Asymptotically Almost Automorphic Solution for Neutral Functional Integro Evolution Equations on Time Scales

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Abstract The script is dedicated to look at the existence, uniqueness with stability consequence of asymptotically almost automorphic ($\mathcal{AAA}$) solution for integro neutral evolution equation on time scales by applying fixed point hypothesis. We give the time scale adaptation of ($\mathcal{AAA}$) functions. Toward the end, a precedent is given for the adequacy of the hypothetical outcomes.

Keywords Asymptotically almost automorphic function · Evolution system · Neutral · Integro · Time scales

1 Introduction

Generally, one study the continuous and discrete cases differently and there are many different sets which are very utilizable. Ergo, this an arduous task that we study differently for all cases. So for evading this type quandary, Hilger, in 1988, [1] present time scales hypothesis which cumulates discrete and continuous investigation. This hypothesis present a robust actualize for applications to populace models, financial matters and quantum material science among others. Thus, managing issues of differential conditions on time scales turns out to be extremely noteworthy and deliberate in the examination field of dynamic frameworks. For more subtle elements of this theme, we allude to the papers [2–4] and the books [5, 6]. These give a glorious portrayal of time scale hypothesis and its apparatus.

Almost automorphy, which is a natural generalization of almost periodicity introduced by Bochner [7]. In [8, 9], the literature of almost automorphy and its applications to differential equations are describe. Recently, the existence of almost automorphic ($\mathcal{AA}$) type solutions for evolution equations has attracted more and

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more attention. The literature of the concept of asymptotically almost automorphy, as a natural extension of almost automorphy, was introduced by N'Guérékata [10]. Now a days, these type of functions have made lots of developments and applications in real life, we refer for more details [11–15].

There are numerous marvels, for example, in the investigation of oscillatory frameworks and in the displaying of a few physical issues, where the theory of neutral differential equations arises [16]. There are many papers on existence of $\mathcal{AAA}$ solution for continuous cases. As per our knowledge, there is no paper on time scale where these type of solution is discussed with neutral functional term in abstract space. The rationale of the present article is discover the existence and uniqueness with stability of $\mathcal{AAA}$ solution for the neutral integro evolution equation on periodic time scale $\mathbb{T}$,

$$\begin{aligned}[y(r) - g(r, y(\varkappa(r)))]^{\Delta} &= A(r)[y(r) - g(r, y(\varkappa(r)))] + \mathcal{P}(r, y(r)) \\ &\quad + \int_{-\infty}^{r} k(r, \sigma(s))h(r, y(s))\Delta s, \end{aligned} \tag{1.1}$$

$r \in \mathbb{T}$. $A(r) : \mathcal{D}(A(r)) \subset Y \to Y$ is a family of linear operators, where Y is Banach space. $|k(r, s)| \leq ce_{\ominus\lambda}(r, s)$, c and λ are positive constant and $\varkappa : \mathbb{T} \to \mathbb{T}$ satisfying $\varkappa(r) \leq r$ for all $r \in \mathbb{T}$. The functions $\mathcal{P} : \mathbb{T} \times Y \to Y$, $g, h : \mathbb{T} \times Y \to Y$ are defined later with specified conditions in next section.

Whatever is left of this article as follows. In Sect. 2, we give basic definitions, results and lemmas. In Sect. 3, using Banach contraction principle, existence and uniqueness of $\mathcal{AAA}$ solution of system (1.1) is discussed. In Sect. 4, some conditions for stability are obtained . In last Sect. 5 a numerical example is shown for potency of hypothetical outcomes.

2 Preliminaries

In this segment, some essential hypothesis and facts for time scales is given which is required for further work.

A time scale, $\mathbb{T}$, is a non empty closed subset of real line. The backward and forward operator is define by $\rho(\zeta) = \sup\{s \in \mathbb{T} : s < \zeta\}$ and $\sigma(\zeta) = \inf\{s \in \mathbb{T} : s > \zeta\}$ respectively. A point ζ is a left dense point and left scattered point when $\rho(\zeta) = \zeta$ and $\rho(\zeta) < \zeta$ respectively with $\zeta > \inf \mathbb{T}$. Also, ζ is right scattered point and right dense when $\sigma(\zeta) > \zeta$ and $\sigma(\zeta) = \zeta$ respectively with $\zeta < \sup \mathbb{T}$. A function $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(\zeta) = \sigma(\zeta) - \zeta$, $\forall \zeta \in \mathbb{T}$, is known as the graininess operator. We will mean the interval $[c, d]_{\mathbb{T}} = \{\zeta \in \mathbb{T} : c \leq \zeta \leq d\}$.

Definition 2.1 If $\Lambda : \mathbb{T} \to \mathbb{R}$ is a function and at left dense points, its left-side limits exist and continuous at right dense points of $\mathbb{T}$ then it is known as rd-continuous. The collection of all rd-continuous functions $\Lambda : \mathbb{T} \to \mathbb{R}$ will be mean by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 Reference [5] A function $q : \mathbb{T} \to \mathbb{R}$ is said to be regressive (positive regressive) if $1 + \mu(\zeta)q(\zeta) \neq 0 (> 0),\ \forall \zeta \in \mathbb{T}$. The collection of regressive (positive regressive) functions is represented by $\mathcal{R}(\mathcal{R}^+)$.

