On Pointwise Convergence of a Family of Nonlinear Integral Operators

Gumrah Uysal and Hemen Dutta

Abstract Let Λ be a non-empty index set consisting of σ indices and σ_0 is allowed to be either accumulation point of Λ or infinity. We assume that the function K_σ, $K_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and for any $\sigma \in \Lambda$ and satisfies some conditions. The main purpose of this work is to investigate the conditions under which Fatou type pointwise convergence is obtained for the operators in the following setting:

$$(T_\sigma f)(x) = \int_{-\infty}^{\infty} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right) \right) dt, \ \ x \in \mathbb{R},$$

where $P_{k,\sigma}$ and $\alpha_{k,\sigma}$ are real numbers satisfying certain conditions, at $p-\mu$-Lebesgue point of function f. The obtained results are used for presenting some theorems for the rate of convergences.

Keywords $p-\mu$-Lebesgue point · Rate of convergence · Lipschitz condition · Unified approach · Nonlinear integral operator

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1 Introduction

In the year 1962, Gadjiev et al. [5] investigated the asymptotic value of the approximation of measurable functions by integral operators of the form:

$$L_{\sigma}\left(f,x\right)=\int\limits_{\mathbb{R}}\sum_{k=1}^{\infty}P_{k,\sigma}f\left(x+\alpha_{k,\sigma}t\right)K_{\sigma}\left(t\right)dt,\ x\in\mathbb{R},\ \sigma\in\Lambda, \tag{1.1}$$

where Λ is a non-empty set of a non-negative real parameters σ, $\alpha_{k,\sigma}$ are assumed to be non-negative real numbers for all values of k and σ with $\sup\limits_{k,\sigma}\left\{\alpha_{k,\sigma}\right\}=\alpha^{*}<\infty$ and $P_{k,\sigma}$ are real numbers satisfying $\sum_{k=1}^{\infty}\left|P_{k,\sigma}\right|\leq M$ (M is independent of σ) and $\sum_{k=1}^{\infty}P_{k,\sigma}=1$ for all $\sigma\in\Lambda$. Also, the kernel function $K_{\sigma}:\mathbb{R}\to\mathbb{R}$ satisfies some certain conditions. The operators of type (1.1) were considered later in the works [16, 17] presenting some theorems concerning convergence in the norms of $L_{1}\left(\mathbb{R}\right)$ and $L_{p}\left(\mathbb{R}\right)$ $(1<p<\infty)$, respectively. In these works, in order to obtain the desired convergence, the new modulus of continuity definitions are given.

In the year 1983, Musielak [11] built a bridge between linear and nonlinear integral operators of convolution type by considering the following setting of integral operators

$$T_{w}f(y)=\int\limits_{G}K_{w}(x-y;f(x))dx,\ y\in G,\ w\in\Lambda, \tag{1.2}$$

where Λ is a non-empty set of indices and K_{w}, $K_{w}:G\times\mathbb{R}\to\mathbb{R}$, for any $w\in\Lambda$, is a kernel function satisfying some conditions including Lipschitz property with respect to its second variable. For some advanced studies concerning approximation by nonlinear integral operators, we refer the reader to [1, 6, 12, 20]. Also, for some works, related to linear integral operators of convolution type, we refer the reader to [2, 3, 7, 14, 21].

In [9], Mamedov handled the following m-singular integral operators

$$L_{\lambda}^{[m]}(f;x)=(-1)^{m+1}\int\limits_{\mathbb{R}}\left[\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt)\right]K_{\lambda}(t)dt, \tag{1.3}$$

where $x\in\mathbb{R}$, $m\geq1$ is a finite natural number and $\lambda\in\Lambda$ which is a non-empty set of non-negative indices, by harnessing m-th finite difference formulas. Under certain conditions, the operators of type (1.1) may be reduced to the operators of type (1.3). Fatou type convergence of nonlinear counterparts of the operators of type (1.3) were studied in [8]. Also, for some studies concerning convergence of m-singular integral operators in different function spaces, we refer the reader to [15, 22]. In 1965, Mamedov [10] studied the saturation classes of linear operators by considering further generalization of the operators of type (1.1), that is, the summation inside the operators runs from $k=-\infty$ to $+\infty$.

