On the New Fractional Operator and Application to Nonlinear Bloch System

J. F. Gómez-Aguilar, Behzad Ghanbari and Ebenezer Bonyah

Abstract In this chapter, we analyze the nonlinear Bloch system with a new fractional operator without singular kernel proposed by Michele Caputo and Mauro Fabrizio. The commensurate and non-commensurate order nonlinear Bloch system is considered. Special solutions using a numerical scheme based in Lagrange interpolations were obtained. We studied the uniqueness and existence of the solutions employing the fixed point theorem. Novel chaotic attractors with total order less than 3 are obtained.

Keywords Fractional calculus · Bloch system · Exponential-decay law · Lagrange interpolation

1 Introduction

The nonlinear Bloch system is a system consisting of three nonlinear ODEs which can be used to model time-dependent nuclear magnetization. These equations are efficient tool to describe the Nuclear Magnetic Resonance (NMR). The dynamic balance between externally applied magnetic fields and also internal sample relaxation times [1] is explained by the Bloch system. Taking advantage of fractional-order differential equations, we model this relaxation as a multiexponential process.

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Fractional order systems are suitable for describing the memory properties of several materials, because it has a stronger memory function. It is well known that, fractional-order derivatives have made great progress in mathematical modeling of the dynamics of complex systems, multi-scale phenomena and viscoelastic materials [2–11]. The fractional derivative with power-law singular kernel imposes an artifical singularity to mathematical models and the memory effects cannot describe accurately. Due to this inconvenience, a new fractional derivative considering the exponential function as non-singular kernel was proposed by Caputo and Fabrizio [12]. This new operator allows to describe more efficiently the memory effect and do not impose artificial singularities as in the old Liouville-Caputo derivative. Several problems in chemical reactions, luminescence, heat transfer, geophysics, physical optics, radioactivity, thermoelectricity, vibrations and electromagnetism are naturally governed by the exponential decay law. These natural phenomena can be studied considered the exponential kernel suggested by Caputo and Fabrizio. Furthermore, this new operator has supplementary properties, it can portray substance heterogeneities and configurations with different scales, which noticeably cannot be managed with the other representations [13–15]. Losada and Nieto in [16] studied the further properties.

Atangana and Baleanu generalized the exponential function and proposed the Mittag-Leffler law as kernel of differentiation [17] arising the Atangana-Baleanu fractional derivative. The fractional-order derivatives with non-singular kernel allows to describe two different waiting times distribution, which is an ideal waiting time distribution as such is observed in nuclear magnetization. The crossover behavior of both operators is due to their capacity of not obeying the classical index-law imposed in fractional calculus. This apparent limitation allows to permits describe more appropriate real world problems [18–21]. In [22], several examples of non-commutative and non-associative problems were presented. The authors justify why the fractional derivatives with non-singular kernel are needed to describe real-world problems. The authors conclude that the commutativity or index-law and semi-group principle are irrelevant in fractional calculus, ending the controversy generated for the use of these fractional-order derivatives.

In recent years, the generalized nonlinear Bloch equation with fractional-order derivatives has attracted great interest of many researchers and scholars in literature [23–30]. A predictor-corrector approach to solve the multi-term time-fractional Bloch equations has been developed in [31]. Also, for some other variants of the equation including Bloch equations with Riemann-Liouville fractional derivative [32–35] or the delay-dependent fractional Bloch equations [36–38].

In this chapter, we apply the new fractional operator with exponential-decay law to the nonlinear Bloch model. We studied the uniqueness and existence of the solutions employing the fixed point theorem. The manuscript is organized as follows. The paper is structured as follows. In Sect. 2, we recall the fractional operators of type Liouville-Caputo sense. In Sect. 3, we formulate the fractional order nonlinear Bloch model, and then the existence and uniqueness of the coupled solutions is proved. We consider numerical simulations in Sect. 4. Finally, we summarize and conclude in Sect. 5.

2 Fractional Operators

Based in the exponential-decay law, the Caputo-Fabrizio fractional operator without singular kernel in Liouville-Caputo sense (CFC) is given by [12]

$$ {}_{0}^{CFC}\mathcal{D}_t^{\gamma}\{f(t)\} = \frac{M(\gamma)}{n-\gamma}\int_0^t \frac{d^n}{dt^n} f(\theta)\exp\Big[-\frac{\gamma}{n-\gamma}(t-\theta)\Big]d\theta, \quad n-1<\gamma\leq n, \tag{1} $$

where $M(\gamma)$ is a normalization function such that $M(0)=M(1)=1$.

