2}}\, n,$$

whereas by Proposition 2, under the assumption $\Lambda_0 := \mathbb{E}(X_0^3\, m^{X_0}) < \infty$,

$$\prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (1+\varepsilon_i)^{(m-1)/2} \geq \frac{c_{16}}{\Lambda_0^{1/2}}\, n$$

(for some constant $c_{16} = c_{16}(m) > 0$ depending only on m; noting that the case $n = 1$ is trivial because $\Lambda_0 \geq \frac{1}{m-1}$). Hence

$$\prod_{i=\lfloor \frac{n}{2} \rfloor+1}^{n} (1+\varepsilon_i)^{(m-1)/2} \leq \frac{c_3^{1/2}\,\Lambda_0^{1/2}}{c_{16}\,\mathbb{P}(X_0=0)^{1/2}}.$$

Consequently, $\varepsilon_n \leq c_{17} \frac{\Lambda_0^{1/2}}{\mathbb{P}(X_0=0)^{1/2}}\, \frac{1}{n}$ with $c_{17} := \frac{2c_3^{1/2}}{c_5 c_{16}}$.

To prove the lower bound, we note that under the assumption

$$\Lambda_0 := \mathbb{E}(X_0^3\, m^{X_0}) < \infty,$$

we have, by (33), $\varepsilon_n \geq \frac{c_{10}}{\max\{G_n'''(m)^{1/2}, 1\}}$; since

$$G_n'''(m) \leq c_{12}\Lambda_0 \prod_{i=0}^{n-1} (1+\varepsilon_i)^{m-1}$$

(see (35)), which is bounded by $c_{12}\Lambda_0\, \frac{c_3}{\mathbb{P}(X_0=0)}\, n^2$ according to Proposition 1, this yields that

$$\varepsilon_n \geq \frac{c_{10}}{\max\left\{(c_{12}c_3)^{1/2} \frac{\Lambda_0^{1/2}}{[\mathbb{P}(X_0=0)]^{1/2}}\, n,\, 1\right\}},$$

as desired. □

9 Some Additional Rigorous Results

Let (b_i) be a numerical sequence. We define its *harmonic limit* as

$$h\text{-}\lim_{i\to\infty} b_i := \lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{b_i}{i} \,,$$

if the latter exists. The h-lim sup and h-lim inf are defined in the same way.

Proposition 3. *Assume* (4)*. If* $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$*, then*

$$h\text{-}\lim_{i\to\infty} [(\mathbb{E}(m^{X_i}) - 1)i] = \frac{2}{m-1} \,.$$

This statement is a (much) weaker version of Conjecture 4b and a complement to Theorem 3. However, it has the advantage that the limiting constant $\frac{2}{m-1}$ appears explicitly in the result.

Proof. We denote as usual $\varepsilon_i := G_i(m) - 1 = \mathbb{E}(m^{X_i}) - 1$. Rewrite the assertions of Propositions 1 and 2 as

$$\begin{aligned}\frac{1}{m-1}\log\frac{c_4}{\Lambda_0} + \frac{2}{m-1}\,\log n &\le \sum_{i=1}^{n} \log(1+\varepsilon_i)\\ &\le \frac{1}{m-1}\log\frac{c_3}{\mathbb{P}(X_0=0)} + \frac{2}{m-1}\,\log n.\end{aligned}$$

Using $\log(1+x) \le x \le \log(1+x) + x^2/2$ for all $x > 0$ we obtain

$$\frac{1}{m-1}\log\frac{c_4}{\Lambda_0} + \frac{2}{m-1}\,\log n \le \sum_{i=1}^{n} \varepsilon_i \le \frac{1}{m-1}\log\frac{c_3}{\mathbb{P}(X_0=0)} + \frac{S}{2} + \frac{2}{m-1}\,\log n$$

where $S := \sum_{i=1}^{\infty} \varepsilon_i^2 < \infty$ by Theorem 3. It follows immediately that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{\varepsilon_i\, i}{i} \;= \lim_{n\to\infty} \frac{1}{\log n} \sum_{i=1}^{n} \varepsilon_i = \frac{2}{m-1}, \tag{36}$$

proving the proposition.

□

Proposition 4. *Assume* (4)*. If* $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$*, then*

$$0 < h\text{-}\liminf_{i\to\infty} [\mathbb{P}(X_i \neq 0) i^2] \le h\text{-}\limsup_{i\to\infty} [\mathbb{P}(X_i \neq 0) i^2] < \infty \,.$$

This result is a weaker version of Conjecture 1, but at least we are able to say something rigorous about the order i^{-2} for $\mathbb{P}(X_i \neq 0)$.

Proof. It is more convenient to use in the calculations the quantity

$$p_i := m^{-1}(1 - G_i(0)^m) \sim \mathbb{P}(X_i \neq 0),\ i \to \infty\,.$$

Recall that (7) with $s = m$ may be written as

$$1 + \varepsilon_{i+1} = \frac{1}{m}(1+\varepsilon_i)^m + \Big(1 - \frac{1}{m}\Big)(1 - mp_i)$$

whereas

$$\varepsilon_{i+1} = \varepsilon_i + \frac{m-1}{2}\,\varepsilon_i^2(1 + o(1)) - (m-1)p_i,\ i \to \infty\,,$$

or equivalently, as in (26),

$$p_i = \frac{\varepsilon_i - \varepsilon_{i+1}}{m-1} + \frac{\varepsilon_i^2}{2}\,(1 + o(1)),\ i \to \infty\,.$$

It follows that, for $n \to \infty$,

$$\sum_{i=1}^{n} \frac{p_i\, i^2}{i} = \sum_{i=1}^{n} [p_i\, i] = \frac{1}{m-1}\Big[- n\varepsilon_{n+1} + \sum_{i=1}^{n} \varepsilon_i \Big] + \sum_{i=1}^{n} \frac{\varepsilon_i^2 i}{2}\,(1 + o(1)).$$

By using (36) and Theorem 3, we immediately obtain

$$h\text{-}\liminf_{i\to\infty} [p_i i^2] \geq \frac{2}{(m-1)^2} + \frac{c_1^2}{2}\,\frac{\mathbb{P}(X_0 = 0)}{\Lambda_0} > 0 \tag{37}$$

and

$$h\text{-}\limsup_{i\to\infty} [p_i i^2] \leq \frac{2}{(m-1)^2} + \frac{c_2^2}{2}\,\frac{\Lambda_0}{\mathbb{P}(X_0 = 0)} < \infty, \tag{38}$$

where c_1 and c_2 are the constants from Theorem 3.

□

Proposition 5. *Assume* (4). *If* $\mathbb{E}(X_0^3\, m^{X_0}) < \infty$, *then*

$$0 < h\text{-}\liminf_{i\to\infty} [\mathbb{E}(X_i) i^2] \leq h\text{-}\limsup_{i\to\infty} [\mathbb{E}(X_i) i^2] < \infty\,.$$

This result is a weaker version of Conjecture 3, but gives a rigorous statement about the order i^{-2} for $\mathbb{E}(X_i)$.

Proof. As in the previous proof, we will use $p_i := m^{-1}(1 - G_i(0)^m)$. We already know that

$$\mathbb{E}(X_{i+1}) = m\mathbb{E}(X_i) - 1 + G_n(0)^m = m\mathbb{E}(X_i) - mp_i,$$

whereas

$$p_i = \mathbb{E}(X_i) - \frac{1}{m}\,\mathbb{E}(X_{i+1}).$$

By summing up, we obtain

$$\sum_{i=1}^{n}[p_i\, i] = \frac{m-1}{m}\sum_{i=2}^{n}[\mathbb{E}(X_i)\, i] + \frac{1}{m}\sum_{i=2}^{n}\mathbb{E}(X_i) + \mathbb{E}(X_1) - \frac{\mathbb{E}(X_{n+1})n}{m}.$$

Notice that the last term is bounded because by Jensen's inequality,

$$\mathbb{E}(X_n)\log m \le \log \mathbb{E}(m^{X_n}) = \log(G_n(m)) \le G_n(m) - 1 \le c_2\, \frac{\Lambda_0^{1/2}}{[\mathbb{P}(X_0=0)]^{1/2}}\, \frac{1}{n},$$

using Theorem 3 at the last step. It follows that

$$\frac{m-1}{m}\, \sum_{i=1}^{n}[\mathbb{E}(X_i)\, i] + O(1) \le \sum_{i=1}^{n}[p_i\, i] \le \sum_{i=1}^{n}[\mathbb{E}(X_i)\, i] + O(1),\ n\to\infty\,.$$

The proof is completed by an application of (37) and (38). □

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Stochastic Duality and Orthogonal Polynomials

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To Charles M. Newman on his 70th birthday

Abstract. For a series of Markov processes we prove stochastic duality relations with duality functions given by orthogonal polynomials. This means that expectations with respect to the original process (which evolves the variable of the orthogonal polynomial) can be studied via expectations with respect to the dual process (which evolves the index of the polynomial). The set of processes include interacting particle systems, such as the exclusion process, the inclusion process and independent random walkers, as well as interacting diffusions and redistribution models of Kipnis–Marchioro–Presutti type. Duality functions are given in terms of classical orthogonal polynomials, both of discrete and continuous variable, and the measure in the orthogonality relation coincides with the process stationary measure.

Keywords: Interacting particle systems · Duality · Orthogonal polynomials · Exclusion process

1 Motivations and Main Results

1.1 Introduction

Duality theory is a powerful tool to deal with stochastic Markov processes by which information on a given process can be extracted from another process, its *dual*. The link between the two processes is provided by a set of so-called *duality functions*, i.e. a set of observables that are functions of both processes and whose expectations, with respect to the two randomness, can be placed in a precise relation (see Definition 1 below). It is the aim of this paper to enlarge the space of duality functions for a series of Markov processes that enjoy the stochastic

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duality property. These novel duality functions are in turn connected to the polynomials that are orthogonal with respect to the inner product provided by the stationary measure of those processes. Thus the paper develops, via a series of examples, a connection between probabilistic objects (duality functions) and orthogonal polynomials. See [4] for a previous example of a process with duality functions in terms of Hermite polynomials. Moreover, in [10,21,41] it is shown how stochastic duality and orthogonal polynomials are connected from an algebraic perspective.

Before going to the statement of the results, it is worth to stress the importance of a duality relation. Duality theory has been used in several different contexts. Originally introduced for interacting particle systems in [45] and further developed in [36], the literature on stochastic duality covers nowadays a host of examples. Duality has been applied – among the others – to boundary driven and/or bulk driven models of transport [5,27,40,46], Fourier's law [23,29], diffusive particle systems and their hydrodynamic limit [20,36], asymmetric interacting particle systems scaling to KPZ equation [1,6,8,9,17,19,26], six vertex models [7,18], multispecies particle models [2,3,33–35], correlation inequalities [25], mathematical population genetics [12,38]. In all such different contexts it is used, in a way or another, the core simplification that duality provides. Namely, in the presence of a dual process, computation of k-point correlation functions of the original process is mapped into the study of the evolution of only k dual particles, thus substantially reducing the difficulty of the problem.

Besides applications, it is interesting to understand the mathematical structure behind duality. This goes back to classical works by Schütz and collaborators [43,44], where the connection between stochastic duality and symmetries of quantum spin chains was pointed out. More recently, the works [13,14,24,34,35] further investigate this framework and provide an algebraic approach to Markov processes with duality starting from a Lie algebra in the symmetric case, and its quantum deformation in the asymmetric one. In this approach duality functions emerge as the intertwiners between two different representations. Therefore one has a constructive theory, in which duality functions arise from representation theory.

An interesting problem is to fully characterize the set of all duality functions. This question was first asked in [38] where it was defined the concept of duality space, i.e. the subspace of all measurable functions on the configuration product space of two Markov processes for which the duality relation (see Definition 1 below) holds. In [38] the dimension of this space is computed for some simple systems and, as far as we know, a general answer is not available in the general case. Although the algebraic approach recalled above yields a duality function from two representations of a (Lie) algebra, it is not clear a-priori if every duality function can be derived from this approach. In this paper we follow a different route. We shall show that duality functions can be placed in relation to orthogonal polynomials.

1.2 Results

We recall the classical definition of stochastic duality.

Definition 1 (Duality of processes). *Let $X = (X_t)_{t\geq 0}$ and $N = (N_t)_{t\geq 0}$ be two continuous time Markov processes with state spaces Ω and Ω^{dual}, respectively. We say that N is* dual *to X with duality function $D : \Omega \times \Omega^{dual} \longmapsto \mathbb{R}$ if*

$$\mathbb{E}_x[D(X_t, n)] = \mathbb{E}_n[D(x, N_t)], \tag{1}$$

for all $x \in \Omega$, $n \in \Omega^{dual}$ and $t \geq 0$. In (1) $\mathbb{E}_x$ (respectively $\mathbb{E}_n$) is the expectation w.r.t. the law of the X process initialized at x (respectively the N process initialized at n). If X and N are the same process, we say that X is self-dual *with self-duality function D.*

Under suitable hypothesis (see [28]), the above definition is equivalent to the definition of duality between Markov generators.

Definition 2 (Duality of generators). *Let L and L^{dual} be generators of the two Markov processes $X = (X_t)_{t\geq 0}$ and $N = (N_t)_{t\geq 0}$, respectively. We say that L^{dual} is* dual *to L with duality function $D : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$ if*

$$[LD(\cdot, n)](x) = [L^{dual} D(x, \cdot)](n) \tag{2}$$

where we assume that both sides are well defined. In the case $L = L^{dual}$ we shall say that the process is self-dual *and the self-duality relation becomes*

$$[LD(\cdot, n)](x) = [LD(x, \cdot)](n). \tag{3}$$

In (2) (resp. (3)) it is understood that L on the lhs acts on D as a function of the first variable x, while L^{dual} (resp. L) on the rhs acts on D as a function of the second variable n. The Definition 2 is easier to work with, so we will always work under the assumption that the notion of duality (resp. self-duality) is the one in Eq. (2) (resp. (3)).

Remark 1. If $D(x, n)$ is duality function between two processes and

$$c : \Omega \times \Omega^{dual} \longrightarrow \mathbb{R}$$

is constant under the dynamics of the two processes then $c(x, n)D(x, n)$ is also duality function. We will always consider duality functions modulo the quantity $c(x, n)$. For instance, the interacting particle systems studied in Sect. 3 conserve the total number of particles and thus c is an arbitrary function of such conserved quantity.

In this paper we shall prove that several Markov processes, for which a duality function is known from the algebraic approach, also admit a different duality function given in terms of polynomials that are orthogonal with respect to the process stationary measure.

The processes that we consider include discrete interacting particle systems (exclusion process, inclusion process and independent random walkers process) as well as interacting diffusions (Brownian momentum process, Brownian energy process) and redistribution models that are obtained via a thermalization limit (Kipnis–Marchioro–Presutti processes). Their generators, that are defined in Sects. 3 and 4, have an algebraic structure from which duality functions have been previously derived [24].

The orthogonal polynomials we use are some of those with hypergeometric structure. More precisely we consider classical orthogonal polynomials, both discrete and continuous, with the exception of discrete Hahn polynomials and continuous Jacobi polynomials. Through this paper we follow the definitions of orthogonal polynomials given in [39].

The added value of linking duality functions to orthogonal polynomials lies on the fact that they constitute an orthogonal basis of the associated Hilbert space. Often in applications [9,13,27] some quantity of interest are expressed in terms of duality functions, for instance the current in interacting particle systems. This is then used in the study of the asymptotic properties and relevant scaling limits. For these reasons it seems reasonable that having an orthogonal basis of polynomials should be useful in those analysis.

The following theorem collects the results of this paper, details and rigorous proofs can be found in Sect. 3 for self-duality and Sect. 4 for duality.

Theorem 1. *For the processes listed below, the following duality relations hold true*

(i) **Self-duality**

Process	*Stationary measure*	*Duality function $D(x,n)$: product of*
Exclusion Process with up to $2j$ particles, SEP(j)	*Binomial*$(2j,p)$	$K_n(x)/\binom{2j}{n}$
Inclusion Process with parameter k, SIP(k)	*Negative Binomial* $(2k,p)$	$M_n(x)\frac{\Gamma(2k)}{\Gamma(2k+n)}$
Independent Random Walkers, IRW	*Poisson*(λ)	$C_n(x)$

where $K_n(x)$ stands for Krawtchouk polynomials, $M_n(x)$ for Meixner polynomials, $C_n(x)$ for Charlier polynomials.

(ii) **Duality**

Process	*Stationary measure*	*Duality function $D(x,n)$: product of*	*Dual Process*
Brownian Momentum Process, BMP	*Gaussian*$(0,\sigma^2)$	$\frac{1}{(2n-1)!!}H_{2n}(x)$	*Inclusion process with* $k=1/4$
Brownian Energy Process with parameter k, BEP(k)	*Gamma*$(2k,\theta)$	$\frac{n!\Gamma(2k)}{\Gamma(2k+n)}L_n^{(2k-1)}(x)$	*Inclusion process with parameter k*
Kipnis–Marchioro–Presutti with parameter k, KMP(k)	*Gamma*$(2k,\theta)$	$\frac{n!\Gamma(2k)}{\Gamma(2k+n)}L_n^{(2k-1)}(x)$	*dual-KMP(k) process with parameter k*

where $H_n(x)$ stands for Hermite polynomials and $L_n^{(2k-1)}(x)$ for generalized Laguerre polynomials.

Remark 2. As can be inferred from the table, the duality function is not, in general, the orthogonal polynomial itself, but a suitable normalization. Moreover, for the process defined on multiple sites the duality functions exhibit a product structure where each factor is in terms of the relevant orthogonal polynomial.

1.3 Comments

Old and New Dualities. It is known that the six processes we consider satisfy the same (self-)duality relation described by the chart above with different (self-)duality functions. New and old (self-)duality functions can be related using the explicit form of the polynomial that appears in the new one via a relation of the following type:

$$D^{new}(x,n) = \sum_{k=0}^{n} d(k,n) D^{old}(x,k)$$

where $d(k,n)$ also depends on the parameter of the stationary measure. If, for example, we consider the self-duality of the Independent Random Walker, where the new self-duality functions are given by the Charlier polynomials and the old self-duality functions are given by the falling factorial, then we have

$$D^{new}(x,n) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \frac{x!}{(x-k)!} \qquad D^{old}(x,n) = \frac{x!}{(x-n)!}$$

so that

$$D^{new}(x,n) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} D^{old}(x,k)$$

where λ is the Poisson distribution parameter.

The Norm Choice. A sequence of orthogonal polynomials $\{p_n(x) : n \in \mathbb{N}\}$ on the interval (a, b) is defined by the choice of a measure μ to be used in the scalar product and a choice of a norm $d_n^2 > 0$

$$\langle p_n, p_m \rangle := \int_a^b p_n(x) p_m(x) d\mu(x) = \delta_{n,m} d_n^2.$$

The main result of this paper states that orthogonal polynomials with a properly chosen *normalization* furnish duality functions for the processes we consider. In our setting the probability measure μ to be used is provided by the marginal stationary measure of the process. The appropriate norm has been found by making the ansatz that new duality functions can be obtained by the Gram–Schmidt orthogonalization procedure initialized with the old duality functions derived from the algebraic approach [24]. Namely, if one starts from a sequence $p_n(x)$ of orthogonal polynomials with norm d_n^2 and claims that $D^{new}(x, n) = b_n p_n(x)$, where b_n is the appropriate normalization, then the previous ansatz yields

$$b_n = \frac{\langle D^{old}(\cdot, n), p_n \rangle}{d_n^2}. \tag{4}$$

General Strategy of the Proof. Once a properly normalized orthogonal polynomial is identified as a candidate duality function, then the proof of the duality statement is obtained via explicit computations. Each proof heavily relies on the *hypergeometric structure* of the polynomials involved. The idea is to use three structural properties of the polynomials family $p_n(x)$, i.e. the differential or difference hypergeometric equation they satisfy, their three terms recurrence relation and the expression for the raising ladder operator. Those properties are then transported to the duality function using the proper rescaling $b_n p_n(x)$. This yields three identities for the duality function, that can be used in the expression of the process generator. Finally, algebraic manipulation allows to verify relation (2) for duality and (3) for self-duality.

Stochastic Duality and Polynomials Duality. Last, we point out that there exists a notion of duality within the context of orthogonal polynomials, see Definition 3.1 in [30]. For example, in case of discrete orthogonal polynomials, the polynomials self-duality can be described by the identity $p_n(x) = p_x(n)$ where x and n take values in the same discrete set. It turns out that Charlier polynomials are self-dual, whereas Krawtchouk and Meixner polynomials, defined as in [39] see also Sect. 3, are not. However, it is possible to rescale them by a constant b_n so that they become self-dual. The constant b_n is indeed provided by (4). It is not clear if the two notions of stochastic duality and polynomials duality are in some relation one to the other. This will be investigated in a future work.

1.4 Paper Organization

The paper is organized as follows. Section 2 is devoted to a (non exhaustive) review of hypergeometric orthogonal polynomials, we recall their key properties that are crucial in proving our results. We closely follow [39] and the expert reader might skip this part without being affected. The original results are presented in Sects. 3 and 4. In Sect. 3 we describe three interacting particle systems that are self-dual and prove the statement of Theorem 1, part (i). Section 4 is dedicated to the duality relations of two diffusion processes and a jump process obtained as thermalization limit of the previous ones and it contains the proof of Theorem 1, part (ii).

2 Preliminaries: Hypergeometric Orthogonal Polynomials

In this section we give a quick overview of the continuous and the discrete hypergeometric polynomials (see [16,30,39,42]) by reviewing some of their structural properties that will be used in the following.

We start by recalling that the hypergeometric orthogonal polynomials arise from an hypergeometric equation, whose solution can be written in terms of an hypergeometric function ${}_rF_s$.

Definition 3 (Hypergeometric function). *The hypergeometric function is defined by the series*

$$ {}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\,\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{x^k}{k!} \tag{5} $$

where $(a)_k$ denotes the Pochhammer symbol defined in terms of the Gamma function as

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. $$

Remark 3. Whenever one of the numerator parameter a_j is a negative integer $-n$, the hypergeometric function ${}_rF_s$ is a finite sum up to n, i.e. a polynomial in x of degree n.

The Continuous Case. Consider the hypergeometric differential equation

$$ \sigma(x)y^{''}(x) + \tau(x)y^{'}(x) + \lambda y(x) = 0 \tag{6} $$

where $\sigma(x)$ and $\tau(x)$ are polynomials of at most second and first degree respectively and λ is a constant. A peculiarity of the hypergeometric equation is that, for all n, $y^{(n)}(x)$, i.e. the n^{th} derivative of a solution $y(x)$, also solves an hypergeometric equation, namely

$$ \sigma(x)y^{(n+2)}(x) + \tau_n(x)y^{(n+1)}(x) + \mu_n y^{(n)}(x) = 0 \tag{7} $$

with

$$ \tau_n(x) = \tau(x) + n\sigma^{'}(x) \tag{8} $$

and

$$ \mu_n = \lambda + n\tau^{'} + \frac{1}{2}n(n-1)\sigma^{''}. $$

We concentrate on a specific family of solutions: for each $n \in \mathbb{N}$, let $\mu_n = 0$, so that

$$\lambda = \lambda_n = -n\tau^{'} - \frac{1}{2}n(n-1)\sigma^{''}$$

and Eq. (7) has a particular solution given by $y^{(n)}(x)$ constant. This implies that $y(x)$ is a polynomial of degree n, called *polynomial of hypergeometric type* (see Remark 3) and denoted by $p_n(x)$. In the following we will assume that those polynomials are of the form

$$p_n(x) = a_n x^n + b_n x^{n-1} + \dots \qquad a_n \neq 0. \tag{9}$$

It is well known [39] that polynomials of hypergeometric type satisfy the orthogonality relation

$$\int_a^b p_n(x)p_m(x)\rho(x)dx = \delta_{n,m}d_n^2(x) \tag{10}$$

for some (possibly infinite) constants a and b and where the function $\rho(x)$ satisfies the differential equation

$$(\sigma\rho)^{'} = \tau\rho. \tag{11}$$

The sequence d_n^2 can be written in terms of $\sigma(x), \rho(x)$ and a_n as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1}(\lambda_n - \lambda_k)} \int_a^b (\sigma(x))^n \rho(x)dx.$$

As a consequence of the orthogonal property the polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{12}$$

where

$$\alpha_n = c_{n+1,n} \qquad \beta_n = c_{n,n} \qquad \gamma_n = c_{n-1,n}$$

with

$$c_{k,n} = \frac{1}{d_k^2}\int_a^b p_k(x)xp_n(x)\rho(x)dx.$$

The coefficients $\alpha_n, \beta_n, \gamma_n$ can be expressed in terms of the squared norm d_n^2 and the leading coefficients a_n, b_n in (9) as [39]

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n}\frac{d_n^2}{d_{n-1}^2}.$$

Finally, we will use the raising operator R that, acting on the polynomials $p_n(x)$, provides the polynomials of degree $n+1$. Such an operator is obtained from the Rodriguez formula, which provides an explicit form for polynomials of hypergeometric type

$$p_n(x) = \frac{B_n(\sigma^n(x)\rho(x))^{(n)}}{\rho(x)} \quad \text{with} \quad B_n = \frac{a_n}{\prod_{k=0}^{n-1}\left(\tau' + \frac{n+k-1}{2}\sigma''\right)}. \tag{13}$$

The expression of the raising operator (see Eq. 1.2.13 in [39]) reads

$$\mathrm{R}p_n(x) = r_n p_{n+1}(x) \tag{14}$$

where

$$\mathrm{R}p_n(x) = \lambda_n \tau_n(x) p_n(x) - n\sigma(x)\tau_n^{'} p_n^{'}(x) \quad \text{and} \quad r_n = \lambda_n \frac{B_n}{B_{n+1}}.$$

Remark that the raising operator increases the degree of the polynomial by one, similarly to the so-called backward shift operator [30]. However the raising operator in (14) does not change the parameters involved in the function ρ, whereas the backward operator increases the degree and lowers the parameters [31].

The Discrete Case. Everything discussed for the continuous case has a discrete analog, where the derivatives are replaced by the discrete difference derivatives. In particular it is worth mentioning that

$$\Delta f(x) = f(x+1) - f(x) \qquad \text{and} \qquad \nabla f(x) = f(x) - f(x-1).$$

The corresponding hypergeometric differential Eq. (6) is the discrete hypergeometric difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda y(x) = 0 \tag{15}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials of second and first degree respectively, λ is a constant. The differential equation solved by the n^{th} discrete derivative of $y(x)$, $y^{(n)}(x) := \Delta^n y(x)$, is the solution of another difference equation of hypergeometric type

$$\sigma(x)\Delta\nabla y^{(n)}(x) + \tau_n \Delta y^{(n)}(x) + \mu_n y^{(n)}(x) = 0 \tag{16}$$

with

$$\tau_n(x) = \tau(x+n) + \sigma(x+n) - \sigma(x)$$

and

$$\mu_n = \lambda + n\tau^{'} + \frac{1}{2}n(n-1)\sigma^{''}.$$

If we impose $\mu_n = 0$, then

$$\lambda = \lambda_n = -n\tau^{'} - \frac{1}{2}n(n-1)\sigma^{''}$$

and $y^{(n)}(x)$ is a constant solution of Eq. (16). Under these conditions, $y(x)$, solution of (15), is a polynomial of degree n, called *discrete polynomial of hypergeometric type* (see Remark 3) and denoted by $p_n(x)$.

The derivation of the orthogonal property is done in a similar way than the one for the continuous case where the integral is replaced by a sum

$$\sum_{x=a}^{b-1} p_n(x) p_m(x) \rho(x) = \delta_{n,m} d_n^2$$

constants a and b can be either finite or infinite and the function $\rho(x)$ is solution of

$$\Delta[\sigma(x)\rho(x)] = \tau(x)\rho(x).$$

The sequence d_n^2 can be written in terms of $\sigma(x), \rho(x)$ and a_n as

$$d_n^2 = \frac{(a_n n!)^2}{\prod_{k=0}^{n-1}(\lambda_n - \lambda_k)} \sum_{x=a}^{b-n-1} \left(\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right)$$

As a consequence of the orthogonal property, the discrete polynomials of hypergeometric type satisfy a three terms recurrence relation

$$xp_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \tag{17}$$

where

$$\alpha_n = c_{n+1,n} \qquad \beta_n = c_{n,n} \qquad \gamma_n = c_{n-1,n}$$

with

$$c_{k,n} = \frac{1}{d_k^2} \sum_{x=a}^{b} p_k(x) x p_n(x).$$

The coefficients $\alpha_n, \beta_n, \gamma_n$ can be expressed in terms of the squared norm d_n^2 and the leading coefficients a_n, b_n in (9) as [39]

$$\alpha_n = \frac{a_n}{a_{n+1}} \qquad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}} \qquad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}.$$

The discrete Rodriguez formula

$$p_n(x) = \frac{B_n}{\rho(x)} \nabla^n \left[\rho(x+n) \prod_{k=1}^{n} \sigma(x+k) \right] \text{ with } B_n = \frac{a_n}{\prod_{k=0}^{n-1} \left(\tau' + \frac{n+k-1}{2} \sigma'' \right)}$$

leads to an expression for the discrete raising operator R (see Eq. 2.2.10 in [39])

$$\mathrm{R} p_n(x) = r_n p_{n+1}(x) \tag{18}$$

where

$$\mathrm{R} p_n(x) = \left[\lambda_n \tau_n(x) - n\tau_n^{'} \sigma(x) \nabla \right] p_n(x) \qquad r_n = \lambda_n \frac{B_n}{B_{n+1}}.$$

We remark again that the raising operator shouldn't be confused with the backward shift operator in [31] which changes the value of parameters of the distribution ρ.

3 Self-duality: Proof of Theorem 1 Part (i)

We consider first self-duality relations for some discrete interacting particle systems. In this case the dual process is an independent copy of the original one. Notice that, even if the original process and its dual are the same, a massive simplification occurs, namely the n-point correlation function of the original process can be expressed by duality in terms of only n dual particles. Thus a problem for many particles, possibly infinitely many in the infinite volume, may be studied via a finite number of dual walkers.

3.1 The Symmetric Exclusion Process, SEP(j)

In the Symmetric Exclusion Process with parameter $j \in \mathbb{N}/2$, denoted by SEP(j), each site can have at most $2j$ particles, jumps occur at rate proportional to the number of particles in the departure site times the number of holes in the arrival site. The special case $j = 1/2$ corresponds to the standard exclusion process with hard core exclusion, i.e. each site can be either full or empty [36]. We consider the setting where the spacial structure is given by the undirected connected graph $G = (V, E)$ with N vertices and edge set E. A particle configuration is denoted by $\mathbf{x} = (x_i)_{i \in V}$ where $x_i \in \{0, \ldots, 2j\}$ is interpreted as the particle number at vertex i. The process generator reads

$$L^{SEP(j)} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i(2j - x_l) \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x}) \right] + (2j - x_i) x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x}) \right]$$

where $\mathbf{x}^{i,l}$ denotes the particle configuration obtained from the configuration $\mathbf{x}$ by moving one particle from vertex i to vertex l and where $f : \{0, 1, \ldots, 2j\}^N \to \mathbb{R}$ is a function in the domain of the generator.

It is easy to verify that the reversible (and thus stationary) measure of the process for every $p \in (0, 1)$ is given by the homogeneous product measure with marginals the Binomial distribution with parameters $2j > 0$ and $p \in (0, 1)$, i.e. with probability mass function

$$\rho(x) = \binom{2j}{x} p^x (1-p)^{2j-x}, \; x \in \{0, 1, \ldots, 2j\}. \tag{19}$$

The orthogonal polynomials with respect to the Binomial distribution are the Krawtchouk polynomials $K_n(x)$ with parameter $2j$ [32]. These polynomials are obtained by choosing in Eq. (15)

$$\sigma(x) = x \qquad \tau(x) = \frac{2jp - x}{1-p} \qquad \lambda_n = \frac{n}{1-p}.$$

Equivalently, Krawtchouk polynomials are polynomial solutions of the finite difference equation

$$x[K_n(x+1) - 2K_n(x) + K_n(x-1)] + \frac{2jp - x}{1-p}[K_n(x+1) - K_n(x)] + \frac{n}{1-p} K_n(x) = 0. \tag{20}$$

They satisfy the orthogonality relation

$$\sum_{x=0}^{2j} K_n(x) K_m(x) \rho(x) = \delta_{n,m} d_n^2$$

with ρ as in (19) and norm in $\ell^2(\{0, 1, \ldots, 2j\}, \rho)$ given by

$$d_n^2 = \binom{2j}{n} p^n (1-p)^n.$$

As a consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = n+1 \qquad \beta_n = n + 2jp - 2np \qquad \gamma_n = p(1-p)(2j-n+1)$$

and the three-point recurrence then becomes

$$xK_n(x) = (n+1)K_{n+1}(x) + (n+2jp-2np)K_n(x) + p(1-p)(2j-n+1)K_{n-1}(x). \tag{21}$$

Furthermore, the raising operator in (18) provides the relation

$$xK_n(x-1) + \frac{p}{1-p}(n+x-2j)K_n(x) = \frac{n-1}{1-p}K_{n+1}(x). \tag{22}$$

Self-Duality for SEP(j). The Krawtchouk polynomials $K_n(x)$ do not satisfy a self-duality relation in the sense of Definition 2. However, the following theorem shows that, with a proper normalization, it is possible to find a duality function related to them.

Theorem 2. *The SEP(j) is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} \frac{n_i!(2j-n_i)!}{2j!} K_{n_i}(x_i) \tag{23}$$

where $K_n(x)$ denotes the Krawtchouk polynomial of degree n.

Remark 4. Since the Krawtchouk polynomials can be rewritten in terms of hypergeometric functions, the self-duality function turns out to be

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} {}_2F_1\left(\begin{matrix} -n_i, -x_i \\ -2j \end{matrix} \middle| \frac{1}{p}\right)$$

where ${}_2F_0$ is the hypergeometric function defined in (5).

Proof. We need to verify the self-duality relation in Eq. (3). Since the generator of the process is a sum of terms acting on two variables only, we shall verify the self-duality relation for two sites, say 1 and 2. We start by writing the action of the SEP(j) generator working on the duality function for these two sites:

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) = x_1(2j-x_2)\big[&D_{n_1}(x_1-1)D_{n_2}(x_2+1) \\ &- D_{n_1}(x_1)D_{n_2}(x_2)\big] \\ &+ (2j-x_1)x_2\big[D_{n_1}(x_1+1)D_{n_2}(x_2-1) \\ &- D_{n_1}(x_1)D_{n_2}(x_2)\big] \end{aligned}$$

rewriting this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) = &\; x_1 D_{n_1}(x_1-1)(2j-x_2)D_{n_2}(x_2+1) \\ &- x_1 D_{n_1}(x_1)(2j-x_2)D_{n_2}(x_2) \\ &+ (2j-x_1)D_{n_1}(x_1+1)x_2 D_{n_2}(x_2-1) \\ &- (2j-x_1)D_{n_1}(x_1)x_2 D_{n_2}(x_2) \end{aligned} \tag{24}$$

we see that we need an expression for the following terms:

$$xD_n(x), \quad xD_n(x-1), \quad (2j-x)D_n(x+1). \tag{25}$$

To get those we first write the difference Eq. (20), the recurrence relation (21) and the raising operator Eq. (22) in terms of $D_n(x)$. This is possible using Eq. (23) that provides $D_n(x)$ as a suitable normalization of $K_n(x)$, i.e.

$$D_n(x) = \frac{n!(2j-n)!}{2j!}K_n(x).$$

Then the first term in (25) is simply obtained from the normalized recurrence relation, whereas the second and third terms are provided by simple algebraic manipulation of the normalized raising operator equation and the normalized difference equation. We get

$$\begin{aligned}
xD_n(x) &= -p(2j-n)D_{n+1}(x) + (n+2pj-2pn)D_n(x) \\
&\quad - n(1-p)D_{n-1}(x) \\
xD_n(x-1) &= -p(2j-n)D_{n+1}(x) + p(2j-2n)D_n(x) \\
&\quad + npD_{n-1}(x) \\
(2j-x)D_n(x+1) &= p(2j-n)D_{n+1}(x) + (1-p)(2j-2n)D_n(x) \\
&\quad - \frac{n}{p}(1-p)^2 D_{n-1}(x).
\end{aligned}$$

These expressions can now be inserted into (24), which then reads:

$$\begin{aligned}
&L^{SEP(j)}D_{n_1}(x_1)D_{n_2}(x_2) \\
&= \left[p(n_1-2j)D_{n_1+1}(x_1) + p(2j-2n_1)D_{n_1}(x_1) + pn_1D_{n_1-1}(x_1)\right] \\
&\quad \times \Big[p(2j-n_2)D_{n_2+1}(x_2) + (1-p)(2j-2n_2)D_{n_2}(x_2) \\
&\qquad - \frac{n_2}{p}(1-p)^2 D_{n_2-1}(x_2)\Big] \\
&\quad + \left[p(2j-n_1)D_{n_1+1}(x_1) - (n_1+2jp-2pn_1)D_{n_1}(x_1)\right. \\
&\qquad \left. + (1-p)n_1D_{n_1-1}(x_1)\right] \\
&\quad \times \left[p(2j-n_2)D_{n_2+1}(x_2) - (n_2+2pj-2pn_2)D_{n_2}(x_2)\right. \\
&\qquad \left. + n_2(1-p)D_{n_2-1}(x_2) + 2jD_{n_2}(x_2)\right] \\
&\quad + [p(n_2-2j)D_{n_2+1}(x_2) + p(2j-2n_2)D_{n_2}(x_2) + pn_2D_{n_2-1}(x_2)] \\
&\quad \times \Big[p(2j-n_1)D_{n_1+1}(x_1) + (1-p)(2j-2n_1)D_{n_1}(x_1) \\
&\qquad - \frac{n_1}{p}(1-p)^2 D_{n_1-1}(x_1)\Big] \\
&\quad + [p(2j-n_2)D_{n_2+1}(x_2) - (n_2+2pj-2pn_2)D_{n_2}(x_2) + (1-p)n_2D_{n_2-1}(x_2)] \\
&\quad \times \left[p(2j-n_1)D_{n_1+1}(x_1) - (n_1+2pj-2pn_1)D_{n_1}(x_1)\right. \\
&\qquad \left. + n_1(1-p)D_{n_1-1}(x_1) + 2jD_{n_1}(x_1)\right].
\end{aligned}$$

Working out the algebra, substantial simplifications are revealed in the above expression. A long but straightforward computation shows that only products of polynomials with degree $n_1 + n_2$ survive. In particular, after simplifications, one is left with

$$\begin{aligned} &L^{SEP(j)} D_{n_1}(x_1) D_{n_2}(x_2) \\ &= n_1(2j - n_2) \left[D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)\right] \\ &\quad + (2j - n_1) n_2 \left[D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)\right] \end{aligned}$$

and the theorem is proved. □

3.2 The Symmetric Inclusion Process, SIP(k)

The Symmetric Inclusion Process with parameter $k > 0$, denoted by SIP(k), is a Markov jump process with unbounded state space where each site can have an arbitrary number of particles. Again, we define the process on an undirected connected $G = (V, E)$ with $|V| = N$. Jumps occur at rate proportional to the number of particles in the departure and the arrival sites, as the generator describes:

$$L^{SIP(k)} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i(2k + x_l) \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x})\right] + x_l(2k + x_i) \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x})\right].$$

Detailed balance is satisfied by a product measure with marginals given by identical Negative Binomial distributions with parameters $2k > 0$ and $0 < p < 1$, i.e. with probability mass function

$$\rho(x) = \binom{2k + x - 1}{x} p^x (1-p)^{2k}, \ x \in \{0, 1, \ldots\}. \tag{26}$$

The polynomials that are orthogonal with respect to the Negative Binomial distribution are the Meixner polynomials $M_n(x)$ with parameter $2k$, first introduced in [37]. Choosing in (15)

$$\sigma(x) = x \qquad \tau(x) = 2kp - x(1-p) \qquad \lambda_n = n(1-p)$$

we have that the Meixner polynomials are solution of the difference equation

$$\begin{aligned} &x\left[M_n(x+1) - 2M_n(x) + M_n(x-1)\right] \\ &\quad + (2kp - x + xp)\left[M_n(x+1) - M_n(x)\right] + n(1-p) M_n(x) = 0. \end{aligned} \tag{27}$$

They satisfy the orthogonal relation

$$\sum_{x=0}^{\infty} M_n(x) M_m(x) \rho(x) = \delta_{m,n} d_n^2$$

with ρ as in (26) and norm in $\ell^2(\mathbb{N}_0, \rho)$ given by

$$d_n^2 = \frac{n! \Gamma(2k + n)}{p^n \Gamma(2k)}$$

where $\Gamma(x)$ is the Gamma function. As consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = \frac{p}{p-1} \qquad \beta_n = \frac{n+pn+2kp}{1-p} \qquad \gamma_n = \frac{n(n-1+2k)}{p-1}$$

which then becomes

$$xM_n(x) = \frac{p}{p-1}M_{n+1}(x) + \frac{n+pn+2kp}{1-p}M_n(x) + \frac{n(n-1+2k)}{p-1}M_{n-1}(x). \quad (28)$$

Furthermore the raising operator in Eq. (18) provides the identity

$$[p(n+2k+x)]M_n(x) - xM_n(x-1) = pM_{n+1}(x). \quad (29)$$

Self-duality for SIP(k). In analogy with the result for the Exclusion process it is possible to find a duality function for the Symmetric Inclusion Process in terms of the Meixner polynomials.

Theorem 3. *The SIP(k) is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} \frac{\Gamma(2k)}{\Gamma(2k+n_i)} M_{n_i}(x_i)$$

where $M_n(x)$ is the Meixner polynomial of degree n.

Remark 5. The self-duality function can be rewritten in terms of hypergeometric function as

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} {}_2F_1\left(\begin{matrix} -n_i, -x_i \\ 2k \end{matrix} \middle| 1-\frac{1}{p}\right).$$

Proof. As was done for the Exclusion Process we verify the self-duality relation in Eq. (3) for two sites, say 1 and 2. The action of the $SIP(k)$ generator working on the self-duality function for two sites is given by

$$\begin{aligned} & L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\ & = x_1(2k+x_2)\left[D_{n_1}(x_1-1)D_{n_2}(x_2+1) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\ & \quad + (2k+x_1)x_2\left[D_{n_1}(x_1+1)D_{n_2}(x_2-1) - D_{n_1}(x_1)D_{n_2}(x_2)\right]. \end{aligned}$$

We rewrite this by factorizing site 1 and 2, i.e.

$$\begin{aligned} L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) = & \; x_1 D_{n_1}(x_1-1)(2k+x_2)D_{n_2}(x_2+1) \\ & - x_1 D_{n_1}(x_1)(2k+x_2)D_{n_2}(x_2) \\ & + (2k+x_1)D_{n_1}(x_1+1)x_2 D_{n_2}(x_2-1) \\ & - (2k+x_1)D_{n_1}(x_1)x_2 D_{n_2}(x_2) \end{aligned} \quad (30)$$

so that we now need an expression for the following terms:

$$xD_n(x), \quad xD_n(x-1), \quad (2k+x)D_n(x+1). \tag{31}$$

To get those, we first write the difference Eq. (27), the recurrence relation (28) and the raising operator Eq. (29) in terms of $D_n(x)$ using

$$D_n(x) = \frac{\Gamma(2k)}{\Gamma(2k+n)} M_n(x).$$

Then the first term in (31) is simply obtained from the normalized recurrence relation, whereas the second and third terms are provided by simple algebraic manipulation of the normalized raising operator equation and the normalized difference equation. We have,

$$\begin{aligned}
xD_n(x) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{n+p(n+2k)}{p-1}D_n(x) \\
&\quad + \frac{n}{p-1}D_{n-1}(x) \\
xD_n(x-1) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{p}{p-1}(2k+2n)D_n(x) \\
&\quad + \frac{p}{p-1}nD_{n-1}(x) \\
(2k+x)D_n(x+1) &= \frac{p}{p-1}(2k+n)D_{n+1}(x) - \frac{1}{p-1}(2k+2n)D_n(x) \\
&\quad + \frac{1}{p-1}nD_{n-1}(x).
\end{aligned}$$

These relations allow us to expand the SIP(k) generator in Eq. (30) as

$$\begin{aligned}
&L^{SIP} D_{n_1}(x_1)D_{n_2}(x_2) \\
&= \left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{p(2k+2n_1)}{p-1}D_{n_1}(x_1) + \frac{pn_1}{p-1}D_{n_1-1}(x_1)\right] \\
&\quad \times \left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{2k+2n_2}{p-1}D_{n_2}(x_2) + \frac{n_2}{p-1}D_{n_2-1}(x_2)\right] \\
&\quad - \left[\frac{p(2k+n_1)}{p-1}D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1}D_{n_1}(x_1) + \frac{n_1}{p-1}D_{n_1-1}(x_1)\right] \\
&\quad \times \left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1}D_{n_2}(x_2)\right. \\
&\qquad \left. + \frac{n_2}{p-1}D_{n_2-1}(x_2) + 2kD_{n_2}(x_2)\right] \\
&\quad + \left[\frac{p(2k+n_2)}{p-1}D_{n_2+1}(x_2) - \frac{p(2k+2n_2)}{p-1}D_{n_2}(x_2) + \frac{pn_2}{p-1}D_{n_2-1}(x_2)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\frac{p(2k+n_1)}{p-1} D_{n_1+1}(x_1) - \frac{2k+2n_1}{p-1} D_{n_1}(x_1) + \frac{n_1}{p-1} D_{n_1-1}(x_1)\right] \\
&- \left[\frac{p(2k+n_2)}{p-1} D_{n_2+1}(x_2) - \frac{n_2+p(n_2+2k)}{p-1} D_{n_2}(x_2) + \frac{n_2}{p-1} D_{n_2-1}(x_2)\right] \\
&\times \left[\frac{p(2k+n_1)}{p-1} D_{n_1+1}(x_1) - \frac{n_1+p(n_1+2k)}{p-1} D_{n_1}(x_1) \right. \\
&\quad \left. + \frac{n_1}{p-1} D_{n_1-1}(x_1) + 2k D_{n_1}(x_1)\right].
\end{aligned}
$$

At this point it is sufficient to notice that the coefficients of products of polynomials with degree different than $n_1 + n_2$ are all zero, so that we are left with

$$
\begin{aligned}
&L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= n_1(2k+n_2)\left[D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)\right] \\
&+ (2k+n_1)n_2\left[D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)\right]
\end{aligned}
$$

and the theorem is proved. □

3.3 The Independent Random Walker, IRW

The Symmetric Independent Random Walkers, denoted IRW, is one of the simplest, yet non-trivial particle system studied in the literature. It consists of independent particles that perform a symmetric continuous time random walk at rate 1. The generator, defined on the undirected connected graph $G = (V, E)$ with N vertices and edge set E, is given by

$$
L^{IRW} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} x_i \left[f(\mathbf{x}^{i,l}) - f(\mathbf{x})\right] + x_l \left[f(\mathbf{x}^{l,i}) - f(\mathbf{x})\right]. \tag{32}
$$

The reversible invariant measure is provided by a product of Poisson distributions with parameter $\lambda > 0$, i.e. with probability mass function

$$
\rho(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \; x \in \mathbb{N}_0. \tag{33}
$$

The orthogonal polynomials with respect the Poisson distribution are the Charlier polynomials $C_n(x)$ [15] with parameter λ. Choosing in Eq. (15)

$$
\sigma(x) = x \qquad \tau(x) = \lambda - x \qquad \lambda_n = n
$$

we obtain the difference equation whose solution is $C_n(x)$

$$
x\left[C_n(x+1) - 2C_n(x) + C_n(x-1)\right] + (\lambda - x)\left[C_n(x+1) - C_n(x)\right] + nC_n(x) = 0. \tag{34}
$$

The orthogonal relation satisfied by Charlier polynomials is

$$\sum_{x=0}^{\infty} C_n(x) C_m(x) \rho(x) = \delta_{m,n} d_n^2$$

where ρ is given in (33) and the norm in $\ell^2(\mathbb{N}_0, \rho)$ is

$$d_n^2 = n! \lambda^{-n}$$

As consequence of the orthogonality they satisfy the recurrence relation (17) with

$$\alpha_n = -\lambda \qquad \beta_n = n + \lambda \qquad \gamma_n = -n$$

which then becomes

$$x C_n(x) = -\lambda C_{n+1}(x) + (n+\lambda) C_n(x) - n C_{n-1}(x). \tag{35}$$

Furthermore, the raising operator in Eq. (18) provides

$$\lambda C_n(x) - x C_n(x-1) = \lambda C_{n+1}(x). \tag{36}$$

Self-duality of IRW. As the following theorem shows, the self-duality relation is given by the Charlier polynomials themselves.

