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Mathematical Modelling, Applied Analysis and Computation

ICMMAAC 2018, Jaipur, India, July 6–8

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Preface

This book is based around the conference *International Conference on Mathematical Modelling, Applied Analysis and Computation (ICMMAAC 2018)*, held at JECRC, Jaipur, from 6 to 8 July 2018. The book contains several useful topics of mathematical modelling, applied analysis and computation having different applications in other scientific areas of research and study. It should be useful for graduate students, researcher and educators interested in diverse areas of research in mathematical sciences including modelling, analysis and computation. The general readers interested in tools and techniques from different areas of mathematical sciences having practical applications in real life should also find the book interesting and useful. The book consists of 20 chapters organized as follows:

Chapter "Certain Banach-Space Operators Acting on the Semicircular Elements Induced by Orthogonal Projections" presents certain Banach-space operators acting on (weighted-) semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections. In particular, it emphasizes in cases where such operators are generated by *-isomorphisms induced by certain shifting processes on the set of all integers. The main results show not only how such Banach-space operators affect the original free-distributional data on (weighted-) semicircular elements, but also how the weighted-semicircular laws are preserved by the operators.

Chapter "Explicit Expressions Related to Degenerate Cauchy Numbers and Their Generating Function" establishes an explicit expression for degenerate Cauchy numbers and finds explicit, meaningful and significant expressions for coefficients in a family of nonlinear differential equations for the generating function of degenerate Cauchy numbers.

Chapter "Statistical Deferred Riesz Summability Mean and Associated Approximation Theorems for Trigonometric Functions" presents an idea of approximation via statistical deferred weighted (Riesz) summability mean for trigonometrical periodic functions defined over a Banach space $C_{2\pi}(\Re)$ and accordingly establishes a new approximation theorem (Korovkin-type). Furthermore, the chapter studies the rate of statistical deferred weighted summability and also establishes another result for the same set of functions by using the modulus of continuity. Finally, it considers a number of special cases and examples to exhibit the relevance of the obtained results and definitions provided in this chapter.

Chapter "On Pointwise Convergence of a Family of Nonlinear Integral Operators" presents some auxiliary theorems concerning existence and pointwise convergence of the certain operators. Then, it presented a Fatou-type convergence theorem for these operators. Finally, the rates of both pointwise and Fatou-type convergences have been established by using the derived results.

Chapter "Existence and Ulam's Type Stability of Integro Differential Equation with Non-instantaneous Impulses and Periodic Boundary Condition on Time Scales" presents existence and stability of integro-differential equation with periodic boundary condition and non-instantaneous impulses on time scales. Banach contraction theorem and nonlinear functional analysis have been used to establish these results. Moreover, to outline the utilization of these outcomes, an example is given.

Chapter "Introduction to Class of Uniformly Fractional Differentiable Functions" presents a new concept of uniformly fractional differentiable functions on an arbitrary interval I of R by using Caputo-type fractional derivative instead of the commonly used first-order derivative. Their interesting properties with few illustrations have been discussed in this chapter.

Chapter "Asymptotically Almost Automorphic Solution for Neutral Functional Integro Evolution Equations on Time Scales" studies the existence, uniqueness with stability consequence of asymptotically almost automorphic (AAA) solution for integro-neutral evolution equation on time scales by applying fixed-point hypothesis. It gives the time scale adaptation of (AAA) functions. Towards the end, a precedent is given for the adequacy of the hypothetical outcomes.

Chapter "An Integral Relation Associated with a General Class of Polynomials and the Aleph Function" reports new finite integral involving two general classes of polynomials with the Aleph function. This integral is supposed to be one of the most universal integrals evaluated until now and act as a key component from which one can obtain as its different special cases, integrals relating a large number of simpler special functions and polynomials. Some useful unique cases of the main outcome have also been discussed in the chapter.

Chapter "On the New Fractional Operator and Application to Nonlinear Bloch System" analyses the nonlinear Bloch system with a new fractional operator without singular kernel proposed by Caputo and Fabrizio. The commensurate and non-commensurate order nonlinear Bloch system is considered. Special solutions using a numerical scheme based on Lagrange interpolations are obtained. It studied the uniqueness and existence of the solutions employing the fixed-point theorem. Novel chaotic attractors with total order less than 3 are obtained.

Chapter "Fractional Order Integration and Certain Integrals of Generalized Multiindex Bessel Function" introduces generalized multiindex Bessel function and some formulas of the Riemann–Liouville fractional integration and differentiation operators. Further, certain integral formulas involving the newly defined generalized multiindex Bessel function have also been derived. It is also proved that such integrals are expressed in terms of the Fox–Wright function. The results presented here are general in nature and easily reducible to new and known results.

Chapter "Fractional Variational Iteration Method for Time Fractional Fourth-Order Diffusion-Wave Equation" applies fractional variational iteration method (FVIM) to solve numerically time-fractional diffusion-wave equation of order four. By using FVIM, a sequence converging rapidly to the exact solution of the fourth-order fractional diffusion-wave equation is obtained. Two test problems are presented to prove the merit of the proposed technique. Plotted graph shows that the numerical solution acquired by employed technique is similar to the exact solution.

Chapter "Analytical Approach to Fractional Navier–Stokes Equations by Iterative Laplace Transform Method" studies iterative Laplace transform scheme to examine fractional Navier–Stokes equations in cylindrical coordinates with initial conditions. The arbitrary ordered derivatives are described in terms of Caputo. By utilizing only the initial conditions, the analytical expressions are derived in the closed form. The results achieved with the aid of the proposed technique are graphically presented.

Chapter "Biological Model of Dengue Spread with Non-Markovian Properties" deals with converting the classical model to fractional model by using the concept of recently established fractional differential operators known as the Caputo-Fabrizio derivative to include into mathematical system the memory and the crossover effects. The new model was subjected to analysis of existence and uniqueness of the system solution to insure the well-posedness of the modified system. Due to the complexity of the new system, a newly introduced numerical scheme was used to solve the system and some numerical simulations were performed to see the effect of the Mittag–Leffler law that brings the crossover effect.

Chapter "Approximate Solution of Higher Order Two Point Boundary Value Problems Using Uniform Haar Wavelet Collocation Method" studies the numerical solution of second- and fourth-order two-point boundary value problems (B.V.P.) based on uniform Haar wavelet. It aims to convert higher order differential equations into a system of differential equations of lower order and then solve it by uniform Haar wavelet, which reduces the time and complexity of the system. The technique introduced in this chapter is easy to apply. The performance of the present method yields more accurate results on increasing the resolution level. To demonstrate the robustness and accuracy of the Haar wavelet collocation method, five problems have been solved and compared with the existing methods present in the literature.

Chapter "Solving Multi-objective Fractional Transportation Problem" deals with optimizing the objective function in the form of one or several ratios subject to some linear constraints. If in multi-objective transportation problem, objective function is in ration of two linear functions under some linear restrictions, then the problem is called multi-objective linear fractional transportation problem. A new method to solve multi-objective linear fractional transportation problem is suggested. Two numerical problems are presented to validate the proposed algorithm.

Chapter "On the Dark and Bright Solitons to the Negative-Order Breaking Soliton Model with (2+1)-Dimensional" studies the complex dynamics of cnoidal waves via the negative-order breaking soliton model with (2+1)-dimensional. This model is arisen in the (2+1)-dimensional interaction of the Riemann wave propagated between y-axis and x-axis. The improved Bernoulli sub-equation function method is used in obtaining some complex and dark solutions with hyperbolic function structure. It presents the interesting contour surfaces along with 2D and 3D graphics of the obtained analytical solutions in this study, plotted by using several computational programmes such as Matlab, Mathematica, and so on.

Chapter "A Reliable Analytical Algorithm for Cubic Isothermal Auto-Catalytic Chemical System" applies an algorithm for the q-homotopy analysis transform method (q-HATM) to solve the Cubic Isothermal Auto-catalytic Chemical System (CIACS). This technique is a combination of the Laplace decomposition method and the homotopy analysis scheme. This method gives the solution in the form of a rapidly convergent series with $\hbar$-curves are employed to determine the intervals of convergent. Averaged residual errors are used to determine the optimal values of $\hbar$. The behaviour of the solutions is shown graphically.

Chapter "Numerical Study of Effects of Adrenal Hormones on Lymphocytes" aims to study a mathematical model to examine the impact of adrenal hormones on the immune system with respect to time evolution and spatial distribution cells in response to hormones concentration. The steady state of the model is studied and found to be uniformly and asymptotically stable subject to the secretion and decay rates of hormones. The numerical experiments using the free diffusion equations further investigates the dynamic behaviour of the "bound" lymphocytes secretion rate of the adrenal hormones induced by psychological stress.

Chapter "Mathematical Modelling of Poor Nutrition in the Human Life Cycle" deals with a mathematical model as a system of nonlinear ordinary differential equations in order to investigate the effects of poor nutrition from conception to adulthood using the poor pregnant woman nutrient status. The steady states are studied and R_0 of poor nutrition in the society are calculated. To keep the society healthy and free of malnutrition, malnourished pregnant females are encouraged to eat foods that contain all the nutrients needed for development. The model is supported with numerical simulation.

Chapter "Characteristics of Homogeneous Heterogeneous Reaction on Flow of Walters' B Liquid Under the Statistical Paradigm" studies the significance of inclined MHD stagnant point flow of Walters B liquid because of stretched surface. Flow phenomenon is studied with Newtonian heating, homogeneous\heterogeneous reactions, Joule heating, and viscous dissipation. The nonlinear PDEs are converted to get nonlinear system of ODEs by invoking suitable transformations and solved by utilizing OHAM. Statistical methodology is used to check the significance and insignificance of the physical parameters via correlation coefficients and probable error. Characteristics of various sundry parameters on velocity, concentration, and temperature fields are studied. Friction and Nusselt numbers are calculated and discussed in detail.

The editors are grateful to the contributors for their cooperation and patience while the manuscripts were being reviewed and processed. The reviewers deserve our sincere gratitude for their wholehearted efforts in evaluating the manuscripts in a timely manner. The editors are also grateful to several colleagues and friends for their kind support during the execution of the task of brining out this volume.

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May 2019

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Certain Banach-Space Operators Acting on the Semicircular Elements Induced by Orthogonal Projections

Ilwoo Cho

Abstract The main purposes of this paper are (i) to construct-and-study (weighted-) semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections, and the Banach $*$-probability space $\mathbb{L}_Q$ generated by these operators, (ii) to establish $*$-isomorphisms on $\mathbb{L}_Q$ from shifting processes on the set $\mathbb{Z}$ of integers, (iii) to consider the $*$-isomorphisms of (ii) as Banach-space operators acting on $\mathbb{L}_Q$ (by regarding the Banach $*$-algebra $\mathbb{L}_Q$ as a Banach space), and (iv) to compare the free-distributional data affected by the operators of (iii) from the original data.

Keywords Free probability · Weighted-semicircular elements · Semicircular elements · Integer-shifts · The integer-shift group · Integer-shift operators · The integer-shift-operator algebra

1991 Mathematics Subject Classification 46L10 · 46L40 · 46L53 · 46L54 · 47L15 · 47L30 · 47L55

1 Introduction

In this paper, certain Banach-space operators acting on (weighted-)semicircular elements induced by mutually orthogonal $|\mathbb{Z}|$-many projections are constructed-and-considered. In particular, we are interested in the cases where such operators are generated by $*$-isomorphisms induced by certain shifting processes on the set $\mathbb{Z}$ of all integers. The main results show not only how such Banach-space operators affect the original free-distributional data on (weighted-)semicircular elements, but also how our weighted-semicircular laws are preserved by the operators.

To study our topics, we (i) construct *weighted-semicircular*, and *semicircular elements* in a certain *Banach $*$-probability space* $\mathfrak{L}_Q(\mathbb{Z})$ induced by a fixed C^*-

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probability space (A, ψ) containing $|\mathbb{Z}|$-many mutually orthogonal projections, (ii) study *free distributions* of certain free reduced words in the Banach $*$-probabilistic sub-structure $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$, generated by the (weighted-)semicircular elements of (i), providing "non-zero" free distributions in $\mathfrak{L}_Q(\mathbb{Z})$, (iii) define suitable shift processes on $\mathbb{Z}$, and (iv) establish certain $*$-homomorphisms on $\mathbb{L}_Q$ induced by the shift processes of (iii), and then (v) compare the free-distributional data of the $*$-homomorphic images of the $*$-homomorphisms in (iv), and the original free-distributional data of (ii).

1.1 Motivations

There are many different approaches to construct semicircular elements (e.g., [1, 4, 6, 12, 15, 19–21]) in topological $*$-probability spaces (e.g., C^*-*probability spaces*, or W^*-*probability spaces*, or *Banach* $*$-*probability spaces*, etc.). The construction of semicircular elements in this paper is highly motivated by those of *weighted-semicircular elements* in certain Banach $*$-probability spaces of [5, 7, 8] induced by p-adic analysis on the *p-adic number fields* $\mathbb{Q}_p$, for *primes* p. So, our construction is different from those of earlier works (also, see [6]).

In [5, 8], we studied weighted-semicircular elements induced by measurable functions on p-adic number fields $\mathbb{Q}_p$ (e.g., [8]), and those from a *free product* Banach $*$-algebra induced from weighted-semicircular elements of [8] (e.g., [5]). The author and Jorgensen applied number-theoretic results (e.g., [10, 18]), and free-probabilistic techniques (e.g., [2–4, 13, 14, 16]) to consider free-probabilistic models of [8], and they realized that there are well-defined *weighted-semicircular elements*. Interestingly, these operators automatically generate corresponding semicircular elements. In [5], the author extended the constructions and the main results of [8] under *free product* over *primes*. The detailed properties of "p-adic" weighted-semicircular elements, and those of corresponding semicircular elements were studied there.

Motivated by [5, 8], the author considered the similar constructions of (weighted-) semicircular elements from arbitrary C^*-probability spaces containing $|\mathbb{Z}|$-many mutually orthogonal projections in [6, 7], by mimicking the constructions of [5, 8]. The main results of [6] show that whenever one can have mutually orthogonal $|\mathbb{Z}|$-many projections in a C^*-probability space, the corresponding weighted-semicircular elements whose weights are characterized by the free-distributional data of the projections; moreover, under suitable (additional) conditions, semicircular elements are well-defined (see short Sects. 3, 4 and 5, below).

In this paper, we are interested in certain Banach-space adjointable operators (in the sense of [9]) acting on weighted-semicircular elements of [6, 7]. Especially, they are induced by certain shift processes on the set $\mathbb{Z}$ of integers, and corresponding well-defined $*$-homomorphisms on the (weighted-)semicircular elements.

1.2 Overview

In Sect. 2, we briefly mention about backgrounds of our proceeding works. In short Sects. 3, 4 and 5, weighted-semicircular elements, and semicircular elements are constructed from mutually orthogonal $|\mathbb{Z}|$-many projections (e.g., [6, 7]).

In Sect. 6, we construct a suitable free-probabilistic, operator-algebraic structure $\mathbb{L}_Q$ generated by our (weighted-)semicircular elements under free product.

In Sect. 7, we define-and-study Banach-space adjointable operators acting on $\mathbb{L}_Q$. In particular, certain shifting processes on $\mathbb{Z}$ are defined in Sect. 7.1, and the corresponding $*$-isomorphisms are determined on $\mathbb{L}_Q$ in Sect. 7.2. We realize that the collection of such $*$-isomorphisms forms a subgroup $\mathfrak{B}$ of the automorphism group $Aut(\mathbb{L}_Q)$ of $\mathbb{L}_Q$. The structure theorem of this group $\mathfrak{B}$ is provided in Sect. 7.2: $\mathfrak{B}$ is group-isomorphic to the infinite cyclic abelian group $(\mathbb{Z}, +)$. We then study how the group $\mathfrak{B}$ generate our Banach-space adjointable operators (in the sense of [9]) on $\mathbb{L}_Q$, and how they affects the free-probabilistic information on $\mathbb{L}_Q$ in Sect. 7.3.