Definition 2.3 Reference [5] Let $\Lambda : \mathbb{T} \to \mathbb{R}$ and $\zeta \in \mathbb{T}$. Δ-derivative, $\Lambda^{\Delta}(\zeta)$ is the number if exist, such that given any $\varepsilon > 0,\ \exists$ a neighbourhood U of ζ such that

$$\left|[\Lambda(\sigma(\zeta)) - \Lambda(s)] - \Lambda^{\Delta}(\zeta)[\sigma(\zeta) - s]\right| \leq \varepsilon|\sigma(\zeta) - s|,\ \forall\, s \in U.$$

Let Λ is rd-continuous; if $\Lambda_*^{\Delta}(\zeta) = \Lambda(\zeta)$, the delta integral is defined by,

$$\int_r^s \Lambda(\zeta)\Delta\zeta = \Lambda_*(s) - \Lambda_*(r), \quad s, r \in \mathbb{T}.$$

Definition 2.4 The exp function on $\mathbb{T}$ is defined as

$$e_q(\tau, \zeta) = \exp\left(\int_\zeta^\tau \xi_{\mu(t)}(q(t))\Delta t\right), \quad \tau, \zeta \in \mathbb{T}, q \in \mathcal{R}.$$

For $b > 0$,

$$\xi_b(Z) = \frac{1}{b} log(1 + Zb).$$

For $b = 0$, $\xi_0(Z) = Z$.

Definition 2.5 Reference [6] Let $q, p \in \mathcal{R}$, define

$$\ominus q = \frac{-q}{1 + \mu q}, \quad q \oplus p = q + p + \mu q p, \quad q \ominus p = q \oplus (\ominus p).$$

Lemma 2.6 *Reference [6] Let us suppose that $p, q \in \mathcal{R}$, then*

1. $e_0(\zeta, r) = 1, \quad e_p(\zeta, \zeta) = 1;$
2. $e_p(\sigma(\zeta), r) = (1 + \mu(\zeta)p)e_p(\zeta, r);$
3. $e_p(\zeta, r) = 1/e_p(r, \zeta) = e_{\ominus p}(r, \zeta);$
4. $e_p(\zeta, r)e_p(r, s) = e_p(\zeta, s);$
5. $e_p(\zeta, r)e_q(\zeta, r) = e_{p \oplus q}(\zeta, r);$
6. $(1/e_p(\zeta, r))^{\Delta} = -p(\zeta)/e_p(\sigma(\zeta), r).$

Lemma 2.7 *Reference [6] Let $q \in \mathcal{R}$ and $b, c, d \in \mathbb{T}$, then*

$$\int_b^c q(\zeta)e_q(d, \sigma(\zeta))\Delta\zeta = e_q(d, b) - e_q(d, c).$$

Lemma 2.8 *Reference [17] For* $0 < \lambda$, $e_{\ominus\lambda}(\zeta, \eta) \leq 1$, $\forall \eta, \zeta \in \mathbb{T}$, *where* $\eta \leq \zeta$.

Definition 2.9 Reference [17] $\mathbb{T}$ is called periodic time scale, if

$$\Pi := \{w \in \mathbb{R} : \zeta \pm w \in \mathbb{T}, \forall \zeta \in \mathbb{T}\} \neq \{0\}.$$

The notations in this section follow as: Y is Banach space with sup norm $\|y\|_\infty = \sup_{r\in\mathbb{T}} \|y(r)\|$. $C(\mathbb{T}, Y)$ contains the collection of continuous functions from $\mathbb{T}$ to Y. $C_0(\mathbb{T}, Y)$ is proper subset of $C(\mathbb{T}, Y)$ containing functions $g : \mathbb{T} \to Y$ which vanish at infinity i.e., $\lim_{|r|\to\infty} \|g(r)\| = 0$ and $C_0(\mathbb{T} \times Y, Y)$ denotes the collection of functions $g : \mathbb{T} \times Y \to Y$ such that $\lim_{|r|\to\infty} \|g(r, y)\| = 0$ uniformly for y in any compact subset of Y.

Definition 2.10 A function $g(r) \in C(\mathbb{T}, Y)$ is called almost automorphic ($\mathcal{AA}$) if for every sequence $(\tau_n') \subset \Pi$, we can extract a subsequence (τ_n) such that

$$g^*(r) := \lim_{n\to\infty} g(r + \tau_n),$$

and

$$\lim_{n\to\infty} g^*(r - \tau_n) = g(r),$$

for each $r \in \mathbb{T}$. We note that the convergence is pointwise. Then, the function g^* not necessarily continuous, but measurable. Moreover, we note if we consider that convergence is uniform on $\mathbb{T}$ instead of pointwise convergence, we get that the function g is almost periodic.

We set $\mathcal{AA}(\mathbb{T}, Y)$ for the collection of all almost automophic functions from $\mathbb{T}$ into Y.

Example 2.11 Let $G : \mathbb{T} \to X$ be a function defined by

$$G(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right).$$

It is $\mathcal{AA}$. However, it not almost periodic because this function is not uniformly continuous on $\mathbb{T}$.

Definition 2.12 A continuous function $g : \mathbb{T} \times Y \to Y$ is called $\mathcal{AA}$ if $g(r, y)$ is $\mathcal{AA}$ in $r \in \mathbb{T}$ uniformly $\forall y$ in any bounded subset of Y.

$\mathcal{AA}(\mathbb{T} \times Y, Y)$ is the collection of all such functions.

Definition 2.13 A continuous function $g : \mathbb{T} \to Y$ is said to be $\mathcal{AAA}$ if $g(r)$ can be decomposed into two parts like that $g(r) = g_1(r) + g_2(r)$, where $g_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$ and $g_2(r) \in C_0(\mathbb{T}, Y)$.