Theorem 4. *The IRW is a self-dual Markov process with self-duality function*

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} C_{n_i}(x_i) \tag{37}$$

where $C_n(x)$ is the Charlier polynomial of degree n.

Remark 6. Reading the Charlier polynomial as hypergeometric function, the duality function then becomes

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} {}_2F_0 \left(\begin{matrix} -n_i, -x_i \\ - \end{matrix} \middle| -\frac{1}{\lambda} \right).$$

Proof. It is clear from (37) that the difference equations, the recurrence relations and the raising operator for $D_n(x)$ are respectively (34), (35) and (36), that we rewrite as:

$$\begin{aligned} D_n(x+1) &= D_n(x) - \frac{n}{\lambda} D_{n-1}(x) \\ x D_n(x) &= -\lambda D_{n+1}(x) + (n+\lambda) D_n(x) - n D_{n-1}(x) \\ x D_n(x-1) &= \lambda D_n(x) - \lambda D_{n+1}(x). \end{aligned}$$

As done before, we use the two particles IRW generator in (32) and the three equations above to check that the self-duality relation holds. We have

$$\begin{aligned}
&L^{IRW} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= x_1 D_{n_1}(x_1-1) D_{n_2}(x_2+1) - x_1 D_{n_1}(x_1) D_{n_2}(x_2) \\
&\quad + D_{n_1}(x_1+1) x_2 D_{n_2}(x_2-1) - D_{n_1}(x_1) x_2 D_{n_2}(x_2) \\
&= [\lambda D_{n_1}(x_1) - \lambda D_{n_1+1}(x_1)] \left[D_{n_2}(x_2) - \frac{n_2}{\lambda} D_{n_2-1}(x_2) \right] \\
&\quad - [-\lambda D_{n_1+1}(x_1) + (n_1+\lambda) D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)] [D_{n_2}(x_2)] \\
&\quad + \left[D_{n_1}(x_1) - \frac{n_1}{\lambda} D_{n_1+1}(x_2) \right] [\lambda D_{n_2}(x_2) - \lambda D_{n_2+1}(x_2)] \\
&\quad - [D_{n_1}(x_1)] [-\lambda D_{n_2+1}(x_2) + (n_2+\lambda) D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)] \,.
\end{aligned}$$

After computing the products and suitable simplifications we get

$$\begin{aligned}
L^{IRW} D_{n_1}(x_1) D_{n_2}(x_2) = n_1 [&D_{n_1-1}(x_1) D_{n_2+1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)] \\
+ n_2 [&D_{n_1+1}(x_1) D_{n_2-1}(x_2) - D_{n_1}(x_1) D_{n_2}(x_2)].
\end{aligned}$$

□

4 Duality: Proof of Theorem 1 Part (ii)

In this last section we show two examples of duality: the initial process is an interacting diffusion, while the dual one is a jump process, which, in particular, turns out to be the SIP process introduced in Sect. 3.2. We also show an example of duality for a redistribution model of Kipnis–Marchioro–Presutti type.

4.1 The Brownian Momentum Proces, BMP

The Brownian Momentum Process (BMP) is a Markov diffusion process introduced in [22]. On the undirected connected graph $G = (V, E)$ with N vertices and edge set E, the generator reads

$$L^{BMP} f(\mathbf{x}) = \sum_{\substack{1 \le i < l \le N \\ (i,l) \in E}} \left(x_i \frac{\partial f}{\partial x_l}(\mathbf{x}) - x_l \frac{\partial f}{\partial x_i}(\mathbf{x}) \right)^2 \tag{38}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a function in the domain of the generator. A configuration is denoted by $\mathbf{x} = (x_i)_{i \in V}$ where $x_i \in \mathbb{R}$ has to be interpreted as a particle momentum. A peculiarity of this process regards its conservation law: if the process is started from the configuration $\mathbf{x}$ then $||\mathbf{x}||_2^2 = \sum_{i=1}^N x_i^2$ is constant during the evolution, i.e. the total kinetic energy is conserved.

The stationary reversible measure of the BMP process is given by a family of product measures with marginals given by independent centered Gaussian random variables with variance $\sigma^2 > 0$. Without loss of generality we will identify a duality function related to Hermite polynomials in the case the variance equals 1/2. The case with a generic value of the variance would be treated in a similar way by using generalized Hermite polynomials. Choosing in (6)

$$\sigma(x) = 1 \qquad \tau(x) = -2x \qquad \lambda_n = 2n$$

we acquire the differential equation satisfied by the Hermite polynomials $H_n(x)$ [39]

$$H_n^{''}(x) - 2xH_n^{'}(x) + 2nH_n(x) = 0. \tag{39}$$

The orthogonal relation they satisfy is

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)\rho(x)dx = \delta_{m,n}d_n^2$$

with density function

$$\rho(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$$

and norm in $L^2(\mathbb{R}, \rho)$ given by

$$d_n^2 = 2^n n!$$

As consequence of the orthogonality they satisfy the recurrence relation (12) with

$$\alpha_n = \frac{1}{2} \qquad \beta_n = 0 \qquad \gamma_n = n$$

which then can be written as

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \tag{40}$$

Furthermore the raising operator in Eq. (14) provides

$$2xH_n(x) - H_n^{'}(x) = H_{n+1}(x). \tag{41}$$

Duality Between BMP and SIP($\frac{1}{4}$). The following theorem has a similar version in [4] and it states the duality result involving the Hermite polynomials.

Theorem 5. *The BMP process is dual to the SIP($\frac{1}{4}$) process through duality function*

$$D_{\mathbf{n}}(\boldsymbol{x}) = \prod_{i=1}^{N} \frac{1}{(2n_i - 1)!!} H_{2n_i}(x_i)$$

where $H_{2n}(x)$ is the Hermite polynomial of degree $2n$.

Remark 7. Reading the Hermite polynomial as hypergeometric function, the duality function then becomes

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{N} \frac{1}{(2n_i - 1)!!} (2x_i)^{2n_i} {}_2F_0 \left(\begin{matrix} -n_i, (-2n_i + 1)/2 \\ - \end{matrix} \middle| -\frac{1}{x_i^2} \right).$$

Proof. Although the proof in [4] can be easily adapted to our case, we show here an alternative proof that follows our general strategy of using the structural properties of Hermite polynomials. It is sufficient, as before, to show the duality relation in Eq. (1) for sites 1 and 2. The action of the BMP generator on duality function reads

$$\begin{aligned}
&L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= (x_1 \partial_{x_2} - x_2 \partial_{x_1})^2 D_{n_1}(x_1) D_{n_2}(x_2) \\
&= x_1^2 D_{n_1}(x_1) D''_{n_2}(x_2) + D''_{n_1}(x_1) x_2^2 D_{n_2}(x_2) - x_1 D'_{n_1}(x_1) D_{n_2}(x_2) \\
&\quad - D_{n_1}(x_1) x_2 D'_{n_2}(x_2) - 2x_1 D'_{n_1}(x_1) x_2 D'_{n_2}(x_2)
\end{aligned}$$

where we use $\partial_{x_i} = \frac{\partial}{\partial x_i}$. We now need the recurrence relation and the raising operator appropriately rewritten in term of the duality function in order to get suitable expression for

$$x^2 D_n(x), \quad D''_n(x), \quad x D'_n(x).$$

This can be done using

$$D_n(x) = \frac{1}{(2n-1)!!} H_{2n}(x)$$

so that

$$x^2 D_n(x) = \frac{1}{4}(2n+1) D_{n+1}(x) + \left(2n + \frac{1}{2}\right) D_n(x) + 2n D_{n-1}(x) \tag{42}$$

$$D_n^{''}(x) = 8n D_{n-1}(x) \tag{43}$$

$$x D_n^{'}(x) = 2n D_n(x) + 4n D_{n-1}(x) \tag{44}$$

where (42) is obtained from iterating twice the recurrence relation in (40), for Eq. (44) we combined (40) and (41) and then (43) is found from the differential Eq. (39) using (44). Proceeding with the substitution into the generator we find

$$\begin{aligned}
&L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= \left(\frac{1}{4}(2n_1+1) D_{n_1+1}(x_1) + \left(2n_1 + \frac{1}{2}\right) D_{n_1}(x_1) + 2n_1 D_{n_1-1}(x_1)\right) \\
&\qquad \times 8n_2 D_{n_2-1}(x_2) \\
&\quad + 8n_1 D_{n_1-1}(x_1) \\
&\qquad \times \left(\frac{1}{4}(2n_2+1) D_{n_2+1}(x_2) + \left(2n_2 + \frac{1}{2}\right) D_{n_2}(x_2) + 2n_2 D_{n_2-1}(x_2)\right) \\
&\quad - (2n_1 D_{n_1}(x_1) + 4n_1 D_{n_1-1}(x_1)) D_{n_2}(x_2) \\
&\qquad - D_{n_1}(x_1) (2n_1 D_{n_2}(x_2) + 4n_2 D_{n_2-1}(x_2)) \\
&\quad - 2 (2n_1 D_{n_1}(x_1) + 4n_1 D_{n_1-1}(x_1)) (2n_2 D_{n_2}(x_2) + 4n_2 D_{n_2-1}(x_2))
\end{aligned}$$

Finally, after appropriate simplification of the terms whose degree is different from $n_1 + n_2$, we get

$$\begin{aligned} &L^{BMP} D_{n_1}(x_1) D_{n_2}(x_2) \\ &= (2n_1+1)2n_2 \left[D_{n_1+1}(x_1)D_{n_2-1}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\ &\quad + 2n_1(2n_2+1)\left[D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{2n_1}(x_1)D_{n_2}(x_2)\right] \\ &= L^{SIP} D_{n_1}(x_1)D_{n_2}(x_2) \end{aligned}$$

which proves the theorem. □

4.2 The Brownian Energy Process, BEP(k)

We now introduce a process, known as Brownian Energy Process with parameter k, BEP(k) in short notation, whose generator is

$$\begin{aligned} L^{BEP(k)} f(\mathbf{x}) = \sum_{1 \le i < l \le N (i,l) \in E} &\left[x_i x_j \left(\frac{\partial}{\partial x_i} f(\mathbf{x}) - \frac{\partial}{\partial x_j} f(\mathbf{x}) \right)^2 \right. \\ &\left. + 2k(x_i - x_j) \left(\frac{\partial}{\partial x_i} f(\mathbf{x}) - \frac{\partial}{\partial x_j} f(\mathbf{x}) \right) \right]. \end{aligned} \tag{45}$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is in the domain of the generator and $\mathbf{x} = (x_i)_{i \in V}$ denotes a configuration of the process with $x_i \in \mathbb{R}^+$ interpreted as a particle energy. The generator in (45) describes the evolution of particle system that exchange their (kinetic) energies. It is easy to verify that the total energy of the system $\sum_{i=1}^N x_i$ is conserved by the dynamic.

In [24] it was shown that the BEP(k) can be obtained from the BMP process once $4k \in \mathbb{N}$ vertical copies of the graph G are introduced. Under these circumstances denoting $z_{i,\alpha}$ the momentum of the i^{th} particle at the α^{th} level, the kinetic energy per (vertical) site is

$$x_i = \sum_{\alpha=1}^{4k} z_{i,\alpha}^2.$$

If we use the above change of variable in the generator of such BMP process on the ladder graph with $4k$ layers, the generator of the BEP(k) is revealed.

The stationary measure of the BEP(k) process is given by a product of independent Gamma distribution with shape parameter $2k$ and scale parameter θ, i.e. with Lebesgue probability mass function

$$\rho(x) = \frac{x^{2k-1} e^{-\frac{x}{\theta}}}{\Gamma(2k)\theta^k}. \tag{46}$$

Without loss of generality we can set $\theta = 1$ so that the polynomials orthogonal with respect to the Gamma distribution are the generalized Laguerre polynomials $L_n^{(2k-1)}(x)$ [39].
Choosing

$$\sigma(x) = x \qquad \tau(x) = 2k - x \qquad \lambda_n = n$$

the differential equation whose generalized Laguerre polynomials are solution becomes

$$x\frac{d^2}{dx^2}L_n^{(2k-1)}(x) + (2k-x)\frac{d}{dx}L_n^{(2k-1)}(x) + nL_n^{(2k-1)}(x) = 0 \tag{47}$$

The orthogonal relation they satisfy is

$$\int_0^{+\infty} L_n^{(2k-1)}(x)L_m^{(2k-1)}(x)\rho(x)dx = \delta_{m,n}d_n^2$$

with mass function as in (46) with $\theta = 1$ and norm in $L^2(\mathbb{R}^+, \rho)$ given by

$$d_n^2 = \frac{\Gamma(n+2k)}{n!\Gamma(2k)}.$$

As consequence of the orthogonality, they satisfy the recurrence relation (12) with

$$\alpha_n = -(n+1) \qquad \beta_n = 2n+2k \qquad \gamma_n = -(n+2k-1)$$

which then can be written as

$$xL_n^{(2k-1)}(x) = -(n+1)L_{n+1}^{(2k-1)}(x) + (2n+2k)L_n^{(2k-1)}(x) - (n+2k-1)L_{n-1}^{(2k-1)}(x) \tag{48}$$

Furthermore, the raising operator in Eq. (14) is given by

$$(2k-x+n)L_n^{(2k-1)}(x) + x\frac{d}{dx}L_n^{(2k-1)}(x) = (n+1)L_{n+1}^{(2k-1)}(x). \tag{49}$$

Duality Between BEP(k) and SIP(k). The duality relation for the Brownian energy process with parameter k is stated below.

Theorem 6. *The BEP(k) process and the SIP(k) process are dual via*

$$D_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^N \frac{n_i!\,\Gamma(2k)}{\Gamma(2k+n_i)} L_{n_i}^{(2k-1)}(x_i) \tag{50}$$

where $L_n^{(2k-1)}(x)$ is the generalized Laguerre polynomial of degree n.

Remark 8. The factor $\Gamma(2k)$ in (50) is not crucial to assess a duality relation, but it allows to write the duality function as the hypergeometric function

$$D_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^N {}_1F_1\left(\begin{matrix}-n_i\\ 2k\end{matrix}\middle|\, x_i\right).$$

Proof. As in the previous cases we notice that the proof can be shown for sites 1 and 2 only, in which case the generator of the BEP acts on

$$\begin{aligned}
&L^{BEP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= \Big[x_1 x_2 \left(\partial_{x_1} - \partial_{x_2}\right)^2 - 2k(x_1 - x_1)(\partial_{x_1} - \partial_{x_2}) \Big] D_{n_1}(x_1) D_{n_2}(x_2) \\
&= (x_1 \partial_{x_1}^2 + 2k\partial_{x_1}) D_{n_1}(x_1) x_2 D_{n_2}(x_2) + x_1 D_{n_1}(x_1)(x_2 \partial_{x_2}^2 + 2k\partial_{x_2}) D_{n_2}(x_2) \\
&\quad - x_1 \partial_{x_1} D_{n_1(x_1)}(x_2 \partial_{x_2} + 2k) D_{n_2}(x_2) - (x_1 \partial_{x_1} + 2k) D_{n_1}(x_1) x_2 \partial_{x_2} D_{n_2}(x_2)
\end{aligned}$$

We seek an expression for

$$x\partial_x^2 D_n + 2k\partial_x D_n, \qquad xD_n, \qquad x\partial_x D_n$$

that can easily be obtained rewriting (47), (48) and (49) for the duality function, using

$$D_n(x) = \frac{n!\,\Gamma(2k)}{\Gamma(2k+n)} L_n^{(2k-1)}(x)$$

so that, after simple manipulation

$$xD_n^{''}(x) + (2k - x)D_n^{'}(x) + nD_n(x) = 0 \tag{51}$$

$$xD_n(x) = -(n + 2k)D_{n+1}(x) + (2n + 2k)D_n(x) - nD_{n-1}(x) \tag{52}$$

$$xD_n^{'}(x) = nD_n(x) - nD_{n-1}(x). \tag{53}$$

Note that plugging (53) into the difference Eq. (51), we get

$$xD_n^{''}(x) + 2kD_n^{'}(x) = -nD_{n-1}(x).$$

Let's now use these information to write explicitly the BEP(k) generator.

$$\begin{aligned}
&L^{BEP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= [-n_1 D_{n_1-1}(x_1)] \left[-(2k + n_2)D_{n_2+1}(x_2) + (2n_2 + 2k)D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)\right] \\
&\quad + \left[-(2k + n_1)D_{n_1+1}(x_1) + (2n_1 + 2k)D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)\right] [-n_2 D_{n_2-1}(x_2)] \\
&\quad - \left[n_1 D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)\right] \left[(n_2 + 2k)D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)\right] \\
&\quad - \left[(n_1 + 2k)D_{n_1}(x_1) - n_1 D_{n_1-1}(x_1)\right] \left[n_2 D_{n_2}(x_2) - n_2 D_{n_2-1}(x_2)\right].
\end{aligned}$$

Expanding products in the above expression we find

$$\begin{aligned}
&L^{BEP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\
&= n_1 D_{n_1-1}(x_1)(n_2 + 2k)D_{n_2+1}(x_2) + (n_1 + 2k)D_{n_1+1}(x_1) n_2 D_{n_2-1}(x_2) \\
&\quad + n_1 D_{n_1}(x_1)(n_2 + 2k)D_{n_2}(x_2) + (n_1 + 2k)D_{n_1+1}(x_1) n_2 D_{n_2-1}(x_2) \\
&\quad + D_{n_1-1}(x_1) D_{n_2}(x_2) \left[-n_1(2n_2 + 2k) + n_1(n_2 + 2k) + n_1 n_2\right] \\
&\quad + D_{n_1}(x_1) D_{n_2-1}(x_2) \left[-(2n_1 + 2k)n_1 + (n_1 + 2k)n_2 + n_1 n_2\right] \\
&\quad + D_{n_1-1}(x_1) D_{n_2-1}(x_2) \left[n_1 n_2 + n_1 n_2 - n_1 n_2 - n_1 n_2\right].
\end{aligned}$$

Noticing that the coefficients of the last three lines are zeros, we finally get

$$\begin{aligned} &L^{BEP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \\ &= n_1(n_2+2k)\left[D_{n_1-1}(x_1)D_{n_2+1}(x_2) - D_{n_1}(x_1)D_{n_2-1}(x_2)\right] \\ &\quad + (n_1+2k)n_2\left[D_{n_1+1}(x_1)D_{n_2-2}(x_2) - D_{n_1}(x_1)D_{n_2}(x_2)\right] \\ &= L^{SIP(k)} D_{n_1}(x_1) D_{n_2}(x_2) \end{aligned} \tag{54}$$

where $L^{SIP(k)}$ works on the dual variables (n_1, n_2). □

4.3 The Kipnis–Marchioro–Presutti Process, KMP(k)

The KMP model was first introduced by Kipnis, Marchioro and Presutti [29] in 1982 as a model of heat conduction that was solved by using a dual process. It is a stochastic model where a continuous non-negative variable (interpreted as energy) is uniformly redistributed among two random particles on a lattice at Poisson random times. A general version with parameter k, that we shall call KMP(k) was defined in [11], by considering a redistribution rule where a fraction p of the total energy is assigned to one particle and the remaining fraction $(1-p)$ to the other particle, with p a Beta$(2k, 2k)$ distributed random variable. Thus the case $k = 1/2$ corresponds to the original KMP model. In [11] it was shown that KMP(k) is in turn related to the Brownian Energy Process with parameter k, as it can be obtained from the BEP(k) via a procedure called "instantaneous thermalization". If the spatial setting remains as described in the previous sections, the generator of the KMP(k) process is

$$\begin{aligned} L^{KMP(k)} f(\mathbf{x}) = \sum_{1\le i<l\le N\, (i,l)\in E} \int_0^1 \Big[f\big(x_1, \ldots, x_{i-1}, p(x_i+x_{i+1}), \\ (1-p)(x_i+x_{i+1}), x_{i+2}, \ldots x_N\big) - f\left(\mathbf{x}\right) \Big] \nu_{2k}(p) dp \end{aligned}$$

where $\nu_{2k}(p)$ is the density function of the Beta distribution with parameters $(2k,\, 2k)$, i.e.

$$\nu_{2k}(p) = \frac{p^{2k-1}(1-p)^{2k-1}\Gamma(4k)}{\Gamma(2k)\Gamma(2k)}, \qquad p \in (0,1).$$

The dual process of KMP(k) [11] is generated by

$$\begin{aligned} L^{\text{dual-}KMP(k)} f(\mathbf{n}) = \sum_{\substack{1\le i<l\le N \\ (i,l)\in E}} \sum_{r=0}^{n_i+n_{i+1}} \Big[f\big(n_1, \ldots, n_{i-1}, r, n_i+n_{i+1}-r, \\ n_{i+2}, \ldots x_N\big) - f\left(\mathbf{n}\right) \Big] \mu_{2k}(r \mid n_i+n_{i+1}) \end{aligned}$$

where $\mu_{2k}(r|C)$ is the mass density function of the Beta Binomial distribution with parameter $(C,\, 2k,\, 2k)$, i.e.

$$\mu_{2k}(r \mid C) = \frac{\binom{2k+r-1}{r}\binom{2k+C-r-1}{C-r}}{\binom{4k+C-1}{C}}, \qquad r \in \{0, 1, \ldots, C\}.$$

This generator is the result of a thermalized limit of the SIP(k) [11]. Our last theorem is stated below.

Theorem 7. *The KMP(k) process duality relation with its dual is established via duality function*

$$D_{\boldsymbol{n}}(\boldsymbol{x}) = \prod_{i=1}^{N} \frac{n_i!\,\Gamma(2k)}{\Gamma(2k+n_i)} L_{n_i}^{(2k-1)}(x_i) \tag{55}$$

where $L_n^{(2k-1)}(x)$ is the generalized Laguerre polynomial of degree n.

Proof. As expected, the duality function is the same as the one for the BEP(k) and SIP(k) duality relation. This shouldn't surprise since BEP(k) and SIP(k) are dual through duality function (55) and the thermalization limit doesn't affect the duality property. Indeed, considering two graph vertices, one has from [11]

$$L^{KMP(k)} f(x_1, x_2) = \lim_{t\to\infty} (e^{tL^{BEP(k)}} - I) f(x_1, x_2)$$

and

$$L^{\text{dual-}KMP(k)} f(n_1, n_2) = \lim_{t\to\infty} (e^{tL^{SIP(k)}} - I) f(n_1, n_2).$$

Thus, combining the previous two equations and (54), the claim follows. □

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Self-Avoiding Walks and Connective Constants

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Dedicated to Chuck Newman in friendship on his 70th birthday

Abstract. The *connective constant* $\mu(G)$ of a quasi-transitive graph G is the asymptotic growth rate of the number of self-avoiding walks (SAWs) on G from a given starting vertex. We survey several aspects of the relationship between the connective constant and the underlying graph G.

- We present upper and lower bounds for μ in terms of the vertex-degree and girth of a transitive graph.
- We discuss the question of whether $\mu \geq \phi$ for transitive cubic graphs (where ϕ denotes the golden mean), and we introduce the Fisher transformation for SAWs (that is, the replacement of vertices by triangles).
- We present strict inequalities for the connective constants $\mu(G)$ of transitive graphs G, as G varies.
- As a consequence of the last, the connective constant of a Cayley graph of a finitely generated group decreases strictly when a new relator is added, and increases strictly when a non-trivial group element is declared to be a further generator.
- We describe so-called graph height functions within an account of 'bridges' for quasi-transitive graphs, and indicate that the bridge constant equals the connective constant when the graph has a unimodular graph height function.
- A partial answer is given to the question of the locality of connective constants, based around the existence of unimodular graph height functions.
- Examples are presented of Cayley graphs of finitely presented groups that possess graph height functions (that are, in addition, harmonic and unimodular), and that do not.
- The review closes with a brief account of the 'speed' of SAW.

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1 Introduction

1.1 Self-avoiding Walks

A *self-avoiding walk* (abbreviated to SAW) on a graph $G = (V, E)$ is a path that visits no vertex more than once. An example of a SAW on the square lattice is drawn in Fig. 1. SAWs were first introduced in the chemical theory of polymerization (see Orr [66] and the book of Flory [22]), and their critical behaviour has attracted the abundant attention since of mathematicians and physicists (see, for example, the book of Madras and Slade [57] and the lecture notes [7]).

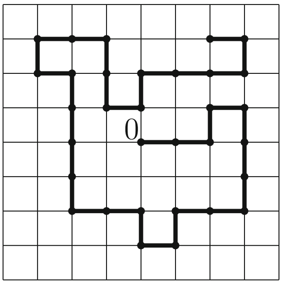

Fig. 1. A 31-step SAW from the origin of the square lattice.

The theory of SAWs impinges on several areas of science including combinatorics, probability, and statistical mechanics. Each of these areas poses its characteristic questions concerning counting and geometry. The most fundamental problem is to count the number of n-step SAWs from a given vertex, and this is the starting point of a rich theory of geometry and phase transition.

Let $\sigma_n(v)$ be the number of n-step SAWs on G starting at the vertex v. The following fundamental theorem of Hammersley asserts the existence of an asymptotic growth rate for $\sigma_n(v)$ as $n \to \infty$. (See Sect. 2.1 for a definition of (quasi-)transitivity.)

Theorem 1 ([41]). *Let $G = (V, E)$ be an infinite, connected, quasi-transitive graph with finite vertex-degrees. There exists $\mu = \mu(G) \in [1, \infty)$, called the* connective constant *of G, such that*

$$\lim_{n\to\infty} \sigma_n(v)^{1/n} = \mu, \qquad v \in V. \tag{1}$$

At the heart of the proof is the observation by Hammersley and Morton [42] that (in the case of a transitive graph) $\log \sigma_n$ is a subadditive function. That is,

$$\sigma_{m+n} \le \sigma_m \sigma_n, \qquad m, n \ge 1. \tag{2}$$

The value $\mu = \mu(G)$ depends evidently on the choice of graph G. Indeed, μ may be viewed as a 'critical point', corresponding, in a sense, to the critical probability of the percolation model, or the critical temperature of the Ising model. Consider the generating function

$$Z_v(x) = \sum_{w \in \Sigma(v)} x^{|w|}, \qquad x \in \mathbb{R}, \tag{3}$$

where $\Sigma(v)$ is the set of finite SAWs starting from a given vertex v, and $|w|$ is the number of edges of w. Viewed as a power series, $Z_v(x)$ has radius of convergence $1/\mu$, and thus a singularity at the point $x = 1/\mu$. Critical exponents may be introduced as in Sect. 1.4.

In this paper we review certain properties of the connective constant $\mu(G)$, in particular exact values (Sect. 1.2), upper and lower bounds (Sect. 2), a sharp lower bound for cubic graphs, and the Fisher transformation (Sect. 3), strict inequalities (Sects. 4, 5), and the locality theorem (Sect. 7). The results summarised here may be found largely in the work of the authors [31–36] and [55]. This review is an expanded and updated version of [30].

Previous work on SAWs tends to have been focussed on specific graphs such as the cubic lattices $\mathbb{Z}^d$ and certain two-dimensional lattices. In contrast, the results of [31–36] are directed at general classes of graphs that are *quasi-transitive*, and often *transitive*. The work reviewed here may be the first systematic study of SAWs on general transitive and quasi-transitive graphs. It is useful to have a reservoir of (quasi-)transitive graphs at one's disposal for the construction and analysis of hypotheses, and to this end the Cayley graphs of finitely generated groups play a significant role (see Sect. 5.1). We note the recent result of Martineau [59] that the set of connective constants of Cayley graphs contains a Cantor space.

Notation for graphs and groups will be introduced when needed. A number of questions are included in this review. The inclusion of a question does not of itself imply either difficulty or importance.

1.2 Connective Constants, Exact Values

For what graphs G is $\mu(G)$ known exactly? There are a number of such graphs, which should be regarded as atypical in this regard. We mention the *ladder* $\mathbb{L}$, the hexagonal lattice $\mathbb{H}$, and the *bridge graph* $\mathbb{B}_\Delta$ with degree $\Delta \ge 2$ of Fig. 2, for which

$$\mu(\mathbb{L}) = \tfrac{1}{2}(1+\sqrt{5}), \quad \mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}, \quad \mu(\mathbb{B}_\Delta) = \sqrt{\Delta - 1}. \tag{4}$$

See [2, p. 184] and [19] for the first two calculations. The third is elementary.

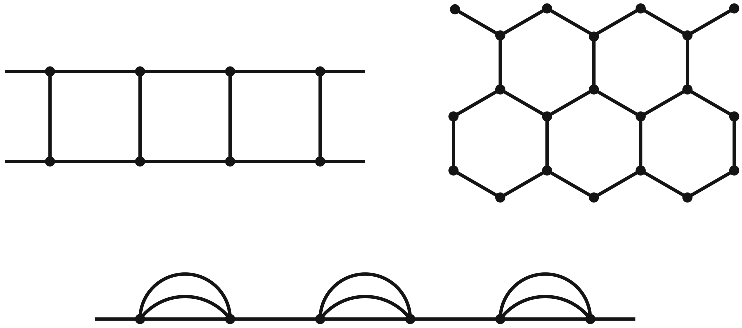

Fig. 2. Three regular graphs: the ladder graph $\mathbb{L}$, the hexagonal tiling $\mathbb{H}$ of the plane, and the bridge graph $\mathbb{B}_\Delta$ (with $\Delta = 4$) obtained from $\mathbb{Z}$ by joining every alternate pair of consecutive vertices by $\Delta - 1$ parallel edges.

In contrast, the value of the connective constant of the square grid $\mathbb{Z}^2$ is unknown, and a substantial amount of work has been devoted to obtaining good bounds. The best rigorous bounds known currently to the authors are those of [45,68], namely (to 5 significant figures)

$$2.6256 \le \mu(\mathbb{Z}^2) \le 2.6792,$$

and more precise numerical estimates are available, including the estimate $\mu \approx 2.63815\ldots$ of [44].

We make some remarks about the three graphs of Fig. 2. There is a correspondence between the Fibonacci sequence and counts of SAWs on the ladder graph $\mathbb{L}$ (see, for example [73]), whereby one obtains that $\mu(\mathbb{L})$ equals the golden ratio $\phi := \frac{1}{2}(1+\sqrt{5})$. We ask in Question 6 whether $\mu(G) \ge \phi$ for all infinite, simple, cubic, transitive graphs, and we discuss evidence for a positive answer to this question.

Amongst a certain class of Δ-regular graphs permitted to possess multiple edges, the bridge graph $\mathbb{B}_\Delta = \sqrt{\Delta - 1}$ is extremal in the sense that $\mu(\mathbb{B}_\Delta)$ is least. See the discussion of Sect. 2.

The proof that $\mu(\mathbb{H}) = \sqrt{2+\sqrt{2}}$ by Duminil-Copin and Smirnov [19] is a very significant recent result. The value $\sqrt{2+\sqrt{2}}$ emerged in the physics literature through work of Nienhuis [65] motivated originally by renormalization group theory. Its proof in [19] is based on the construction of an observable with certain properties of discrete holomorphicity, complemented by a neat use of the bridge decomposition introduced by Hammersley and Welsh [43]. The bridge decomposition has been used since to prove the locality theorem for connective constants (see Sect. 6 and Theorem 14).

1.3 Three Problems on the Square Lattice

There are a number of beautiful open problems associated with SAWs and connective constants, of which we select three. Our first problem is to prove that a random n-step SAW from the origin of $\mathbb{Z}^2$ converges, when suitably rescaled, to the Schramm–Loewner curve $\mathrm{SLE}_{8/3}$. This important conjecture has been discussed and formalized by Lawler, Schramm, and Werner [52].

Question 1. Does a uniformly distributed n-step SAW from the origin of $\mathbb{Z}^2$ converge, when suitably rescaled, to the random curve $\mathrm{SLE}_{8/3}$?

Recent progress in this direction was made by Gwynne and Miller [40], who proved that a SAW on a random quadrangulation converges to $\mathrm{SLE}_{8/3}$ on a certain Liouville-gravity surface.

There is an important class of results usually referred to as the 'pattern theorem'. In Kesten's original paper [49] devoted to $\mathbb{Z}^2$, a *proper internal pattern* $\mathcal{P}$ is defined as a finite SAW with the property that, for any $k \geq 1$, there exists a SAW containing at least k translates of $\mathcal{P}$. The pattern theorem states that: for a given proper internal pattern $\mathcal{P}$, there exists $a > 0$ such that the number of n-step SAWs from the origin 0, containing fewer than an translates of $\mathcal{P}$, is exponentially smaller than the total $\sigma_n := \sigma_n(0)$.

The lattice $\mathbb{Z}^2$ is bipartite, in that its vertices can be coloured black or white in such a way that every edge links a black vertex and a white vertex. The pattern theorem may be used to prove for this bipartite graph that

$$\lim_{n\to\infty} \frac{\sigma_{n+2}}{\sigma_n} = \mu^2.$$

The proof is based on a surgery of SAWs that preserves the parity of their lengths. The following stronger statement has been open since Kesten's paper [49], see the discussion at [57, p. 244].

Question 2. Is it the case for SAWs on $\mathbb{Z}^2$ that $\sigma_{n+1}/\sigma_n \to \mu$?

Hammersley's Theorem 1 establishes the existence of the connective constant for any infinite *quasi-transitive* graph. It is easy to construct examples of (non-quasi-transitive) graphs for which the limit defining μ does not exist, and it is natural to enquire of the situation for a random graph. For concreteness, we consider here the infinite cluster I of bond percolation on $\mathbb{Z}^2$ with edge-density $p > \frac{1}{2}$ (see [26]).

Question 3. Let $\sigma_n(v)$ be the number of n-step SAWs on I starting at the vertex v. Does the limit $\mu(v) := \lim_{n\to\infty} \sigma_n(v)^{1/n}$ exist a.s., and satisfy $\mu(v) = \mu(w)$ a.s. on the event $\{v, w \in I\}$?

Discussions of issues around this question, including of when $\mu(v) = p\mu(\mathbb{Z}^2)$ a.s. on the event $\{v \in I\}$, may be found in papers of Lacoin [50, 51]. The SAW problem on (deterministic) weighted graphs is considered in [38] (see Sect. 6.3).

1.4 Critical Exponents for SAWs

'Critical exponents' play a significant role in the theory of phase transitions. Such exponents have natural definitions for SAWs on a given graph, as summarised next. The reader is referred to [7, 57] and the references therein for general accounts of critical exponents for SAWs. The three exponents that have received most attention in the study of SAWs are as follows.

We consider only the case of SAWs in Euclidean spaces, thus excluding, for example, the hyperbolic space of [58]. Suppose for concreteness that there exists a periodic, locally finite embedding of G into $\mathbb{R}^d$ with $d \geq 2$, and no such embedding into $\mathbb{R}^{d-1}$. The case of general G has not been studied extensively, and most attention has been paid to the hypercubic lattice $\mathbb{Z}^d$.

The Critical Exponent γ. It is believed (when $d \neq 4$) that the generic behaviour of $\sigma_n(v)$ is given by:

$$\sigma_n(v) \sim A_v n^{\gamma-1}\mu^n, \qquad \text{as } n \to \infty, \text{ for } v \in V, \tag{5}$$

for constants $A_v > 0$ and $\gamma \in \mathbb{R}$. The value of the 'critical exponent' γ is believed to depend on d only, and not further on the choice of graph G. Furthermore, it is believed (and largely proved, see the account in [57]) that $\gamma = 1$ when $d \geq 4$. In the borderline case $d = 4$, (5) should hold with $\gamma = 1$ and subject to the correction factor $(\log n)^{1/4}$. (See the related work [6] on weakly self-avoiding walk.)

The Critical Exponent η. Let $v, w \in V$, and

$$Z_{v,w}(x) = \sum_{n=0}^{\infty} \sigma_n(v,w)x^n, \qquad x > 0,$$

where $\sigma_n(v,w)$ is the number of n-step SAWs with endpoints v, w. It is known under certain circumstances that the generating functions $Z_{v,w}$ have radius of convergence μ^{-1} (see [57, Cor. 3.2.6]), and it is believed that there exists an exponent η and constants $A'_v > 0$ such that

$$Z_{v,w}(\mu^{-1}) \sim A'_v d_G(v,w)^{-(d-2+\eta)}, \qquad \text{as } d_G(v,w) \to \infty, \tag{6}$$

where $d_G(v,w)$ is the graph-distance between v and w. Furthermore, η satisfies $\eta = 0$ when $d \geq 4$.

The Critical Exponent ν. Let $\Sigma_n(v)$ be the set of n-step SAWs from v, and let π_n be chosen at random from $\Sigma_n(v)$ according to the uniform probability measure. Let $\|\pi\|$ be the graph-distance between the endpoints of a SAW π. It is believed (when $d \neq 4$) that there exists an exponent ν (the so-called *Flory exponent*) and constants $A''_v > 0$, such that

$$\mathbb{E}(\|\pi_n\|^2) \sim A''_v n^{2\nu}, \qquad v \in V. \tag{7}$$

As above, this should hold for $d = 4$ subject to the inclusion of the correction factor $(\log n)^{1/4}$. It is believed that $\nu = \frac{1}{2}$ when $d \geq 4$.

The exponent ν is an indicator of the geometry of an n-step SAW π chosen with the uniform measure. In the *diffusive* case, we have $\nu = \frac{1}{2}$, whereas in the *ballistic* case (with $\|\pi_n\|$ typically of order n), we have $\nu = 1$. We return to this exponent in Sect. 9.

The three exponents γ, η, ν are believed to be related through the so-called *Fisher relation* $\gamma = \nu(2-\eta)$. The definitions (5), (6), (7) may be weakened to logarithmic asymptotics, in which case we say they hold *logarithmically*.

2 Bounds for Connective Constants

We discuss upper and lower bounds for connective constants in this section, beginning with some algebraic background.

2.1 Transitivity of Graphs

The automorphism group of the graph $G = (V, E)$ is denoted $\mathrm{Aut}(G)$, and the identity automorphism is written $\mathbf{1}$. The expression $\mathcal{A} \le \mathcal{B}$ means that $\mathcal{A}$ is a subgroup of $\mathcal{B}$, and $\mathcal{A} \trianglelefteq \mathcal{B}$ means that $\mathcal{A}$ is a normal subgroup.

A subgroup $\Gamma \le \mathrm{Aut}(G)$ is said to *act transitively* on G if, for $v, w \in V$, there exists $\gamma \in \Gamma$ with $\gamma v = w$. It is said to *act quasi-transitively* if there exists a finite set W of vertices such that, for $v \in V$, there exist $w \in W$ and $\gamma \in \Gamma$ with $\gamma v = w$. The graph is called *transitive* (respectively, *quasi-transitive*) if $\mathrm{Aut}(G)$ acts transitively (respectively, quasi-transitively) on G.

An automorphism γ is said to *fix* a vertex v if $\gamma v = v$. The *stabilizer* of $v \in V$ is the subgroup

$$\mathrm{Stab}_v := \{\gamma \in \mathrm{Aut}(G) : \gamma v = v\}.$$

The subgroup Γ is said to *act freely* on G (or on the vertex-set V) if $\Gamma \cap \mathrm{Stab}_v = \{\mathbf{1}\}$ for $v \in V$.

Let $\mathcal{G}$ (respectively, $\mathcal{Q}$) be the set of infinite, simple, locally finite, transitive (respectively, quasi-transitive), rooted graphs, and let $\mathcal{G}_\Delta$ (respectively, $\mathcal{Q}_\Delta$) be the subset comprising Δ-regular graphs. We write $\mathbf{1} = \mathbf{1}_G$ for the root of the graph G.

2.2 Bounds for μ in Terms of Degree

Let G be an infinite, connected, Δ-regular graph. How large or small can $\mu(G)$ be? It is trivial by counting non-backtracking walks that $\sigma_n(v) \le \Delta(\Delta-1)^{n-1}$, whence $\mu(G) \le \Delta - 1$ with equality if G is the Δ-regular tree. It is not difficult to prove the strict inequality $\mu(G) < \Delta - 1$ when G is quasi-transitive and contains a cycle (see [33, Thm 4.2]). Lower bounds are harder to obtain.

A multigraph is called *loopless* if each edge has distinct endvertices.

Theorem 2 **([33, Thm 4.1]).** *Let G be an infinite, connected, Δ-regular, transitive, loopless multigraph with $\Delta \ge 2$. Then $\mu(G) \ge \sqrt{\Delta - 1}$ if either*

(a) *G is simple, or*
(b) *G is non-simple and $\Delta \le 4$.*

Note that, for the (non-simple) bridge graph $\mathbb{B}_\Delta$ with $\Delta \ge 2$, we have the equality $\mu(\mathbb{B}_\Delta) = \sqrt{\Delta - 1}$.

Here is an outline of the proof of Theorem 2. A SAW is called *forward-extendable* if it is the initial segment of some infinite SAW. Let $\sigma_n^{\mathrm{F}}(v)$ be the number of forward-extendable n-step SAWs starting at v. Theorem 2 is proved by showing as follows that

$$\sigma_{2n}^{\mathrm{F}}(v) \ge (\Delta - 1)^n. \tag{8}$$

Let π be a (finite) SAW from v, with final endpoint w. For a vertex $x \in \pi$ satisfying $x \ne w$, and an edge $e \notin \pi$ incident to x, the pair (x, e) is called π-*extendable* if there exists an infinite SAW starting at v whose initial segment traverses π until x, and then traverses e.

First, it is proved subject to a certain condition Π that, for any $2n$-step forward-extendable SAW π, there are at least $n(\Delta - 2)$ π-extendable pairs. Inequality (8) may be deduced from this statement.

The second part of the proof is to show that graphs satisfying either (a) or (b) of the theorem satisfy condition Π. It is fairly simple to show that (b) suffices, and it may well be reasonable to extend the conclusion to include values of $\Delta \ge 5$.

Question 4. Is it the case that $\mu(G) \ge \sqrt{\Delta - 1}$ in the non-simple case of Theorem 2(b) with $\Delta \ge 5$?

The growth rate μ^{F} of the number of forward-extendable SAWs has been studied further by Grimmett, Holroyd, and Peres [29]. They show that $\mu^{\mathrm{F}} = \mu$ for any infinite, connected, quasi-transitive graph, with further results involving the numbers of backward-extendable and doubly-extendable SAWs.

Question 5. Let $\Delta \ge 3$. What is the sharp lower bound $\mu_{\min}(\Delta) := \inf\{\mu(G) : G \in \mathcal{G}_\Delta\}$? How does $\mu_{\min}(\Delta)$ behave as $\Delta \to \infty$?

This question is considered in Sect. 3.2 when $\Delta = 3$, and it is asked in Question 6 whether or not $\mu_{\min}(3) = \phi$, the golden mean. The lower bound $\mu \ge \sqrt{\Delta - 1}$ of Theorem 2(a) may be improved as follows when G is non-amenable.

Let P be the transition matrix of simple random walk on $G = (V, E)$, and let I be the identity matrix. The *spectral bottom* of $I - P$ is defined to be the largest λ with the property that, for all $f \in \ell^2(V)$,

$$\big\langle f(I - P)f \big\rangle \ge \lambda \langle f, f \rangle. \tag{9}$$

It may be seen that $\lambda(G) = 1 - \rho(G)$ where $\rho(G)$ is the spectral radius of P (see [56, Sect. 6], and [72] for an account of the spectral radius). It is known that G is a non-amenable if and only if $\rho(G) < 1$, which is equivalent to $\lambda(G) > 0$. This was proved by Kesten [47,48] for Cayley graphs of finitely-presented groups, and extended to general transitive graphs by Dodziuk [16] (see also the references in [56, Sect. 6.10]).

Theorem 3 **([36, Thm 6.2]).** *Let $G \in \mathcal{G}_\Delta$ with $\Delta \geq 3$, and let $\lambda = \lambda(G)$ be the above spectral bottom. The connective constant satisfies*

$$\mu(G) \geq (\Delta - 1)^{\frac{1}{2}(1+c\lambda)}, \tag{10}$$

where $c = \Delta(\Delta - 1)/(\Delta - 2)^2$.

2.3 Upper Bounds for μ in Terms of Degree and Girth

The *girth* of a simple graph is the length of its shortest cycle. Let $\mathcal{G}_{\Delta,g}$ be the subset of $\mathcal{G}_\Delta$ containing graphs with girth g.

Theorem 4 **([34, Thm 7.4]).** *For $G \in \mathcal{G}_{\Delta,g}$ where $\Delta, g \geq 3$, we have that $\mu(G) \leq y$ where $\zeta := 1/y$ is the smallest positive real root of the equation*

$$(\Delta - 2)\frac{M_1(\zeta)}{1 + M_1(\zeta)} + \frac{M_2(\zeta)}{1 + M_2(\zeta)} = 1, \tag{11}$$

with

$$M_1(\zeta) = \zeta, \qquad M_2(\zeta) = 2(\zeta + \zeta^2 + \cdots + \zeta^{g-1}). \tag{12}$$

The upper bound y is sharp, and is achieved by the free product graph $F := K_2 * K_2 * \cdots * K_2 * \mathbb{Z}_g$, with $\Delta - 2$ copies of the complete graph K_2 on two vertices and one copy of the cycle $\mathbb{Z}_g$ of length g.

The proof follows quickly by earlier results of Woess [72], and Gilch and Müller [24]. By [72, Thm 11.6], every $G \in \mathcal{G}_{\Delta,g}$ is covered by F, and by [24, Thm 3.3], F has connective constant $1/\zeta$.

3 Cubic Graphs and the Golden Mean

A graph is called *cubic* if it is regular with degree $\Delta = 3$. Cubic graphs have the property that every edge-self-avoiding cycle is also vertex-self-avoiding. We assume throughout this section that $G = (V, E) \in \mathcal{Q}_3$, and we write $\phi := \frac{1}{2}(1 + \sqrt{5})$ for the golden mean.

3.1 The Fisher Transformation

Let $v \in V$, and recall that v has degree 3 by assumption. The so-called *Fisher transformation* acts at v by replacing it by a triangle, as illustrated in Fig. 3. The Fisher transformation has been valuable in the study of the relations between Ising, dimer, and general vertex models (see [13, 21, 53, 54]), and also in the calculation of the connective constant of the Archimedean lattice $(3, 12^2)$ (see, for example, [28, 39, 46]). The Fisher transformation may be applied at every vertex of a cubic graph, of which the hexagonal and square/octagon lattices are examples.

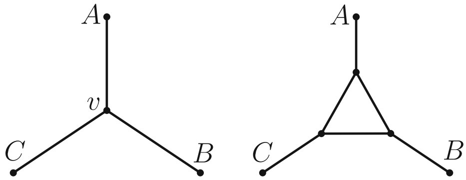

Fig. 3. The Fisher transformation of the star.

Let G be, in addition, quasi-transitive. By Theorem 1, G has a well-defined connective constant $\mu = \mu(G)$ satisfying (1). Write $F(G)$ for the graph obtained by applying the Fisher transformation at every vertex of G. The automorphism group of G induces an automorphism subgroup of $F(G)$, so that $F(G)$ is quasi-transitive and has a well-defined connective constant. It is noted in [31], and probably elsewhere also, that the connective constants of G and $F(G)$ have a simple relationship. This conclusion, and its iteration, are given in the next theorem.

Theorem 5 ([31, Thm 3.1]). *Let $G \in \mathcal{Q}_3$, and consider the sequence $(G_k : k = 0, 1, 2, \dots)$ given by $G_0 = G$ and $G_{k+1} = F(G_k)$.*

(a) *The connective constants $\mu_k := \mu(G_k)$ satisfy $\mu_k^{-1} = g(\mu_{k+1}^{-1})$ where $g(x) = x^2 + x^3$.*
(b) *The sequence μ_k converges monotonely to the golden mean ϕ, and*

$$-\left(\frac{4}{7}\right)^k \leq \mu_k^{-1} - \phi^{-1} \leq \left(\frac{2}{7-\sqrt{5}}\right)^k, \qquad k \geq 1.$$

The idea underlying part (a) is that, at each vertex v visited by a SAW π on G_k, one may replace that vertex by either of the two paths around the 'Fisher triangle' of G_{k+1} at v. Some book-keeping is necessary with this argument, and this is best done via the generating functions (3).

A similar argument may be applied in the context of a 'semi-cubic' graph.