The Caputo-Fabrizio fractional integral is defined below [16]

$$ {}_{0}^{CF}I_t^{\gamma} f(t) = \frac{2(1-\gamma)}{M(\gamma)(2-\gamma)}f(t) + \frac{2\gamma}{M(\gamma)(2-\gamma)}\int_0^t f(s)ds. \qquad t\geq 0. $$

where,

$$ M(\gamma) = \frac{2}{2-\gamma}, \qquad 0<\gamma<1. \tag{2} $$

Losada and Nieto [16] analyzed more properties of this newly presented fractional operator.

3 Bloch System with Non-singular Kernel

The nonlinear Bloch system [36] in Caputo-Fabrizio-Caputo sense is given by

$$ \begin{aligned} {}_{0}^{CFC}\mathcal{D}_t^{\gamma_1}x(t) &= \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t), \\ {}_{0}^{CFC}\mathcal{D}_t^{\gamma_2}y(t) &= -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t), \\ {}_{0}^{CFC}\mathcal{D}_t^{\gamma_3}z(t) &= y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t)-1), \end{aligned} \tag{3} $$

with initial conditions

$$ x(t) = x(0), \qquad y(t) = y(0), \qquad z(t) = z(0). \tag{4} $$

System (3) can be made more realistic as the nuclear magnetization as a function of time should not follow the same fractional order dynamics. For this reason, we introducing three different orders of the fractional-differential operators $\gamma_i \in (0, 1]$

for $i = 1, 2, 3$. The system (3) is called commensurate when $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, otherwise is called non-commensurate (for this case, the total order of the system is then changed from 3 to the sum of each particular order).

Existence of the coupled solutions.
We investigate the numerical results predicted by the fractional model given by the system (3). Firstly start to investigate the existence and uniqueness of the solutions. By using the fixed-point theorem, we define the existence of the solution. First, transform system (3) into an integral equation as follows

$$x(t) - x(0) = {}_0^{CF} I_t^{\gamma_1}\Big[\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\Big],$$

$$y(t) - y(0) = {}_0^{CF} I_t^{\gamma_2}\Big[- \zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\Big],$$

$$z(t) - z(0) = {}_0^{CF} I_t^{\gamma_3}\Big[y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\Big]. \quad (5)$$

On using the definition (2), we get

$$\begin{aligned} x(t) = x_0 &+ \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big\{\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\Big\} \\ &+ \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t \Big[\zeta y(s) + \varrho z(s)(x(s)\sin(\varphi) - y(s)\cos(\varphi)) - \frac{1}{\Psi_2}x(s)\Big]ds, \end{aligned} \quad (6)$$

$$\begin{aligned} y(t) = y_0 &+ \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big\{ - \zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\Big\} \\ &+ \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t \Big[- \zeta x(s) - z(s) + \varrho z(s)(y(s)\sin(\varphi) + x(s)\cos(\varphi)) - \frac{1}{\Psi_2}y(s)\Big]ds, \end{aligned} \quad (7)$$

$$\begin{aligned} z(t) = z_0 &+ \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big\{y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\Big\} \\ &+ \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t \Big[y(s) - \varrho\sin(\varphi)(x(s)^2 + y(s)^2) - \frac{1}{\Psi_1}(z(s) - 1)\Big]ds. \end{aligned} \quad (8)$$

Now, we consider the following kernels

$$G_1(t, x(t)) = \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t),$$

$$G_2(t, y(t)) = -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t), \quad (9)$$

$$G_3(t, z(t)) = y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1).$$

Now, we prove that the kernels G_1, G_2 and G_3 satisfy the Lipschitz condition. To achieve we first prove this condition for each kernel proposed. We start with the kernel 1. Let x and X be two functions, using the Cauchy's inequality, then we assess the following

$$||G_1(t, x(t)) - G_1(t, X(t))|| \leq \left|\left|\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\right|\right|. \quad (10)$$

