Existence and Ulam's Type Stability of Integro Differential Equation with Non-instantaneous Impulses and Periodic Boundary Condition on Time Scales

Vipin Kumar and Muslim Malik

Abstract The present manuscript is dedicated to the study of existence and stability of integro differential equation with periodic boundary condition and non-instantaneous impulses on time scales. Banach contraction theorem and non-linear functional analysis have been used to established these results. Moreover, to outline the utilization of these outcomes an example is given.

Keywords Existence · Stability · Time scales · Non-instantaneous impulses

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1 Introduction

There are many physical models which are subject to sudden changes in its states, such rapid changes are known as impulsive response. In the current hypothesis, there are two types of impulsive system, one is instantaneous and another one is known as non-instantaneous impulsive system. In the instantaneous impulsive system, the duration of these abrupt changes is very little correlation to the duration of the whole process, for example pulses, stuns and cataclysmic events [7, 16], while in the non-instantaneous impulses, the duration of these changes continues over a finite time interval. For the initial studies related with the existence, uniqueness, and controllability of non-instantaneous impulsive systems of integer and fractional order, we refer to [10, 15, 18, 21] and the references cited therein. Further, stability analysis of dynamical systems becomes an important research area and various form of

V. Kumar · M. Malik (✉)
School of Basic Sciences, Indian Institute of Technology Mandi, Mandi, HP, India
e-mail: muslim@iitmandi.ac.in

V. Kumar
e-mail: math.vipinkumar219@gmail.com

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stabilities have been developed including Lyapunov stability, Mittag-Leffler function and exponential for dynamical equations. Moreover, an interesting type of stability was introduced by Ulam and Hyers is known as Ulam-Hyers stability which is highly useful in numerical analysis and optimization for dynamical equations. The Ulam-Hyper's stability for many dynamical equations of integer and fractional order has been studied in lots of articles [4, 5, 25, 26].

In 1988, Hilger presented the time scales calculus. The investigation of analytics on time scales incorporates the continuous and discrete analysis, therefore the investigation of dynamical system on time scales has picked up an awesome consideration and numerous scientists have discovered the uses of time scales in heat transfer system [19], population dynamics [28] and economics [11, 12]. For more details about time scales one can refer the book [8, 9] and papers [2, 3, 17]. Further over the most recent couple of years, many authors talked about the existence, uniqueness and stability of dynamical system on time scales [1, 6, 13, 14, 20, 22–24, 27]. Particularly, Geng [13], presented the concepts of lower and upper solutions for a PBVP on time scales.

According as far as anyone is concerned, there is no manuscript which examined the existence, uniqueness and stability investigation of integro differential equations with non-instantaneous impulses on time scales. Spurred by the above actualities, we take the differential equations with periodic boundary condition and non-instantaneous impulses on time scale of the form:

$$\begin{aligned} v^{\Delta}(\theta) &= \mathcal{C}\left(\theta, v(\theta), \int_0^{\theta} h(\theta,\tau,v(\tau))\Delta\tau\right), \quad \theta \in \cup_{k=0}^{l}(\lambda_k,\theta_{k+1}]_{\mathbb{T}}, \\ v(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \theta\in(\theta_k,\lambda_k]_{\mathbb{T}},\ k=1,2,\ldots,l,\ q\in(0,1) \\ v(0) &= v(T) \end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\theta_k, \lambda_k \in \mathbb{T}$ are right dense points with $0 = \lambda_0 = \theta_0 < \theta_1 < \lambda_1 < \theta_2 < \cdots \lambda_l < \theta_{l+1} = T$, $v(\theta_k^-) = \lim_{h\to 0^+} v(\theta_k - h)$, $v(\theta_k^+) = \lim_{h\to 0^+} v(\theta_k + h)$, represent the left and right limits of $v(\theta)$ at $\theta = \theta_k$. The functions $\boldsymbol{g}_k(\theta, v(\theta_k^-)) \in C(I,\mathbb{R})$ represent non-instantaneous impulses during the intervals $(\theta_k,\lambda_k]_{\mathbb{T}},\ k = 1,2,\ldots,l$, so impulses at θ_k have some duration, namely on intervals $(\theta_k,\lambda_k]_{\mathbb{T}}$. $\mathcal{C} : I = [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathcal{Q}\times\mathbb{R}\to\mathbb{R}$ are given functions, where $\mathcal{Q} = \{(\theta,\tau) \in I\times I : 0 \le \tau \le \theta \le T\}$.

Throughout the manuscript, we impose

$$\mathcal{M}(v(\theta)) = \int_0^{\theta} h(\theta,\tau,v(\tau))\Delta\tau.$$

The structure of the manuscript is as: In second section, we give preliminaries, fundamental definitions, useful lemmas and some important results. In the subsequent sections, the main results of the manuscript are discussed. Finally, an example is given to outline the utilization of these outcomes.

2 Preliminaries

Below, we give basic notations, fundamental definitions and useful lemmas. Let $(X, \|.\|)$ be a Banach space. $C(I, \mathbb{R})$ be the set of all continuous functions. In order to define the solution of the Eq. (1.1), we define the space $PC(I, \mathbb{R})$ of piecewise continuous functions defined as $PC(I, \mathbb{R}) = \{v : I \to \mathbb{R} : v \in C(\theta_k, \theta_{k+1}]_{\mathbb{T}}, \mathbb{R}), k = 0, 1, \ldots, l$ and there exists $v(\theta_k^-)$ and $v(\theta_k^+)$, $k = 1, 2, \ldots, l$ with $v(\theta_k^-) = v(\theta_k)\}$. It can be seen easily that $PC(I, \mathbb{R})$ is a Banach space with the TZ-norm

$$\|v\|_\Omega = \sup_{\theta\in[a,b]} \frac{\|v(\theta)\|}{e_\Omega(\theta, a)}, \text{ for some } \Omega \in \mathcal{R}^+.$$