In Sect. 8, we re-characterize the free-distributional data of Sect. 7.3 with help of the group-isomorphic relation of Sect. 7.2. Also, group dynamical systems of $\mathfrak{B}$ are studied.

2 Preliminaries

Readers can review fundamental analytic-and-combinatorial free probability theory from [17, 19] (and the cited papers therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical *measure theory* and *statistics*. The classical *independence* is replaced by so-called the *freeness*, by replacing measures on sets to linear functionals on algebras. It has various applications not only in pure mathematics (e.g., [2–4, 12, 14, 15]), but also in related fields (e.g., [5–8, 13, 16, 20, 21]).

In particular, we use combinatorial free-probabilistic approach of *Speicher* (e.g., [17]). *Free moments* and *free cumulants* of operators will be computed without introducing detailed concepts. Also, *free product* (in the sense of [17, 19]) will be used without precise introduction.

3 Fundamental Settings

In this section, we establish backgrounds of our proceeding works. Let (B, φ) be a topological $*$-probability space (a C^*-probability space, or a W^*-probability space, or a Banach $*$-probability space, etc), where B is a topological $*$-algebra (a C^*-algebra, resp., a W^*-algebra, resp., a Banach $*$-algebra, etc), and φ is a (bounded or unbounded) linear functional on B.

An operator a of B is said to be a *free random variable*, whenever it is regarded as an element of (B, φ). As usual in *operator theory*, an operator a is said to be *self-adjoint*, if $a^* = a$ in B, where a^* is the *adjoint of* a (e.g., [11]).

Definition 3.1 A self-adjoint free random variable a is said to be weighted-semicircular in (B, φ) with its weight $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (or, in short, t_0-semicircular), if a satisfies the free cumulant computations,

$$k_n(a, \ldots, a) = \begin{cases} k_2(a, a) = t_0 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_n(\ldots)$ is the free cumulant on B in terms of φ under the Möbius inversion of [17].

If $t_0 = 1$ in (3.1), the 1-semicircular element a is said to be semicircular in (B, φ), i.e., a is semicircular in (B, φ), if a satisfies

$$k_n(a, \ldots, a) = \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

By the Möbius inversion of [17], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \overset{def}{=} \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, and

$$c_k = \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{k!(k+1)!}$$

are the k-th Catalan numbers for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Similarly, a self-adjoint free random variable a is semicircular in (B, φ), if and only if a is 1-semicircular in (B, φ), if and only if

$$\varphi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

by (3.3), for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3).

So, we will use the t_0-semicircularity (3.1) (or the semicircularity (3.2)) and its characterization (3.3) (resp., (3.4)) alternatively from below.

If a is a self-adjoint free random variable in (B, φ), then

$$\text{the } \textit{free moments } \left(\varphi(a^n)\right)_{n=1}^{\infty},$$

and

$$\text{the}\,\textit{free cumulants}\ \ (k_n(a, \ldots, a))_{n=1}^{\infty}$$

provide equivalent free-distributional data of a in (B, φ) (e.g., [17]). Indeed, the *Möbius inversion* makes us have

$$\varphi(a^n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} k_{|V|}(a, \ldots, a) \right),$$

and

$$k_n(a, \ldots, a) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \theta} \varphi(a^{|V|}) \right) \mu_\pi,$$

where $NC(n)$ is the *lattice* consisting of all *noncrossing partitions* over $\{1, \ldots, n\}$, and "$V \in \pi$" means "V is a *block of* π," and where

$$\mu_\pi = \mu(\pi, 1_n)$$

the Möbius functional value at $(\pi, 1_n)$, where 1_n is the maximal partition of $NC(n)$ consisting of only one block, for all $n \in \mathbb{N}$.

In the rest of this section, we fix a C^*-probability space (A, ψ), and assume that there are $|\mathbb{Z}|$-many projections $\{q_j\}_{j \in \mathbb{Z}}$ in the C^*-algebra A, i.e., the operators q_j satisfy

$$q_j^* = q_j = q_j^2 \text{ in } A,$$

for all $j \in \mathbb{Z}$. Assume further that these projections $\{q_j\}_{j \in \mathbb{Z}}$ are *mutually orthogonal* from each other in A, in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \text{ for all } i, j \in \mathbb{Z}, \tag{3.5}$$

where δ is the *Kronecker delta*.

Now, we fix the family $\{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections (3.5) of A, and we denote it by $\mathbf{Q}$, i.e.,

$$\mathbf{Q} = \{q_j : j \in \mathbb{Z}\} \text{ in } A, \tag{3.6}$$

satisfying (3.5).

Remark 3.1 One can have such a C^*-algebraic structure A containing a family $\mathbf{Q}$ in the sense of (3.6), naturally, or artificially. Clearly, in the settings of [5, 8], one can naturally take such structures.

Suppose there is a C^*-algebra A_0 containing a family $\mathbf{Q}_N = \{q_1, \ldots, q_N\}$ of mutually orthogonal N-many projections $q_1, \ldots, q_N$, for $N \in \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Then, under suitable direct product, or tensor product, or free product of copies of A_0 with product topology, one can construct a C^*-algebra A containing a family $\mathbf{Q}$ with

$|\mathbb{Z}|$-many mutually orthogonal projections, where $\mathbf{Q}_0$ is contained in $\mathbf{Q}$, and every projection of $\mathbf{Q}$ is unitarily equivalent to a projection of $\mathbf{Q}_0$ in A.

And let Q be the C^*-subalgebra of A generated by the family $\mathbf{Q}$ of (3.6),

$$Q \overset{def}{=} C^* (\mathbf{Q}) \subseteq A. \tag{3.7}$$

Then it is easy to check that:

Proposition 3.1 *Let Q be a C^*-subalgebra (3.7) of a C^*-algebra A, generated by* $\mathbf{Q}$ *of (3.6). Then*

$$Q \overset{*\text{-}iso}{=} \underset{j \in \mathbb{Z}}{\oplus} \left(\mathbb{C} \cdot q_j\right) \overset{*\text{-}iso}{=} \mathbb{C}^{\oplus |\mathbb{Z}|}, \tag{3.8}$$

in A.

Proof The proof of (3.8) is straightforward by the mutual-orthogonality (3.5) of the generator set $\mathbf{Q}$ of Q in A. ■

Define now linear functionals ψ_j on the C^*-algebra Q by

$$\psi_j (q_i) = \delta_{ij} \psi (q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.9}$$

for all $j \in \mathbb{Z}$, where ψ is the linear functional of the fixed C^*-probability space (A, ψ). The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.9) are well-defined on Q by the structure theorem (3.8).

Assumption Let (A, ψ) be a fixed C^*-probability space, and let Q be the C^*-subalgebra (3.7) of A. In the rest of this paper, we further assume that

$$\psi (q_j) \in \mathbb{C}^{\times}, \text{ for all } j \in \mathbb{Z}.$$

□

By (3.7) and (3.8), if $T \in Q$, then

$$T = \sum_{j \in \mathbb{Z}} t_j q_j \qquad (\text{with } t_j \in \mathbb{C}),$$

and hence,

$$\psi_j(T) = t_j \psi (q_j),$$

by (3.9), for all $j \in \mathbb{Z}$.

Definition 3.2 The C^*-probability spaces $\left(Q, psi_j\right)$ are called the j-th C^*-probability spaces of Q in a given C^*-probability space (A, ψ), where Q is in the sense of (3.7), and ψ_j are in the sense of (3.9), for all $j \in \mathbb{Z}$.

Now, let's define bounded linear transformations c and a acting on the C^*-algebra Q, by linear morphisms satisfying

$$c\left(q_j\right) = q_{j+1}, \text{ and } a\left(q_j\right) = q_{j-1}, \tag{3.10}$$

for all $j \in \mathbb{Z}$. Then c and a are well-defined bounded linear operators "on Q." One can understand they are *Banach-space operators* in the *operator space* $B(Q)$ consisting of all bounded linear transformations acting on Q, by regarding Q as a Banach space equipped with its C^*-norm (e.g., [9]).

Definition 3.3 We call these Banach-space operators c and a of (3.10), the *creation*, respectively, the *annihilation* on Q.

The creation c and the annihilation a on Q are indeed well-defined because of the structure theorem (3.8) of Q. Define now a new Banach-space operator l on Q by

$$l = c + a \in B(Q). \tag{3.11}$$

Definition 3.4 We call the Banach-space operator $l \in B(Q)$ of (3.11), the *radial operator on Q*.

By the definition (3.11), one has

$$l\left(\sum_{j \in \mathbb{Z}} t_j q_j\right) = \sum_{j \in \mathbb{Z}} t_j \left(q_{j+1} + q_{j-1}\right), \text{ on } Q.$$

Now, define a closed subspace $\mathfrak{L}$ of $B(Q)$ by

$$\mathfrak{L} \overset{def}{=} \overline{\mathbb{C}[\{l\}]}^{\|.\|}, \tag{3.12}$$

generated by the radial operator l of (3.11), where the operator norm $\|.\|$ on the operator space $B(Q)$ is defined to be

$$\|T\| = \sup\{\|Tq\|_Q : \|q\|_Q = 1\},$$

for all $T \in B(Q)$, where $\|.\|_Q$ is the C^*-*norm on Q* (inherited from the C^*-norm on A), and where $\overline{X}^{\|.\|}$ mean the *operator-norm closures* of subsets X of the *operator space* $B(Q)$ (e.g., [9]). It is not difficult to check that, by the definition (3.12), this subspace $\mathfrak{L}$ forms an algebra in the vector space $B(Q)$, i.e., it forms a Banach algebra.

On this Banach algebra $\mathfrak{L}$ of (3.12), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n l^n\right)^* = \sum_{n=0}^{\infty} \overline{t_n} l^n \text{ in } \mathfrak{L}, \tag{3.13}$$

where $\overline{z}$ mean the *conjugates of* $z \in \mathbb{C}$.

Then this operation (3.13) becomes a well-defined *adjoint on* the Banach algebra $\mathfrak{L}$ of (3.12) (e.g., [11]), and hence, every element of $\mathfrak{L}$ is *adjointable* in $B(Q)$ (e.g., [9]). So, the algebra $\mathfrak{L}$ forms a *Banach* $*$*-algebra* in $B(Q)$ with the adjoint (3.13).

Definition 3.5 We call the Banach $*$-algebra $\mathfrak{L}$ of (3.12), the *radial* (*Banach* $*$*-)algebra on* Q (or, in the operator space $B(Q)$).

Now, let $\mathfrak{L}$ be the radial algebra on Q. Define the *tensor product Banach* $*$*-algebra* $\mathfrak{L}_Q$,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q, \tag{3.14}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of Banach $*$-algebras.

Since $\mathfrak{L}$ is a Banach $*$-algebra, and Q is a C^*-algebra, the tensor product $\mathfrak{L}_Q$ of (3.14) is a well-defined Banach $*$-algebra under product topology.

Definition 3.6 We call the tensor product Banach $*$-algebra $\mathfrak{L}_Q$ of (3.14), the radial projection (Banach $*$-)algebra on Q.

4 Weighted-Semicircular Elements Induced by Q

Throughout this section, let's fix the settings of Sect. 3. We here construct weighted-semicircular elements induced by the family $\mathbf{Q}$ of mutually orthogonal projections generating the radial projection algebra $\mathfrak{L}_Q$ of (3.14). Let (Q, ψ_j) be the j-th C^*-probability spaces of Q in (A, ψ), where ψ_j are in the sense of (3.9), for all $j \in \mathbb{Z}$.

Remark that, if u_j are the generating operators of $\mathfrak{L}_Q$,

$$u_j \overset{def}{=} l \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = \left(l \otimes q_j\right)^n = l^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^n = q_j$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Then one can construct a linear functional φ_j on $\mathfrak{L}_Q$ by a linear morphism satisfying that

$$\varphi_j\left((l \otimes q_i)^n\right) \overset{def}{=} \psi_j\left(l^n(q_i)\right), \tag{4.2}$$

for all $n \in \mathbb{N}$, for all $i, j \in \mathbb{Z}$.

These linear functionals φ_j of (4.2) are well-defined by (3.8), (3.12) and (3.14), for all $j \in \mathbb{Z}$.

Definition 4.1 We call the Banach $*$-probability spaces

$$\left(\mathfrak{L}_Q, \varphi_j\right), \textit{for all } j \in \mathbb{Z}, \tag{4.3}$$

the j-th (*Banach-$*$-*)*probability spaces on* Q.

Observe that, if c and a are the creation, respectively, the annihilation on Q of (3.10), then

$$ca = 1_Q = ac, \text{ the identity operator on } Q. \tag{4.4}$$

Indeed, for any generators $q_j \in \mathbf{Q}$ of Q,

$$ca\left(q_j\right) = c\left(a\left(q_j\right)\right) = c\left(q_{j-1}\right) = q_{j-1+1} = q_j,$$

and

$$ac\left(q_j\right) = a\left(c\left(q_j\right)\right) = a\left(q_{j+1}\right) = q_{j+1-1} = q_j,$$

for all $j \in \mathbb{Z}$. More generally, one has

$$\begin{aligned} c^n a^n = 1_Q = a^n c^n, &\text{ for all } n \in \mathbb{N}, \quad \text{and} \\ c^{n_1} a^{n_2} = a^{n_2} c^{n_1}, &\text{ for all } n_1, n_2 \in \mathbb{N}, \end{aligned} \tag{4.4)$'$ ($$

by (4.4).

Thus, one obtains that

$$l^n = (c+a)^n = \sum_{k=0}^{n} \binom{n}{k} c^k a^{n-k}, \tag{4.5}$$

for all $n \in \mathbb{N}$, by (4.4)$'$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \forall k \leq n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Note that, for any $n \in \mathbb{N}$,

$$l^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} c^k a^{n-k}, \tag{4.6}$$

by (4.5). So, the formula (4.6) does not contain 1_Q-terms by (4.4) and (4.4)$'$.

Note also that, for any $n \in \mathbb{N}$, one has

$$l^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} c^k a^{n-k} = \binom{2n}{n} c^n a^n + [\text{Rest terms}], \tag{4.7}$$

by (4.5). So, l^{2n} contains $\binom{2n}{n}$-many 1_Q-terms by (4.4)′ and (4.7).

Proposition 4.1 *Let l be the radial operator (3.1) on Q. Then*

(4.8) l^{2n-1} *does not contain* $1_Q -$ *terms in* $\mathfrak{L}$,

(4.9) l^{2n} *contains* $\binom{2n}{n} \cdot 1_Q$ *in* $\mathfrak{L}$.

Proof The statements (4.8) and (4.9) are proven by (4.6), respectively, by (4.7). ■

Remark that, since

$$u_j^n = \left(l \otimes q_j\right)^n = l^n \otimes q_j,$$

one has

$$\varphi_j \left(u_j^{2n-1}\right) = \psi_j \left(l^{2n-1} \left(q_j\right)\right) = 0, \tag{4.10}$$

for all $n \in \mathbb{N}$, by (3.9) and (4.8).

Similarly, we have

$$\varphi_j \left(u_j^{2n}\right) = \psi_j \left(l^{2n} \left(q_j\right)\right) = \psi_j \left(\binom{2n}{n} q_j + [\text{Rest terms}]\right)$$

by (4.7)

$$= \binom{2n}{n} \psi_j \left(q_j\right) = \binom{2n}{n} \psi \left(q_j\right),$$

by (3.9) and (4.9). I.e.,

$$\varphi_j \left(u_j^{2n}\right) = \binom{2n}{n} \psi \left(q_j\right), \tag{4.11}$$

for all $n \in \mathbb{N}$.

Thus, one obtains the following free-distributional data on the j-th probability space $\left(\mathfrak{L}_Q, \varphi_j\right)$, for $j \in \mathbb{Z}$.