Let Λ be a non-empty index set consisting of σ indices. Here, σ_0 is allowed to be either accumulation point of Λ or infinity. We assume that the function $K_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and satisfies some conditions. The main purpose of this work is to investigate the conditions under which Fatou type pointwise convergence is obtained for the operators in the following setting:

$$(T_\sigma f)(x) = \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right) \right) dt, \ x \in \mathbb{R}, \tag{1.4}$$

where $\alpha_{k,\sigma}$ are positive real numbers with finite supremum value for all values of k and σ, that is, $\sup_{k,\sigma} \{\alpha_{k,\sigma}\} = \alpha^* < \infty$, $P_{k,\sigma}$ are real numbers with $\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \le M$ (M is independent of σ) and $\sum_{k=1}^{\infty} P_{k,\sigma} = 1$ for all $\sigma \in \Lambda$, at $p - \mu$-Lebesgue point of function f as $(x, \sigma) \to (x_0, \sigma_0)$. As in [5, 17], we also suppose that f has a majorant function, that is, there exists a function φ satisfying $|f(x)| \le \varphi(x) < \infty$ for all $x \in \mathbb{R}$. Here, $L_p(\mathbb{R})$ $(1 \le p < \infty)$ will denote the space of all measurable functions f for which the Lebesgue integral of $|f|^p$ has finite value on $\mathbb{R}$. The obtained results are used for presenting some theorems for the rate of convergences. Here, the operators of type (1.4) are obtained by incorporating the operators of type (1.1) and (1.2).

The paper is organized as follows: In Sect. 2, we introduce fundamental notions. In Sect. 3, we give some auxiliary theorems concerning existence and pointwise convergence of the operators of type (1.4). In Sect. 4, we present a Fatou type convergence theorem for these operators. In Sect. 5, we establish the rates of both pointwise and Fatou type convergences by using the results obtained in the previous two sections.

2 Preliminaries

The following definition is obtained by incorporating the characterization of function μ given by Gadjiev [7] which helps to generalize well-known Lebesgue point definition and the idea used in [16, 17] in order to create new modulus of continuity definitions. For some other μ-Lebesgue point characterizations, we refer the reader to [3, 8, 14, 15, 22].

Definition 1 Let $\delta_0 \in \mathbb{R}^+$ be a fixed number. A point $x \in \mathbb{R}$ satisfying the following relations:

$$\lim_{h \to 0^+} \left(\frac{1}{\mu(h)} \int_0^h \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x + \alpha_{k,\sigma} t\right) - f(x) \right] \right|^p dt \right)^{\frac{1}{p}} = 0, \tag{2.1}$$

$$\lim_{h\to 0^+}\left(\frac{1}{\mu(h)}\int_{-h}^{0}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x+\alpha_{k,\sigma}t\right)-f\left(x\right)\right]\right|^{p}dt\right)^{\frac{1}{p}}=0, \tag{2.2}$$

where $0 < h \le \delta_0$ and relations (2.1) and (2.2) are independent of the choice of $\sigma \in \Lambda$, is called $p-\mu$-Lebesgue point of f $(1 \le p < \infty)$. Here, $\mu : \mathbb{R} \to \mathbb{R}$ is an increasing and absolutely continuous function on $[0, \delta_0]$ with $\mu\,(0) = 0$.

The following definition is obtained by incorporating kernel properties used in the works [5, 17, 20]. Also, usage of Lipschitz condition is due by Musielak [11].