A closed non-empty subset of real number is called time scales $\mathbb{T}$. A time scale interval is defined as $[i, m]_{\mathbb{T}} = \{\theta \in \mathbb{T} : i \leq \theta \leq m\}$, accordingly, we define $(i, m)_{\mathbb{T}}$, $[i, m)_{\mathbb{T}}$ and so on. Now onwards, we used a time scale interval $[i, m]$ instead of $[i, m]_{\mathbb{T}}$. Also, now onward if max $\mathbb{T}$ exists, then we take $\mathbb{T}^k = \mathbb{T}\backslash\{\max \mathbb{T}\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T}^k \to \mathbb{T}$ is defined by $\sigma(\theta) := \inf\{r \in \mathbb{T} : r > \theta\}$ with the substitution $\inf\{\phi\} = \sup \mathbb{T}$ and the graininess function $\mu : \mathbb{T}^k \to [0, \infty)$ is define as $\mu(\theta) := \sigma(\theta) - \theta, \forall\theta \in \mathbb{T}^k$.

Definition 2.1 Let $z : \mathbb{T} \to \mathbb{R}$ and $\theta \in \mathbb{T}^k$. The delta derivative $z^\Delta(\theta)$ is the number (when it exists) such that given any $\epsilon > 0$, there is a neighbourhood U of θ such that

$$|[z(\sigma(\theta)) - z(\tau)] - z^\Delta(\theta)[\sigma(\theta) - \tau]| \leq \epsilon|\sigma(\theta) - \tau|, \quad \forall\, \tau \in U.$$

Definition 2.2 Function Z is said to be antiderivative of $z : \mathbb{T} \to \mathbb{R}$ provided $Z^\Delta(\theta) = z(\theta)$ for each $\theta \in \mathbb{T}^k$, then the delta integral is defined by

$$\int_{\theta_0}^{\theta} z(\zeta)\Delta\zeta = Z(\theta) - Z(\theta_0).$$

A function $z : \mathbb{T} \to \mathbb{R}$ is called rd-continuous on $\mathbb{T}$, if z has finite left-sided limits at points $\theta \in \mathbb{T}$ with $\sup\{r \in \mathbb{T} : r < \theta\} = \theta$ and z is continuous at points $\theta \in \mathbb{T}$ with $\sigma(\theta) = \theta$. The collection of all rd-continuous functions $z : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive (positive regressive) if $1 + \mu(\theta)p(\theta) \neq 0(> 0), \forall\theta \in \mathbb{T}$ and the set of all regressive (positive regressive) functions are denoted by $\mathcal{R}(\mathcal{R}^+)$.

Definition 2.4 The generalized exponential function is defined as

$$e_p(\theta, r) = \exp\left(\int_r^\theta \xi_{\mu(\zeta)}(p(\zeta))\Delta\zeta\right), \quad \theta, r \in \mathbb{T},\ p \in \mathcal{R},$$

where $\xi_{\mu(\beta)}(p(\beta))$ is given by

$$\xi_{\mu(\beta)}(\varkappa) = \begin{cases} \frac{1}{\mu(\beta)} Log(1 + \mu(\beta)\varkappa), & \text{if } \mu(\beta) \neq 0. \\ \varkappa, & \text{if } \mu(\beta) = 0. \end{cases}$$

Lemma 2.5 ([17]) *Let* $\theta_1, \theta_2 \in \mathbb{T}$, *such that* $\theta_1 \leq \theta_2$ *and* $z : \mathbb{R} \to \mathbb{R}$ *be a non-decreasing continuous function. Then,*

$$\int_{\theta_1}^{\theta_2} z(\zeta)\Delta\zeta \leq \int_{\theta_1}^{\theta_2} z(\zeta)d\zeta. \tag{2.1}$$

Lemma 2.6 *Let* $\boldsymbol{g} : I \to \mathbb{R}$ *be a right dense continuous function. Then, for any* $k = 1, 2, \ldots, l$, *the solution of the following problem*

$$\begin{aligned} &v^\Delta(\theta) = \boldsymbol{g}(\theta), \quad \theta \in \cup_{k=0}^l (\lambda_k, \theta_{k+1}], \\ &v(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^\theta (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l, \\ &v(0) = v(T), \end{aligned}$$

is given by the following integral equation

$$\begin{aligned} v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, v(\theta_l^-))\Delta\zeta + \int_{\lambda_l}^T \boldsymbol{g}(\zeta)\Delta\zeta + \int_0^\theta \boldsymbol{g}(\zeta)\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1], \\ v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^\theta (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \forall\, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l, \\ v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta + \int_{\lambda_k}^\theta \boldsymbol{g}(\zeta)\Delta\zeta, \quad \forall\, \theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l. \end{aligned}$$

(H1): The non-linear function $\mathcal{C} : J_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ J_1 = \cup_{k=0}^l [\lambda_k, \theta_{k+1}]$ is continuous and $\exists$ positive constants $L_{\mathcal{C}_1},\ L_{\mathcal{C}_2}$ such that

$$|\mathcal{C}(\theta, v_1, v_2) - \mathcal{C}(\theta, w_1, w_2)| \leq L_{\mathcal{C}_1}|v_1 - w_1| + L_{\mathcal{C}_2}|v_2 - w_2|,$$
$$\forall\, \theta \in I,\ v_j, w_j \in \mathbb{R},\ j = 1, 2.$$

Also, $\exists$ positive constants $C_{\mathcal{C}},\ M_{\mathcal{C}}$ and $N_{\mathcal{C}}$ such that

$$|\mathcal{C}(\theta, v, w)| \leq C_{\mathcal{C}} + M_{\mathcal{C}}|v| + N_{\mathcal{C}}|w|, \quad \forall\, \theta \in I,\ v, w \in \mathbb{R}.$$

(H2): $h : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists$ positive constant L_h such that

$$|h(\theta, \tau, v) - h(\theta, \tau, w)| \leq L_h|v - w|, \quad \forall\, \theta, \tau \in \mathcal{Q},\ v, w \in \mathbb{R}.$$