Theorem 6 ([31, Thm 3.3]). *Let G be an infinite, connected, bipartite graph with vertex-sets coloured black and white, and suppose the coloured graph is quasi-transitive, and every black vertex has degree 3. Let $\widetilde{G}$ be the graph obtained by applying the Fisher transformation at each black vertex. The connective constants μ and $\widetilde{\mu}$ of G and $\widetilde{G}$, respectively, satisfy $\mu^{-2} = h(\widetilde{\mu}^{-1})$, where $h(x) = x^3 + x^4$.*

Example 1. Take $G = \mathbb{H}$, the hexagonal lattice with connective constant $\mu = \sqrt{2+\sqrt{2}} \approx 1.84776$, see [19]. The ensuing lattice $\widetilde{\mathbb{H}}$ is illustrated in Fig. 4, and its connective constant $\widetilde{\mu}$ satisfies $\mu^{-2} = h(\widetilde{\mu}^{-1})$, which may be solved to obtain $\widetilde{\mu} \approx 1.75056$.

We return briefly to the critical exponents of Sect. 1.4. In [31, Sect. 3], reasonable definitions of the three exponents γ, η, ν are presented, none of which depend on the existence of embeddings into $\mathbb{R}^d$. Furthermore, it is proved that the values of the exponents are unchanged under the Fisher transformation.

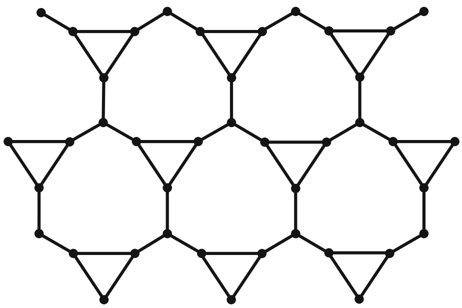

Fig. 4. The lattice $\widetilde{\mathbb{H}}$ is derived from the hexagonal lattice $\mathbb{H}$ by applying the Fisher transformation at alternate vertices. Its connective constant $\widetilde{\mu}$ is a root of the equation $x^{-3} + x^{-4} = 1/(2+\sqrt{2})$.

3.2 Bounds for Connective Constants of Cubic Graphs

Amongst cubic graphs, the 3-regular tree T_3 has largest connective constant $\mu(T_3) = 2$. It is an open problem to determine the sharp lower bound on $\mu(G)$ for $G \in \mathcal{G}_3$. Recall the ladder graph $\mathbb{L}$ of Fig. 2, with $\mu(\mathbb{L}) = \phi$.

Question 6. [34, Qn 1.1] Is it the case that $\mu(G) \geq \phi$ for $G \in \mathcal{G}_3$?

Even in the case of graphs with small girth, the best general lower bounds known so far are as follows.

Theorem 7 **(**[34, Thms 7.1, 7.2]**).**

(a) *For* $G \in \mathcal{G}_{3,3}$*, we have that*
$$\mu(G) \geq x, \tag{13}$$
where $x \in (1,2)$ *satisfies*
$$\frac{1}{x^2} + \frac{1}{x^3} = \frac{1}{\sqrt{2}}. \tag{14}$$

(b) *For* $G \in \mathcal{G}_{3,4}$*, we have that*
$$\mu(G) \geq 12^{1/6}. \tag{15}$$

The sharp upper bounds for $\mathcal{G}_{3,3}$ and $\mathcal{G}_{3,4}$ are those of Theorem 4, and they are attained respectively by the Fisher graph of the 3-regular tree, and of the degree-4 tree (in which each vertex is replaced by a 4-cycle).

Question 6 is known to have a positive answer for various classes of graph, including so-called TLF-planar graphs (see [34,69]). The word *plane* means a simply connected Riemann surface without boundaries. An *embedding* of a graph $G = (V,E)$ in a plane $\mathcal{P}$ is a function $\eta : V \cup E \to \mathcal{P}$ such that η restricted to V is an injection and, for $e = \langle u, v\rangle \in E$, $\eta(e)$ is a C^1 image of $[0,1]$. An embedding is ($\mathcal{P}$-)*planar* if the images of distinct edges are disjoint except possibly at their endpoints, and a graph is ($\mathcal{P}$-)*planar* if it possesses a ($\mathcal{P}$-)planar embedding. An embedding is *topologically locally finite* (*TLF*) if the images of the vertices have no accumulation point, and a connected graph is called *TLF-planar* if it possesses a planar TLF embedding. Let $\mathcal{T}_\Delta$ denote the class of transitive, TLF-planar graphs with vertex-degree Δ.

Theorem 8 **([34]).** *Let $G \in \mathcal{T}_3$ be infinite. Then $\mu(G) \geq \phi$.*

Two techniques are used repeatedly in the proof. The first is to construct an injection from the set of SAWs on the ladder graph $\mathbb{L}$ that move either rightwards or vertically, into the set of SAWs on a graph G' derived from G. A large subclass of $\mathcal{T}_3$ may be treated using such a construction. The remaining graphs in $\mathcal{T}_3$ require detailed analyses using a variety of transformations of graphs including the Fisher transformation of Sect. 3.1.

A second class of graphs for which $\mu \geq \phi$ is given as follows. The definition of a transitive graph height function is deferred to Definition 1.

Theorem 9 **([34, Thm 3.1]).** *We have that $\mu(G) \geq \phi$ for any cubic graph $G \in \mathcal{G}_3$ that possesses a transitive graph height function.*

This theorem covers all Cayley graphs of finitely presented groups with strictly positive first Betti numbers (see Sect. 5.1 and [34, Example 3]). Cayley graphs are introduced in Sect. 5.1.

4 Strict Inequalities for Connective Constants

4.1 Outline of Results

Consider a probabilistic model on a graph G, such as the percolation or random-cluster model (see [27]). There is a parameter (perhaps 'density' p or 'temperature' T) and a 'critical point' (usually written p_c or T_c). The numerical value of the critical point depends on the choice of graph G. It is often important to understand whether a systematic change in the graph causes a *strict* change in the value of the critical point. A general approach to this issue was presented by Aizenman and Grimmett [1] and developed further in [5,12,25] and [26, Chap. 3]. The purpose of this section is to review work of [32] directed at the corresponding question for self-avoiding walks.

Let G be a subgraph of G', and suppose each graph is quasi-transitive. It is trivial that $\mu(G) \leq \mu(G')$. Under what conditions does the strict inequality $\mu(G) < \mu(G')$ hold? Two sufficient conditions for the strict inequality are reviewed here. This is followed in Sect. 5 with a summary of consequences for Cayley graphs.

The results of this section apply to *transitive* graphs. Difficulties arise under the weaker assumption of quasi-transitivity.

4.2 Quotient Graphs

Let $G = (V, E) \in \mathcal{G}$. Let $\Gamma \le \mathrm{Aut}(G)$ act transitively, and let $\mathcal{A} \trianglelefteq \Gamma$ (we shall discuss the non-normal case later). There are several ways of constructing a quotient graph $G/\mathcal{A}$, the strongest of which (for our purposes) is given next. The set of neighbours of a vertex $v \in V$ is denoted by ∂v.

We denote by $\vec{G} = (\overline{V}, \vec{E})$ the *directed* quotient graph $G/\mathcal{A}$ constructed as follows. Let $\approx$ be the equivalence relation on V given by $v_1 \approx v_2$ if and only if there exists $\alpha \in \mathcal{A}$ with $\alpha v_1 = v_2$. The vertex-set $\overline{V}$ comprises the equivalence classes of $(V, \approx)$, that is, the orbits $\overline{v} := \mathcal{A}v$ as v ranges over V. For $v, w \in V$, we place $|\partial v \cap \overline{w}|$ directed edges from $\overline{v}$ to $\overline{w}$ (if $\overline{v} = \overline{w}$, these edges are directed loops).

Example 2. Let G be the square lattice $\mathbb{Z}^2$ and let $m \ge 1$. Let Γ be the set of translations of $\mathbb{Z}^2$, and let $\mathcal{A}$ be the normal subgroup of Γ generated by the map that sends (i, j) to $(i + m, j)$. The quotient graph $G/\mathcal{A}$ is the square lattice 'wrapped around a cylinder', with each edge replaced by two oppositely directed edges.

Since $\vec{G}$ is obtained from G by a process of identification of vertices and edges, it is natural to ask whether the strict inequality $\mu(\vec{G}) < \mu(G)$ is valid. Sufficient conditions for this strict inequality are presented next.

Let $L = L(G, \mathcal{A})$ be the length of the shortest SAW of G with (distinct) endpoints in the same orbit. Thus, for example, $L = 1$ if $\vec{G}$ possesses a directed loop. A group is called *trivial* if it comprises the identity only.

Theorem 10 ([32, Thm 3.8]**).** *Let Γ act transitively on G, and let $\mathcal{A}$ be a non-trivial, normal subgroup of Γ. The connective constant $\vec{\mu} = \mu(\vec{G})$ satisfies $\vec{\mu} < \mu(G)$ if: either*

(a) *$L \ne 2$, or*
(b) *$L = 2$ and either of the following holds:*
  (i) *G contains some 2-step SAW $v\ (= w_0), w_1, w_2\ (= v')$ satisfying $\overline{v} = \overline{v}'$ and $|\partial v \cap \overline{w}_1| \ge 2$,*
  (ii) *G contains some SAW $v\,(= w_0), w_1, w_2, \dots, w_l\,(= v')$ satisfying $\overline{v} = \overline{v}'$, $\overline{w}_i \ne \overline{w}_j$ for $0 \le i < j < l$, and furthermore $v' = \alpha v$ for some $\alpha \in \mathcal{A}$ which fixes no w_i.*

Remark 1. In the situation of Theorem 10, can one *calculate* an explicit $R = R(G, \mathcal{A}) < 1$ such that $\mu(\vec{G})/\mu(G) < R$? The answer is (in principle) positive under a certain condition, namely that the so-called 'bridge constant' of G equals its connective constant. Bridges are discussed in Sect. 6, and it is shown in Theorem 13 that the above holds when G possesses a so-called 'unimodular graph height function' (see Definition 1). See also [32, Thm 3.11] and [37, Remark 4.5].

We call $\mathcal{A}$ *symmetric* if

$$|\partial v \cap \overline{w}| = |\partial w \cap \overline{v}|, \qquad v, w \in V.$$

Consider the special case $L = 2$ of Theorem 10. Condition (i) of Theorem 10(b) holds if $\mathcal{A}$ is symmetric, since $|\partial w \cap \overline{v}| \geq 2$. Symmetry of $\mathcal{A}$ is implied by unimodularity, for a definition of which we refer the reader to [32, Sect. 3.5] or [56, Sect. 8.2].

Example 3. Conditions (i)–(ii) of Theorem 10(b) are necessary in the case $L = 2$, in the sense illustrated by the following example. Let G be the infinite 3-regular tree with a distinguished end ω. Let Γ be the set of automorphisms that preserve ω, and let $\mathcal{A}$ be the normal subgroup generated by the interchanges of the two children of any given vertex v (and the associated relabelling of their descendants). The graph $\vec{G}$ is isomorphic to that obtained from $\mathbb{Z}$ by replacing each edge by two directed edges in one direction and one in the reverse direction. It is easily seen that $L = 2$, but that neither (i) nor (ii) holds. Indeed, $\mu(\vec{G}) = \mu(G) = 2$.

The proof of Theorem 10 follows partly the general approach of Kesten in his pattern theorem, see [49] and [57, Sect. 7.2]. Any n-step SAW $\vec{\pi}$ in the directed graph $\vec{G}$ lifts to a SAW π in the larger graph G. The idea is to show there exists $a > 0$ such that 'most' such $\vec{\pi}$ contain at least an sub-SAWs for which the corresponding sub-walks of π may be replaced by SAWs on G. Different subsets of these sub-SAWs of $\vec{G}$ give rise to different SAWs on G. The number of such subsets grows exponentially in n, and this introduces an exponential 'entropic' factor in the counts of SAWs.

Unlike Kesten's proof and its later elaborations, these results apply in the general setting of transitive graphs, and they utilize algebraic and combinatorial techniques.

We discuss next the assumption of normality of $\mathcal{A}$ in Theorem 10. The (undirected) simple quotient graph $\overline{G} = (\overline{V}, \overline{E})$ may be defined as follows even if $\mathcal{A}$ is not a normal subgroup of Γ. As before, the vertex-set $\overline{V}$ is the set of orbits of V under $\mathcal{A}$. Two distinct orbits $\mathcal{A}v$, $\mathcal{A}w$ are declared adjacent in $\overline{G}$ if there exist $v' \in \mathcal{A}v$ and $w' \in \mathcal{A}w$ with $\langle v', w' \rangle \in E$. We write $\overline{G} = G_{\mathcal{A}}$ to emphasize the role of $\mathcal{A}$.

The relationship between the *site percolation* critical points of G and $G_{\mathcal{A}}$ is the topic of a conjecture of Benjamini and Schramm [10], which appears to make the additional assumption that $\mathcal{A}$ acts freely on V. The last assumption is stronger than the assumption of unimodularity.

We ask for an example in which the non-normal case is essentially different from the normal case.

Question 7. Let Γ be a subgroup of $\mathrm{Aut}(G)$ acting transitively on G. Can there exist a non-normal subgroup $\mathcal{A}$ of Γ such that: (i) the quotient graph $G_{\mathcal{A}}$ is transitive, and (ii) there exists no normal subgroup $\mathcal{N}$ of some transitively acting Γ' such that $G_{\mathcal{A}}$ is isomorphic to $G_{\mathcal{N}}$? Might it be relevant to assume that $\mathcal{A}$ acts freely on V?

We return to connective constants with the following question, inspired in part by [10].

Question 8. Is it the case that $\mu(G_{\mathcal{A}}) < \mu(G)$ under the assumption that $\mathcal{A}$ is a non-trivial (not necessarily normal) subgroup of Γ acting freely on V, such that $G_{\mathcal{A}}$ is transitive?

The proof of Theorem 10 may be adapted to give an affirmative answer to Question 8 subject to a certain extra condition on $\mathcal{A}$, see [32, Thm 3.12]. Namely, it suffices that there exists $l \in \mathbb{N}$ such that $G_{\mathcal{A}}$ possesses a cycle of length l but G has no cycle of this length.

4.3 Quasi-Transitive Augmentations

We consider next the systematic addition of new edges, and the effect thereof on the connective constant. Let $G = (V, E) \in \mathcal{G}$. From G, we derive a second graph $G' = (V, E')$ by adding further edges to E, possibly in parallel to existing edges. *We assume that E is a proper subset of E'.*

Theorem 11 ([32, Thm 3.2]). *Let $\Gamma \leq \mathrm{Aut}(G)$ act transitively on G, and let $\mathcal{A} \leq \Gamma$ satisfy either or both of the following.*

(a) *$\mathcal{A}$ is a normal subgroup of Γ acting quasi-transitively on G.*
(b) *The index $[\Gamma : A]$ is finite.*

If $\mathcal{A} \leq \mathrm{Aut}(G')$, then $\mu(G) < \mu(G')$.

Example 4. Let $\mathbb{Z}^2$ be the square lattice, with $\mathcal{A}$ the group of its translations. The triangular lattice $\mathbb{T}$ is obtained from $\mathbb{Z}^2$ by adding the edge $e = \langle 0, (1,1) \rangle$ together with its images under $\mathcal{A}$, where 0 denotes the origin. Since $\mathcal{A}$ is a normal subgroup of itself, it follows that $\mu(\mathbb{Z}^2) < \mu(\mathbb{T})$. This example may be extended to augmentations by other periodic families of new edges, as explained in [32, Example 3.4].

Remark 2. In the situation of Theorem 11, can one calculate an $R > 1$ such that $\mu(G')/\mu(G) > R$? As in Remark 1, the answer is positive when G' has a unimodular graph height function.

A slightly more general form of Theorem 11 is presented in [32]. Can one dispense with the assumption of normality in Theorem 11(a)?

Question 9. Let Γ act transitively on G, and let $\mathcal{A}$ be a subgroup of Γ that acts quasi-transitively on G. If $\mathcal{A} \leq \mathrm{Aut}(G')$, is it necessarily the case that $\mu(G) < \mu(G')$?

A positive answer would be implied by an affirmative answer to the following question.

Question 10. Let $G \in \mathcal{G}$, and let $\mathcal{A} \leq \mathrm{Aut}(G)$ act quasi-transitively on G. When does there exist a subgroup Γ of $\mathrm{Aut}(G)$ acting transitively on G such that $\mathcal{A} \leq \Gamma$ and $\Gamma \cap \mathrm{Stab}_v \leq \mathcal{A}$ for $v \in V$?

See [32, Prop. 3.6] and the further discussion therein.

5 Connective Constants of Cayley Graphs

5.1 Cayley Graphs

Let Γ be an infinite group with identity element $\mathbf{1}$ and finite generator-set S, where S satisfies $S = S^{-1}$ and $\mathbf{1} \notin S$. Thus, Γ has a presentation as $\Gamma = \langle S \mid R \rangle$ where R is a set of relators. The group Γ is called *finitely generated* since $|S| < \infty$, and *finitely presented* if, in addition, $|R| < \infty$.

The Cayley graph $G = G(\Gamma, S)$ is defined as follows. The vertex-set V of G is the set of elements of Γ. Distinct elements $g, h \in V$ are connected by an edge if and only if there exists $s \in S$ such that $h = gs$. It is easily seen that G is connected, and Γ acts transitively by left-multiplication. It is standard that Γ acts freely, and hence G is unimodular and therefore symmetric. Accounts of Cayley graphs may be found in [4] and [56, Sect. 3.4].

Reference is occasionally made here to the *first Betti number* of Γ. This is the power of $\mathbb{Z}$, denoted $B(\Gamma)$, in the abelianization $\Gamma/[\Gamma, \Gamma]$. (See [35, Remark 4.2].)

5.2 Strict Inequalities for Cayley Graphs

Theorems 10 and 11 have the following implications for Cayley graphs. Let $s_1 s_2 \cdots s_l = \mathbf{1}$ be a relation. This relation corresponds to the closed walk

$$(\mathbf{1}, s_1, s_1 s_2, \dots, s_1 s_2 \cdots s_l = \mathbf{1})$$

of G passing through the identity $\mathbf{1}$. Consider now the effect of adding a further relator. Let $t_1, t_2, \dots, t_l \in S$ be such that $\rho := t_1 t_2 \cdots t_l$ satisfies $\rho \neq \mathbf{1}$, and write $\Gamma_\rho = \langle S \mid R \cup \{\rho\} \rangle$. Then Γ_ρ is isomorphic to the quotient group $\Gamma/\mathcal{N}$ where $\mathcal{N}$ is the normal subgroup of Γ generated by ρ.

Theorem 12 **(**[32, Corollaries 4.1, 4.3]**).** *Let $G = G(\Gamma, S)$ be the Cayley graph of the infinite, finitely presented group $\Gamma = \langle S \mid R \rangle$.*

(a) *Let $G_\rho = G(\Gamma_\rho, S)$ be the Cayley graph obtained by adding to R a further non-trivial relator ρ. Then $\mu(G_\rho) < \mu(G)$.*
(b) *Let $w \in \Gamma$ satisfy $w \neq \mathbf{1}$, $w \notin S$, and let $\overline{G}_w$ be the Cayley graph of the group obtained by adding w (and w^{-1}) to S. Then $\mu(G) < \mu(\overline{G}_w)$.*

As noted in Remarks 1 and 2, non-trivial bounds may in principle be calculated for the ratios of the two connective constants in case (a) (respectively, case (b)) whenever G (respectively, $\overline{G}_w$) has a unimodular graph height function.

Example 5. The *square/octagon* lattice, otherwise known as the Archimedean lattice $(4, 8^2)$, is the Cayley graph of the group with generator set $S = \{s_1, s_2, s_3\}$ and relator set

$$R = \{s_1^2, s_2^2, s_3^2, s_1 s_2 s_1 s_2, s_1 s_3 s_2 s_3 s_1 s_3 s_2 s_3\}.$$

(See [32, Fig. 3].) By adding the further relator $s_2s_3s_2s_3$, we obtain a graph isomorphic to the ladder graph of Fig. 2, whose connective constant is the golden mean ϕ.

By Theorem 12(a), the connective constant μ of the square/octagon lattice is strictly greater than $\phi = 1.618\ldots$. The best lower bound currently known appears to be $\mu > 1.804\ldots$, see [45].

Example 6. The square lattice $\mathbb{Z}^2$ is the Cayley graph of the abelian group with $S = \{a, b\}$ and $R = \{aba^{-1}b^{-1}\}$. We add a generator ab (and its inverse), thus adding a diagonal to each square of $\mathbb{Z}^2$. Theorem 12(b) implies the standard inequality $\mu(\mathbb{Z}^2) < \mu(\mathbb{T})$ of Example 4.

6 Bridges

6.1 Bridges and Graph Height Functions

Various surgical constructions are useful in the study of self-avoiding walks, of which the most elementary involves concatenations of so-called 'bridges'. Bridges were introduced by Hammersley and Welsh [43] in the context of the hypercubic lattice $\mathbb{Z}^d$. An *n-step bridge* on $\mathbb{Z}^d$ is a self-avoiding walk $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ such that

$$\pi_0(1) < \pi_m(1) \le \pi_n(1), \qquad 0 < m \le n,$$

where $x(1)$ denotes the first coordinate of a vertex $x \in \mathbb{Z}^d$.

The significant property of bridges is as follows: given two bridges $\pi = (0, x_1, \ldots, x_m)$, $\pi' = (0, y_1, \ldots, y_n)$ starting at 0, the concatenation $\pi \cup [x_m + \pi']$ is an $(m+n)$-step bridge from 0. It follows that the number b_n of n-step bridges from 0 satisfies

$$b_{m+n} \ge b_m b_n, \tag{16}$$

whence the *bridge constant* $\beta(\mathbb{Z}^d) := \lim_{n\to\infty} b_n^{1/n}$ exists. Since $b_n \le \sigma_n$, it is trivial that $\beta(\mathbb{Z}^d) \le \mu(\mathbb{Z}^d)$. Using a surgery argument, Hammersley and Welsh proved amongst other things that $\beta = \mu$ for $\mathbb{Z}^d$.

In this section, we discuss the bridge constant for transitive graphs, therein introducing the graph height functions that will be useful in the discussion of locality in Sect. 7.

First we define a graph height function, and then we use such a function to define a bridge.

Definition 1 ([37, Defn 3.1]). *Let $G = (V, E) \in \mathcal{Q}$ with root labelled* **1**.

(i) *A* graph height function *on G is a pair $(h, \mathcal{H})$ such that:*
  (a) *$h : V \to \mathbb{Z}$, and $h(\mathbf{1}) = 0$,*
  (b) *$\mathcal{H} \le \mathrm{Aut}(G)$ acts quasi-transitively on G such that h is $\mathcal{H}$-*difference-invariant*, in the sense that*

$$h(\alpha v) - h(\alpha u) = h(v) - h(u), \qquad \alpha \in \mathcal{H},\ u, v \in V,$$

(c) *for* $v \in V$, *there exist* $u, w \in \partial v$ *such that* $h(u) < h(v) < h(w)$.

(ii) *A graph height function* $(h, \mathcal{H})$ *is called* transitive *(respectively,* unimodular*) if the action of* $\mathcal{H}$ *is transitive (respectively, unimodular).*

The reader is referred to [56, Chap. 8] and [37, eqn (3.1)] for discussions of unimodularity.

6.2 The Bridge Constant

Let $(h, \mathcal{H})$ be a graph height function of the graph $G \in \mathcal{Q}$. A *bridge* $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ is a SAW on G satisfying

$$h(\pi_0) < h(\pi_m) \le h(\pi_n), \qquad 0 < m \le n.$$

Let b_n be the number of n-step bridges π from $\pi_0 = \mathbf{1}$. Using quasi-transitivity, it may be shown (similarly to (16)) that the limit $\beta = \lim_{n\to\infty} b_n^{1/n}$ exists, and β is called the *bridge constant*. Note that β depends on the choice of graph height function.

The following is proved by an extension of the methods of [43].

Theorem 13 ([37, Thm 4.3]). *Let* $G \in \mathcal{Q}$ *possess a unimodular graph height function* $(h, \mathcal{H})$. *The associated bridge constant* $\beta = \beta(G, h, \mathcal{H})$ *satisfies* $\beta = \mu(G)$.

In particular, the value of β does not depend on the choice of *unimodular* graph height function.

6.3 Weighted Cayley Graphs

A natural extension of the theory of self-avoiding walks is to *edge-weighted* graphs. Let $G = (V, E)$ be an infinite graph, and let $\phi : E \to [0, \infty)$. The *weight* of a SAW π traversing the edges $e_1, e_2, \ldots, e_n$ is defined as

$$w_\phi(\pi) := \prod_{i=1}^{n} \phi(e_i).$$

One may ask about the asymptotic behaviour of the sum of the $w_\phi(\pi)$ over all SAWs π with length n starting at a given vertex. The question is more interesting when G is not assumed locally finite, since the number σ_n of SAWs from a given vertex may then be infinite. Some conditions are needed on the pair (G, ϕ), and these are easiest stated when G is a Cayley graph.

Let $\Gamma = \langle S \mid R \rangle$ be an infinite, finitely presented group, with a Cayley graph G (we do not assume that G is locally finite). Let $\phi : \Gamma \to [0, \infty)$ be such that

(a) $\phi(\mathbf{1}) = 0$,
(b) ϕ is *symmetric* in that $\phi(\gamma) = \phi(\gamma^{-1})$ for $\gamma \in \Gamma$,
(c) ϕ is *summable* in that $\sum_{\gamma \in \Gamma} \phi(\gamma) < \infty$.

The aggregate weights of SAWs on G have been studied in [38]. It turns out to be useful to consider a generalized notion of the length $l(\pi)$ of a SAW π, and there is an interaction between l and ϕ. Subject to certain assumptions (in particular, Γ is assumed virtually indicable), it is shown that the bridge and connective constants are equal. This yields a continuity theorem for the connective constants of weighted graphs.

7 Locality of Connective Constants

7.1 Locality of Critical Values

The locality question for SAWs may be stated as follows: for which families of rooted graphs is the value of the connective constant $\mu = \mu(G)$ determined by the graph-structure of large bounded neighbourhoods of the root of G? Similar questions have been asked for other systems including the percolation model, see [9,60].

Let $G \in \mathcal{Q}$. The *ball* $S_k = S_k(G)$, with radius k, is the rooted subgraph of G induced by the set of its vertices within distance k of the root $\mathbf{1}$. For $G, G' \in \mathcal{Q}$, we write $S_k(G) \simeq S_k(G')$ if there exists a graph-isomorphism from $S_k(G)$ to $S_k(G')$ that maps $\mathbf{1}$ to $\mathbf{1}'$. Let

$$K(G, G') = \max\big\{k : S_k(G) \simeq S_k(G')\big\}, \qquad G, G' \in \mathcal{G},$$

and $d(G, G') = 2^{-K(G,G')}$. The corresponding metric space was introduced by Babai [3]; see also [11,15].

Question 11. Under what conditions on $G \in \mathcal{Q}$ and $\{G_n\} \subseteq \mathcal{Q}$ is it the case that

$$\mu(G_n) \to \mu(G) \quad \text{if} \quad K(G, G_n) \to \infty?$$

The locality property of connective constants turns out to be related in a surprising way to the existence of *harmonic* graph height functions (see Theorem 16).

7.2 Locality Theorem

Let $G \in \mathcal{Q}$ support a graph height function $(h, \mathcal{H})$. There are two associated integers d, r defined as follows. Let

$$d = d(h) = \max\big\{|h(u) - h(v)| : u, v \in V,\ u \sim v\big\}. \tag{17}$$

If $\mathcal{H}$ acts transitively, we set $r = 0$. Assume $\mathcal{H}$ does not act transitively, and let $r = r(h, \mathcal{H})$ be the infimum of all r such that the following holds. Let $o_1, o_2, \dots, o_M$ be representatives of the orbits of $\mathcal{H}$. For $i \neq j$, there exists $v_j \in \mathcal{H}o_j$ such that $h(o_i) < h(v_j)$, and a SAW from o_i to v_j, with length r or less, all of whose vertices x, other than its endvertices, satisfy $h(o_i) < h(x) < h(v_j)$.

For $D \geq 1$ and $R \geq 0$, let $\mathcal{Q}_{D,R}$ be the subset of $\mathcal{Q}$ containing graphs which possess a unimodular graph height function $(h, \mathcal{H})$ with $d(h) \leq D$ and $r(h, \mathcal{H}) \leq R$.

Theorem 14 (Locality Theorem, [37, Thm 5.1]**).** *Let $G \in \mathcal{Q}$. Let $D \ge 1$ and $R \ge 0$, and let $G_n \in \mathcal{Q}_{D,R}$ for $n \ge 1$. If $K(G, G_n) \to \infty$ as $n \to \infty$, then $\mu(G_n) \to \mu(G)$.*

The rationale of the proof is as follows. Consider for simplicity the case of transitive graphs. Since $\log \sigma_n$ is a subadditive sequence (see (2)), we have that $\mu \le \sigma_n^{1/n}$ for $n \ge 1$. Similarly, by (16), $\log \beta_n$ is superadditive, so that $\beta \ge b_n^{1/n}$ for $n \ge 1$. Therefore,

$$b_n^{1/n} \le \beta \le \mu \le \sigma_n^{1/n}, \qquad n \ge 1. \tag{18}$$

Now, b_n and σ_n depend only on the ball $S_n(G)$. If G is such that $\beta = \mu$, then their shared value can be approximated, within any prescribed accuracy, by counts of bridges and SAWs on bounded balls. By Theorem 13, this holds if G supports a unimodular graph height function.

7.3 Application to Cayley Graphs

Let $\Gamma = \langle S \mid R \rangle$ be finitely presented with Cayley graph $G = G(\Gamma, S)$. Let $t \in \Gamma$ have infinite order. We present an application of the Locality Theorem 14 to the situation in which a new relator t^n is added. Let G_n be the Cayley graph of the group $\Gamma_n = \langle S \mid R \cup \{t^n\} \rangle$.

Theorem 15 ([35, Thm 6.1])**.** *If the first Betti number of Γ satisfies $B(\Gamma) \ge 2$, then $\mu(G_n) \to \mu(G)$ as $n \to \infty$.*

8 Existence of Graph Height Functions

We saw in Sects. 6 and 7 that, subject to the existence of certain unimodular graph height functions, the equality $\beta = \mu$ holds, and a locality result follows. In addition, there exists a terminating algorithm for calculating μ to any degree of precision (see [37]). In this section, we identify certain classes of graphs that possess unimodular graph height functions.

For simplicity in the following, we restrict ourselves to Cayley graphs of finitely generated groups.

8.1 Elementary Amenable Groups

The class EG of elementary amenable groups was introduced by Day in 1957, [14], as the smallest class of groups that contains the set EG_0 of all finite and abelian groups, and is closed under the operations of taking subgroups, and of forming quotients, extensions, and directed unions. Day noted that every group in EG is amenable (see also von Neumann [64]). Let EFG be the set of infinite, finitely generated members of EG.

Theorem 16 **([36, Thm 4.1]).** *Let* $\Gamma \in \text{EFG}$*. There exists a normal subgroup* $\mathcal{H} \trianglelefteq \Gamma$ *with* $[\Gamma : \mathcal{H}] < \infty$ *such that any locally finite Cayley graph* G *of* Γ *possesses a graph height function of the form* $(h, \mathcal{H})$ *which is both unimodular and harmonic.*

Note that the graph height function $(h, \mathcal{H})$ of the theorem is *harmonic*. It has a further property, namely that $\mathcal{H} \trianglelefteq \Gamma$ has finite index, and $\mathcal{H}$ acts on Γ by left-multiplication. Such a graph height function is called *strong*.

The proof of Theorem 16 has two stages. Firstly, by a standard algebraic result, there exist $\mathcal{H} \trianglelefteq \Gamma$ such that: $|\Gamma/\mathcal{H}| < \infty$, and $\mathcal{H}$ is *indicable* in that there exists a surjective homomorphism $F : \mathcal{H} \to \mathbb{Z}$. At the second stage, we consider a random walk on the Cayley graph G, and set $h(\gamma) = \mathbb{E}_\gamma(F(H))$, where H is the first hitting point of $\mathcal{H}$ viewed as a subset of vertices. That h is harmonic off $\mathcal{H}$ is automatic, and on $\mathcal{H}$ because the action of $\mathcal{H}$ is unimodular.

The conclusion of Theorem 16 is in fact valid for the larger class of infinite, finitely generated, virtually indicable groups (see [38, Thm 3.2]).

8.2 Graphs with No Graph Height Function

There exist transitive graphs possessing no graph height function, and examples include the (amenable) Cayley graph of the Grigorchuk group, and the (non-amenable) Cayley graph of the Higman group (see [36, Thms 5.1, 8.1]). This may be deduced from the next theorem.

Theorem 17 **([36, Cor. 9.2]).** *Let* $\Gamma = \langle S \mid R \rangle$ *where* $|S| < \infty$*, and let* Π *be the subgroup of permutations of* S *that preserve* Γ *up to isomorphism. Let* G *be a Cayley graph of* Γ *satisfying* $\text{Stab}_1 = \Pi$*, where* Π *is viewed as a subgroup of* $\text{Aut}(G)$*.*

(a) *If* Γ *is a torsion group, then* G *has no graph height function.*
(b) *Suppose* Γ *has no proper, normal subgroup with finite index. If* G *has graph height function* $(h, \mathcal{H})$*, then* (h, Γ) *is also a graph height function.*

The point is that, when $\text{Stab}_{\mathbf{1}} = \Pi$, every automorphism of G is obtained by a certain composition of an element of Γ and an element of Π. The Grigorchuk group is a torsion group, and part (a) applies. The Higman group Γ satisfies part (b), and is quickly seen to have no graph height function of the form (h, Γ). (A graph height function of the form (h, Γ) is called a *group height function* in [35, Sect. 4].)

9 Speed, and the Exponent ν

For simplicity in this section, we consider only transitive rooted graphs G. Let π_n be a random n-step SAW from the root of G, chosen according to the uniform measure on the set Σ_n of such walks. What can be said about the graph-distance $\|\pi_n\|$ between the endpoints of π_n?

We say that SAW on G has *positive speed* if there exist $c, \alpha > 0$ such that

$$\mathbb{P}(\|\pi_n\| \le cn) \le \mathrm{e}^{-\alpha n}, \qquad n \ge 0.$$

While stronger than the natural definition through the requirement of exponential decay to 0, this is a useful definition for the results of this section. When G is infinite, connected, and quasi-transitive, and SAW on G has positive speed, it is immediate that

$$Cn^2 \le \mathbb{E}(\|\pi_n\|^2) \le n^2,$$

for some $C > 0$; thus (7) holds (in a slightly weaker form) with $\nu = 1$.

For SAW on $\mathbb{Z}^d$ it is known, [18], that SAW does not have positive speed, and that that $\mathbb{E}(\|\pi_n\|^2)/n^2 \to 0$ as $n \to \infty$ (cf. (7)). Complementary 'delocalization' results have been proved in [17], for example that, for $\epsilon > 0$ and large n,

$$\mathbb{P}(\|\pi_n\| = 1) \le n^{-\frac{1}{4}+\epsilon},$$

and it is asked there whether

$$\mathbb{P}(\|\pi_n\| = x) \le n^{-\frac{1}{4}+\epsilon}, \qquad x \in \mathbb{Z}^d.$$

We pose the following question for non-amenable graphs.

Question 12. [18] Is it the case that, for any non-amenable Cayley graph G of an infinite, finitely generated group, SAW on G has positive speed?

Progress towards this question may be summarised as follows. By bounding the number of SAWs by the number of non-backtracking paths, Nachmias and Peres [63, eqn (2.3)] have proved that that, for a non-amenable, transitive graph G satisfying

$$(\Delta - 1)\rho < \mu, \tag{19}$$

SAW on G has positive speed. Here, Δ is the vertex degree, ρ is the spectral radius of simple random walk on G (see the discussion around (9)), and μ is the connective constant.

It is classical (see, for example, [26, eqn (1.13)]) that $\mu p_\mathrm{c} \ge 1$, where p_c is the critical probability of bond percolation on G. Inequality (19) is therefore implied by the stronger inequality

$$(\Delta - 1)\rho p_\mathrm{c} < 1. \tag{20}$$

These inequalities (19)–(20) are useful in several settings.

A. [63] There exist $\rho_0 < 1$ and $g_0 < \infty$ such that, if G is a non-amenable, transitive graph with spectral radius less than ρ_0 and girth at least g_0, then (20) holds, and hence SAW on G has positive speed.
B. [67] Let $S \subset \Gamma$ be a finite symmetric generating set of an infinite group Γ, and $\Delta = |S|$. Let $S^{(k)}$ be the multiset of cardinality Δ^k comprising all elements $g \in \Gamma$ with length not exceeding k in the word metric given by S,

each such element included with multiplicity equal to the number of such ways of expressing g. The set $S^{(k)}$ generates Γ. By [67, Proof of Thm 2],

$$\rho(G, S^{(k)}) p_c(G, S^{(k)}) \Delta^k \to 0 \qquad \text{as } k \to \infty,$$

where $(G, S^{(k)})$ denotes the (possibly non-simple) Cayley graph of Γ with generator-set $S^{(k)}$. Inequality (20) follows for sufficiently large k

C. By the argument of [67, Prop. 1], (20) holds if

$$\rho < \frac{\Delta^2 + \Delta - 1}{\Delta^2 + (\Delta - 1)^2}.$$

This holds when $\rho < \frac{1}{2}$, irrespective of Δ.

D. Thom [71] has shown that, for any finitely generated, non-amenable group Γ and any $\epsilon > 0$, there exists a finite symmetric generating set S such that the corresponding Cayley graph $G = G(\Gamma, S)$ has $\rho < \epsilon$. By the above, SAW on G has positive speed.

E. [70] Let $i = i(G)$ be the edge-isoperimetric constant of an infinite, transitive graph G. Since $p_c \le (1 + i)^{-1}$ and $\rho^2 \le 1 - (i/\Delta)^2$ ([61, Thm 2.1(a)]), inequality (20) holds whenever

$$(\Delta - 1) \frac{\sqrt{1 - (i/\Delta)^2}}{1 + i} < 1.$$

It is sufficient that $i/\Delta > 1/\sqrt{2}$.

F. [20] It is proved that

$$\rho \le \frac{\sqrt{8\Delta - 16} + 3.47}{\Delta},$$

when G is planar. When combined with C above, for example, this implies that SAW has positive speed on any transitive, planar graph with sufficiently large Δ. A related inequality for hyperbolic tesselations is found in [20, Thm 7.4].

We turn next to graphs embedded in the hyperbolic plane. It was proved by Madras and Wu [58] that SAWs on most regular tilings of the hyperbolic plane have positive speed. Note that the graphs treated in [58] are both (vertex-)transitive and edge-transitive (unlike the graphs in F above). It was proved by Benjamini [8] that SAWs on the seven regular planar triangulations of the hyperbolic plane have mean displacement bounded beneath by a linear function.

We turn finally to a discussion of the number of ends of a transitive graph. The number of *ends* of an infinite, connected graph $G = (V, E)$ is the supremum over all finite subsets $W \subset V$ of the number of infinite components that remain after deletion of W. An infinite, finitely generated group Γ is said to have k ends if some locally finite Cayley graph (and hence all such Cayley graphs) has k ends. Recall from [62, Prop. 6.2] that a transitive graph G has $k \in \{1, 2, \infty\}$

and, moreover, if $k = 2$ (respectively, $k = \infty$) then G is amenable (respectively, non-amenable).

Infinite, connected, (quasi-)transitive graphs with two or more ends have been studied by Li [55, Thm 1.3], who has proved, subject to two conditions, that SAW has positive speed. The approach of the proof (see [55]) is to consider a finite 'cutset' W with the property that many SAWs cross S to another component of $G \setminus W$ and never cross back. The pattern theorem is a key element in the proof. These results may be applied, for example, to a cylindrical quotient graph of $\mathbb{Z}^d$ (see [2,23]), and the infinite free product of two finite, transitive, connected graphs. Here are two corollaries for Cayley graphs.

Theorem 18. ([55, Thms 1.7, 1.8]). *Let Γ be an infinite, finitely generated group with two or more ends.*

(a) *If Γ has infinitely many ends, and G is a locally finite Cayley graph, there exists $c > 0$ such that*

$$\limsup_{n\to\infty} \left|\{\pi \in \Sigma_n(\mathbf{1}) : \|\pi\| \geq cn\}\right|^{1/n} = \mu.$$

(b) *There exists some locally finite, Cayley graph G such that SAW on G has positive speed.*

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Front Propagation and Quasi-Stationary Distributions: Two Faces of the Same Coin

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Dedicated to Chuck Newman on the occasion of his 70th birthday.

Abstract. We analyze the connection between front propagation and quasi-stationary distributions in translation invariant one-dimensional Markov processes. We describe the link between them through the microscopic models known as Branching Brownian Motion with selection and Fleming–Viot.

Keywords: Selection principle · Quasi-stationary distributions · Branching brownian motion with selection · Traveling waves

1 Introduction

A selection mechanism in front propagation can be thought of as follows: a certain phenomenology is described through an equation that admits an infinite number of traveling-wave solutions, but there is only one which has a physical meaning, the one with minimal velocity. Under mild assumptions on initial conditions, the solution converges to this minimal-velocity traveling wave. The most remarkable example of this fact is the celebrated F–KPP equation (for Fisher, Kolmogorov–Petrovskii–Piskunov)

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + rg(v), \quad x \in \mathbb{R},\, t > 0, \\ v(0,x) &= v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

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Assume for simplicity that g has the form $g(s) = s^2 - s$, but this can be generalized up to some extent. We also restrict ourselves to initial data v_0 that are distribution functions of probability measures in $\mathbb{R}$. The equation was introduced in 1937 [15,24] as a model for the evolution of a genetic trait and since then, has been widely studied.

Both Fisher and Kolmogorov, Petrovskii and Piskunov proved independently that this equation admits an infinite number of traveling wave solutions (TW) of the form $v(t,x) = w_c(x - ct)$ that travel at velocity c. This fact is somehow unexpected from the modeling point of view.

Fisher proposed a way to overcome this difficulty, related to the probabilistic representation given later on by McKean [29], weaving links between solutions to (1) and Branching Brownian Motion. The general principle behind is that microscopic effects should be taken into account to properly describe the physical phenomena. With a similar point of view in mind, Brunet, Derrida and coauthors [8–11] started in the nineties a study of the effect of microscopic noise in front propagation for Eq. (1) and related models, which resulted in a huge number of works that study the change in the behavior of the front when microscopic effects are taken into account. These works include both numerical and heuristic arguments [8–11,22] as well as rigorous proofs [4,5,13,26]. Before that, Bramson et al. [7] gave the first rigorous proof of a microscopic model for (1) that has a unique velocity for every initial condition. They also prove that these velocities converge in the macroscopic scale to the minimum velocity of (1), and call this fact a *microscopic selection principle*, as opposed to the macroscopic selection principle stated above, that holds for solutions of the hydrodynamic equation.

Consider a Markov process $X = (X_t,\, t \geq 0)$, killed at some state or region that we call 0, defined on certain filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The absorption time is defined by $\tau = \inf\{t > 0\colon X_t \in 0\}$. The conditioned evolution at time t is defined by

$$\mu_t^{\gamma}(\cdot) := \mathbb{P}_{\gamma}(X_t \in \cdot | \tau > t).$$

Here γ denotes the initial distribution of the process. A probability measure ν is said to be a quasi-stationary distribution (QSD) if $\mu_t^{\nu} = \nu$ for all $t \geq 0$.

The *Yaglom limit* is a probability measure ν defined by

$$\nu := \lim_{t \to \infty} \mu_t^{\delta_x},$$

if it exists and does not depend on x. It is known that if the Yaglom limit exists, then it is a QSD. A general principle is that the Yaglom limit *selects* the minimal QSD, i.e. the Yaglom limit is the QSD with minimal mean absorption time. This fact has been proved for a wide class of processes that include birth and death process, Galton–Watson processes [32], random walks [21] and Brownian Motion [27] among others.

It is a typical situation that there is an infinite number of quasi-stationary distributions, but the *Yaglom limit* (the limit of the conditioned evolution of the process started from a deterministic initial condition) selects the minimal one, i.e. the one with minimal expected time of absorption.

This description reveals that similar phenomena occur in both contexts (TW and QSD). The purpose of this note is to show why and how are they related. We first explain the link through the example of Brownian Motion and then we show how to extend this relation to more general Lévy processes. This article has essentially no proofs. In the companion paper [20] we give rigorous proofs for the existence of a precise bijection between TW and QSD in the context of one-dimensional Lévy processes.

2 Macroscopic Models

We elaborate on the two macroscopic models we study: front propagation and QSD.

2.1 Front Propagation in the F–KPP

Since the seminal papers [15,24], Eq. (1) has received a huge amount of attention for several reasons. Among them, it is one of the simplest models explaining several phenomena that are expected to be universal. For instance, it admits a continuum of traveling wave solutions that can be parametrized by their velocity c. More precisely, for each $c \in [\sqrt{2r}, +\infty)$ there exists a function $w_c : \mathbb{R} \to [0,1]$ such that

$$v(t,x) = w_c(x - ct)$$

is a solution to (1). For $c < \sqrt{2r}$, there is no traveling wave solution, [1,24]. Hence $c^* = \sqrt{2r}$ represents the minimal velocity and w_{c^*} the minimal traveling wave. Moreover, if v_0 verifies for some $0 < b < \sqrt{2r}$

$$\lim_{x\to\infty} \mathrm{e}^{bx}(1 - v_0(x)) = a > 0,$$

then

$$\lim_{t\to\infty} v(t, x + ct) = w_c(x), \quad \text{for } c = r/b + \frac{1}{2}b, \tag{2}$$

see [29–31]. If the initial measure has compact support (or fast enough decay at infinity), the solution converges to the minimal traveling wave and the domain of attraction and velocity of each traveling wave is determined by the tail of the initial distribution. A smooth traveling wave solution of (1) that travels at velocity c is a solution to

$$\frac{1}{2}w'' + cw + r(w^2 - w) = 0. \tag{3}$$

2.2 Quasi-Stationary Distributions

For Markov chains in finite state spaces, the existence and uniqueness of QSDs as well as the convergence of the conditioned evolution to this unique QSD for every initial measure follows from Perron–Frobenius theory. The situation is more delicate for unbounded spaces as there can be zero, one, or an infinite number of QSD. Among those distributions, the *minimal* QSD is the one that minimizes $\mathbb{E}_\nu(\tau)$. Here $\mathbb{E}$ denotes expectation respect to $\mathbb{P}$.

The presence of an infinite number of quasi-stationary distributions might be anomalous from the modeling point of view, in the sense that no physical nor biological meaning has been attributed to them. The reason for their presence here and in the front propagation context is similar: when studying for instance population or gene dynamics through the conditioned evolution of a Markov process, we implicitly consider an infinite population and microscopic effects are lost.

So, as Fisher suggests, in order to avoid the undesirable infinite number of QSD, we should take into account microscopic effects. A natural way to do this is by means of interacting particle systems. We discuss this in Sect. 3.

Brownian Motion with drift Quasi-stationary distributions for Brownian Motion with constant drift towards the origin are studied in [27, 28]. We briefly review here some of the results of these papers and refer to them for the details.

