Approximate Solution of Higher Order Two Point Boundary Value Problems Using Uniform Haar Wavelet Collocation Method

Akmal Raza and Arshad Khan

Abstract An efficient collocation method is proposed for the numerical solution of second and fourth order two-point boundary value problems (B.V.P.) based on uniform Haar wavelet. We have converted higher order differential equations into a system of differential equations of lower order and then solve it by uniform Haar wavelet, which reduces the time and complexity of the system. The technique introduced here is easy to apply. The performance of the present method yield more accurate results on increasing the resolution level. To demonstrate the robustness and accuracy of the Haar wavelet collocation method, five problems have been solved and compared with the existing methods present in the literature [1–6].

Keywords Haar wavelet · Collocation points

2000 Mathematics Subject Classification: 65M99 · 65N35 · 65N55 · 65L10

1 Introduction

Wavelet Analysis is a new development in the field of Mathematics. Wavelets were introduced in seismology to provide a time localisation to seismic analysis. Wavelet theory involves representing square integrable functions in terms of simple wavelet functions at different scale and positions. The fundamental idea of wavelet is translation and scaling according to the need [7–10]. The best property of wavelet is compact support, which is boom for the numerical solution of differential equations. Meanwhile in numerical analysis, wavelet methods have become an important tool for solution of differential and integral equations that has been discussed in many research papers with different approaches such as Galerkin method, finite

A. Raza (✉) · A. Khan
Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
e-mail: akmalrazataqvi@gmail.com

A. Khan
e-mail: akhan2@jmi.ac.in

J. Singh et al. (eds.), *Mathematical Modelling, Applied Analysis and Computation*, Springer Proceedings in Mathematics & Statistics 272,
https://doi.org/10.1007/978-981-13-9608-3_14

element method, finite difference method, filter-bank method, adaptive method etc. [7, 8, 11–15]. One of the best and easiest wavelet in wavelet theory is Haar wavelet. Gaussian and Legendre wavelet is also applied in treatment of Numerical solutions of differential equations but lots of numerical difficulties appeared on using these wavelets. The detection of singularities, local high frequencies, irregular structures and transient phenomena exhibited by analyzed function is possible on using wavelets. Use of orthogonal functions to construct the solution of differential equations was initially established in 1995 by Chen and Hsiao [14]. During the last two decades different types of functions have been applied to find the approximate solution of differential equations. But Haar wavelet gives the desirable results for such types of problems due to its simplicity, orthogonality and compact support.

2 Multiresolution Analysis and Haar Wavelet

Definition: A multiresolution analysis consists of a sequence $\{V_j : j \in Z\}$ of embedded closed subspace of $L^2(R)$ that satisfy the following properties:

1. **Increasing**: $V_j \subset V_{j+1} : j \in Z$
2. **Density**: $\overline{\bigcup_{j\in Z} V_j} = L^2(R)$
3. **Separation**: $\bigcap_{j\in Z} V_j = \{0\}$
4. **Scaling**: $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$
5. **Orthonormal basis**: $\exists$ a scaling function $\phi \in V_0$ such that $\{\phi_{0,k}(t) = \phi(t-k) : k \in Z\}$ is an orthonormal basis for V_0.

Haar Wavelet: Haar function was discovered long before the wavelet was introduced by Hungarian Mathematician Alfred Haar in 1909. Haar is the simplest orthonormal wavelet with compact support [16].
The Haar wavelet family for $t \in [0, 1]$ is defined as follows:

$$h_u(t) = \begin{cases} 1, & \xi_1(u) \le t < \xi_2(u) \\ -1, & \xi_2(u) \le t < \xi_3(u) \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where u indicates the wavelet number and

$\xi_1(u) = \frac{k}{m}$, $\xi_2(u) = \frac{k+0.5}{m}$, $\xi_3(u) = \frac{k+1}{m}$
$m = 2^j$, $j = 0, 1, 2..., J$, and integer $k = 0, 1..., m-1$.

Also J indicates the level of resolution and k represents the translation parameter. Index u is calculated as $u = m + k + 1$ which is true for $u \ge 2$.
For $u = 1$ the Haar wavelet is given by

$$h_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Because of constant and piecewise nature of Haar wavelet, derivative vanishes. Due to lack of differentiability authors move towards integration approach instead of differentiation [14].