Also, $\exists$ positive constants $C_h,\ M_h$ such that

$$|h(\theta, \tau, v)| \leq C_h + M_h|v|, \quad \forall\, \theta, \tau \in \mathcal{Q},\ v \in \mathbb{R}.$$

(H3): The functions $\boldsymbol{g}_k : I_k \times \mathbb{R} \to \mathbb{R},\ I_k = [\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$ are continuous and $\exists$ a positive constant $L_{\boldsymbol{g}}$ such that

$$|\boldsymbol{g}_k(\theta, v) - \boldsymbol{g}_k(\theta, w)| \leq L_{\boldsymbol{g}}|v - w|, \quad \forall\, v, w \in \mathbb{R}, \theta \in I_k,\ k = 1, 2, \ldots, l.$$

Also, $\exists$ a positive constant $M_{\boldsymbol{g}}$ such that $|\boldsymbol{g}_k(\theta, v)| \leq M_{\boldsymbol{g}}, \quad \forall\, \theta \in I_k$ and $v \in \mathbb{R}$.

(H4): $\max_{1 \leq k \leq l}\left(e_\Omega(T, \lambda_k)\left(\frac{M_{\mathcal{C}}}{\Omega} + \frac{N_{\mathcal{C}} M_h}{\Omega^2}\right)\right) < 1.$

3 Existence and Uniqueness

Theorem 3.1 *Let the assumptions **(H1)–(H4)** are holds, then Eq. (1.1) has a unique solution provided,*

$$e_\Omega(T, \lambda_l)\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right) < 1.$$

Proof Consider a subset $\mathcal{D} \subseteq PC(I, \mathbb{R})$ such that

$$\mathcal{D} = \{v \in PC(I, \mathbb{R}) : \|v\|_\Omega \leq \beta\},$$

where

$$\beta = \max_{1 \leq k \leq l}\left(\frac{\dfrac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + N_{\mathcal{C}} C_h (T^2 + \theta_1^2)}{1 - (1 + e_\Omega(T, \lambda_k))\left(\dfrac{M_{\mathcal{C}}}{\Omega} + \dfrac{N_{\mathcal{C}} M_h}{\Omega^2}\right)}\right).$$

Now, define an operator $\Pi : \mathcal{D} \to \mathcal{D}$ given by

$$(\Pi v)(\theta) = \int_0^\theta \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, v(\theta_l^-))\Delta\zeta$$
$$+ \int_{\lambda_l}^{T} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1],$$

$$(\Pi v)(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \forall\, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l,$$

$$(\Pi v)(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta + \int_{\lambda_k}^{\theta} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta,$$
$$\forall\, \theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l.$$

The proof of this theorem are divided into two steps.
Step 1: To use the Banach contraction theorem, we have to show that $\Pi : \mathcal{D} \to \mathcal{D}$. For this, we are taking three cases as follows:
Case 1: For $\theta \in (\lambda_k, \theta_{k+1}]$, $k = 1, 2, \ldots, l$ and $v \in \mathcal{D}$, we have:

$$|(\Pi v)(\theta)| \le \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} |\boldsymbol{g}_k(\zeta, v(\theta_k^-))|\Delta\zeta + \int_{\lambda_k}^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta$$
$$\le \frac{M_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1}\Delta\zeta + \int_{\lambda_k}^{\theta} (C_{\mathcal{C}} + M_{\mathcal{C}}|v(\zeta)| + N_{\mathcal{C}}|\mathcal{M}(v(\zeta))|)\Delta\zeta$$
$$\le \frac{M_{\boldsymbol{g}}(\lambda_k - \theta_k)^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h \theta_{k+1})(\theta_{k+1} - \lambda_k)$$
$$+ \left(M_{\mathcal{C}}\beta + \frac{N_{\mathcal{C}} M_h \beta}{\Omega}\right) \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k)\Delta\zeta$$
$$\le \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h T)T + \frac{M_{\mathcal{C}}\beta e_\Omega(\theta, \lambda_k)}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta e_\Omega(\theta, \lambda_k)}{\Omega^2}.$$

Hence,

$$\|\Pi v\|_\Omega \le \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h T)T + \frac{M_{\mathcal{C}}\beta}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta}{\Omega^2}. \tag{3.1}$$

Case 2: For $\theta \in [0, \theta_1]$ and $v \in \mathcal{D}$, we have:

$$
\begin{aligned}
|(\Pi v)(\theta)| \leq & \int_0^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} |\boldsymbol{g}_l(\zeta, v(\theta_l^-))|\Delta\zeta \\
& + \int_{\lambda_l}^{T} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta \\
& \leq \frac{M_{\boldsymbol{g}}(\lambda_l - \theta_l)^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T - \lambda_l) + M_{\mathcal{C}}\beta \int_{\lambda_l}^{T} e_{\Omega}(\zeta, \lambda_l)\Delta\zeta + N_{\mathcal{C}} C_h T (T - \lambda_l) \\
& + \frac{N_{\mathcal{C}} M_h \beta}{\Omega} \int_{\lambda_l}^{T} e_{\Omega}(\zeta, \lambda_l)\Delta\zeta + C_{\mathcal{C}}\theta_1 + N_{\mathcal{C}} C_h \theta_1^2 \\
& + \left(M_{\mathcal{C}}\beta + \frac{N_{\mathcal{C}} M_h \beta}{\Omega} \right) \int_0^{\theta} e_{\Omega}(\zeta, 0)\Delta\zeta \\
& \leq \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + \frac{M_{\mathcal{C}}\beta e_{\Omega}(T, \lambda_l)}{\Omega} + N_{\mathcal{C}} C_h (T^2 + \theta_1^2) \\
& + \frac{N_{\mathcal{C}} M_h \beta e_{\Omega}(T, \lambda_l)}{\Omega^2} + \frac{M_{\mathcal{C}}\beta e_{\Omega}(\theta, 0)}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta e_{\Omega}(\theta, 0)}{\Omega^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\Pi v\|_{\Omega} \leq & \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + \frac{M_{\mathcal{C}}\beta e_{\Omega}(T, \lambda_l)}{\Omega} + N_{\mathcal{C}} C_h (T^2 + \theta_1^2) \\
& + \frac{N_{\mathcal{C}} M_h \beta e_{\Omega}(T, \lambda_l)}{\Omega^2} + \frac{M_{\mathcal{C}}\beta}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta}{\Omega^2}.
\end{aligned} \tag{3.2}
$$

