Quantum Dynamics and Frequency Shift of a Periodically Driven Multi-photon Anharmonic Oscillator

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Abstract. The Hamiltonian and hence the relevant equations of motion involving the dynamics of a driven classical anharmonic oscillator with 2m–th anharmonicity are framed. By neglecting the nonsynchronous energy terms, we derive the model of a driven multi-photon (2m–th) quantum anharmonic oscillator. The dynamical nature of the field operators is expressed in terms of the coupling constant, excitation number and the driven parameter. The solutions presented here are fully analytical and are exact in nature.

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1 Introduction

The basic physics is easily understood in terms of the physical models. Of course, the model of a harmonic oscillator (HO) is perhaps the most useful one among them. The model of a simple harmonic oscillator is realized when a particle moves under the action of a restoring force. In spite of the wide applications of the SHO model, it is inadequate when we come across with the real physical situations. For a real physical system, the inclusion of damping and/or anharmonicities in the model of HO are inevitable. In addition to these, the oscillator may also be put under the action of an external force and hence the model of a forced (driven) oscillator. It is true that the presence of damping in the model of an SHO is not a serious problem as long as we are interested in the classical regime. On the other hand, the presence of damping in the model of a SHO makes the problem a nontrivial one. Therefore, the damped quantum harmonic oscillator requires special attention. In this short communication, we will ignore the presence of damping if any. Because of the wide range of applications and of the fundamental nature of the problem, the problems of anharmonic oscillator have attracted people from various branches of physics [1–8].

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2 Driven Classical Anharmonic Oscillator

The Hamiltonian of a driven classical anharmonic oscillator of unit mass is given by

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{\lambda}{2m} q^{2m} + fq \tag{1}$$

where q and p are the classical position and momentum, respectively. The parameter λ is a small positive constant and will be termed as anharmonic constant. The frequency of the oscillator governed by the (1) is assumed to be 1. The m ($m \geq 2$) being a positive integer involving the order of anharmonicity. From our choice of the Hamiltonian, it appears that we will be investigating for the anharmonic oscillators of having even orders of anharmonicities. It is attributed because of the fact that most of the time we are interested in the matter–field interaction to discuss the anharmonic effects. The medium with inversion symmetry automatically eliminates the even orders of nonlinear susceptibilities and hence the even orders of nonlinear polarization. Again, the odd orders of nonlinear polarization contribute to the even orders of nonlinear terms in the model Hamiltonian. By virtue of symmetry, the entire gaseous medium is inversion symmetric. Therefore, the choice of the Hamiltonian is quite consistent with the matter–field interaction. Now, the equations of motion corresponding to the Hamiltonian (1) follows from the Hamiltonian's equation. Therefore, we have

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q - \lambda q^{2m-1} - f \end{aligned} \tag{2}$$

The above two equations (2) are combined to obtain a second-order inhomogeneous nonlinear differential equation. The corresponding equation is given by

$$\ddot{q} + q + \lambda q^{2m-1} = -f \tag{3}$$

In absence of driven term (i.e., $f = 0$) and for $m = 2$, the (3) corresponds to the equation of motion of a classical quartic anharmonic oscillator. Unfortunately, the classical quartic anharmonic oscillator does not give a closed-form analytical solution. Of course, for small values of λ, the classical quartic anharmonic oscillator is solved approximately [6]. In absence of anharmonic and driven terms, the Hamiltonian reduces to the Hamiltonian of a simple (one dimensional) harmonic oscillator. Now, the quantum mechanical counterpart of the Hamiltonian (1) is obtained by the replacement of the classically conjugate position $q(t)$ and momentum $p(t)$ by their corresponding operators. During the passage from classical anharmonic oscillator governed by the Hamiltonian (1) to the corresponding quantum mechanical oscillator, the fundamental commutation relation between the position and momentum operators should be respected.