Similarly for the second and third cases, we have

$$\begin{aligned}||G_2(t, y(t)) - G_2(t, Y(t))|| \leq \bigg|\bigg| &- \zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) \\ &+ x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\bigg|\bigg|,\end{aligned}$$

$$||G_3(t, z(t)) - G_3(t, Z(t))|| \leq \left|\left|y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\right|\right|, \quad (11)$$

consider the following recursive formula, we have

$$x_{(n)}(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}G_1(t, x_{(n-1)}) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t G_1(s, x_{(n-1)})ds,$$

$$y_{(n)}(t) = \frac{2(1-\gamma_2)}{(M(\gamma_2)(2-\gamma_2)}G_2(t, y_{(n-1)}) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t G_2(s, y_{(n-1)})ds,$$

$$z_{(n)} = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}G_3(t, z_{(n-1)}) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t G_3(s, z_{(n-1)})ds. \quad (12)$$

Applying the norm and the triangular inequality, we get

$$
\begin{aligned}
||a_{(n)}(t)|| &= ||x_{(n)}(t) - X_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}||G_1(t, x_{(n-1)}(t)) - G_1(t, X_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Bigg|\Bigg|\int_0^t \Big[G_1(s, x_{(n-1)}(s)) - G_1(s, X_{(n-2)}(s))\Big]\Bigg|\Bigg|ds, \\
||b_{(n)}(t)|| &= ||y_{(n)}(t) - Y_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}||G_2(t, y_{(n-1)}(t)) - G_2(t, Y_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Bigg|\Bigg|\int_0^t \Big[G_2(s, y_{(n-1)}(s)) - G_2(s, Y_{(n-2)}(s))\Big]\Bigg|\Bigg|ds, \\
||c_{(n)}(t)|| &= ||z_{(n)}(t) - Z_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}||G_3(t, z_{(n-1)}(t)) - G_3(t, Z_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Bigg|\Bigg|\int_0^t \Big[G_3(s, z_{(n-1)}(s)) - G_3(s, Z_{(n-2)}(s))\Big]\Bigg|\Bigg|ds,
\end{aligned}
\tag{13}
$$

where,

$$
x_{(n)}(t) = \sum_{m=0}^{\infty} a_m(t); \qquad y_{(n)}(t) = \sum_{m=0}^{\infty} b_m(t); \qquad z_{(n)}(t) = \sum_{m=0}^{\infty} c_m(t). \tag{14}
$$

Since the kernels satisfies the Lipschitz condition, we have

$$
\begin{aligned}
||a_{(n)}(t)|| = ||x_{(n)}(t) - X_{(n-1)}(t)|| &\leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1||x_{(n-1)}(t) - X_{(n-2)}(t)|| \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\int_0^t ||x_{(n-1)}(s) - X_{(n-2)}(s)||ds,
\end{aligned}
$$

$$
\begin{aligned}
||b_{(n)}(t)|| = ||y_{(n)}(t) - Y_{(n-1)}(t)|| &\leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3||y_{(n-1)}(t) - Y_{(n-2)}(t)|| \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4\int_0^t ||y_{(n-1)}(s) - Y_{(n-2)}(s)||ds,
\end{aligned}
$$

$$||c_{(n)}(t)|| = ||z_{(n)}(t) - Z_{(n-1)}(t)|| \leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)} \Delta_5 ||z_{(n-1)}(t) - Z_{(n-2)}(t)|| + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \Delta_6 \int_0^t ||z_{(n-1)}(s) - Z_{(n-2)}(s)|| ds. \tag{15}$$

Considering system (13) bounded, we have proven that the kernels satisfy Lipschitz condition, therefore following the results obtained in (13) using the recursive technique, we get the following relation

$$||a_{(n)}(t)|| \leq ||x(0)|| + \left\{ \left\{ \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)} \Delta_1 \right\}^n + \left\{ \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \Delta_2 t \right\}^n \right\},$$

$$||b_{(n)}(t)|| \leq ||y(0)|| + \left\{ \left\{ \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)} \Delta_3 \right\}^n + \left\{ \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \Delta_4 t \right\}^n \right\},$$

$$||c_{(n)}(t)|| \leq ||z(0)|| + \left\{ \left\{ \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)} \Delta_5 \right\}^n + \left\{ \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \Delta_6 t \right\}^n \right\}. \tag{16}$$