Case 3: For $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$ and $v \in \mathcal{D}$, we can easily get:

$$
\|\Pi v\|_{\Omega} = \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)}. \tag{3.3}
$$

After summarizing the above inequalities (3.1)–(3.3), we get:

$$
\|\Pi v\|_{\Omega} \leq \beta.
$$

Therefore, $\Pi : \mathcal{D} \to \mathcal{D}$.
Step 2: In this step, we will show that the operator Π is a contracting operator. Here also, we are taking three cases as follows:

Case 1: For any $v, w \in \mathcal{D},\ \theta \in (\lambda_k, \theta_{k+1}], k = 1, 2, \ldots, l$, we have:

$$\begin{aligned}
|(\Pi v)(\theta) - (\Pi w)(\theta)| &\leq \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} |\boldsymbol{g}_k(\zeta, v(\theta_k^-)) - \boldsymbol{g}_k(\zeta, w(\theta_k^-))| \Delta\zeta \\
&\quad + \int_{\lambda_k}^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(w(\zeta)))| \Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} \frac{(\lambda_k - \zeta)^{q-1} |v(\theta_k^-) - w(\theta_k^-)| e_\Omega(\theta_k^-, \theta_k)}{e_\Omega(\theta_k^-, \theta_k)} \Delta\zeta \\
&\quad + L_{\mathcal{C}_1} \int_{\lambda_k}^{\theta} \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, \lambda_k)}{e_\Omega(\zeta, \lambda_k)} \Delta\zeta \\
&\quad + L_{\mathcal{C}_2} \int_{\lambda_k}^{\theta} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))| \Delta\zeta \\
&\leq \frac{\|v - w\|_\Omega L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k)(\lambda_k - \theta_k)^q}{\Gamma(q+1)} \\
&\quad + L_{\mathcal{C}_1} \|v - w\|_\Omega \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k) \Delta\zeta \\
&\quad + \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k) \Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k)(\lambda_k - \theta_k)^q \|v - w\|_\Omega}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1} e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega} \\
&\quad + \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}$$

Thus, we have:

$$\|\Pi v - \Pi w\|_\Omega \leq \left[\frac{L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k) T^q}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2} \right] \|v - w\|_\Omega. \tag{3.4}$$

Case 2: For any $v, w \in \mathcal{D}, \theta \in [0, \theta_1]$, we have:

$$
\begin{aligned}
|(\Pi v)(\theta) - (\Pi w)(\theta)| \leq & \int_0^\theta |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))|\Delta\zeta \\
& + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} |\boldsymbol{g}_l(\zeta, v(\theta_l^-)) - \boldsymbol{g}_l(\zeta, w(\theta_l^-))|\Delta\zeta \\
& + \int_{\lambda_l}^{T} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))|\Delta\zeta \\
\leq & \frac{L_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} \frac{(\lambda_l - \zeta)^{q-1} |v(\theta_l^-) - w(\theta_l^-)| e_\Omega(\theta_l^-, \theta_l)}{e_\Omega(\theta_l^-, \theta_l)} \Delta\zeta \\
& + L_{\mathcal{C}_1} \int_{\lambda_l}^{T} \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, \lambda_l)}{e_\Omega(\zeta, \lambda_l)} \Delta\zeta \\
& + L_{\mathcal{C}_2} \int_{\lambda_l}^{T} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))|\Delta\zeta \\
& + L_{\mathcal{C}_1} \int_0^\theta \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, 0)}{e_\Omega(\zeta, 0)} \Delta\zeta \\
& + L_{\mathcal{C}_2} \int_0^\theta |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))|\Delta\zeta \\
\leq & \frac{L_{\boldsymbol{g}} e_\Omega(\theta_l^-, \theta_l)(\lambda_l - \theta_l)^q \|v - w\|_\Omega}{\Gamma(q+1)} + L_{\mathcal{C}_1} \|v - w\|_\Omega \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta \\
& + \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta + L_{\mathcal{C}_1} \|v - w\|_\Omega \int_0^\theta e_\Omega(\zeta, 0)\Delta\zeta \\
& + \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_0^\theta e_\Omega(\zeta, 0)\Delta\zeta \\
\leq & \frac{L_{\mathcal{C}_2} L_h e_\Omega(T, \lambda_l) \|v - w\|_\Omega}{\Omega^2} + \frac{L_{\boldsymbol{g}} e_\Omega(\theta_l^-, \theta_l)(\lambda_l - \theta_l)^q \|v - w\|_\Omega}{\Gamma(q+1)} \\
& + \frac{\|v - w\|_\Omega L_{\mathcal{C}_1} e_\Omega(T, \lambda_l)}{\Omega} + \frac{L_{\mathcal{C}_1} e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega} \\
& + \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Therefore,

$$\|\Pi v-\Pi w\|_{\Omega}\leq\left[\frac{L_{g}e_{\Omega}(\theta_{l}^{-},\theta_{l})T^{q}}{\Gamma(q+1)}+(1+e_{\Omega}(T,\lambda_{l}))\left(\frac{L_{\mathcal{C}_1}}{\Omega}+\frac{L_{\mathcal{C}_2}L_h}{\Omega^2}\right)\right]\|v-w\|_{\Omega}. \tag{3.5}$$