Theorem 4.2 *Fix $j \in \mathbb{Z}$, and let $u_k = l \otimes q_k$ be the k-th generating operators of the j-th probability space $(\mathfrak{L}_Q, \varphi_j)$, for all $k \in \mathbb{Z}$, for $j \in \mathbb{Z}$. Then*

$$\varphi_j \left(u_k^n\right) = \delta_{j,k} \omega_n \left(\left(\frac{n}{2} + 1\right) \psi \left(q_j\right)\right) c_{\frac{n}{2}}, \tag{4.12}$$

where ω_n are in the sense of (3.3) for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers for all $k \in \mathbb{N}$.

Proof First, take the j-th generating operator u_j in the j-th probability space $\left(\mathfrak{L}_Q, \varphi_j\right)$, for $j \in \mathbb{Z}$. By (4.10) and (4.11), one can get that:

$$\varphi_j\left(u_j^{2n-1}\right) = 0,$$

and

$$\begin{aligned}\varphi_j\left(u_j^{2n}\right) &= \binom{2n}{n}\psi\left(q_j\right) = \left(\frac{n+1}{n+1}\right)\binom{2n}{n}\psi\left(q_j\right) \\ &= \left((n+1)\psi\left(q_j\right)\right)\left(\frac{1}{n+1}\binom{2n}{n}\right) \\ &= \left((n+1)\psi\left(q_j\right)\right)c_n,\end{aligned}$$

for all $n \in \mathbb{N}$. So,

$$\varphi_j\left(u_j^n\right) = \omega_n\left((n+1)\psi(q_j)\right)c_n, \text{ for all } n \in \mathbb{N}.$$

Assume now that $k \neq j$ in $\mathbb{Z}$. Then, by the definition (4.2) of φ_j (and by the definition (3.9) of ψ_j),

$$\varphi_j\left(u_k^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the formula (4.12) holds. ■

Motivated by (4.12), we define a linear morphism,

$$E_{j,Q} : \mathfrak{L}_Q \to \mathfrak{L}_Q$$

by a surjective linear transformation satisfying

$$E_{j,Q}\left(u_i^n\right) \overset{def}{=} \begin{cases} \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}u_j^n & \text{if } i = j \\ \\ 0_{\mathfrak{L}_Q}, \text{ the zero operator of } \mathfrak{L}_Q & \text{otherwise,}\end{cases} \tag{4.13}$$

for all $n \in \mathbb{N}$, $i, j \in \mathbb{Z}$, where $[\frac{n}{2}]$ mean the *minimal integers* greater than or equal to $\frac{n}{2}$, for example,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

The linear transformations $E_{j,Q}$ of (4.13) are well-defined bounded linear transformations on $\mathfrak{L}_Q$, because of the cyclicity (3.12) of the tensor factor $\mathfrak{L}$ of $\mathfrak{L}_Q$, and the structure theorem (3.8) of the other tensor factor Q of $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

Define now new linear functionals τ_j on $\mathfrak{L}_Q$ by

$$\tau_j \overset{def}{=} \varphi_j \circ E_{j,Q} \text{ on } \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.14}$$

where φ_j are in the sense of (4.2), and $E_{j,Q}$ are in the sense of (4.13).

Definition 4.2 The well-defined Banach $*$-probability spaces

$$\mathfrak{L}_Q(j) \overset{denote}{=} \left(\mathfrak{L}_Q, \tau_j\right) \tag{4.15}$$

are called the j-th filtered (Banach-$*$-)probability spaces of $\mathfrak{L}_Q$, where τ_j are the linear functionals (4.14) on $\mathfrak{L}_Q$, for all $j \in \mathbb{Z}$.

On the j-th filtered probability space $\mathfrak{L}_Q(j)$ of (4.15), One can get that

$$\begin{aligned}\tau_j\left(u_j^n\right) &= \varphi_j\left(E_{j,Q}\left(u_j^n\right)\right)\\ &= \varphi_j\left(\frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\left(u_j^n\right)\right) = \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\varphi_j\left(u_j^n\right)\\ &= \frac{\psi(q_j)^{n-1}}{([\frac{n}{2}]+1)}\omega_n\left(\left(\frac{n}{2}+1\right)\psi\left(q_j\right)\right)c_{\frac{n}{2}},\end{aligned}$$

by (4.12), i.e.,

$$\tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}}, \tag{4.16}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$, where ω_n are in the sense of (3.3).

Lemma 4.3 *Let $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ be the j-th filtered probability space of $\mathfrak{L}_Q$, for an arbitrarily fixed $j \in \mathbb{Z}$. Then*

$$\tau_j\left(u_i^n\right) = \delta_{j,i}\left(\omega_n \psi(q_j)^n c_{\frac{n}{2}}\right), \tag{4.17}$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$, for all $i \in \mathbb{Z}$.

Proof If $i = j$ in $\mathbb{Z}$, then the free-momental data (4.17) holds true by (4.16), for all $n \in \mathbb{N}$.

If $i \neq j$ in $\mathbb{Z}$, then, by the very definition (4.13) of the j-th filterization $E_{j,Q}$, and also by the definition (4.2) of φ_j,

$$\tau_j\left(u_i^n\right) = 0, \text{ for all } n \in \mathbb{N}.$$

Therefore, the free-distributional data (4.17) holds true, for all $i \in \mathbb{Z}$. ∎

The following theorem is proven by the above free-distributional data (4.17) in terms of the weighted-semicircularity characterization (3.3) of the weighted-semicircularity (3.1).

Theorem 4.4 *Let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space $\left(\mathfrak{L}_Q, \tau_j\right)$ of $\mathfrak{L}_Q$, for $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the "j-th" generating operator of $\mathfrak{L}_Q$. Then u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$.*

Proof First of all, the operator u_j is self-adjoint in $\mathfrak{L}_Q$ (for all $j \in \mathbb{Z}$). Indeed,

$$u_j^* = \left(l \otimes q_j\right)^* = l \otimes q_j = u_j$$

(for all $j \in \mathbb{Z}$) by (3.13).

Let's fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the j-th generating operator of the j-th filtered probability space $\mathfrak{L}_Q(j)$. Then, by (4.17), we have that

$$\tau_j\left(u_j^n\right) = \omega_n \left(\psi\left(q_j\right)^2\right)^{\frac{n}{2}} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, and where c_k are the k-th Catalan numbers, for all $k \in \mathbb{N}_0$.

Therefore, by the characterization (3.3) of the weighted-semicircularity (3.1), this self-adjoint element u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$. ■

The above theorem shows that, for any $j \in \mathbb{Z}$, the j-th generating operator u_j is $\psi(q_j)^2$-semicircular in the j-th filtered probability space $\mathfrak{L}_Q(j)$ of Q, by (4.17). Meanwhile, also by (4.17), one can verify the following result, too.

Theorem 4.5 *Let $u_i = l \otimes u_i$ be the i-th generating operators of the j-th filtered probability space $\mathfrak{L}_Q(j)$, for all $j \neq i \in \mathbb{Z}$. Then u_i have the zero free distribution in $\mathfrak{L}_Q(j)$.*

Proof Let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space for a fixed $j \in \mathbb{Z}$, and assume $i \neq j$ in $\mathbb{Z}$. Consider the i-th generating operators u_i of $\mathfrak{L}_Q(j)$. It is shown already that u_i are self-adjoint in $\mathfrak{L}_Q$, and hence, the free distributions of u_i are completely characterized by the free-momental sequences

$$\left(\tau_j(u_i^n)\right)_{n=1}^{\infty} = (0, 0, 0, \ldots),$$

the zero sequence, by (4.17). It guarantees that the free distributions of $u_i \in \mathfrak{L}_Q(j)$ are the zero free distribution, for all $j \neq i \in \mathbb{Z}$. ■

The above two theorems characterize the free-probabilistic information of the generators $\{u_i\}_{i \in \mathbb{Z}}$ of our j-th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. From below, we focus on "non-zero" free-distributional data on $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$.

By the Möbius inversion of [17], if u_i are the i-th generating operators of the j-th filtered probability space $\mathfrak{L}_Q(j)$, then

$$k_n^j(u_i, \ldots, u_i) = \begin{cases} \delta_{j,i}\psi\left(q_j\right)^2 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{4.18}$$

for all $n \in \mathbb{N}$, and $i \in \mathbb{Z}$, by (4.17), where $k_n^j(\ldots)$ is the free cumulant on $\mathfrak{L}_Q$ with respect to the linear functional τ_j, for $j \in \mathbb{Z}$.

5 Semicircular Elements Induced by Q

As in Sect. 4, let $\mathfrak{L}_Q(j)$ be the j-th filtered probability space of Q for $j \in \mathbb{Z}$. Then the j-th generating operator $u_j = l \otimes q_j$ of $\mathfrak{L}_Q$ is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$, satisfying that

$$\begin{gathered} \tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}}, \qquad \text{equivalently,} \\ k_n^j\left(u_j, \ldots, u_j\right) = \begin{cases} \psi(q_j)^2 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases} \end{gathered} \tag{5.1}$$

for all $n \in \mathbb{N}$, by (4.17) and (4.18).

By the weighted-semicircularity (5.1), one may/can obtain the following semicircular element of $\mathfrak{L}_Q(j)$ (under an additional condition); let

$$U_j \overset{def}{=} \frac{1}{\psi(q_j)} u_j \in \mathfrak{L}_Q(j), \tag{5.2}$$

for $j \in \mathbb{Z}$. Recall that we assumed $\psi(q_k) \in \mathbb{C}^\times$, for all $k \in \mathbb{Z}$, and hence, the above operator U_j of (5.2) is well-defined in $\mathfrak{L}_Q(j)$.

Theorem 5.1 *Let $U_j = \frac{1}{\psi(q_j)} u_j$ be a free random variable (5.2) of the j-th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$, where u_j is the j-th generating operator $l \otimes q_j$ of $\mathfrak{L}_Q$. If*

$$\psi(q_j) \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\} \text{ in } \mathbb{C}^\times,$$

then U_j is semicircular in $\mathfrak{L}_Q(j)$.

Proof Fix $j \in \mathbb{Z}$, and assume $\psi(q_j) \in \mathbb{R}^\times$ in $\mathbb{C}^\times$. Then

$$U_j^* = \left(\frac{1}{\psi(q_j)} u_j\right)^* = U_j,$$

by the self-adjointness of u_j in $\mathfrak{L}_Q$, because $\psi(q_j) \in \mathbb{R}^\times$.

Consider now that

$$
\begin{aligned}
k_n^j\left(\underbrace{U_j, U_j, \ldots, U_j}_{n\text{-times}}\right) &= k_n^j\left(\tfrac{1}{\psi(q_j)}u_j, \ldots, \tfrac{1}{\psi(q_j)}u_j\right) \\
&= \left(\tfrac{1}{\psi(q_j)}\right)^n k_n^j\left(\underbrace{u_j, u_j, \ldots, u_j}_{n\text{-times}}\right),
\end{aligned}
\tag{5.3}
$$

for all $n \in \mathbb{N}$, by the bimodule map property of free cumulants (e.g., [17]). Thus, by (5.3), one has that

$$
\begin{aligned}
k_n^j\left(U_j, \ldots, U_j\right) &= \begin{cases} \left(\frac{1}{\psi(q_j)}\right)^2 k_2^j(u_j, u_j) & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } n = 2 \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
\tag{5.4}
$$

by the $\psi(q_j)^2$-semicircularity (5.1) of u_j in $\mathfrak{L}_Q(j)$.

Therefore, by (5.4) and (3.2), the self-adjoint free random variable U_j is semicircular in $\mathfrak{L}_Q(j)$. ∎

The above theorem shows that, from our $\psi(q_j)^2$-semicircular elements $u_j = l \otimes q_j$ in $\mathfrak{L}_Q(j)$, the corresponding semicircular elements $U_j = \frac{1}{\psi(q_j)} u_j$ are canonically obtained in the j-th filtered probability space $\mathfrak{L}_Q(j)$, whenever $\psi(q_j) \in \mathbb{R}^{\times}$ in $\mathbb{C}$, for $j \in \mathbb{Z}$.

Assumption From below, for convenience, we will automatically assume that

$$\psi(q_j) \in \mathbb{R}^{\times} \text{ in } \mathbb{C}, \text{ for } q_j \in \mathbf{Q},$$

for all $j \in \mathbb{Z}$. □

6 The Free Filterization $\underset{j \in \mathbb{Z}}{\star} \mathfrak{L}_Q(j)$ of Q

Let (A, ψ) be a fixed C^*-probability space containing a family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections q_j satisfying

$$\psi(q_j) \in \mathbb{R}^{\times}, \text{ for all } j \in \mathbb{Z},$$

and let $\mathfrak{L}_Q(j)$ be the corresponding j-th filtered probability space of Q, for all $j \in \mathbb{Z}$.

For the system

$$\{\mathfrak{L}_Q(j) : j \in \mathbb{Z}\}$$

of Banach $*$-probability spaces, define the *free product Banach $*$-probability space* $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\begin{aligned}\mathfrak{L}_Q(\mathbb{Z}) &\overset{denote}{=} \left(\mathfrak{L}_Q(\mathbb{Z}), \tau\right) \\ &\overset{def}{=} \underset{j\in\mathbb{Z}}{\star} \mathfrak{L}_Q(j) = \left(\underset{j\in\mathbb{Z}}{\star} \mathfrak{L}_{Q,j}, \underset{j\in\mathbb{Z}}{\star} \tau_j\right),\end{aligned} \tag{6.1}$$

with

$$\mathfrak{L}_Q(\mathbb{Z}) = \underset{j\in\mathbb{Z}}{\star} \mathfrak{L}_{Q,j}, \text{ with } \mathfrak{L}_{Q,j} = \mathfrak{L}_Q, \forall j \in \mathbb{Z},$$

and

$$\tau = \underset{j\in\mathbb{Z}}{\star} \tau_j \text{ on } \mathfrak{L}_Q(\mathbb{Z}).$$

For more about free product $*$-probability spaces, see [17, 19].

Definition 6.1 Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free product Banach $*$-probability space (6.1) of the system $\{\mathfrak{L}_Q(j)\}_{j\in\mathbb{Z}}$ of all j-th filtered probability spaces of Q. Then it is said to be the free filterization of $Q \subset (A, \psi)$.

Now, construct two subsets $\mathcal{X}$ and $\mathcal{S}$ of $\mathfrak{L}_Q(\mathbb{Z})$,

$$\begin{aligned}\mathcal{X} &= \{u_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}, \quad \text{and} \\ \mathcal{S} &= \{U_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}.\end{aligned} \tag{6.2}$$

Recall that a subset $\mathcal{Y}$ of an arbitrary topological $*$-probability space (B, φ) is said to be a *free family*, if all elements of $\mathcal{Y}$ are free from each other in (B, φ). Also, a free family $\mathcal{Y}$ is called a *free (weighted-)semicircular family* in (B, φ), if this family $\mathcal{Y}$ is not only a free family in (B, φ), but also a subset of B whose elements are (weighted-)semicircular in (B, φ). (e.g., [7, 19]).

Theorem 6.1 *Let $\mathcal{X}$ and $\mathcal{S}$ be in the sense of (6.2) in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of (6.1).*

(6.3) The family $\mathcal{X}$ is a free weighted-semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$.
(6.4) The family $\mathcal{S}$ is a free semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$.

Proof Let $\mathcal{X}$ be in the sense of (6.2) in $\mathfrak{L}_Q(\mathbb{Z})$. All elements u_j of $\mathcal{X}$ are taken from mutually distinct free blocks $\mathfrak{L}_Q(j)$ of $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$, and hence, they are free from each other in $\mathfrak{L}_Q(\mathbb{Z})$. Thus, this family $\mathcal{X}$ is a free family in $\mathfrak{L}_Q(\mathbb{Z})$. Moreover, every element u_j is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(j)$ by (4.17) and (4.18). So, the powers u_j^n of each self-adjoint operator $u_j \in \mathcal{X}$ are again contained in the free block $\mathfrak{L}_Q(j)$

as free reduced words with their lengths-1 in $\mathfrak{L}_Q(\mathbb{Z})$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$. Thus, we have

$$\tau\left(u_j^n\right) = \tau_j\left(u_j^n\right) = \omega_n \psi(q_j)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$. It shows that each element $u_j \in \mathcal{X}$ is $\psi(q_j)^2$-semicircular in $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$. Therefore, the family $\mathcal{X}$ of (6.2) is a free weighted-semicircular family in $\mathfrak{L}_Q(\mathbb{Z})$. Equivalently, the statement (6.3) holds.