Definition 2 Let $1 \le p < \infty$ and σ_0 be an accumulation point of non-empty index set Λ or infinity. A family K consisting of the functions $K_\sigma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where $K_\sigma(\vartheta, u)$ has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and for any $\sigma \in \Lambda$ and the following conditions hold:

(a) $K_\sigma(\vartheta, 0) = 0$, for every $\vartheta \in \mathbb{R}$ and $\sigma \in \Lambda$.
(b) There exists a function $L_\sigma : \mathbb{R} \to \mathbb{R}_0^+$ whose Lebesgue integral has finite value on $\mathbb{R}$ for any $\sigma \in \Lambda$ such that the following inequality:

$$|K_\sigma(t, u) - K_\sigma(t, v)| \le L_\sigma(t)\,|u - v|$$

holds for every $t \in \mathbb{R}$, $u, v \in \mathbb{R}$ and $\sigma \in \Lambda$.
(c) For every $u \in \mathbb{R}$, we have $\lim\limits_{\sigma\to\sigma_0}\left|\int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma}u\right)dt - u\right| = 0$.
(d) $\lim\limits_{\sigma\to\sigma_0}\left[\int_{|t|>\xi} L_\sigma\,(t)\,dt\right] = 0$ for every $\xi > 0$.
(e) $\lim\limits_{\sigma\to\sigma_0}\left[\int_{|t|>\xi} \eta^p\,(\alpha^* t)\,L_\sigma\,(t)\,dt\right] = 0$ for every $\xi > 0$.
(f) For a certain real number $\delta_1 > 0$, the function $L_\sigma(t)$ is non-decreasing on $(-\delta_1, 0]$ and non-increasing on $[0, \delta_1)$ with respect to t, for any $\sigma \in \Lambda$.
(g) $\int_{\mathbb{R}} \eta\,(\alpha^* t)\,L_\sigma\,(t)\,dt \le N_1 < \infty$ and $\int_{\mathbb{R}} L_\sigma\,(t)\,dt \le N_2 < \infty$ for all $\sigma \in \Lambda$ (N_1 and N_2 are independent of $\sigma \in \Lambda$).

Here,

$$\eta\,(t) = \sup_{\substack{x\in\mathbb{R}\\ |y|\le t}}\frac{\varphi\,(x+y)}{\varphi\,(x)} < \infty,$$

where $\varphi : \mathbb{R} \to \mathbb{R}^+$ and φ is an aforementioned majorant function. Throughout this manuscript, we assume that K_σ satisfies above conditions.

3 Existence of the Operators and Pointwise Convergence

Theorem 1 *Let $1 \le p < \infty$ such that there exists a positive function $\varphi \in L_p(\mathbb{R})$ satisfying $|f(x)| \le \varphi(x) < \infty$ for all $x \in \mathbb{R}$. Then, the functions $T_\sigma f \in L_p(\mathbb{R})$ and the inequality*

$$\|T_\sigma f\|_{L_p(\mathbb{R})} \leq M \|\varphi\|_{L_p(\mathbb{R})} \int_{\mathbb{R}} \eta\left(\alpha^* t\right) L_\sigma(t)\, dt,$$

holds for every $\sigma \in \Lambda$.

Proof First, under the hypotheses, the convergence of the series

$$\sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)$$

is guaranteed for all fixed $t \in \mathbb{R}$ (for details, see [5, 17]), that is,

$$\begin{aligned} \left|\sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right| &\leq \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left|f\left(x + \alpha_{k,\sigma} t\right)\right| \\ &\leq M\varphi(x)\, \eta\left(\alpha^* t\right). \end{aligned}$$

Let $p = 1$. By conditions (a) and (b), we may write

$$\begin{aligned} \|T_\sigma f\|_{L_1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt \right| dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} L_\sigma(t) \left|M\varphi(x)\, \eta\left(\alpha^* t\right)\right| dt dx. \end{aligned}$$

In view of Fubini theorem (see, e.g., [2]), we obtain the desired result, that is,

$$\|T_\sigma f\|_{L_1(\mathbb{R})} \leq M \|\varphi\|_{L_1(\mathbb{R})} \int_{\mathbb{R}} \eta\left(\alpha^* t\right) L_\sigma(t)\, dt.$$