The integration of Haar wavelet has been obtained from [13] and given as follows:

$$I_1 h_u(t) = \begin{cases} t - \xi_1(u), & \xi_1(u) \le t < \xi_2(u) \\ \xi_3(u) - t, & \xi_2(u) \le t < \xi_3(u) \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The double integration of Haar wavelet can be given as follows:

$$I_2 h_u(t) = \begin{cases} \frac{1}{2}(t - \xi_1(u))^2, & \xi_1(u) \le t < \xi_2(u) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3(u) - t)^2, & \xi_2(u) \le t < \xi_3(u) \\ \frac{1}{4m^2}, & \xi_3(u) \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

Proceeding in similar manner the nth integration of Haar wavelet can be written as:

$$I_n h_u(t) = \begin{cases} 0, & if \quad t < \xi_1(u) \\ \frac{1}{n!}[t - \xi_1(u)]^n, & \xi_1(u) \le t < \xi_2(u) \\ \frac{1}{n!}[(t - \xi_1(u))^n - 2(t - \xi_2(u))^n], & \xi_2(u) \le t < \xi_3(u) \\ \frac{1}{n!}[(t - \xi_1(u))^n - 2(t - \xi_2(u))^n + (t - \xi_3(u))^n], & \xi_3(u) \le t \end{cases} \tag{2.5}$$

Now consider, any square integrable function $f(t) \in L^2[0, 1]$, can be approximated by the dialation and translation of Haar wavelet [12, 13]

$$f(t) = \sum_{u=1}^{N} a_u h_u(t) \tag{2.6}$$

The Haar wavelet coefficients a_u are calculated as

$$a_u =< y(t), h_u(t) >= \int_0^1 y(t) \cdot \overline{h_u(t)} dt. \tag{2.7}$$

The collocation points are given as

$$X(u) = \frac{2u - 1}{m}, u = 1, 2, ..., m. \tag{2.8}$$

The matrix of Haar wavelet with respect to the collocation points is given as

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The matrix of integral and double integral of Haar wavelet with respect to the collocation points are given as:

$$I_1H = \frac{1}{16}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, I_2H = \frac{1}{512}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

3 Methods of Solution

3.1 Method for Solving Second Order Differential Equations

Consider a second order differential equation

$$y'' = \phi(t, y, y') \tag{3.1}$$

with boundary conditions

$$y(0) = \alpha, y(1) = \beta. \tag{3.2}$$

Let us suppose that

$$y'(t) = z(t) \Rightarrow y''(t) = z'(t) \tag{3.3}$$

$$y'(t) = \sum_{u=1}^{N} a_u h_u(t) \tag{3.4}$$

$$z'(t) = \sum_{u=1}^{N} b_u h_u(t). \tag{3.5}$$

Now integrating Eqs. (3.4) and (3.5) with respect to t from 0 to t we get

$$y(t) = \sum_{u=1}^{N} a_u I_1 h_u(t) + y(0) \tag{3.6}$$

and

$$z(t) = \sum_{u=1}^{N} b_u I_1 h_u(t) + z(0). \tag{3.7}$$

Substituting the values from Eqs. (3.3–3.7) in (3.1), we get the following system of equations

$$\sum_{u=1}^{N} b_u h_u(t) = \phi(t, \sum_{u=1}^{N} a_u I_1 h_u(t) + y(0), \sum_{u=1}^{N} b_u I_1 h_u(t) + z(0)) \tag{3.8}$$

Solving the above system of equation and find out the unknown Haar wavelet coefficient a_u and b_u with the help of Eq. (3.3) and then put in Eq. (3.6) to get the approximate solution of the differential equation.