Therefore, Eq. (16) exists and is continuous. Nonetheless, to show that the above functions are a system of solutions of Eq. (3), we assume

$$x(t) = x_{(n)}(t) - \Xi_{1(n)}(t); \quad y(t) = y_{(n)}(t) - \Xi_{2(n)}(t); \quad z(t) = z_{(n)}(t) - \Xi_{3(n)}(t), \tag{17}$$

where $\Xi_{1(n)}$, $\Xi_{2(n)}$ and $\Xi_{3(n)}$ are reminder terms of series solution. Thus

$$x(t) - X_{(n)}(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)} G_1(t, x - \Xi_{1(n)}(t)) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \int_0^t G_1(s, x - \Xi_{1(n)}(s)) ds,$$

$$y(t) - Y_{(n)}(t) = \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)} G_2(t, y - \Xi_{2(n)}(t)) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \int_0^t G_2(s, y - \Xi_{2(n)}(s)) ds,$$

$$z(t) - Z_{(n)}(t) = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)} G_3(t, z - \Xi_{3(n)}(t)) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \int_0^t G_3(s, z - \Xi_{3(n)}(s)) ds. \tag{18}$$

Applying the norm on both sides and using the Lipschitz condition, we get

$$\left\|x(t) - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}G_1(t, x(t)) - x(0) - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t G_1(s, x(s))ds\right\|$$

$$\leq ||\Xi_{1(n)}(t))|| + \left\{\frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1 + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2 t\right\}||\Xi_{1(n)}(t)||,$$

$$\left\|y(t) - \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}G_2(t, y(t)) - y(0) - \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t G_2(s, y(s))ds\right\|$$

$$\leq ||\Xi_{2(n)}(t))|| + \left\{\frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3 + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4 t\right\}||\Xi_{2(n)}(t)||,$$

$$\left\|z(t) - \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}G_3(t, z(t)) - z(0) - \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t G_3(s, z(s))ds\right\|$$

$$\leq ||\Xi_{3(n)}(t))|| + \left\{\frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5 + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6 t\right\}||\Xi_{3(n)}(t)||. \quad (19)$$

On taking the limit $n \to \infty$ of Eq. (19), we get

$$x(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}G_1(t, x(t)) + x(0) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t G_1(s, x(s))ds,$$

$$y(t) = \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}G_2(t, y(t)) + y(0) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t G_2(s, y(s))ds,$$

$$z(t) = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}G_3(t, z(t)) + z(0) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t G_3(s, z(s))ds. \quad (20)$$

Uniqueness of the solutions.

We assume that we can find another solutions for Eq. (3); say $x(t)$, $y(t)$ and $z(t)$; then

$$x(t) - X(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big[G_1(t, x(t)) - G_1(t, X(t))\Big] + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t \Big[G_1(s, x(s)) - G_1(s, X(s))\Big]ds,$$

$$y(t) - Y(t) = \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big[G_2(t, y(t)) - G_2(t, Y(t))\Big] + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t \Big[G_2(s, y(s)) - G_2(s, Y(s))\Big]ds,$$

$$z(t) - Z(t) = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big[G_3(t, z(t)) - G_3(t, Z(t))\Big] + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t \Big[G_3(s, z(s)) - G_3(s, Z(s))\Big]ds. \tag{21}$$

Apply the norm both sides of Eq. (21), we have

$$||x(t) - X(t)|| \leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big[\Big|\Big|G_1(t, x(t)) - G_1(t, X(t))\Big|\Big|\Big] + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t \Big[\Big|\Big|G_1(s, x(s)) - G_1(s, X(s))\Big|\Big|\Big]ds,$$

$$||y(t) - Y(t)|| \leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big[\Big|\Big|G_2(t, y(t)) - G_2(t, Y(t))\Big|\Big|\Big] + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t \Big[\Big|\Big|G_2(s, y(s)) - G_2(s, Y(s))\Big|\Big|\Big]ds,$$

$$||z(t) - Z(t)|| \leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big[\Big|\Big|G_3(t, z(t)) - G_3(t, Z(t))\Big|\Big|\Big] + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t \Big[\Big|\Big|G_3(s, z(s)) - G_3(s, Z(s))\Big|\Big|\Big]ds, \tag{22}$$

considering the Lipschitz condition, having the fact in mind that the solutions are bounded, we get

$$||x(t) - X(t)|| \leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 + \left\{\frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right\}^n,$$

$$||y(t) - Y(t)|| \leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3\xi_3 + \left\{\frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4\xi_4 t\right\}^n,$$

$$||z(t) - Z(t)|| \leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5\xi_5 + \left\{\frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6\xi_6 t\right\}^n, \tag{23}$$

this is true for any n.