$\mathcal{AAA}(\mathbb{T}, Y)$ is the collection of all such functions.

Definition 2.14 A continuous function $g : \mathbb{T} \times Y \to Y$ is said to be $\mathcal{AAA}$ in r uniformly for y in any compact subset of Y if $g(r, y)$ can be decomposed into two parts like that $g(r, y) = g_1(r, y) + g_2(r, y)$, where $g_1 \in \mathcal{AA}(\mathbb{T} \times Y, Y)$ and $g_2(r) \in C_0(\mathbb{T} \times Y, Y)$.

We set $\mathcal{AAA}(\mathbb{T} \times Y, Y)$ is the collection of all such functions.

Example 2.15 Let $\chi : \mathbb{T} \to Y$ be a function such that

$$\chi(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right) + e^{-|r|}.$$

This function is $\mathcal{AAA}$ as first part belongs to $\mathcal{AA}(\mathbb{T}, Y)$ and second part belongs to $C_0(\mathbb{T}, Y)$.

Example 2.16 Let $\wp : \mathbb{T} \times Y \to Y$ be a function such that

$$\wp(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right)\cos y + \frac{1}{1 + r^2}\sin y.$$

This function is $\mathcal{AAA}$ in $r \in \mathbb{T}$ for each $y \in Y$ because first part belongs to $\mathcal{AA}(\mathbb{T} \times Y, Y)$ and second part belongs to $C_0(\mathbb{T} \times Y, Y)$.

Lemma 2.17 *If $g_1, g_2, g \in \mathcal{AAA}(\mathbb{T}, Y)$, then:*

- $g_1 + g_2 \in \mathcal{AAA}(\mathbb{T}, Y)$;
- $\lambda g \in \mathcal{AAA}(\mathbb{T}, Y)$, *for any scalar* λ;
- *If $\alpha \in \mathbb{R}$ is a constant then, $g_\alpha \in \mathcal{AAA}(\mathbb{T}, Y)$, where $g_\alpha : \mathbb{T} \to Y$ define as $g_\alpha(\cdot) = g(\cdot + \alpha)$;*
- *The range $R_g = \{g(r) : r \in \mathbb{T}\}$ is relatively compact of Y, thus g is bounded with respect to norm.*

Definition 2.18 A function $g(r, s)$ is said to be bi-$\mathcal{AA}$ if for every sequence $\tau_n' \subset \Pi$, there is a subsequence τ_n and a function $g^*(r, s)$ such that the translation of g converge to g^*, that is $\|g(r + \tau_n, s + \tau_n) - g^*(r, s)\| \to 0$ as $n \to \infty$ and $\|g^*(r - \tau_n, s - \tau_n) - g(r, s)\| \to 0$ as $n \to \infty$, $\forall\, r, s \in \mathbb{T}$.

We set $bi\mathcal{AA}(\mathbb{T} \times \mathbb{T}, Y)$ is the collection of all such functions.

Remark 2.19 Exponential function on time scale is bi $\mathcal{AA}$ function.

Lemma 2.20 *The decomposition of $\mathcal{AAA}$ function $g = g_1 + g_2$, where $g_1 \in \mathcal{AA}(\mathbb{T}, Y)$ and $g_2 \in C_0(\mathbb{T}, Y)$ is unique i.e., $g = g_1 \oplus g_2$.*

Proof From the definition, we can easily observe $g_1(\mathbb{T}) \subset \overline{g(\mathbb{T})}$. Assume that $g = g_1 + g_2$ and $g = h_1 + h_2$ then $0 = (g_1 - h_1) + (g_2 - h_2) \in \mathcal{AAA}(\mathbb{T}, Y)$, where $(g_1 - h_1) \in \mathcal{AA}(\mathbb{T}, Y)$ and $(g_2 - h_2) \in C_0(\mathbb{T}, Y)$. In view of above result $g_1 - h_1 = 0$. Consequently, $g_2 - h_2 = 0$, i.e., $g_1 = h_1$ and $g_2 = h_2$.

Lemma 2.21 *The space $\mathcal{AAA}(\mathbb{T}, Y)$ is a Banach space with sup norm*

$$\|g\|_\infty = \sup_{r\in\mathbb{T}} \|g(r)\|.$$

Proof Consider $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{AAA}(\mathbb{T}, Y)$. We can express uniquely $g_n = f_n + h_n$, where f_n is a sequence in $\mathcal{AA}(\mathbb{T}, Y)$ and h_n is in $C_0(\mathbb{T}, Y)$. From Lemma 2.20, we see $\|f_n - f_m\|_\infty \leq \|g_n - g_m\|_\infty$. We deduce from here that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $AA(\mathbb{T}, Y)$. So, $h_n = g_n - f_n$ is Cauchy sequence in $C_0(\mathbb{T}, Y)$. We conclude that $f_n \to f \in \mathcal{AA}(\mathbb{T}, Y)$ and $h_n \to h \in C_0(\mathbb{T}, Y)$ and finally $g_n \to f + h \in \mathcal{AAA}(\mathbb{T}, Y)$.