Similarly, one can verify that the family $\mathcal{S}$ of (6.2) is a free family in $\mathfrak{L}_Q(\mathbb{Z})$, because U_j are the scalar-products $\frac{1}{\psi(q_j)} u_j$ of u_j in the free family $\mathcal{X}$ of $\mathfrak{L}_Q(\mathbb{Z})$, for all $j \in \mathbb{Z}$. So, the semicircularity (5.4) of U_j's guarantees that this free family $\mathcal{S}$ is a free semicircular family in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$, i.e., the statement (6.4) holds. ■

By (4.17) and (4.18), the only "j-th" generating operators u_j of the free blocks $\mathfrak{L}_Q(j)$ provide non-zero free distributions on $\mathfrak{L}_Q(\mathbb{Z})$ by (6.1). Thus, we now restrict our interests to the Banach $*$-subalgebra $\mathbb{L}_Q$ of the free filterization $\mathfrak{L}_Q(\mathbb{Z})$, whose elements have possible non-zero free distributions.

Definition 6.2 Let $\mathfrak{L}_Q(\mathbb{Z})$ be the free filterization of Q. Define a Banach $*$-subalgebra $\mathbb{L}_Q$ of $\mathfrak{L}_Q(\mathbb{Z})$ by

$$\mathbb{L}_Q \overset{def}{=} \overline{\mathbb{C}\left[\mathcal{X}\right]}, \tag{6.5}$$

where $\mathcal{X}$ is the free weighted-semicircular family (6.3) in $\mathfrak{L}_Q(\mathbb{Z})$, and $\overline{Y}$ are the Banach-topology closures of the subsets Y of $\mathfrak{L}_Q(\mathbb{Z})$. Construct the Banach $*$-probability space,

$$\mathbb{L}_Q \overset{denote}{=} \left(\mathbb{L}_Q, \tau = \tau \mid_{\mathbb{L}_Q}\right), \tag{6.6}$$

as a free-probabilistic sub-structure of $\mathfrak{L}_Q(\mathbb{Z}) = \left(\mathfrak{L}_Q(\mathbb{Z}), \tau\right)$.

We call the Banach $*$-algebra $\mathbb{L}_Q$ of (6.5), or the Banach $*$-probability space $\mathbb{L}_Q$ of (6.6), the semicircular (free-sub-)filterization of $\mathfrak{L}_Q(\mathbb{Z})$.

By the definitions (6.5) and (6.6), the operators of the semicircular filterization $\mathbb{L}_Q$ are the free random variables in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$, having "possible" non-zero free distributions. In particular, all free reduced words of $\mathfrak{L}_Q(\mathbb{Z})$ in $\mathcal{X}$ (and hence, elements of $\mathbb{L}_Q$) have non-zero free distributions in $\mathfrak{L}_Q(\mathbb{Z})$, by (4.17) and (4.18).

Theorem 6.2 *Let $\mathbb{L}_Q$ be the semicircular filterization (6.5) in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$. Then*

$$\begin{aligned}\mathbb{L}_Q &\overset{def}{=} \overline{\mathbb{C}\left[\mathcal{X}\right]} = \overline{\mathbb{C}[\mathcal{S}]} \\ &\overset{*\text{-}iso}{=} \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{u_j\}]} \overset{*\text{-}iso}{=} \overline{\mathbb{C}\left[\underset{j\in\mathbb{Z}}{\star}\{u_j\}\right]},\end{aligned} \tag{6.7}$$

in $\mathfrak{L}_Q(\mathbb{Z})$, where "$\overset{*\text{-iso}}{=}$" means "being Banach-$*$-isomorphic," and where $(\star)$ in the first $*$-isomorphic relation of (6.7) means the free-probabilistic free product of [17, 19], and $(\star)$ in the second $*$-isomorphic relation of (6.7) is the pure-algebraic free product inducing noncommutative free words in $\mathcal{X}$.

Proof The free weighted-semicircular family $\mathcal{X}$ of (6.3) can be re-written by

$$\mathcal{X} = \{\psi(q_j)U_j \in \mathfrak{L}_Q(j) : j \in \mathbb{Z}\}$$

in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of Q, where U_j are the semicircular elements $\frac{1}{\psi(q_j)}u_j$ of the free semicircular family $\mathcal{S}$ of (6.4). Therefore,

$$\overline{\mathbb{C}[\mathcal{X}]} = \overline{\mathbb{C}[\mathcal{S}]} \text{ in } \mathfrak{L}_Q(\mathbb{Z}).$$

It shows that the first (set-)equality $(=)$ of (6.7) holds.

By the definition (6.5) of $\mathbb{L}_Q$, it is generated by the free family $\mathcal{X}$, and hence, the first $*$-isomorphic relation of (6.7) holds in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ by (6.1).

Since

$$\mathbb{L}_Q \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{u_j\}]} \text{ in } \mathfrak{L}_Q(\mathbb{Z}),$$

every element T of $\mathbb{L}_Q$ is a limit of linear combinations of free reduced words (in the sense of [17, 19]). Also, all (pure-algebraic) free words in $\mathcal{X}$ have their unique free-reduced-word forms under operator-product on $\mathfrak{L}_Q(\mathbb{Z})$. Furthermore, if we have a free (reduced) word

$$W = \prod_{l=1}^{N} u_{j_l} \text{ in } \mathcal{X},$$

then, as an operator, its adjoint W^* satisfies

$$W^* = \prod_{l=1}^{N} u_{j_{N-l+1}} \text{ in } \mathbb{L}_Q,$$

by the self-adjointness of $u_j \in \mathcal{X}$. Therefore, the second $*$-isomorphic relation of (6.7) holds, too. ∎

The equality $\mathbb{L}_Q = \overline{\mathbb{C}[\mathcal{S}]}$ in (6.7) shows that the name, the semicircular filterization, is well-fit for the Banach $*$-probabilistic sub-structure $\mathbb{L}_Q$ of (6.6) in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$.

Theorem 6.3 *Let $\mathbb{L}_Q$ be the semicircular filterization (6.5) in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of Q, and let*

$$X = \prod_{l=1}^{N} u_{j_l}^{n_l} \in \mathbb{L}_Q, \textit{ for } n_1, \ldots, n_N \in \mathbb{N},$$

where the integer sequence $(j_1, \ldots, j_N)$ *is alternating in* $\mathbb{Z}$ *(if* $N > 1$ *in* $\mathbb{N}$*) in the sense that:*

$$j_1 \neq j_2, j_2 \neq j_3, \ldots, j_{N-1} \neq j_N \text{ in } \mathbb{Z}.$$

(6.8) If $N = 1$*, then*

$$\tau\left(X^n\right) = \tau\left((X^*)^n\right) = \omega_{nn_1} \psi(q_{j_1})^{nn_1} c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

(6.9) If $N > 1$ *and if* $j_1, \ldots, j_N$ *are mutually distinct from each other in* $\mathbb{Z}$ *then*

$$\tau(X) = \tau(X^*) = \prod_{l=1}^{N} \left(\omega_{n_l} \psi(q_{j_l})^{n_l} c_{\frac{n_l}{2}}\right).$$

Proof Suppose first that $N = 1$, and $X = u_{j_1}^{n_1} \in \mathbb{L}_Q$. Then, by (6.7), this operator X is a free reduced word with its length-1, contained in the free block $\overline{\mathbb{C}[\{u_{j_1}\}]}$ of the semicircular filterization $\mathbb{L}_Q$. So, the operators $X^n = u_{j_1}^{nn_1}$ are free reduced words with their lengths-1 in $\overline{\mathbb{C}[\{u_{j_1}\}]}$ embedded in $\mathbb{L}_Q$. Thus, one can get that

$$\tau(X^n) = \tau_{j_1}\left(u_{j_1}^{nn_1}\right) = \omega_{nn_1} \psi(q_{j_1})^{nn_1} c_{\frac{nn_1}{2}},$$

for all $n \in \mathbb{N}$, by the $\psi(q_{j_1})^2$-semicircularity of $u_{j_1} \in \mathcal{X}$.

By the self-adjointness of u_{j_1}, one also has that $X = X^*$ in $\mathbb{L}_Q$. Therefore, the statement (6.8) holds.

Assume now that $N > 1$ in $\mathbb{N}$. Then, by the assumption that $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, the operators X and X^* form the free reduced words with their lengths-N in $\mathbb{L}_Q$ (e.g., [17, 19]). Moreover, since $j_1, \ldots, j_N$ are assumed to be mutually distinct in $\mathbb{Z}$, one has that

$$\tau(X) = \prod_{l=1}^{N} \tau_{j_l}\left(u_{j_l}^{n_l}\right) = \prod_{l=1}^{N} \left(\omega_{n_l} \psi(q_{j_l})^{n_l} c_{\frac{n_l}{2}}\right)$$

by the weighted-semicircularity of $u_{j_1}, \ldots, u_{j_N} \in \mathcal{X}$ in $\mathfrak{L}_Q(\mathbb{Z})$

$$= \prod_{l=1}^{N} \tau_{j_{N-l+1}}\left(u_{j_{N-l+1}}^{n_{N-j+1}}\right) = \tau\left(\prod_{l=1}^{N} u_{j_{N-l+1}}^{n_{N-l+1}}\right) = \tau(X^*). \tag{6.10}$$

So, the statement (6.9) holds, by (6.10). ■

The above theorem characterizes free distributions of free reduced words of the semicircular filterization $\mathbb{L}_Q$ in the free weighted-semicircular family $\mathcal{X}$.

Observation 6.1 Let $\mathcal{S}$ be the free semicircular family (6.3) in $\mathfrak{L}_Q(\mathbb{Z})$. In the above theorem, if we replace $u_{j_l} \in \mathcal{X}$ to $U_{j_l} \in \mathcal{S}$, for $l = 1, \ldots, N$, then the similar free-distributional data can be obtained by replacing $\psi(q_{j_l})$ in the formulas (6.8), (6.9) and (6.10) to 1, under similar conditions, for all $l = 1, \ldots, N$. I.e.,

$$\begin{aligned} &\tau\left(U_{j_1}^n\right) = \tau\left(\left(U_{j_1}^*\right)^n\right) = \omega_n c_{\frac{n}{2}}, \ \forall n \in \mathbb{N}, \quad \text{and} \\ &\tau\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right) = \tau\left(\left(\prod_{l=1}^{N} U_{j_l}\right)^*\right) = \prod_{l=1}^{N}\left(\omega_{n_l} c_{\frac{n_l}{2}}\right), \end{aligned} \tag{6.11}$$

etc. (also, see [6, 7]). □

7 Shifts on $\mathbb{Z}$ and Integer-Shifts on $\mathbb{L}_Q$

In this section, let (A, ψ) be a fixed C^*-probability space containing a family $\mathbf{Q} = \{q_j\}_{j\in\mathbb{Z}}$ of mutually-orthogonal projections q_j's having

$$\psi(q_j) \in \mathbb{R}^\times, \ \text{for all } j \in \mathbb{Z},$$

and let $\mathbb{L}_Q$ be the semicircular filterization of the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of $Q = C^*(\mathbf{Q})$.

7.1 (±)-Shifts on $\mathbb{Z}$

Let $\mathbb{Z}$ be the set of all integers. Define bijective functions h_+ and h_- on $\mathbb{Z}$ by

$$\begin{aligned} h_+(j) &= j+1, \quad \text{and} \\ h_-(j) &= j-1, \end{aligned} \tag{7.1.1}$$

for all $j \in \mathbb{Z}$.

Then, for these bijections $h_\pm$ of (7.1.1), one can construct the following bijections $h_\pm^{(n)}$ on $\mathbb{Z}$,

$$h_\pm^{(n)} = \underbrace{h_\pm \circ h_\pm \circ \cdots \circ h_\pm}_{n\text{-times}}, \tag{7.1.2}$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with identities, $h_\pm^{(1)} = h_\pm$, and

$$h_\pm^{(0)} = id_{\mathbb{Z}}, \ \text{the identity map on } \mathbb{Z},$$

satisfying

$$id_{\mathbb{Z}}(j) = j, \text{ for all } j \in \mathbb{Z},$$

where $(\circ)$ is the usual functional composition.

By (7.1.2),

$$h_{\pm}^{(n)}(j) = j \pm n, \text{ for all } j \in \mathbb{Z},$$

for all $n \in \mathbb{N}_0$.

Definition 7.1 Let $h_{\pm}^{(n)}$ be in the sense of (7.1.2), for all $n \in \mathbb{N}_0$. Then we call $h_{\pm}^{(n)}$, the n-$(\pm)$-shifts on $\mathbb{Z}$. If $n = 1$, then the 1-$(\pm)$-shifts $h_{\pm}$ of (7.1.1) are simply said to be $(\pm)$-shifts on $\mathbb{Z}$. Of course, if $n = 0$ in $\mathbb{N}_0$, then 0-$(\pm)$-shifts are identified to be the identity map $id_{\mathbb{Z}}$ on $\mathbb{Z}$.

From these shifting processes $h_{\pm}^{(n)}$ on $\mathbb{Z}$, we construct certain $*$-isomorphisms on the semicircular filterization $\mathbb{L}_Q$.

7.2 *Integer-Shifts on* $\mathbb{L}_Q$

Let $\mathbb{L}_Q$ be the semicircular filterization in the free filterization $\mathfrak{L}_Q(\mathbb{Z})$ of Q, and let $h_{\pm}^{(n)}$ be n-$(\pm)$-shifts on $\mathbb{Z}$, for all $n \in \mathbb{N}_0$. In this section, by using these shifts, certain $*$-isomorphisms $\beta_{\pm}^{(n)}$ on $\mathbb{L}_Q$ are constructed, and we study how the $*$-isomorphisms act on $\mathbb{L}_Q$, for $n \in \mathbb{N}$.

Define a "multiplicative" bounded linear transformation $\beta_{\pm}$ on $\mathbb{L}_Q$ by a morphism satisfying that:

$$\beta_{\pm}\left(U_j\right) = U_{h_{\pm}(j)}, \tag{7.2.1}$$

for $U_j \in \mathcal{S}$, for all $j \in \mathbb{Z}$.

Remark that, by (6.7), the free semicircular family $\mathcal{S}$ of (6.4) is the generator set of $\mathbb{L}_Q$. So, by (6.6), the above multiplicative linear transformation $\beta_{\pm}$ of (7.2.1) is well-defined on $\mathbb{L}_Q$. By (7.2.1), we obtain the following computations.

Lemma 7.1 *Let* $Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \in \mathbb{L}_Q$*, for* $U_{j_1}, \ldots, U_{j_N} \in \mathcal{S}$*, and* $n_1, \ldots, n_N \in \mathbb{N}$*, for* $N \in \mathbb{N}$*. Then*

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} U_{j_l \pm 1}^{n_l}. \tag{7.2.2}$$

Proof Let Y be given as above in $\mathbb{L}_Q$. Then, by the multiplicativity of the linear transformations $\beta_{\pm}$ of (7.2.1), one has that

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} \beta_{\pm}\left(U_{j_l}^{n_l}\right) = \prod_{l=1}^{N} \left(\beta_{\pm}\left(U_{j_l}\right)\right)^{n_l} = \prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}.$$

Therefore, the formula (7.2.2) holds. ■

Now, let $u_{j_1}, \ldots, u_{j_N} \in \mathcal{X}$ be weighted-semicircular elements generating $\mathbb{L}_Q$, for $N \in \mathbb{N}$, and let

$$X = \prod_{l=1}^{N} u_{j_l}^{n_l}, \text{ for } n_1, \ldots, n_N \in \mathbb{N}.$$

Then

$$\beta_{\pm}(X) = \beta_{\pm}\left(\left(\prod_{l=1}^{N} \psi(q_{j_l})^{n_l}\right)\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right)\right)$$

since

$$U_{j_l} = \frac{1}{\psi(q_{j_l})} u_{j_l} \in \mathcal{S} \Longleftrightarrow u_{j_l} = \psi(q_{j_l}) U_{j_l} \in \mathcal{X}$$

in $\mathbb{L}_Q$, and hence, the above equality goes to

$$\begin{aligned} &= \left(\prod_{l=1}^{N} \psi(q_{j_l})^{n_l}\right) \beta_{\pm}\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right) \\ &= \left(\prod_{l=1}^{N} \psi(q_{j_l})^{n_l}\right)\left(\prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}\right), \end{aligned}$$

by (7.2.2).