Now, we prove the theorem for the case $1 < p < \infty$. By conditions (a) and (b), we may write

$$\begin{aligned} \|T_\sigma f\|_{L_p(\mathbb{R})} &= \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} L_\sigma(t) \left|M\varphi(x)\, \eta\left(\alpha^* t\right)\right| dt \right)^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Now, applying generalized Minkowski inequality to the last inequality above (see, e.g., [19]), we have

$$\begin{aligned}\|T_\sigma f\|_{L_p(\mathbb{R})} &\leq \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} L_\sigma^p(t) \left| M\varphi(x)\, \eta\left(\alpha^* t\right) \right|^p dx \right)^{\frac{1}{p}} dt \\ &= M \int\limits_{\mathbb{R}} L_\sigma(t)\, \eta\left(\alpha^* t\right) dt \left(\int\limits_{\mathbb{R}} |\varphi(x)|^p dx \right)^{\frac{1}{p}} \\ &= M \|\varphi\|_{L_p(\mathbb{R})} \int\limits_{\mathbb{R}} \eta\left(\alpha^* t\right) L_\sigma(t)\, dt.\end{aligned}$$

The desired result follows from condition (g). Thus the proof is completed. □

Now, we give a theorem concerning pointwise convergence of the operators of type (1.4).

Theorem 2 *Suppose that there exists a positive function* φ *satisfying* $|f(x)| \leq \varphi(x) < \infty$ *for all* $x \in \mathbb{R}$*. If* $x_0 \in \mathbb{R}$ *is a* $p - \mu$*-Lebesgue point of the function* f $(1 \leq p < \infty)$*, then*

$$\lim_{\sigma \to \sigma_0} |(T_\sigma f)(x_0) - f(x_0)| = 0$$

provided that $\sigma \in \Lambda_1 \subseteq \Lambda$ *on which the function*

$$\int\limits_{-\delta}^{\delta} \left| \{\mu(|t|)\}_t' \right| L_\sigma(t) dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$*, is bounded as* σ *tends to* σ_0*.*

Proof We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $|I_\sigma(x_0)| = |(T_\sigma f)(x_0) - f(x_0)|$.

From (c), we can write

$$\begin{aligned}|I_\sigma(x_0)| = \Bigg| &\int\limits_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \\ &+ \int\limits_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt - \int\limits_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt \Bigg|.\end{aligned}$$

Using (b), we may easily get

$$|I_\sigma(x_0)| \le \int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| L_\sigma(t)\, dt + \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0) \right) dt - f(x_0) \right|.$$

Since whenever A and B being positive numbers the inequality $(A+B)^p \le 2^p(A^p + B^p)$ holds (see, e.g., [13]), we have

$$\begin{aligned} |I_\sigma(x_0)|^p &\le 2^p \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| L_\sigma(t)\, dt \right)^p \\ &\quad + 2^p \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0) \right) dt - f(x_0) \right|^p \\ &= 2^p \left(I_1 + I_2 \right). \end{aligned}$$

By (c), I_2 tends to zero as σ tends to σ_0. Next, applying Hölder's inequality (see [13]) to the integral I_1 and condition (g), we obtain

$$\begin{aligned} I_1 &= \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| \left(L_\sigma(t)\right)^{\frac{1}{p}} \left(L_\sigma(t)\right)^{\frac{1}{q}} dt \right)^p \\ &\le \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma(t)\, dt \right) \left(\int_{\mathbb{R}} L_\sigma(t)\, dt \right)^{\frac{p}{q}} \\ &\le (N_2)^{\frac{p}{q}} \int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma(t)\, dt \\ &= (N_2)^{\frac{p}{q}} I_{11}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since $x_0 \in \mathbb{R}$ is a $p-\mu$-Lebesgue point of the function f in view of relations (2.1) and (2.2) for all $\varepsilon > 0$, there exists a number $\delta > 0$ such that the following inequalities hold there:

$$\int_0^h \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p dt \le \varepsilon^p \mu(h), \tag{3.1}$$

$$\int_{-h}^{0} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f\left(x_0\right) \right] \right|^p dt \leq \varepsilon^p \mu\left(h\right), \tag{3.2}$$

where $0 < h \leq \delta$ provided that $0 < \delta < \min\left\{\delta_0, \delta_1\right\}$.