Lemma 2.22 *Let $g : \mathbb{T} \times Y \to Y,\ (r, y) \to g(r, y) \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ in $r \in \mathbb{T}$, for each $y \in Y$ and assume that g satisfies Lipschitz condition i.e.,*

$$\|g(r, y) - g(r, y^*)\| \leq L\|y - y^*\|,$$

for all $y, y^* \in Y$ and for every $r \in \mathbb{T}$, where $L > 0$ is constant. Then $G : \mathbb{T} \to Y$ given by $G(\cdot) = g(\cdot, y(\cdot))$ is $\mathcal{AAA}$ provided $y : \mathbb{T} \to Y$ is $\mathcal{AAA}$.

Proof Since $g, y \in \mathcal{AAA}$, then we can decompose as

$$g = g_1 + g_2, \quad y = y_1 + y_2,$$

where $g_1 \in \mathcal{AA}(\mathbb{T} \times Y, Y),\ g_2 \in C_0(\mathbb{T} \times Y, Y),\ y_1 \in \mathcal{AA}(\mathbb{T}, Y),\ y_2 \in C_0(\mathbb{T}, Y)$. We can write

$$g(r, y(r)) = g_1(r, y_1(r)) + g(r, y(r)) - g(r, y_1(r)) + g_2(r, y_1(r))$$

By *Lemma* 3.3 in [18] $g_1(r, y_1(r)) \in \mathcal{AA}(\mathbb{T}, Y)$. Noticing that $\|g(r, y(r)) - g(r, y_1(r))\| \leq L\|y_2(r)\| \to 0$ as $|r| \to \infty$. Hence $g(r, y(r)) - g(r, y_1(r)) \in C_0(\mathbb{T}, Y)$. Now, since $\overline{\{y_1(r),\ r \in \mathbb{T}\}}$ is compact set of Y, $g_2(r, y_1(r)) \in C_0(\mathbb{T}, Y)$. In conclusion, $g(r, y(r)) \in \mathcal{AAA}(\mathbb{T}, Y)$.

Definition 2.23 A continuous function $y : \mathbb{T} \to Y$ is called $\mathcal{AAA}$ solution of system (1.1) on $\mathbb{T}$ if $y(r)$ is and satisfies $\mathcal{AAA}$

$$y(r) = S(r, a)[y(a) - g(a, y(\varkappa(a)))] + g(r, y(\varkappa(r))) + \int_a^r S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s \\ + \int_a^r S(r, \sigma(s)) \int_{-\infty}^s k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s, \qquad \forall\, r \geq a \in \mathbb{T}. \tag{2.1}$$

3 Main Result

To prove main result of this manuscript, we assume the following assumptions which are further mandatory:

A1. The system

$$y^{\Delta}(r) = A(r)y(r), \quad s \le r, \quad r, s \in \mathbb{T},$$

has an evolution family of operators $\{S(r, s) : s \le s,\ s, r \in \mathbb{T}\}$. $S(r, s)$ is asymptotically stable i.e., $\exists$ constants $R_0, \omega > 0$ satisfying

$$\|S(r, s)\| \le R_0 e_{\ominus\omega}(r, s)$$

for all $r, s \in \mathbb{T}$ with $r \ge s$.

A2. For any sequence $\{\tau_n'\}_{n\in\mathbb{N}} \subset \Pi$, we can find a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that for any $\varepsilon > 0, \exists N \in \mathbb{N}$,

$$||S(r+\tau_n, s+\tau_n) - S(r, s)|| \le \varepsilon e_{\ominus\omega(r,s)} \quad \text{and} \quad ||S(r-\tau_n, s-\tau_n) - S(r, s)|| \le \varepsilon e_{\ominus\omega(r,s)},$$

$\forall n > N,\ \forall r, s \in \mathbb{T}, r \ge s.$

A3. $g \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist constant $L_g > 0$ such that

$$\|g(r, y) - g(r, x)\| \le L_g\|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

A4. $h \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist a constant $L_h > 0$ such that

$$\|h(r, y) - h(r, x)\| \le L_h\|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

A5. $\mathcal{P} \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist a constant $L_{\mathcal{P}} > 0$ such that

$$\|\mathcal{P}(r, y) - \mathcal{P}(r, x)\| \le L_{\mathcal{P}}\|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

Lemma 3.1 *Suppose $\xi \in \mathcal{AAA}(\mathbb{T}, Y)$ holds, $\Xi(\eta) : \mathbb{T} \to Y$ defined by*

$$\Xi(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi(\zeta)\Delta\zeta, \quad \eta \in \mathbb{T},$$

is $\mathcal{AAA}(\mathbb{T}, Y)$.

Proof Since $\xi \in \mathcal{AAA}(\mathbb{T}, Y)$. So, we can decompose it as $\xi(\eta) = \xi_1(\eta) + \xi_2(\eta)$, where $\xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$ and $\xi_2(\eta) \in C_0(\mathbb{T}, Y)$. Now,

$$\Xi(\eta) = \Xi_1(\eta) + \Xi_2(\eta),$$

where $\Xi_1(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1(\zeta)\Delta\zeta$ and $\Xi_2(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta$. To proof complete, we have to prove $\Xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$, $\Xi_2(\eta) \in C_0(\mathbb{T}, Y)$. Since $\xi_1 \in \mathcal{AA}(\mathbb{T}, Y)$, there exists ξ_1^* and a subsequence $\{\tau_n\} \subset \Pi$ for each sequence $\{\tau_n'\}$ such that

$$\lim_{n\to\infty} \|\xi_1(\eta + \tau_n) - \xi_1^*(\eta)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\xi_1^*(\eta - \tau_n) - \xi_1(\eta)\| = 0. \tag{3.1}$$