Case 3: Similarly, for $\theta\in(\theta_k,\lambda_k],\ k=1,2,\ldots,l$, we get:

$$|(\Pi v)(\theta)-(\Pi w)(\theta)|\leq\frac{L_{g}e_{\Omega}(\theta_{k}^{-},\theta_{k})T^{q}}{\Gamma(q+1)}\|v-w\|_{\Omega}.$$

Therefore,

$$\|\Pi v-\Pi w\|_{\Omega}\leq\frac{L_{g}T^{q}}{e_{\Omega}(\theta_k,\theta_k^{-})\Gamma(q+1)}\|v-w\|_{\Omega}. \tag{3.6}$$

After summarizing the inequalities (3.4)–(3.6), we get:

$$\|\Pi v-\Pi w\|_{\Omega}\leq L_{\Pi}\|v-w\|_{\Omega},$$

where

$$L_{\Pi}=\max_{1\leq k\leq l}\left[\frac{L_{g}T^{q}e_{\Omega}(\theta_{k}^{-},\theta_{k})}{\Gamma(q+1)}+(1+e_{\Omega}(T,\lambda_{l}))\left(\frac{L_{\mathcal{C}_1}}{\Omega}+\frac{L_{\mathcal{C}_2}L_h}{\Omega^2}\right)\right].$$

Hence, for sufficiently large Ω, Π is a strict contraction mapping. Therefore, Π has a unique fixed point and that fixed point is the solution of the taken Eq. (1.1). □

Let us consider a special case when $\mathcal{C}\left(\theta,v(\theta),\int_0^{\theta}h(\theta,\tau,v(\tau))\Delta\tau\right)=\mathcal{P}(\theta,v)+\int_0^{\theta}h(\theta,\tau,v(\tau))\Delta\tau$ then (1.1) becomes:

$$\begin{aligned}
v^{\Delta}(\theta)&=\mathcal{P}(\theta,v)+\int_0^{\theta}h(\theta,\tau,v(\tau))\Delta\tau,\quad \theta\in\cup_{k=0}^{l}(\lambda_k,\theta_{k+1}],\\
v(\theta)&=\frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta,v(\theta_k^{-}))\Delta\zeta,\quad \theta\in(\theta_k,\lambda_k],\ k=1,2,\ldots,l,\\
v(0)&=v(T).
\end{aligned} \tag{3.7}$$

(H5): $\mathcal{P}:J_1\times\mathbb{R}\to\mathbb{R}$ is a non-linear continuous function and $\exists$ a positive constant $L_{\mathcal{P}}$ such that

$$|\mathcal{P}(\theta, v) - \mathcal{P}(\theta, w)| \leq L_{\mathcal{P}}|v - w|, \quad \forall\, \theta \in I,\ v, w \in \mathbb{R}.$$

Also, ∃ positive constants $C_{\mathcal{P}}$ and $M_{\mathcal{P}}$ such that

$$|\mathcal{P}(\theta, v)| \leq C_{\mathcal{P}} + M_{\mathcal{P}}|v|, \quad \forall\, \theta \in I,\ v \in \mathbb{R}.$$

(H6): $\max_{1\leq k\leq l}\left(e_{\Omega}(T, \lambda_k)\left(\frac{M_{\mathcal{P}}}{\Omega} + \frac{M_h}{\Omega^2}\right)\right) < 1.$

Corollary 3.2 *If the assumptions (**H2**)–(**H3**) and (**H5**)–(**H6**) are holds, then the Eq. (3.7) has a unique solution, provided*

$$e_{\Omega}(T, \lambda_l)\left(\frac{L_{\mathcal{P}}}{\Omega} + \frac{L_h}{\Omega^2}\right) < 1.$$

4 Hyer-Ulam's Stability

For $\epsilon > 0$, $\psi \geq 0$, and nondecreasing $\varphi \in PC(I, \mathbb{R}^+)$, consider the below inequalities

$$\begin{cases} |w^{\Delta}(\theta) - \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta)))| \leq \epsilon, \quad \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]. \\ \left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases} \tag{4.1}$$

$$\begin{cases} |w^{\Delta}(\theta) - \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta)))| \leq \epsilon\varphi(\theta), \quad \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]. \\ \left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon\psi, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases} \tag{4.2}$$

Definition 4.1 ([25]) Equation (1.1) is called Hyer's-Ulam stable if there exists a positive constant $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}$ such that for $\epsilon > 0$ and for each solution w of inequality (4.1), there exist a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \leq H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}\epsilon, \quad \forall\, \theta \in I.$$

Definition 4.2 ([25]) Equation (1.1) is said to be generalized Hyer's-Ulam stable if there exists $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(0) = 0$ such that for each solution w of inequalities (4.1), there exists a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \le \mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(\epsilon), \quad \forall\, \theta \in I.$$

Remark 4.3 Definition (4.1) $\implies$ Definition (4.2).

Definition 4.4 ([25]) Equation (1.1) is said to be Hyers-Ulam-Rassias stable w.r.t (φ, ψ), if there exists $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g, \varphi)}$ such that for $\epsilon > 0$ and for each solution w of inequality (4.2), there exist a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \le H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g, \varphi)}\epsilon(\varphi(\theta), \psi), \quad \forall\, \theta \in I.$$

Remark 4.5 A function $w \in PC(I, \mathbb{R})$ is a solution of inequality (4.1) if and only if there is $\mathsf{G} \in PC(I, \mathbb{R})$ and a sequence $\mathsf{G}_k,\ k = 1, 2, \ldots, l$, such that

(a) $|\mathsf{G}(\theta)| \le \epsilon, \forall\, \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]$ and $|\mathsf{G}_k| \le \epsilon,\ \forall\, \theta \in (\theta_k, \lambda_k], k = 1, 2, \ldots, l.$
(b) $w^{\Delta}(\theta) = \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta))) + \mathsf{G}(\theta),\ \theta \in (\lambda_k, \theta_{k+1}],\ k = 0, 1, \ldots, l.$
(c) $w(\theta) = \dfrac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l.$