3.2 Method for Solving Fourth Order Differential Equations

Consider the fourth order ordinary linear differential equation of the form.

$$y'''' = \phi(t, y, y', y'', y''') \tag{3.9}$$

with boundary conditions

$$y(0) = a, y(1) = b, y''(0) = c, y''(1) = d.$$

Let us suppose that

$$y''(t) = z(t), \tag{3.10}$$

$$y'''(t) = z'(t), \tag{3.11}$$

$$y''''(t) = z''(t). \tag{3.12}$$

$$\text{and} \qquad y''(t) = \sum_{u=1}^{N} a_u h_u(t). \tag{3.13}$$

On integrating Eq. (3.13) from 0 to t with respect to t we get

$$y'(t) = \sum_{u=1}^{N} a_u P_{u,1}(t) + y'(0) \tag{3.14}$$

Again integrating Eq. (3.14) from 0 to t, with respect to t we get

$$y(t) = \sum_{u=1}^{N} a_u P_{u,2}(t) + ty'(0) + y(0) \tag{3.15}$$

Also we assume that

$$z'' = \sum_{u=1}^{N} b_u h_u(t) \tag{3.16}$$

On integrating Eq. (3.16) from 0 to t we get,

$$z'(t) = \sum_{u=1}^{N} b_u P_{u,1}(t) + z'(0) \tag{3.17}$$

Again integrating Eq. (3.17) from 0 to t we get

$$z(t) = \sum_{u=1}^{N} b_u P_{u,2}(t) + tz'(0) + z(0) \tag{3.18}$$

We can find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y''''(0)$ from the boundary conditions. Now put Eqs. (3.10–3.18) in (3.9), we get the following system of equation

$$\begin{aligned}\sum_{u=1}^{N} b_u h_u(t) = \phi(t, &\sum_{u=1}^{N} a_u P_{u,2}(t) + t.y'(0) + y(0), \sum_{u=1}^{N} a_u P_{u,1}(t) + y'(0),\\ &\sum_{u=1}^{N} b_u P_{u,2}(t) + t.z'(0) + z(0), \sum_{u=1}^{N} b_u P_{u,1}(t) + z'(0))\end{aligned} \tag{3.19}$$

Find the value of the vector a_u and then put these values in the Eq. (3.15) to get the Haar approximate solution of the required differential equation.

4 Numerical Examples

In this section we have tested five problems to demonstrate the accuracy and effectiveness of proposed method.

Problem 1 Consider the second order differential equations [1]

$$y'' = 100y, \tag{4.1}$$

with boundary conditions

$$y(0) = y(1) = 1. \tag{4.2}$$

Exact solution of the problem is

$$y = \frac{\cos h(10t - 5)}{\cos h5} \tag{4.3}$$

Obtained maximum absolute errors for different resolutions are given in Table 1 and graph for J = 4 is given in Fig. 1.

Problem 2 Consider Dirichlet problem given in [4]:

$$-y'' = (\frac{537}{10}\pi)^2 \sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi)^2 \sin(\frac{23}{10}\pi t), \tag{4.4}$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0, \qquad t \in [0, 1]. \tag{4.5}$$

Exact solution is

$$y = (\frac{537}{10}\pi)\sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi)\sin(\frac{23}{10}\pi t) \qquad . \tag{4.6}$$

Table 1 Maximum absolute error for Problem 1

Level of resolution J	Our method	[1]
3	1.6719e−04	1.2800e−03
4	2.2854e−05	3.0700e−04
5	2.9131e−06	–
10	8.4470e−11	–