The system given by Eq. (3) has a unique solution if the below condition holds.

$$\left(1 - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right) \geq 0. \tag{24}$$

If the condition (24) holds, then

$$||x(t) - X(t)||\left(1 - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right) \leq 0, \tag{25}$$

implies that $||x(t) - X(t)|| = 0$. Then we get, $x(t) = X(t)$.

Employing the same way, we have

$$x(t) = X(t); \qquad y(t) = Y(t); \qquad z(t) = Z(t). \tag{26}$$

Therefore, we verified the uniqueness of coupled-solutions.

Now we propose a numerical solution of the nonlinear Bloch system considering the fractional derivative of Caputo-Fabrizio in Liouville-Caputo sense using the numerical scheme proposed by Atangana and Toufik in [39].

First we consider the following fractional differential equation with fading memory

$${}_{0}^{CFC}\mathscr{D}_t^{\alpha} y(t) = f(t, y(t)), \tag{27}$$

using the fundamental theorem of fractional calculus we obtain the solution of the above equation [39]

$$y_{n+1} = y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f(t_n, y_n) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f(t_{n-1}, y_{n-1}) \tag{28}$$

Again, we apply the numerical scheme (28) to have a numerical solution to Eq. (3) in Caputo-Fabrizio-Caputo sense

$$
\begin{aligned}
x_{n+1}(t) &= x_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_1(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_1(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), \\
y_{n+1} &= y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_2(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_2(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), \\
z_{n+1} &= z_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_3(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_3(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}),
\end{aligned}
\tag{29}
$$

where,

$$
\begin{aligned}
f_1(t, x(t), y(t), z(t)) &:= \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2} x(t), \\
f_2(t, x(t), y(t), z(t)) &:= -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2} y(t), \\
f_3(t, x(t), y(t), z(t)) &:= y(t) - \varrho \sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1).
\end{aligned}
\tag{30}
$$

In the next section, we consider Eq. (29) for obtain several numerical solutions considering different values of the fractional order γ arbitrarily chosen.

4 Numerical Simulations

Numerical solutions of the system (3) have been depicted in Fig. 1a–f and Fig. 2a–f for the commensurate and non-commensurate order system, respectively. The parameter values used in the simulations are $\zeta = 1.26$, $\varrho = 10$, $\varphi = 0.7764$, $\Psi_1 = 0.5$, $\Psi_2 = 0.25$ and the initial conditions are $x(t) = 0.1$, $y(t) = 0.1$ and $z(t) = 0.1$. The step size used in evaluating the approximate solution was $h = 0.0001$.