Now, we consider I_{11}. It is easy to see that the following equality holds:

$$\begin{aligned} I_{11} &= \left\{ \int_{|t|>\delta} + \int_{-\delta}^{\delta} \right\} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma\left(t\right) dt \\ &= I_{111} + I_{112}. \end{aligned}$$

For the integral I_{111}, we can write

$$\begin{aligned} I_{111} &= \int_{|t|>\delta} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma\left(t\right) dt \\ &\leq 2^p \int_{|t|>\delta} \left| \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right) \right|^p L_\sigma\left(t\right) dt \\ &\quad + 2^p \left| f\left(x_0\right) \right|^p \int_{|t|>\delta} L_\sigma\left(t\right) dt \\ &= 2^p \left(I_{1111} + I_{1112}\right). \end{aligned}$$

Under the hypotheses, we observe that

$$\begin{aligned} I_{1111} &= \int_{|t|>\delta} \left| \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right) \right|^p L_\sigma\left(t\right) dt \\ &= \int_{|t|>\delta} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \frac{f\left(x_0 + \alpha_{k,\sigma} t\right)}{\varphi\left(x_0 + \alpha_{k,\sigma} t\right)} \frac{\varphi\left(x_0 + \alpha_{k,\sigma} t\right)}{\varphi\left(x_0\right)} \varphi\left(x_0\right) \right|^p L_\sigma\left(t\right) dt \\ &\leq M^p \varphi^p\left(x_0\right) \int_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma\left(t\right) dt. \end{aligned}$$

Using (e) and (d), I_{1111} tends to 0 and I_{1112} tends to 0 as σ tends to σ_0, respectively.

Lastly, we have to show that I_{112} tends to 0 as σ tends to σ_0.

Obviously, I_{112} may be written in the form

$$I_{112} = \left\{ \int_{-\delta}^{0} + \int_{0}^{\delta} \right\} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma\left(t\right) dt$$

$$= I_{1121} + I_{1122}.$$

For I_{1121}, let us define

$$F(t) := \int_{t}^{0} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} v\right) - f(x_0) \right] \right|^{p} dv.$$

By (3.2) the inequality

$$F(t) \leq \varepsilon^{p} \mu(-t) \tag{3.3}$$

holds for every δ satisfying $0 < \delta < \min\{\delta_0, \delta_1\}$. In view of (3.3) and following similar strategy as in [7, 14], we have

$$|I_{1121}| \leq \varepsilon^{p} \int_{-\delta}^{0} \mu'(-t) L_{\sigma}(t)\, dt.$$

Similarly,

$$|I_{1122}| \leq \varepsilon^{p} \int_{0}^{\delta} \mu'(t) L_{\sigma}(t)\, dt.$$

Incorporating above results, we have

$$|I_{112}| \leq \varepsilon^{p} \int_{-\delta}^{\delta} \left| \{\mu(|t|)\}_t' \right| L_{\sigma}(t)\, dt.$$

The remaining part follows from the arbitrariness of ε and boundedness of $\int_{-\delta}^{\delta} \left| \{\mu(|t|)\}_t' \right| L_{\sigma}(t)\, dt$ as σ tends to σ_0. This completes the proof. □

4 Main Theorem

In this section we will prove the Fatou type pointwise convergence of the operators of type (1.4). For the original description, we refer the reader to Fatou [4]. Some related works may be found in [5, 8, 14, 18]. For this purpose, we suppose that for a sufficiently small number $\delta > 0$ such that the function Ω_{δ} given as

$$\Omega_{\delta}(x, \sigma) = \int_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_{\sigma}(t)\, dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, is bounded on the set defined as

$$Z_{C,\delta} = \{(x,\sigma) \in \mathbb{R} \times \Lambda_1 : \Omega_\delta(x,\sigma) < C\},$$

where C is positive constant which can be made arbitrarily small, as (x,σ) tends to (x_0, σ_0). Here, this set is given before the theorem, but it can be given inside the theorem up to desire.