3 Driven Quantum Anharmonic Oscillator

In order to obtain the quantum mechanical counterpart of the Hamiltonian (1), we replace the classical canonical position (q) and momentum (p) by their equivalent operators. Therefore, we have

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2} + \frac{\lambda}{2m}\,\hat{q}^{2m} + f\hat{q} \tag{4}$$

In the Hamiltonian (4), we assumed that the mass and the frequency of the oscillator are unity. During the passage from classical-driven anharmonic oscillator to its quantum mechanical counterpart, we need to the impose the following equal time commutation relation [9,10]:

$$[\hat{q}(t),\, \hat{p}(t)] = [\hat{q}(0),\, \hat{p}(0)] = i \tag{5}$$

where $\hbar = 1$. Now, in order to differentiate between the *c-numbers* and the operators, we use the caret for the operators. Based on the relation (5), we define the usual relations connecting the position and momentum operators with those of the dimensionless annihilation ($\hat{a}$) and creation ($\hat{a}^\dagger$) operators

$$\begin{aligned} \hat{a}(t) &= \tfrac{1}{\sqrt{2}}\left(\hat{q}(t) + i\hat{p}(t)\right) \\ \hat{a}^\dagger(t) &= \tfrac{1}{\sqrt{2}}\left(\hat{q}(t) - i\hat{p}(t)\right) \end{aligned} \tag{6}$$

Hence, we have

$$\begin{aligned} \hat{q}(t) &= \tfrac{1}{\sqrt{2}}\left(\hat{a}(t) + \hat{a}^\dagger(t)\right) \\ \hat{p}(t) &= -\tfrac{i}{\sqrt{2}}\left(\hat{a}(t) - \hat{a}^\dagger(t)\right) \end{aligned} \tag{7}$$

Therefore, it follows that the annihilation operator $\hat{a}(t)$ and creation operator $\hat{a}^\dagger(t)$ obey the following commutation relation:

$$[\hat{a}(t),\, \hat{a}^\dagger(t)] = 1 \tag{8}$$

In the later part of this article, we shall simply use $\hat{a}$ and $\hat{a}^\dagger$ instead of $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ respectively. Now, the Hamiltonian follows as

$$\hat{H} = \left(a^\dagger a + \frac{1}{2}\right) + \frac{\lambda}{2m}\,\frac{1}{2^m}\left(\hat{a} + \hat{a}^\dagger\right)^{2m} + f\left(\hat{a} + \hat{a}^\dagger\right) \tag{9}$$

In order to obtain the equations of motion involving the operators $\hat{q}$ and $\hat{p}$, we use the Heisenberg operator equation of motion. Therefore, we have

$$\dot{a} = -ia - \frac{i\lambda}{2^m}\left(\hat{a} + \hat{a}^\dagger\right)^{2m-1} - if \tag{10}$$

where, we have made use of the commutation relation $\left[a, g(a, a^\dagger)\right] = \frac{\partial g(a,a^\dagger)}{\partial a^\dagger}$. The equation of motion for the field operator $a^\dagger$ may easily be obtained by taking the Hermitian conjugate of the (10). In order to study the dynamical behavior of the field operators, we need to solve the differential equation (10). Unfortunately, the differential equation is nonlinear and is involving the noncommuting operators. Therefore, a numerical solution or an approximate analytical solution may be explored. The differential equation involving the noncommuting operators are certainly a nontrivial problem if at all possible. Therefore, in the present article, we rely on the so-called rotating wave approximation for solving the above differential equation. The RWA entails us to remove the nonconserving energy terms. In order to do so, we express the normal ordered form of $\left(\hat{a}+\hat{a}^{\dagger}\right)^{2m-1}$

$$\left(\hat{a}+\hat{a}^{\dagger}\right)^{2m-1}=\sum_{r=0}^{m-1}(2r-1)!!\begin{pmatrix}2m-1\\2r\end{pmatrix}:\left(\hat{a}+\hat{a}^{\dagger}\right)^{2m-2r-1}: \tag{11}$$