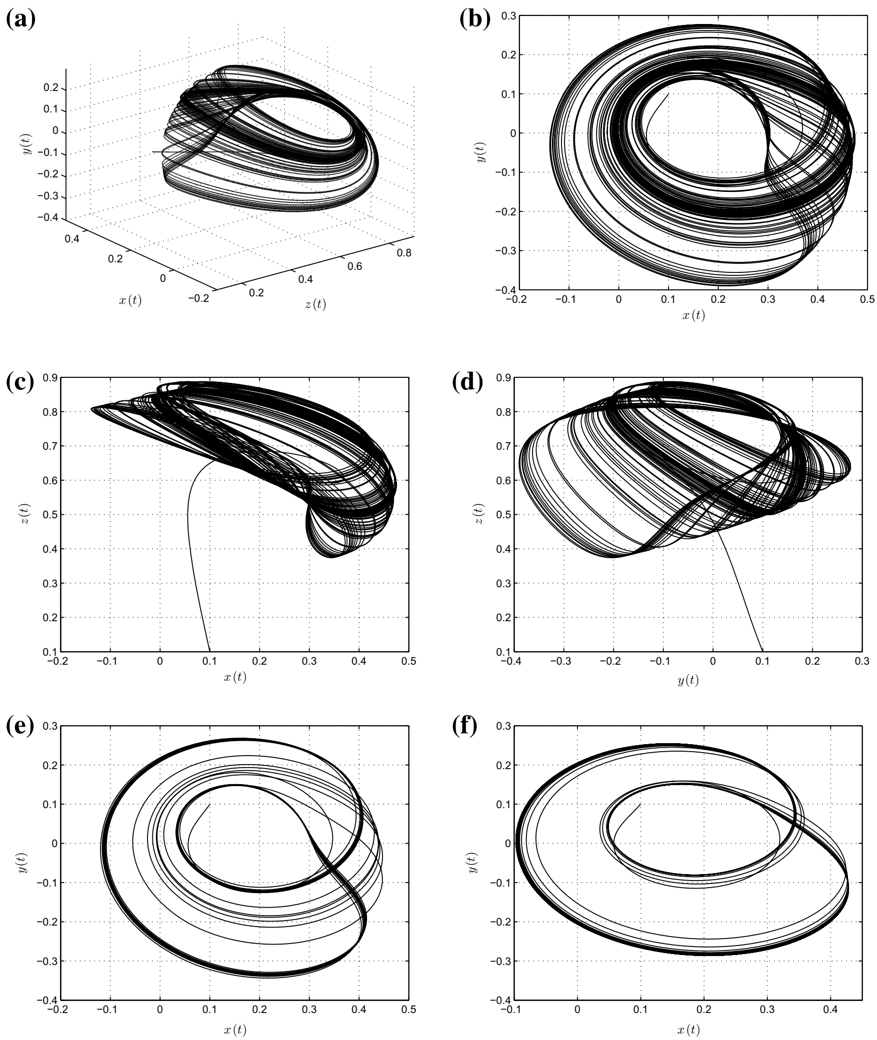

Fig. 1 Numerical simulation for the commensurate nonlinear Bloch system with non-singular kernel. In **a–d** projections of chaos for $\gamma = 0.95$. In **e–f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma = 0.92$ and $\gamma = 0.87$, respectively

Numerical solutions of the system (3) have been depicted in Fig. 3a–f and Fig. 4a–f for the commensurate and non-commensurate order system, respectively. The parameter values used in the simulations are $\zeta = -1.26$, $\varrho = 35$, $\varphi = 0.173$, $\Psi_1 = 5$, $\Psi_2 = 2.5$ and the initial conditions are $x(t) = 0.1$, $y(t) = 0.1$ and $z(t) = 0.1$. The step size used in evaluating the approximate solution was $h = 0.0001$.

