Introduction to Class of Uniformly Fractional Differentiable Functions

Krunal B. Kachhia and Jyotindra C. Prajapati

Abstract In this paper, authors introduced new concept of uniformly fractional differentiable functions on an arbitrary interval I of R by using Caputo-type fractional derivative instead of the commonly used first-order derivative. Their interesting properties with few illustrations have been discussed in this paper.

Keywords Uniformly differentiable functions · Uniformly continuous functions · Uniformly fractional differentiable functions · Caputo fractional derivative

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1 Introduction

The fractional calculus is a theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n-fold integration. We shall explain the result connected to classical analysis, namely uniformly differential functions given by Patel [1], can be extended to fractional calculus, i.e they can be generalized by replacing the first order the first derivatives and integrals, respectively, by derivatives and integrals of non-integer. The uniformly differentiable function can be defined as:

Definition 1 Let I be an interval in R. A differentiable function $f : I \to R$ is uniformly differentiable, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in I$ satisfying $|x - y| < \delta$,

K. B. Kachhia
Department of Mathematical Sciences, P. D. Patel Institute of Applied Sciences, Charotar University of Science and Technology, Changa, Anand 388421, Gujarat, India
e-mail: krunalmaths@hotmail.com

J. C. Prajapati (✉)
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388120, Gujarat, India
e-mail: drjyotindra18@gmail.com

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$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon \tag{1}$$

and

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon \tag{2}$$

The collection of all uniformly differentiable functions on I will be denoted by $UD(I)$. The class of uniformly differentiable function has connection with class of uniformly continuous functions which are well-known class of functions in classical analysis. The uniform continuous defined by Apostol [2] as:

Definition 2 A function $f : I \to R$ is uniformly continuous function on interval I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$|f(x) - f(y)| < \epsilon \tag{3}$$

Definition 3 The Caputo fractional derivative of order α defined by Caputo [3] as

$${}^{C}D^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha-1}}\, d\tau \ (n-1 < \alpha < n) \tag{4}$$

The following theorem is given by Diethelm [4].

Theorem 4 *Let* $0 < \alpha \leq 1$, $a < b$ *and* $f \in C[a, b]$ *be such that* ${}^{C}D^{\alpha}(f) \in C[a, b]$. *Then there exist* $\xi \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{(b-a)^{\alpha}} = \frac{1}{\Gamma(\alpha)}\, {}^{C}D^{\alpha}(f(\xi)) \tag{5}$$

Also some properties of Local fractional calculus was studied by Yang [5] and Yang and Gao [6]. Kachhia and Prajapati [7] introduced concept of functions of bounded fractional differential variation using the Caputo-type fractional derivative.

Definition 5 A Caputo fractional differentiable function f is absolutely fractional differentiable function on interval I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for an collection of pairwise disjoint intervals $\{(a_i, b_i)\}$ in I satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$,

$$\sum_{i=1}^{n} \left| \Gamma(\alpha) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(a_i)) \right| < \epsilon \tag{6}$$

and

$$\sum_{i=1}^{n} \left| \Gamma(\alpha) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(b_i)) \right| < \epsilon \tag{7}$$

where $0 < \alpha \leq 1$.

The Hölder continuous function defined by Gilberg and Trudinger [8] as:

Definition 6 A function $f : R \to C$ is said to be Hölder continuous if for all $x, y \in R$, there are non-negative real constants M, α such that

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}$$

2 Uniformly Fractional Differentiable Functions

In this section, authors introduced the new concept of uniformly factional differentiable functions as:

Definition 7 Let I be an interval in R. A Caputo fractional differentiable function f is uniformly fractional differentiable function on I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in I$ satisfying $|x - y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \epsilon \tag{8}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| < \epsilon \tag{9}$$

where $0 < \alpha \leq 1$.

If we take $\alpha = 1$, then Eqs. (8) and (9) reduces to Eqs. (1) and (2) respectively. The collection of all uniformly fractional differentiable functions on I will be denoted by $UFD(I)$.