Now, by the above Remark 4.5, we have:

$$\begin{cases} w^{\Delta}(\theta) = \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta))) + \mathsf{G}(\theta),\ \theta \in (\lambda_k, \theta_{k+1}],\ k = 0, 1, \ldots, l, \\ w(\theta) = \dfrac{1}{\Gamma(q)} \displaystyle\int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases}$$

From Lemma 2.6, one can find that the solution w with $w(0) = w(T)$ of the above equation is given by

$$\begin{aligned} w(\theta) &= \int_0^{\theta} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, w(\theta_l^-))\Delta\zeta + \mathsf{G}_l \\ &\quad + \int_{\lambda_l}^{T} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1], \end{aligned}$$

$$w(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \quad \forall\, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l,$$

$$\begin{aligned} w(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k + \int_{\lambda_k}^{\theta} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta, \\ \forall\, \theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l. \end{aligned}$$

Therefore, for $\theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int\limits_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta - \int\limits_{\lambda_k}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right|$$

$$\leq |\mathsf{G}_k| + \int\limits_{\lambda_k}^{\theta}|\mathsf{G}(\zeta)|\Delta\zeta \leq \epsilon(1+T).$$

Also, for $\theta \in [0, \theta_1]$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int\limits_{\theta_l}^{\lambda_l}(\lambda_l-\zeta)^{q-1}\boldsymbol{g}_l(\zeta, w(\theta_l^-))\Delta\zeta - \int\limits_{\lambda_l}^{T}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right.$$

$$\left. - \int\limits_{0}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right| \leq |\mathsf{G}_l| + \int\limits_{\lambda_l}^{T}|\mathsf{G}(\zeta)|\Delta\zeta + \int\limits_{0}^{\theta}|\mathsf{G}(\zeta)|\Delta\zeta$$

$$\leq \epsilon(1+2T).$$

Similarly, for $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int\limits_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon.$$

We have similar remark for the inequality (4.2).

Theorem 4.6 *If the assumptions of Theorem 3.1 are holds, then the Eq. (1.1) is Hyer-Ulam stable.*

Proof Let $w \in PC(I, \mathbb{R})$ be the solution of inequality (4.1) and $v \in PC(I, \mathbb{R})$ be a unique solution of the Eq. (1.1). Therefore, for $\theta \in (\lambda_k, \theta_{k+1}]$, $k = 1, 2, \ldots, l$, we have:

$$|w(\theta) - v(\theta)| \leq \left|w(\theta) - \int\limits_{\lambda_k}^{\theta}\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta\right| - \frac{1}{\Gamma(q)}\int\limits_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta$$

$$\leq \left|w(\theta) - \frac{1}{\Gamma(q)}\int\limits_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right.$$

$$\left. - \int\limits_{\lambda_k}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right|$$

$$+ \left|\frac{1}{\Gamma(q)}\int\limits_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}(\boldsymbol{g}_k(\zeta, w(\theta_k^-)) - \boldsymbol{g}_k(\zeta, v(\theta_k^-)))\Delta\zeta\right|$$

$$
\begin{aligned}
&+\left|\int_{\lambda_k}^{\theta}(\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))-\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(v(\zeta))))\Delta\zeta\right| \\
&\leq \epsilon(1+T)+\frac{L_g}{\Gamma(q)}\int_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}|w(\theta_k^-)-v(\theta_k^-)|\Delta\zeta \\
&+ L_{\mathcal{C}_1}\int_{\lambda_k}^{\theta}|w(\zeta)-v(\zeta)|\Delta\zeta + L_{\mathcal{C}_2}\int_{\lambda_k}^{\theta}|\mathcal{M}(w(\zeta))-\mathcal{M}(v(\zeta))|\Delta\zeta \\
&\leq \epsilon(1+T)+\frac{L_g e_\Omega(\theta_k^-,\theta_k)(\lambda_k-\theta_k)^q\|v-w\|_\Omega}{\Gamma(q+1)} \\
&+ \frac{L_{\mathcal{C}_1}e_\Omega(\theta,\lambda_k)\|v-w\|_\Omega}{\Omega}+\frac{L_{\mathcal{C}_2}L_h e_\Omega(\theta,\lambda_k)\|v-w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Hence,

$$
\|w-v\|_\Omega \leq \epsilon(1+T)+\left[\frac{L_g e_\Omega(\theta_k^-,\theta_k)T^q}{\Gamma(q+1)}+\frac{L_{\mathcal{C}_1}}{\Omega}+\frac{L_{\mathcal{C}_2}L_h}{\Omega^2}\right]\|v-w\|_\Omega. \quad (4.3)
$$

Also, for $\theta \in [0, \theta_1]$, we have:

$$
\begin{aligned}
|w(\theta)-v(\theta)| &\leq \left|w(\theta)-\int_0^{\theta}\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta\right| \\
&- \frac{1}{\Gamma(q)}\int_{\theta_l}^{\lambda_l}(\lambda_l-\zeta)^{q-1}\boldsymbol{g}_l(\zeta, v(\theta_l^-))\Delta\zeta-\int_{\lambda_l}^{T}\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta \\
&\leq \epsilon(1+2T)+\frac{L_g}{\Gamma(q)}\int_{\theta_l}^{\lambda_l}(\lambda_l-\zeta)^{q-1}|v(\theta_l^-)-w(\theta_l^-)|\Delta\zeta \\
&+ L_{\mathcal{C}_1}\int_{\lambda_l}^{T}|v(\zeta)-w(\zeta)|\Delta\zeta+L_{\mathcal{C}_2}\int_{\lambda_l}^{T}|\mathcal{M}(v(\zeta))-\mathcal{M}(w(\zeta))|\Delta\zeta \\
&+ L_{\mathcal{C}_1}\int_{0}^{\theta}|v(\zeta)-w(\zeta)|\Delta\zeta+L_{\mathcal{C}_2}\int_{0}^{\theta}|\mathcal{M}(v(\zeta))-\mathcal{M}(w(\zeta))|\Delta\zeta \\
&\leq \epsilon(1+2T)+\frac{L_g e_\Omega(\theta_l^-,\theta_l)(\lambda_l-\theta_l)^q\|v-w\|_\Omega}{\Gamma(q+1)}+\frac{L_{\mathcal{C}_1}e_\Omega(T,\lambda_l)\|v-w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2}L_h e_\Omega(T,\lambda_l)\|v-w\|_\Omega}{\Omega^2}+\frac{L_{\mathcal{C}_1}e_\Omega(\theta,0)\|v-w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2}L_h e_\Omega(\theta,0)\|v-w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Thus,