Theorem 3 *Suppose such that there exists a positive function φ satisfying $|f(x)| \leq \varphi(x) < \infty$ for all $x \in \mathbb{R}$. If $x_0 \in \mathbb{R}$ is a $p-\mu$-Lebesgue point of the function f $(1 \leq p < \infty)$, then*

$$\lim_{(x,\sigma)\to(x_0,\sigma_0)} |(T_\sigma f)(x) - f(x_0)| = 0$$

provided that $(x,\sigma) \in Z_{C,\delta}$.

Proof We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $0 < |x_0 - x| < \frac{\delta}{2}$ for a given $0 < \delta < \min\{\delta_0, \delta_1\}$.

Now, set $I_\sigma(x) = |(T_\sigma f)(x) - f(x_0)|$. Let us write

$$\begin{aligned}
|I_\sigma(x)| &= \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right| \\
&= \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt - K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt \right. \\
&\quad \left. + \ K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|I_\sigma(x)|^p &\leq 2^p \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right] \right| L_\sigma(t)\, dt \right)^p \\
&\quad + 2^p \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|^p \\
&= 2^p \{I_1 + I_2\}.
\end{aligned}$$

Clearly, by Theorem 2, $I_2 \to 0$ as σ tends to σ_0.

The following inequality holds for I_1:

$$\begin{aligned} I_1 &= \left(\int\limits_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right] \right| L_\sigma(t)\, dt \right)^p \\ &\leq \left(\left\{ \int\limits_{|t|>\delta} + \int\limits_{-\delta}^{\delta} \right\} \sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_\sigma(t)\, dt \right)^p \\ &\leq 2^p \left(\int\limits_{|t|>\delta} \sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_\sigma(t)\, dt \right)^p \\ &\quad + 2^p \left(\int\limits_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_\sigma(t)\, dt \right)^p \\ &= 2^p \left(I_{11} + I_{12} \right). \end{aligned}$$

Applying Hölder's inequality to I_{11} and using condition (g), we have

$$\begin{aligned} I_{11} &\leq \left(\int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \right)^p \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right|^p L_\sigma(t)\, dt \right) \left(\int\limits_{\mathbb{R}} L_\sigma(t)\, dt \right)^{\frac{p}{q}} \\ &\leq \left(\int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \right)^p \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right|^p L_\sigma(t)\, dt \right) (N_2)^{\frac{p}{q}} \\ &= I_{111} (N_2)^{\frac{p}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that

$$\begin{aligned} I_{111} &\leq 2^p \int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \right)^p \left| f\left(x + \alpha_{k,\sigma} t\right) \right|^p L_\sigma(t)\, dt \\ &\quad + 2^p \int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \right)^p \left| f\left(x_0 + \alpha_{k,\sigma} t\right) \right|^p L_\sigma(t)\, dt. \end{aligned}$$

Following same strategy as in Theorem 2, we have

$$\begin{aligned} I_{111} &\leq 2^p M^p \varphi^p(x) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\, dt \\ &\quad + 2^p M^p \varphi^p(x_0) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\, dt \\ &= I_{1111} + I_{1112}. \end{aligned}$$

Using condition (e), $I_{1111} \to 0$ and $I_{1112} \to 0$ as $(x, \sigma) \to (x_0, \sigma_0)$. The result follows from the hypothesis on the integral $I_{12.}$ Thus the proof is completed. □

5 Rate of Convergence

Theorem 4 *Suppose that the hypotheses of Theorem 2 are satisfied. Let*

$$\Delta(\sigma, \delta) = \int_{-\delta}^{\delta} \left| \{\mu(|t|)\}_t' \right| L_\sigma(t)dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$*, and the following conditions are satisfied:*

(i) $\Delta(\sigma, \delta)$ tends to 0 as $\sigma \to \sigma_0$ for some $\delta > 0$.