For $c > 0$ we consider a one-dimensional Brownian Motion $X = (X_t)_{t\geq 0}$ with drift $-c$ defined by $X_t = B_t - ct$. Here $(B_t)_{t\geq 0}$ is a one dimensional Wiener process defined in the standard Wiener space. We use $\mathbb{P}_x$ for the probability defined in this space such that $(B_t)_{t\geq 0}$ is Brownian Motion started at x and $\mathbb{E}_x$ for expectation respect to $\mathbb{P}_x$. Define the hitting time of zero, when the process is started at $x > 0$ by $\tau_x(c) = \inf\{t > 0\colon X_t = 0\}$ and denote with P_t^c the sub-Markovian semigroup defined by

$$P_t^c f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{\{\tau_x(c)>t\}}). \tag{4}$$

In this case, differentiating (4) and after some manipulation it can be seen that the conditioned evolution μP_t has a density $u(t,\cdot)$ for every $t > 0$ and verifies

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t,x) + c\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\frac{\partial u}{\partial x}(t,0)u(t,x), \quad t > 0, x > 0, \\ u(t,0) &= u(t,+\infty) = 0, \quad t > 0. \end{aligned} \tag{5}$$

It is easy to check that if ν is a QSD, the hitting time of zero, started with ν is an exponential variable of parameter r and hence ν is a QSD if and only if there exists $r > 0$ such that

$$\nu P_t^c = \mathrm{e}^{-rt}\nu, \quad \text{for any} \quad t > 0. \tag{6}$$

Differentiating (4) and using the semigroup property and (6) we get that ν is a QSD if and only if

$$\int \left(\frac{1}{2}f'' - cf'\right)\mathrm{d}\nu = -r\int f\,\mathrm{d}\nu, \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+). \tag{7}$$

Integrating by parts we get that the density w of ν must verify

$$\frac{1}{2}w'' + cw' + rw = 0. \tag{8}$$

Solutions to this equation with initial condition $w(0) = 0$ are given by

$$w(x) = \begin{cases} m\mathrm{e}^{-cx} \sin(\sqrt{2r - c^2}x) & r > \frac{c^2}{2}, \\ mx\mathrm{e}^{-cx} & r = \frac{c^2}{2}, \\ m\mathrm{e}^{-cx} \sinh(\sqrt{c^2 - 2r}x) & r < \frac{c^2}{2}. \end{cases}$$

Here m is a normalizing constant. Observe that w defines an integrable density function if and only if $0 < r \le c^2/2$ (or equivalently, $c \ge \sqrt{2r}$). One can thus parametrize the set of QSDs by their eigenvalues r, $\{\nu_r\colon 0 < r \le c^2/2\}$. For each r, the distribution function of ν_r, $v(x) = \int_0^x w(y)\,\mathrm{d}y$ is a monotone solution of (8) with boundary conditions

$$v(0) = 0, \qquad v(+\infty) = 1. \tag{9}$$

3 Particle Systems

In this section we introduce two particle systems. The first one is known as Branching Brownian Motion (BBM) with selection of the N right-most particles (N-BBM). As a consequence of the link between BBM and F–KPP that we describe below, this process can be thought of as a microscopic version of F–KPP. The second one is called Fleming–Viot and was introduced by Burdzy, Ingemar, Holyst and March [12], in the context of Brownian Motion in a d-dimensional bounded domain. It is a slight variation of the original one introduced by Fleming and Viot [16]. The first interpretation of this process as a microscopic version of a conditioned evolution is due to Ferrari and Marić [14].

3.1 BBM and F–KPP Equation

One-dimensional supercritical Branching Brownian Motion is a well-understood object. Particles diffuse following standard Brownian Motion started at the origin and branch at rate 1 into two particles. As already underlined, its connection with the F–KPP equation and traveling waves was pointed out by McKean in the seminal paper [29]. Denote with N_t the number of particles alive at time $t \ge 0$ and $\xi_t(1) \le \cdots \le \xi_t(N_t)$ the position of the particles enumerated from left to right. McKean's representation formula states that if $0 \le v_0(x) \le 1$ and we start the process with one particle at 0 (i.e. $N(0) = 1$, $\xi_0(1) = 0$), then

$$v(t, x) := \mathbb{E}\left(\prod_{i=1}^{N_t} v_0(\xi_t(i) + x)\right)$$

is the solution of (1). Of special interest is the case where the initial condition is the Heaviside function $v_0 = \mathbf{1}\{[0, +\infty)\}$ since in this case

$$v(t, x) = \mathbb{P}(\xi_t(1) + x > 0) = \mathbb{P}(\xi_t(N_t) < x).$$

This identity as well as various martingales obtained as functionals of this process have been widely exploited to obtain the precise behavior of solutions of (1), using analytic as well as probabilistic tools.

3.2 *N*-BBM and Durrett–Remenik Equation

Consider now a variant of BBM where the N right-most particles are selected. In other words, each time a particle branches, the left-most one is killed, keeping the total number of particles constant.

This process was introduced by Brunet and Derrida [8,9] as part of a family of models of branching-selection particle systems to study the effect of microscopic noise in front propagation.

Durrett and Remenik [13] considered a slightly different process in the Brunet–Derrida class: N-BRW. The system starts with N particles. Each particle gives rise to a child at rate one. The position of the child of a particle at $x \in \mathbb{R}$ is $x + y$, where y is chosen according to a probability distribution with density ρ, which is assumed symmetric and with finite expectation. After each birth, the $N + 1$ particles are sorted and the left-most one is deleted, in order to keep always N particles. They prove that if at time zero the particles are distributed according to independent variables with distribution $u_0(x)\mathrm{d}x$, then the empirical measure of this system converges to a deterministic probability measure ν_t for every t, which is absolutely continuous with density $u(t, \cdot)$, a solution of the following free-boundary problem

$$\begin{gathered}\textit{Find } (\gamma, u) \textit{ such that}\\ \frac{\partial u}{\partial t}(t,x) = \int_{-\infty}^{\infty} u(t,y)\rho(x-y)\,\mathrm{d}y \quad \forall x > \gamma(t),\\ \int_{\gamma(t)}^{\infty} u(t,y)\,\mathrm{d}y = 1, \quad u(t,x) = 0, \quad \forall x \le \gamma(t),\\ u(0,x) = u_0(x).\end{gathered} \tag{10}$$

They also find all the traveling wave solutions for this equation. Just as for the BBM, there exists a minimal velocity $c^* \in \mathbb{R}$ such that for $c \ge c^*$ there is a unique traveling wave solution with speed c and no traveling wave solution with speed c for $c < c^*$. The value c^* and the behavior at infinity of the traveling waves can be computed explicitly in terms of the Laplace transform of the random walk. In Sect. 4 we show that these traveling waves correspond to QSDs of drifted random walks.

It follows from renewal arguments that for each N, the process seen from the left-most particle is ergodic, which in turn implies the existence of a velocity v_N at which the empirical measure travels for each N. Durrett and Remenik prove that these velocities are increasing and converge to c^* as N goes to infinity.

We can interpret this fact as a *weak selection principle*: the microscopic system has a unique velocity for each N (as opposed to the limiting equation) and the velocities converge to the minimal velocity of the macroscopic equation. The word "weak" here refers to the fact that only convergence of the velocities is proved, but not convergence of the empirical measures in equilibrium.

In view of these results, the same theorem is expected to hold for a N-BBM that branches at rate r. In this case the limiting equation is conjectured to be given by

Find (γ, u) *such that*

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t,x) + ru(t,x) \quad \forall x > \gamma(t), \\ \int_{\gamma(t)}^{\infty} u(t,y)\,\mathrm{d}y = 1, &\quad u(t,x) = 0, \quad \forall x \le \gamma(t), \\ u(0,x) &= u_0(x). \end{aligned} \tag{11}$$

The empirical measures in equilibrium are expected to converge to the minimal traveling wave.

Traveling waves. Let us look at the traveling wave solutions $u(t,x) = w(x - ct)$ of (11). Plugging in the equation we see that they must verify

$$\frac{1}{2}w'' + cw' + rw = 0, \quad w(0) = 0, \quad \int_0^{\infty} w(y)\,\mathrm{d}y = 1, \tag{12}$$

which is exactly (8). Note nevertheless that in (12) the parameter r is part of the data of the problem (the branching rate) and c is part of the unknown (the velocity), while in (8) the situation is reversed: c is data (the drift) and r unknown (the absorption rate under the QSD). However, we have the following relation

$$\begin{array}{c} c \text{ is a minimal velocity for } r \\ \text{in (12)} \end{array} \iff \begin{array}{c} r \text{ is a maximal absorption rate for } c \\ \text{in (8)} \end{array}$$

Observe also that $1/r$ is the mean absorption time for the QSD associated to r and hence, if r is maximal, the associated QSD is minimal. So the minimal QSD for Brownian Motion in $\mathbb{R}_+$ and the minimal velocity traveling wave of (11) are one and the same. They are given by

$$u_{c^*(r)}(x) = u_{r^*(c)} = 2r^* x \mathrm{e}^{-\sqrt{2r^*}} = (c^*)^2 x \mathrm{e}^{-c^* x},$$

which is the one with fastest decay at infinity.

Again, the distribution function v of u is a monotone solution to the same problem but with boundary conditions given by $v(0) = 0$, $v(+\infty) = 1$.

3.3 Fleming–Viot and QSD

The Fleming–Viot process can be thought of as a microscopic version of conditioned evolutions. Its dynamics are built with a continuous time Markov process $X = (X_t,\, t \ge 0)$ taking values in a metric space $\Lambda \cup \{0\}$, that we call the *driving process*. We assume that 0 is absorbing in the sense that

$$\mathbb{P}_{\delta_0}(X_t = 0) = 1, \qquad \forall t \ge 0.$$

As before, we use τ for the absorption time

$$\tau = \inf\{t > 0\colon X_t \notin \Lambda\},$$

and P_t for the sub-Markovian semigroup defined by

$$P_t f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}\{\tau > t\}).$$

For a given $N \geq 2$, the Fleming–Viot process is an interacting particle system with N particles. We use $\xi_t = (\xi_t(1), \ldots, \xi_t(N)) \in \Lambda^N$ to denote the state of the process, $\xi_t(i)$ denotes the position of particle i at time t. Each particle evolves according to X and independently of the others unless it hits 0, at which time, it chooses one of the $N-1$ particles in Λ uniformly and takes its position. The genuine definition of this process is not obvious and in fact is not true in general. It can be easily constructed for processes with bounded jumps to 0, but is much more delicate for diffusions in bounded domains [6,17] and it does not hold for diffusions with a strong drift close to the boundary of Λ.

Here we are also interested in the empirical measure of the process

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_t(i)}. \tag{13}$$

Its evolution mimics the conditioned evolution: the mass lost from Λ is redistributed in Λ proportionally to the mass at each state. Hence, as N goes to infinity, we expect to have a deterministic limit given by the conditioned evolution of the driving process X, i.e.

$$\mu_t^N(A) \to \mathbb{P}(X_t \in A | \tau > t) \qquad (N \to \infty).$$

This is proved in [33] by the martingale method in great generality. See also [18] for a proof based on sub and super-solutions for PDEs and correlations inequalities. A much more subtle question is the ergodicity of the process for fixed N and the behavior of these invariant measures as $N \to \infty$. As a general principle it is expected that

Conjecture 1. If the driving process X has a Yaglom limit ν, then the Fleming–Viot process driven by X is ergodic, with (unique) invariant measure λ^N and the empirical measures (13) distributed according to λ^N converge to ν.

We refer to [18] for an extended discussion on this issue. This conjecture has been proved to be true for subcritical Galton–Watson processes, where a continuum of QSDs arises [2] and also for certain birth and death processes [34].

We have again here a *microscopic selection principle*: whereas there exists an infinite number of QSDs, when microscopic effects are taken into account (through the dynamics of the Fleming–Viot process), there is a unique stationary distribution for the empirical measure, which selects asymptotically the minimal QSD of the macroscopic model.

When the driving process is a one dimensional Brownian Motion with drift $-c$ towards the origin as in Sect. 2.2, the proof of the whole picture remains open, but the ergodicity of FV for fixed N has been recently proved [3, 23].

So, from [33, Theorem 2.1] we have that for every $t > 0$, μ_t^N converges as $N \to \infty$ to a measure μ_t with density $u(t, \cdot)$ satisfying (5). The open problem is to prove a similar statement in equilibrium. Observe that u is a stationary solution of (5) if and only if it solves (8) for some $r > 0$. Hence, although Eq. (11) and (5) are pretty different, stationary solutions to (5) coincide with traveling waves of (11).

3.4 Choose the Fittest and F–KPP Equation

We introduce now the last microscopic model that is useful to explain the relation between all these phenomena. In this subsection $\xi_t = (\xi_t(1), \dots, \xi_t(N))$ will denote the state of the $N-$ particle system that we describe below. In this model particles perform independent standard Brownian Motion. Additionally each particle has a Poisson process with rate 1/2 and when this process rings, the particle chooses uniformly among the other particles to form a pair. Among this pair of particles, the particle with the smaller position adopts the position of the other one.

We consider again the empirical measure of this process as in (13) and denote with $v^N(x, t)$ the cumulative distribution function

$$v^N(x, t) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{\xi_t^i \leq x\} \tag{14}$$

It is proved in [19] that if there is a distribution function v_0 such that $v_N(\cdot, 0) \to v_0$ uniformly, in probability, then for all $t > 0$

$$\lim_{N \to \infty} \|v^N(\cdot, t) - v(\cdot, t)\|_\infty = 0, \qquad \text{in probability.} \tag{15}$$

Here v is the solution of the F–KPP equation (1). Moreover, for each N it is easily seen that the process is ergodic. This implies the existence of an asympotic velocity

$$v_N = \lim_{t \to \infty} \frac{\xi_t(i)}{t},$$

independent of i. It is also proved in [19] that, as $N \to \infty$, $v_N \nearrow \sqrt{2r}$, the minimal velocity of (1).

Summing up

1. The link between N-BBM and Fleming Viot, in the Brownian Motion case is clear. Both processes evolve according to N independent Brownian Motions and branch into two particles. At branching times, the left-most particle is eliminated (selection) to keep the population size constant. The difference is that while N-BBM branches at a constant rate Nr, Fleming–Viot branches

each time a particle hits 0. This explains why in the limiting equation for N-BBM the branching rate is data and the velocity is determined by the system while in the hydrodynamic equation for Fleming–Viot the velocity is data and the branching rate is determined by the system.
2. The empirical measure of N-BBM is expected to converge in finite time intervals to the solution of (11). This is supported by the results of [13] where the same result is proved for random walks.
3. The empirical measure of Fleming–Viot driven by Brownian Motion converges in finite time intervals to the solution of (5).
4. Both N-BBM seen from the left-most particle and FV are ergodic and their empirical measure in equilibrium is expected to converge to the deterministic measure given by the minimal solution of (12).
   Note though that while for N-BBM r is data and minimality refers to c, for Fleming–Viot c is data and minimality refers to $1/r$ (microscopic selection principle).
5. $u(t, x) = w(x - ct)$ is a traveling wave solution of (11) if and only if w is the density of a QSD for Brownian Motion with drift $-c$ and eigenvalue $-r$.
6. c is minimal for r (in (12)) if and only if $1/r$ is minimal for c. So, we can talk of a "minimal solution of (12)", which is both a minimal QSD and a minimal velocity traveling wave.
7. The microscopic selection principle is conjectured to hold in both cases, with the same limit, but a rigorous proof is still unavailable.

4 Traveling Waves and QSD for Lévy Processes

Let $X = (X_t,\, t \geq 0)$ be a Lévy process with values in $\mathbb{R}$, defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, Laplace exponent $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\mathbb{E}(\mathrm{e}^{\theta X_t}) = \mathrm{e}^{\psi(\theta)t}$ and generator $\mathcal{L}$. This centered Lévy process plays the role of Brownian Motion in the previous sections. Now, for $c > 0$ we consider the *drifted process* X^c given by $X^c_t = X_t - ct$. It is immediate to see that the Laplace exponent of X^c is given by $\psi_c(\theta) = \psi(\theta) - c\theta$, that C_0^2 is contained in the domain of the generator $\mathcal{L}_c$ of X^c, and that $\mathcal{L}_c f = \mathcal{L} f - cf'$. Recall that the forward Kolmogorov equation for X is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}_x(f(X_t)) = \mathcal{L}f(x),$$

while the forward Kolmogorov (or Fokker–Plank) equation for the density u (which exists since $\sigma > 0$) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t, x) = \mathcal{L}^* u(t, \cdot)(x).$$

Here $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$. As in the Brownian case, we consider

- A branching Lévy process (BLP) $(\bar{\xi}_t(1), \dots, \bar{\xi}_t(N_t))$ driven by $\mathcal{L}^*$.
- A branching Lévy process with selection of the N rightmost particles (N-BLP), driven by $\mathcal{L}$.
- A Fleming–Viot process driven by $\mathcal{L}_c$ (FV).

We focus on the last two processes. For a detailed account on BLP, we refer to [25].

Let us just mention that the F–KPP equation can be generalized in this context to:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \mathcal{L}^* v + r g(v), \quad x \in \mathbb{R},\, t > 0, \\ v(0,x) &= v_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{16}$$

A characterization of the traveling waves as well as sufficient conditions of existence are then provided in [20,25].

For N-BLP we expect (but a proof is lacking) that the empirical measure converges to a deterministic measure whose density is the solution of the generalized Durrett–Remenik equation

$$\begin{aligned} &\textit{Find } (\gamma, u) \quad \textit{such that} \\ &\frac{\partial u}{\partial t}(t,x) = \mathcal{L}^* u(t,x) + r u(t,x), \quad x > \gamma(t), \\ &\int_{\gamma(t)}^{\infty} u(t,y)\,\mathrm{d}y = 1, \quad u(t,x) = 0, \quad x \leq \gamma(t), \\ &u(0,x) = u_0(x), \qquad x \geq 0. \end{aligned} \tag{17}$$

Existence and uniqueness of solutions to this problem have to be examined.

We show below the existence of traveling wave solutions for this equation under mild conditions on $\mathcal{L}$ based on the existence of QSDs.

Concerning FV, it is known [33] that the empirical measure converges to the deterministic process given by the conditioned evolution of the process, which has a density for all times and verifies

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \mathcal{L}^* u(t,x) + c\frac{\partial u}{\partial x}(t,x) - u(t,x)\int_0^{\infty} \mathcal{L}^* u(t,y)\mathrm{d}y \quad t > 0, x > 0, \\ &u(t,0) = u(t,+\infty) = 0, \quad t > 0, \end{aligned} \tag{18}$$

Proposition 1. *The following statements are equivalent:*

- *The probability measure ν with density w is a QSD for X^c with eigenvalue $-r$,*
- *$u(t,x) = w(x - ct)$ is a traveling wave solution with speed c for the free-boundary problem (17), with parameter r.*

Proof. Denote $\langle f, g\rangle = \int f(x)g(x)\mathrm{d}x$. A QSD ν for X^c with eigenvalue $-r$ is a solution of the equation

$$\langle \nu \mathcal{L}_c + r\nu, f\rangle = 0, \forall f \in C_0^2.$$

Using that $\mathcal{L}_c^* f = \mathcal{L}^* f + c f'$ and writing that ν has density w, we obtain that

$$\mathcal{L}^* w + c w' + r w = 0,$$

which in turn is clearly equivalent to w being a traveling wave solution with speed c for (17). □

In the companion paper [20] we prove that the picture that we described for the Brownian Motion case holds in more generality. Our result states that, under mild conditions, for given $r, c > 0$ we have that there exists a QSD ν_r for X^c with eigenvalue $-r$ if and only if there is a traveling wave w_c for (16) that travels at velocity c and moreover, ν_r is minimal for c if and only if w_c is minimal for r. Jointly with Proposition 1 this also proves that the existence of a traveling wave for (16) is also equivalent to the existence of a traveling wave for the Durrett–Remenik equation (17).

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A Monotonicity Property for Once Reinforced Biased Random Walk on $\mathbb{Z}^d$

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Dedicated to Chuck Newman after 70 years of life and 50 years of massive contributions to probability. Bring on NYU Waiheke!

Abstract. We study once-reinforced biased random walk on $\mathbb{Z}^d$. We prove that for sufficiently large bias, the speed $v(\beta)$ is monotone decreasing in the reinforcement parameter β in the region $[0, \beta_0]$, where β_0 is a small parameter depending on the underlying bias. This result is analogous to results on Galton–Watson trees obtained by Collevecchio and the authors.

Keywords: Once-reinforced random walk · Reinforced random walk · Large bias · Coupling

1 Introduction

Reinforced random walks have been studied extensively since the introduction of the (linearly)-reinforced random walk of Coppersmith and Diaconis [9]. In this paper we study (a biased version of) once-reinforced random walk, which was introduced by Davis [10] as a possible simpler model of reinforcement to understand. While there have been recent major advances in the understanding of linearly reinforced walks on $\mathbb{Z}^d$ (see e.g. [1,11,29,30] and the references therein), rather less is known about once-reinforced walks on $\mathbb{Z}^d$.

When the underlying random walk is biased we expect the once-reinforced random walk to be ballistic in the direction of the bias. On regular trees with d offspring per vertex, the underlying walk has a drift away from the root when $d \geq 2$ and the ballisticity of the once-reinforced walk is a well known result due to Durrett, Kesten and Limic [12] (see [6] for a softer proof, and [8,25] for further results). When one introduces an additional bias on the tree (letting children have initial weight α), one must first clarify what one means by once-reinforcement. Indeed, ballisticity depends on the sign of $d\alpha - 1$ in the setting of

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additive once-reinforcement, but a different criterion reveals itself in the setting of *multiplicative* once-reinforcement (see [7], and also [25]). It is also shown in [7], for sufficiently large d and sufficiently small $\beta_0 > 0$, depending on d, that the speed of the walk away from the root is monotone in the reinforcement $\beta \in [0, \beta_0]$. In this paper we prove analogous monotonicity results on $\mathbb{Z}^d$, when the underlying bias is sufficiently large.

1.1 The Model

Fix $d \geq 2$. Let $\mathcal{E}_+ = (e_i)_{i=1,\dots,d}$ denote the canonical basis on $\mathbb{Z}^d$, and $\mathcal{E} = \{e \in \mathbb{Z}^d : |e| = 1\}$ denote the set of neighbours of the origin in $\mathbb{Z}^d$. Let $\mathcal{E}_- = \mathcal{E} \setminus \mathcal{E}_+$.

Given $\boldsymbol{\alpha} = (\alpha_e)_{e\in\mathcal{E}} \in \mathbb{R}_+^{\mathcal{E}}$ and $\beta > 0$, we define a once-edge-reinforced random walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ with natural filtration $(\mathcal{F}_n)_{n\in\mathbb{Z}_+}$ (i.e. $\mathcal{F}_n = \sigma(X_k : k \leq n)$) as follows. Set $X_0 = 0$ almost surely. For any $n \geq 0$, let $E_n = \{[X_{i-1}, X_i] : 1 \leq i \leq n\}$ denote the set of non-oriented edges crossed by $\boldsymbol{X}$ up to time n. Define

$$W_e(n) := \alpha_e(1 + \beta \mathbb{1}_{\{[X_n, X_n+e]\in E_n\}}). \tag{1}$$

The walk jumps to a neighbor $X_n + e$ with conditional probability given by

$$\mathbb{P}_\beta\left(X_{n+1} = X_n + e \middle| \mathcal{F}_n\right) = \frac{W_e(n)}{\sum_{e'\in\mathcal{E}} W_{e'}(n)}. \tag{2}$$

Together, (1) and (2) correspond to *multiplicative* once-edge reinforcement, with reinforcement parameter β. See Sect. 1.2 for other related models.

Without loss of generality we may assume that the direction of bias (if any) is in the positive coordinate direction for every coordinate. Moreover, we assume that every direction has positive weight. Thus, up to a rescaling of parameters, without loss of generality $\alpha_e \geq 1$ for every $e \in \mathcal{E}$.

Condition D. *For every $e \in \mathcal{E}_+$, $\alpha_e \geq \alpha_{-e} \geq 1$.*

Let $\alpha_+ = \sum_{e\in\mathcal{E}_+} \alpha_e$ and $\alpha_- = \sum_{e\in\mathcal{E}_-} \alpha_e$, and let $\alpha = \alpha_+ + \alpha_- = \sum_{e\in\mathcal{E}} \alpha_e$.

Our results depend upon a modification of an argument of [2] involving a coupling with a 1-dimensional biased random walk. This general coupling approach has also been utilised on $\mathbb{Z}^d$ for biased random walk on random conductances in [3] (see [7] as well).

For this reason, we will assume the following with $\kappa \gg 1$.

Condition κ. *The parameters $\boldsymbol{\alpha}$ and β_0 are such that $\beta_0 \leq 1/\alpha_+$ and*

$$\frac{\alpha_+}{(1+\beta_0)^2\alpha_-^2} > \kappa. \tag{3}$$

If Condition κ holds with $\kappa = 1$ then a simple comparison with 1-dimensional biased random walk shows that the walker is *ballistic* in direction $\ell_+ := \sum_{e\in\mathcal{E}_+} e$ (i.e. $\liminf_{n\to\infty} n^{-1} X_n \cdot \ell_+ > 0$). In particular the walk is *transient* in direction ℓ_+ (i.e. $\liminf_{n\to\infty} X_n \cdot \ell_+ = \infty$). Combined with regeneration arguments (see e.g. [31,32]), this allows one to prove that there exists $v \in \mathbb{R}^d$ with $v \cdot \ell_+ > 0$ such that $\mathbb{P}(\lim_{n\to\infty} n^{-1}X_n = v) = 1$. Since the norm $\|x\| := \sum_{i=1}^d |x_i|$ is a continuous function on $\mathbb{R}^d$ we also have that

$$\|v\| = \lim_{n\to\infty} n^{-1}\|X_n\|, \qquad \text{almost surely.}$$

Note that if $\boldsymbol{X}$ is a nearest neighbour walk on $\mathbb{Z}^d$ then $\|\boldsymbol{X}\|$ is a nearest neighbour walk on $\mathbb{Z}_+$. Therefore if $v \cdot e \geq 0$ for each $e \in \mathcal{E}_+$ then $v \cdot \ell_+ = \|v\|$. Assuming Condition D, it is *intuitively obvious* that if v exists then $v \cdot e \geq 0$ for each $e \in \mathcal{E}_+$. We conjecture that this is true however it does not seem easy to prove.

Conjecture 1. Assume Condition D. Then for each $\beta > 0$ there exists $v = v_\beta \in \mathbb{R}^d$ with $v \cdot e \geq 0$ for each $e \in \mathcal{E}_+$ such that $\mathbb{P}(n^{-1}X_n \to v) = 1$.

Note that it is not at all obvious that $v \cdot e$ should be strictly positive for all β when the underlying random walk (i.e. $\beta = 0$) has a positive speed in direction e. In particular on regular trees there are settings where the underlying random walk is ballistic but the multiplicative-once-reinforced walk is recurrent [7]. On the other hand, we believe that on $\mathbb{Z} \times F$ where F is a finite graph $v_\beta \cdot e_1 > 0$ whenever $v_0 \cdot e_1 > 0$.

We make the following conjecture about the behaviour of v_β as β varies.

Conjecture 2. Assume Condition D. Then for β_0 such that Condition κ holds with $\kappa \geq 1$, $v_\beta \cdot \ell_+$ is strictly decreasing in $\beta \leq \beta_0$.

There are at least 3 different strategies for proving monotonicity of the speed for random walks on $\mathbb{Z}^d$: coupling, expansion, and Girsanov transformation methods, with the former usually being the weapon of choice, where possible. When $d = 1$ there is a rather general coupling method [18,22] for proving monotonicity, however this argument completely breaks down when $d \geq 2$. We are not aware of any current technology that lets one resolve Conjecture 2 for all $\kappa \geq 1$.

If the bias in direction ℓ_+ is sufficiently large then for small reinforcements the reinforced walker still has a large bias in direction ℓ_+ (e.g. when Condition κ holds for large κ). Thus, what the walker sees locally almost all of the time is a single reinforced edge in some direction $-e \in \mathcal{E}_-$, and no reinforced edges in directions in $\mathcal{E}_+$. In this case it is again *intuitively obvious* that the speed of the reinforced version of the walk in direction ℓ_+ is decreasing in β for small β. Actually proving this is non-trivial.

We will prove a version of this result assuming one of the following:

Condition S. *The parameters $\boldsymbol{\alpha}$ satisfy $\alpha_e = \alpha_+/d$ and $\alpha_{-e} = \alpha_-/d$ for each $e \in \mathcal{E}_+$.*

Note that Condition S (which is a symmetry condition) together with Condition κ (for $\kappa \geq 1$) implies Condition D. Condition S implies that the true direction of bias of the underlying random walk is ℓ_+. At the other extreme, the following condition (with $\kappa \gg 1$) implies that the true direction of bias of the underlying random walk is almost in direction e_1.

Condition e_1. *The parameters* $\boldsymbol{\alpha}$ *and* β_0 *are such that* $\beta_0 \le 1/\alpha_+$ *and satisfy*

$$\frac{\alpha_{e_1}}{(1+\beta_0)^2(\alpha-\alpha_{e_1})^2} > \kappa.$$

Note that this implies Condition κ.

Curiously, with the technique that we employ, it seems considerably harder to prove our results for parameters $\boldsymbol{\alpha}$ between the two extremes given by Conditions S and e_1.

Our main results are the following Theorems, which verify that if either Condition S or Condition e_1 (for small ε) hold for large drifts and small reinforcement, increasing the reinforcement slows the walker down.

Theorem 1. *There exists* $\kappa_0 < \infty$ *such that if Condition S holds and if Condition* κ *holds for* κ_0, $\boldsymbol{\alpha}$ *and* β_0, *then for all* $\beta \le \beta' \le \beta_0$,

$$(v_\beta - v_{\beta'}) \cdot \ell_+ > 0 \quad \text{and} \quad \|v_\beta\| > \|v_{\beta'}\|.$$

(Note that the second conclusion in Theorem 1 is immediate from the first and Condition S.)

Theorem 2. *Suppose that Condition D holds. There exists* $\kappa_0 < \infty$ *such that if Condition* e_1 *holds for* κ_0, $\boldsymbol{\alpha}$ *and* β_0 *then for all* $\beta \le \beta' \le \beta_0$,

$$(v_\beta - v_{\beta'}) \cdot \ell_+ > 0.$$

Note that if $\alpha_- = 0$ then there is nothing to prove in either case. Otherwise $\alpha_e > 0$ for some $e \in \mathcal{E}_-$ in which case by Condition D we have that

$$\alpha_- \ge 1. \tag{4}$$

1.2 Discussion

We have restricted ourselves to multiplicative-once-edge-reinforced random walk. In the general $\boldsymbol{\alpha}$ setting one can describe a rather general once-reinforcement scheme as follows. Recall that E_n is the set of edges crossed by the walk up to time n and let $V_n = \{X_0, \dots, X_n\}$ denote the set of vertices visited up to time n. Then, given a parameter set $(\boldsymbol{\alpha}, \boldsymbol{\alpha}^V, \boldsymbol{\alpha}^E) \in (0,\infty)^{2d\times 3}$, define the law of a walk $\mathbb{P}_{\boldsymbol{\alpha},\boldsymbol{\alpha}^V,\boldsymbol{\alpha}^E}$ via (2) and the edge weights:

$$W_e(n) = \alpha_e \mathbb{1}_{\{X_n+e\notin V_n\}} + \alpha_e^V \mathbb{1}_{\{X_n+e\in V_n, [X_n,X_n+e]\notin E_n\}} + \alpha_e^E \mathbb{1}_{\{[X_n,X_n+e]\in E_n\}}. \tag{5}$$

This general setting includes multiplicative-once-edge-reinforced random walk (this is the choice $\alpha_e^V = \alpha_e$ and $\alpha_e^E = (1+\beta)\alpha_e$ for each $e \in \mathcal{E}$), additive-once-edge-reinforced random walk (the choice $\alpha_e^V = \alpha_e$ and $\alpha_e^E = \alpha_e + \beta$ for each $e \in \mathcal{E}$), and vertex-reinforced versions of these models (by setting $\alpha_e^V = \alpha_e^E$

for each $e \in \mathcal{E}$). It also includes hybrid models where the weight of a directed edge depends on whether the undirected edge has been traversed, and otherwise whether the other endvertex of the edge has been visited before.

The increased level of difficulty (for the coupling technique) that we encounter in studying models with $\boldsymbol{\alpha}$ that are not at the "extremes" given by either Conditions S or e_1, seems (at first glance) to persist for expansion methods (see e.g. [15–17,23]). We have not investigated the possibility of proving such results using Girsanov transformation methods (see e.g. [27,28]).

In the setting of reinforcement for an unbiased walk, the velocity $\boldsymbol{v}$ is zero, but one imagines that (e.g. in the case of once-reinforcement), monotonicity in β still holds for various quantities such as $\mathbb{E}[|X_n|^2]$, $\mathbb{E}[\|X_n\|]$ and the expected number of visits to 0 up to time n. Results of this kind hold in the elementary setting where one only keeps track of the most recently traversed edge (but can in fact fail in this setting with more general reinforcement schemes than once-reinforcement), see e.g. [19,20].

There is also a substantial literature on monotonicity (or lack thereof) for random walks in random graphs, where one is often interested in the monotonicity (or lack thereof) of the speed of the walk in some parameter defining the bias of the walk or the structure of the underlying graph (see e.g. [2–5,7,13,21,26]).

2 Preliminary Results

We will need the concept of regeneration times. Let $\boldsymbol{Y}$ denote a nearest neighbour simple random walk on $\mathbb{Z}$ with probability $p > 1/2$ of stepping to the right. By the law of large numbers $n^{-1}Y_n \to 2p-1 > 0$ almost surely. Moreover it is easily computed that

$$\mathbb{P}(\inf_{n\geq 0} Y_n \geq Y_0) = p^{-1}(2p-1). \tag{6}$$

We say that $N \in \mathbb{Z}_+$ is a *regeneration time* of Y if $\sup_{n<N} Y_n < Y_N \leq \inf_{n\geq N} Y_n$. Letting $\mathcal{D}_Y$ denote the set of regeneration times for Y, and $D_0 = \{0 \in \mathcal{D}_Y\}$ we see from (6) that $\mathbb{P}(D_0) = p^{-1}(2p-1) > 0$. In fact $|\mathcal{D}_Y| = \infty$ almost surely and we write $(\tau_i)_{i\in\mathbb{N}} = \mathcal{D}_Y \cap \mathbb{N}$ (with $\tau_i < \tau_{i+1}$ for each i) for the ordered strictly positive elements of $\mathcal{D}_Y$. Set $\tau_0 = 0$ (this may or may not be a regeneration time).

More generally, for a walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ we say that $N \in \mathbb{Z}_+$ is a *regeneration time* of $\boldsymbol{X}$ in the direction of $x \in \mathbb{R}^d \setminus \{o\}$ if $\sup_{n<N} X_n \cdot x < X_N \cdot x \leq \inf_{n\geq N} Y_n \cdot x$. For $n \geq 1$ let $\Delta_n^{\boldsymbol{X}} = X_n - X_{n-1}$.

The following is Lemma 3.1 of [7].

Lemma 1 (Lemma 3.1 of [7]). *Suppose that $\mathbf{Z}'$ and $\mathbf{Z}$ are nearest neighbour walks on $\mathbb{Z}$ and that $\boldsymbol{Y}$ is a nearest neighbour simple random walk on $\mathbb{Z}$ with $\mathbb{P}(Y_1 = 1) = p > 1/2$, all on the same probability space such that:*

(i) $\Delta_n^{\mathbf{Z}'} = \Delta_n^{\mathbf{Z}} = 1$ *whenever* $\Delta_n^{\boldsymbol{Y}} = 1$,

then the regeneration times τ_i *of* $\boldsymbol{Y}$ *are also regeneration times for* $\mathbf{Z}$ *and* $\mathbf{Z}'$*. Moreover if*

(ii) $(Z'_{\tau_{i+1}} - Z'_{\tau_i})_{i\in\mathbb{N}}$ *are i.i.d. random variables and* $(Z_{\tau_{i+1}} - Z_{\tau_i})_{i\in\mathbb{N}}$ *are i.i.d. random variables, with* $Z'_{\tau_{i+1}} - Z'_{\tau_i}$ *and* $Z_{\tau_{i+1}} - Z_{\tau_i}$ *being independent of* $(\tau_k : k \le i)$ *for each* i*, and*
(iii) $\mathbb{E}\left[Z_{\tau_1} - Z'_{\tau_1}\middle|D_0\right] > 0$.

Then there exist $v > v' > 0$ *such that* $\mathbb{P}(n^{-1}Z_n \to v, n^{-1}Z'_n \to v') = 1$.

Let $\overline{\mathbb{P}}(\cdot) = \mathbb{P}(\cdot|D_0)$. Let $\mathfrak{B} = \left\{1 \le i < \tau_1 : \Delta_i^{\boldsymbol{Y}} = -1\right\}$ denote the set of times before τ_1 when $\boldsymbol{Y}$ takes a step back. The following statement (and its proof) is a trivial modification of Lemma 3.2 of [7].

Lemma 2. *Let* $\mathbf{Z}, \mathbf{Z}'$ *be nearest neighbour walks on* $\mathbb{Z}$*, and* $\boldsymbol{Y}$ *a biased random walk on* $\mathbb{Z}$ *satisfying assumption Lemma 1 (i). Suppose also that* $\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} < 0\big) = 0$ *and*

$$\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} \ge 1\big) > \sum_{k=2}^{\infty} 2k\overline{\mathbb{P}}(|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0). \tag{7}$$

Then (iii) of Lemma 1 holds. Therefore if the assumption of Lemma 1(ii) also holds then Lemma 1 holds.

To prove Theorem 1 it therefore suffices to prove the following theorem.

Theorem 3. *There exists* $\kappa_0 < \infty$ *such that if Condition* κ *holds for* κ_0*,* $\boldsymbol{\alpha}$ *and* β_0*, and Condition S holds, then for all* $\beta \le \beta' \le \beta_0$*, there exists a probability space on which the conditions of Lemma 1 hold for* $\mathbf{Z} = \boldsymbol{X}(\beta) \cdot \ell_+$ *and* $\mathbf{Z}' = \boldsymbol{X}(\beta') \cdot \ell_+$*.*

Similarly, to prove Theorem 2 it suffices to prove the following.

Theorem 4. *Suppose that Condition D holds. There exists* $\kappa_0 < \infty$ *such that if Condition* e_1 *holds for* κ_0*,* $\boldsymbol{\alpha}$ *and* β_0 *then for all* $\beta \le \beta' \le \beta_0$*, there exists a probability space on which the conditions of Lemma 1 hold for* $\mathbf{Z} = \boldsymbol{X}(\beta) \cdot \ell_+$ *and* $\mathbf{Z}' = \boldsymbol{X}(\beta') \cdot \ell_+$*.*

In the next section we construct the probability spaces relevant to Theorems 3 and 4.

3 The Coupling

This section adapts an argument of Ben Arous, Fribergh and Sidoravicius [2].

In the construction of the coupling and verification of its properties we will use various jargon as follows.

We say that a walk $\boldsymbol{Y}$ on $\mathbb{Z}$ jumps *forward* (at time $n+1$) if $\Delta_{n+1}^{\boldsymbol{Y}} = 1$. Otherwise, we say that Y jumps *backwards*.

For a walk $\boldsymbol{X}$ on $\mathbb{Z}^d$ (which will have the law of a once reinforced biased random walk on Z^d), we will say that $\boldsymbol{X}$ jumps *forward* (at time $n+1$) if $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{E}_+$ and *backwards* otherwise. We will write that a (non-oriented) edge e of $\mathbb{Z}^d$ is *reinforced* at a given time n if it has already been crossed by X, i.e. if $e \in E_n$. We will use the notation $\mathcal{I}_n := \{e \in \mathcal{E} : [X_n, X_n + e] \in E_n\}$, and say that $\boldsymbol{X}$ jumps *on its trace* (at time $n+1$) if $\Delta^{\boldsymbol{X}}_{n+1} \in \mathcal{I}_n$ (i.e. it jumps through an edge which is already reinforced), otherwise we say that it jumps *out of its trace*. We call *local environment* of $\boldsymbol{X}$ at time n the collection of weights of the edges adjacent to X_n.

Recall that we have defined $\mathcal{E}_+ = (e_i)_{i=1,\dots,d}$. We extend the notation to $e_{i+d} = -e_i \in \mathcal{E}_-$ for any $i \in \{1, \dots, d\}$. Moreover, we use the shorthand $\alpha_i = \alpha_{e_i}$ for any $i \in \{1, \dots, 2d\}$. For each $I \subset [2d]$, for any $i \in [2d]$ and for each $\beta \geq 0$, define

$$p^{(\beta)}_{i,I} := \frac{\alpha_i(1 + \mathbb{1}_{\{i \in I\}}\beta)}{\sum_{j=1}^{2d} \alpha_j(1 + \mathbb{1}_{\{j \in I\}}\beta)}.$$

Note that if $\boldsymbol{X}$ is a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter β, then $\mathbb{P}(\Delta^{\mathbf{X}^{(\beta)}}_{n+1} = e_i | \mathcal{F}_n) = p^{(\beta)}_{i,\mathcal{I}_n(\beta)}$ a.s.

Note that if $\beta' > \beta$ then for any $I \subset [2d]$ we have

$$\begin{aligned} p^{(\beta')}_{i,I} &\geq p^{(\beta)}_{i,I}, \text{ for any } i \in I, \\ p^{(\beta')}_{i,I} &\leq p^{(\beta)}_{i,I}, \text{ for any } i \notin I, \\ p^{(\beta')}_{i,I} \wedge p^{(\beta)}_{i,I} &\geq p^Y_i, \text{ for any } i \in [d], \end{aligned}$$

where

$$p^Y_i := \frac{\alpha_i}{\alpha + \beta'(\alpha - \alpha_i)}. \tag{8}$$

Let us also define

$$p^Y := \sum_{i=1}^{d} \alpha_i, \;\; q^Y := 1 - p^Y, \text{ and } p^Y_{i+d} := 0, \; \forall i \in [d]. \tag{9}$$

Note that from summing (8) and using Condition κ and (4) we have $\beta_0 \leq 1/\alpha_+$ and $\alpha_+/\alpha_- > \kappa$, and therefore

$$p^Y > \frac{\alpha_+}{(1+\beta_0)\alpha} = 1 - \frac{\beta_0\alpha_+}{(1+\beta_0)\alpha} - \frac{\alpha_-}{\alpha} \geq 1 - \frac{2}{\kappa}. \tag{10}$$

In particular, p^Y can be taken arbitrarily close to 1 under Condition κ for large κ.

The main idea is to couple three walks, $\boldsymbol{X}, \boldsymbol{X}', \boldsymbol{Y}$ satisfying the conditions of Theorems 3 and 4:

(1) $\boldsymbol{X} = \mathbf{X}^{(\beta)}$ a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter β;
(2) $\boldsymbol{X}' = \mathbf{X}^{(\beta')}$ a biased ORRW on $\mathbb{Z}^d$ with reinforcement parameter $\beta' > \beta$;
(3) $\mathbf{Y}$ a biased random walk on $\mathbb{Z}$.

In particular (see (7)) we must be able to control how each walk can make gains on the other in direction ℓ_+. To this end, we will say that $\boldsymbol{X}$ and $\boldsymbol{X}'$ are *still coupled at time* n if $(X_k)_{k\le n} = (X'_k)_{k\le n}$, and that they decouple at time $n+1$ if also $X_{n+1} \neq X'_{n+1}$. We will write $\delta = \inf\{n : X_n \neq X'_n\}$ to denote the decoupling time. We call *discrepancy* (at time n) the difference $X_n - X'_n$. We say that the decoupling creates a *negative discrepancy* if $(X_\delta - X'_\delta)\cdot \ell_+ < 0$.

3.1 The Dynamics

Fix $\beta' > \beta \ge 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space on which $(U_i)_{i\ge 1}$ is an i.d.d. collection of $U[0,1]$ random variables. We will set $X_0 = X'_0 = o \in \mathbb{Z}^d$ and $Y_0 = 0$, and (X_n, X'_n, Y_n) will be $\mathcal{G}_n = \sigma(U_k : k \le n)$-measureable. We will use $'$ notation to denote quantities depending on $\boldsymbol{X}'$ e.g. $E'_n = \{[X'_{k-1}, X'_k] : k \le n\}$ and $\mathcal{I}'_n := \{e \in \mathcal{E} : [X'_n, X'_n + e] \in E'_n\}$.

The coupling is given by the following rules, that we will explain in Sect. 3.2. Firstly,

(0_Y) for any $n \in \mathbb{N}$, $Y_n = \sum_{i=1}^n \left(\mathbb{1}_{\{U_i > q^Y\}} - \mathbb{1}_{\{U_i \le q^Y\}}\right)$.

The above takes care of the marginal distribution of $\boldsymbol{Y}$. Next, regardless of the environment at time n,

(0_X) If $U_{n+1} \in \left(1 - \sum_{j=1}^i p_j^Y, 1 - \sum_{j=1}^{i-1} p_j^Y\right]$ for $i \in [d]$, then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_i$;

Otherwise we define the joint increments inductively, considering separately the cases when the two walks $\boldsymbol{X}$, $\boldsymbol{X}'$ have $(*)$ the same local environments or $(**)$ different local environments.

$(=)$ Suppose that $\mathcal{I}_n = \mathcal{I}'_n$, and let $\mathcal{I} = \mathcal{I}_n$. Denote $k = |\mathcal{I}|$, write $r_1 < \cdots < r_k$ for the elements of $\mathcal{I}$ listed in increasing order and $\bar{r}_1 < \cdots < \bar{r}_{2d-k}$ for the elements of $[2d] \setminus \mathcal{I}$ in increasing order. Then,

- $(=_{\mathcal{I}})$ If $U_{n+1} \in \left(q^Y - \sum_{j=1}^i \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right), q^Y - \sum_{j=1}^{i-1}\left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right)\right]$ for $i \in [k]$, then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_{r_i}$;
- $(=_{\mathcal{I}^c})$ If

$$U_{n+1} \in \left(q^Y - \sum_{j=1}^k \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right) - \sum_{j=1}^i \left(p_{\bar{r}_j,\mathcal{I}}^{(\beta')} - p_{\bar{r}_j}^Y\right),\right.$$
$$\left. q^Y - \sum_{j=1}^k \left(p_{r_j,\mathcal{I}}^{(\beta)} - p_{r_j}^Y\right) - \sum_{j=1}^{i-1}\left(p_{\bar{r}_j,\mathcal{I}}^{(\beta')} - p_{\bar{r}_j}^Y\right)\right],$$

  then $\Delta_{n+1}^{\boldsymbol{X}} = \Delta_{n+1}^{\boldsymbol{X}'} = e_{\bar{r}_i}$;
- $(=_{\mathcal{I},\mathcal{I}^c})$ If $U_{n+1} \in \left(0, 1 - \sum_{j=1}^k p_{r_j,\mathcal{I}}^{(\beta)} - \sum_{j=1}^{2d-k} p_{\bar{r}_j,\mathcal{I}}^{(\beta')}\right]$ then

(i) if

$$U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p^{(\beta')}_{r_j,\mathcal{I}} - p^{(\beta)}_{r_j,\mathcal{I}} \right), \sum_{j=1}^{i} \left(p^{(\beta')}_{r_j,\mathcal{I}} - p^{(\beta)}_{r_j,\mathcal{I}} \right) \right]$$

for $i \in \{1, \ldots, k\}$, then $\Delta^{\boldsymbol{X}'}_{n+1} = e_{r_i}$;

(ii) if

$$U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p^{(\beta)}_{\bar{r}_j,\mathcal{I}} - p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \right), \sum_{j=1}^{i} \left(p^{(\beta)}_{\bar{r}_j,\mathcal{I}} - p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \right) \right]$$

for $i \in \{1, \ldots, 2d-k\}$, then $\Delta^{\boldsymbol{X}}_{n+1} = e_{\bar{r}_j}$.

($\neq$) Now, if $\mathcal{I}'_n \neq \mathcal{I}_n$then we follow:

($\neq_{X'}$) if

$$U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p^{(\beta')}_{j,\mathcal{I}} - p^{Y}_{j} \right), \sum_{j=1}^{i} \left(p^{(\beta')}_{j,\mathcal{I}} - p^{Y}_{j} \right) \right]$$

for $i \in [2d]$ then $\Delta^{\boldsymbol{X}'}_{n+1} = e_i$;

($\neq_X$) if

$$U_{n+1} \in \left(\sum_{j=1}^{i-1} \left(p^{(\beta)}_{j,\mathcal{I}} - p^{Y}_{j} \right), \sum_{j=1}^{i} \left(p^{(\beta)}_{j,\mathcal{I}} - p^{Y}_{j} \right) \right]$$

for $i \in [2d]$ then $\Delta^{\boldsymbol{X}}_{n+1} = e_i$.

From now on, we denote $(\mathcal{G}_n)$ the natural filtration generated by the sequence (U_n) and note that the three walks are measurable with respect to this filtration.