Now, corresponding to ξ_1^*, let us define $\Xi_1^*(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(s))\xi_1^*(s)ds$. Now, we compute

$$\begin{aligned}
\|\Xi_1(\eta + \tau_n) - \Xi_1^*(\eta)\| &= \Big\| \int_{-\infty}^{\eta+\tau_n} k(\eta + \tau_n, \sigma(\zeta))\xi_1(\zeta)\Delta\zeta - \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1^*(\zeta)\Delta\zeta \Big\| \\
&= \Big\| \int_{-\infty}^{\eta} k(\eta + \tau_n, \sigma(\zeta) + \tau_n)\xi_1(\zeta + \tau_n)\Delta\zeta - \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1^*(\zeta)\Delta\zeta \Big\| \\
&\leq c \int_{-\infty}^{\eta} \Big\| e_{\ominus\lambda}(\eta + \tau_n, \sigma(\zeta) + \tau_n) - e_{\ominus\lambda}(\eta, \sigma(\zeta)) \Big\| \big\| \xi_1(\zeta + \tau_n) \big\| \Delta\zeta \\
&\quad + c \int_{-\infty}^{\eta} e_{\ominus\lambda}(\eta, \sigma(\zeta)) \big\| \xi_1(\zeta + \tau_n) - \xi_1^*(\zeta) \big\| \Delta\zeta \\
&\leq c\|\xi_1\|_\infty \int_{-\infty}^{\eta} \Big\| e_{\ominus\lambda}(\eta + \tau_n, \sigma(\zeta) + \tau_n) - e_{\ominus\lambda}(\eta, \sigma(\zeta)) \Big\| \\
&\quad + \frac{c(1 + \bar{\mu}\lambda)}{\lambda} \sup_{\eta\in\mathbb{T}} \big\| \xi_1(\eta + \tau_n) - \xi_1^*(\eta) \big\|,
\end{aligned}$$

where $\bar{\mu} = \sup_{\eta\in\mathbb{T}} \mu(\eta)$. From Remark 2.19 and Eq. 3.1, we have $\lim_{n\to\infty} \|\Xi_1(\eta + \tau_n) - \Xi_1^*(\eta)\| = 0$. Using the similar arguments, we get $\lim_{n\to\infty} \|\Xi_1^*(\eta - \tau_n) - \Xi_1(\eta)\| = 0$. Hence $\Xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$.

Now, since $\xi_2(\eta) \in C_0(\mathbb{T}, Y)$ then $\forall \varepsilon > 0$, $\exists$ a constant $R > 0$ such that

$$\|\xi_2(\eta)\| < \varepsilon, \quad |\eta| > R. \tag{3.2}$$

which yields that

$$\begin{aligned}
\|\Xi_2(\eta)\| &= \Big\| \int_{-\infty}^{R} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta + \int_{R}^{\eta} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta \Big\| \\
&\leq c\|\xi_2\|_\infty \int_{-\infty}^{R} e_{\ominus\lambda}(\eta, \sigma(\zeta))\Delta\zeta + \varepsilon c \int_{R}^{\eta} e_{\ominus\lambda}(\eta, \sigma(\zeta))\Delta\zeta
\end{aligned}$$

$$\leq c\|\xi_2\|_\infty \frac{(1+\bar{\mu}\lambda)}{\lambda} e_\lambda(R, |\eta|) + \varepsilon c \frac{(1+\bar{\mu}\lambda)}{\lambda}[1 - e_{\ominus\lambda}(\eta, R)]$$
$$\leq c\|\xi_2\|_\infty \frac{(1+\bar{\mu}\lambda)}{\lambda} e^{\lambda(R-|\eta|)} + \varepsilon c \frac{(1+\bar{\mu}\lambda)}{\lambda}$$
$$\lim_{|\eta|\to\infty} \|\Xi_2(\eta)\| = 0.$$

Therefore, $\Xi_2(\eta) \in C_0(\mathbb{T}, Y)$. □

Lemma 3.2 *Let* $P \in \mathcal{AAA}(\mathbb{T}, Y)$ *and suppose* (A1)–(A2) *is satisfied. If* $\mathfrak{P} : \mathbb{T} \to Y$ *is defined by*

$$\mathfrak{P}(r) = \int_{-\infty}^{r} S(r, \sigma(s))P(s)\Delta s, \quad r \in \mathbb{T},$$

then $\mathfrak{P}(\cdot) \in \mathcal{AAA}(\mathbb{T}, Y)$.

Proof Since $P \in \mathcal{AAA}(\mathbb{T}, Y)$. So, we can decompose it as $P(r) = P_1(r) + P_2(r)$, where $P_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$ and $P_2(r) \in C_0(\mathbb{T}, Y)$. Now,

$$\mathfrak{P}(r) = \mathfrak{P}_1(r) + \mathfrak{P}_2(r)$$

where $\mathfrak{P}_1(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_1(s)\Delta s$ and $\mathfrak{P}_2(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_2(s)\Delta\zeta$. To proof complete, we have to prove $\mathfrak{P}_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$, $\mathfrak{P}_2(r) \in C_0(\mathbb{T}, Y)$. Since $P_1 \in \mathcal{AA}(\mathbb{T}, Y)$ there exists P_1^* and a subsequence $\{\tau_n\} \subset \Pi$ for each sequence $\{\tau_n'\}$ such that

$$\lim_{n\to\infty} \|P_1(r+\tau_n) - P_1^*(r)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|P_1^*(r-\tau_n) - P_1(r)\| = 0 \tag{3.3}$$