where the notation :: stands for the normal ordered form, the binomial coefficient $\begin{pmatrix}x\\y\end{pmatrix}=\frac{x!}{(x-y)!y!}$ and $(2r-1)!!=(2r-1)(2r-3)....3.1$. Now, we make a binomial expansion of the term $\left(:\left(\hat{a}+\hat{a}^{\dagger}\right)^{2m-2r-1}:\right)$ under normal ordered form. We have $2m-2r$ number of terms. In this expansion, we have two middle terms. Out of these two terms, the middle term $t_{m-r+1}=\begin{pmatrix}2m-2r-1\\m-r+1\end{pmatrix}\hat{a}^{\dagger m-r}\hat{a}^{m-r+1}$ will be the synchronous one. Now, we neglect all the terms except the synchronous one in the expansion of $\left(\hat{a}+\hat{a}^{\dagger}\right)^{2m-1}$ in the (10). Hence, the (10) assumes the following form:

$$\dot{\hat{a}}=-i[\hat{a},\hat{H}]=-i\hat{O}\hat{a}-if \tag{12}$$

where, the operator $\hat{O}(t)$ is given by

$$\hat{O}(t)=1+\tfrac{\lambda}{2^m}\sum_{r=0}^{m-1}(2r-1)!!\begin{pmatrix}2m-1\\2r\end{pmatrix}\begin{pmatrix}2m-2r-1\\m-r+1\end{pmatrix}\hat{a}^{\dagger m-r}\hat{a}^{m-r} \tag{13}$$

The equation (12) of the field operator $\hat{a}$ corresponds the equation of motion of the driven m-photon anharmonic oscillator. As a matter of fact, the m-photon anharmonic oscillator is found useful for investigating squeezing [11–13], phase properties [14], Wigner function [15], and the kth power squeezing of the radiation field [16]. However, the driven m-photon anharmonic oscillator is still unexplored. For $f=0$, the operator $\hat{O}(t)$ is a constant of motion. The operator $\hat{O}(t)$ is a function of number operator $a^{\dagger}a$. This is quite reasonable since the conservation of number is guaranteed by the use of RWA and hence the removal of nonsynchronous terms from the equations of motion. The operator $\hat{O}$ has the dimension of frequency and hence may be called as a frequency operator. In absence of anharmonicity ($\lambda=0$), the operator $\hat{O}$ reduces to a unit operator. Essentially, the anharmonic part of the Hamiltonian (4) is responsible for the shift of the frequency of the oscillator (here we assumed 1). In other words, the frequency normalization takes place through the nonlinear term. The complete solution of the differential equation (12) can only be obtained if the functional form of $f(t)$ is explicitly known. The corresponding solution will follow as

$$e^{-i\hat{O}t}A - e^{-i\hat{O}t}i\int f(t)e^{i\hat{O}t}dt \tag{14}$$

where A is the integration constant to be evaluated under initial condition. Taking the Hermitian conjugate of the above solution (14), we obtain the the solution for the creation operator $\hat{a}^\dagger$. Therefore, the dynamical behavior of the driven anharmonic oscillator under the rotating wave approximation are completely solved provided the integral (14) is evaluated for physically acceptable function $f(t)$. As an example, we assume that the oscillator is periodically driven. Therefore, we have

$$f(t) = f(0)\cos\omega t \tag{15}$$

Finally, the analytical expressions for the annihilation operator (14) assumes the following form:

$$\hat{a}(t) = e^{-i\hat{O}t}\hat{a}(0) + f(0)\left(\hat{O}^2 - \omega^2\right)^{-1}\left[\hat{O}e^{-i\hat{O}t} - \hat{O}\cos\omega t + i\omega\sin\omega t\right] \tag{16}$$

It is possible to obtain the creation operator $a^\dagger$ from the (16). Assuming $f(0)$ as real, we have