3.2 Properties of the Coupling

Let us explain the coupling defined in Sect. 3.1. The proof that the marginals of $\mathbf{Y}$, $\mathbf{X}^{(\beta)}$ and $\mathbf{X}^{(\beta')}$ have the correct distributions is left to the reader. Let us simply emphasize that

$$\begin{aligned} 0 &\leq q^Y - \sum_{j=1}^{k} \left(p^{(\beta)}_{r_j,\mathcal{I}} - p^{Y}_{r_j} \right) - \sum_{j=1}^{d-k} \left(p^{(\beta')}_{\bar{r}_j,\mathcal{I}} - p^{Y}_{\bar{r}_j} \right) \\ &= 1 - \sum_{j=1}^{k} p^{(\beta)}_{r_j,\mathcal{I}} - \sum_{j=1}^{d-k} p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \\ &= \sum_{j=1}^{d-k} \left(p^{(\beta)}_{\bar{r}_j,\mathcal{I}} - p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \right) = \sum_{j=1}^{k} \left(p^{(\beta')}_{r_j,\mathcal{I}} - p^{(\beta)}_{r_j,\mathcal{I}} \right). \end{aligned}$$

For $I \subset [2d]$ let $\alpha_I = \sum_{i\in I} \alpha_i$ (so e.g. $\alpha_+ = \alpha_{[d]}$). We note that

$$\sum_{j=1}^{k} p^{(\beta)}_{r_j,\mathcal{I}} = \frac{\alpha_{\mathcal{I}}(1+\beta)}{\alpha_{\mathcal{I}}(1+\beta)+\alpha_{\mathcal{I}^c}} \tag{11}$$

$$\sum_{j=1}^{d-k} p^{(\beta')}_{\bar{r}_j,\mathcal{I}} = \frac{\alpha_{\mathcal{I}^c}}{\alpha_{\mathcal{I}}(1+\beta')+\alpha_{\mathcal{I}^c}}, \tag{12}$$

and therefore that the quantity in the first interval of $(=_{\mathcal{I},\mathcal{I}^c})$ is

$$1-\sum_{j=1}^{k} p^{(\beta)}_{r_j,\mathcal{I}} - \sum_{j=1}^{d-k} p^{(\beta')}_{\bar{r}_j,\mathcal{I}} = \frac{\alpha_{\mathcal{I}}\alpha_{\mathcal{I}^c}(\beta'-\beta)}{(\alpha_{\mathcal{I}}(1+\beta)+\alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta')+\alpha_{\mathcal{I}^c})}. \tag{13}$$

So the first discrepancy by the walk will give us a (small) factor of $(\beta'-\beta)$.

Similarly, letting $\mathcal{I}_+ = \mathcal{I}\cap[d]$ and $\mathcal{I}^c_+ = \mathcal{I}^c\cap[d]$ we see that the union of the intervals in $(=_{\mathcal{I},\mathcal{I}^c})$ (i) over $i \le |\mathcal{I}_+|$ gives the interval

$$\left(0, \frac{\alpha_{\mathcal{I}_+}\alpha_{\mathcal{I}^c}(\beta'-\beta)}{(\alpha_{\mathcal{I}}(1+\beta)+\alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta')+\alpha_{\mathcal{I}^c})}\right]. \tag{14}$$

Similarly, the union over $i \le |\mathcal{I}^c_+|$ of the intervals in $(=_{\mathcal{I},\mathcal{I}^c})$ (ii) gives

$$\left(0, \frac{\alpha_{\mathcal{I}^c_+}\alpha_{\mathcal{I}}(\beta'-\beta)}{(\alpha_{\mathcal{I}}(1+\beta)+\alpha_{\mathcal{I}^c})(\alpha_{\mathcal{I}}(1+\beta')+\alpha_{\mathcal{I}^c})}\right]. \tag{15}$$

Here are the main properties satisfied under the coupling:

(P1) Whenever $\mathbf{Y}$ jumps forward, so do $\boldsymbol{X}$ and $\boldsymbol{X}'$, and they take the same step that is independent of the local environment (this holds by (0_Y) and (0_X) of the coupling);

(P2) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment, if $\boldsymbol{X}$ jumps on its trace then $\boldsymbol{X}'$ also jumps on its trace and takes the same step (this holds by $(\neq_{\mathcal{I}})$ (and (0_X)) of the coupling);

(P3) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment, if $\boldsymbol{X}'$ jumps out of its trace then $\boldsymbol{X}$ also jumps out of its trace and takes the same step (this holds by $(\neq_{\mathcal{I}^c})$ (and (0_X)) of the coupling);

(P4) When $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment and if the walks decouple, then $\boldsymbol{Y}$ jumps backwards, $\boldsymbol{X}$ jumps out of its trace and $\boldsymbol{X}'$ jumps on its trace (this holds by $(=_{\mathcal{I},\mathcal{I}^c})$ (and (0_Y)) of the coupling).

Items $(\neq_X)$ and $(\neq_{X'})$ in the coupling are dealing with the case when $\boldsymbol{X}$ and $\boldsymbol{X}'$ do not have the same local environment and $\mathbf{Y}$ jumps backwards. For this case, we only define simple rules in order for the marginals to have the good distributions.

Common Regeneration Structure. Let $\mathbf{Z} = \boldsymbol{X} \cdot \ell_+$ and $\mathbf{Z}' = \boldsymbol{X}' \cdot \ell_+$. By property (P1) our coupling satisfies Lemma 1(i). This implies that if t is a regeneration time for the walk $\mathbf{Y}$, then it is also a regeneration time for $\boldsymbol{X}$ and $\boldsymbol{X}'$. Recall that $\tau_1, \tau_2, ...$, denotes the sequence of positive regeneration times of $\mathbf{Y}$. Recall that D_0 is the event on which 0 is a regeneration time, and we defined $\overline{\mathbb{P}}[\cdot] := \mathbb{P}[\cdot|D_0]$. Using that $\mathbf{Y}$ is a biased random walk on $\mathbb{Z}$ with probability to jump on the right equal to p^Y, these regeneration times are well defined as soon as $p^Y > 1/2$ and (6) shows that $\mathbb{P}(D_0) = (2p^Y - 1)/p^Y =: p_\infty$.

Using classical arguments on regeneration times and taking advantage of the common regeneration structure, we obtain the following result.

Proposition 1. *For any $\beta' > \beta > 0$ and $(\alpha_i)_{i=1,\dots,2d}$ such that $p^Y > 1/2$, we have that, under $\mathbb{P}$,*

$$\left(Y_{\tau_{k+1}} - Y_{\tau_k}, X_{\tau_{k+1}} - X_{\tau_k}, X'_{\tau_{k+1}} - X'_{\tau_k}, \tau_{k+1} - \tau_k\right),\ k \geq 0$$

are independent and (except for $k = 0$) have the same distribution as

$$\left(Y_{\tau_1}, X_{\tau_1}, X'_{\tau_1}, \tau_1\right)$$

under $\overline{\mathbb{P}}$.

Since this result is classical (see e.g. [26,32], or [14]) and intuitively clear, we only give a sketch proof.

Sketch of Proof (of Proposition 1). Suppose that $t \in \mathcal{D}_Y$, i.e. t is a regeneration time for $\boldsymbol{Y}$. Then $\Delta^{\boldsymbol{Y}}_{t+1} = 1$ so both $\boldsymbol{X}$ and $\boldsymbol{X}'$ take a forward step which is chosen independent of the environment. Moreover, whenever $(X_{t+n} - X_t) \cdot \ell_+ = 0$ or $(X'_{t+n} - X'_t) \cdot \ell_+ = 0$ we must have that $Y_{t+n} = Y_t$ and therefore again $\Delta^{\boldsymbol{Y}}_{t+n+1} = 1$ and both $\boldsymbol{X}$ and $\boldsymbol{X}'$ take a forward step independent of the environment. On the other hand, whenever both $(X_{t+n} - X_t) \cdot \ell_+ > 0$ and $(X_{t+n} - X_t) \cdot \ell_+ > 0$, we have that neither $\boldsymbol{X}$ nor $\boldsymbol{X}'$ are incident to an edge reinforced before time t. This shows that for any possible paths $\vec{y}_t, \vec{x}_t, \vec{x'}_t$, the conditional distribution of $(Y_{t+n} - Y_t, X_{t+n} - X_t, X'_{t+n} - X'_t)$ given $((Y_k)_{k\leq t}, (X_k)_{k\leq t}, (X'_k)_{k\leq t}) = (\vec{y}_t, \vec{x}_t, \vec{x'}_t)$ and $t \in \mathcal{D}_Y$ does not depend on t or $(\vec{y}_t, \vec{x}_t, \vec{x'}_t)$ and the result follows. □

4 Proofs of Theorem 3 and Theorem 4

Proposition 1 implies that item (ii) of Lemma 1 is satisfied by $\mathbf{Y}$, $\mathbf{Z}' = \boldsymbol{X}' \cdot \ell_+$ and $\mathbf{Z} = \boldsymbol{X} \cdot \ell_+$. Hence, to prove Theorem 3 and Theorem 4, we only need to check the requirements of Lemma 2.

Define the first time before τ_1 the walks $\boldsymbol{X}$ and $\boldsymbol{X}'$ decouple as

$$\delta_1 = \inf\left\{i \leq \tau_1 :\ X_i \neq X'_i\right\}.$$

Note that $\delta_1 = \infty$ if the walks do not decouple before τ_1.

Proposition 2. *For any $\beta' > \beta > 0$, and α such that $p^Y > 1/2$, we have that*

$$\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} < 0\big) = 0.$$

Proof. The fact that $p^Y > 1/2$ guarantees that τ_1 and $\overline{\mathbb{P}}$ are well defined.
Note that, by property (P1), at any time $n \in \mathbb{N}$ such that $\Delta_n^{\mathbf{Y}} = 1$, we have that $\Delta_n^{\mathbf{Z}} = \Delta_n^{\mathbf{Z}'} = 1$ and thus $Z_{n+1} - Z'_{n+1} = Z_n - Z'_n$.
On the event $\{|\mathfrak{B}| = 1\}$, there exists only one time $0 < n_b < \tau_1$ such that $\Delta_{n_b}^{\mathbf{Y}} = -1$ and thus $Z_{\tau_1} - Z'_{\tau_1} = (\Delta_{n_b}^{\mathbf{Z}} - \Delta_{n_b}^{\mathbf{Z}'}) \cdot \ell_+$. At time $n_b - 1$, $\boldsymbol{X}$ and $\boldsymbol{X}'$ are still coupled and thus have the same local environment, which is such that $\mathcal{I}_{n_b-1} = \mathcal{I}_{n_b-1} = \{e_{d+i}\}$ for some $i \in [d]$. Thus by (P4) in order for the walks to decouple on this step we must have $\delta_{n_b}^{\boldsymbol{X}'} = e_{d+i}$, which cannot create a negative discrepancy. □

The last result together with the two following Propositions respectively imply Theorem 3 and Theorem 4 by Lemma 2 and Lemma 1.

Proposition 3. *There exists $\kappa_0 < \infty$ such that if Condition κ holds for κ_0, $\boldsymbol{\alpha}$ and β_0, and Condition S holds, then for all $0 < \beta < \beta' < \beta_0$, inequality (7) is satisfied.*

Proposition 4. *Suppose that Condition D holds. There exist $\kappa_0 < \infty$ and $\beta_0 > 0$ such that if Condition e_1 holds for κ_0, $\boldsymbol{\alpha}$ and β_0, then for all $0 < \beta < \beta' < \beta_0$, inequality (7) is satisfied.*

4.1 Bounds on the Decoupling Events

Lemma 3. *Assume that $\alpha_{i+d} > 0$ for any $i \in [d]$. There exists $C_0 > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \le \beta < \beta' < \beta_0$,*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} = 2\right) \ge C_0 \frac{\beta' - \beta}{\alpha_+}.$$

Proof. Let $A = \big\{|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} = 2\big\}$ and note that

$$\overline{\mathbb{P}}(A) = \frac{1}{p_\infty} \mathbb{P}\left(A, D_0\right).$$

Now, let us describe the following scenario (which is in fact the only possible event on which the walks decouple) such that $A \cap D_0$ holds:

(i) $(Y_1, Y_2, Y_3, Y_4) = (1, 0, 1, 2)$ and $\tau_1 = 4$, so also D_0 occurs;
(ii) $\Delta_1^{\boldsymbol{X}} = \Delta_1^{\boldsymbol{X}'} \in \mathcal{E}_+$, then $\boldsymbol{X}'$ steps back onto its trace ($\Delta_2^{X'} = -\Delta_1^{X'}$), while $\boldsymbol{X}$ steps forward ($\Delta_2^{\boldsymbol{X}} \in \mathcal{E}_+$).

Now, following this scenario and conditional on $\Delta_1^{\boldsymbol{X}} = \Delta_1^{\boldsymbol{X}'} = e_i$, we have (from $(=_{\mathcal{I},\mathcal{I}^c})$) that

$$\mathbb{P}\left(\Delta_2^{\boldsymbol{X}} \in \mathcal{E}_+, \Delta_2^{\boldsymbol{X}'} = e_{i+d} \middle| \mathcal{F}_2\right) = \frac{\alpha_+}{\alpha + \beta\alpha_{i+d}} - \frac{\alpha_+}{\alpha + \beta'\alpha_{i+d}} \tag{16}$$

$$= \frac{(\beta' - \beta)\alpha_+\alpha_{i+d}}{(\alpha + \beta\alpha_{i+d})(\alpha + \beta'\alpha_{i+d})} \tag{17}$$

$$\geq \frac{(\beta' - \beta)\alpha_+}{(\alpha_+ + (1 + \beta')\alpha_-)^2} \tag{18}$$

$$= \frac{(\beta' - \beta)}{\alpha_+}\left(\frac{\alpha_+}{(\alpha_+ + (1 + \beta')\alpha_-)}\right)^2 \tag{19}$$

$$\geq \frac{(\beta' - \beta)\kappa_0^2}{\alpha_+(\kappa_0 + 1)^2} \geq c\frac{\beta' - \beta}{\alpha_+}. \tag{20}$$

Hence, summing over $i \in [d]$ and using Condition D we obtain

$$\overline{\mathbb{P}}(A) \geq \frac{1}{p_\infty} \times p^Y \times cd\frac{\beta' - \beta}{\alpha_+} \times \left(p^Y\right)^2 \times p_\infty.$$

We conclude noting that for $\kappa_0 > 4$, $p^Y > 1/2$ (by (10)). $\square$

Lemma 4. *There exists $C_1 > 0$ such that: If Condition S holds then there exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \leq \beta < \beta' < \beta_0$,*

$$\overline{\mathbb{P}}\left[|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \leq \frac{C_1(\beta' - \beta)}{\alpha_+\kappa_0}.$$

Proof. Assume that, at a given time n, the walks are still coupled. Let $J(\mathcal{I})$ denote the right hand side of (13) and $J_1(\mathcal{I})$ and $J_2(\mathcal{I})$ denote the right hand sides of the intervals in (14) and (15) respectively. Under Condition S these quantities can be written as $J(k_+, k_-)$, $J_1(k_+, k_-)$ and $J_2(k_+, k_-)$, corresponding to the values above for given k_+ and k_-.

Then we can write item $(=_{\mathcal{I},\mathcal{I}^c})$ as:

$(=_{\mathcal{I},\mathcal{I}^c,S})$ If $U_{n+1} \in (0, J(k_+, k_-)]$ then
- (i) if $U_{n+1} \in (0, J_1(k_+, k_-)]$ then $\Delta_{n+1}^{\boldsymbol{X}'} \in \mathcal{E}_+$ and $\Delta_{n+1}^{\boldsymbol{X}'} \in \mathcal{E}_-$ otherwise;
- (ii) if $U_{n+1} \in (0, J_2(k_+, k_-)]$ then $\Delta_{n+1}^{\boldsymbol{X}} \in \mathcal{E}_+$ and $\Delta_{n+1}^{\boldsymbol{X}} \in \mathcal{E}_-$ otherwise.

On the event $\{|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0, D_0\}$, the walk $\mathbf{Y}$ can only do one of the following:

1. $E_1 = \{\{Y_0, \ldots, Y_7\} = \{0, 1, 2, 1, 0, 1, 2, 3\}, \tau_1 = 7\}$;
2. $E_2 = \{\{Y_0, \ldots, Y_6\} = \{0, 1, 0, 1, 0, 1, 2\}, \tau_1 = 6\}$;
3. $E_3 = \{\{Y_0, \ldots, Y_7\} = \{0, 1, 0, 1, 2, 1, 2, 3\}, \tau_1 = 7\}$.

Recall the remarks from Sect. 3.2. In particular, recall that, by (P1), when $\boldsymbol{Y}$ steps forward, $\boldsymbol{X}$ and $\boldsymbol{X}'$ take the same step. Moreover, if $\boldsymbol{X}$ and $\boldsymbol{X}'$ have the same local environment with only one reinforced edge and if this edge is in one of the directions of $\mathcal{E}_-$, then, by (P4), $\boldsymbol{X}$ and $\boldsymbol{X}'$ cannot decouple creating a negative discrepancy. It follows that no negative discrepancy can be created the first time $\boldsymbol{Y}$ steps back, so the magnitude of any negative discrepancy can only be 2. It also follows that on $(E_2 \cap \{\delta = 4\}) \cup E_3$, $\boldsymbol{X}$ and $\boldsymbol{X}'$ cannot create any negative discrepancy.

It therefore remains to consider the cases $E_1 \cap \{\delta = 3\}$, $E_2 \cap \{\delta = 2\}$, and $E_1 \cap \{\delta = 4\}$.

On $(E_1 \cap \{\delta = 3\}) \cup (E_2 \cap \{\delta = 2\})$, $\boldsymbol{X}'$ jumps backwards on its trace at time δ and one of the following happens:

- At time δ, $\boldsymbol{X}$ jumps forward, hence $Z_\delta - Z'_\delta = 2$, thus $Z_{\tau_1} - Z'_{\tau_1} \geq 0$ and the walks cannot create any negative discrepancy before the regeneration time.
- At time δ, the $\boldsymbol{X}$ jumps backwards out of the trace, hence $Z_\delta - Z'_\delta = 0$ but the two walks do not have the same local environment anymore. Then, the second time $\mathbf{Y}$ jumps backwards, hence the two walks can create a negative discrepancy.

Using the item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(ii) above (and (13),(15)) in the case $k_+ = 0$ and $k_- = 1$, we have (by bounding only the conditional probability of the step taken at time δ) that

$$
\begin{aligned}
&\mathbb{P}\left((E_1 \cap \{\delta = 3\}) \cup (E_2 \cap \{\delta = 2\}), Z_{\tau_1} - Z'_{\tau_1} < 0\right) \\
&\leq J(0,1) - J_2(0,1) = \frac{(d-1)(\alpha_-)^2(\beta' - \beta)}{(\alpha_-\beta + d\alpha)(\alpha_-\beta' + d\alpha)} \\
&\leq (\beta' - \beta)\left(\frac{\alpha_-}{\alpha}\right)^2 \leq \frac{(\beta' - \beta)\alpha_+}{\alpha^2 \kappa_0} \leq \frac{(\beta' - \beta)}{\alpha_+ \kappa_0},
\end{aligned}
\tag{21}
$$

where we have used Condition κ for the penultimate inequality.

Now, we want to study the probability of the event $\{E_1, \delta = 4\}$ and consider the cases when the discrepancy can be negative or not. One of the following happens:

- If $\Delta_3^{\boldsymbol{X}} = \Delta_3^{\boldsymbol{X}'} \in \mathcal{E}_+$ then, the local environment $\mathcal{I}_3$ is made of one single reinforced edge, in the direction $\mathcal{E}_-$, hence $\boldsymbol{X}$ and $\boldsymbol{X}'$ cannot decouple creating a negative discrepancy;
- At time 3, both $\boldsymbol{X}$ and $\boldsymbol{X}'$ can jump backward on their trace. This corresponds to the event $E_{1,1} := \{E_1, \delta = 4, \Delta_3^{\boldsymbol{X}} = -\Delta_2^{\boldsymbol{X}}\}$, which is treated below;
- At time 3, both $\boldsymbol{X}$ and $\boldsymbol{X}'$ can jump backward out of their trace. This corresponds to the event $E_{1,2} := \{E_1, \delta = 4, \Delta_3^{\boldsymbol{X}} \in \mathcal{E}_- \setminus \{-\Delta_2^{\boldsymbol{X}}\}\}$, which is treated below.

On the event $E_{1,1}$, the local environment $\mathcal{I}_3$ of the walks is made of one reinforced edge in some direction in $\mathcal{E}_+$ and one reinforced edge in some direction in $\mathcal{E}_-$.

By property (P4), in order to decouple, $\boldsymbol{X}'$ has to jump on its trace and $\boldsymbol{X}$ out of its trace. More precisely, in order to create a negative discrepancy, $\boldsymbol{X}'$ has to jump forward on its trace and $\boldsymbol{X}$ backwards out of its trace. Now, note that, in the case $k_+ = k_- = 1$, $J_1(k_+, k_-) = J_2(k_+, k_-)$ so the two intervals of item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(i) and item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(ii) are equal. Hence, on $E_{1,1}$, we have that $\Delta_\delta^{\boldsymbol{X}} \cdot \ell_+ = \Delta_\delta^{\boldsymbol{X}'} \cdot \ell_+$. This implies that Therefore, we have that

$$\mathbb{P}\left(E_{1,1}, Z_{\tau_1} - Z'_{\tau_1}\right) = 0. \tag{22}$$

Note that this is not necessarily true if we do not assume Condition S, with the prescribed ordering of $\mathcal{I}$ and $\mathcal{I}^c$.

On the event $E_{1,2}$, the local environment $\mathcal{I}_3$ of the walks is made of one single reinforced edge in some direction of $\mathcal{E}_+$. As the walks decouple, we have that $\boldsymbol{X}'$ jumps forward on the trace and $\boldsymbol{X}$ jumps out of the trace. The walks create a negative discrepancy only if, furthermore, $\boldsymbol{X}$ jumps backwards: the probability of this step is given by item $(=_{\mathcal{I},\mathcal{I}^c,S})$ above. Recalling that, at time 3, $\mathbf{Y}$ steps backwards and using the item $(=_{\mathcal{I},\mathcal{I}^c,S})$-(ii) above (and (13),(15)) in the case $k_+ = 1$ and $k_- = 0$, we have that

$$\begin{aligned}
\mathbb{P}\left[E_{1,2}, Z_{\tau_1} - Z'_{\tau_1} < 0\right] &\leq q^Y \times (J(1,0) - J_2(1,0)) \\
&\leq q^Y \frac{d\alpha_- \alpha_+ (\beta' - \beta)}{(\alpha_+\beta + d\alpha)(\alpha_+\beta' + d\alpha)} \\
&\leq C \frac{(1+\beta_0)\alpha_-}{\alpha_+} \times (\beta' - \beta) \times \frac{\alpha_-}{\alpha_+} \\
&\leq C(\beta' - \beta) \left(\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^2 \leq C \frac{(\beta' - \beta)}{\alpha_+ \kappa_0},
\end{aligned}$$

where we used that, under Condition κ ,

$$q^Y \leq 2 \frac{(1+\beta_0)\alpha_-}{\alpha_+}.$$

This, together with (21) and (22), implies the conclusion. □

Lemma 5. *There exist $C_2, C_2' > 0$ such that: There exists κ_0 such that for α, β_0 satisfying Condition κ for κ_0, and all β, β' satisfying $0 \leq \beta < \beta' < \beta_0$, and all $k \geq 2$,*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right) \leq C_2' k (\beta' - \beta) \left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right) \left(C_2 \frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1}.$$

Proof. Denote $A_k = \left\{|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right\}$.
Recall that δ is the time of decoupling. Note that $\mathbf{Y}$ necessarily starts by jumping forward and that the two last steps before τ_1 are also necessarily forward (otherwise this contradicts the definition of τ_1).

Following [2, Lemma 4.1] and adjusting it to the definition of the regeneration times that we use here, one can prove (see e.g. Lemma 3.3 of [7]) that

$|\mathfrak{B}| = k \Rightarrow \tau_1 \leq 3k+1$, almost surely, i.e. $\overline{\mathbb{P}}\big(\{|\mathfrak{B}| = k\} \setminus \{\tau_1 \leq 3k+1\}\big) = 0$. Using this fact, we have that

$$\overline{\mathbb{P}}(A_k) \leq p_\infty^{-1}\mathbb{P}\big(2 \leq \delta \leq 3k, |\{1 \leq n \leq 3k-2 : \Delta^{\boldsymbol{Y}}_{n+1} = -1\}| \geq k\big). \tag{23}$$

On the event $\{\delta = n+1\}$ the walks are still coupled at time n and thus, in particular, $\mathcal{I} := \mathcal{I}_n = \mathcal{I}'_n$.

Denote $k = |\mathcal{I}|$, $\mathcal{I} = \{r_1, \ldots, r_k\}$ and $\{1, \ldots, 2d\} \setminus \mathcal{I} = \{\bar{r}_1, \ldots, \bar{r}_{2d-k}\}$. Then the event of decoupling at time $n+1$ is controlled as follows, according to the item $(=_{\mathcal{I},\mathcal{I}^c})$ of the coupling,

$$\begin{aligned}
\{\delta = n+1\} &= \{\delta > n\} \cap \left\{ U_{n+1} \leq 1 - \sum_{j=1}^{k} p^{(\beta)}_{r_j,\mathcal{I}} - \sum_{j=1}^{d-k} p^{(\beta')}_{\bar{r}_j,\mathcal{I}} \right\} \\
&= \{\delta > n\} \cap \left\{ U_{n+1} \leq 1 - \frac{(1+\beta)\alpha_{\mathcal{I}}}{\alpha + \beta\alpha_{\mathcal{I}}} - \frac{\alpha_{\mathcal{I}^c}}{\alpha + \beta'\alpha_{\mathcal{I}}} \right\} \\
&= \{\delta > n\} \cap \left\{ U_{n+1} \leq \frac{\alpha_{\mathcal{I}^c}\alpha_{\mathcal{I}}(\beta' - \beta)}{(\alpha + \beta\alpha_{\mathcal{I}})(\alpha + \beta'\alpha_{\mathcal{I}})} \right\} \\
&\subset \left\{ U_{n+1} \leq (\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right) \right\},
\end{aligned}$$

where we have used the fact that either $1 \in \mathcal{I}$ or $1 \in \mathcal{I}^c$ to obtain the last relation.

Thus, we have

$$\begin{aligned}
\overline{\mathbb{P}}(A_k) &\leq p_\infty^{-1} \sum_{n=2}^{3k-1} \mathbb{P}\Big(U_n \leq (\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right), \\
&\qquad\qquad \big|\{j \in \{1, \ldots, 3k-1\} \setminus \{n\} : U_j \leq q^Y\}\big| \geq k-1 \Big) \\
&\leq ck(\beta' - \beta)\Big(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\Big) \\
&\qquad\qquad \times \mathbb{P}\big(\exists B \subset [3k-2] : |B| = k-1, U_j \leq q^Y \forall j \in B\big) \\
&\leq ck(\beta' - \beta)\Big(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\Big)\binom{3k-2}{k-1}(q^Y)^{k-1} \\
&\leq c'k(\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right)\left(c'' q^Y\right)^{k-1} && (24) \\
&\leq c'k(\beta' - \beta)\left(\frac{\alpha - \alpha_1}{\alpha} \wedge 1\right)\left(\frac{c'''(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1}, && (25)
\end{aligned}$$

where we used the inequality $e^{11/12}(n/e)^n \leq n! \leq e(n/e)^n$, which holds for any $n \geq 1$, see [24], and we also used that

$$q^Y \leq C\frac{(1+\beta_0)\alpha_-}{\alpha_+} \text{ and } p_\infty \geq 1/2.$$

□

Lemma 6. *There exist* $C_3 > 0$ *such that: There exists* κ_0 *such that for* α, β_0 *satisfying Condition* κ *for* κ_0, *and all* β, β' *satisfying* $0 \le \beta < \beta' < \beta_0$, *we have that*

$$\sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \le \frac{C_3(\beta' - \beta)}{\alpha_+ \kappa_0}.$$

Proof. Using Lemma 5, we have that

$$\begin{aligned}\sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] &\le C(\beta' - \beta)\sum_{k=3}^{\infty} k^2 \left(C_2 \frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^{k-1} \\ &\le C'(\beta' - \beta)\left(\frac{(1+\beta_0)\alpha_-}{\alpha_+}\right)^2 \left(1 - \frac{C_2}{\kappa_0}\right)^{-3} \\ &\le C' \frac{\beta' - \beta}{\alpha_+ \kappa_0},\end{aligned}$$

where we have used Condition κ for large κ_0. □

Lemma 7. *There exist* $C_4 > 0$ *such that: There exists* κ_0 *such that for* α, β_0 *satisfying Condition* e_1 *and Condition* D *then, for all* $0 < \beta < \beta' < \beta_0$, *we have that*

$$\overline{\mathbb{P}}\left(|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right) \le C_4 \frac{\beta' - \beta}{\alpha_+ \kappa_0}.$$

Proof. This is a direct consequence of Condition e_1 and Lemma 5 for $k = 2$. □

4.2 Proofs of Proposition 3 and Proposition 4

Recall that Propositions 3 and 4 respectively imply Theorem 3 and Theorem 4.

Proof of Proposition 3. By Lemma 3, Lemma 4 and Lemma 6, there exists $\kappa_0 < \infty$ such that if Condition κ holds for κ_0, $\boldsymbol{\alpha}$ and β_0 and if Condition S holds, then, for all $0 < \beta < \beta' < \beta_0$, we have that

$$\begin{aligned}&\sum_{k=2}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \\ &\le 4\overline{\mathbb{P}}\left[|\mathfrak{B}| = 2, Z_{\tau_1} - Z'_{\tau_1} < 0\right] + \sum_{k=3}^{\infty} 2k\overline{\mathbb{P}}\left[|\mathfrak{B}| = k, Z_{\tau_1} - Z'_{\tau_1} < 0\right] \\ &\le C\frac{(\beta' - \beta)}{\alpha_+ \kappa_0} \le \frac{C'}{\kappa_0}\overline{\mathbb{P}}\big(|\mathfrak{B}| = 1, Z_{\tau_1} - Z'_{\tau_1} \ge 1\big).\end{aligned}$$

Hence, we can conclude that inequality (7) is satisfied as soon as κ_0 is large enough, which proves the Proposition. □

Proof of Proposition 4. Suppose Condition D holds. There exist $\kappa_0 < \infty$ and $\beta_0 > 0$ such that if Condition e_1 holds for κ_0, $\boldsymbol{\alpha}$ and β_0, then, for all $0 < \beta < \beta' < \beta_0$, we conclude exactly as in the previous proof that (7) holds by Lemma 3, Lemma 7 and Lemma 6. □

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Velocity Estimates for Symmetric Random Walks at Low Ballistic Disorder

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We dedicate this work to Chuck Newman on the occasion of his 70th birthday

Abstract. We derive asymptotic estimates for the velocity of random walks in random environments which are perturbations of the simple symmetric random walk but have a small local drift in a given direction. Our estimates complement previous results presented by Sznitman in [16] and are in the spirit of expansions obtained by Sabot in [11].

Keywords: Random walk in random environment · Asymptotic expansion · Green function

1 Introduction and Main Results

The mathematical derivation of explicit formulas for fundamental quantities of the model of random walk in a random environment is a challenging problem. For quantities like the velocity, the variance or the invariant measure of the environment seen from the random walk, few results exist (see the review [12] for the case of the Dirichlet environment and for expansions see [4,11]). In [11],

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Sabot derived an asymptotic expansion for the velocity of the random walk at low disorder under the condition that the local drift of the perturbed random walk is linear in the perturbation parameter. As a corollary one can deduce that, in the case of perturbations of the simple symmetric random walk, the velocity is equal to the local drift with an error which is cubic in the perturbation parameter. In this article we explore up to which extent this expansion can be generalized to perturbations which are not necessarily linear in the perturbation parameter and we exhibit connections with previous results of Sznitman about ballistic behavior [16].

Fix an integer $d \geq 2$ and for $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ let $|x| := |x_1| + \cdots + |x_d|$ denote its l^1-norm. Let $V := \{x \in \mathbb{Z}^d : |x|_1 = 1\}$ be the set of canonical vectors in $\mathbb{Z}^d$ and $\mathcal{P}$ denote the set of all probability vectors $\vec{p} = (p(e))_{e \in V}$ on V, i.e. such that $p(e) \geq 0$ for all $e \in V$ and also $\sum_{e \in V} p(e) = 1$. Furthermore, let us consider the product space $\Omega := \mathcal{P}^{\mathbb{Z}^d}$ endowed with its Borel σ-algebra $\mathcal{B}(\Omega)$. We call any $\omega = (\omega(x))_{x \in \mathbb{Z}^d} \in \Omega$ an *environment*. Notice that, for each $x \in \mathbb{Z}^d$, $\omega(x)$ is a probability vector on V, whose components we will denote by $\omega(x, e)$ for $e \in V$, i.e. $\omega(x) = (\omega(x, e))_{e \in V}$. The *random walk in the environment* ω starting from $x \in \mathbb{Z}^d$ is then defined as the Markov chain $(X_n)_{n \in \mathbb{N}_0}$ with state space $\mathbb{Z}^d$ which starts from x and is given by the transition probabilities

$$P_{x,\omega}(X_{n+1} = y + e | X_n = y) = \omega(y, e),$$

for all $y \in \mathbb{Z}^d$ and $e \in V$. We will denote its law by $P_{x,\omega}$. We assume throughout that the space of environments Ω is endowed with a probability measure $\mathbb{P}$, called the *environmental law*. We will call $P_{x,\omega}$ the *quenched law* of the random walk, and also refer to the semi-direct product $P_x := \mathbb{P} \otimes P_{x,\omega}$ defined on $\Omega \times \mathbb{Z}^{\mathbb{N}}$ as the *averaged* or *annealed law* of the random walk. In general, we will call the sequence $(X_n)_{n \in \mathbb{N}_0}$ under the annealed law a *random walk in a random environment* (RWRE) with environmental law $\mathbb{P}$. Throughout the sequel, we will always assume that the random vectors $(\omega(x))_{x \in \mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$. Furthermore, we shall also assume that $\mathbb{P}$ is *uniformly elliptic*, i.e. that there exits a constant $\kappa > 0$ such that for all $x \in \mathbb{Z}^d$ and $e \in V$ one has

$$\mathbb{P}(\omega(x, e) \geq \kappa) = 1.$$

Given $l \in \mathbb{S}^{d-1}$, we will say that our random walk $(X_n)_{n \in \mathbb{N}_0}$ is *transient in direction* l if

$$\lim_{n \to \infty} X_n \cdot l = +\infty \quad P_0 - a.s.,$$

and say that it is *ballistic in direction* l if it satisfies the stronger condition

$$\liminf_{n \to \infty} \frac{X_n \cdot l}{n} > 0 \qquad P_0 - a.s.$$

Any random walk which is ballistic with respect to some direction l satisfies a law of large numbers (see [5] for a proof of this fact), i.e. there exists a deterministic vector $\vec{v} \in \mathbb{R}^d$ with $\vec{v} \cdot l > 0$ such that

$$\lim_{n \to +\infty} \frac{X_n}{n} = \vec{v} \qquad P_0 - a.s..$$

This vector $\vec{v}$ is known as the *velocity* of the random walk.

Throughout the following we will fix a certain direction, say

$$e_1 := (1, 0, \ldots, 0) \in \mathbb{S}^{d-1}$$

for example, and study transience/ballisticity only in this fixed direction. Thus, whenever we speak of transience or ballisticity of the RWRE it will be understood that it is with respect to this given direction e_1. However, we point out that all of our results can be adapted and still hold for any other direction.

For our main results, we will consider environmental laws $\mathbb{P}$ which are small perturbations of the simple symmetric random walk. More precisely, we will work with environmental laws $\mathbb{P}$ supported on the subset $\Omega_\epsilon \subseteq \Omega$ for $\epsilon > 0$ sufficiently small, where

$$\Omega_\epsilon := \left\{ \omega \in \Omega : \left| \omega(x, e) - \frac{1}{2d} \right| \leq \frac{\epsilon}{4d} \text{ for all } x \in \mathbb{Z}^d \text{ and } e \in V \right\}. \tag{1}$$

Notice that if $\mathbb{P}$ is supported on Ω_ϵ for some $\epsilon \leq 1$ then it is uniformly elliptic with constant

$$\kappa = \frac{1}{4d}. \tag{2}$$

Since we wish to focus on RWREs for which there is ballisticity in direction e_1, it will be necessary to impose some condition. If for $x \in \mathbb{Z}^d$ we define the *local drift of the RWRE at site* x as the random vector

$$\vec{d}(x) := \sum_{e \in V} \omega(x, e) e$$

then, for the walk to be ballistic in direction e_1, one could expect that it is enough that $\lambda := \mathbb{E}[\vec{d}(0)] \cdot e_1$ is positive, where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$ (notice that all local drift vectors $(\vec{d}(x))_{x \in \mathbb{Z}^d}$ are i.i.d. so that it suffices to consider only the local drift at 0). Nevertheless, as shown in [3], there exist examples of environments for which the expectation of the local drift is positive in a given direction, while the velocity has a negative component in that direction. Therefore, we will impose two different stronger conditions on the local drift, specifying exactly how small we allow λ to be. The first condition is the *quadratic local drift condition.*

Quadratic Local Drift Condition (QLD)**.** Given $\epsilon \in (0, 1)$, we say that the environmental law $\mathbb{P}$ satisfies the quadratic local drift condition $(\text{QLD})_\epsilon$ if $\mathbb{P}(\Omega_\epsilon) = 1$ and, furthermore,

$$\lambda := \mathbb{E}(\vec{d}(0)) \cdot e_1 \geq \epsilon^2.$$

Our second condition, the *local drift condition*, is weaker for dimensions $d \geq 3$.

Local Drift Condition (LD)**.** Given $\eta, \epsilon \in (0, 1)$, we say that an environmental law $\mathbb{P}$ satisfies the local drift condition $(\text{LD})_{\eta,\epsilon}$ if $\mathbb{P}(\Omega_\epsilon) = 1$ and, furthermore,

$$\lambda := \mathbb{E}[\vec{d}(0)] \cdot e_1 \geq \epsilon^{\alpha(d)-\eta}, \tag{3}$$

where

$$\alpha(d) := \begin{cases} 2 & \text{if } d = 2 \\ 2.5 & \text{if } d = 3 \\ 3 & \text{if } d \geq 4. \end{cases} \tag{4}$$

Observe that for $d = 2$ and any $\epsilon \in (0,1)$ condition $(\mathrm{LD})_{\eta,\epsilon}$ implies $(\mathrm{QLD})_\epsilon$ for all $\eta \in (0,1)$, whereas if $d \geq 3$ and $\eta \in (0, \frac{1}{2})$ it is the other way round, $(\mathrm{QLD})_\epsilon$ implies $(\mathrm{LD})_{\eta,\epsilon}$. It is known that for every $\eta \in (0,1)$ there exists $\epsilon_0 = \epsilon(d,\eta)$ such that any RWRE with an environmental law $\mathbb{P}$ satisfying $(\mathrm{LD})_{\eta,\epsilon}$ for some $\epsilon \in (0, \epsilon_0)$ is ballistic. Indeed, for $d \geq 3$ this was proved by Sznitman in [16], whereas the case $d = 2$ is proven in [9], and is also direct consequence of Theorem 2 stated below. Therefore, any RWRE with an environmental law which satisfies $(\mathrm{LD})_{\eta,\epsilon}$ for ϵ small enough is such that P_0-a.s. the limit

$$\vec{v} := \lim_{n\to\infty} \frac{X_n}{n}$$

exists and is different from 0. Our first result is the following.

Theorem 1. *Given any $\eta \in (0,1)$ and $\delta \in (0,\eta)$ there exists some*

$$\epsilon_0 = \epsilon_0(d, \eta, \delta) \in (0,1)$$

such that, for every $\epsilon \in (0, \epsilon_0)$ and any environmental law satisfying $(LD)_{\eta,\epsilon}$, the associated RWRE is ballistic with a velocity $\vec{v}$ which verifies

$$0 < \vec{v} \cdot e_1 \leq \lambda + c_0 \epsilon^{\alpha(d)-\delta} \tag{5}$$

for some constant $c_0 = c_0(d,\eta,\delta) > 0$. We abbreviate (5) by writing $0 < \vec{v} \cdot e_1 \leq \lambda + O_{d,\eta,\delta}(\epsilon^{\alpha(d)-\delta})$.

Our second result is concerned with RWREs with an environmental law satisfying (QLD).

Theorem 2. *There exists $\epsilon_0 \in (0,1)$ depending only on the dimension d such that for all $\epsilon \in (0,\epsilon_0)$ and any environmental law satisfying $(QLD)_\epsilon$, the associated RWRE is ballistic with a velocity $\vec{v}$ which verifies*

$$|\vec{v} \cdot e_1 - \lambda| \leq \frac{\epsilon^2}{d}.$$

Combining both results we immediately obtain the following corollary.

Corollary 1. *Given $\delta \in (0,1)$ there exists some $\epsilon_0 = \epsilon_0(d,\delta) \in (0,1)$ such that, for all $\epsilon \in (0,\epsilon_0)$ and any environmental law satisfying $(QLD)_\epsilon$, the associated RWRE is ballistic with a velocity $\vec{v}$ which verifies*

$$\lambda - \frac{\epsilon^2}{d} \leq \vec{v} \cdot e_1 \leq \lambda + O_{d,\delta}(\epsilon^{\alpha(d)-\delta}).$$

Observe that for dimension $d = 2$ all the information given by Theorem 1 and Corollary 1 is already contained in Theorem 2, whereas this is not so for dimensions $d \geq 3$. To understand better the meaning of our results, let us give some background. First, for $x \in \mathbb{Z}^d$ and $e \in V$ let us rewrite our weights $\omega(x, e)$ as

$$\omega(x, e) = \frac{1}{2d} + \epsilon \xi_\epsilon(x, e), \tag{6}$$

where

$$\xi_\epsilon(x, e) := \frac{1}{\epsilon}\left(\omega(x, e) - \frac{1}{2d}\right).$$

Notice that if $\mathbb{P}(\Omega_\epsilon) = 1$ then $\mathbb{P}$-almost surely we have $|\xi_\epsilon(x, e)| \leq \frac{1}{4d}$ for all $x \in \mathbb{Z}^d$ and $e \in V$. In [11], Sabot considers a fixed $p_0 \in \Omega$ together with an i.i.d. sequence of bounded random vectors $\xi = (\xi(x))_{x \in \mathbb{Z}^d} \subseteq [-1, 1]^V$, where each $\xi(x) = (\xi(x, e))_{e \in V}$ satisfies $\xi(x, e) \in [-1, 1]$ for all $e \in V$ and $\sum_{e \in V} \xi(x, e) = 0$. Then, he defines for each $\epsilon > 0$ the random environment ω on any $x \in \mathbb{Z}^d$ and $e \in V$ as

$$\omega(x, e) := p_0(e) + \epsilon \xi(x, e).$$

In the notation of (6), this corresponds to choosing $p_0(e) = \frac{1}{2d}$ and $\xi_\epsilon(x, e) := \xi(x, e)$ not depending on ϵ. Under the assumption that the local drift associated to this RWRE does not vanish, it satisfies Kalikow's condition [8], and therefore it has a non-zero velocity $\vec{v}$. Sabot then proves that this velocity satisfies the following expansion: for any small $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(d, \delta) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ one has that

$$\vec{v} = \vec{d_0} + \epsilon \vec{d_1} + \epsilon^2 \vec{d_2} + O_{d,\delta}\left(\epsilon^{3-\delta}\right), \tag{7}$$

where

$$\vec{d_0} := \sum_{e \in V} p_0(e) e, \qquad \vec{d_1} := \sum_{e \in V} \mathbb{E}[\xi(0, e)] e,$$

and

$$\vec{d_2} := \sum_{e \in V}\left(\sum_{e' \in V} C_{e,e'} J_{e'}\right) e,$$

with

$$C_{e,e'} := \mathrm{Cov}(\xi(0, e), \xi(0, e')) \qquad \text{and} \qquad J_e := g_{p_0}(e, 0) - g_{p_0}(0, 0).$$

Here $g_{p_0}(x, y)$ denotes the Green's function of a random walk with jump kernel p_0. It turns out that for the particular case in which p_0 is the jump kernel of a simple symmetric random walk (which is the choice we make in this article), we have that $\vec{d_0} = 0$ and also $\vec{d_2} = 0$. In particular, for this case we have $\lambda = \epsilon \vec{d_1} \cdot e_1 = O(\epsilon)$ and

$$\vec{v} \cdot e_1 = \lambda + O_{d,\delta}\left(\epsilon^{3-\delta}\right). \tag{8}$$

Even though this expansion was only shown valid in the regime $\lambda = O(\epsilon)$, from it one can guess that, at least at a formal level, the random walk should be ballistic whenever $\lambda \geq \epsilon^{3-\eta}$ for any $\eta > \delta$. This was established previously by Sznitman from [16] for dimensions $d \geq 4$, but remains open for dimensions $d = 2$ and $d = 3$ (see also [6, 10] for generalizations of Sznitman's result from [16]). In this context, our results show that under condition (LD), which is always weaker than the $\lambda = O(\epsilon)$ assumption in [11], for $d = 2$ the random walk is indeed ballistic and expansion (8) is still valid up to the second order (Theorem 2), whereas for $d \geq 3$ we show that at least an upper estimate compatible with the right-hand side of (8) holds for the velocity (Theorem 1).

The proof of Theorem 1 is rather different from the proof of the velocity expansion (7) of [11], and is based on a mixture of renormalization methods together with Green's functions estimates, inspired in methods presented in [2, 16]. As a first step, one establishes that the averaged velocity of the random walk at distances of the order ϵ^{-4} is precisely equal to the average of the local drift with an error of order $\epsilon^{\alpha(d)-\delta}$. To do this, essentially it is shown that a right approximation for the behavior of the random walk at distances ϵ^{-1} is that of a simple symmetric random walk, so that one has to find a good estimate for the probability to move to the left or to the right of a rescaled random walk moving on a grid of size ϵ^{-1}. This last estimate is obtained through a careful approximation of the Green's function of the random walk, which involves comparing it with its average by using a martingale method. This is a crucial step which explains the fact that one loses precision in the error of the velocity in dimensions $d = 2$ and $d = 3$ compared with $d \geq 4$. As a final result of these computations, we obtain that the polynomial condition of [2] holds. In the second step, we use a renormalization method to derive the upper bound for the velocity, using the polynomial condition proved in the first step as a seed estimate. The proof of Theorem 2 is somewhat simpler, and is based on a generalization of Kalikow's formula proved in [11] and a careful application of Kalikow's criteria for ballisticity.

The article is organized as follows. In Sect. 2 we introduce the general notation and establish some preliminary facts about the RWRE model, including some useful Green's function estimates. In Sect. 3, we prove Theorem 2. In Sect. 4, we obtain the velocity estimates for distances of order ϵ^{-4} which is the first step in the proof of Theorem 1. Finally, in Sect. 5 we finish the proof of Theorem 1 through the renormalization argument described above.

2 Preliminaries

In this section we introduce the general notation to be used throughout the article and also review some basic facts about RWREs which we shall need later.

2.1 General Notation

Given any subset $A \subset \mathbb{Z}^d$, we define its (outer) boundary as

$$\partial A := \{x \in \mathbb{Z}^d - A : |x - y| = 1 \text{ for some } y \in A\}.$$

Also, we define the first exit time of the random walk from A as

$$T_A := \inf\{n \geq 0 : X_n \notin A\}.$$

In the particular case in which $A = \{b\} \times \mathbb{Z}^{d-1}$ for some $b \in \mathbb{Z}$, we will write T_b instead of T_A, i.e.

$$T_b := \inf\{n \geq 0 : X_n \cdot e_1 = b\}.$$

Throughout the rest of this paper $\epsilon > 0$ will be treated as a fixed variable. Also, we will denote generic constants by $c_1, c_2, \ldots$. However, whenever we wish to highlight the dependence of any of these constants on the dimension d or on η, we will write for example $c_1(d)$ or $c_1(\eta, d)$ instead of c_1. Furthermore, for the sequel we will fix a constant $\theta \in (0,1)$ to be determined later and define

$$L := 2[\theta\epsilon^{-1}] \tag{9}$$

where $[\cdot]$ denotes the (lower) integer part and also

$$N := L^3, \tag{10}$$

which will be used as length quantifiers. In the sequel we will often work with slabs and boxes in $\mathbb{Z}^d$, which we introduce now. For each $M \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $l \in \mathbb{S}^{d-1}$ we define the slab

$$U_{l,M}(x) := \left\{y \in \mathbb{Z}^d : -M \leq (y-x) \cdot l < M\right\}. \tag{11}$$

Whenever $l = e_1$ we will suppress l from the notation and write $U_M(x)$ instead. Similarly, whenever $x = 0$ we shall write U_M instead of $U_M(0)$ and abbreviate $U_L(0)$ simply as U for L as defined (9). Also, for each $M \in \mathbb{N}$ and $x \in \mathbb{Z}^d$, we define the box

$$B_M(x) := \left\{y \in \mathbb{Z}^d : -\frac{M}{2} < (y-x) \cdot e_1 < M \right.$$
$$\left. \text{and } |(y-x) \cdot e_i| < 25M^3 \text{ for } 2 \leq i \leq d\right\} \tag{12}$$

together with its *frontal side*

$$\partial_+ B_M(x) := \{y \in \partial B_{M,M'}(x) : (y-x) \cdot e_1 \geq M\},$$

its *back side*

$$\partial_- B_M(x) := \left\{y \in \partial B_{M,M'}(x) : (y-x) \cdot e_1 \leq -\frac{M}{2}\right\},$$

its *lateral side*

$$\partial_l B_M(x) := \left\{y \in \partial B_{M,M'}(x) : |(y-x) \cdot e_i| \geq 25M^3 \text{ for some } 2 \leq i \leq d\right\},$$

and, finally, its *middle-frontal part*

$$B^*_M(x) := \Big\{ y \in B_M(x) : \frac{M}{2} \leq (y-x)\cdot e_1 < M \\ \text{and } |(y-x)\cdot e_i| < M^3 \text{ for } 2 \leq i \leq d \Big\}$$

together with its corresponding *back side*

$$\partial_- B^*_M(x) := \left\{ y \in B^*_M(x) : (y-x)\cdot e_1 = \frac{M}{2} \right\}.$$

As in the case of slabs, we will use the simplified notation $B_M := B_M(0)$ and also $\partial_i B_M := \partial_i B_M(0)$ for $i = +, -, l$, with the analogous simplifications for $B^*_M(0)$ and its back side.