Corollary 7.2 *Let* $X = \prod_{l=1}^{N} u_{j_l}^{n_l} \in \mathbb{L}_Q$*, for* $u_{j_1}, \ldots, u_{j_N} \in \mathcal{X}$ *in* $\mathbb{L}_Q$*, for* $n_1, \ldots, n_N, N \in \mathbb{N}$*. Then*

$$\begin{aligned} \beta_{\pm}(X) &= \left(\prod_{l=1}^{N} \psi(q_{j_l})^{n_l}\right)\left(\prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}\right) \\ &= \left(\prod_{l=1}^{N} \psi(q_{j_l})^{n_l}\right)\left(\beta_{\pm}\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right)\right), \end{aligned} \tag{7.2.2$'$}$$

in $\mathbb{L}_Q$*, where* $U_{j_l} = \frac{1}{\psi(q_{j_l})} u_{j_l} \in \mathcal{S}$ *in* $\mathbb{L}_Q$*, for all* $l = 1, \ldots, N$*.*

Proof The proof of (7.2.2)′ is done by (7.2.2). ■

By (7.2.2) and (7.2.2)′, one can realize that the free-reduced-word-ness on $\mathbb{L}_Q$ is preserved by that on the set $\beta_{\pm}(\mathbb{L}_Q)$. Indeed, if an arbitrary N-tuple $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, then the N-tuples $(h_{\pm}(j_1), \ldots, h_{\pm}(j_N))$ are alternating in $\mathbb{Z}$, too, for all $N \in \mathbb{N}$. It guarantees that $\beta_{\pm}$ preserves the freeness on the semicircular filterization

$\mathbb{L}_Q$. So, if the operators Y and X are in the sense of the above lemma, respectively, of the above corollary, and if we further assume they are free reduced words with their lengths-N in $\mathbb{L}_Q$, with

$$j_1 \neq j_2,\ j_2 \neq j_3, \ldots, j_{N-1} \neq j_N \text{ in } \mathbb{Z},$$

then the images

$$\beta_{\pm}(Y), \text{ and } \beta_{\pm}(X)$$

are again free reduced words with their lengths-N in $\mathbb{L}_Q$.

Theorem 7.3 *Let $\beta_{\pm}$ be the multiplicative linear transformations (7.2.1) on $\mathbb{L}_Q$. Then they are $*$-isomorphisms on $\mathbb{L}_Q$.*

Proof By (6.5), (6.6) and (6.7), all elements of the semicircular filterization $\mathbb{L}_Q$ are the limits of linear combinations of free reduced words in the free semicircular family $\mathcal{S}$ of (6.4). So, let's focus on free reduced words of $\mathbb{L}_Q$ in $\mathcal{S}$.

Let $(j_1, \ldots, j_N)$ be an alternating N-tuple in $\mathbb{Z}$ for $N \in \mathbb{N}$, and

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l}, \text{ for } n_1, \ldots, n_N \in \mathbb{N}.$$

By the alternating-ness of $(j_1, \ldots, j_N)$, the above operator Y is a free reduced word with its length-N in $\mathbb{L}_Q$ by (6.7).

Then, by (7.2.2),

$$\beta_{\pm}(Y) = \prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l}, \tag{7.2.3}$$

where $h_{\pm}$ are the $(\pm)$-shifts (7.1.1) on $\mathbb{Z}$.

By the bijectivity of $h_{\pm}$, the relation (7.2.3) guarantees the bijectivity of $\beta_{\pm}$ on $\mathbb{L}_Q$. I.e., these multiplicative linear transformations $\beta_{\pm}$ of (7.2.1) are generator-preserving, and hence, they are bounded and bijective on $\mathbb{L}_Q$.

Consider now that if Y is as above, then

$$\beta_{\pm}(Y^*) = \beta_{\pm}\left(\prod_{l=1}^{N} U_{j_{N-j+1}}^{n_{N-l+1}}\right)$$

by the self-adjointness of $U_{j_1}, \ldots, U_{j_N}$

$$= \prod_{l=1}^{N} U_{h_{\pm}(j_{N-l+1})}^{n_{N-l+1}}$$

by (7.2.2)

$$= \left(\prod_{l=1}^{N} U_{h_{\pm}(j_l)}^{n_l} \right)^* = (\beta_{\pm}(Y))^* . \tag{7.2.4}$$

So,

$$\beta_{\pm}(T^*) = (\beta_{\pm}(T))^* , \text{ for all } T \in \mathbb{L}_Q,$$

by (7.2.4), under linearity.

Therefore, the bounded multiplicative linear transformations $\beta_{\pm}$ of (7.2.1) are both bijective, and adjoint-preserving on $\mathbb{L}_Q$, equivalently, they are well-defined $*$-isomorphisms on $\mathbb{L}_Q$. ■

From the above theorem, one can realize that the $(\pm)$-shifts $h_{\pm}$ on $\mathbb{Z}$ induce the corresponding $*$-isomorphisms $\beta_{\pm}$ on $\mathbb{L}_Q$.

Definition 7.2 Let $\beta_{\pm}$ be the $*$-isomorphisms (7.2.1) on the semicircular filterization $\mathbb{L}_Q$, induced by the $(\pm)$-shifts $h_{\pm}$ of (7.1.1) on $\mathbb{Z}$. Then they are said to be $(\pm)$-integer-shift(-$*$-isomorphism)s on $\mathbb{L}_Q$.

These two $*$-isomorphisms $\beta_{\pm}$ satisfy the following identity relation on $\mathbb{L}_Q$.

Proposition 7.4 *Let $\beta_{\pm}$ be the $(\pm)$-integer-shifts (7.2.1) on $\mathbb{L}_Q$. Then*

$$\beta_+\beta_- = 1_{\mathbb{L}_Q} = \beta_-\beta_+ \text{ on } \mathbb{L}_Q, \tag{7.2.5}$$

where $1_{\mathbb{L}_Q}$ is the identity map on $\mathbb{L}_Q$, satisfying

$$1_{\mathbb{L}_Q}(T) = T, \text{ for all } T \in \mathbb{L}_Q.$$

Proof As we discussed above, it suffices to consider the cases where we have free reduced words

$$Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \text{ of } \mathbb{L}_Q, \text{ for } n_1, \ldots, n_N \in \mathbb{N},$$

for $N \in \mathbb{N}$, where $U_{j_l} \in \mathcal{S}$, for $l = 1, \ldots, N$, and $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, by (7.2.2), (7.2.2)$'$, and (6.7).

Observe that

$$\begin{aligned} \beta_+\beta_-(Y) &= \beta_+ \left(\prod_{l=1}^{N} U_{h_-(j_l)}^{n_l} \right) = \beta_+ \left(\prod_{l=1}^{N} U_{j_l-1}^{n_l} \right) \\ &= \prod_{l=1}^{N} U_{h_+(j_l-1)}^{n_l} = \prod_{l=1}^{N} U_{j_l-1+1}^{n_l} = Y, \end{aligned}$$

similarly,

$$\beta_-\beta_+(Y) = Y.$$

Therefore, for any arbitrary operators $S \in \mathbb{L}_Q$,

$$\beta_+\beta_-(S) = \beta_-\beta_+(S) \text{ in } \mathbb{L}_Q.$$

Therefore, the identity (7.2.5) holds. ∎

Let $\beta_\pm$ be the ($\pm$)-integer-shifts on $\mathbb{L}_Q$. Then one can construct $*$-isomorphisms $\beta_\pm^n$,

$$\beta_\pm^n = \underbrace{\beta_\pm\beta_\pm \cdot \cdots \cdot \beta_\pm}_{n\text{-times}} \text{ on } \mathbb{L}_Q, \tag{7.2.6}$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with identity,

$$\beta_+^0 = 1_{\mathbb{L}_Q} = \beta_-^0.$$

Since $\beta_\pm$ and $1_{\mathbb{L}_Q}$ are $*$-isomorphisms, the morphisms $\beta_\pm^n$ are well-defined $*$-isomorphisms on $\mathbb{L}_Q$, too, for all $n \in \mathbb{N}_0$.

Definition 7.3 Let $\beta_\pm^n$ be the $*$-isomorphisms (7.2.6) on the semicircular filterization $\mathbb{L}_Q$, for all $n \in \mathbb{N}_0$. Then they are called the n-($\pm$)-(integer-)shifts on $\mathbb{L}_Q$, for all $n \in \mathbb{N}_0$.

By (7.2.5) and (7.2.6), one obtains the following relations on the system $\{\beta_\pm^n : n \in \mathbb{N}_0\}$ of $*$-isomorphisms.

Theorem 7.5 *Let $\beta_\pm^n$ be the n-($\pm$)-shifts on the semicircular filterization $\mathbb{L}_Q$, for $n \in \mathbb{N}_0$. Then they satisfy*

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_-^{n_2}\beta_+^{n_1} = \begin{cases} 1_{\mathbb{L}_Q} & \textit{if } n_1 = n_2 \\ \beta_+^{n_1-n_2} & \textit{if } n_1 > n_2 \\ \beta_-^{n_2-n_1} & \textit{if } n_1 < n_2, \end{cases} \tag{7.2.7}$$

on $\mathbb{L}_Q$, for all $n_1, n_2 \in \mathbb{N}_0$. Also,

$$\beta_+^{n_1}\beta_+^{n_2} = \beta_+^{n_1+n_2}, \textit{ and } \beta_-^{n_1}\beta_-^{n_2} = \beta_-^{n_1+n_2}, \tag{7.2.8}$$

on $\mathbb{L}_Q$, for all $n_1, n_2 \in \mathbb{N}_0$.

Proof By the identity (7.2.5), two $*$-isomorphisms β_+ and β_- are not only commutative on $\mathbb{L}_Q$, but also their products $\beta_+\beta_-$ and $\beta_-\beta_+$ become the identity $*$-isomorphism $1_{\mathbb{L}_Q}$ on $\mathbb{L}_Q$. So, for any $n_1, n_2 \in \mathbb{N}_0$,

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_-^{n_2}\beta_+^{n_1} \text{ on } \mathbb{L}_Q.$$

Thus, let's focus on the $*$-isomorphisms $\beta_+^{n_1}\beta_-^{n_2}$, for arbitrarily fixed $n_1, n_2 \in \mathbb{N}_0$. Suppose first that $n_1 = n_2 = n$ in $\mathbb{N}_0$. Then, by (7.2.5),

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^n\beta_-^n = (\beta_+\beta_-)^n = \left(1_{\mathbb{L}_Q}\right)^n = 1_{\mathbb{L}_Q}. \tag{7.2.9}$$

Assume now that $n_1 > n_2$ in $\mathbb{N}_0$. Then

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^{n_1-n_2}\beta_+^{n_2}\beta_-^{n_2} = \beta_+^{n_1-n_2}, \tag{7.2.10}$$

on $\mathbb{L}_Q$, by (7.2.9).

Similar to (7.2.10), if $n_1 < n_2$ in $\mathbb{N}_0$, then

$$\beta_+^{n_1}\beta_-^{n_2} = \beta_+^{n_1}\beta_-^{n_1}\beta_-^{n_2-n_1} = \beta_-^{n_2-n_1}, \tag{7.2.11}$$

on $\mathbb{L}_Q$.

So, the formula (7.2.7) is proven by (7.2.9), (7.2.10) and (7.2.11).

For any free generators $U_j \in \mathcal{S}$ of $\mathbb{L}_Q$ (by (6.7)), one can get that

$$\begin{aligned}
\beta_+^{n_1}\beta_+^{n_2}\left(U_j^n\right) &= \beta_+^{n_1}\left(U_{j+n_2}^n\right) \\
&= U_{j+n_1+n_2}^n = \beta_+^{n_1+n_2}\left(U_j^n\right), \quad \text{and} \\
\beta_-^{n_1}\beta_-^{n_2}\left(U_j^n\right) &= \beta_-^{n_1}\left(U_{j-n_2}^n\right) = U_{j-n_2-n_1}^n \\
&= U_{j-(n_1+n_2)}^n = \beta_-^{n_1+n_2}\left(U_j^n\right),
\end{aligned} \tag{7.2.12}$$

for all $j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, for all $n_1, n_2 \in \mathbb{N}_0$.

Therefore, the formula (7.2.8) holds on $\mathbb{L}_Q$ by (7.2.2), (7.2.2)$'$, and (7.2.12). ■

The above relations (7.2.7) and (7.2.8) can be re-expressed as follows;

$$\begin{gathered}
\beta_{e_1}^{n_1}\beta_{e_2}^{n_2} = \beta_{e_2}^{n_2}\beta_{e_1}^{n_1} = \beta_{sgn(e_1n_1+e_2n_2)}^{|e_1n_1+e_2n_2|} \text{ on } \mathbb{L}_Q, \quad \text{with} \\
sgn(e_1n_1 + e_2n_2) = \begin{cases} + \text{ if } e_1n_1 + e_2n_2 \geq 0 \\ - \text{ if } e_1n_1 + e_2n_2 < 0, \end{cases}
\end{gathered} \tag{7.2.13}$$

for all $e_1, e_2 \in \{\pm\}$, and $n_1, n_2 \in \mathbb{N}_0$, where sgn in (7.2.13) is the sign map on $\mathbb{Z}$,

$$sgn(j) \overset{def}{=} \begin{cases} + & \text{if } j \geq 0 \\ - & \text{if } j < 0, \end{cases}$$

for all $j \in \mathbb{Z}$, and $|.|$ is the absolute value on $\mathbb{Z}$.

From below, we use the re-expression (7.2.13) for the results (7.2.7) and (7.2.8) for convenience.

Now, consider the system $\mathfrak{B}$ of n-$(\pm)$-shifts $\beta_\pm^n$ on $\mathbb{L}_Q$, i.e.,

$$\mathfrak{B} = \{\beta_\pm^n\}_{n \in \mathbb{N}_0}. \tag{7.2.14}$$

Let $Aut(\mathbb{L}_Q)$ be the group,

$$Aut\left(\mathbb{L}_Q\right) = \left(\left\{ \alpha : \mathbb{L}_Q \to \mathbb{L}_Q \,\middle|\, \begin{array}{c} \alpha \text{ are} \\ *\text{-isomorphisms} \\ \text{on } \mathbb{L}_Q \end{array} \right\}, \cdot \right) \tag{7.2.15}$$

consisting of all $*$-isomorphisms on $\mathbb{L}_Q$, where the operation $(\cdot)$ means the product (or compositions) of $*$-isomorphisms. We call $Aut(\mathbb{L}_Q)$ of (7.2.15), the ($*$-) *automorphism group on* $\mathbb{L}_Q$. (Recall that $*$-isomorphisms on a $*$-algebra are called $*$-*automorphisms*.)

By the construction (7.2.14), the system $\mathfrak{B}$ is definitely a "subset" of the automorphism group $Aut(\mathbb{L}_Q)$ of (7.2.15). Note that the operation $(\cdot)$ is closed on $\mathfrak{B}$, in the sense that:

$$\left(\beta_{e_1}^{n_1}, \beta_{e_2}^{n_2}\right) \in \mathfrak{B} \times \mathfrak{B} \longmapsto \beta_{e_1}^{n_1}\beta_{e_2}^{n_2} \in \mathfrak{B}, \tag{7.2.16}$$

for all $e_1, e_2 \in \{\pm\}$, and $n_1, n_2 \in \mathbb{N}_0$, by (7.2.13).