Now, corresponding to P_1^*, let us define $\mathfrak{P}_1^*(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)ds$. Now, we compute

$$\|\mathfrak{P}_1(r+\tau_n) - \mathfrak{P}_1^*(r)\| = \Big\| \int_{-\infty}^{r+\tau_n} S(r+\tau_n, \sigma(s))P_1(s)\Delta s - \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)\Delta s \Big\|$$
$$= \Big\| \int_{-\infty}^{r} S(r+\tau_n, \sigma(s)+\tau_n)P_1(s+\tau_n)\Delta s - \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)\Delta s \Big\|$$
$$\leq \int_{-\infty}^{r} \Big\|S(r+\tau_n, \sigma(s)+\tau_n) - S(r, \sigma(s))\Big\| \Big\|P_1(s+\tau_n)\Big\|\Delta s$$
$$+ \int_{-\infty}^{r} \|S(r, \sigma(s))\| \Big\|P_1(s+\tau_n) - P_1^*(s)\Big\|\Delta s$$
$$\leq \|P_1\|_\infty \frac{\varepsilon(1+\bar{\mu}\omega)}{\omega} + \frac{R_0(1+\bar{\mu}\omega)}{\omega} \sup_{r\in\mathbb{T}} \Big\|P_1(r+\tau_n) - P_1^*(r)\Big\|.$$

From Eq. 3.3, we have $\lim_{n\to\infty} \|\mathfrak{P}_1(r+\tau_n) - \mathfrak{P}_1^*(r)\| = 0$. Using the similar arguments, we get $\lim_{n\to\infty} \|\mathfrak{P}_1^*(r-\tau_n) - \mathfrak{P}_1(r)\| = 0$. Hence $\mathfrak{P}_1(r) \in AA(\mathbb{T}, Y)$.

Now, analogously to the previous lemma proof we can easily find $\lim_{|r|\to\infty} \|\mathfrak{P}_2(r)\| = 0$. Hence $\mathfrak{P}_2(\cdot) \in C_0(\mathbb{T}, Y)$. □

Now we are prepare for our main result which gives the unique $\mathcal{AAA}$ solution of system (1.1).

Theorem 3.3 *Let us assumptions* $(A1)$–$(A5)$ *hold, the system (1.1) has a unique* $\mathcal{AAA}$ *solution* $y : \mathbb{T} \to Y$ *provided*

$$\Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) < 1. \tag{3.4}$$

Proof Firstly, let us define a nonlinear operator

$$(\mathcal{G}y)(r) = g(r, y(\varkappa(r))) + \int_{-\infty}^{r} S(r,\sigma(s))\mathcal{P}(s, y(s))\Delta s$$
$$+ \int_{-\infty}^{r} S(r,\sigma(s)) \int_{-\infty}^{s} k(s,\sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s.$$

From the assumptions, Lemmas 2.17, 2.22, 3.1 and 3.2, we conclude that the operator $\mathcal{G}$ is from $AAA(\mathbb{T}, Y)$ into $AAA(\mathbb{T}, Y)$ which is Banach space from Lemma 2.21. To prove the remaining part, suppose $y, x \in \mathcal{AAA}(\mathbb{T}, Y)$, then

$$\begin{aligned}
&\|(\mathcal{G}y)(r) - (\mathcal{G}x)(r)\| \\
&\le \|g(r, y(\varkappa(r))) - g(r, x(\varkappa(r)))\| + \Big\| \int_{-\infty}^{r} S(r,\sigma(s))[\mathcal{P}(s,y(s)) - \mathcal{P}(s,x(s))]\Delta s \Big\| \\
&+ \Big\| \int_{-\infty}^{r} S(r,\sigma(s)) \int_{-\infty}^{s} k(s,\sigma(\zeta))[h(\zeta,y(\zeta)) - h(\zeta,x(\zeta))]\Delta\zeta\Delta s \Big\| \\
&\le L_g\|y(\varkappa(r)) - x(\varkappa(r))\| + K_0 L_{\mathcal{P}} \int_{-\infty}^{r} e_{\ominus\omega}(r,\sigma(s))\|y(s) - x(s)\|\Delta s \\
&+ R_0 c L_h \int_{-\infty}^{r} e_{\ominus\omega}(r,\sigma(s)) \int_{-\infty}^{s} e_{\ominus\lambda}(s,\sigma(\zeta))\|y(\zeta) - x(\zeta)\|\Delta\zeta\Delta s \\
&\le \Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) \sup_{r\in\mathbb{T}} \|y(r) - x(r)\|
\end{aligned}$$

where $\bar{\mu} = \sup_{r\in\mathbb{T}} \mu(r)$.

$$\|(\mathcal{G}y) - (\mathcal{G}x)\|_\infty = M\|y - x\|_\infty,$$

where $M = \left(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\right)$. According to condition (3.4), $M < 1$ which implies $\mathcal{G}$ is a contraction mapping. Therefore using the Banach contraction theorem, we get a unique fixed point $y(r)$ in $\mathcal{AAA}(\mathbb{T}, Y)$ such that $\mathcal{G}y = y$ that is

$$y(r) = g(r, y(\varkappa(r))) + \int_{-\infty}^{r} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s + \int_{-\infty}^{r} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s$$

for all $r \in \mathbb{T}$. If we let $a \in \mathbb{T}$, then

$$y(a) = g(a, y(\varkappa(a))) + \int_{-\infty}^{a} S(a, \sigma(s))\mathcal{P}(s, y(s))\Delta s + \int_{-\infty}^{a} S(a, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s$$

using evolution operator property $S(r, t)S(t, s) = S(r, s),\ s \le t \le r.$

$$S(r, a)y(a) = S(r, a)g(a, y(\varkappa(a))) + \int_{-\infty}^{a} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s + \int_{-\infty}^{a} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s$$

$$S(r, a)[y(a) - g(a, y(\varkappa(a)))] = y(r) - g(r, y(\varkappa(r))) - \int_{a}^{r} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s - \int_{a}^{r} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s.$$