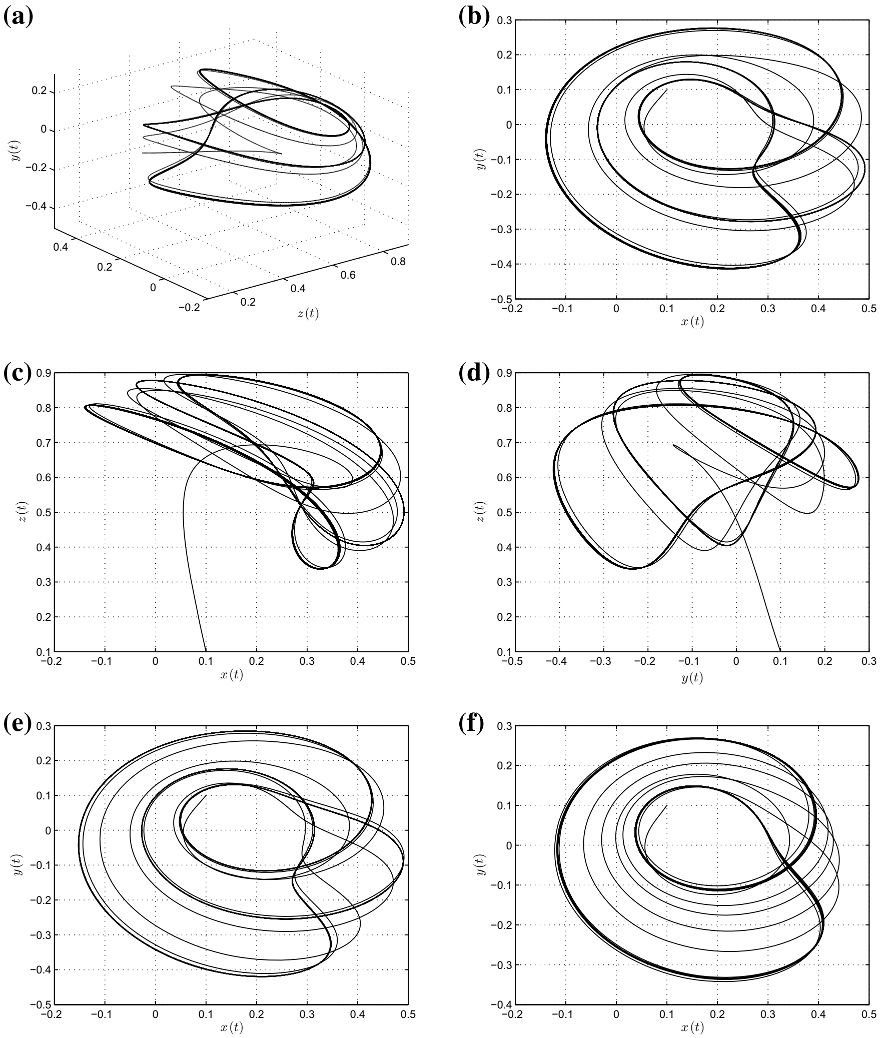

Fig. 2 Numerical simulation for the non-commensurate nonlinear Bloch system with non-singular kernel. In **a**–**d** projections of chaos for $\gamma_1 = 1, \gamma_2 = 0.95$ and $\gamma_3 = 1$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma_1 = 0.94$, $\gamma_2 = 1$ and $\gamma_3 = 1$ and $\gamma_1 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 0.92$, respectively

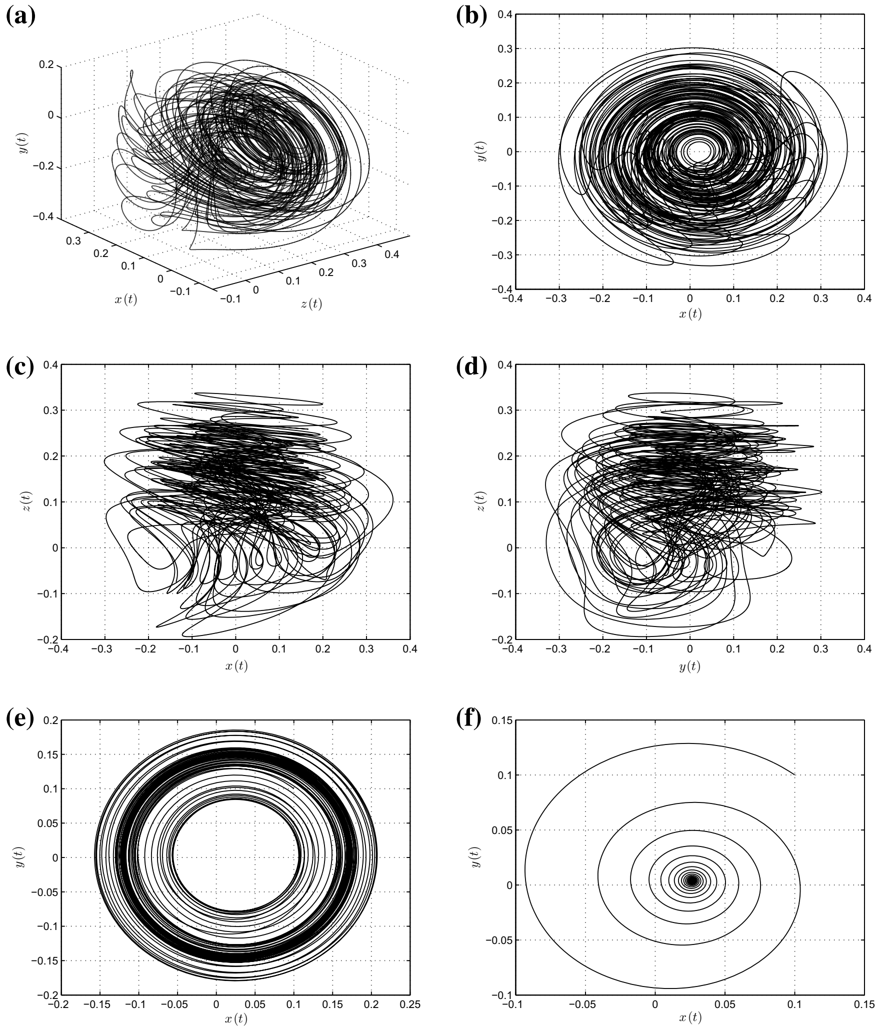

Fig. 3 Numerical simulation for the commensurate nonlinear Bloch system with with non-singular kernel. In **a**–**d** projections of chaos for $\gamma = 0.95$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma = 0.92$ and $\gamma = 0.87$, respectively