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\hat{O}t} + f(0)\left(\hat{O}^2 - \omega^2\right)^{-1}\left[\hat{O}e^{i\hat{O}t} - \hat{O}\cos\omega t - i\omega\sin\omega t\right] \tag{17}$$

where the Hermitian nature of the frequency operator $\hat{O}$ is utilized. Certainly, the solutions for the field operators involving the (16) and (17) correspond the physical model of a multiphoton driven anharmonic oscillator. Interestingly, the dynamics of the driven anharmonic oscillator corresponding to the Hamiltonian (9) is completely known. The exact nature of the solution (16) is certainly useful for investigating quantum statistical properties of the radiation field coupled to the oscillator. It is of interest to calculate the shift of the frequency of the oscillator by calculating the matrix elements of $< n|\hat{a}(t)|n+1 >$ in terms of the number state basis $|n >$. It is clear that the shift of the frequency of the oscillator is governed by the anharmonic parameter Before we go further, we check the equal time commutation relation between the annihilation and creation operators. This can easily be established since the time derivative of the following commutator vanishes:

$$\frac{d}{dt}\left[a(t), a^\dagger(t)\right] = 0 \tag{18}$$

In calculating the above relation (18), we have made use of the (12) and it's Hermitian conjugate. Now, we construct the dimensionless quadrature operators which are happened to be position and momentum operators involving the (7). Therefore, we have

$$\begin{aligned}\hat{q}(t) &= \tfrac{1}{\sqrt{2}}\left(e^{-i\hat{O}t}\hat{a}(0) + a^\dagger(0)e^{i\hat{O}t} + 2f(0)\hat{O}\left(\hat{O}^2 - \omega^2\right)^{-1}\left\{\cos\hat{O}t - \cos\omega t\right\}\right)\\ \hat{p}(t) &= -\tfrac{i}{\sqrt{2}}\left(e^{-i\hat{O}t}\hat{a}(0) - a^\dagger(0)e^{i\hat{O}t} - 2if(0)\left(\hat{O}^2 - \omega^2\right)^{-1}\left\{\hat{O}\sin\hat{O}t - \omega\sin\omega t\right\}\right)\end{aligned} \tag{19}$$

In addition to the fundamental academic interests, the present solution will find potential applications in the investigation of the quantum statistical properties of the radiation field coupled to a nonlinear medium. However, the present exact solution is obtained at the cost of the sacrifice of nonconserving energy (i.e., RWA) terms in the original Hamiltonian (13).

4 Conclusion

In the present investigation, we provide the solution of the driven anharmonic oscillator. The complete solution of the driven anharmonic oscillator requires the knowledge of the functional form of the driven parameter. The present solution is explored under the RWA and is certainly based on the complete analytical approach. The present solution will find applications in the investigation of quantum statistical properties of the radiation fields and in the dynamics of the trapped ions/atoms in an MOT. These quantum statistical properties include squeezing, higher ordered squeezing, photon antibunching, and photon statistics.

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References

1. Nayfeh, A.H.: Introduction to Perturbation Techniques. Wiley, New York (1981)
2. Nayfeh, A.H., Mook, D.T.: Non-linear Oscillations. Wiley, New York (1979)
3. Bellman, R.: Methods of Nonlinear Analysis, vol. 1, p. 198. Academic Press, New York (1970)
4. Ross, S.L.: Differential Equation, 3rd edn. Wiley, New York (1984)
5. Mandal, S.: Phys. Lett. A **299**, 531–542 (2002)
6. Mandal, S.: J. Phys. A **31**, L501–L505 (1998)
7. Pathak, A., Mandal, S.: Phys. Lett. A **286**, 261–276 (2001)
8. Bender, C.M., Bettencourt, L.M.A.: Phys. Rev. Lett **77**, 4114–4117 (1996)
9. Gasiorowicz, S.: Quantum Physics, p. 271. Wiley, New York (1979)
10. Schiff, L.I.: Quantum Mechanics, vol. 3, p. 176. Mc. Graw Hill Book Company, New York (1987)
11. Gerry, C.C.: Phys. Lett. A **124**, 237–239 (1987)
12. Buzek, V.: Phys. Lett. A **136**, 188–192 (1989)
13. Tanas, R.: Phys. Lett. A **141**, 217–220 (1989)
14. Paprzycka, M., Tanas, R.: Quant. Opt. **4**, 331 (1992)
15. Kheruntsyan, K.V.: J. Opt. B Quantum Semiclass. Opt. **1225** (1999)
16. Buzek, V., Jex, I.: Phys. Rev. A **41**, 4079 (1990)