Theorem 8 *A function f is uniformly fractional differentiable function on an interval I if and only if ${}^{C}D^{\alpha}(f)$ is uniformly continuous on I.*

Proof Let $f : I \to R$ be uniformly fractional differentiable. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \frac{\epsilon}{2} \tag{10}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| < \frac{\epsilon}{2} \tag{11}$$

Now for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left|{}^{C}D^{\alpha}(f(x)) - {}^{C}D^{\alpha}(f(y))\right| = \left|{}^{C}D^{\alpha}(f(x)) - \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) + \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| \tag{12}$$

We get

$$\left|{}^{C}D^{\alpha}(f(x)) - {}^{C}D^{\alpha}(f(y))\right| \leq \left|{}^{C}D^{\alpha}(f(x)) - \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)\right| + \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| \tag{13}$$

By using Eqs. (10) and (11), we obtain

$$|{}^{C}D^{\alpha}(f(b_i)) - {}^{C}D^{\alpha}(f(a_i))| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{14}$$

Hence ${}^{C}D^{\alpha}(f)$ is a uniformly continuous on I.
Conversely suppose that ${}^{C}D^{\alpha}(f)$ is uniformly continuous on I. Let $\epsilon > 0$ be given. Then there exist a $\delta > 0$ such that for any x, y in I satisfying $|x-y| < \delta$,

$$\left|{}^{C}D^{\alpha}(f(x)) - {}^{C}D^{\alpha}(f(y))\right| < \epsilon \tag{15}$$

Then from Theorem 4, there exist $c \in (y, x)$ such that

$$f(x) - f(y) = \frac{{}^{C}D^{\alpha}(f(x))(x-y)^{\alpha}}{\Gamma(\alpha)} \tag{16}$$

Since $|c-y| < \delta$, for any $\epsilon > 0$, there exist a $\delta > 0$ such that for any x, y in I

$$|{}^{C}D^{\alpha}(f(c)) - {}^{C}D^{\alpha}(f(y))| < \epsilon \tag{17}$$

By using Eq. (16)

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| < \epsilon \tag{18}$$

Again $|x-c| < \delta$, then for any $\epsilon > 0$, there exist a $\delta > 0$ such that for any x, y in I

$$\sum_{i=1}^{n}\left|{}^{C}D^{\alpha}(f(x)) - {}^{C}D^{\alpha}(f(c))\right| < \epsilon \tag{19}$$

By using Eq. (16)

$$\sum_{i=1}^{n}\left|{}^{C}D^{\alpha}(f(x))-\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)\right|<\epsilon \tag{20}$$

Therefore f is uniformly fractional differentiable on I.

Example 9 The $\frac{1}{2}$ order Caputo derivative of function $f(t)=t$ is $2\sqrt{\frac{t}{\pi}}$ which is uniformly continuous on [0, c]. Then by Theorem 8 uniformly fractional differentiable functions on [0, c] of order $\frac{1}{2}$.

In fact, using Theorem 8, several examples of uniformly fractional differentiable functions can be constructed.
The following is motivated by the principle that differentiability implies continuity.

Theorem 10 *If f is uniformly fractional differentiable function on an interval I, then f is uniformly continuous on I.*

Proof Since a function $f: I\rightarrow R$ is uniformly fractional differentiable, then if for any $\epsilon>0$, there is a $\delta>0$ such that for any x, y in I satisfying $|x-y|<\delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(x))\right|<\epsilon \tag{21}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(y))\right|<\epsilon \tag{22}$$

Since ${}^{C}D^{\alpha}(f)$ is bounded on I, so there exit $M>0$ such that

$$|{}^{C}D^{\alpha}(f(t))|\leq M\ (\forall\, t\in I) \tag{23}$$

Take $\delta_0=\min\{(\delta)^{\frac{1}{\alpha}},(\frac{\epsilon}{\epsilon+M})^{\frac{1}{\alpha}}\}$. Let $x, y\in I$ satisfying $|x-y|<\delta_0$.
Now

$$|f(x)-f(y)|\leq \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)(x-y)^{\alpha}-{}^{C}D^{\alpha}(f(x))(x-y)^{\alpha}+{}^{C}D^{\alpha}(f(x))(x-y)^{\alpha}\right| \tag{24}$$

Therefore

$$|f(x)-f(y)|\leq \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(x))\right||x-y|^{\alpha}+|{}^{C}D^{\alpha}(f(x))||x-y|^{\alpha} \tag{25}$$

Finally

$$|f(x) - f(y)| < \delta_0\epsilon + M\delta_0 = \delta_0(\epsilon + M) < \epsilon \tag{26}$$

Hence f is an uniformly continuous on I.

Theorem 11 *Every absolutely fractional differentiable function on I is uniformly fractional differentiable on I.*

Proof Since $f : I \to R$ is an absolutely fractional differentiable. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any finite collection of pairwise disjoint intervals $\{(a_i, b_i)\}$ in I satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$,

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(a_i))\right| < \epsilon \tag{27}$$

and

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(b_i))\right| < \epsilon \tag{28}$$

In particular

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \epsilon \tag{29}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| < \epsilon \tag{30}$$

Hence f is uniformly fractional differentiable function on I.