Clearly, by (7.2.8), one can get that

$$\left(\beta_e^{n_1}\beta_e^{n_2}\right)\beta_e^{n_3} = \beta_e^{n_1+n_2+n_3} = \beta_e^{n_1}\left(\beta_e^{n_2}\beta_e^{n_3}\right), \tag{7.2.17}$$

for all $e \in \{\pm\}$, and $n_1, n_2, n_3 \in \mathbb{N}_0$.

Observe now that

$$\begin{aligned} \left(\beta_+^{n_1}\beta_-^{n_2}\right)\beta_+^{n_3} &= \beta_{\sigma(n_1,n_2)}^{|n_1-n_2|}\beta_+^{n_3} = \beta_{\sigma(|n_1-n_2|,n_3)}^{||n_1-n_2|-n_3|}, \quad \text{and} \\ \beta_+^{n_1}\left(\beta_-^{n_2}\beta_+^{n_3}\right) &= \beta_+^{n_1}\beta_{\sigma(n_2,n_3)}^{|n_2-n_3|} = \beta_{\sigma(n_1,|n_2-n_3|)}^{|n_1-|n_2-n_3||}, \end{aligned} \tag{7.2.18}$$

by (7.2.7) (and (7.2.13)), for all $n_1, n_2, n_3 \in \mathbb{N}_0$, where

$$\sigma(n,k) \overset{def}{=} sgn(n-k), \text{ for all } n, k \in \mathbb{N}_0,$$

in (7.2.18).

Consider two positive quantities a_1 and a_2,

$$\begin{aligned} a_1 &= ||n_1 - n_2| - n_3|, \quad \text{and} \\ a_2 &= |n_1 - |n_2 - n_3||, \end{aligned} \tag{7.2.19}$$

for $n_1, n_2, n_3 \in \mathbb{N}_0$.

If either $n_1 \leq n_2 \leq n_3$, or $n_1 \geq n_2 \geq n_3$ in $\mathbb{N}_0$, then

$$a_1 = |n_2 - n_1 - n_3| = a_2; \tag{7.2.20}$$

and if either $n_1 \leq n_3 \leq n_2$, or $n_1 \geq n_3 \geq n_2$ in $\mathbb{N}_0$, then

$$a_1 = |n_2 - n_1 - n_3| = a_2; \tag{7.2.21}$$

and if either $n_2 \leq n_3 \leq n_1$, or $n_2 \geq n_3 \geq n_1$ in $\mathbb{N}_0$, then

$$a_1 = |n_1 - n_2 - n_3| = a_2, \tag{7.2.22}$$

where a_1 and a_2 are the quantities (7.2.19).

Lemma 7.6 *Let* $\mathfrak{B} = \{\beta_{\pm}^{n}\}_{n \in \mathbb{N}_0}$ *be the system (7.2.14). Then*

$$\left(\beta_{e_1}^{n_1} \beta_{e_2}^{n_2}\right) \beta_{e_3}^{n_3} = \beta_{e_1}^{n_1} \left(\beta_{e_2}^{n_2} \beta_{e_3}^{n_3}\right) \text{ on } \mathbb{L}_Q, \tag{7.2.23}$$

for all $e_1, e_2, e_3 \in \{\pm\}$*, and* $n_1, n_2, n_3 \in \mathbb{N}_0$.

Proof By (7.2.17), we have

$$\left(\beta_{e}^{n_1} \beta_{e}^{n_2}\right) \beta_{e}^{n_3} = \beta_{e}^{n_1} \left(\beta_{e}^{n_2} \beta_{e}^{n_3}\right) \text{ on } \mathbb{L}_Q,$$

for all $e \in \{\pm\}$, and $n_1, n_2, n_3 \in \mathbb{N}_0$.

By (7.2.18), (7.2.20), (7.2.21) and (7.2.22),

$$\begin{aligned} \left(\beta_{+}^{n_1} \beta_{-}^{n_2}\right) \beta_{+}^{n_3} &= \beta_{sgn(|n_1-n_2|-n_3)}^{||n_1-n_2|-n_3|} = \beta_{sgn(a_1')}^{a_1} \\ &= \beta_{sgn(a_2')}^{a_2} = \beta_{sgn(n_1-|n_2-n_3|)}^{|n_1-|n_2-n_3||} \\ &= \beta_{+}^{n_1} \left(\beta_{-}^{n_2} \beta_{+}^{n_3}\right), \end{aligned} \tag{7.2.24}$$

on $\mathbb{L}_Q$, for all $n_1, n_2, n_3 \in \mathbb{N}_0$, where $a_1 = \left|a_1'\right|$ and $a_2 = \left|a_2'\right|$ are in the sense of (7.2.19), and sgn is the sign map on $\mathbb{Z}$ in (7.2.13).

Similar to (7.2.24), one can obtain that

$$\begin{aligned} \left(\beta_{-}^{n_1} \beta_{+}^{n_2}\right) \beta_{-}^{n_3} &= \beta_{sgn(|n_1-n_2|-n_3)}^{||n_1-n_2|-n_3|} \\ &= \beta_{sgn(n_1-|n_2-n_3|)}^{|n_1-|n_2-n_3||} = \beta_{-}^{n_1} \left(\beta_{+}^{n_2} \beta_{-}^{n_3}\right), \end{aligned} \tag{7.2.25}$$

on $\mathbb{L}_Q$, for all $n_1, n_2, n_3 \in \mathbb{N}_0$.

Therefore, the formula (7.2.23) holds on $\mathfrak{B}$, by (7.2.17), (7.2.24) and (7.2.25). ■

By the above lemma, we obtain the following structure theorem of the system $\mathfrak{B}$ of (7.2.14) in the automorphism group $Aut(\mathbb{L}_Q)$.

Theorem 7.7 *Let* $\mathfrak{B}$ *be the subset (7.2.14) of the automorphism group* $Aut(\mathbb{L}_Q)$ *of (7.2.15). Then* $\mathfrak{B}$ *is a subgroup of* $Aut(\mathbb{L}_Q)$.

Proof Let $\mathfrak{B}$ be in the sense of (7.2.14). Then, by (7.2.16), the operation $(\cdot)$ is closed on $\mathfrak{B}$. So, the algebraic pair $\mathfrak{B} = (\mathfrak{B}, \cdot)$ is well-constructed as an algebraic substructure of $Aut(\mathbb{L}_Q)$. By (7.2.23), this operation is associative on $\mathfrak{B}$, and hence, it forms a semigroup. Since

$$\beta_+^0 = 1_{\mathbb{L}_Q} = \beta_-^0 in \mathfrak{B},$$

and since

$$\beta_e^n \cdot 1_{\mathbb{L}_Q} = \beta_e^n = 1_{\mathbb{L}_Q} \cdot \beta_e^n,$$

for all $e \in \{\pm\}$, and $n \in \mathbb{N}_0$, the semigroup $\mathfrak{B}$ contains its $(\cdot)$-identity $1_{\mathbb{L}_Q}$. Thus, it forms a monoid.

Finally, by (7.2.7), all elements $\beta_\pm^n \in \mathfrak{B}$ have their unique $(\cdot)$-inverses $\beta_\mp^n \in \mathfrak{B}$, such that

$$\beta_+^n \beta_-^n = 1_{\mathbb{L}_Q} = \beta_-^n \beta_+^n \text{ on } \mathbb{L}_Q,$$

for all $n \in \mathbb{N}_0$, i.e.,

$$\left(\beta_\pm^n\right)^{-1} = \beta_\mp^n \text{ on } \mathbb{L}_Q, \text{ for all } n \in \mathbb{N}_0,$$

where x^{-1} mean the group-inverses of x. So, this monoid $\mathfrak{B}$ forms a group.

Therefore, the system $\mathfrak{B}$ is a subgroup of the automorphism group $Aut(\mathbb{L}_Q)$. ■

By the above theorem, the system $\mathfrak{B}$ of (7.2.14) is a group. As a group, $\mathfrak{B}$ satisfies the following group-property.

Theorem 7.8 *Let $\mathfrak{B}$ be the subgroup (7.2.14) of the automorphism group $Aut(\mathbb{L}_Q)$. Then $\mathfrak{B}$ is group-isomorphic to the infinite abelian cyclic group $\mathbb{Z} = (\mathbb{Z}, +)$. I.e.,*

$$\mathfrak{B} \overset{Group}{=} (\mathbb{Z}, +), \tag{7.2.26}$$

where "$\overset{Group}{=}$" means "being group-isomorphic."

Proof Define now a function $\Phi : \mathbb{Z} \to \mathfrak{B}$ by

$$\Phi : j \in \mathbb{Z} \longmapsto \beta_{sgn(j)}^{|j|} \in \mathfrak{B}, \tag{7.2.27}$$

where sgn is the sign map on $\mathbb{Z}$ (for example, $\Phi(2) = \beta_+^2$, and $\Phi(-3) = \beta_-^3$, etc.), with identity,

$$0 \in \mathbb{Z} \longmapsto 1_{\mathbb{L}_Q} = \beta_\pm^0 \in \mathfrak{B}.$$

It is not hard to check that this function Φ of (7.2.27) is a well-defined bijection from $\mathbb{Z}$ onto $\mathfrak{B}$, by (7.2.14). Consider now that

$$\begin{aligned}\Phi(j_1 + j_2) = \beta_{sgn(j_1+j_2)}^{|j_1+j_2|} &= \beta_{sgn(j_1)}^{|j_1|} \beta_{sgn(j_2)}^{|j_2|} \\ &= \Phi(j_1)\Phi(j_2),\end{aligned} \tag{7.2.28}$$

in $\mathfrak{B}$, by (7.2.13), for all $j_1, j_2 \in \mathbb{Z}$.

So, the bijection Φ of (7.2.27) is a group-homomorphism by (7.2.28), equivalently, it is a group-isomorphism from $\mathbb{Z}$ onto $\mathfrak{B}$. Therefore, the group-isomorphic relation (7.2.26) holds true. ■

The above theorem characterizes the group-structure of the subgroup $\mathfrak{B} = \{\beta_{\pm}^{n}\}_{n \in \mathbb{N}_0}$ of the automorphism group $Aut(\mathbb{L}_Q)$. I.e., $\mathfrak{B}$ is an infinite cyclic abelian group.

Definition 7.4 Let $\mathfrak{B}$ be the subgroup (7.2.14) of the automorphism group $Aut(\mathbb{L}_Q)$. We call $\mathfrak{B}$, the integer-shift (sub)group (of $Aut(\mathbb{L}_Q)$ acting) on $\mathbb{L}_Q$.

7.3 Free Distributions on $\mathbb{L}_Q$ Under the Action of $\mathfrak{B}$

Let $\mathfrak{B}$ be the integer-shift group (7.2.14) acting on the semicircular filterization $\mathbb{L}_Q$ of Q, which is an infinite abelian cyclic subgroup of the automorphism group $Aut(\mathbb{L}_Q)$, by (7.2.26). In this section, we consider how our $*$-isomorphisms $\beta_{\pm}^{n} \in \mathfrak{B}$ affects the free probability on the semicircular filterization $\mathbb{L}_Q$. To do that, we fix $n_0 \in \mathbb{N}_0$ arbitrarily throughout this section, and fix the corresponding n_0-($\pm$)-shifts $\beta_{\pm}^{n_0}$ in $\mathfrak{B}$, and construct new linear functionals $\tau_{\pm(n_0)}$ on $\mathbb{L}_Q$,

$$\begin{gathered} \tau_{\pm(n_0)} \overset{def}{=} \tau \circ \beta_{\pm}^{n_0} \text{ on } \mathbb{L}_Q, \quad \text{i.e.,} \\ \tau_{+(n_0)} = \tau \circ \beta_{+}^{n_0}, \text{ and } \tau_{-(n_0)} = \tau \circ \beta_{-}^{n_0}, \end{gathered} \tag{7.3.1}$$

on $\mathbb{L}_Q$, where τ is the linear functional of (6.6).

Since $\beta_{\pm}^{n_0} \in \mathfrak{B}$ are well-defined $*$-isomorphisms, and τ is a linear functional on $\mathbb{L}_Q$, the morphism $\tau_{\pm(n_0)}$ of (7.3.1) are well-determined bounded linear functionals on $\mathbb{L}_Q$.

Proposition 7.9 *Suppose $n_0 = 0$ in $\mathbb{N}_0$, and hence, $\beta_{\pm}^{n_0} = \beta_{\pm}^{0} = 1_{\mathbb{L}_Q}$ is the group-identity of the integer-shift group $\mathfrak{B}$. Then*

$$\tau_{\pm(0)} = \tau \circ \beta_{\pm}^{0} = \tau, \tag{7.3.2}$$

on the semicircular filterization $\mathbb{L}_Q$.

Proof The identity (7.3.2) is trivial by (7.3.1) and (7.2.7). ■

By (7.3.2), we are not interested in the case where $n_0 = 0$ in $\mathbb{N}_0$. So, if there is no confusion, we will automatically assume below that $n_0 \in \mathbb{N}$ in $\mathbb{N}_0$, and $\tau_{\pm(n_0)}$ are the corresponding linear functionals (7.3.1) on $\mathbb{L}_Q$.

Definition 7.5 Let $\mathbb{L}_Q$ be the semicircular filterization (as a Banach $*$-algebra), and let $\tau_{\pm(n_0)}$ be the linear functionals (7.3.1) on $\mathbb{L}_Q$. The pairs

$$\mathbb{L}_Q^{+(n_0)} = \left(\mathbb{L}_Q, \tau_{+(n_0)}\right), \text{ and} \\ \mathbb{L}_Q^{-(n_0)} = \left(\mathbb{L}_Q, \tau_{-(n_0)}\right) \tag{7.3.3}$$

are called the n_0-(+)-shifted (Banach-)$*$-probability space of $\mathbb{L}_Q$, respectively, the n_0-($-$)-shifted (Banach-)$*$-probability space of $\mathbb{L}_Q$.

Define operators $X, Y \in \mathbb{L}_Q$ by

$$X = \prod_{l=1}^{N} u_{j_l}^{n_l}, \text{ and } Y = \prod_{l=1}^{N} U_{j_l}^{n_l} \text{ in } \mathbb{L}_Q \tag{7.3.4}$$

where $u_{j_l} \in \mathcal{X}$ are $\psi(q_{j_l})^2$-semicircular elements, and $U_{j_l} = \frac{1}{\psi(q_{j_l})} u_{j_l} \in \mathcal{S}$ are the corresponding semicircular elements, generating the semicircular filterization $\mathbb{L}_Q$ (by (6.7)), for all $l = 1, \ldots, N$, for $N \in \mathbb{N}$.

Theorem 7.10 *Let X and Y be in the sense of (7.3.4), as free random variables of the n_0-($\pm$)-shifted probability spaces $\mathbb{L}_Q^{\pm(n_0)}$ of (7.3.3).*

(7.3.5) If $N = 1$ in $\mathbb{N}$, then

$$\tau_{\pm(n_0)}\left(X^n\right) = \omega_{nn_1} \psi\left(q_{j_l}\right)^{nn_1} c_{\frac{nn_1}{2}},$$

and

$$\tau_{\pm(n_0)}\left(Y^n\right) = \omega_{nn_1} c_{\frac{nn_1}{2}},$$

for all $n \in \mathbb{N}$.