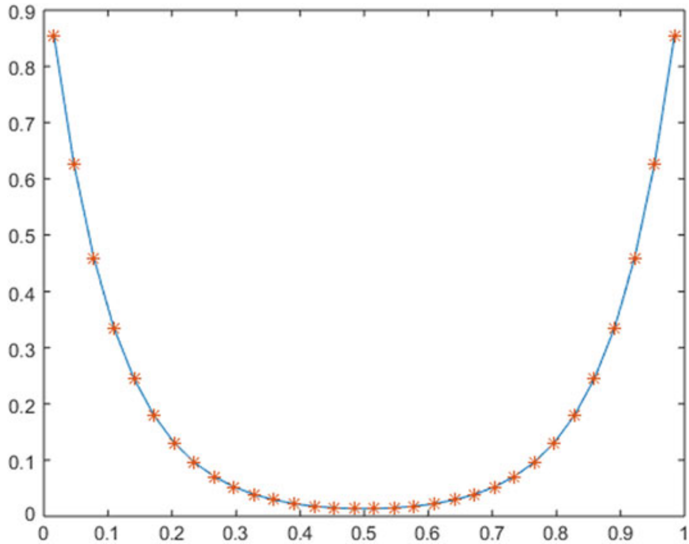

Fig. 1 Exact and Haar solution of Problem 1 for J = 4

Table 2 Maximum absolute error for Problem 2

Level of resolution	J = 6	7	8	9
Our method	9.5472e−04	1.2490e−04	1.5790e−05	1.9750e−06
[4]	1.2100e−02	1.3260e−02	1.0820e−04	7.3580e−06

Obtained maximum absolute errors for different resolutions are given in Table 2 and graph for $J = 6$ is given in Fig. 2.

Problem 3 Consider Dirichlet problem given in [4]:

$$-y'' + y = [1 + (\frac{537}{10}\pi)^2]\sin(\frac{537}{10}\pi t) + [1 + (\frac{23}{10}\pi)^2]\sin(\frac{23}{10}\pi t), \quad (4.7)$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0. \quad t \in [0, 1] \quad (4.8)$$

Exact solution is

$$y = (\frac{537}{10}\pi)\sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi)\sin(\frac{23}{10}\pi t) \quad . \quad (4.9)$$

Obtained maximum absolute errors for different resolutions are given in Table 3 and graph is given in Fig. 3.

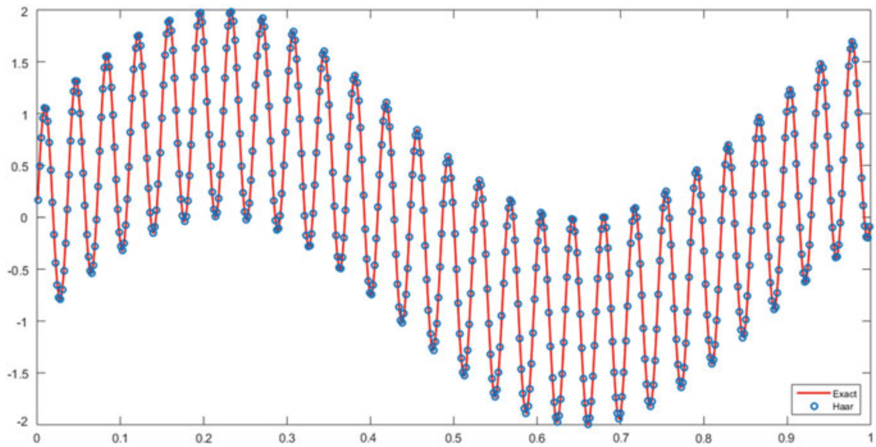

Fig. 2 Exact and Haar solution of Problem 2

Table 3 Maximum absolute error for Problem 3

Level of resolution	J = 6	7	8	9
Our method	9.4584e−04	1.2457e−04	1.5741e−05	1.9690e−06
[4]	1.2100e−2	1.3260e−3	1.0820e−4	7.3590e−6