From last equality, we find that system (1.1) has a unique $\mathcal{AAA}$ solution, given by (2.1). □

4 Stability Result

Definition 4.1 A solution y is called stable, if for any arbitrary $0 < \varepsilon$, there exists $0 < \delta$ such that

$$\|y(r) - \overline{y}(r)\| < \varepsilon, \quad \forall r \geq a, \quad r, a \in \mathbb{T}$$

whenever $\|y(a) - \overline{y}(a)\| < \delta$, where $\overline{y}$ is the solution of System (1.1) with initial condition $\overline{y}(a) \in Y$.

Theorem 4.2 *If the conditions of Theorem 3.3 satisfies, system (1.1) has a unique stable $\mathcal{AAA}$ mild solution.*

Proof By Theorem 3.3, we get that problem (1.1) has a unique $\mathcal{AAA}$ mild solution whose integral form is given by,

$$y(r) = S(r,a)[y(a) - g(a, y(\varkappa(a)))] + g(r, y(\varkappa(r))) + \int_a^t S(r,\sigma(s))\mathcal{P}(s, y(s))\Delta s$$
$$+ \int_a^t S(r,\sigma(s)) \int_{-\infty}^s k(s,\sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s,$$

for $\forall r > a \in \mathbb{T}$. Now, let us suppose that $y(r)$ is $\mathcal{AAA}$ solution of the system (1.1) and $\overline{y}(r)$ is another solution of the system (1.1).

$$\begin{aligned}
&\|y(r) - \overline{y}(r)\| \\
&\quad \leq \left\| S(r,a)[y(a) - \overline{y}(a)] \right\| + \left\| S(r,a)[g(a, y(\varkappa(a))) - g(a, \overline{y}(\varkappa(a))] \right\| \\
&+ \left\| \int_a^r S(r,\sigma(s))[\mathcal{P}(s, y(s)) - \mathcal{P}(s, \overline{y}(s))]\Delta s \right\| + \left\| g(r, y(\varkappa(r))) - g(r, \overline{y}(\varkappa(r)) \right\| \\
&\quad + \left\| \int_a^r S(r,\sigma(s)) \int_{-\infty}^s k(s,\sigma(\zeta))[h(\zeta, y(\zeta)) - h(\zeta, \overline{y}(\zeta))]\Delta\zeta\Delta s \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq R_0(1 + L_g)e_{\ominus\omega}(r,a)\|y(a) - \overline{y}(a)\| + L_g\|y(\varkappa(r)) - \overline{y}(\varkappa(r))\| \\
&\quad + \left(K_0 L_{\mathcal{P}} + \frac{K_0 c L_h(1 + \bar{\mu}\lambda)}{\lambda}\right) \int_a^r e_{\ominus\omega}(r,\sigma(s)) \sup_{s\in\mathbb{T}} \|y(s) - \overline{y}(s)\|\Delta s \\
&\leq R_0(1 + L_g)\|y(a) - \overline{y}(a)\| + \\
&\quad \left(L_g + \frac{K_0 L_{\mathcal{P}}(1 + \bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1 + \bar{\mu}\lambda)(1 + \bar{\mu}\omega)}{\lambda\omega}\right) \sup_{r\in\mathbb{T}} \|y(r) - \overline{y}(r)\|
\end{aligned}$$

$$\|y-\overline{y}\|_\infty - M\|y-\overline{y}\|_\infty \le R_0(1+L_g)\|y(a)-\overline{y}(a)\|$$

$$\|y-\overline{y}\|_\infty \le \frac{R_0(1+L_g)\|y(a)-\overline{y}(a)\|}{1-M}$$

where $\frac{R_0(1+L_g)}{1-M} > 0$, choose a $\delta > 0$ such that $\delta < \frac{\varepsilon(1-M)}{R_0(1+L_g)}$, then

$$\|y-\overline{y}\| < \varepsilon.$$

From Definition 4.1, the system (1.1) is stable. □

5 Example

Here, we give an example on different different time scale which shows the fruitfulness of results obtained in previous sections.