2.2 Ballisticity Conditions

For the development of the proof of our results, it will be important to recall a few ballisticity conditions, namely, Sznitman's (T) and (T') conditions introduced in [14, 15] and also the polynomial condition presented in [2]. We do this now, considering only ballisticity in direction e_1 for simplicity.

Conditions (T) and (T'). Given $\gamma \in (0,1]$ we say that condition $(T)_\gamma$ is satisfied (in direction e_1) if there exists a neighborhood V of e_1 in $\mathbb{S}^{d-1}$ such that for every $l' \in V$ one has that

$$\limsup_{M \to +\infty} \frac{1}{M^\gamma} \log P_0 \left(X_{T_{U_{l',M}}} \cdot l' < 0 \right) < 0. \tag{13}$$

As a matter of fact, Sznitman originally introduced a condition $(T)_\gamma$ which is slightly different from the one presented here, involving an asymmetric version of the slab $U_{l',M}$ in (13) and an additional parameter $b > 0$ which modulates the asymmetry of this slab. However, it is straightforward to check that Sznitman's original definition is equivalent to ours, so we omit it for simplicity.

Having defined the conditions $(T)_\gamma$ for all $\gamma \in (0,1]$, we will say that:

- (T) is satisfied (in direction e_1) if $(T)_1$ holds,
- (T') is satisfied (in direction e_1) if $(T)_\gamma$ holds for all $\gamma \in (0,1)$.

It is clear that (T) implies (T'), while the other implication was only very recently proved in [7], showing that both conditions are in fact equivalent.

Condition $(P)_K$. Given $K \in \mathbb{N}$ we say that the polynomial condition $(P)_K$ holds (in direction e_1) if for some $M \geq M_0$ one has that

$$\sup_{x \in B^*_M} P_x \left(X_{T_{B_M}} \notin \partial_+ B_M \right) \leq \frac{1}{M^K},$$

where

$$M_0 := \exp\left\{100 + 4d(\log\kappa)^2\right\} \tag{14}$$

where κ is the uniform ellipticity constant, which in our present case can be taken as $\kappa = \frac{1}{4d}$, see (2). It is well-known that both (T') and $(P)_K$ imply ballisticity in direction e_1, see [2, 15]. Furthermore, in [2] it is shown that

$$\begin{aligned}(P)_K \text{ holds for some } K \geq 15d+5 &\iff (T') \text{ holds}\\ &\iff (T)_\gamma \text{ holds for some } \gamma \in (0,1).\end{aligned}$$

Together with the result from [7], we obtain that conditions $(T), (T')$ and the validity of $(P)_K$ for some $K \geq 15d+5$ are all equivalent.

2.3 Green's Functions and Operators

Let us now introduce some notation we shall use related to the Green's functions of the RWRE and of the simple symmetric random walk (SSRW).

Given a subset $B \subseteq \mathbb{Z}^d$, the Green's functions of the RWRE and SSRW killed upon exiting B are respectively defined for $x, y \in B \cup \partial B$ as

$$g_B(x,y,\omega) := E_{x,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n = y\}}\right) \quad \text{and} \quad g_{0,B}(x,y) := g_B(x,y,\omega_0),$$

where ω_0 is the corresponding weight of the SSRW, given for all $x \in \mathbb{Z}^d$ and $e \in V$ by

$$\omega_0(x,e) = \frac{1}{2d}.$$

Furthermore, if $\omega \in \Omega$ is such that $E_{x,\omega}(T_B) < +\infty$ for all $x \in B$, we can define the corresponding Green's operator on $L^\infty(B)$ by the formula

$$G_B[f](x,\omega) := \sum_{y \in B} g_B(x,y,\omega) f(y).$$

Notice that g_B, and therefore also G_B, depends on ω only though its restriction $\omega|_B$ to B. Finally, it is straightforward to check that if B is a slab as defined in (11) then both g_B and G_B are well-defined for all environments $\omega \in \Omega_\epsilon$ with $\epsilon \in (0,1)$.

3 Proof of Theorem 2

The proof of Theorem 2 has several steps. We begin by establishing a law of large numbers for the sequence of hitting times $(T_n)_{n \in \mathbb{N}}$.

3.1 Law of Large Numbers for Hitting Times

We now show that, under the condition $(P)_K$, the sequence of hitting times $(T_n)_{n\in\mathbb{N}}$ satisfies a law of large numbers with the inverse of the velocity in direction e_1 as its limit.

Proposition 1. *If $(P)_K$ is satisfied for some $K \geq 15d+5$ then P_0-a.s. we have that*

$$\lim_{n\to\infty} \frac{E_0(T_n)}{n} = \lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{\vec{v}\cdot e_1} > 0, \tag{15}$$

where $\vec{v}$ is the velocity of the corresponding RWRE.

To prove Proposition 1, we will require the following lemma and its subsequent corollary.

Lemma 1. *If $(P)_K$ holds for some $K \geq 15d+5$ then there exists $c_1 > 0$ such that for each $n \in \mathbb{N}$ and all $a > \frac{1}{v\cdot e_1}$ one has that*

$$P_0\left(\frac{T_n}{n} \geq a\right) \leq \frac{1}{c_1} \exp\left\{-c_1\left(\left(\log\left(a - \frac{1}{\vec{v}\cdot e_1}\right)\right)^{\frac{2d-1}{2}} + (\log(n))^{\frac{2d-1}{2}}\right)\right\}. \tag{16}$$

Proof. By Berger, Drewitz and Ramírez [2], we know that since $(P)_K$ holds for $K \geq 15d+5$, necessarily (T') must also hold. Now, a careful examination of the proof of Theorem 3.4 in [15] shows that the upper bound in (16) is satisfied. □

Corollary 2. *If $(P)_K$ holds for some $K \geq 15d+5$ then $\left(\frac{T_n}{n}\right)_{n\in\mathbb{N}}$ is uniformly P_0-integrable.*

Proof. Note that, by Lemma 1, for $K > \frac{1}{v\cdot e_1}$ and $n \geq 2$ we have that

$$\begin{aligned}\int_{\{\frac{T_n}{n}\geq K\}} \frac{T_n}{n} dP_0 &\leq \sum_{k=K}^{\infty} (k+1) P_0\left(\frac{T_n}{n} \geq k\right) \\ &\leq \frac{1}{c_1} \sum_{k=K}^{\infty} (k+1) e^{-c_1(\log(k-1/(\vec{v}\cdot e_1)))^{(2d-1)/2} - c_1(\log(2))^{(2d-1)/2}}.\end{aligned}$$

From here it is clear that, since $d \geq 2$, we have

$$\lim_{K\to\infty}\left[\sup_{n\geq 1} \int_{\{\frac{T_n}{n}\geq K\}} \frac{T_n}{n} dP_0\right] = 0$$

which shows the uniform P_0-integrability. □

Let us now see how to obtain Proposition 1 from Corollary 2. Since $(P)_K$ holds for $K \geq 15d+5$, by Berger, Drewitz and Ramírez [2] we know that the position of the random walk satisfies a law of large numbers with a velocity $\vec{v}$ such that $\vec{v} \cdot e_1 > 0$. Now, note that for any $\varepsilon > 0$ one has

$$P_0\left(\left|\frac{n}{T_n} - \vec{v}\cdot e_1\right| \geq \varepsilon\right) = P_0\left(\left|\frac{X_{T_n}\cdot e_1}{T_n} - \vec{v}\cdot e_1\right| \geq \varepsilon\right) \leq \sum_{k=n}^{\infty} P_0\left(\left|\frac{X_k\cdot e_1}{k} - \vec{v}\cdot e_1\right| \geq \varepsilon\right) \leq \sum_{k=n}^{\infty} e^{-C(\log k)^{\frac{2d-1}{2}}},$$

where in the last inequality we have used the slowdown estimates for RWREs satisfying (T') proved by Sznitman in [15] (see also the improved result of Berger in [1]). Hence, by Borel–Cantelli we conclude that P_0-a.s.

$$\lim_{n\to\infty} \frac{n}{T_n} = \vec{v}\cdot e_1,$$

from where the second equality of (15) immediately follows. The first one is now a direct consequence of the uniform integrability provided by Corollary 2.

3.2 Introducing Kalikow's Walk

Given a nonempty connected strict subset $B \subsetneq \mathbb{Z}^d$, for $x \in B$ we define *Kalikow's walk* on B (starting from x) as the random walk starting from x which is killed upon exiting B and has transition probabilities determined by the environment $\omega_B \in \mathcal{P}^B$ given by

$$\omega_B^x(y,e) := \frac{\mathbb{E}(g_B(x,y,\omega)\omega(y,e))}{\mathbb{E}(g_B(x,y,\omega))}. \tag{17}$$

It is straightforward to check that by the uniform ellipticity of $\mathbb{P}$ we have $0 < \mathbb{E}(g_B(x,y,\omega)) < +\infty$ for all $y \in B$, so that the environment ω_B^x is well-defined. In accordance with our present notation, we will denote the law of Kalikow's walk on B by P_{x,ω_B^x} and its Green's function by $g_B(x,\cdot,\omega_B^x)$. The importance of Kalikow's walk, named after S. Kalikow who originally introduced it in [8], lies in the following result which is a slight generalization of Kalikow's formula [8] and of the statement of it given in [11].

Proposition 2. *If $B \subsetneq \mathbb{Z}^d$ is connected then for any $x \in B$ with $P_{x,\omega_B^x}(T_B < +\infty) = 1$ we have*

$$\mathbb{E}(g_B(x,y)) = g_B(x,y,\omega_B^x) \tag{18}$$

for all $y \in B \cup \partial B$.

Proof. The proof is similar to that of [11, Proposition 1], but we include it here for completeness. First, let us observe that for any $\omega \in \Omega_\epsilon$ and $y \in B \cup \partial B$ we have by the Markov property

$$\begin{aligned} g_B(x,y,\omega) &= E_{x,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n=y\}}\right) \\ &= \sum_{n=0}^{\infty} P_{x,\omega}\left(X_n=y, T_B \geq n\right) \\ &= \mathbb{1}_{\{x\}}(y) + \sum_{n=1}^{\infty}\sum_{e\in V} P_{x,\omega}\left(X_{n-1}=y-e, X_n=y, T_B > n-1\right) \\ &= \mathbb{1}_{\{x\}}(y) + \sum_{e\in V}\sum_{n=1}^{\infty} P_{x,\omega}\left(X_{n-1}=y-e, T_B > n-1\right)\omega(y-e,e) \\ &= \mathbb{1}_{\{x\}}(y) + \sum_{e\in V} \mathbb{1}_B(y-e) g_B(x,y-e,\omega)\omega(y-e,e), \end{aligned}$$

so that

$$\mathbb{E}(g_B(x,y)) = \mathbb{1}_{\{x\}}(y) + \sum_{e\in V\,:\,y-e\in B} \mathbb{E}(g_B(x,y-e))\omega_B^x(y-e,e).$$

Similarly, if for each $k \in \mathbb{N}_0$ we define

$$g_B^{(k)}(x,y,\omega_B^x) := E_{x,\omega_B^x}\left(\sum_{n=0}^{T_B\wedge k} \mathbb{1}_{\{X_n=y\}}\right)$$

then by the same reasoning as above we obtain

$$g_B^{(k+1)}(x,y,\omega_B^x) = \mathbb{1}_{\{x\}}(y) + \sum_{e\in V\,:\,y-e\in B} g_B^{(k)}(x,y-e,\omega_B^x)\omega_B^x(y-e,e). \tag{19}$$

In particular, we see that for all $k \in \mathbb{N}_0$

$$\begin{aligned} &\mathbb{E}(g_B(x,y)) - g_B^{(k+1)}(x,y,\omega_B^x) \\ &\qquad = \sum_{e\in V\,:\,y-e\in B}\Big(\mathbb{E}(g_B(x,y-e)) - g_B^{(k)}(x,y-e,\omega_B^x)\Big)\,\omega_B^x(y-e,e) \end{aligned}$$

which, since ω_B^x is nonnegative and also $g_B^{(0)}(x,y,\omega_B^x) = \mathbb{1}_{\{x\}}(y) \leq \mathbb{E}(g_B(x,y))$ for every $y \in B\cup\partial B$, by induction implies that $g_B^{(k)}(x,y,\omega_B^x) \leq \mathbb{E}(g_B(x,y))$ for all $k \in \mathbb{N}_0$. Therefore, by letting $k \to +\infty$ in this last inequality we obtain

$$g_B(x,y,\omega_B^x) \leq \mathbb{E}(g_B(x,y)) \tag{20}$$

for all $y \in B\cup\partial B$. In particular, this implies that

$$P_{x,\omega_B^x}(T_B < +\infty) = \sum_{y\in\partial B} g_B(x,y,\omega_B^x) \leq \sum_{y\in\partial B}\mathbb{E}(g_B(x,y)) = P_x(T_B < +\infty) \leq 1. \tag{21}$$

Thus, if $P_{x,\omega_B^x}(T_B < +\infty) = 1$ then both sums on (21) are in fact equal which, together with (20), implies that

$$g_B(x, y, \omega_B^x) = \mathbb{E}(g_B(x, y))$$

for all $y \in \partial B$. Finally, to check that this equality also holds for every $y \in B$, we first notice that for any $y \in B \cup \partial B$ we have by (19) that

$$g_B(x, y, \omega_B^x) = \mathbb{1}_{\{x\}}(y) + \sum_{e \in V \,:\, y-e \in B} g_B(x, y-e, \omega_B^x)\omega_B^x(y-e, e)$$

so that if $y \in B \cup \partial B$ is such that $\mathbb{E}(g_B(x, y)) = g_B(x, y, \omega_B^x)$ then

$$0 = \sum_{e \in V \,:\, y-e \in B} \left(\mathbb{E}(g_B(x, y-e)) - g_B(x, y-e, \omega_B^x)\right) \omega_B^x(y-e, e).$$

Hence, by the nonnegativity of ω_B^x and (20) we conclude that if $y \in B \cup \partial B$ is such that (18) holds then (18) also holds for all $z \in B$ of the form $z = y - e$ for some $e \in V$. Since we already have that (18) holds for all $y \in \partial B$ and B is connected, by induction one can obtain (18) for all $y \in B$. □

As a consequence of this result, we obtain the following useful corollary, which is the original formulation of Kalikow's formula [8].

Corollary 3. *If $B \subsetneq \mathbb{Z}^d$ is connected then for any $x \in B$ such that $P_{x,\omega_B^x}(T_B < +\infty) = 1$ we have*

$$E_x(T_B) = E_{x,\omega_B^x}(T_B)$$

and

$$P_x(X_{T_B} = y) = P_{x,\omega_B^x}(X_{T_B} = y)$$

for all $y \in \partial B$.

Proof. This follows immediately from Proposition 2 upon noticing that, by definition of g_B, we have on the one hand

$$E_x(T_B) = \sum_{y \in B} \mathbb{E}(g_B(0, y)) = \sum_{y \in B} g_B(x, y, \omega_B^x) = E_{x,\omega_B^x}(T_B)$$

and, on the other hand, for any $y \in \partial B$

$$P_x(X_{T_B} = y) = \mathbb{E}(g_B(x, y)) = g_B(x, y, \omega_B^x) = P_{x,\omega_B^x}(X_{T_B} = y).$$

□

Proposition 1 shows that in order to obtain bounds on $\vec{v} \cdot e_1$, the velocity in direction e_1, it might be useful to understand the behavior of the expectation $E_0(T_n)$ as n tends to infinity, provided that the polynomial condition $(P)_K$ indeed holds for K sufficiently large. As it turns out, Corollary 3 will provide a way in which to verify the polynomial condition together with the desired bounds for $E_0(T_n)$ by means of studying the killing times of certain auxiliary Kalikow's walks. To this end, the following lemma will play an important role.

Lemma 2. *If given a connected subset $B \subsetneq \mathbb{Z}^d$ and $x \in B$ we define for each $y \in B$ the drift at y of the Kalikow's walk on B starting from x as*

$$\vec{d}_{B,x}(y) := \sum_{e \in V} \omega_B^x(y,e)e$$

where ω_B^x is the environment defined in (17), then

$$\vec{d}_{B,x}(y) = \frac{\mathbb{E}\left(\frac{\vec{d}(y,\omega)}{\sum_{e \in V} \omega(x,e) f_{B,x}(y,y+e,\omega)}\right)}{\mathbb{E}\left(\frac{1}{\sum_{e \in V} \omega(x,x+e) f_{B,x}(y,y+e,\omega)}\right)}.$$

where $f_{B,x}$ is given by

$$f_{B,x}(y,z,\omega) := \frac{P_{z,\omega}(T_B \leq H_y)}{P_{x,\omega}(H_y < T_B)}$$

and $H_y := \inf\{n \in \mathbb{N}_0 : X_n = y\}$ denotes the hitting time of y.

Proof. Observe that if for $y, z \in B \cup \partial B$ and $\omega \in \Omega$ we define

$$g(y,z,\omega) := P_{z,\omega}(H_y < T_B)$$

then by the strong Markov property we have for any $y \in B$

$$\begin{aligned}\mathbb{E}(g_B(x,y,\omega)) &= \mathbb{E}\left(E_{x,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n=y\}}\right)\right) \\ &= \mathbb{E}\left(g(y,x,\omega)E_{y,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n=y\}}\right)\right).\end{aligned}$$

Now, under the law $P_{y,\omega}$, the total number of times $n \in \mathbb{N}_0$ in which the random walk X is at y before exiting B is a geometric random variable with success probability

$$p := \sum_{e \in V} \omega(y,y+e)(1 - g(y,y+e,\omega)),$$

so that

$$E_{y,\omega}\left(\sum_{n=0}^{T_B} \mathbb{1}_{\{X_n=y\}}\right) = \frac{1}{\sum_{e \in V} \omega(y,y+e)(1-g(y,y+e,\omega))}.$$

It follows that

$$\mathbb{E}(g_B(x,y,\omega)) = \mathbb{E}\left(\frac{1}{\sum_{e \in V} \omega(y,y+e) f_{B,x}(y,y+e,\omega)}\right)$$

where $f_{B,x}$ is defined as

$$f_{B,x}(y,z,\omega) := \frac{1-g(y,z,\omega)}{g(y,x,\omega)} = \frac{P_{z,\omega}(T_B \leq H_y)}{P_{x,\omega}(H_y < T_B)}.$$

By proceeding in the same manner, we also obtain

$$\mathbb{E}(g_B(x,y,\omega)\omega(y,e)) = \mathbb{E}\left(\frac{\omega(y,e)}{\sum_{e\in V}\omega(y,y+e)f_{B,x}(y,y+e,\omega)}\right),$$

so that

$$\vec{d}_{B,x}(y) = \frac{\mathbb{E}\left(\frac{\vec{d}(y,\omega)}{\sum_{e\in V}\omega(y,y+e)f_{B,x}(y,y+e,\omega)}\right)}{\mathbb{E}\left(\frac{1}{\sum_{e\in V}\omega(y,y+e)f_{B,x}(y,y+e,\omega)}\right)}.$$

□

As a consequence of Lemma 2, we obtain the following key estimates on the drift of Kalikow's walk.

Proposition 3. *If $\mathbb{P}$ satisfies $(QLD)_\epsilon$ for some $\epsilon \in (0,1)$ then for any connected subset $B \subsetneq \mathbb{Z}^d$ and $x \in B$ we have*

$$\sup_{y\in B}\left[|\vec{d}_{B,x}(y)\cdot e_1 - \lambda|\right] \le \frac{\epsilon^2}{d}.$$

Proof. First, let us decompose

$$\mathbb{E}\left(\frac{\vec{d}(y,\omega)\cdot e_1}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)}\right) = \mathbb{E}\left(\frac{(\vec{d}(y,\omega)\cdot e_1)_+ - (\vec{d}(y,\omega)\cdot e_1)_-}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)}\right).$$

Now, notice that since $\mathbb{P}(\Omega_\epsilon)=1$ we have

$$\begin{aligned}\frac{1}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)} &\le \frac{1}{\sum_{e\in V}\left(\frac{1}{2d}-\frac{\epsilon}{4d}\right)f_{B,x}(y,y+e,\omega)}\\ &\le \frac{2d}{\sum_{e\in V}f_{B,x}(y,y+e,\omega)}\cdot\frac{1}{1-\frac{\epsilon}{2}}\end{aligned}$$

and also

$$\frac{1}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)} \ge \frac{2d}{\sum_{e\in V}f_{B,x}(y,y+e,\omega)}\cdot\frac{1}{1+\frac{\epsilon}{2}}.$$

In particular, we obtain that

$$\begin{aligned}&\mathbb{E}\left(\frac{(\vec{d}(y,\omega)\cdot e_1)_\pm}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)}\right)\\ &\qquad\le \frac{2d}{1-\frac{\epsilon}{2}}\cdot\mathbb{E}\left(\frac{1}{\sum_{e\in V}f_{B,x}(y,y+e,\omega)}\cdot(\vec{d}(y,\omega)\cdot e_1)_\pm\right)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\frac{(\vec{d}(y,\omega)\cdot e_1)_\pm}{\sum_{e\in V}\omega(y,e)f_{B,x}(y,y+e,\omega)}\right)\\ &\qquad\ge \frac{2d}{1+\frac{\epsilon}{2}}\cdot\mathbb{E}\left(\frac{1}{\sum_{e\in V}f_{B,x}(y,y+e,\omega)}\cdot(\vec{d}(y,\omega)\cdot e_1)_\pm\right).\end{aligned}$$

Furthermore, since $\sum_{e\in V} f_{B,x}(y,y+e,\omega)$ is independent of $\omega(y)$ it follows that

$$\mathbb{E}\left(\frac{1}{\sum_{e\in V} f_{B,x}(y,y+e,\omega)}\cdot(\vec{d}(y,\omega)\cdot e_1)_\pm\right) \\ = \mathbb{E}\left(\frac{1}{\sum_{e\in V} f_{B,x}(y,y+e,\omega)}\right)\cdot\mathbb{E}\left((\vec{d}(y,\omega)\cdot e_1)_\pm\right).$$

Finally, by combining this with the previous estimates, a straightforward calculation yields

$$\lambda - \frac{4}{4-\epsilon^2}\cdot\epsilon\cdot\mathbb{E}(|\vec{d}(y,\omega)\cdot e_1|) + \frac{2\epsilon^2}{4-\epsilon^2}\cdot\lambda \\ \leq \vec{d}_{B,x}(y) \leq \lambda + \frac{4}{4-\epsilon^2}\cdot\epsilon\cdot\mathbb{E}(|\vec{d}(y,\omega)\cdot e_1|) + \frac{2\epsilon^2}{4-\epsilon^2}\cdot\lambda$$

Since $\mathbb{P}(\Omega_\epsilon) = 1$ implies that $|\lambda| \leq \mathbb{E}(|\vec{d}(y,\omega)\cdot e_1|) \leq \frac{\epsilon}{2d}$, by recalling that $\epsilon < 1$ we conclude that if $(\text{QLD})_\epsilon$ is satisfied then

$$\lambda - \frac{2}{3d}\cdot\epsilon^2 + \frac{2}{3}\epsilon^4 \leq \vec{d}_{B,x}(x) \leq \lambda + \frac{2}{3d}\cdot\epsilon^2 + \frac{1}{3d}\cdot\epsilon^3$$

from where the result immediately follows. □

3.3 (QLD) Implies $(P)_K$

Having the estimates from the previous section, we are now ready to prove the following result.

Proposition 4. *If $\mathbb{P}$ verifies $(QLD)_\epsilon$ for some $\epsilon \in (0,1)$ then $(P)_K$ is satisfied for any $K \geq 15d+5$.*

Proposition 4 follows from the validity under (QLD) of the so-called *Kalikow's condition.* Indeed, if we define the coefficient

$$\varepsilon_{\mathcal{K}} := \inf\{\vec{d}_{B,0}(y)\cdot e_1 : B \subsetneq \mathbb{Z}^d \text{ connected with } 0\in B,\ y\in B\}.$$

then Kalikow's condition is said to hold whenever $\varepsilon_{\mathcal{K}} > 0$. It follows from [17, Theorem 2.3], [13, Proposition 1.4] and [15, Corollary 1.5] that Kalikow's condition implies condition (T). On the other hand, from the discussion in Sect. 2.2 we know that (T) implies $(P)_K$ for $K \geq 15d+5$ so that, in order to prove Proposition 4, it will suffice to check the validity of Kalikow's condition. But it follows from Proposition 3 that, under $(\text{QLD})_\epsilon$ for some $\epsilon \in (0,1)$, for each connected $B \subsetneq \mathbb{Z}^d$ and $y \in B$ we have

$$\vec{d}_{B,0}(y) \geq \lambda - \frac{\epsilon^2}{d} \geq \frac{d-1}{d}\epsilon^2 > 0$$

so that Kalikow's condition is immediately satisfied and thus Proposition 4 is proved.

Alternatively, one could show Proposition 4 by checking the polynomial condition directly by means of Kalikow's walk. Indeed, if κ denotes the uniform ellipticity constant of $\mathbb{P}$ then $\omega_B^x(y,e) \geq \kappa$ for all connected subsets $B \subsetneq \mathbb{Z}^d$, $x, y \in B$ and $e \in V$. In particular, it follows from this that $P_{x,\omega^x_{B_M}}(T_{B_M} < +\infty) = 1$ for all $x \in B_M$ and boxes B_M as in Sect. 2. Corollary 3 then shows that, in order to obtain Proposition 4, it will suffice to prove the following lemma.

Lemma 3. *If $\mathbb{P}$ verifies (QLD)$_\epsilon$ for some $\epsilon \in (0, \frac{1}{\sqrt{2(d-1)}})$ then for each $K \in \mathbb{N}$ we have*

$$\sup_{x \in B^*_M} P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \notin \partial_+ B_M\right) \leq \frac{1}{M^K}$$

if $M \in \mathbb{N}$ is taken sufficiently large (depending on K).

Proof. Notice that for each $x \in B^*_M$ we have

$$P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \notin \partial_+ B_M\right) \leq P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \in \partial_l B_M\right) + P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \in \partial_- B_M\right) \quad (22)$$

so that it will suffice to bound each term on the right-hand side of (22) uniformly in B^*_M.

To bound the first term, we define the quantities

$$n_+ := \#\{n \in \{1,\ldots,T_{B_M}\} : X_n - X_{n-1} = e_1\}$$

and

$$n_- := \#\{n \in \{1,\ldots,T_{B_M}\} : X_n - X_{n-1} = -e_1\}.$$

and notice that on the event $\{X_{T_{B_M}} \in \partial_l B_M\}$ we must have $n_+ - n_l \leq \frac{M}{2}$ since otherwise X would reach $\partial_+ B_M$ before $\partial_l B_M$. Furthermore, on this event we also have that $T_{B_M} \geq 24M^3$ since, by definition of B^*_M, starting from any $x \in B^*_M$ it takes X at least $24M^3 + 1$ steps to reach $\partial_l B_M$. It then follows that

$$P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \in \partial_l B_M\right) = \sum_{n=24M^3}^{\infty} P_{x,\omega^x_{B_M}}\left(n_+ - n_- \leq \frac{M}{2}, T_{B_M} = n\right). \quad (23)$$

Now, observe that the right-hand side of (23) can be bounded from above by

$$\sum_{n=24M^3}\left[P_{x,\omega^x_{B_M}}\left(n_+ + n_- \leq \kappa N, T_{B_M} = n\right) + P_{x,\omega^x_{B_M}}\left(n_+ + n_- > \kappa n, n_+ - n_- \leq \frac{M}{2}, T_{B_M} = n\right)\right].$$

But since for all $y \in B_M$ and $e \in V$ we have $\omega^x_{B_M}(y,e) \geq \kappa = \frac{1}{4d}$ by the uniform ellipticity of $\mathbb{P}$ and, furthermore, by Proposition 3

$$\frac{\omega^x_{B_M}(y,e_1) - \omega^x_{B_M}(y,-e_1)}{\omega^x_{B_M}(y,e_1) + \omega^x_{B_M}(y,-e_1)} = \frac{\vec{d}_{B_M,x}(y) \cdot e_1}{\omega^x_{B_M}(y,e_1) + \omega^x_{B_M}(y,-e_1)} \geq \frac{\lambda - \frac{\epsilon^2}{d}}{2\kappa} \geq 2(d-1)\epsilon^2 > 0, \quad (24)$$

it follows by coupling with a suitable random walk (with i.i.d. steps) that for $n \geq 24M^3$ and $\epsilon < \frac{1}{\sqrt{2(d-1)}}$ (so as to guarantee that $\frac{1}{2} + (d-1)\epsilon^2 \in (0,1)$) we have

$$P_{x,\omega^x_{B_M}}(n_+ + n_- \leq \kappa n, T_{B_M} = n) \leq F(\kappa n; n, 2\kappa)$$

and

$$P_{x,\omega^x_{B_M}}\left(n_+ + n_- > \kappa n, n_+ - n_- \leq \frac{M}{2}, T_{B_M} = n\right) \\ \leq F\left(\frac{\kappa}{2}n + \sqrt[3]{n}; \kappa n, \frac{1}{2} + (d-1)\epsilon^2\right)$$

where $F(t; k, p)$ denotes the cumulative distribution function of a (k,p)-Binomial random variable evaluated at $t \in \mathbb{R}$. By using Chernoff's bound which states that for $t \leq np$

$$F(t; n, p) \leq \exp\left\{-\frac{1}{2p} \cdot \frac{(np-t)^2}{n}\right\}$$

we may now obtain the desired polynomial decay for this term, provided that M is large enough (as a matter of fact, we get an exponential decay in M, with a rate which depends on κ and ϵ).

To deal with second term in the right-hand side of (22), we define the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}_0}$ by setting

$$\begin{cases} \tau_0 := 0 \\ \tau_{n+1} := \inf\{n > \tau_n : (X_n - X_{n-1}) \cdot e_1 \neq 0\} \wedge T_{B_M} \end{cases}$$

and consider the auxiliary chain $Y = (Y_k)_{k\in\mathbb{N}_0}$ given by

$$Y_k := X_{\tau_k} \cdot e_1$$

It follows from its definition and (24) that Y is a one-dimensional random walk with a probability of jumping right from any position which is at least $\frac{1}{2} + (d-1)\epsilon^2$. Now recall that, for any random walk on $\mathbb{Z}$ starting from 0 with nearest-neighbor jumps which has a probability $p \neq \frac{1}{2}$ of jumping right from any position, given $a, b \in \mathbb{N}$ the probability $E(-a, b, p)$ of this walk exiting the interval $[-a, b]$ through $-a$ is exactly

$$E(-a, b, p) = \frac{1 - \left(\frac{1-p}{p}\right)^b}{1 - \left(\frac{1-p}{p}\right)^{a+b}} \cdot \left(\frac{1-p}{p}\right)^a.$$

Thus, we obtain that

$$P_{x,\omega^x_{B_M}}\left(X_{T_{B_M}} \in \partial_- B_M\right) \leq E\left(-M, \frac{M}{2}, \frac{1}{2} + (d-1)\epsilon^2\right) \\ = \frac{1 - \left(\frac{1-2(d-1)\epsilon^2}{1+2(d-1)\epsilon^2}\right)^{\frac{M}{2}}}{1 - \left(\frac{1-2(d-1)\epsilon^2}{1+2(d-1)\epsilon^2}\right)^{\frac{3}{2}M}} \cdot \left(\frac{1-2(d-1)\epsilon^2}{1+2(d-1)\epsilon^2}\right)^M$$

from where the desired polynomial decay (in fact, exponential) for this second term now follows. This concludes the proof. □

3.4 Finishing the Proof of Theorem 2

We now show how to conclude the proof of Theorem 2.

First, we observe that by Proposition 4 the polynomial condition $(P)_K$ holds for all $K \geq 15d + 5$ if $(\mathrm{QLD})_\epsilon$ is satisfied for ϵ sufficiently small, so that by Proposition 1 we have in this case that

$$\lim_{n\to+\infty} \frac{E_0(T_n)}{n} = \frac{1}{\vec{v}\cdot e_1}.$$

On the other hand, it follows from Proposition 3 that if for each $n \in \mathbb{N}$ we define the hyperplane

$$B_n := \{x \in \mathbb{Z}^d : x \cdot e_1 \leq n\},$$

then $P_{0,B_n^0}(T_{B_n} < +\infty) = 1$. Indeed, Proposition 2 yields that $\inf_{y\in B_n} \vec{d}_{B_n,0}(y) \cdot e_1 > \frac{1}{d}\epsilon^2 > 0$ which, for example by suitably coupling Kalikow's walk with a one-dimensional walk with i.i.d. steps and a drift $\frac{1}{d}\epsilon^2$ to the right, yields our claim. Hence, by Corollary 3 we obtain that

$$\lim_{n\to+\infty} \frac{E_{0,\omega^0_{B_n}}(T_n)}{n} = \frac{1}{\vec{v}\cdot e_1}. \tag{25}$$

Now, noting that for each $n \in \mathbb{N}$ the stopped process $M^{(n)} = (M_k^{(n)})_{k\in\mathbb{N}_0}$ defined as

$$M_k^{(n)} := X_{k\wedge T_n} - \sum_{j=1}^{k\wedge T_n} \vec{d}_{S_n,0}(X_{j-1})$$

is a mean-zero martingale under $P_{0,\omega^0_{S_n}}$, by Proposition 3 and the optional stopping theorem for $M^{(n)}$, we conclude that

$$n = E_{0,\omega^0_{S_n}}(X_{T_n}\cdot e_1) \leq \left(\lambda + \frac{\epsilon^2}{d}\right) E_{0,\omega^0_{S_n}}(T_n)$$

and analogously that

$$n = E_{0,\omega^0_{S_n}}(X_{T_n}\cdot e_1) \geq \left(\lambda - \frac{\epsilon^2}{d}\right) E_{0,\omega^0_{S_n}}(T_n).$$

Together with (25), these inequalities imply that

$$|\vec{v}\cdot e_1 - \lambda| \leq \frac{\epsilon^2}{d}$$

from where the result now follows.

4 Proof of Theorem 1 (Part I): Seed Estimates

We now turn to the proof of Theorem 1. Let us observe that, having already proven Theorem 2 which is a stronger statement for dimension $d = 2$, it suffices to show Theorem 1 only for $d \geq 3$. The main element in the proof of this result will be a renormalization argument, to be carried out in Sect. 5. In this section, we establish two important estimates which will serve as the input for this renormalization scheme. More precisely, this section is devoted to proving the following result. As noted earlier, we assume throughout that $d \geq 3$.

Theorem 3. *If $d \geq 3$ then for any $\eta \in (0,1)$ and $\delta \in (0,\eta)$ there exist $c_2, c_3, c_4 > 0$ and $\theta_0 \in (0,1)$ depending only on d, η and δ such that if:*

i. The constant θ from (9) is chosen smaller than θ_0,
ii. $(LD)_{\eta,\epsilon}$ is satisfied for ϵ sufficiently small depending only on d, η, δ and θ,

then

$$\sup_{x \in B^*_{NL}} P_x\left(X_{T_{B_{NL}}} \notin \partial_+ B_{NL}\right) \leq e^{-c_2 \epsilon^{-1}} \tag{26}$$

and

$$\mathbb{P}\left(\left\{\omega \in \Omega : \sup_{x \in \partial_- B^*_{NL}} \left| \frac{E_{x,\omega}\left(T_{B_{NL}}\right)}{NL/2} - \frac{1}{\lambda} \right| > \frac{c_4}{\lambda^2} \epsilon^{\alpha(d)-\delta} \right\}\right) \leq e^{-c_3 \epsilon^{-\delta}}. \tag{27}$$

We divide the proof of this result into a number of steps, each occupying a separate subsection.

4.1 (LD) Implies $(P)_K$

The first step in the proof will be to show (26). Notice that, in particular, (26) tells us that for any $K \geq 1$ the polynomial condition $(P)_K$ is satisfied if ϵ is sufficiently small. This fact will also be important later on. The general strategy to prove (26) is basically to exploit the estimates obtained in [16] to establish the validity of the so-called effective criterion. First, let us consider the box B given by

$$B := (-NL, NL) \times \left(-\frac{1}{4}(NL)^3, \frac{1}{4}(NL)^3\right)^{d-1} \tag{28}$$

and define all its different boundaries $\partial_i B$ for $i = +, -, l$ by analogy with Sect. 2.1. Observe that if for $x \in B^*_{NL}$ we consider $B(x) := B + x$, i.e the translate of B centered at x, then by choice of B we have that for any $\omega \in \Omega$

$$P_{x,\omega}\left(X_{T_{B_{NL}}} \notin \partial_+ B_{NL}\right) \leq P_{x,\omega}\left(X_{T_{B(x)}} \notin \partial_+ B(x)\right). \tag{29}$$

Thus, from the translation invariance of $\mathbb{P}$ it follows that to obtain (26) it will suffice to show that

$$P_0\left(X_{T_B} \notin \partial_+ B\right) \leq e^{-c_2 \epsilon^{-1}} \tag{30}$$

for some constant $c_2 = c_2(d,\eta) > 0$ if ϵ is sufficiently small. To do this, we will exploit the results developed in [16, Section 4]. Indeed, if for $\omega \in \Omega$ we define

$$q_B(\omega) := P_{0,\omega}\left(X_{T_B} \notin \partial_+ B\right) \quad \text{and} \quad \rho_B(\omega) := \frac{q_B(\omega)}{1-q_B(\omega)}$$

then observe that

$$P_0(X_{T_B} \notin \partial_+ B) = \mathbb{E}(q_B) \le \mathbb{E}(\sqrt{q_B}) \le \mathbb{E}(\sqrt{\rho_B}).$$

But the results from [16, Section 4] show that there exists a constant $c > 0$ and $\theta_0(d), \tilde{\epsilon}_0(d,\eta) > 0$ such that if $(\mathrm{LD})_{\eta,\epsilon}$ is satisfied for $\epsilon \in (0,\tilde{\epsilon}_0)$ and L from (9) is given by $L = 2[\theta\epsilon^{-1}]$ with $\theta \in (0,\theta_0)$ then

$$\mathbb{E}(\sqrt{\rho_B}) \le \frac{80}{c\epsilon^{\alpha(d)-\eta}L}\exp\left(-\frac{c}{20}\epsilon^{\alpha(d)-\eta}NL\right) + 2d\exp\left(NL\left[\frac{\log 4d}{2} - 50\frac{\log 4d}{\log 2}\left(\frac{3}{4}-\frac{7}{100}\right)^2\right]\right).$$

However, since for $\epsilon < \frac{\theta}{2}$ we have that

$$\epsilon^{\alpha(d)-\eta}NL \ge 16\epsilon^{\alpha(d)-\eta}(\theta\epsilon^{-1}-1)^4 \ge \theta^4\epsilon^{\alpha(d)-\eta-4} \ge \theta^4\epsilon^{-(1+\eta)}$$

together with

$$\epsilon^{\alpha(d)-\eta}L \ge \theta\epsilon^{\alpha(d)-\eta-1} \ge \theta\epsilon$$

and

$$\frac{\log 4d}{2} - 50\frac{\log 4d}{\log 2}\left(\frac{3}{4}-\frac{7}{100}\right)^2 < 0,$$

it is straightforward to check that if $\epsilon < \epsilon_0(d,\eta,\theta)$ then (30) is satisfied.

4.2 Exit Measure from Small Slabs

The second step is to obtain a control on the probability that the random walk exits the slab U "to the right". For this we will follow to some extent Section 3 of Sznitman [16]. We begin by giving two estimates: first, a bound for the (annealed) expectation of $G_U(\vec{d}(0)\cdot e_1)$ in terms of the annealed expectation of T_U, and then a bound in $\mathbb{P}$-probability for the fluctuations of $E_{0,\omega}(T_U)$ around its mean $E_0(T_U)$.

Proposition 5. *If $d \ge 3$ and $\epsilon \in (0, \frac{1}{8d})$ then there exist positive constants c_5, c_6, c_7 and c_8 such that if $\epsilon, \theta \in (0,1)$ are such that $L \ge 2$ and $\epsilon L \le c_5$ then one has*

$$\left|\mathbb{E}\left(G_U[\vec{d}\cdot e_1](0)\right) - \lambda E_0(T_U)\right| \le c_6\epsilon\log L, \tag{31}$$

and also

$$c_7L^2 \le E_0\left(T_U\right) \le c_8L^2. \tag{32}$$

Furthermore, given any $\eta \in (0,1)$ there exists $\epsilon_0 = \epsilon_0(d,\eta,\theta) \in (0,1)$ such that if (LD)$_{\eta,\epsilon}$ is satisfied for $\epsilon \in (0,\epsilon_0)$ then

$$\mathbb{E}\left(G_U[\vec{d}\cdot e_1](0)\right) \geq \frac{2}{5}d\lambda L^2. \tag{33}$$

Proof. A careful inspection of the proof of [16, Proposition 3.1] yields the estimates (31) and (33). On the other hand, inequalities (2.28) and (3.6) of [16] give us the bounds in (32). □

The next estimate we shall need is essentially contained in Proposition 3.2 of [16], which gives a control on the difference between the random variable $G_U(\vec{d}(0)\cdot e_1)$ and its expectation for $d \geq 3$. We include it here for completeness and refer to [16] for a proof.

Proposition 6. *If $d \geq 3$ then there exist constants $c_9, c_{10} > 0$ such that if $\epsilon,\theta,\alpha \in (0,1)$ satisfy $L \geq 2$ and $\epsilon L < \frac{1}{2}\cdot\frac{1-\alpha}{2-\alpha}\cdot c_9$, one has for all $u \geq 0$ that*

$$\mathbb{P}\left[\left|G_U[\vec{d}(\cdot,\omega)\cdot e_1](0) - \mathbb{E}(G_U[\vec{d}\cdot e_1](0))\right| \geq u\right] \leq c_{10}\exp\left\{-\frac{u^2}{c_{\alpha,L}}\right\}, \tag{34}$$

where

$$c_{\alpha,L} := c_{11}\epsilon^2\sum_{y\in U} g_{0,U}(0,y)^{2/(2-\alpha)}$$

for some constant $c_{11} = c_{11}(d) > 0$ and

$$c_{\alpha,L} \leq \begin{cases} c_{1,2}\epsilon^2 L^{1+(2(1-\alpha)/(2-\alpha))} & \text{for } d=3 \\ c_{1,2}\epsilon^2 L^{(4(1-\alpha)/(2-\alpha))} & \text{for } d=4 \\ c_{1,2}\epsilon^2 & \text{for } d\geq 5 \text{ and } \alpha \geq \frac{4}{5} \end{cases}$$

for some $c_{1,2} = c_{1,2}(\alpha,d) > 0$.

Finally, we establish a control of the fluctuations of the quenched expectation $E_{0,\omega}(T_U)$ analogous to the one obtained in Proposition 6.

Proposition 7. *If $d \geq 3$ then for any $\alpha \in [0,1)$ and $\epsilon,\theta \in (0,1)$ with $L \geq 2$ and $\epsilon L < \frac{1}{2}\cdot\frac{1-\alpha}{2-\alpha}\cdot c_9$ where c_9 is the constant from Proposition 6, one has for all $u \geq 0$ that*

$$\mathbb{P}\left[|E_{0,\omega}(T_U) - E_0(T_U)| \geq u\right] \leq c_{12}\exp\left\{-\frac{u^2}{c'_{\alpha,L}}\right\} \tag{35}$$

for some $c_{12} = c_{12}(d) > 0$, where

$$c'_{\alpha,L} := c_{13}\sum_{y\in U} g_{0,U}(0,y)^{2/(2-\alpha)}$$

for some constant $c_{13} = c_{13}(d) > 0$ *and*

$$c'_{\alpha,L} \leq \begin{cases} c'_{1,2} L^{1+(2(1-\alpha)/(2-\alpha))} & \text{for } d = 3 \\ c'_{1,2} L^{(4(1-\alpha)/(2-\alpha))} & \text{for } d = 4 \\ c'_{1,2} & \text{for } d \geq 5 \text{ and } \alpha \geq \frac{4}{5} \end{cases}$$

for some $c'_{1,2} = c'_{1,2}(\alpha, d) > 0$.