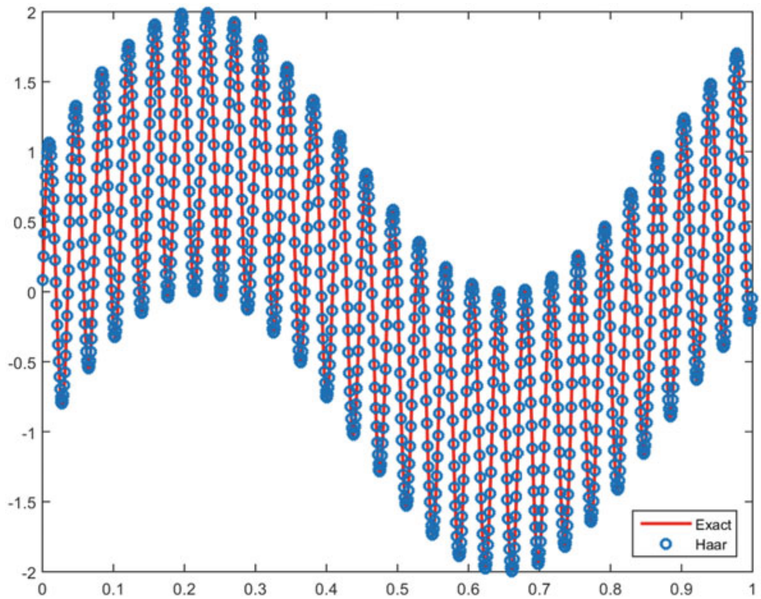

Fig. 3 Exact and Haar solution of Problem 3 for J = 9

Problem 4 Let us assume the fourth order B.V.P. given in [3]

$$y'''' + ty = -(t^3 + 7t + 8)e^t, \quad t \in [0, 1] \tag{4.10}$$

with boundary conditions

$$y(0) = y(1) = 0, \tag{4.11}$$

$$y''(t) = 1 \quad when \quad t = 0, \tag{4.12}$$

$$y''(t) = -4e. \quad when \quad t = 1 \tag{4.13}$$

Exact solution of the problem is

$$y(t) = t(1-t)e^t. \tag{4.14}$$

Obtained maximum absolute errors for different resolutions are given in Table 4 and graph is given in Fig. 4.

Table 4 Maximum absolute errors for Problem 4

Level of resolution J	Our method	[3]
2	5.2806e−04	4.5900e−04
3	9.4372e−05	1.9000e−4
4	1.6728e−05	5.2300e−05
5	2.9591e−06	–
6	5.2319e−07	–

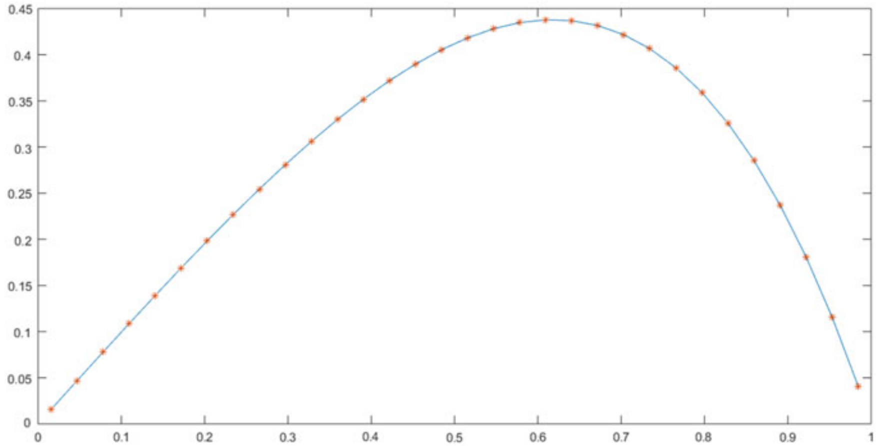

Fig. 4 Exact and Haar solution of Problem 4 for J = 4

Problem 5 Let us assume the fourth order B.V.P. given in [3]

$$y'''' - y = -4(2t\cos t + 3\sin t), \quad t \in [0, 1] \tag{4.15}$$

with boundary conditions

$$y(0) = y(1) = 0, \quad y''(t) = 0, \quad when \quad t = 0, y''(1) = 4\cos 1 + 2\sin 1. \tag{4.16}$$

Exact solution of the problem is

$$y(t) = (t^2 - 1)\sin t. \tag{4.17}$$

Obtained maximum absolute errors for different resolutions are given in Table 5 and graph is given in Fig. 5.

Table 5 Maximum absolute errors for Problem 5

Level of resolution J	Our method	[3]
2	5.6800e−04	6.6600e−04
3	9.7369e−05	1.6500e−04
4	1.4350e−05	4.1200e−05
5	1.5638e−06	–