Proposition 12 *If f is uniformly fractional differential function on I and if ${}^{C}D_a^{\alpha}(f)$ is bounded on I, then f is Hölder continuous on I.*

Proof Let f is uniformly fractional differential function. Then for x, y in I satisfying $|x - y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \epsilon \tag{31}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y))\right| < \epsilon \tag{32}$$

Since ${}^{C}D^{\alpha}(f)$ is bounded on I, so there exit $M > 0$ such that

$$|{}^{C}D^{\alpha}(f(t))| \leq M \ (\forall\, t \in I) \tag{33}$$

Now

$$\begin{aligned}|f(x)-f(y)| &= \left|\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right)(x-y)^\alpha - {}^C D_a^\alpha(f(y))(x-y)^\alpha + {}^C D_a^\alpha(f(y))(x-y)^\alpha\right| \\ &\leq \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D_a^\alpha(f(y))\right| |x-y|^\alpha + |{}^C D_a^\alpha(f(y))||x-y|^\alpha \\ &\leq (\epsilon + M)|x-y|^\alpha\end{aligned}$$

Hence f is Hölder continuous function on R.

Theorem 13 *The space $UFD(I)$ of uniformly fractional differentiable functions on interval I is a vector space with pointwise operations.*

Proof Let $f, g \in UFD(I)$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x-y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(x))\right| < \frac{\epsilon}{2} \tag{34}$$

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(x))\right| < \frac{\epsilon}{2} \tag{35}$$

$$\left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(g(x))\right| < \frac{\epsilon}{2} \tag{36}$$

and

$$\left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(g(x))\right| < \frac{\epsilon}{2} \tag{37}$$

Now for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x-y| < \delta$,

$$\begin{aligned}&\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha((f+g)(x))\right| = \\ &\quad\left|\Gamma(\alpha)\left(\frac{(f(x)+g(x))-(f(y)+g(y))}{(x-y)^\alpha}\right) - {}^C D^\alpha((f(x)+(g(x))\right|\end{aligned} \tag{38}$$

Then

$$\begin{aligned}&\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha((f+g)(x))\right| \leq \\ &\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(x))\right| + \left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(g(x))\right|\end{aligned} \tag{39}$$

By using Eqs. (34) and (35) the Eq. (39) reduces to

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(x)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((f+g)(x))\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \tag{40}$$

Similarly by using Eqs. (36) and (37) we obtain

$$\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((f+g)(y))\right|<\epsilon \tag{41}$$

Hence $f+g\in UFD(I)$. Now let $f\in UFD(I)$ and $k\in C$. Then for any $\epsilon>0$, there is a $\delta>0$ such that for any x, y in I satisfying $|x-y|<\delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(x))\right|<\frac{\epsilon}{k}, \tag{42}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(y))\right|<\frac{\epsilon}{k}, \tag{43}$$

Now for any $\epsilon>0$, there is a $\delta>0$ such that for any x, y in I satisfying $|x-y|<\delta$,

$$\begin{aligned}\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(x))\right|=\\ \left|k\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-k\,{}^{C}D^{\alpha}((f(x))\right|\end{aligned} \tag{44}$$

By using Eq. (42) the above equation reduces to

$$\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(x))\right|<k\frac{\epsilon}{k}=\epsilon \tag{45}$$

Similarly by using Eq. (43) we obtain

$$\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(y))\right|<\epsilon \tag{46}$$

Thus $kf\in UFD(I)$.
Therefore the space $UFD(I)$ of uniformly fractional differentiable functions on I is a vector space with pointwise operations.

References

1. Patel, M.R.: Uniformly differentiable functions. M.Sc. Research Project, Sardar Patel University, 2009–10
2. Apostol, T.M.: Mathematical Analysis, 2nd edn. Narosa Publishing House (1997)
3. Caputo, M.: Linear models of dissipation whose Q is almost frequency independent II. Geophys. J. R. Astron. Soc. **13**, 529–539 (1967)
4. Diethelm, K.: The mean value theorems and a Nagumo-type uniquness theorem for Caputo's fractional calculus. Fract. Calc. Appl. Anal. **15**(2) (2012)
5. Yang, X.J.: A short note on local fractional calculus of functions of one variable. J. Appl. Libr. Inf. Sci. (JALIS) **1**(1), 1–12 (2012)
6. Yang, X.J., Gao, G.: The fundamentals of local fractional derivative of the one-variable non-differentiable functions. World Sci-Tech. R and D **31**(5), 920–921 (2009)
7. Prajapati, J.C., Kachhia, K.B.: Functions of bounded fractional differential variation a new concept. Georgian Math. J. **23**(3), 417–427 (2016)
8. Gilberg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, New York (2001)