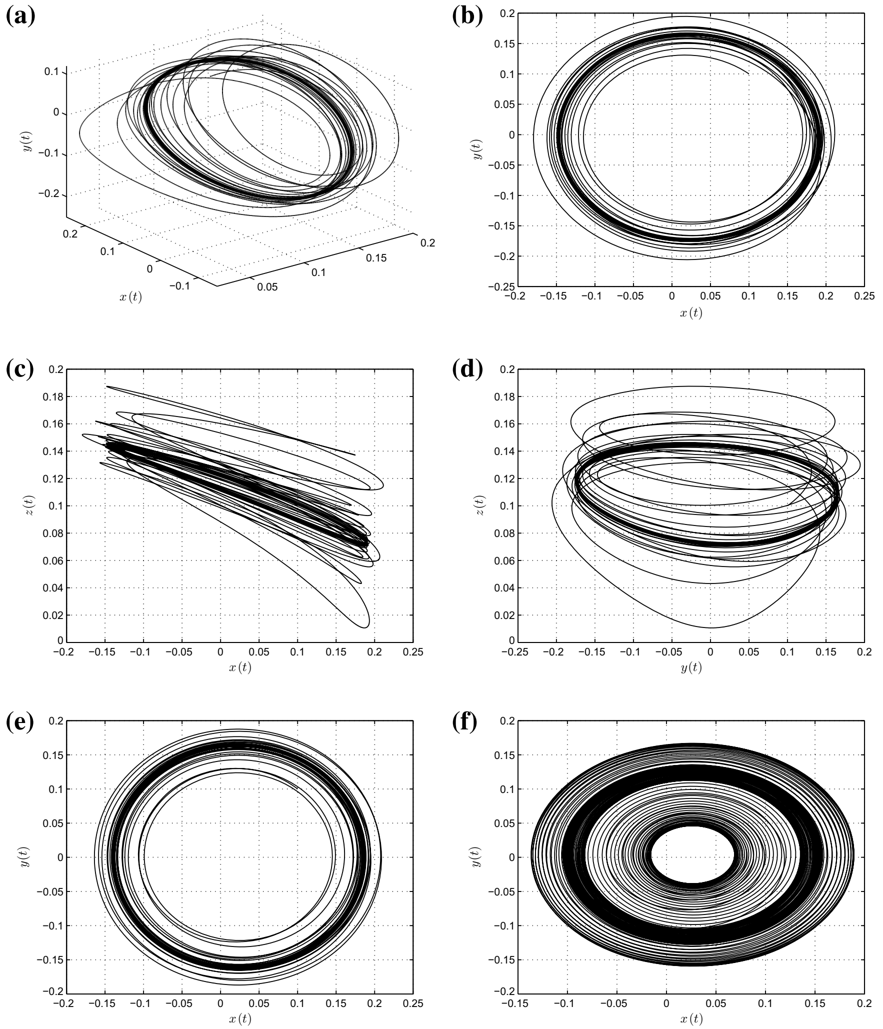

Fig. 4 Numerical simulation for the non-commensurate nonlinear Bloch system with with non-singular kernel. In **a**–**d** projections of chaos for $\gamma_1 = 1$, $\gamma_2 = 0.95$ and $\gamma_3 = 1$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma_1 = 0.94$, $\gamma_2 = 1$ and $\gamma_3 = 1$ and $\gamma_1 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 0.92$, respectively

5 Conclusions

In this chapter, we used the new definition of fractional operator with an exponential kernel proposed by Caputo and Fabrizio. This new operator can describe material heterogeneities and structures with different scales, which cannot be handling with the classical theories. To further apply this operator, we have modified the nonlinear Bloch equation with feedback. We prove the existence and uniqueness of the coupled-solutions. The numerical results for nonlinear Bloch with non-singular kernel shows that with decreases the order of time-fractional operator ($\gamma \to 0$), several irregular attractors are formed and the model exhibit transient chaos. The characteristics of the alternative model, in contrast with the classical model, memory properties, the nuclear magnetization or other independent quantities are considered.

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