$$\|w-v\|_{\Omega} \leq \epsilon(1+2T) + \left[\frac{L_{\boldsymbol{g}} e_{\Omega}(\theta_l^-, \theta_l) T^q}{\Gamma(q+1)} + (1+e_{\Omega}(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right]\|v-w\|_{\Omega}. \tag{4.4}$$

Similarly, for $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$, we can easily find that

$$\begin{aligned} |w(\theta) - v(\theta)| &\leq \left| w(\theta) - \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta-\zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-)) \Delta\zeta \right| \\ &\leq \epsilon + \frac{L_{\boldsymbol{g}} (\lambda_k - \theta_k)^q e_{\Omega}(\theta_k^-, \theta_k) \|w-v\|_{\Omega}}{\Gamma(q+1)}. \end{aligned}$$

Therefore,

$$\|w-v\|_{\Omega} \leq \epsilon + \frac{L_{\boldsymbol{g}} T^q}{e_{\Omega}(\theta_k, \theta_k^-)\Gamma(q+1)} \|v-w\|_{\Omega}. \tag{4.5}$$

After summarizing the above inequalities (4.3)–(4.5), we get:

$$\begin{aligned} \|w-v\|_{\Omega} \leq \epsilon(1+2T) &+ \left[\frac{L_{\boldsymbol{g}} e_{\Omega}(\theta_l^-, \theta_l) T^q}{\Gamma(q+1)} + (1+e_{\Omega}(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right] \\ &\times \ \|v-w\|_{\Omega},\ \forall\, \theta \in I. \end{aligned}$$

Hence,

$$\|w-v\|_{\Omega} \leq H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)} \epsilon, \quad \theta \in I,$$

where $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)} = \dfrac{1+2T}{1-L_{\Pi}} > 0$. Thus, the Eq. (1.1) is Ulam-Hyer's stable. Moreover, if we put $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(\epsilon) = H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}\epsilon$, $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(0) = 0$, then the Eq. (1.1) is generalized Ulam-Hyer's stable. □

(H7): There exists a $\delta_{\varphi} > 0$ such that $\int_0^{\theta} \varphi(\zeta)\Delta\zeta \leq \delta_{\varphi}\varphi(\theta),\ \forall\, \theta \in I$.

The following theorem is the consequence of the Theorem 4.6.

Theorem 4.7 *If the conditions of Theorem 3.1 and* ***(H7)*** *are holds, then the Eq. (1.1) is Hyer's-Ulam-Rassias stable.*

5 Example

Consider the following equation with impulses on $\mathbb{T}$, $(0, 3/5, 4/5, 1 \in \mathbb{T})$

$$v^{\Delta}(\theta) = \frac{5 + |v(\theta)|}{20e^{\theta+3}(1 + |v(\theta)|)} + \frac{1}{10}\int_0^{\theta} \frac{\theta\tau^2 \sin(v(\tau))}{e^{\tau+5}}\Delta\tau, \quad \theta \in I' = [0, 1]_{\mathbb{T}} \setminus (\theta_1, \lambda_1]_{\mathbb{T}},$$

$$v(\theta) = \frac{1}{\Gamma(q)}\int_{\theta_1}^{\theta} \frac{(\theta - \zeta)^{q-1}(1 + \zeta^2 \sin(v(\theta_1^-)))}{15}\Delta\zeta, \quad \theta \in (\theta_1, \lambda_1]_{\mathbb{T}}, \tag{5.1}$$

$$v(0) = v(1).$$

Set,

$$\mathcal{C}(\theta, v, w) = \frac{5 + |v(\theta)|}{20e^{\theta+3}(1 + |v(\theta)|)} + \frac{1}{10}w, \ \theta \in I', \ v, w \in \mathbb{R},$$

$$h(\theta, \tau, v) = \frac{\theta\tau^2 \sin(v(\tau))}{e^{\tau+5}}, \ \forall\, \theta, \tau \in I', v \in \mathbb{R},$$

and

$$\boldsymbol{g}_1(\theta, v) = \frac{1 + \theta^2 \sin(v(\theta_1^-))}{15}, \ \theta \in (\theta_1, \lambda 1], \ v \in \mathbb{R}.$$

Then, $\forall\, \theta, \tau \in I = [0, 1], \ v, w, x, y \in \mathbb{R}$, we have:

$$\begin{aligned} |f(\theta, v, w) - f(\theta, x, y)| &\le \frac{1}{20e^3}|v - x| + \frac{1}{10}|w - y|, \\ |f(\theta, v, w)| &\le \frac{5 + |v|}{20e^3} + \frac{1}{10}|w|, \\ |\boldsymbol{g}_1(\theta, v) - \boldsymbol{g}_1(\theta, w)| &\le \frac{1}{15}|v - w|, \ |h(\theta, \tau, v)| \le \frac{1}{e^5} + \frac{1}{e^5}|v|, \\ |h(\theta, \tau, v) - h(\theta, \tau, w)| &\le \frac{1}{e^5}|v - w|. \end{aligned}$$

Hence, the assumptions (H1)–(H4) are holds with $L_{\mathcal{C}_1} = \frac{1}{20e^3}$, $L_{\mathcal{C}_2} = \frac{1}{10}$, $C_{\mathcal{C}} = \frac{5}{20e^3}$, $M_{\mathcal{C}} = \frac{1}{20e^3}$, $N_{\mathcal{C}} = \frac{1}{10}$, $L_h = \frac{1}{e^5}$, $C_h = \frac{1}{e^5}$, $M_h = \frac{1}{e^5}$, $L_g = \frac{1}{15}$, $M_g = \frac{2}{15}$. Also, for $l = 1, \ \theta_1 = 3/5, \ \lambda_1 = 4/5, \ T = 1, \ \Omega = 10$, the condition

$$e_{\Omega}(T, \lambda_1)\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right) = 0.0039 \ (<1)$$

holds. Thus, from Theorems 3.1 and 4.6, Eq. (5.1) has a Ulam Hyer's stable solution which is unique.