(7.3.6) Let $N > 1$ in $\mathbb{N}$, and let the integer-sequence $(j_1, \ldots, j_N)$ be alternating in $\mathbb{Z}$. Assume further that $j_1, \ldots, j_N$ are mutually distinct in $\mathbb{Z}$. Then

$$\tau_{\pm(n_0)}(X) = \prod_{l=1}^{N} \left(\omega_{n_l} \psi\left(q_{j_l}\right)^{n_l} c_{\frac{n_l}{2}}\right) = \tau_{\pm(n_0)}\left(X^*\right)$$

and

$$\tau_{\pm(n_0)}(Y) = \prod_{l=1}^{N} \left(\omega_{n_l} c_{\frac{n_l}{2}}\right) = \tau_{\pm(n_0)}(Y^*).$$

Proof First assume that $N = 1$, and hence, $X = u_{j_1}^{n_1}$, and $Y = U_{j_1}^{n_1}$ in the n_0-($\pm$)-shifted probability spaces $\mathbb{L}_Q^{\pm(n_0)}$. Then both X^n and Y^n are regarded as the free reduced words with their lengths-1 in $\mathbb{L}_Q$, for all $n \in \mathbb{N}$, by (6.5) and (6.7). Thus,

$$\begin{aligned} \tau_{\pm(n_0)}\left(\left(u_{j_1}^{n_1}\right)^n\right) &= \psi(q_{j_1})^{nn_1} \tau\left(\beta_{\pm}^{n_0}(U_{j_1}^{nn_1})\right) \\ &= \psi(q_{j_1})^{nn_1} \tau_{j_1 \pm n_0}\left(U_{j_1 \pm n_0}^{nn_1}\right) \\ &= \omega_{nn_1} \psi\left(q_{j_1}\right)^{nn_1} c_{\frac{nn_1}{2}}, \end{aligned}$$

by (7.2.2)$'$, and

$$\tau_{\pm(n_0)}\left(\left(U_{j_1}^{n_1}\right)^n\right) = \tau\left(U_{j_1\pm n_0}^{nn_1}\right) = \omega_{nn_1} c_{\frac{nn_1}{2}},$$

by (7.2.2) and (7.2.3), for all $n \in \mathbb{N}$, by the semicircularity of $U_{j_1\pm n_0} \in \mathcal{S}$, in the semicircular filterization $\mathbb{L}_Q$. Therefore, the statement (7.3.5) holds.

Assume now that $N > 1$ in $\mathbb{N}$, and the operators X and Y are in the sense of (7.3.4) in $\mathbb{L}_Q$. By the condition that $(j_1, \ldots, j_N)$ is alternating in $\mathbb{Z}$, these operators X and Y are free reduced words with their lengths-N in $\mathbb{L}_Q$, by (6.5) and (6.7). Moreover, since $j_1, \ldots, j_N$ are assumed to be mutually distinct in $\mathbb{Z}$, one can get that

$$\tau_{\pm(n_0)}(X) = \psi_X \tau\left(\beta_{\pm}^{n_0}\left(\prod_{l=1}^{N} U_{j_l}^{n_l}\right)\right) = \psi_X \tau\left(\prod_{l=1}^{N} U_{j_l\pm n_0}^{n_l}\right)$$

by (7.2.2)$'$, and (7.2.3)

$$\begin{aligned} &= \psi_X \prod_{l=1}^{N} \tau_{j_l\pm n_0}\left(U_{j_l\pm n_0}^{n_l}\right) \\ &= \prod_{l=1}^{N}\left(\omega_{n_l}\psi\left(q_{j_l\pm n_0}\right)^{n_l} c_{\frac{n_l}{2}}\right), \end{aligned} \tag{7.3.7}$$

by the semicircularity of $U_{j_l\pm n_0} \in \mathcal{S}$, where

$$\psi_X = \prod_{l=1}^{N} \psi(q_{j_l})^{n_l} \in \mathbb{R}^{\times} in \mathbb{C}.$$

Similar to (7.3.7),

$$\tau_{\pm(n_0)}(Y) = \prod_{l=1}^{N}\left(\omega_{n_l} c_{\frac{n_l}{2}}\right),$$

by (7.2.2), (7.2.3) and by the semicircularity of $U_{j_l\pm n_0} \in \mathcal{S}$. Remark that, by the self-adjointness of u_{j_l} and U_{j_l}, for all $l = 1, \ldots, N$,

$$X^* = \prod_{l=1}^{N} u_{j_{N-l+1}}^{n_{N-l+1}} \text{ and } Y^* = \prod_{l=1}^{N} U_{j_{N-l+1}}^{n_{N-l+1}}$$

are free reduced words with their lengths-N in $\mathbb{L}_Q$, with mutually distinct $j_N, j_{N-1}, \ldots, j_1$ in $\mathbb{Z}$. Thus,

$$\tau_{\pm(n_0)}(X^*) = \tau_{\pm(n_0)}(X),$$

and

$$\tau_{\pm(n_0)}(Y^*) = \tau_{\pm(n_0)}(Y).$$

Therefore, the statement (7.3.6) holds true. ■

The above theorem shows how the original free distributional data on the semicircular filterization $\left(\mathbb{L}_Q, \tau\right)$ are affected by the n_0-($\pm$)-shift $\beta_{\pm}^{n_0}$ on $\mathbb{L}_Q$. Compare the free-distributional data (6.8), (6.9), and the above results (7.3.5), (7.3.6).

Corollary 7.11 *Let $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$ be the semicircular filterization, and $\mathbb{L}_Q^{\pm(n_0)} = (\mathbb{L}_Q, \tau_{\pm(n_0)})$, the n_0-($\pm$)-shifted $*$-probability spaces (7.3.3) of $\mathbb{L}_Q$.*

(7.3.8) The semicircular law on $\mathbb{L}_Q$ induced by $U_j \in \mathcal{S}$ on $\mathbb{L}_Q$ is preserved to that on $\mathbb{L}_Q^{\pm(n_0)}$.

(7.3.9) The $\psi(q_j)^2$-semicircular laws induced by $u_j \in \mathcal{X}$ on $\mathbb{L}_Q$ are preserved to be the $\psi\left(q_j\right)^2$-semicircular laws on $\mathbb{L}_Q^{\pm(n_0)}$.

Proof Now, let $U_j \in \mathcal{S}$ be a semicircular element $\frac{1}{\psi(q_j)} u_j$ in the semicircular filterization $\mathbb{L}_Q$, for $u_j \in \mathcal{X}$, for $j \in \mathbb{Z}$. By understanding it as a free random variable in the n_0-($\pm$)-shifted $*$-probability spaces $\mathbb{L}_Q^{\pm(n_0)}$, one has that

$$\tau_{\pm(n_0)}\left(U_j^n\right) = \tau\left(U_{j\pm n_0}^n\right) = \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by (7.3.5), (7.3.6) and (7.3.7). It means that the self-adjoint free random variable U_j is semicircular in $\mathbb{L}_Q^{\pm(n_0)}$, too. I.e., the semicircular law on $\mathbb{L}_Q$ induced by $U_j \in \mathcal{S}$ is preserved to be the semicircular law on $\mathbb{L}_Q^{\pm(n_0)}$ induced by $U_{j\pm n_0} \in \mathcal{S}$. Therefore, the statement (7.3.8) holds.

Now, consider the $\psi(q_j)^2$-semicircular element $u_j = \psi(q_j)U_j \in \mathcal{X}$ in the semicircular filterization $\mathbb{L}_Q$, and regard it as a self-adjoint free random variable in the n_0-($\pm$)-shifted $*$-probability spaces $\mathbb{L}_Q^{\pm(n_0)}$. Then

$$\tau_{\pm(n_0)}\left(u_j^n\right) = \psi(q_j)^n \tau\left(U_{j\pm n_0}^n\right) = \omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by (7.2.2)$'$, (7.3.5), (7.3.6) and (7.3.7).

It shows that the $\psi(q_j)^2$-semicircular law on $\mathbb{L}_Q$ induced by $u_j \in \mathcal{X}$ is preserved to be the $\psi(q_j)^2$-semicircular laws on $\mathbb{L}_Q^{\pm(n_0)}$ induced by

$$\beta_{\pm}^{n_0}\left(u_j\right) = \psi(q_j) U_{j\pm n_0} \in \mathbb{L}_Q,$$

respectively, where $U_{j\pm n_0} \in \mathcal{S}$. So, the statement (7.3.9) holds true. ■

As we have seen above, the (weighted-)semicircular law(s) induced by our free semicircular family $(\mathcal{X}\cup)\mathcal{S}$ on the semicircular filterization $\mathbb{L}_Q$ is (are) preserved to be the "same" (weighted-)semicircular law(s) induced by $(\mathcal{X}\cup)\mathcal{S}$ on the n_0-($\pm$)-shifted $*$-probability spaces $\mathbb{L}_Q^{\pm(n_0)}$ of (7.3.3), by (7.3.8) and (7.3.9).

Definition 7.6 Let (B_1, φ_1) and (B_2, φ_2) be arbitrary topological $*$-probability spaces. We say that they are free-($*$-)isomorphic, if (i) B_1 and B_2 are $*$-isomorphic via a $*$-isomorphism $\Omega : B_1 \to B_2$, and (ii)

$$\varphi_2\left(\Omega(a)\right) = \varphi_1(a), \text{ for all } a \in (B_1, \varphi_1),$$

where $\Omega(a) \in (B_2, \varphi_2)$. In other words, (B_1, φ_1) and (B_2, φ_2) are free-isomorphic, if and only if they are equivalent in the sense of Voiculescu (e.g., [19]).

By (7.3.8), (7.3.9) and (6.7), we obtain the following theorem.

Theorem 7.12 *Let $\mathbb{L}_Q$ be the semicircular filterization, and let $\mathbb{L}_Q^{\pm(n_0)}$ be the n_0-($\pm$)-$*$-probability spaces (7.3.3) of $\mathbb{L}_Q$. Then they are free-isomorphic from each other.*

Proof By the structure theorem (6.7) of $\mathbb{L}_Q$, it suffices to show that free reduced words in the generator set $\mathcal{S}$, our free semicircular family, of $\mathbb{L}_Q$ preserves their free distributions to those of $\mathbb{L}_Q^{\pm(n_0)}$, under the identity operator on $\mathbb{L}_Q$. But, by (7.3.5), (7.3.6), (7.3.8) and (7.3.9), free distributions of free generators of $\mathbb{L}_Q$ are preserved to be the same free distributions of $\mathbb{L}_Q^{\pm(n_0)}$, under the identity operators,

$$I_+ : \mathbb{L}_Q \to \mathbb{L}_Q^{+(n_0)}, \text{ and } I_- : \mathbb{L}_Q \to \mathbb{L}_Q^{-(n_0)},$$

where

$$I_\pm(T) = T \in \mathbb{L}_Q^{\pm(n_0)}, \text{ for all } T \in \mathbb{L}_Q.$$

Therefore, the Banach $*$-probability spaces $\mathbb{L}_Q$ and $\mathbb{L}_Q^{+(n_0)}$ (resp., $\mathbb{L}_Q$ and $\mathbb{L}_Q^{-(n_0)}$) are free-isomorphic. And hence, $\mathbb{L}_Q^{+(n_0)}$ and $\mathbb{L}_Q^{-(n_0)}$ are free-isomorphic, too. ∎

The above theorem fully characterize how each $*$-isomorphism β of the integer-shift group $\mathfrak{B}$ affects the free probability on the semicircular filterization $\mathbb{L}_Q$. By acting $\beta \in \mathfrak{B}$ on $\mathbb{L}_Q$, the free probability on $\mathbb{L}_Q$ is preserved to that on $\beta(\mathbb{L}_Q) = \mathbb{L}_Q$.

8 Actions of $\mathfrak{B}$ on $\mathbb{L}_Q$

Let $\mathbb{L}_Q = (\mathbb{L}_Q, \tau)$ be the semicircular filterization, and let $\mathfrak{B}$ be the integer-shift group acting on $\mathbb{L}_Q$, an infinite abelian cyclic subgroup of the automorphism group $Aut(\mathbb{L}_Q)$. In Sect. 7.3, we showed how $\mathfrak{B}$ acts on $\mathbb{L}_Q$, and how it preserves the original free-distributional data on $\mathbb{L}_Q$, by (7.3.8) and (7.3.9).

Corollary 8.1 *Let $\mathfrak{B}$ be the integer-shift group acting on the semicircular filterization $\mathbb{L}_Q$, and let*

$$h = \beta_{e_1}^{n_1}\beta_{e_2}^{n_2}\dots\beta_{e_N}^{n_N} \in \mathfrak{B}, \tag{8.1}$$

where $(e_1, \dots, e_N) \in \{\pm\}^N$, and $n_1, \dots, n_N \in \mathbb{N}^N$, for $N \in \mathbb{N}$. Then

$$\tau\left(\left(h(u_j)\right)^n\right) = \omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}} = \tau\left(u_j^n\right), \quad \textit{and}$$
$$\tau\left(\left(h(U_j)\right)^n\right) = \omega_n c_{\frac{n}{2}} = \tau\left(U_j^n\right), \tag{8.2}$$

for all $n \in \mathbb{N}$.

Proof Let h be a $*$-isomorphism (8.1) in the integer-shift group $\mathfrak{B}$, acting on $\mathbb{L}_Q$. Since $\mathfrak{B}$ is a well-defined group, and $\mathfrak{B} = \{\beta_{\pm}^n\}_{n \in \mathbb{N}_0}$, set-theoretically, there exist unique $e \in \{\pm\}$, and $n_0 \in \mathbb{N}_0$, such that

$$h = \beta_e^{n_0} \in \mathfrak{B}.$$

Indeed, one can take

$$e_1 n_1 + e_2 n_2 + \cdots + e_N n_N \in \mathbb{Z}, \qquad \text{where}$$
$$e_l n_l = \begin{cases} n_l & \text{if } e_l = + \\ -n_l & \text{if } e_l = -, \end{cases} \tag{8.3}$$

for all $l = 1, \ldots, N$. Then

$$e = sgn\left(e_1 n_1 + e_2 n_2 + \cdots + e_N n_N\right) \in \{\pm\} \qquad \text{and}$$
$$n_0 = |e_1 n_1 + e_2 n_2 + \cdots + e_N n_N| \in \mathbb{N}_0, \tag{8.4}$$

by (7.2.6) and (8.3).

Therefore, one obtains that

$$\tau\left(\left(h(u_j)\right)^n\right) = \tau\left(h(u_j^n)\right) = \tau\left(\beta_e^{n_0}(u_j^n)\right)$$
$$= \psi(q_j)^n \tau\left(U_{jen_0}^n\right) = \omega_n \psi\left(q_j\right)^n c_{\frac{n}{2}},$$

by (8.4), for all $n \in \mathbb{N}$. Therefore, the free-distributional data in (8.2) hold. ■

Now, let's re-consider the structure theorem (7.2.6) of our integer-shift group $\mathfrak{B}$. By (7.2.6), one can directly act the group $\mathbb{Z}$ on the semicircular filterization $\mathbb{L}_Q$. Let $j \in \mathbb{Z}$. Then

$$j = e_j n_j \text{ in } \mathbb{Z}, \qquad \text{with}$$
$$e_j = sgn(j) \in \{\pm\}, \text{ and } n_j = |j| \in \mathbb{N}_0. \tag{8.5}$$

For instance, if $j = 3$, then $e_j = +$, and $n_j = 3$; if $j = -2$, then $e_j = -$, and $n_j = 2$; if $j = 0$, then $e_j = +$, or $-$, and $n_j = 0$, etc.

By regarding every integer $j \in \mathbb{Z}$ as its unique expression (8.5), one can define an action α of $\mathbb{Z}$ acting on $\mathbb{L}_Q$ by

$$\alpha : j \in \mathbb{Z} \longmapsto \beta_{e_j}^{n_j} \in \mathfrak{B} \subset Aut(\mathbb{L}_Q), \tag{8.6}$$

where e_j and n_j for an integer j are in the sense of (8.5). Then, by (7.2.6), the above action α of (8.6) is well-defined.