Consider the PDE on general periodic time scales $\mathbb{T}$,

$$\begin{aligned}
&\frac{\partial}{\Delta_1 r}U(r,y)\\
&= \frac{\partial^2}{\Delta_2 x^2}U(r,y) + \frac{\partial}{\Delta_1 r}\Big[\frac{1}{250}\sin\Big(\frac{1}{1+\sin r+\sin\sqrt{2}r}\Big)\sin U(r,y) + \frac{1}{250}e^{-|r|}\cos U(r,y)\Big]\\
&\quad + \frac{1}{250}\cos\Big(\frac{1}{1+\sin r+\cos\sqrt{2}r}\Big)\cos U(r,y) + \frac{1}{250}\frac{1}{1+r^2}\sin U(r,y)\\
&\quad + \int_{-\infty}^{r} e_{-\frac{1}{4}}(r,\sigma(s))\Big[\cos\sqrt{2}s\sin U(s,y) + \frac{1}{1+s^2+s^4}\cos U(s,y)\Big]\Delta s, \qquad y\in[0,\pi]_{\mathbb{T}}
\end{aligned} \tag{5.1}$$

$$U(r,0) = U(r,\pi) = 0, \quad r\in\mathbb{T},$$

Let $\vartheta(r) = U(r,\cdot)$, we consider the operator A by

$$A\vartheta = \frac{\partial^2}{\Delta_2 y^2}\vartheta, \quad \vartheta\in\mathcal{D}(A) = \{\mathbb{H}_0^1[0,\pi]_{\mathbb{T}} \cap \mathbb{H}_0^2[0,\pi]_{\mathbb{T}}\}.$$

As the similar argument of Sect. 3.1 in [19] and in [20], any one can simply find that the evolution system $\{S(r,s) : r \ge s\}$ satisfies $||S(r,s)|| \le e_{\ominus\frac{1}{2}}(r,s), r \ge s,$ with $R_0 = 1$ and $\omega = \frac{1}{2}$. On based of above things, system (5.1) can be converted in form as (1.1) and satisfied all assumptions with $L_g, L_h, L_{\mathcal{P}} = \frac{1}{125},\ c = 1,\ \lambda = \frac{1}{4}$. Now, it remains to check one condition for different different time scales.

Case1: If $\mathbb{T} = \mathbb{R}$, then $\bar{\mu} = 0$, hence

$$\left(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\right) = 0.104 < 1.$$

Case2: If $\mathbb{T} = \mathbb{Z}$, then $\bar{\mu} = 1$, hence

$$\left(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\right) = 0.176 < 1.$$

Case3: If $\mathbb{T} = 2\mathbb{Z}$, then $\bar{\mu} = 2$, hence

$$\left(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\right) = 0.264 < 1.$$

In all of cases, we find that all conditions of Theorems 3.3 and 4.2 satisfy, so we derive that problem (5.1) has a unique stable $\mathcal{AAA}$ solution.

References

1. Hilger, S.: Ein Makettenkalkul mit Anwendung auf Zentrumsmannigfaltigkeiten. Ph.D. thesis, Universitt Wurzburg (1988)
2. Bohner, M., Peterson, A.: A survey of exponential functions on time scales. Cubo Mat. Educ. **3**(2), 285–301 (2001)
3. Agarwal, R., Bohner, M., O'Regan, D., Peterson, A.: Dynamic equations on time scales: a survey. J. Comput. Appl. Math. **141**(1–2), 1–26 (2002)
4. Agarwal, R., Bohner, M.: Basic calculus on time scales and some of its applications. Resultate der Mathematik **35**(1), 3–22 (1999)
5. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkh*ä*user, Basel (2001)
6. Bohner, M., Peterson, A. (eds).: Advances in Dynamic Equations on Time Scales. Birkh*ä*user, Boston, MA (2003)
7. Bochner, S.: Continuous mappings of almost automorphic and almost periodic functions. Proc. Nat. Acad. Sci. **52**(4), 907–910 (1964)
8. N'Gu*é*r*é*kata, G.M.: Topics in Almost Automorphy. Springer Science & Business Media (2007)
9. Diagana, T.: Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces. Springer, New York (2013)
10. N'Gu*é*r*é*kata, G.M.: Sur les solutions presque automorphes d'*é*quations differentielles abstraiters. Ann. Sci. Math. Qu*é*bec **1**, 69–79 (1981)
11. Bugajewski, D., Gaston, M.: N'Gu*é*r*é*kata, On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces. Nonlinear Anal. **59**(8), 1333–1345 (2004)
12. Diagana, T., Hern*á*ndez, E., Jose, P.C., dos Santos.: Existence of asymptotically almost automorphic solutions to some abstract partial neutral integro-differential equations. Nonlinear Anal. **71**, 248–257 (2009)
13. Diagana, T., Gaston, M., N'Gu*é*r*é*kata, Minh, N.V.: Almost automorphic solutions of evolution equations. Proc. Am. Math. Soc. **132**(11), 3289–3298 (2004)
14. Kavitha, V., Abbas, S., Murugesu, R.: Asymptotically almost automorphic solutions of fractional order neutral integro-differential equations. Bull. Malays. Math. Sci. Soc. **39**(3), 1075–1088 (2016)

15. Dhama, S., Abbas, S.: Existence and stability of square-mean almost automorphic solution for neutral stochastic evolution equations with Stepanov-like terms on time scales. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemticas, pp. 1–20 (2018)
16. Rubanik, V.P.: Oscillations of Quasilinear Systems with Retardation. Nauka, Moscow (1969)
17. Lizama, C., Mesquita, J.G.: Almost automorphic solutions of non-autonomous difference equations. J. Math. Anal. Appl. **407**(2), 339–349 (2013)
18. Wang, C., Li, Y.: Weighted pseudo almost automorphic functions with applications to abstract dynamic equations on time scales. Ann. Polonici Math. **3**(108) (2013)
19. Jackson, B.: Partial dynamic equations on time scales. J. Comput. Appl. Math. **186**(2), 391–415 (2006)
20. Li, Y., Wang, P.: Almost periodic solution for neutral functional dynamic equations with Stepanov-almost periodic terms on time scales. Discret. Contin. Dyn. Syst. Ser. **S**(10.3) (2017)