(ii) For every $\xi > 0$, we have

$$\int_{|t|>\xi} L_\sigma(t)\, dt = \mathbf{o}(\Delta(\sigma, \delta))$$

as $\sigma \to \sigma_0$.

(iii) For every $\xi > 0$ and $1 \le p < \infty$, we have

$$\int_{|t|>\xi} \eta^p \left(\alpha^* t\right) L_\sigma(t)\, dt = \mathbf{o}(\Delta(\sigma, \delta))$$

as $\sigma \to \sigma_0$.

(iv) Letting $\sigma \to \sigma_0$, we have

$$\left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0) \right) dt - f(x_0) \right|^p = \mathbf{o}(\Delta(\sigma, \delta)).$$

Then, at each $p - \mu$-Lebesgue point of f $(1 \le p < \infty)$, we have

$$|(T_\sigma f)(x_0) - f(x_0)|^p = \mathbf{o}(\Delta(\sigma, \delta))$$

as σ tends to σ_0.

Proof By the hypotheses of Theorem 2, we have

$$
\begin{aligned}
|(T_\sigma f)(x_0) - f(x_0)|^p \le \varepsilon^p 2^p (N_2)^{\frac{p}{q}} \int_{-\delta}^{\delta} \left|\{\mu(|t|)\}_t'\right| L_\sigma(t)\,dt. \\
+ 2^{2p} \varphi^p(x_0) (N_2)^{\frac{p}{q}} M^p \int_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt. \\
+ 2^{2p} (N_2)^{\frac{p}{q}} |f(x_0)|^p \int_{|t|>\delta} L_\sigma(t)\,dt \\
+ 2^p \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt - f(x_0) \right|^p.
\end{aligned}
$$

The proof follows from $(i)-(iv)$. □

Theorem 5 *Suppose that the hypotheses of Theorem 3 are satisfied. Let*

$$
\Omega_\delta(x,\sigma) = \int_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left|f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right)\right| L_\sigma(t)\,dt,
$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, *and the following conditions are satisfied:*

(i) $\Omega_\delta(x,\sigma)$ tends to 0 as (x,σ) tends to (x_0,σ_0) for some $\delta > 0$.
(ii) For every $\xi > 0$ and $1 \le p < \infty$, we have

$$
\int_{|t|>\xi} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt = \mathbf{o}(\Omega_\delta(x,\sigma))
$$

as (x,σ) tends to (x_0,σ_0).
(iii) Letting $(x,\sigma) \to (x_0,\sigma_0)$,

$$
\left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|^p = \mathbf{o}\left(\Omega_\delta(x,\sigma)\right).
$$

Then, at each $p-\mu$-Lebesgue point of f $(1 \le p < \infty)$, we have

$$
|(T_\sigma f)(x) - f(x_0)|^p = \mathbf{o}\left(\Omega_\delta(x,\sigma)\right)
$$

as (x,σ) tends to (x_0,σ_0).

Proof Under the hypotheses of Theorem 3, we may write

$$\begin{aligned}|(T_\sigma f)(x) - f(x_0)|^p \le{}& (N_2)^{\frac{p}{q}} 2^{3p} M^p \varphi^p(x) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt \\ &+ (N_2)^{\frac{p}{q}} 2^{3p} M^p \varphi^p(x_0) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt \\ &+ 2^{2p} \left(\int\limits_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right)\right| L_\sigma(t)\,dt \right)^p \\ &+ 2^p \left| \int\limits_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|^p .\end{aligned}$$

By conditions $(i)-(iii)$, we obtain the desired result. This completes the proof. □

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