Proof. We follow the proof of [16, Proposition 3.2], using the martingale method introduced there. Let us first enumerate the elements of U as $\{x_n : n \in \mathbb{N}\}$. Now define the filtration

$$\mathcal{G}_n := \begin{cases} \sigma(\omega(x_1), \ldots, \omega(x_n)) & \text{if } n \geq 1 \\ \{\emptyset, \Omega\} & \text{if } n = 0 \end{cases}$$

and also the bounded $\mathcal{G}_n$-martingale $(F_n)_{n\in\mathbb{N}_0}$ given for each $n \in \mathbb{N}_0$ by

$$F_n := \mathbb{E}\left(G_U[\mathbf{1}](0)|\mathcal{G}_n\right)$$

where $\mathbf{1}$ is the function constantly equal to 1, i.e. $\mathbf{1}(x) = 1$ for all $x \in \mathbb{Z}^d$. Observe that

$$G_U[\mathbf{1}](0, \omega) = \mathbb{E}_{0,\omega}(T_U)$$

by definition of G_U. Thus, if we prove that for all $n \in \mathbb{N}$

$$|F_n - F_{n-1}| \leq c_{14} g_{0,U}(0, x_n)^{\frac{1}{2-\alpha}} =: \gamma_n \tag{36}$$

for some $c_{14} = c_{14}(d) > 0$ then, since $F_0 = E_0(T_U)$ and $F_\infty = E_{0,\omega}(T_U)$, by using Azuma's inequality and the bound for $c_{\alpha,L}$ in Proposition 6 (see the proof of [16, Proposition 3.2] for further details) we obtain (35) at once. In order to prove (36), for each $n \in \mathbb{N}$ and all environments $\omega, \omega' \in \Omega_\epsilon$ coinciding at every x_i with $i \neq n$ define

$$\Gamma_n(\omega, \omega') := G_U[\mathbf{1}](0, \omega') - G_U[\mathbf{1}](0, \omega).$$

Since $F_n - F_{n-1}$ can be expressed as an integral of $\Gamma_n(\omega, \omega')$ with respect to ω and ω', it is enough to prove that $\Gamma_n(\omega, \omega')$ is bounded from above by the constant γ_n from (36). To do this, we introduce for $u \in [0, 1]$ the environment ω_u defined for each $i \in \mathbb{N}$ by

$$\omega_u(x_i) = (1 - u) \cdot \omega(x_i) + u \cdot \omega'(x_i).$$

If we set

$$H_{x_n} := \inf\{j \geq 0 : X_j = x_n\} \text{ and } \overline{H}_{x_n} := \inf\{j \geq 1 : X_j = x_n\}$$

then, by the strong Markov property for the stopping time H_{x_n}, a straightforward computation yields that

$$G_U[\mathbf{1}](0, \omega_u) = E_{0,\omega_u}\left(H_{x_n} \wedge (T_U - 1) + 1\right) + P_{0,\omega_u}(H_{x_n} < T_U) E_{x_n,\omega_u}(T_U). \tag{37}$$

Similarly, by the strong Markov property for the stopping time $\overline{H}_{x_n}$ we have

$$E_{x_n,\omega_u}(T_U) = E_{x_n,\omega_u}(\overline{H}_{x_n} \wedge (T_U - 1) + 1) + P_{x_n,\omega_u}(\overline{H}_{x_n} < T_U) E_{x_n,\omega_u}(T_U), \quad (38)$$

so that

$$\begin{aligned} G_U[\mathbf{1}](0,\omega_u) &= E_{0,\omega_u}(H_{x_n} \wedge (T_U - 1) + 1) \\ &\quad + \frac{P_{0,\omega_u}(H_{x_n} < T_U)}{P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U)} E_{x_n,\omega_u}(\overline{H}_{x_n} \wedge (T_U - 1) + 1). \end{aligned} \quad (39)$$

Notice that $P_{0,\omega_u}(H_{x_n} < T_U)$ and the first term in the right-hand side of (39) do not depend on u. Furthermore, by the Markov property for time $j = 1$, we have

$$P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U) = \sum_{e \in V} \omega_u(x_n, e) P_{x_n+e,\omega_u}(H_{x_n} > T_U)$$

and

$$E_{x_n,\omega_u}(\overline{H}_{x_n} \wedge (T_U - 1) + 1) = \sum_{e \in V} \omega_u(x_n, e)(1 + E_{x_n+e,\omega_u}(H_{x_n} \wedge (T_U - 1) + 1)),$$

so that differentiating $G_U[\mathbf{1}](0,\omega_u)$ with respect to u yields

$$\partial_u G_U[\mathbf{1}](0,\omega_u) = \frac{P_{0,\omega_u}(H_{x_n} < T_U)}{P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U)} \sum_{e \in V} (\omega'(x_n, e) - \omega(x_n, e))(A_e - B_e)$$

where

$$A_e := 1 + E_{x_n+e,\omega_u}(H_{x_n} \wedge (T_U - 1) + 1)$$

and

$$B_e := \frac{P_{x_n+e,\omega_u}(H_{x_n} > T_U)}{P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U)} E_{x_n,\omega_u}(\overline{H}_{x_n} \wedge (T_U - 1) + 1).$$

Now, by a similar argument to the one used to obtain (37) and (38), we have that

$$\begin{aligned} E_{x_n+e,\omega_u}(H_{x_n} \wedge & \ (T_U - 1) + 1) \\ & = G_U[\mathbf{1}](x_n + e, \omega_u) - P_{x_n+e,\omega_u}(\overline{H}_{x_n} < T_U) G_U[\mathbf{1}](x_n, \omega_u) \end{aligned}$$

and

$$P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U) G_U[\mathbf{1}](x_n, \omega_u) = E_{x_n,\omega_u}(\overline{H}_{x_n} \wedge (T_U - 1) + 1),$$

from which we conclude that

$$A_e - B_e = 1 + \nabla_e G_U[\mathbf{1}](x_n, \omega_u),$$

where for any bounded $f : \mathbb{Z}^d \to \mathbb{R}$, $x \in \mathbb{Z}^d$ and $\omega \in \Omega$ we write

$$\nabla_e G_U[f](x, \omega_u) := G_U[f](x + e, \omega_u) - G_U[f](x, \omega_u).$$

Furthermore, by the proof of [16, Proposition 3.2] we have

$$\frac{P_{0,\omega_u}(H_{x_n} < T_U)}{P_{x_n,\omega_u}(\overline{H}_{x_n} > T_U)} = g_U(0, x_n, \omega_u) \leq c' g_{0,U}(0, x_n)^{\frac{1}{2-\alpha}}$$

and

$$|\nabla_e G_U[\mathbf{1}](x_n, \omega_u)| \leq c'' L$$

where L is the quantifier from (9). Since for $\omega', \omega \in \Omega_\epsilon$ we have

$$\sum_{e \in V} |\omega'(x_n, e) - \omega(x_n, e)| \leq \epsilon,$$

we conclude that for all $u \in [0, 1]$

$$|\partial_u G_U[\mathbf{1}](0, \omega_u)| \leq c' g_{0,U}(0, x_n)^{\frac{1}{2-\alpha}} \epsilon(1 + c'' L) \leq c_{14} g_{0,U}(0, x_n)^{\frac{1}{2-\alpha}}$$

since $\epsilon \leq 1$ and $\epsilon L < \frac{1}{2} \cdot \frac{1-\alpha}{2-\alpha} \cdot c_9 \leq \frac{c_9}{4}$ by hypothesis. From this estimate (36) immediately follows, which concludes the proof. □

4.3 Exit Measures from Small Slabs Within a Seed Box

The next step in the proof is to show that, on average, the random walk starting from any $z \in B_{NL}$ sufficiently far away from $\partial_l B_{NL}$ moves at least $\pm L$ steps in direction e_1 before reaching $\partial_l B_{NL}$. The precise estimate we will need is contained in the following proposition.

Proposition 8. *There exist three positive constants $c_{15}, c_{16} = c_{16}(d)$ and $\epsilon_0 = \epsilon_0(d)$ verifying that if $\epsilon, \theta \in (0, 1)$ are such that $L \geq 2$, $\epsilon L \leq c_{16}$ and $\epsilon \in (0, \epsilon_0)$, then one has that*

$$\sup_{\omega \in \Omega_\epsilon} \left| E_{z,\omega}(T_{U_L(z)}) - E_{z,\omega}(T_{U_L(z)} \wedge T_{\partial_l B_{NL}}) \right| \leq e^{-c_{15} L} \tag{40}$$

for all $z \in B'_{NL}$, where

$$B'_{NL} := \left\{ z \in B_{NL} : \sup_{2 \leq i \leq d} |z \cdot e_i| \leq 25(NL)^3 - N \right\}.$$

To prove Proposition 8 we will require the following two lemmas related to the exit time T_U. The first lemma gives a uniform bound on the second moment of T_U.

Lemma 4. *There exist constants $c_{17}, c_{18} = c_{18}(d) > 0$ such that if $\epsilon, \theta \in (0, 1)$ are taken such that $L \geq 2$ and $\epsilon L \leq c_{18}$ then one has that*

$$\sup_{z \in \mathbb{Z}^d,\, \omega \in \Omega_\epsilon} E_{z,\omega}(T^2_{U_L(z)}) \leq c_{17} L^4.$$

Proof. Let us fix $x \in \mathbb{Z}^d$ and write $U_z := U_L(z)$ in the sequel for simplicity. Notice that

$$\begin{aligned} E_{z,\omega}(T_{U_z}^2) &= E_{z,\omega}\left(\left(\sum_{x\in U_z}\sum_{j=0}^{\infty}\mathbb{1}_x(X_j)\mathbb{1}(j<T_{U_z})\right)^2\right) \\ &= 2\sum_{x\in U_z}\sum_{y\in U_z}\sum_{j=0}^{\infty}\sum_{k>j}^{\infty}E_{z,\omega}\left(\mathbb{1}_x(X_j)\mathbb{1}_y(X_k)\mathbb{1}(k<T_{U_z})\mathbb{1}(j<T_{U_z})\right) \\ &\quad + E_{z,\omega}\left(T_{U_z}\right). \end{aligned} \tag{41}$$

Now, by the Markov property, for each $j<k$ we have that

$$\begin{aligned} &E_{z,\omega}\left(\mathbb{1}_x(X_j)\mathbb{1}_y(X_k)\mathbb{1}(k<T_{U_z})\mathbb{1}(j<T_{U_z})\right) \\ &\qquad = E_{z,\omega}\left(\mathbb{1}_x(X_j)\mathbb{1}(j<T_{U_z})\right)E_{x,\omega}\left(\mathbb{1}_y(X_i)\mathbb{1}(i<T_{U_z})\right), \end{aligned}$$

where $i := k-j$. Substituting this back into (41), we see that

$$\begin{aligned} &E_{z,\omega}(T_{U_z}^2) \\ &\le 2\sum_{x\in U_z}\sum_{y\in U_z}\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}E_{z,\omega}\left(\mathbb{1}_x(X_j)\mathbb{1}(j<T_{U_z})\right)E_{x,\omega}\left(\mathbb{1}_y(X_i)\mathbb{1}(i<T_{U_z})\right) \\ &\quad + E_{z,\omega}(T_{U_z}) \\ &\le 2\sum_{x\in U_z}\sum_{j=0}^{\infty}E_{z,\omega}\left(\mathbb{1}_x(X_j)\mathbb{1}(j<T_{U_z})\right)E_{x,\omega}\left(T_{U_z}\right) + E_{z,\omega}(T_{U_z}) \\ &= 2E_{z,\omega}(T_{U_z})\left(\sup_{y\in\mathbb{Z}^d}E_{y,\omega}(T_{U_z})\right) + E_{z,\omega}(T_{U_z}) \\ &\le 2c^2L^4 + cL^2 \end{aligned}$$

for some constant $c>0$, where for the last line we have used inequality (2.28) of Sznitman in [16], which says that

$$\sup_{z,y\in\mathbb{Z}^d,\omega\in\Omega_\epsilon}E_{y,\omega}(T_{U_z}) \le cL^2$$

whenever $L\ge 2$ and $\epsilon L\le c_{18}(d)$. From this the result immediately follows. □

Our second auxiliary lemma states that, with overwhelming probability, the random walk starting from any $x\in B_{NL}$ far enough from $\partial_l B_{NL}$ is very likely to move at least $\pm L$ steps in direction e_1 before reaching $\partial_l B_{NL}$.

Lemma 5. *There exist constants $c_{19}, c_{20}>0$ such that if $\epsilon,\theta\in(0,1)$ satisfy $L\ge 2$ and $\epsilon L\le c_{20}$ then for any $a\in(0,25(NL)^3)$ and $z\in B_{NL}$ verifying $\sup_{2\le i\le d}|z\cdot e_i|\le a$ one has*

$$\sup_{\omega\in\Omega_\epsilon}P_{z,\omega}(T_{\partial_l B_{NL}}\le T_{U_L(z)}) \le 2e^{-c_{19}\frac{25(NL)^3-a}{L^2}}.$$

Proof. Note that for any $z \in B_{NL}$ with $\sup_{2\le i\le d}|z\cdot e_i| \le a$ one has that

$$P_{z,\omega}(T_{\partial_l B_{NL}} \le T_{U_L(z)}) \le P_{z,\omega}(T_{U_L(z)} \ge 25(NL)^3 - a).$$

Furthermore, by Proposition 2.2 in [16], there exist constants $c_{19}, c_{20} > 0$ such that if $\epsilon L \le c_{20}$ then for any $z \in \mathbb{Z}^d$

$$\sup_{x\in\mathbb{Z}^d,\omega\in\Omega_\epsilon} E_{x,\omega}\left(e^{\frac{c_{19}}{L^2}T_{U_L(z)}}\right) \le 2.$$

Hence, by the exponential Tchebychev inequality we conclude that

$$P_{z,\omega}(T_{\partial_l B_{NL}} \le T_{U_L(z)}) \le e^{-c_{19}\frac{25(NL)^3-a}{L^2}} E_{z,\omega}\left(e^{\frac{c_{19}}{L^2}T_{U_L(z)}}\right) \le 2e^{-c_{19}\frac{25(NL)^3-a}{L^2}}.$$

□

We are now ready to prove Proposition 8. Indeed, notice that

$$\begin{aligned} E_{z,\omega}(T_{U_L(z)}) &= E_{z,\omega}(T_{U_L(z)}\mathbb{1}(T_{U_L(z)} < T_{\partial_l B_{NL}})) \\ &\quad + E_{z,\omega}(T_{U_L(z)}\mathbb{1}(T_{U_L(z)} \ge T_{\partial_l B_{NL}})) \\ &= E_{z,\omega}((T_{U_L(z)} \wedge T_{\partial_l B_{NL}})\mathbb{1}(T_{U_L(z)} < T_{\partial_l B_{NL}})) \\ &\quad + E_{z,\omega}(T_{U_L(z)}\mathbb{1}(T_{U_L(z)} \ge T_{\partial_l B_{NL}})) \\ &\le E_{z,\omega}(T_{U_L(z)} \wedge T_{\partial_l B_{NL}}) + E_{z,\omega}(T_{U_L(z)}\mathbb{1}(T_{U_L(z)} \ge T_{\partial_l B_{NL}})). \end{aligned}$$

Hence, since $\sup_{2\le i\le d}|z\cdot e_i| \le 25(NL)^3 - N$ for any $z \in B'_{NL}$, by the Cauchy–Schwarz inequality and Lemmas 4 and 5 it follows that

$$\begin{aligned} |E_{z,\omega}(T_{U_L(z)}) - E_{z,\omega}(T_{U_L(z)} \wedge T_{\partial_l B_{NL}})| &\le \sqrt{E_{z,\omega}(T^2_{U_L(z)})P_{z,\omega}(T_{U_L(z)} \ge T_{\partial_l B_{NL}})} \\ &\le \sqrt{2c_{17}}L^2 e^{-\frac{c_{19}}{2}L}. \end{aligned}$$

From this estimate, taking $c_{16} := \min\{c_{18}, c_{20}\}$ and ϵ sufficiently small yields (40).

4.4 Renormalization Scheme to Obtain a Seed Estimate

Our next step is to derive estimates on the time spent by the random walk on slabs of size NL.

Let us fix $\omega \in \Omega$ and define two sequences $W = (W_k)_{k\in\mathbb{N}_0}$ and $V = (V_k)_{k\in\mathbb{N}_0}$ of stopping times, by setting $W_0 = 0$ and then for each $k \in \mathbb{N}_0$

$$W_{k+1} := \inf\{n > W_k : |(X_n - X_{W_k})\cdot e_1| \ge L\} \quad \text{and} \quad V_k := W_k \wedge T_{B_{NL}}.$$

Now, consider the random walks $Y = (Y_k)_{k\in\mathbb{N}_0}$ and $Z = (Z_k)_{k\in\mathbb{N}_0}$ defined for $k \in \mathbb{N}_0$ by the formula

$$Y_k := X_{W_k} \tag{42}$$

and

$$Z_k := X_{V_k}. \tag{43}$$

Notice that at each step, the random walk Y jumps from x towards some y with $(y - x) \cdot e_1 \geq L$, i.e. it exits the slab $U_L(x)$ “to the right”, with probability $\hat{p}(x, \omega)$, where

$$\hat{p}(x, \omega) := P_{x,\omega}\left(T_{x \cdot e_1 + L} < T_{x \cdot e_1 - L}\right).$$

Observe also that $\hat{p}$ verifies the relation

$$\hat{p}(x, \omega) = \frac{1}{2} + \frac{1}{2L} G_{U_L(x)}[\vec{d} \cdot e_1](x, \omega) \tag{44}$$

which follows from an application of the optional sampling theorem to the P_ω-martingale $(M_{k,\omega})_{k \in \mathbb{N}}$ given by

$$M_{k,\omega} = X_k - \sum_{j=0}^{k-1} \vec{d}(X_j, \omega).$$

Now, for each $p \in [0, 1]$ let us couple $Y^{(e_1)} := (Y_k \cdot e_1)_{k \in \mathbb{N}_0}$ with a random walk $y^{(p)} := (y_k^{(p)})_{k \in \mathbb{N}_0}$ on $\mathbb{Z}$, which starts at 0 and in each step jumps one unit to the right with probability p and one to the left with probability $1 - p$, in such a way that both $Y^{(e_1)}$ and $y^{(p)}$ jump together in the rightward direction with the largest possible probability, i.e. for any $k \geq 0$, $x \in \mathbb{Z}^d$ and $m \in \mathbb{Z}$

$$P_\omega(Y_{k+1}^{(e_1)} \geq x \cdot e_1 + L,\ y_{k+1}^{(p)} = m + 1 | Y_k = x,\ y_k^{(p)} = m) = \min\{\hat{p}(x, \omega), p\}.$$

The explicit construction of such a coupling is straightforward, so we omit the details. Call this the *coupling to the right* of $Y^{(e_1)}$ and $y^{(p)}$. Now, consider the random walks $y^- := y^{(p_-)}$ and $y^+ := y^{(p_+)}$, where

$$p_- := \left(\frac{1}{2} + \frac{\mathbb{E}(G_U[\vec{d} \cdot e_1](0)) - \epsilon^{\alpha(d)-2-\delta}}{2L}\right) \vee 0 \tag{45}$$

and

$$p_+ := \left(\frac{1}{2} + \frac{\mathbb{E}(G_U[\vec{d} \cdot e_1](0)) + \epsilon^{\alpha(d)-2-\delta}}{2L}\right) \wedge 1, \tag{46}$$

and assume that they are coupled with $Y^{(e_1)}$ to the right. Let us call E_0^- and E_0^+ the expectations defined by their respective laws. Next, for each $M \in \mathbb{N}$ define the stopping times $\mathcal{T}_M^Y$, S_M^+ and S_M^- given by

$$\mathcal{T}_M^Y := \inf\{k \geq 0 : Y_k \cdot e_1 \geq LM\} \quad \text{and} \quad \mathcal{S}_M^\pm := \inf\left\{k \geq 0 : y_k^\pm \geq M\right\}.$$

Finally, if for each subset $A \subset \mathbb{Z}^d$ we define the stopping time

$$\mathcal{T}_A^Z := \inf\{k \geq 0 : Z_k \notin A\},$$

we have the following control on the expectation of $\mathcal{T}_{B_{NL}}^Z$.

Proposition 9. *If $d \geq 3$ then for any given $\eta \in (0,1)$ and $\delta \in (0,\eta)$ there exist $c_{21}, c_{22} > 0$ and $\theta_0 \in (0,1)$ depending only on d, η and δ such that if:*

i. The constant θ from (9) is chosen smaller than θ_0,
ii. $(LD)_{\eta,\epsilon}$ is satisfied for ϵ sufficiently small depending only on d, η, δ and θ,

then for any $z \in \partial_- B^*_{NL}$ we have

$$\mathbb{P}\Bigg(\Bigg\{\omega \in \Omega : \frac{N/2}{2p_+ - 1} - e^{-c_{22}\epsilon^{-1}} \leq E_{z,\omega}\left(\mathcal{T}^Z_{B'_{NL}}\right) \leq E_{z,\omega}\left(\mathcal{T}^Z_{B_{NL}}\right) \leq \frac{N/2}{2p_- - 1}\Bigg\}\Bigg) \geq 1 - e^{-c_{21}\epsilon^{-\delta}}.$$

Proof. Define the event

$$\mathcal{B} := \bigcap_{x \in B_{NL}} \left\{\omega \in \Omega : \left| G_U[\vec{d}\cdot e_1](x,\omega) - \mathbb{E}(G_U[\vec{d}\cdot e_1](x))\right| \leq \epsilon^{\alpha(d)-2-\delta}\right\}. \quad (47)$$

Let us observe that for any $\omega \in \mathcal{B}$ we have $p_- \leq \hat{p}(x,\omega)$ for all $x \in B_{NL}$. In particular, since $Y^{(e_1)}$ is coupled to the right with y^-, if $Y^{(e_1)}_0 = \frac{NL}{2}$ and $y^-_0 = 0$ then for any $\omega \in \mathcal{B}$ we have

$$Y^{(e_1)}_k \geq L y^-_k + \frac{NL}{2}$$

for all $0 \leq k \leq \mathcal{T}^Z_{B_{NL}}$ so that, in particular, for any $\omega \in \mathcal{B}$

$$\mathcal{T}^Z_{B_{NL}} \leq S^-_{\frac{N}{2}}$$

and thus

$$E_{z,\omega}\left(\mathcal{T}^Z_{B_{NL}}\right) \leq E^-_0(S^-_{\frac{N}{2}}).$$

Similarly, since $Y^{(e_1)}$ is coupled to the right with y^+ and $\hat{p}(x,\omega) \leq p_+$ for all $x \in B_{NL}$ when $\omega \in \mathcal{B}$, if $y^+_0 = 0$ then for any $\omega \in \mathcal{B}$ we have

$$Y^{(e_1)}_k \leq L y^+_k + \frac{NL}{2}$$

for all $0 \leq k \leq \mathcal{T}^Z_{B_{NL}}$, so that for any such ω on the event $\{\mathcal{T}^Y_{NL} = \mathcal{T}^Z_{B'_{NL}}\}$ we have

$$\mathcal{T}^Z_{B'_{NL}} \geq S^+_{\frac{N}{2}}.$$

Therefore, we see that for each $z \in \partial_- B^*_{NL}$

$$\begin{aligned} E_{z,\omega}\left(\mathcal{T}^Z_{B'_{NL}}\right) &= E_{z,\omega}\left(\mathcal{T}^Z_{B'_{NL}} \mathbb{1}(\mathcal{T}^Z_{B'_{NL}} < \mathcal{T}^Y_{NL})\right) + E_{z,\omega}\left(\mathcal{T}^Z_{B'_{NL}} \mathbb{1}(\mathcal{T}^Y_{NL} = \mathcal{T}^Z_{B'_{NL}})\right) \\ &\geq E_{z,\omega}\left(S^+_{\frac{N}{2}} \mathbb{1}(\mathcal{T}^Y_{NL} = \mathcal{T}^Z_{B'_{NL}})\right) \\ &= E^+_0\left(S^+_{\frac{N}{2}}\right) - E_{z,\omega}\left(S^+_{\frac{N}{2}} \mathbb{1}(\mathcal{T}^Z_{B'_{NL}} < \mathcal{T}^Y_{NL})\right). \end{aligned} \quad (48)$$

Now, by the Cauchy–Schwarz inequality we have that

$$E_{z,\omega}\left(S^+_{\frac{N}{2}}\mathbb{1}(\mathcal{T}^Z_{B'_{NL}}<\mathcal{T}^Y_{NL})\right)\leq\sqrt{E^+_0\left(\left(S^+_{\frac{N}{2}}\right)^2\right)P_{z,\omega}\left(X_{T_{B'_{NL}}}\notin\partial_+B'_{NL}\right)}$$
$$\leq\sqrt{E^+_0\left(\left(S^+_{\frac{N}{2}}\right)^2\right)P_{z,\omega}\left(X_{T_{B(z)}}\notin\partial_+B(z)\right)}\quad(49)$$

where $B(z):=B+z$ for B as defined in (28) and, to obtain the last inequality, we have repeated the same argument used to derive (29) but for B'_{NL} instead of B_{NL} (which still goes through if $L\geq 2$). On the other hand, using the fact that the sequences $M^{\pm}=(M^{\pm}_n)_{n\in\mathbb{N}_0}$ and $N^{\pm}=(N^{\pm}_n)_{n\in\mathbb{N}_0}$ given for each $n\in\mathbb{N}_0$ by

$$M^{\pm}_n=y^{\pm}_n-n(2p^{\pm}-1)$$

and

$$N^{\pm}_n=\left(y^{\pm}_n-n(2p^{\pm}-1)\right)^2-n(1-(2p^{\pm}-1)^2)$$

are all martingales with respect to the natural filtration generated by their associated random walks, and also that by Proposition 5 if ϵ is sufficiently small (depending on d,θ,η and δ)

$$2p^{\pm}-1=\frac{1}{L}(\mathbb{E}(G_U[\vec{d}\cdot e_1](0))\pm\epsilon^{\alpha(d)-\eta/2})>0$$

since $(\mathrm{LD})_{\eta,\epsilon}$ is satisfied and $\delta<\eta$, we conclude that

$$E^{\pm}_0(S^{\pm}_{\frac{N}{2}})=\frac{N/2}{2p^{\pm}-1},$$

and

$$E^+_0\left(\left(S^+_N\right)^2\right)=\frac{(N/2)^2}{(2p^+-1)^2}+\frac{(N/2)}{2p^+-1}(1-(2p^+-1)^2)\leq C_+N^2$$

if $\epsilon\in(0,1)$, where $C_+>0$ is a constant depending on p^+. Inserting these bounds in (48) and (49), we conclude that for $\omega\in\mathcal{B}$ one has

$$\frac{N/2}{2p^+-1}-\sqrt{C_+N^2P_{z,\omega}\left(X_{T_{B(z)}}\notin\partial_+B(z)\right)}\leq E_{z,\omega}\left(\mathcal{T}^Z_{B'_{NL}}\right)$$
$$\leq E_{z,\omega}\left(\mathcal{T}^Z_{B_{NL}}\right)\leq\frac{N/2}{2p^--1}.\quad(50)$$

But, by the proof of (26) in Sect. 4.1 and Markov's inequality, we have that

$$\mathbb{P}\left(P_{z,\omega}\left(X_{T_{B(z)}}\notin\partial_+B(z)\right)\geq e^{-\frac{1}{2}c_2\epsilon^{-1}}\right)\leq e^{\frac{1}{2}c_2\epsilon^{-1}}P_z\left(X_{T_{B(z)}}\notin\partial_+B(z)\right)$$
$$\leq\exp\left(-\frac{1}{2}c_2\epsilon^{-1}\right),\quad(51)$$

where $c_2 = c_2(d, \eta) > 0$ is the constant from (26). Furthermore, Proposition 6 implies that θ from (9) can be chosen so that for any ϵ sufficiently small (depending on d, δ and θ)

$$\mathbb{P}(\mathcal{B}^c) \leq C(d)(NL)^{3(d-1)+1} \exp\left(-c\epsilon^{-\delta}\right) \tag{52}$$

for some constants $C(d), c > 0$. Combining the estimates (51) and (52) with the inequalities in (50), we conclude the proof. □

4.5 Proof of (27)

We conclude this section by giving the proof of (27). The proof has two steps: first, we express the expectation $E_{x,\omega}(T_{B_{NL}})$ for $x \in \partial_- B^*_{NL}$ in terms of the Green's function of Z and the quenched expectation of $T_{U_L} \wedge T_{B_{NL}}$, and then combine this with the estimates obtained in the previous subsections to conclude the result. The first step is contained in the next lemma.

Lemma 6. *If we define $\mathcal{Z} := \{z \in B_{NL} : z \cdot e_1 = kL \text{ for some } k \in \mathbb{Z}\}$ and the Green's function*

$$g_Z(x, y, \omega) := \sum_{i=0}^{\infty} \mathbb{E}_{x,\omega}(\mathbb{1}_{\{y\}}(Z_i)\mathbb{1}_{\{i<\mathcal{T}^Z_{B_{NL}}\}}),$$

where Z is the random walk in (43), then for any $x \in \partial_- B^*_{NL}$ we have that

$$E_{x,\omega}\left(T_{B_{NL}}\right) = \sum_{z\in\mathcal{Z}} g_Z(x, z, \omega) E_{z,\omega}\left(T_{U_L(z)} \wedge T_{B_{NL}}\right). \tag{53}$$

Proof. Note that

$$\begin{aligned}
E_{x,\omega}\left(T_{B_{NL}}\right) &= \sum_{y\in B_{NL}} E_{x,\omega}\left(\sum_{n=0}^{\infty} \mathbb{1}_{\{y\}}(X_n)\mathbb{1}_{\{n<T_{B_{NL}}\}}\right) \\
&= \sum_{y\in B_{NL}} \sum_{i=0}^{\infty} E_{x,\omega}\left(\sum_{n=W_i}^{W_i-1} \mathbb{1}_{\{y\}}(X_n)\mathbb{1}_{\{n<T_{B_{NL}}\}}\right) \\
&= \sum_{y\in B_{NL}} \sum_{i=0}^{\infty} E_{x,\omega}\Bigg(\mathbb{1}_{\{W_i<T_{B_{NL}}\}} E_{X_{W_i},\omega} \\
&\qquad \times \left(\sum_{n=0}^{W_1-1} \mathbb{1}_{\{y\}}(X_n)\mathbb{1}_{\{n<T_{B_{NL}}\}}\right)\Bigg) \\
&= \sum_{y\in B_{NL}} \sum_{i=0}^{\infty} \sum_{z\in B_{NL}} E_{z,\omega}\left(\sum_{n=0}^{W_1-1} \mathbb{1}_{\{y\}}(X_n)\mathbb{1}_{\{n<T_{B_{NL}}\}}\right) \\
&\qquad \times E_{x,\omega}\left(\mathbb{1}_{\{z\}}(Y_i)\mathbb{1}_{\{W_i<T_{B_{NL}}\}}\right) \\
&= \sum_{z\in\mathcal{Z}} g_Z(x, z, \omega) E_{z,\omega}\left(T_{U_L(z)} \wedge T_{B_{NL}}\right),
\end{aligned}$$

where in the third equality we have used the Markov property for X valid under the probability P_ω and, in the last one, that Y visits only sites in $\mathcal{Z}$ before the time $T_{B_{NL}}$. □

Now, to continue with the proof let us define the event

$$\mathcal{A}_2 := \bigcap_{z\in\mathcal{Z}} \left\{\omega \in \Omega : \left|E_{z,\omega}(T_{U_L(z)}) - E_0(T_{U_L})\right| \le \epsilon^{-\alpha^*(d)-\delta}\right\},$$

where

$$\alpha^*(d) := 3-\alpha(d) = \begin{cases} 0.5 & \text{if } d=3 \\ 0 & \text{if } d \ge 4. \end{cases} \tag{54}$$

By Lemma 6, Proposition 5 and (53) we have for any $x \in \partial_- B^*_{NL}$ and $\omega \in \mathcal{A}_2$ that

$$\begin{aligned} E_{x,\omega}(T_{B_{NL}}) &\le \sum_{z\in\mathcal{Z}} g_Z(x,z,\omega)E_{z,\omega}(T_{U_L(z)}) \\ &\le \sum_{z\in\mathcal{Z}} g_Z(x,z,\omega)(E_0(T_{U_L}) + \epsilon^{-\alpha^*(d)-\eta}) \\ &\le E_{x,\omega}(\mathcal{T}^Z_{B_{NL}})(E_0(T_{U_L}) + \epsilon^{-\alpha^*(d)-\delta}) \\ &\le E_{x,\omega}(\mathcal{T}^Z_{B_{NL}}) \left(\frac{1}{\lambda}\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) + \frac{c_6}{\lambda}\epsilon\log L + \epsilon^{-\alpha^*(d)-\delta}\right) \end{aligned}$$

if $\epsilon, \theta \in (0,1)$ are taken such that $L \ge 2$ and $\epsilon L \le c_5$. In a similar manner, since for every $z \in \mathcal{Z}$ we have $T_{U_L(z)} \wedge T_{B_{NL}} = T_{U_L(z)} \wedge T_{\partial_l B_{NL}}$, by using also Proposition 8 we obtain that

$$\begin{aligned} E_{x,\omega}(T_{B_{NL}}) &\ge \sum_{z\in\mathcal{Z}\cap B'_{NL}} g_Z(x,z,\omega)E_{z,\omega}\left(T_{U_L(z)} \wedge T_{B_{NL}}\right) \\ &\ge \sum_{z\in\mathcal{Z}\cap B'_{NL}} g_Z(x,z,\omega)(E_{z,\omega}(T_{U_L(z)}) - e^{-c_{15}L}) \\ &\ge \sum_{z\in\mathcal{Z}\cap B'_{NL}} g_Z(x,z,\omega)\left(E_0(T_{U_L}) - \epsilon^{-\alpha^*(d)-\delta} - e^{-c_{15}L}\right) \\ &\ge E_{x,\omega}(\mathcal{T}^Z_{B'_{NL}})\left(E_0(T_{U_L}) - \epsilon^{-\alpha^*(d)-\delta} - e^{-c_{15}L}\right) \\ &\ge E_{x,\omega}(\mathcal{T}^Z_{B'_{NL}}) \\ &\qquad \times \left(\frac{1}{\lambda}\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) - \frac{c_6}{\lambda}\epsilon\log L - \epsilon^{-\alpha^*(d)-\delta} - e^{-c_{15}L}\right) \end{aligned}$$

for any $\omega \in \mathcal{A}_2$ provided that $\epsilon, \theta \in (0,1)$ are taken such that $L \ge 2$, $\epsilon L \le c_{16}$ and $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is the one from Proposition 8. Next, consider the event

$$\begin{aligned} \mathcal{A}_3 := \Bigg\{\omega \in \Omega : \frac{N/2}{2p^+ - 1} - e^{-c_{22}\epsilon^{-1}} &\le E_{x,\omega}\left(\mathcal{T}^Z_{B'_{NL}}\right) \\ &\le E_{x,\omega}\left(\mathcal{T}^Z_{B_{NL}}\right) \le \frac{N/2}{2p_- - 1}\Bigg\}, \end{aligned}$$

where $p^{\pm}$ are those defined in (45) and (46), respectively. Since $2p^{\pm}-1>0$ by $(\mathrm{LD})_{\eta,\epsilon}$, we see that for $\omega \in \mathcal{A}_2 \cap \mathcal{A}_3$

$$\begin{aligned} E_{x,\omega}(T_{B_{NL}}) &\leq \frac{N/2}{2p^- - 1}\left(\frac{1}{\lambda}\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) + \frac{c_6}{\lambda}\epsilon \log L + \epsilon^{-\alpha^*(d)-\delta}\right) \\ &\leq \frac{NL/2}{\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) - \epsilon^{\alpha(d)-2-\delta}} \\ &\qquad \times \left(\frac{1}{\lambda}\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) + \frac{c_6}{\lambda}\epsilon \log L + \epsilon^{-\alpha^*(d)-\delta}\right) \\ &= \frac{NL}{2}\left(\frac{1}{\lambda}\left(1 + \frac{\epsilon^{\alpha(d)-2-\delta} + c_6\epsilon\log L + \lambda\epsilon^{-\alpha^*(d)-\delta}}{\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) - \epsilon^{\alpha(d)-2-\delta}}\right)\right). \end{aligned}$$

Furthermore, if ϵ is chosen sufficiently small (depending on η, δ and θ) so as to guarantee that $L \geq 2$ together with

$$\frac{1}{\theta^2}\cdot \epsilon^{\eta-\delta} < \frac{1}{5}d$$

then by Proposition 5 we have $\mathbb{E}(G_U[\vec{d}\cdot e_1](0)) - \epsilon^{\alpha(d)-2-\delta} \geq \frac{2}{5}d\lambda L^2 - \lambda\epsilon^{-2+\eta-\delta} \geq \frac{1}{5}\lambda L^2$, so that

$$\begin{aligned} \frac{E_{x,\omega}(T_{B_{NL}})}{NL/2} - \frac{1}{\lambda} &\leq \frac{5}{\lambda^2 L^2}\left(\epsilon^{\alpha(d)-2-\delta} + c_6\epsilon\log L + \lambda\epsilon^{-\alpha^*(d)-\delta}\right) \\ &\leq \frac{5}{\lambda^2}\left(\frac{1}{\theta^2}\epsilon^{\alpha(d)-\delta} + 2c_6\frac{\log L}{L^3} + \frac{1}{2d\theta^2}\epsilon^{-\alpha^*(d)+3-\delta}\right) \\ &\leq \frac{C(d,\theta)}{\lambda^2}\epsilon^{\alpha(d)-\delta} \end{aligned}$$

if ϵ is taken sufficiently small depending on δ, where:

i. To obtain the second inequality we have used that $\theta\epsilon^{-1} \leq L \leq 2\epsilon^{-1}$ whenever $\epsilon < \theta$ and also that the inequality $\lambda \leq \frac{\epsilon}{2d}$ holds in our case since $\mathbb{P}(\Omega_\epsilon) = 1$.
ii. For the third inequality we have used that $L^{-3}\log L \leq \theta^{-3}\epsilon^{3-\delta} \leq \theta^{-3}\epsilon^{\alpha(d)-\delta}$ when ϵ is sufficiently small so as to guarantee that $\epsilon < \theta$ and $\log L \leq \epsilon^{-\delta}$.

By performing also the analogous computation but for the lower bound instead, we conclude that if θ, ϵ are chosen appropriately then for any $\omega \in \mathcal{A}_2 \cap \mathcal{A}_3$ and $x \in \partial_- B^*_{NL}$ we have

$$\left|\frac{E_{x,\omega}(T_{B_{NL}})}{NL/2} - \frac{1}{\lambda}\right| \leq \frac{c_4}{\lambda^2}\epsilon^{\alpha(d)-\delta}.$$

We can now finish the proof by using Propositions 7 and 9 to obtain an exponential upper bound of the form $e^{-c_3\epsilon^{-\delta}}$ for the probability $\mathbb{P}(\mathcal{A}_2^c \cup \mathcal{A}_3^c)$.

5 Proof of Theorem 1 (Part II): The Renormalization Argument

We now finish the proof of Theorem 1 by using the results established in Sects. 3.1 and 4. To conclude, we only need to show the following proposition.

Proposition 10. *If $d \geq 3$ then for any given $\eta > 0$ and $\delta \in (0, \eta)$ there exists $\epsilon_0 = \epsilon_0(d, \eta, \delta) > 0$ such that if $(LD)_{\eta,\epsilon}$ holds for $\epsilon \in (0, \epsilon_0)$ then we have P_0-a.s. that*

$$\liminf_{n\to\infty} \frac{E_0(T_n)}{n} \geq \frac{1}{\lambda} + \frac{1}{\lambda^2} O_{d,\eta,\delta}\left(\epsilon^{\alpha(d)-\delta}\right). \tag{55}$$

Indeed, let us recall from Sect. 4.1 that if our RWRE satisfies $(\mathrm{LD})_{\eta,\epsilon}$ for ϵ sufficiently small so as to guarantee that $NL \geq M_0$ and $(NL)^{-(15d+5)} \geq e^{-c_2\epsilon^{-1}}$, where M_0 and c_2 are respectively the constants from (14) and (26), then the polynomial condition $(P)_{15d+5}$ is satisfied and therefore, by Proposition 1, we have that our RWRE is ballistic with velocity $\vec{v} \in \mathbb{R}^d - \{0\}$ verifying

$$\lim_{n\to+\infty} \frac{E_0(T_n)}{n} = \frac{1}{\vec{v}\cdot e_1} > 0.$$

Together with (55), this implies that

$$\frac{1}{\vec{v}\cdot e_1} \geq \frac{1}{\lambda} + \frac{1}{\lambda^2} O_{d,\eta,\delta}\left(\epsilon^{\alpha(d)-\delta}\right).$$

Taking the reciprocal of this inequality then yields Theorem 1. Thus, the remainder of the section is devoted to the proof of Proposition 10.

5.1 The Renormalization Scheme

The general strategy to prove Proposition 10 will be to apply a renormalization argument similar to the one developed by Berger, Drewitz and Ramírez in [2] to show that the polynomial condition $(P)_K$ for K sufficiently large implies condition (T') in [15]. We outline the construction of the different scales involved in the argument below.

We start by introducing two sequences $(N_k)_{k\in\mathbb{N}_0}$ and $(N'_k)_{k\in\mathbb{N}_0}$ specifying the size of each scale. These sequences will depend on ϵ and are defined by fixing first

$$N_0 := NL$$

and then for each $k \in \mathbb{N}_0$ setting

$$N_k := a_k N'_k \quad \text{and} \quad N'_{k+1} := b_k N'_k,$$

where $(a_k)_{k\in\mathbb{N}_0}$ and $(b_k)_{k\in\mathbb{N}_0}$ are two sequences of natural numbers to be chosen appropriately. Observe that, with this definition, for each $k \in \mathbb{N}_0$ we have

$$N_{k+1} = \alpha_k N_k$$

for $\alpha_k := \frac{a_{k+1}}{a_k} b_k$. For the renormalization argument to work, we will require $(a_k)_{k\in\mathbb{N}_0}$ and $(b_k)_{k\in\mathbb{N}_0}$ to satisfy the following conditions:

C1. $a_0 = 2$, i.e. $N_0' : \frac{NL}{2}$.
C2. $(a_k)_{k\in\mathbb{N}_0}$ is increasing.
C3. $a_k \leq \frac{1}{22} b_k$ for all $k \in \mathbb{N}_0$, i.e. $N_k \leq \frac{1}{22} N_{k+1}'$ for all k.
C4. $\sup_{k\in\mathbb{N}_0} \frac{\log \alpha_k}{a_k} < +\infty$
C5. For each $k \in \mathbb{N}$ one has that

$$\frac{2}{a_k} + \frac{1}{12} \cdot \frac{\log a_{k-1}}{a_{k-1}} + \frac{NL}{\alpha_{k-1}} < \frac{1}{(k+1)^2}.$$

C6. There exists a constant $c_* > 0$ (independent of k and ϵ) such that for all $j \in \mathbb{N}$

$$\sum_{i=1}^{j} \log \alpha_{i-1} \leq c_* j^2 \log \epsilon^{-1}.$$

C7. There exists a constant $c^* > 0$ (independent of k and ϵ) such that

$$\prod_{k=1}^{\infty} \left(1 - 8\frac{a_{k-1}}{b_{k-1}}\right) \geq 1 - c^* \epsilon^3.$$

Notice that, in particular, (C1), (C2) and (C3) together imply that $a_k \leq \alpha_k$ and $\alpha_k \geq 22$ for all k. One possible choice of sequences is given for each $k \in \mathbb{N}_0$ by

$$a_{k+1} := (k+1+K)^3 \quad \text{and} \quad b_k := a_k(k+1+K)^2,$$

for $K := 22[\epsilon^{-6}]$. Indeed, (C1), (C2) and (C3) are simple to verify if $\epsilon \in (0,1)$. On the other hand, we have that

$$\alpha_k = a_{k+1}(k+1+K)^2 = (k+1+K)^5$$

so that (C4) is also satisfied because $\frac{\log(k+1+K)}{k+K} \to 0$ as $k \to +\infty$. Moreover, since we have $K \geq 22$ by definition, if $k \in \mathbb{N}$ then

$$\frac{2}{a_k} = \frac{1}{(k+K)^2} \cdot \frac{1}{22} < \frac{1}{3} \cdot \frac{1}{(k+1)^2},$$

$$\frac{1}{12} \cdot \frac{\log a_{k-1}}{a_{k-1}} \leq \frac{1}{12} \wedge \left(\frac{1}{12} \cdot \frac{1}{(k+K)^2} \cdot \frac{3\log(k+K)}{k+K}\right) \leq \frac{1}{3} \cdot \frac{1}{(k+1)^2}$$

and

$$\frac{NL}{\alpha_{k-1}} \leq \frac{16\epsilon^{-4}}{(k+1+K)^5} \leq \frac{1}{3} \cdot \frac{1}{(k+1)^2}$$

if ϵ is sufficiently small so as to guarantee that $\frac{16}{K} \leq \frac{1}{3}$, so that (C5) follows at once. Furthermore, for each $j \in \mathbb{N}$ one has that

$$\begin{aligned}
\sum_{i=1}^{j} \log \alpha_{i-1} &\leq 5 \sum_{i=1}^{j} \log(i+K) \\
&\leq 5 \sum_{i=1}^{j} (\log i + \log(K+1)) \\
&\leq 5(j^2 + j\log(K+1)) \leq 5j^2 \log(K+1)
\end{aligned}$$

from where (C6) easily follows provided that ϵ is sufficiently small. Finally, since $\log(1-x) \geq -2x^2$ for $x \leq \frac{1}{2}$, we obtain

$$\sum_{k=1}^{\infty} \log\left(1 - 8\frac{a_{k-1}}{b_{k-1}}\right) = \sum_{k=1}^{\infty} \log\left(1 - \frac{8}{(k+K)^2}\right)$$
$$\geq \sum_{k=1}^{\infty} \frac{128}{(k+K)^4}$$
$$\geq -\frac{128}{22} \cdot \frac{1}{\lceil \epsilon^{-6} \rceil} \cdot \sum_{k=1}^{\infty} \frac{1}{k^3} \geq -c^* \epsilon^6$$

for $c^* = \frac{128}{11} \sum_{k=1}^{\infty} \frac{1}{k^3}$, from which (C7) readily follows.

Next, we introduce the concept of boxes of scale $k \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we say that a set $Q_k \subseteq \mathbb{Z}^d$ is a box of scale k (or simply *k-box* to abbreviate) if it is of the form $Q_k = B_{N_k}(x)$ for some $x \in \mathbb{Z}^d$, where for $M \in \mathbb{N}$ the box $B_M(x)$ is defined as in (12). For any k-box Q_k we define its boundaries $\partial_i Q_k$ for $i = +, -, l$ as in Sect. 2.1. However, for our current purposes we will need to consider a different definition of its middle-frontal part. Indeed, for any given k-box $Q_k = B_{N_k}(x)$ we define its *middle-frontal k-part* as

$$\tilde{Q}_k := \{y \in B_{N_k}(x) : N_k - N'_k \leq (y-x)\cdot e_1 < N_k, \\ |(y-x)\cdot e_i| < N_k^3 \text{ for } 2 \leq i \leq d\}$$

together with its corresponding *back side*

$$\partial_- \tilde{Q}_k := \left\{ y \in \tilde{Q}_k : (y-x)\cdot e_1 = N_k - N'_k \right\}.$$

Observe that for 0-boxes this definition coincides with the previous one of plain middle-frontal parts.

For the sequel it will be necessary to introduce for each $k \in \mathbb{N}_0$ the partition $\mathcal{C}_k = (C_k^{(z)})_{z\in\mathbb{Z}^d}$ of $\mathbb{Z}^d$ by middle-frontal k-parts defined as

$$C_k^{(z)} := \{y \in \mathbb{Z}^d : z_1 N'_k \leq y_1 < (z_1+1)N'_k, \\ z_i(2N_k^3 - 1) \leq y_i < (z_i+1)(2N_k^3-1) \text{ for } 2 \leq i \leq d\}.$$

Given this partition $\mathcal{C}_k$, for each $x \in \mathbb{Z}^d$ we define

i. $z(x)$ as the unique element of $\mathbb{Z}^d$ such that $x \in C_k^{(z(x))}$.
ii. $Q_k(x)$ as the unique k-box having $C_k^{(z(x))}$ as its middle-frontal k-part.
iii. $U_k(x)$ as the symmetric slab around x given by

$$U_k(x) := \bigcup_{z : |(z - z(x))\cdot e_1| \leq \frac{3}{2}a_k - 1} C_k^{(z)}$$

together with its corresponding (inner) boundaries

$$\partial_- U_k(x) := \left\{ y \in U_k(x) : y_1 = \left(z(x) - \left(\frac{3}{2}a_k - 1\right)\right) N'_k \right\}$$

and

$$\partial_+ U_k(x) := \left\{ y \in U_k(x) : y_1 = \left(z(x) + \left(\frac{3}{2} a_k - 1 \right) \right) N'_k \right\}.$$

Observe that, with this particular choice of boundaries, we have $\partial_- Q_k(x) \subseteq \partial_- U_k(x)$.

Finally, we need to introduce the notion of good and bad k-boxes. Given $\omega \in \Omega$, $k \in \mathbb{N}_0$ and $\epsilon > 0$, we will say that:

- A 0-box Q_0 is (ω, ϵ)-*good* if it satisfies the estimates

$$\inf_{x \in \tilde{Q}_0} P_{x,\omega}(X_{T_{Q_0}} \in \partial_+ Q_0) \geq 1 - e^{-\frac{c_2}{2}\epsilon^{-1}} \tag{56}$$

  and

$$\inf_{x \in \partial_- \tilde{Q}_0} E_{x,\omega}(T_{Q_0}) > \left(\frac{1}{\lambda} - \frac{c_4}{\lambda^2} \epsilon^{\alpha(d)-\delta} \right) N'_0, \tag{57}$$

  where c_2, c_4 are the constants from Theorem 3. Otherwise, we will say that Q_0 is (ω, ϵ)-*bad*.
- A $(k+1)$-box Q_{k+1} is (ω, ϵ)-*good* if there exists a k-box Q'_k such that all k-boxes intersecting Q_{k+1} but not Q'_k are necessarily (ω, ϵ)-good. Otherwise, we will say that Q_{k+1} is (ω, ϵ)-*bad*.

The following lemma, which is a direct consequence of the seed estimates proved in Theorem 3, states that all 0-boxes are good with overwhelming probability.

Lemma 7. *Given $\eta \in (0,1)$ there exist positive constants c_{23} and θ_0 depending only on d and η such that if:*

i. The constant θ from (9) is chosen smaller than θ_0,
ii. $(LD)_{\eta,\epsilon}$ is satisfied for ϵ sufficiently small depending only on d, η and θ,

then for any 0-box Q_0 we have that

$$\mathbb{P}(\{\omega \in \Omega : Q_0 \text{ is } (\omega, \epsilon)\text{-bad}\}) \leq e^{-c_{23} N_0^{\frac{\delta}{4}}}.$$

Proof. Notice that, by translation invariance of $\mathbb{P}$, it will suffice to consider the case of $Q_0 = B_{NL}$. In this case, (27) implies that the probability of (57) not being satisfied is bounded from above by

$$e^{-\frac{c_3}{2} N_0^{\frac{\delta}{4}}}, \tag{58}$$

since $N_0^{\frac{\delta}{4}} = L^\delta \leq 2^\delta \cdot \theta^\delta \cdot \epsilon^{-\delta} \leq 2\epsilon^{-\delta}$. On the other hand, by Markov's inequality and (26) we have

$$\mathbb{P}\left(\sup_{x \in \tilde{Q}_0} P_{x,\omega}(X_{T_{Q_0}} \notin \partial_+ Q_0) > e^{-\frac{c_2}{2}\epsilon^{-1}} \right) \leq e^{\frac{c_2}{2}\epsilon^{-1}} \sum_{x \in \tilde{Q}_0} P_x(X_{T_{Q_0}} \notin \partial_+ Q_0) \leq |\tilde{Q}_0| e^{-\frac{c_2}{2}\epsilon^{-1}}. \tag{59}$$

Combining (58) with (59) yields the result. □

Even though the definition of good k-box is different for $k \geq 1$, it turns out that such k-boxes still satisfy analogues of (56) and (57). The precise estimates are given in Lemmas 8 and 10 below.