6 Conclusion

In this manuscript, we have successfully established the existence of a unique solution for the system (1.1) by using the Banach contraction theorem and nonlinear functional analysis. Also, we established the Ulam-Hyer's stability of the taken problem (1.1). To illustrate the application of obtained results, we have given an example.

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References

1. Abass, S.: Qualitative analysis of dynamic equations on time scales. Electron. J. Differ. Equ. **2018**(51), 1–13 (2018)
2. Agarwal, R.P., Bohner, M.: Basic calculus on time scales and some of its applications. Results Math. **35**(1–2), 3–22 (1999)
3. Agarwal, R.P., Bohner, M., O'Regan, D., Peterson, A.: Dynamic equations on time scales: a survey. J. Comput. Appl. Math. **141**, 1–26 (2002)
4. Agarwal, R.P., Hristova, S., O'Regan, D.: Caputo fractional differential equations with non-instantaneous impulses and strict stability by Lyapunov functions. Filomat **31**(16), 5217–5239 (2017)
5. Abbas, S., Benchohra, M., Ahmed, A., Zhou, Y.: Some stability concepts for abstract fractional differential equations with not instantaneous impulses. Fixed Point Theory **18**(1), 3–15 (2017)
6. András, S., Mészáros, A.R.: Ulam-Hyers stability of dynamic equations on time scales via Picard operators. Appl. Math. Comput. **219**(9), 4853–4864 (2013)
7. Benchohra, M., Henderson, J., Ntouyas, S.K.: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
8. Bohner, M., Peterson, A.: Dynamic Equations on Time Scales. Birkhuser, Basel (2001)
9. Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. Springer Science and Business Media (2002)
10. Feĉkan, M., Wang, J.R.: A general class of impulsive evolution equations. Topol. Methods Nonlinear Anal. **46**(2), 915–933 (2015)
11. Ferhan, A.M., Biles, D.C., Lebedinsky, A.: An application of time scales to economics. Math. Comput. Model. **43**(7–8), 718–726 (2006)
12. Ferhan, A.M., Uysal, F.: A production-inventory model of HMMS on time scales. Appl. Math. Lett. **21**(3), 236–243 (2008)
13. Geng, F., Xu, Y., Zhu, D.: Periodic boundary value problems for first-order impulsive dynamic equations on time scales. Nonlinear Anal. Theory Methods Appl. **69**(11), 4074–4087 (2008)
14. Guan, W., Li, D.G., Ma, S.H.: Nonlinear first-order periodic boundary-value problems of impulsive dynamic equations on time scales. Electron. J. Differ. Equ. **2012**(198), 1–8 (2012)
15. Hernández, E., O'Regan, D.: On a new class of abstract impulsive differential equations. Proc. Am. Math. Soc. **141**(5), 1641–1649 (2013)

16. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations, vol. 6. World Scientific (1989)
17. Liu, H., Xiang, X.: A class of the first order impulsive dynamic equations on time scales. Nonlinear Anal. Theory Methods Appl. **69**(9), 2803–2811 (2008)
18. Malik, M., Kumar, A., Fečkan, M.: Existence, uniqueness and stability of solutions to second order nonlinear differential equations with non-instantaneous impulses. J. King Saud Univ. Sci. **30**(2), 204–213 (2018)
19. Naidu, D.: Singular perturbations and time scales in control theory and applications: an overview. Dyn. Contin. Discrete Impuls. Syst. Ser. B **9**, 233–278 (2002)
20. O$\hat{p}$uz, A.D., Topal, F. S.: Symmetric positive solutions for the systems of higher-order boundary value problems on time scales. Adv. Pure Appl. Math. **8**(4), 285–292 (2017)
21. Pandey, D.N., Das, S., Sukavanam, N.: Existence of solution for a second-order neutral differential equation with state dependent delay and non-instantaneous impulses. Int. J. Nonlinear Sci. **18**(2), 145–155 (2014)
22. Shen, Y.: The Ulam stability of first order linear dynamic equations on time scales. Results Math. **72**(4), 1881–1895 (2017)
23. Su, Y.H., Feng, Z.: Variational approach for a p-Laplacian boundary value problem on time scales. Appl. Anal. **97**(13), 2269–2287 (2018)
24. Tokmak Fen, F., Karaca, I.Y.: Existence of positive solutions for a second-order p-Laplacian impulsive boundary value problem on time scales. Bull. Iran. Math. Soc. **43**(6), 1889–1903 (2017)
25. Wang, J.R., Fečkan, M., Zhou, Y.: Ulams type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**(1), 258–264 (2012)
26. Wang, J.R., Li, X.: A uniform method to Ulam-Hyers stability for some linear fractional equations. Mediter. J. Math. **13**, 625–635 (2016)
27. Zhang, X., Zhu, C.: Periodic boundary value problems for first order dynamic equations on time scales. Adv. Differ. Equ. **1**, 76 (2012)
28. Zhuang, K.: Periodic solutions for a stage-structure ecological model on time scales. Electron. J. Differ. Equ. **2007**(88), 1–7 (2007)