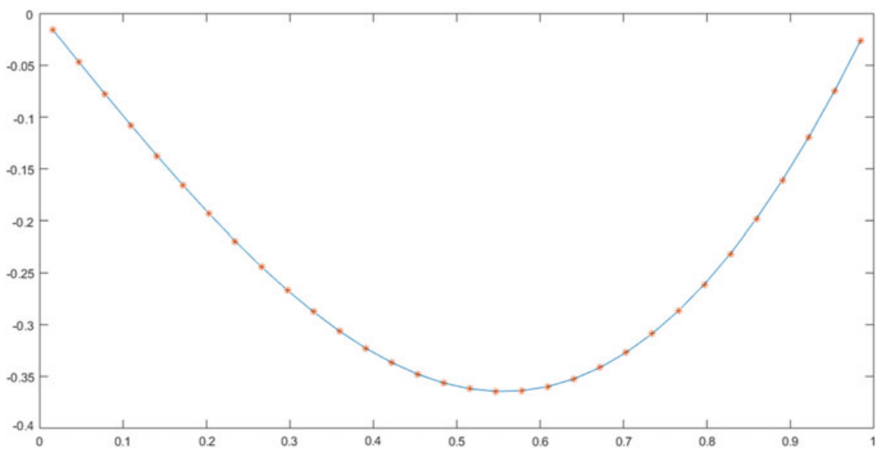

Fig. 5 Exact and Haar solution of Problem 5 for J = 4

5 Conclusion

We have converted second order differential equation into system of first order and fourth order differential equations into system of second order of differential equation, which is easy to solve to get the approximations of higher order two point boundary value problems. Haar wavelet collocation method has been applied on second and fourth order two point B.V.P. We have compared our results with the existing method given in [1, 3–6] which shows that our results are better.

Acknowledgements The authors would like to express their sincere thanks to Professor Khalil Ahmad for his valuable suggestions which greatly improved the quality of the paper.
First author is also very thankful to the UGC, New Delhi, for providing the financial support as UGC-CSIR JRF.

References

1. Khan, A.: Parametric cubic spline solution of two point boundary value problems. Appl. Math. Comput. **154**, 175–182 (2004)
2. Khan, A., Aziz, T.: The numerical solution of third-order boundary-value problems using quintic splines. Appl. Math. Comput. **137**, 253–260 (2003)
3. Khan, A., Khandelwal, P.: Non-polynomial sextic spline approach for the solution of fourth-order boundary value problems. Appl. Math. Comput. **218**, 3320–3329 (2011)
4. Jia, R.-Q., Liu, S.-T.: Wavelet bases of hermite cubic splines on the interval. Adv. Comput. Math. **25**, 23–39 (2006)
5. Chang, P., Piau, P.: Simple procedure for the designation of haar wavelet matrices for differential equations. In: Proceedings of the International MultiConference of Engineers and Computer Scientists, IMECS 2008, vol. 2, pp. 19–21 March, 2008. Hong Kong
6. Li, Z., Wang, Y., Tan, F.: The solution of a class of third-order boundary value problems by the reproducing kernel method. In: Hindawi Publishing Corporation Abstract and Applied Analysis (2012)
7. Ahmad, K., Shah, F.A.: Introduction to Wavelets with Applications. Real World Education Publishers, New Delhi (2013)
8. Nievergelt, Y.: Wavelets Made Easy. Springer (1999). 978-1-4612-0573-9
9. Mallat, S.: A Wavelet Tour of Signal Processing. Academic Press, New York (2009)
10. Chen, C.T.: Signals and Systems. Oxford University Press, New York (2004)
11. Daubechies, I.: Ten Lectures on Wavelets. SIAM, Philadelphia (1992)
12. Debnath, L., Shah, F.A.: Wavelet Transform and Their Application. Birkhauser, New York, NY (2015)
13. Lepik, U., Hein, H.: Haar Wavelet with Applications. Springer (2014)
14. Chen, C.F., Hsiao, C.H.: Haar wavelet method for solving lumped and distributed-parameter system. IEEE Proc.-Control Theory Appl. **144**, 87-94 (1997)
15. Islam, S., Aziz, I., Sarler, B.: The numerical solution of second order boundary value problems by collocation method with Haar wavelets. Math Comput. Model. **50**, 1577–1590 (2010)
16. Raza, A., Khan, A.: Haar wavelet series solution for solving neutral delay differential equations. J. King Saud Univ.-Sci., Elsevier (2018). https://doi.org/10.1016/j.jksus.2018.09.013