Lemma 8. *Given any $\eta \in (0,1)$ there exists $\epsilon_0 > 0$ satisfying that for each $\epsilon \in (0,\epsilon_0)$ there exists a sequence $(d_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{R}_{>0}$ depending on d, η, δ and ϵ such that for each $k \in \mathbb{N}_0$ the following holds:*

i. $d_k \geq \Xi_k d_0$, where $\Xi_k \in (0,1)$ is given by

$$\Xi_k := \prod_{j=1}^{k}\left(1 - \frac{1}{(j+1)^2}\right),$$

with the convention that $\prod_{j=1}^{0} := 1$.

ii. If Q_k is a (ω,ϵ)-good k-box then

$$\inf_{x\in\tilde{Q}_k} P_{x,\omega}(X_{T_{Q_k}} \in \partial_+ Q_k) \geq 1 - e^{-d_k N_k}. \tag{60}$$

Proof. First, observe that if for $k = 0$ we take

$$d_0 := \frac{c_2}{2\epsilon N_0} \tag{61}$$

then condition (i) holds trivially since $\Xi_0 = \frac{1}{2}$ and (ii) also holds by definition of (ω,ϵ)-good 0-box. Hence, let us assume that $k \geq 1$ and show that (60) is satisfied for any fixed (ω,ϵ)-good k-box Q_k. To this end, for each $x \in \tilde{Q}_k$ we write

$$P_{x,\omega}(X_{T_{Q_k}} \notin \partial_+ Q_k) \leq P_{x,\omega}(X_{T_{Q_k}} \in \partial_l Q_k) + P_{x,\omega}(X_{T_{Q_k}} \in \partial_- Q_k). \tag{62}$$

We will show that if ϵ is sufficiently small (not depending on k) and there exists $d_{k-1} > 0$ satisfying that:

i'. $d_{k-1} \geq \Xi_{k-1} d_0$,

ii'. For any (ω,ϵ)-good $(k-1)$-box Q_{k-1} and all $y \in \tilde{Q}_{k-1}$

$$\max\{P_{y,\omega}(X_{T_{Q_{k-1}}} \in \partial_l Q_{k-1}), P_{y,\omega}(X_{T_{Q_{k-1}}} \in \partial_- Q_{k-1})\} \leq e^{-d_{k-1}N_{k-1}},$$

then there also exists $d_k > 0$ with $d_k \geq \Xi_k d_0$ such that for all $x \in \tilde{Q}_k$

$$\max\{P_{x,\omega}(X_{T_{Q_k}} \in \partial_l Q_k), P_{x,\omega}(X_{T_{Q_k}} \in \partial_- Q_k)\} \leq \frac{1}{2} e^{-d_k N_k}. \tag{63}$$

From this, an inductive argument using that (i') and (ii') hold for d_0 as in (61) will yield the result. We estimate each term on the left-hand side of (63) separately, starting with the leftmost one.

For this purpose, we recall the partition $\mathcal{C}_{k-1}$ introduced in the beginning of this subsection and define a sequence of stopping times $(\kappa_j)_{j\in\mathbb{N}_0}$ by fixing $\kappa_0 := 0$ and then for $j \in \mathbb{N}_0$ setting

$$\kappa_{j+1} := \inf\{n > \kappa_j : X_n \notin Q_{k-1}(X_{\kappa_j})\}.$$

Having defined the sequence $(\kappa_j)_{j\in\mathbb{N}_0}$ we consider the rescaled random walk $Y = (Y_j)_{j\in\mathbb{N}_0}$ given by the formula

$$Y_j := X_{\kappa_j \wedge T_{Q_k}}. \tag{64}$$

Now, since Q_k is (ω,ϵ)-good, there exists a $(k-1)$-box Q'_{k-1} such that all $(k-1)$-boxes intersecting Q_k but not Q'_{k-1} are also (ω,ϵ)-good. Define then $\mathcal{B}_{Q'_{k-1}}$ as the collection of all $(k-1)$-boxes which intersect Q'_{k-1} and also set $\mathcal{Q}'_{k-1}$ as the smallest horizontal slab S of the form

$$S = \{z \in \mathbb{Z}^d : \exists\, y \in Q'_{k-1} \text{ with } |(z-y)\cdot e_i| < M \text{ for all } 2 \le i \le d\}$$

which contains $\mathcal{B}_{Q'_{k-1}}$. Observe that, in particular, any $(k-1)$-box which does not intersect $\mathcal{Q}'_{k-1}$ is necessarily (ω,ϵ)-good. Next, we define the stopping times m_1, m_2 and m_3 as follows:

- m_1 is the first time that Y reaches a distance larger than $7N_k^3$ from both $\mathcal{Q}'_{k-1}$ and $\partial_l Q_k$, the lateral sides of the box Q_k.
- m_2 is the first time that Y exits the box Q_k.
- $m_3 := \inf\{j > m_1 : Y_j \in \mathcal{Q}'_{k-1}\}$.

Note that on the event $\{X_{T_{Q_k}} \in \partial_l Q_k\}$ we have $P_{x,\omega}$-a.s. $m_1 < m_2 < +\infty$ so that the stopping time

$$m' := m_2 \wedge m_3 - m_1$$

is well-defined. Furthermore, notice that on the event $\{X_{T_{Q_k}} \in \partial_l Q_k\}$ for each $m_1 < j < m' + m_1$ (such j exist because $m' > 1$, see (65) below) we have that at time κ_j our random walk X is exiting $Q_{k-1}(X_{\kappa_{j-1}})$. This box is necessarily good since it cannot intersect $\mathcal{Q}'_{k-1}$, being $j < m_3$. Moreover, X can exit this box $Q_{k-1}(X_{\kappa_{j-1}})$ either through its back, front or lateral sides. Hence, let us define n_-, n_+ and n_l as the respective number of such back, frontal and lateral exits, i.e. for $i = -, +, l$ define

$$n_i := \#\{m_1 < j < m' + m_1 : X_{\kappa_j} \in \partial_i Q_{k-1}(X_{\kappa_{j-1}})\}.$$

Furthermore, set n_+^* as the number of pairs of consecutive frontal exits, i.e.

$$n_+^* := \#\{m_1 < j < m' + m_1 - 1 : X_{\kappa_i} \in \partial_+ Q_{k-1}(X_{\kappa_{j-1}}) \text{ for } i = j, j+1\}.$$

Note that with any pair of *consecutive* frontal exits the random walk moves at least a distance N'_{k-1} to the right direction e_1, since it must necessarily traverse the entire width of some $C_{k-1}^{(z)}$. Similarly, with any back exit the random walk can move at most a distance $\frac{3}{2}N_{k-1}$ to the left in direction e_1, which is the width of any $(k-1)$-box. Therefore, since our starting point $x \in \tilde{Q}_k$ is at a distance not greater than N'_k from $\partial_+ Q_k$, we conclude that on the event $\{X_{T_{Q_k}} \in \partial_l Q_k\}$ one must have

$$N'_{k-1} \cdot n_+^* - \frac{3}{2} N_{k-1} \cdot n_- \leq N'_k.$$

On the other hand, by definition of m_1 it follows that

$$m' \geq \frac{7N_k^3}{25N_{k-1}^3} = \frac{7}{25}\alpha_{k-1}^3 =: m''_k. \tag{65}$$

Furthermore, observe that $n_+ + n_- + n_l = m' - 1$ and also that $n_+ - n_+^* \leq n_- + n_l$ since $n_+ - n_+^*$ is the number of frontal exits which were followed by a back or lateral exit. Thus, since $N_{k-1} \geq 2N'_{k-1}$ by assumption, from the above considerations we obtain that

$$\frac{N'_k}{N'_{k-1}} + 3\frac{N_{k-1}}{N'_{k-1}} \cdot (n_- + n_l) \geq m' - 1.$$

From here, a straightforward computation using the definition of N_j and N'_j for $j \geq 0$ shows that

$$n_- + n_l \geq \frac{1}{3a_{k-1}} \cdot (m' - m''_k) + M_k$$

where

$$M_k := \frac{1}{3a_{k-1}} \left(\frac{7}{25}\alpha_{k-1}^3 - b_{k-1} - 1 \right) \geq \frac{1}{15a_{k-1}} \alpha_{k-1}^3$$

since $b_{k-1} \leq \alpha_{k-1}$ and $1 \leq \alpha_{k-1} \leq \frac{1}{25}\alpha_{k-1}^3$. Thus, by conditioning on the value of $m' - m''_k$ it follows that

$$\begin{aligned} P_{x,\omega}(X_{T_{Q_k}} \in \partial_l Q_k) &\leq P_{x,\omega}\left(n_- + n_l \geq \frac{1}{3a_{k-1}} \cdot (m' - m''_k) + M_k \right) \\ &\leq \sum_{N \geq 0} P\left(U_N \geq \frac{1}{3a_{k-1}} \cdot N + M_k \right), \end{aligned}$$

where each U_N is a Binomial random variable of parameters $n := m''_k + N$ and $p_k := e^{-d_{k-1}N_{k-1}}$. Using the simple bound $P(U_N \geq r) \leq p_k^r 2^{N+m''_k}$ for $r \geq 0$ yields

$$\begin{aligned} P_{x,\omega}(X_{T_{Q_k}} \in \partial_l Q_k) &\leq \left[\frac{1}{1 - 2p_k^{\frac{1}{3a_{k-1}}}} \right] p_k^{M_k} 2^{m''_k} \\ &\leq \left[\frac{1}{1 - 2p_k^{\frac{1}{3a_{k-1}}}} \right] e^{-d_{k-1}N_{k-1}M_k + m''_k \log 2}. \end{aligned}$$

Now, since

$$\inf_{k \in \mathbb{N}_0} \Xi_k = \prod_{j=1}^{\infty} \left(1 - \frac{1}{(j+1)^2} \right) = \frac{1}{2},$$

it follows that $d_{k-1}N'_{k-1} \geq \frac{1}{4}d_0N_0$ because one then has $d_{k-1} \geq \Xi_{k-1}d_0 \geq \frac{1}{2}d_0$ and $N'_{k-1} \geq \frac{1}{2}N_0$. Hence, we obtain that

$$p_k^{\frac{1}{3a_{k-1}}} = e^{-\frac{1}{3}d_{k-1}N'_{k-1}} \leq e^{-\frac{1}{12}d_0N_0} \leq e^{-\frac{c_2}{24}\epsilon^{-1}} \tag{66}$$

and also

$$\begin{aligned} -d_{k-1}N_{k-1}M_k + m''_k \log 2 &= -d_{k-1}N_k \left(\frac{1}{3a_{k-1}\alpha_{k-1}} M_l - \frac{7}{25} \frac{\log 2}{d_{k-1}N_{k-1}} \alpha_{k-1}^2 \right) \\ &\leq -d_{k-1}N_k \left(\left(\frac{1}{15a_{k-1}} - \frac{28}{25} \frac{\log 2}{d_0N_0a_{k-1}} \right) \alpha_{k-1}^2 \right) \\ &\leq -d_{k-1}N_k \left(\left(\frac{1}{15} - \frac{56}{25} \frac{\log 2}{c_2}\epsilon \right) \frac{\alpha_{k-1}^2}{a_{k-1}} \right) \\ &\leq -d_{k-1}N_k \left(\left(\frac{1}{15} - \frac{56}{25} \frac{\log 2}{c_2}\epsilon \right) \alpha_{k-1} \right) \end{aligned} \tag{67}$$

since $a_{k-1} \leq \alpha_{k-1}$. Thus, if ϵ is taken sufficiently small so as to guarantee that

$$\frac{1}{1 - 2e^{-\frac{c_2}{24}\epsilon^{-1}}} \leq 2 \quad \text{and} \quad \frac{1}{15} - \frac{56}{25}\frac{\log 2}{c_2} \cdot \epsilon \geq \frac{1}{16}$$

then, since $\alpha_{k-1} \geq 16$ and $\epsilon \in (0,1)$ by construction, we conclude that

$$\begin{aligned} P_{x,\omega}(X_{T_{Q_k}} \in \partial_l Q_k) \leq 2e^{-d_{k-1}N_k} &\leq \frac{1}{2}\exp\left\{-d_{k-1}N_k + \log 4\right\} \\ &\leq \frac{1}{2}\exp\left\{-d_{k-1}N_k\left(1 - \frac{\log 4}{d_{k-1}N_k}\right)\right\} \\ &\leq \frac{1}{2}\exp\left\{-d_{k-1}N_k\left(1 - \frac{\log 4}{a_k} \cdot \frac{1}{d_{k-1}N'_k}\right)\right\} \\ &\leq \frac{1}{2}\exp\left\{-d_{k-1}N_k\left(1 - \frac{\log 4}{a_k} \cdot \frac{4}{d_0N_0}\right)\right\} \\ &\leq \frac{1}{2}\exp\left\{-d_{k-1}N_k\left(1 - \frac{\log 4}{a_k} \cdot \epsilon \cdot \frac{8}{c_2}\right)\right\} \\ &\leq \frac{1}{2}e^{-\hat{d}_kN_k}, \end{aligned} \tag{68}$$

for $\hat{d}_k > 0$ given by the formula

$$\hat{d}_k := d_{k-1}\left(1 - \frac{1}{a_k}\right),$$

provided that ϵ is also small enough so as to guarantee that

$$\frac{8\log 4}{c_2} \cdot \epsilon < 1.$$

We turn now to the bound of the remaining term in the left-hand side of (63). Consider once again the partition $\mathcal{C}_{k-1}$ and notice that if $X_0 = x \in \tilde{Q}_k$ then, by construction, we have $Q_{k-1}(x) \subseteq U_{k-1}(x)$. We can then define a sequence $Z = (Z_n)_{n \in \mathbb{N}_0} \subseteq \mathbb{R}$ as follows:

i. First, define $\kappa'_0 := 0$ and for each $j \in \mathbb{N}$ set

$$\kappa'_j := \inf\{n > \kappa'_{j-1} : X_n \in \partial_- U_{k-1}(X_{\kappa'_{j-1}}) \cup \partial_+ U_{k-1}(X_{\kappa'_{j-1}})\}.$$

ii. Having defined the sequence $(\kappa'_j)_{j \in \mathbb{N}_0}$, for each $j \in \mathbb{N}_0$ define

$$Z_j := z(X_{\kappa'_j \wedge T_{Q_k}}) \cdot e_1.$$

The main idea behind the construction of Z is that:

- Z starts inside the one-dimensional interval $[l_k, r_k]$, where

$$l_k = \min\{z \cdot e_1 : C^{(z)}_{k-1} \cap Q_k \neq \emptyset\} \quad \text{and} \quad r_k = \max\{z \cdot e_1 : C^{(z)}_{k-1} \cap Q_k \neq \emptyset\},$$

  and moves inside this interval until the random walk X first exits Q_k. Once this happens, Z remains at its current position forever afterwards.
- Until X first exits Q_k, the increments of Z are symmetric, i.e. $Z_{j+1} - Z_j = \pm\left(\frac{3}{2}a_{k-1} - 1\right)$ for all j with $\kappa'_{j+1} < T_{Q_k}$.
- Given that $X_{\kappa'_j} = y \in Q_k$, if X exits $Q_{k-1}(y)$ through its back side then $X_{\kappa'_{j+1}} \in \partial_- U_{k-1}(y)$, so that $Z_{j+1} - Z_k = -\left(\frac{3}{2}a_{k-1} - 1\right)$.

Thus, it follows that

$$P_{x,\omega}(X_{T_{Q_k}} \in \partial_- Q_k) \leq P_{x,\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}) \tag{69}$$

where $T^Z_{l_k}$ and $\overline{T}^Z_{r_k}$ respectively denote the hitting times for Z of the sets $(-\infty, l_k]$ and $(r_k, +\infty)$. To bound the right-hand side of (69), we need to obtain a good control over the jumping probabilities of the random walk Z. These will depend on whether the corresponding slab U_{k-1} which Z is exiting at each given time contains a (ω, ϵ)-bad $(k-1)$-box or not. More precisely, since Q_k is (ω, ϵ)-good we know that there exists some $(k-1)$-box $\overline{Q}'_{k-1}$ such that all $(k-1)$-boxes which intersect Q_k but not $\overline{Q}'_{k-1}$ are necessarily (ω, ϵ)-good. Define then

$$\begin{cases} L_{k-1} := \min\{z \cdot e_1 : C^{(z)}_{k-1} \cap \overline{Q}_{k-1} \neq \emptyset\} - 2a_{k-1} \\ R_{k-1} := \max\{z \cdot e_1 : C^{(z)}_{k-1} \cap \overline{Q}_{k-1} \neq \emptyset\} + 2a_{k-1} \end{cases}$$

and observe that, with this definition, if $y \in \mathbb{Z}^d$ satisfies $y \in C^{(z)}_{k-1}$ for some $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$ with $z_1 \notin [L_{k-1}, R_{k-1}]$ then all $(k-1)$-boxes contained in the slab $U_{k-1}(y)$ are necessarily good. From this observation and the uniform ellipticity, it follows that the probability of Z jumping right from a given position $z_1 \in [l_k, r_k]$ is bounded from below by

$$p_{k-1}(z_1) := \begin{cases} (1 - e^{-d_{k-1}N_{k-1}})^{\frac{3}{2}a_{k-1}-1} & \text{if } z_1 \notin [L_{k-1}, R_{k-1}] \\ \kappa^{\frac{3}{2}N_{k-1}} & \text{if } z_1 \in [L_{k-1}, R_{k-1}]. \end{cases}$$

Hence, if we write $T^Z_{[L_{k-1},R_{k-1}]}$ to denote the hitting time of $[L_{k-1},R_{k-1}]$ and $\Theta_Z := \{T^Z_{l_k} < \overline{T}^Z_{r_k}\}$ then we can decompose

$$P_{x,\omega}(\Theta_Z) = P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} = +\infty\}) + P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}). \tag{70}$$

Now, recall that if $(W_j)_{j\in\mathbb{N}_0}$ is a random walk on $\mathbb{Z}$ starting from 0 with nearest-neighbor jumps which has probability $p \neq \frac{1}{2}$ of jumping right then, given $a,b \in \mathbb{N}$, the probability $E(-a,b,p)$ of exiting the interval $[-a,b]$ through $-a$ is exactly

$$E(-a,b,p) = (1-p)^a \cdot \frac{p^b-(1-p)^b}{p^{a+b}-q^{a+b}} \leq \frac{(1-p)^a}{p^{a+b}-(1-p)^{a+b}} =: \overline{E}(-a,b,p).$$

Furthermore, if $a,b \in \mathbb{N}$ are such that

$$\frac{N_k-2N'_k}{N_{k-1}} \leq a \leq 2\cdot\frac{N_k}{N_{k-1}} \quad \text{and} \quad a+b \leq 4\cdot\frac{N_k}{N_{k-1}}$$

then for $p'_{k-1} := (1-e^{-d_{k-1}N_{k-1}})^{\frac{3}{2}a_{k-1}-1}$ and ϵ sufficiently small (but not depending on k) one has

$$\overline{E}(-a,b,p'_{k-1}) \leq 2e^{-\tilde{d}_k N_k}. \tag{71}$$

for

$$\tilde{d}_k := d_{k-1}\left(1-\frac{2}{a_k}-\frac{1}{12}\cdot\frac{\log a_{k-1}}{a_{k-1}}\right) \geq d_{k-1}\left(1-\frac{1}{(k+1)^2}\right) > 0.$$

Indeed, by Bernoulli's inequality which states that $(1-p)^n \geq 1-np$ for all $n\in\mathbb{N}$ and $p\in(0,1)$, for ϵ sufficiently small so as to guarantee that $\frac{32}{c_2}\cdot\epsilon < \frac{1}{12}$ we have that

$$\begin{aligned}
(1-p'_{k-1})^a &\leq \left(\left(\frac{3}{2}a_{k-1}-1\right)e^{-d_{k-1}N_{k-1}}\right)^a \\
&\leq \exp\left\{-ad_{k-1}N_{k-1} - 2a\log a_{k-1}\right\} \\
&\leq \exp\left\{-d_{k-1}N_k\left(a\frac{N_{k-1}}{N_k} + 2a\frac{\log a_{k-1}}{d_{k-1}N_k}\right)\right\} \\
&\leq \exp\left\{-d_{k-1}N_k\left(\frac{N_k-2N'_k}{N_k} - 4\frac{\log a_{k-1}}{d_{k-1}N_{k-1}}\right)\right\} \\
&\leq \exp\left\{-d_{k-1}N_k\left(1-\frac{2}{a_k} - 16\cdot\frac{\log a_{k-1}}{a_{k-1}}\cdot\frac{1}{d_0N_0}\right)\right\} \\
&\leq \exp\left\{-d_{k-1}N_k\left(1-\frac{2}{a_k} - \frac{32}{c_2}\cdot\epsilon\cdot\frac{\log a_{k-1}}{a_{k-1}}\right)\right\} \leq e^{-\tilde{d}_kN_k}
\end{aligned}$$

where we use that $\frac{3}{2} \leq a_{k-1}$ in the second line and $d_{k-1}N'_{k-1} \geq \frac{1}{4}d_0N_0$ in the second-to-last one. Similarly, by (C4) we can take ϵ sufficiently small so as to guarantee that

$$\frac{32}{c_2}\cdot\epsilon\cdot\sup_{j\in\mathbb{N}}\left(\frac{\log\alpha_{j-1}}{a_{j-1}}\right) < \frac{1}{2},$$

in which case we have that

$$\begin{aligned}(p'_{k-1})^{a+b} &\geq 1 - 2a_{k-1}(a+b)e^{-d_{k-1}N_{k-1}}\\ &\geq 1 - \exp\left\{-d_{k-1}N_{k-1}\left(1 - \frac{\log 2a_{k-1}(a+b)}{d_{k-1}N_{k-1}}\right)\right\}\\ &\geq 1 - \exp\left\{-d_{k-1}N_{k-1}\left(1 - \frac{16}{c_2}\cdot\epsilon\cdot\left(\frac{\log a_{k-1}}{a_{k-1}} + \frac{\log \alpha_{k-1}}{a_{k-1}}\right)\right)\right\}\\ &\geq 1 - \exp\left\{-d_{k-1}N_{k-1}\left(1 - \frac{32}{c_2}\cdot\epsilon\cdot\frac{\log \alpha_{k-1}}{a_{k-1}}\right)\right\}\\ &\geq 1 - \exp\left\{-\frac{c_2}{16}\epsilon^{-1}\right\}, \end{aligned} \tag{72}$$

where we have used that $2 \leq a_{k-1}$ and $4 \leq \alpha_{k-1}$ to obtain the third line. Finally, we have

$$\begin{aligned}(1-p'_{k-1})^{a+b} \leq 1 - p'_{k-1} &\leq \left(\frac{3}{2}a_{k-1} - 1\right)e^{-d_{k-1}N_{k-1}}\\ &\leq 2a_{k-1}(a+b)e^{-d_{k-1}N_{k-1}} \leq \exp\left\{-\frac{c_2}{16}\epsilon^{-1}\right\}\end{aligned}$$

where, for the last inequality, we have used the bound (72). Hence, by choosing ϵ sufficiently small (independently of k) so as to guarantee that

$$(p'_{k-1})^{a+b} - (1-p'_{k-1})^{a+b} \geq \frac{1}{2},$$

we obtain (71).

With this, from the considerations made above it follows that

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} = +\infty\}) \leq \overline{E}(-a, b, p'_{k-1})$$

for

$$a := \left[\frac{(z(x)\cdot e_1) - l_k}{\frac{3}{2}a_{k-1} - 1}\right] \quad \text{and} \quad b := \left[\frac{r_k - (z(x)\cdot e_1)}{\frac{3}{2}a_{k-1} - 1}\right] + 1,$$

where $[\cdot]$ here denotes the (lower) integer part. Recalling that the width in direction e_1 of any $C^{(z)}_{k-1}$ is exactly N'_{k-1} and also that $N'_{j-1} \leq N_{j-1} \leq \frac{1}{8}N'_j$ holds for all $j \in \mathbb{N}$, by using the fact that $x \in \tilde{Q}_k$ it is straightforward to check that

$$\begin{aligned}\frac{N_k - N'_k}{N_{k-1}} &\leq \frac{\frac{3}{2}N_k - N'_k - N'_{k-1} - (\frac{3}{2}N_{k-1} - N'_{k-1})}{\frac{3}{2}N_{k-1} - N'_{k-1}}\\ &\leq a \leq \frac{\frac{3}{2}N_k + N'_{k-1}}{\frac{3}{2}N_{k-1} - N'_{k-1}} \leq \frac{N_k + N'_k}{N_{k-1}} \leq 2\cdot\frac{N_k}{N_{k-1}}\end{aligned}$$

and

$$a + b \leq a + \frac{N'_k + \frac{3}{2}N_{k-1}}{\frac{3}{2}N_{k-1} - N'_{k-1}} \leq 2\cdot\frac{N_k + N'_k}{N_{k-1}} \leq 4\cdot\frac{N_k}{N_{k-1}}$$

so that (71) in this case yields

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} = +\infty\}) \leq 2e^{-\tilde{d}_k N_k}. \tag{73}$$

To bound the remaining term in the right-hand side of (70), we separate matters into two cases: either $z(x) \cdot e_1 \leq R_{k-1}$ or $z(x) \cdot e_1 > R_{k-1}$. Observe that if $z(x) \cdot e_1 \leq R_{k-1}$ and we define

$$\begin{cases} l(x) := \inf\left\{j \geq 0 : z(x) \cdot e_1 - j\left(\frac{3}{2}a_{k-1} - 1\right) < L_{k-1}\right\} < +\infty \\ z_l(x) := z(x) \cdot e_1 - l(x)\left(\frac{3}{2}a_{k-1} - 1\right) \end{cases}$$

then Z necessarily visits the site $z_l(x)$ on the event $\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}$. On the other hand, if $z(x) \cdot e_1 > R_{k-1}$ and we define

$$\begin{cases} r(x) := \sup\left\{j \geq 0 : z(x) \cdot e_1 - j\left(\frac{3}{2}a_{k-1} - 1\right) > R_{k-1}\right\} < +\infty \\ z_r(x) := z(x) \cdot e_1 - r(x)\left(\frac{3}{2}a_{k-1} - 1\right) \end{cases}$$

then Z necessarily visits the site $z_r(x)$ on the event $\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}$. In the first case, by the strong Markov property we can bound

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq P^Z_{z_l(x),\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}).$$

where $P^Z_{z_l(x),\omega}$ denotes the quenched law of Z starting from $z_l(x)$. Using the strong Markov property once again, we can check that

$$P^Z_{z_l(x),\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}) \leq \frac{P^Z_{z_l(x),\omega}(D^-)}{P^Z_{z_l(x),\omega}(D^- \cup D^+)} \leq \frac{P^Z_{z_l(x),\omega}(D^-)}{P^Z_{z_l(x),\omega}(D^+)}$$

where

$$D^- := \{T^Z_{l_k} < H^Z_{z_l(x)}\} \quad \text{and} \quad D^+ := \{\overline{T}^Z_{r_k} < H^Z_{z_l(x)}\}$$

and we define $H^Z_y := \inf\{j > 1 : Z_j = y\}$ for each $y \in \mathbb{Z}$. Now, by forcing Z to always jump right, using that $(r_k - R_{k-1}) \cdot N'_{k-1} \leq N'_k$ holds whenever $z(x) \cdot e_1 \leq R_{k-1}$ and also that

$$|R_{k-1} - L_{k-1}| \leq \frac{3}{2}a_{k-1} + 2 + 4a_{k-1} \leq 8a_{k-1}$$

we obtain

$$P^Z_{z_l(x),\omega}(D^+) \geq \kappa^{8N_{k-1}}\left(p'_{k-1}\right)^{\frac{N'_k}{\frac{3}{2}N_{k-1} - N'_{k-1}}} \geq \kappa^{8N_{k-1}}\left(p'_{k-1}\right)^{\frac{N_k}{N}k-1} \geq \frac{1}{2}\kappa^{8N_{k-1}},$$

where we have used (72) to obtain the last inequality. On the other hand, by the Markov property at time $j = 1$, we have that

$$P^Z_{z_l(x),\omega}(T^Z_{l_k} < H^Z_{z_l(x)}) \leq \overline{E}(-a', b', p'_{k-1})$$

for

$$a' := \left[\frac{z_l(x) - l_k}{\frac{3}{2}a_{k-1} - 1}\right] \quad \text{and} \quad b' := 1.$$

Using the facts that $x \in \tilde{Q}_k$, $z(x) \cdot e_1 \leq R_{k-1}$, $|R_{k-1} - L_{k-1}| \leq 8a_{k-1}$ and $N'_{k-1} \leq N_{k-1} \leq \frac{1}{22}N'_k$, it is easy to check that

$$\frac{N_k - N'_k}{N_{k-1}} \leq a' \leq 2 \cdot \frac{N_k}{N_{k-1}} \quad \text{and} \quad a + b \leq 4 \cdot \frac{N_k}{N_{k-1}},$$

so that (71) immediately yields

$$P^Z_{z_l(x),\omega}(D^-) \leq 2e^{-\tilde{d}_k N_k},$$

and thus

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq 4\kappa^{-8N_{k-1}} e^{-\tilde{d}_k N_k}. \tag{74}$$

It remains only to treat the case in which $z(x)\cdot e_1 > R_{k-1}$. Recall that in this case we had that Z necessarily visits $z_r(x)$ so that, by the strong Markov property, we have

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq P^Z_{z(x)\cdot e_1,\omega}(T^Z_{z_r(x)} < \overline{T}^Z_{r_k}) \cdot P^Z_{z_r(x),\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}).$$

Notice that, by proceeding as in the previous cases, we obtain

$$P^Z_{z(x)\cdot e_1,\omega}(T^Z_{z_r(x)} < \overline{T}^Z_{r_k}) \leq \overline{E}(-a'', b, p'_{k-1})$$

for

$$a'' := \left[\frac{(z(x) \cdot e_1) - z_r(x)}{\frac{3}{2}a_{k-1} - 1}\right] \quad \text{and} \quad b := \left[\frac{r_k - (z(x) \cdot e_1)}{\frac{3}{2}a_{k-1} - 1}\right] + 1.$$

Now, we have two options: either $|l_k - z_r(x)| \leq 11a_{k-1}$ or $|l_k - z_r(x)| > 11a_{k-1}$. In the first case, we have that

$$a'' \leq a := \left[\frac{(z(x) \cdot e_1) - l_k}{\frac{3}{2}a_{k-1} - 1}\right] \leq a'' + 1 + \frac{11a_{k-1}}{\frac{3}{2}a_{k-1} - 1} \leq a'' + 9$$

so that, by the bound previously obtained on a and $a + b$, we conclude that

$$\frac{N_k - 2N'_k}{N_{k-1}} \leq a'' \leq 2 \cdot \frac{N_k}{N_{k-1}} \quad \text{and} \quad a'' + b \leq 4 \cdot \frac{N_k}{N_{k-1}},$$

which implies that

$$\begin{aligned} P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) &\leq P^Z_{z(x)\cdot e_1,\omega}(T^Z_{z_r(x)} < T^Z_{r_k}) \\ &\leq \overline{E}(-a'', b, p'_{k-1}) \leq 2e^{-\tilde{d}_k N_k}. \end{aligned} \tag{75}$$

On the other hand, if $|l_k - z_r(x)| > 11a_{k-1}$ then, since $|z_r(x) - z_l(x)| < 11a_{k-1}$ holds because $|R_{k-1} - L_{k-1}| \leq 8a_{k-1}$, the walk Z starting from $z_r(x)$ must necessarily visit $z_l(x)$ if it is to reach $(-\infty, l_k]$ before $(r_k, +\infty)$. Therefore, using the strong Markov property we obtain that

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq P^Z_{z(x)\cdot e_1,\omega}(T^Z_{z_r(x)} < \overline{T}^Z_{r_k}) \cdot P^Z_{z_l(x),\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}).$$

Since it still holds that $1 \leq a'' + b \leq a + b \leq 4 \cdot \frac{N_k}{N_{k-1}}$ in this case, then

$$(p'_{k-1})^{a''+b} - (1 - p'_{k-1})^{a''+b} \geq (p'_{k-1})^{4\cdot \frac{N_k}{N_{k-1}}} - (1 - p'_{k-1}) \geq \frac{1}{2}$$

so that

$$P^Z_{z(x),\omega}(T^Z_{z_r(x)} < T^Z_{r_k}) \leq \overline{E}(-a'', b, p'_{k-1}) \leq 2(1 - p'_{k-1})^{a''}.$$

On the other hand, as before we have

$$P^Z_{z_l(x),\omega}(T^Z_{l_k} < \overline{T}^Z_{r_k}) \leq \frac{P^Z_{z_l(x),\omega}(D^-)}{P^Z_{z_l(x),\omega}(D^+)}$$

but now the distance of $z_l(x)$ from the edges l_k and r_k has changed. Indeed, one now has the bounds

$$P^Z_{z_l(x),\omega}(D^+) \geq \kappa^{8N_{k-1}}(p'_{k-1})^{\left[\frac{r_k - l_k}{\frac{3}{2}a_{k-1}-1}\right]+1} \geq \kappa^{8N_{k-1}}(p'_{k-1})^{4\cdot\frac{N_k}{N_{k-1}}} \geq \frac{1}{2}\kappa^{8N_{k-1}}$$

and

$$P^Z_{z_l(x),\omega}(D^-) \leq \overline{E}(-\hat{a}, b', p'_{k-1})$$

for

$$\hat{a} := \left[\frac{z_l(x) - l_k}{\frac{3}{2}a_{k-1} - 1}\right] \quad \text{and} \quad b' := 1.$$

Since clearly $\hat{a} + b' \leq a + b \leq 4 \cdot \frac{N_k}{N_{k-1}}$ because $z(x) \cdot e_1 \geq z_l(x)$ by definition, we obtain that

$$(p'_{k-1})^{\hat{a}+b'} - (1 - p'_{k-1})^{\hat{a}+b'} \geq \frac{1}{2},$$

so that

$$P^Z_{z_l(x),\omega}(D^-) \leq 2(1 - p'_{k-1})^{\hat{a}}.$$

We conclude that

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq 8\kappa^{-8N_{k-1}}(1 - p'_{k-1})^{a''+\hat{a}}.$$

Now, recalling that $11a_{k-1} > |z_r(x) - z_l(x)|$, we see that

$$\begin{aligned}
4 \cdot \frac{N_k}{N_{k-1}} \geq a'' + \hat{a} &\geq \frac{(z(x) \cdot e_1) - z_r(x) + z_l(x) - l_k - 2\left(\frac{3}{2}a_{k-1} - 1\right)}{\frac{3}{2}a_{k-1} - 1} \\
&\geq \frac{(z(x) \cdot e_1) - l_k - \left(\frac{3}{2}a_{k-1} - 1\right)}{\frac{3}{2}a_{k-1} - 1} - \frac{z_r(x) - z_l(x)}{\frac{3}{2}a_{k-1} - 1} - 1 \\
&\geq \frac{N_k - N'_k}{N_{k-1}} - 9 \\
&\geq \frac{N_k - 2N'_k}{N_{k-1}}
\end{aligned}$$

so that

$$P_{x,\omega}(\Theta_Z \cap \{T^Z_{[L_{k-1},R_{k-1}]} < +\infty\}) \leq 8\kappa^{-8N_{k-1}} e^{-\tilde{d}_k N_k}. \quad (76)$$

In conclusion, gathering (73), (74), (75) and (76) yields

$$P_{x,\omega}(X_{T_{Q_k}} \in \partial_- Q_k) \leq 10\kappa^{-8N_{k-1}} e^{-\tilde{d}_k N_k} \leq \frac{1}{2} e^{-d'_k N_k},$$

where

$$d'_k := \tilde{d}_k - 8\log 20\kappa^{-1} \cdot \frac{1}{\alpha_{k-1}}.$$

Together with (68), this gives (63) for $d_k := \min\{\hat{d}_k, d'_k\}$. It only remains to check that $d_k \geq \Xi_k d_0$. To see this, first notice that (C5) implies that

$$\hat{d}_k = d_{k-1}\left(1 - \frac{1}{a_k}\right) \geq d_{k-1}\left(1 - \frac{2}{a_k}\right) \geq d_{k-1}\left(1 - \frac{1}{(k+1)^2}\right) \geq \Xi_k d_0$$

since $d_{k-1} \geq \Xi_{k-1} d_0$. Thus, it will suffice to check that $d'_k \geq \Xi_k d_0$ holds if ϵ is sufficiently small. This will follow once again from (C5). Indeed, if ϵ is such that $\frac{32\log 20\kappa^{-1}}{c_2} \cdot \epsilon < 1$ then we have that

$$\begin{aligned} d'_k :&= d_{k-1}\left(1 - \frac{2}{a_k} - \frac{1}{2}\cdot\frac{\log a_{k-1}}{a_{k-1}} - \frac{8\log 20\kappa^{-1}}{d_{k-1}}\cdot\frac{1}{\alpha_{k-1}}\right) \\ &= d_{k-1}\left(1 - \frac{2}{a_k} - \frac{1}{12}\cdot\frac{\log a_{k-1}}{a_{k-1}} - \frac{32\log 20\kappa^{-1}}{c_2}\cdot\epsilon\cdot\frac{NL}{\alpha_{k-1}}\right) \\ &\geq d_{k-1}\left(1 - \frac{1}{(k+1)^2}\right) \geq \Xi_k d_0. \end{aligned}$$

This shows that $d_k \geq \Xi_k d_0$ and thus concludes the proof. □

Lemma 9. *Given any $\eta \in (0,1)$ and $\delta \in (0,\eta)$ there exists $\epsilon_0 = \epsilon_0(d,\eta,\delta) > 0$ such that if Q_k is a (ω,ϵ)-good k-box for some $\epsilon \in (0,\epsilon_0)$ and $k \in \mathbb{N}_0$ then*

$$\inf_{x\in\partial_-\tilde{Q}_k} E_{x,\omega}(T_{Q_k}) > \left(\frac{1}{\lambda} - \frac{c_4}{\lambda}\epsilon^{\alpha(d)-\delta}\right) N'_k \left[\prod_{j=1}^{k}\left(1 - 8\frac{a_{j-1}}{b_{j-1}}\right)\right]^2 \quad (77)$$

with the convention that $\prod_{j=1}^{0} := 1$.

Proof. We will prove (77) by induction on $k \in \mathbb{N}_0$. Notice that (77) holds for $k = 0$ by definition of (ω,ϵ)-good 0-box. Thus, let us assume that $k \geq 1$ and that (77) holds for (ω,ϵ)-good $(k-1)$-boxes. Consider a (ω,ϵ)-good k-box Q_k and let $x \in \partial_-\tilde{Q}_k$. Observe that if for $j = 0,\ldots,b_{k-1}$ we define the stopping times

$$O_j := \inf\{n \in \mathbb{N}_0 : (X_n - X_0)\cdot e_1 = jN'_{k-1}\} \wedge T_{Q_k}$$

then $T_{Q_k} = \sum_{j=1}^{b_{k-1}} O_j - O_{j-1}$. Furthermore, if for each j we define $Y_j := X_{T_{O_j}}$ then it follows from the strong Markov property that

$$\begin{aligned} E_{x,\omega}(T_{Q_k}) &= \sum_{j=1}^{b_{k-1}} E_{x,\omega}(O_j - O_{j-1}) \\ &\geq \sum_{j=1}^{b_{k-1}} E_{x,\omega}((O_j - O_{j-1})\mathbb{1}_{\{O_{j-1}<T_{Q_k}\}}) \\ &\geq \sum_{j=1}^{b_{k-1}} E_{x,\omega}(E_{Y_{j-1},\omega}(O_1)\mathbb{1}_{\{O_{j-1}<T_{Q_k},\, d(Y_j,\partial_l Q_k)>25N_{k-1}^3\}}) \\ &\geq \sum_{j=1}^{b_{k-1}} E_{x,\omega}(E_{Y_{j-1},\omega}(T_{\hat{Q}_{k-1}(Y_{j-1})})\mathbb{1}_{\{X_{T_{Q'_k}}\in\partial_+ Q'_k\}}), \end{aligned} \tag{78}$$

where $d(\cdot,\partial_l Q_k)$ denotes the distance to the lateral side $\partial_l Q_k$ and we define the box Q'_k as

$$Q'_k := \{y \in Q_k : d(y,\partial_l Q_k) > 25N_{k-1}^3\},$$

together with its frontal side

$$\partial_+ Q'_k := \partial_+ Q_k \cap Q'_k$$

and the $(k-1)$-box $\hat{Q}_{k-1}(y)$ for any $y \in \mathbb{Z}^d$ through the formula

$$\hat{Q}_{k-1}(y) := B_{N_{k-1}}(y - (N_{k-1} + N'_{k-1})e_1).$$

Observe that if $Y_j \in Q'_k$ then $\hat{Q}_{k-1}(Y_j) \subseteq Q_k$ so that

$$E_{Y_j,\omega}(O_1) \geq E_{Y_j,\omega}(T_{\hat{Q}_{k-1}(Y_j)}),$$

which explains how we obtained (78). Now, since there can be at most $|R_{k-1} - L_{k-1}| \leq 8a_{k-1}$ boxes of the form $\hat{Q}_{k-1}(Y_{j-1})$ for $j = 1,\ldots,b_{k-1}$ which are (ω,ϵ)-bad, it follows from the inductive hypothesis that

$$\begin{aligned} E_{x,\omega}(T_{Q_k}) \geq \left(\frac{1}{\lambda} - \frac{c_4}{\lambda}\epsilon^{\alpha(d)-\delta}\right) \; N'_k \left[\prod_{j=1}^{k-1}\left(1 - 8\frac{a_{j-1}}{b_{j-1}}\right)\right]^2 \\ \times \left(1 - 8\frac{a_{k-1}}{b_{k-1}}\right) P_{x,\omega}\left(X_{T_{Q'_k}} \in \partial_+ Q'_k\right). \end{aligned} \tag{79}$$

But, by performing a careful inspection of the proof of Lemma 8, one can show that

$$P_{x,\omega}\left(X_{T_{Q'_k}} \in \partial_+ Q'_k\right) \geq 1 - e^{-\frac{1}{4}d_0 N_k}$$

so that, using that $e^{-x} \leq \frac{1}{x}$ for $x \geq 1$ and also that $N_k \geq N'_k \geq b_{k-1}N'_{k-1} \geq \frac{b_{k-1}}{a_{k-1}}N_{k-1} \geq \frac{b_{k-1}}{a_{k-1}}N_0$, for $\epsilon < c_2$ we obtain

$$\begin{aligned} P_{x,\omega}\left(X_{T_{Q'_k}} \in \partial_+ Q'_k\right) &\geq 1 - e^{-\frac{1}{4}d_0 N_k} \\ &\geq 1 - \frac{4}{d_0 N_k} \\ &\geq 1 - \frac{4}{d_0 N_0} \cdot \frac{a_{k-1}}{b_{k-1}} \\ &= 1 - \frac{8}{c_2} \cdot \epsilon \cdot \frac{a_{k-1}}{b_{k-1}} \geq 1 - 8\frac{a_{k-1}}{b_{k-1}} \end{aligned}$$

which, combined with (79), yields (77). □

Finally, we need the following estimate concerning the probability of a k-box being (ω, ϵ)-bad.

Lemma 10. *Given $\eta \in (0,1)$ and $\delta \in (0,\eta)$ there exists θ_0 depending only on d, η and δ such that if:*

i. The constant θ from (9) is chosen smaller than θ_0,
ii. $(LD)_{\eta,\epsilon}$ is satisfied for ϵ sufficiently small depending only on d, η, δ and θ,

then there exists $c_{24} = c_{24}(d, \eta, \delta, \theta, \epsilon)$ such that for all $k \in \mathbb{N}_0$ and any k-box Q_k one has

$$\mathbb{P}(\{\omega \in \Omega : Q_k \text{ is } (\omega,\epsilon)\text{-bad}\}) \leq e^{-c_{24}2^k}.$$

Proof. For each $k \in \mathbb{N}_0$ and $\epsilon > 0$ define

$$q_k(\epsilon) := \mathbb{P}(\{\omega : Q_k \text{ is } (\omega,\epsilon)\text{-bad}\})$$

Notice that q_k does not depend on the particular choice of Q_k due to the translation invariance of $\mathbb{P}$. We will show by induction on $k \in \mathbb{N}_0$ that

$$q_k \leq e^{-m_k 2^k} \tag{80}$$

for m_k given by

$$m_k := c_{23} N_0^{\frac{\delta}{4}} - 12d \sum_{j=1}^{k} \frac{\log N_j}{2^j}$$

with the convention that $\sum_{j=1}^{0} := 0$. From (80), the result will follow once we show that $\inf_k m_k > 0$.

First, observe that (80) holds for $k = 0$ by Lemma 7. Therefore, let us assume that $k \geq 1$ and (80) holds for $k-1$. Notice that if Q_k is (ω,ϵ)-bad then necessarily there must be at least two (ω,ϵ)-bad $(k-1)$-boxes which intersect Q_k but not each other. Since the number of $(k-1)$-boxes which can intersect Q_k is at most

$$\frac{3}{2}N_k \cdot \left(50N_k^3\right)^{d-1} \leq (2N_k)^{6d},$$

then by the union bound and the product structure of $\mathbb{P}$ we conclude that

$$q_k \le (2N_k)^{6d} q_{k-1}^2 \le \exp\left\{6d \log 2N_k - m_{k-1}2^k\right\} \le e^{-m_k 2^k}.$$

Thus, it only remains to check that

$$\inf_k m_k = c_{23} N_0^{\frac{\delta}{4}} - 12d \sum_{j=1}^{\infty} \frac{\log N_j}{2^j} > 0. \tag{81}$$

But notice that by (C6) we have that

$$\sum_{j=1}^{\infty} \frac{\log N_j}{2^j} \le \log N_0 + \sum_{j=1}^{\infty} \left(\frac{1}{2^j} \sum_{i=1}^{j} \log \alpha_{i-1} \right) \le c \log \epsilon^{-1}$$

for some constant $c > 0$, from where (81) follows if ϵ sufficiently small (depending on δ and θ). □

Let us now see how to deduce Proposition 10 from Lemmas 9 and 10. For each $k \in \mathbb{N}_0$ consider the k-box given by

$$Q_k := \left(-\frac{3}{2} N_k + N_k', N_k' \right) \times \left(-25N_k^3, 25N_k^3 \right)^{d-1}.$$

Using the probability estimate on Lemma 10, the Borel–Cantelli lemma then implies that if ϵ, θ are chosen appropriately small then for $\mathbb{P}$-almost every ω the boxes Q_k are all (ω, ϵ)-good except for a finite amount of them. In particular, by Lemma 9 we have that for $\mathbb{P}$-almost every ω

$$\liminf_{k \to +\infty} \frac{E_{0,\omega}(T_{N_k'})}{N_k'} \ge \left(\frac{1}{\lambda} - \frac{c_4}{\lambda} \epsilon^{\alpha(d)-\delta} \right) \left[\prod_{j=1}^{\infty} \left(1 - 8 \frac{a_{j-1}}{b_{j-1}} \right) \right]^2$$

By Fatou's lemma, the former implies that

$$\liminf_{k \to +\infty} \frac{E_0(T_{N_k'})}{N_k'} \ge \left(\frac{1}{\lambda} - \frac{c_4}{\lambda} \epsilon^{\alpha(d)-\delta} \right) \left[\prod_{j=1}^{\infty} \left(1 - 8 \frac{a_{j-1}}{b_{j-1}} \right) \right]^2$$

which in turn, since $\lim_{n \to +\infty} \frac{E_0(T_n)}{n}$ exists by Proposition 1, yields that

$$\liminf_{n \to +\infty} \frac{E_0(T_n)}{n} \ge \left(\frac{1}{\lambda} - \frac{c_4}{\lambda} \epsilon^{\alpha(d)-\delta} \right) \left[\prod_{j=1}^{\infty} \left(1 - 8 \frac{a_{j-1}}{b_{j-1}} \right) \right]^2.$$

Recalling now that by (C7) we have

$$\prod_{j=1}^{\infty} \left(1 - 8 \frac{a_{j-1}}{b_{j-1}} \right) = 1 + O(\epsilon^3),$$

we conclude the result.

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