Recall now that if G is a group, and $\mathcal{A}$ is a topological $*$-algebra, and if there exists a group-action γ of G acting on $\mathcal{A}$, i.e., $\gamma(g) \in Hom(\mathcal{A})$, for all $g \in G$, where $Hom(\mathcal{A})$ is the ($*$-)*homomorphism group* consisting of all $*$-homomorphisms on $\mathcal{A}$, satisfying

$$\gamma(g_1 g_2) = \gamma(g_1)\gamma(g_2) \text{ on } \mathcal{A}, \ \forall g_1, g_2 \in G,$$

then the mathematical triple

$$(G, \mathcal{A}, \gamma)$$

is called the *group* (*topological*-$*$-)*dynamical system of* G *acting on* $\mathcal{A}$ *via a group-action* γ. In particular, if $\mathcal{A}$ is a C^*-algebra, or a W^*-algebra (von Neumann algebra), or a Banach $*$-algebra, then the triple is said to be a *group* C^**-dynamical system*, respectively, a *group* W^**-dynamical system*, respectively, a *group Banach* $*$*-dynamical system* (or a *group* B^**-dynamical system*), etc. Remark that the automorphism group $Aut(\mathcal{A})$ is a subgroup of the homomorphism group $Hom(\mathcal{A})$. So, there are well-defined group B^*-dynamical systems,

$$\left(\mathbb{Z}, \mathbb{L}_Q, \alpha\right),$$

where α is in the sense of (8.6), and

$$\left(\mathfrak{B}, \mathbb{L}_Q, \beta\right),$$

with

$$\beta : \beta_e^n \in \mathfrak{B} \longmapsto \beta_e^n \in Aut(\mathbb{L}_Q), \tag{8.7}$$

for all $e \in \{\pm\}$, and $n \in \mathbb{N}_0$.

Theorem 8.2 *Let* $\mathfrak{B}$ *be the integer-shift group embedded in the automorphism group* $Aut(\mathbb{L}_Q)$ *of the semicircular filterization* $\mathbb{L}_Q$*, and let* $\mathbb{Z} = (\mathbb{Z}, +)$ *be the infinite abelian cyclic group. Then the group* B^**-dynamical systems*

$$\left(\mathfrak{B}, \mathbb{L}_Q, \beta\right) \text{ and } \left(\mathbb{Z}, \mathbb{L}_Q, \alpha\right)$$

are equivalent in the sense that: (i) $\mathfrak{B}$ *and* $\mathbb{Z}$ *are group-isomorphic via a group-isomorphism* $\Omega : \mathbb{Z} \to \mathfrak{B}$*, and (ii)* $\beta\left(\Omega(j)\right) = \alpha(j)$ *on* $\mathbb{L}_Q$*, for all* $j \in \mathbb{Z}$*, where* α *and* β *are in the sense of (8.6) and (8.7), respectively. I.e.,*

$$\left(\mathfrak{B}, \mathbb{L}_Q, \beta\right) \overset{equi}{=} \left(\mathbb{Z}, \mathbb{L}_Q, \alpha\right). \tag{8.8}$$

Proof Let $\left(\mathfrak{B}, \mathbb{L}_Q, \beta\right)$ and $\left(\mathbb{Z}, \mathbb{L}_Q, \alpha\right)$ be the above group B^*-dynamical systems, where α and β are in the sense of (8.6), respectively, (8.7).

By (7.2.6), there exists a group-isomorphism $\Omega : \mathbb{Z} \to \mathfrak{B}$,

$$\Omega(j) = \beta_{e_j}^{n_j} \in \mathfrak{B}, \text{ for all } j \in \mathbb{Z},$$

where $j = e_j n_j$ in the sense of (8.5).

By (8.6) and (8.7), one has that

$$\beta\left(\Omega(j)\right) = \beta\left(\beta_{e_j}^{n_j}\right) = \beta_{e_j}^{n_j} = \alpha(j), \text{ on } \mathbb{L}_Q,$$

for all $j \in \mathbb{Z}$. Therefore, these two group B^*-dynamical systems $\left(\mathfrak{B}, \mathbb{L}_Q, \beta\right)$ and $\left(\mathbb{Z}, \mathbb{L}_Q, \alpha\right)$ are equivalent. ∎

The above theorem shows that the group action α of $\mathbb{Z}$ acting on $\mathbb{L}_Q$ affects the free probability on the semicircular filterization $\mathbb{L}_Q$ just like the group action β of $\mathfrak{B}$.

Observation 8.1 All same results of Sect. 7.3 are re-obtained, if we replace $\beta_e^n \in \mathfrak{B}$ to $\alpha(en) \in \alpha(\mathbb{Z})$, for all $e \in \{\pm\}$ and $n \in \mathbb{N}_0$, where α and β are in the sense of (8.6) and (8.7), respectively. The proof is done by the equivalence (8.8). □

9 Banach-Space Operators on $\mathbb{L}_Q$ Preserving Free Probability

Let (A, ψ) be a fixed C^*-probability space containing the family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections, and let Q be the C^*-subalgebra $C^*(\mathbf{Q})$ of A, and let $\mathbb{L}_Q$ be the corresponding semicircular filterization of Q. Also, let $\mathfrak{B}$ be the integer-shift group in the automorphism group $Aut(\mathbb{L}_Q)$, which is group-isomorphic to $\mathbb{Z} = (\mathbb{Z}, +)$. Let X be an arbitrary *Banach space*, and let $B(X)$ be the operator space consisting of all bounded linear transformations, the Banach-space operators, on X (e.g., [9]). Let I_X be the identity operator on X,

$$I_X(x) = x, \text{ for all } x \in X.$$

A Banach-space operator $T \in B(X)$ is said to be *invertible on* X, if there exists a unique Banach-space operator $T^{-1} \in B(X)$, such that

$$TT^{-1} = I_X = T^{-1}T \text{ on } X.$$

The operator T^{-1} is called the *inverse* (*operator*) *of* T on the Banach space X. Now, for our semicircular filterization $\mathbb{L}_Q$, we consider the operator space $B(\mathbb{L}_Q)$ by regarding the Banach $*$-algebra $\mathbb{L}_Q$ as a Banach space. Then the integer-shift group

$\mathfrak{B}$ is contained in $B(\mathbb{L}_Q)$, as a subset, because all integer-shifts of $\mathfrak{B}$ are bounded (multiplicative) linear transformations on $\mathbb{L}_Q$. Equivalently, every group element β_e^n of $\mathfrak{B}$ is a Banach-space operator on $\mathbb{L}_Q$, for all $e \in \{\pm\}$, $n \in \mathbb{N}_0$.

Theorem 9.1 *Let $\mathbb{L}_Q$ be our semicircular filterization of Q. There exists an invertible Banach-space operator T in the operator space $B(\mathbb{L}_Q)$ such that T preserves the free probability on $\mathbb{L}_Q$.*

Proof The proof of this theorem is done by construction. Indeed, let β_e^n be the n-(e)-shift in the integer-shift group $\mathfrak{B}$ contained in the operator space $B(\mathbb{L}_Q)$, which is a Banach-space operator on $\mathbb{L}_Q$, for $e \in \{\pm\}$ and $n \in \mathbb{N}_0$. Then this Banach-space operator β_e^n is invertible with its inverse $\left(\beta_e^n\right)^{-1}$,

$$\left(\beta_e^n\right)^{-1} = \beta_{-e}^n \in \mathfrak{B} \subset B(\mathbb{L}_Q),$$

where β_{-e}^n is the group-inverse of β_e^n in $\mathfrak{B}$, satisfying

$$\left(\beta_e^n\right)\left(\beta_e^n\right)^{-1} = 1_{\mathbb{L}_Q} = \left(\beta_e^n\right)^{-1}\left(\beta_e^n\right),$$

on $\mathbb{L}_Q$, where $1_{\mathbb{L}_Q}$ is the identity operator on the Banach space $\mathbb{L}_Q$ (which is also the group-identity of $\mathfrak{B}$).

Moreover, this operator β_e^n preserves the free-distributional data of all free reduced words in the free semicircular family $\mathcal{S}$ generating the Banach space $\mathbb{L}_Q$, by (7.3.8), (7.3.9) and (8.2). Therefore, by (6.5), (6.6) and (6.7), the free probability on $\mathbb{L}_Q$ is preserved by β_e^n. ■

The above theorem illustrates that there are sufficiently many invertible Banach-space operators on $\mathbb{L}_Q$ preserving free probability on $\mathbb{L}_Q$.

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Explicit Expressions Related to Degenerate Cauchy Numbers and Their Generating Function

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Abstract In the paper, by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, the authors establish an explicit expression for degenerate Cauchy numbers and find explicit, meaningful, and significant expressions for coefficients in a family of nonlinear differential equations for the generating function of degenerate Cauchy numbers.

Keywords Explicit expression · Degenerate Cauchy number · Coefficient · Nonlinear differential equation · Generating function · Bell polynomial of the second kind · Faà di Bruno formula

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1 Motivations and Main Results

It is well known [4, 7, 8, 35, 39] that the Cauchy numbers of the first kind C_n can be generated by

$$\frac{t}{\ln(1+t)}=\sum_{n=0}^{\infty}C_n\frac{t^n}{n!}.$$

In [2], degenerate Cauchy numbers $C_n(\lambda)$ were defined by

$$\frac{\lambda\big[e^{[(1+t)^\lambda-1]/\lambda}-1\big]}{(1+t)^\lambda-1}=\sum_{n=0}^{\infty}C_n(\lambda)\frac{t^n}{n!}.$$

Because

$$\lim_{\lambda\to0}\frac{(1+t)^\lambda-1}{\lambda}=\ln(1+t)\quad\text{or}\quad\lim_{\lambda\to0}e^{[(1+t)^\lambda-1]/\lambda}=1+t,$$

it follows that

$$\lim_{\lambda\to0}C_n(\lambda)=C_n,\quad n\ge0. \tag{1}$$

In [2, Theorem 2.1], it was established that the family of nonlinear differential equations

$$(1+t)^n\big[(1+t)^\lambda-1\big]^nF_\lambda^{(n)}(t)=F_\lambda(t)\sum_{i=1}^{2n}a_i(n,\lambda)(1+t)^{i\lambda}+\sum_{i=1}^{2n-1}b_i(n,\lambda)(1+t)^{i\lambda} \tag{2}$$

for $n\in\mathbb{N}$ has the same solution

$$F_\lambda(t)=\frac{e^{[(1+t)^\lambda-1]/\lambda}-1}{(1+t)^\lambda-1}, \tag{3}$$

where $a_i(n,\lambda)$ for $1\le i\le 2n$ and $b_i(n,\lambda)$ for $1\le i\le 2n-1$ are uniquely determined by

$$\begin{aligned}
a_1(n,\lambda)&=-\frac{1}{\lambda}\langle n-1-\lambda\rangle_{n+1},\quad a_2(n,\lambda)=\langle n-1-2\lambda\rangle_{n-1}\\
&\quad-\frac{1}{\lambda}\sum_{i=0}^{n-2}[\lambda-(\lambda+1)(n-i)]\langle n-i-2-\lambda\rangle_{n-i}\langle n-1-2\lambda\rangle_i,\\
a_i(n,\lambda)&=[(i-1)\lambda-(\lambda+1)n]a_i(n-1,\lambda)\\
&\quad+a_{i-2}(n-1,\lambda)+(n-1-i\lambda)a_i(n-1,\lambda),\quad 3\le i\le 2n-2,\\
a_{2n-1}(n,\lambda)&=\frac{1}{2}n[(\lambda-1)(n-1)-2(\lambda+1)],
\end{aligned}$$

$$
\begin{aligned}
a_{2n}(n,\lambda)&=1, b_1(n,\lambda)=\langle n-1-\lambda\rangle_{n-1},\\
b_i(n,\lambda)&=[(i-1)\lambda-(\lambda+1)(n-1)]b_{i-1}(n-1,\lambda)\\
&\qquad + a_{i-1}(n-1,\lambda)+(n-1-i\lambda)b_i(n-1,\lambda),\quad 2\le i\le 2n-3,\\
b_{2n-2}(n,\lambda)&=(\lambda-1)\binom{n-1}{2}-2(n-1)-\lambda,\quad b_{2n-1}(n,\lambda)=1
\end{aligned}
$$

in terms of the falling factorials

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge 1;\\ 1, & n=0.\end{cases}$$

It is clear that the generating function $F_\lambda(t)$ defined by (3) satisfies

$$\lim_{\lambda\to 0}[\lambda F_\lambda(t)]=\frac{t}{\ln(1+t)}.$$

It is obvious that

(1) the above expressions for $a_i(n,\lambda)$ and $b_i(n,\lambda)$ are recursive and can not be computed easily;
(2) the original proof of [2, Theorem 2.1] is inductive, recursive, and long;
(3) there was no any application given in [2].

In this paper, by virtue of the Faà di Bruno formula (9) and two identities (10) and (11) for the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, we will establish an explicit expression (4) for degenerate Cauchy numbers $C_n(\lambda)$ and find explicit, meaningful, and significant expressions (7) and (8) for coefficients $a_i(n,\lambda)$ and $b_i(n,\lambda)$ in the family of nonlinear differential equations (2) related to the generating function $F_\lambda(t)$ of degenerate Cauchy numbers $C_n(\lambda)$.

Our main results can be stated as the following theorems.

Theorem 1 *For $n\ge 0$, degenerate Cauchy numbers $C_n(\lambda)$ and the Cauchy numbers C_n can be explicitly and respectively computed by*

$$C_n(\lambda)=\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!\lambda^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\ell\lambda\rangle_n \tag{4}$$

and

$$C_n=\sum_{k=0}^{n}\frac{s(n,k)}{k+1}. \tag{5}$$

Theorem 2 *For $n \in \mathbb{N}$, the generating function $F_\lambda(t)$ and its derivatives of degenerate Cauchy numbers $C_n(\lambda)$ satisfy*

$$F_\lambda^{(n)}(t) = \frac{F_\lambda(t)}{(1+t)^n\big[1-(1+t)^\lambda\big]^n} \sum_{i=1}^{2n} \alpha_i(n,\lambda)(1+t)^{i\lambda} + \sum_{i=1}^{2n-1} \beta_i(n,\lambda)(1+t)^{i\lambda} \tag{6}$$

with

$$\alpha_i(n,\lambda) = \sum_{\substack{k+m=i\\ 1\le k\le n\\ 0\le m\le n}} (-1)^m A_k(n,\lambda) \sum_{\ell=0}^{\min\{n-m,k\}} \frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m} \tag{7}$$

for $1 \le i \le 2n$ and

$$\beta_i(n,\lambda) = \sum_{\substack{k+m=i\\ 1\le k\le n\\ 0\le m\le n-1}} (-1)^{m+1} A_k(n,\lambda) \sum_{\ell=0}^{\min\{k-1,n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell-1}{m} \tag{8}$$

for $1 \le i \le 2n-1$, where

$$A_k(n,\lambda) = \frac{(-1)^k}{\lambda^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell\lambda - q).$$

2 Lemmas

In order to obtain our main results, we need the following lemmas.

Lemma 1 ([1, pp. 134 and 139]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n-k+1\\ \ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n-k+1} i\ell_i=n\\ \sum_{i=1}^{n-k+1} \ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!} \prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big). \tag{9}$$

Lemma 2 ([1, p. 135]). *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \,\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \tag{10}$$

where a and b are any complex numbers.

Lemma 3 *For $n \ge k \ge 0$ and $\lambda, \alpha \in \mathbb{C}$, we have*

$$\mathrm{B}_{n,k}\left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\right) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda) \tag{11}$$

or, equivalently,

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\langle\alpha\ell\rangle_n. \tag{12}$$

Proof This explicit formula (11) was first established in [25, Remark 1] and then was applied in [16, Sect. 2], [17, First proof of Theorem 2], [19, Lemma 2.2], [22, Remark 6.1], [23, Lemma 4], and [32, Lemma 2.6]. The formula (12) and the equivalence were presented in [33, Theorems 2.1 and 4.1].

3 Proofs of Theorems 1 and 2

We are now in a position to prove our main results.

Proof of Theorem 1 For $n \ge 0$, applying $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$ and $f(u) = \frac{e^u-1}{u}$ to (9) and making use of (10) and (11) in sequence arrive at

$$\begin{aligned}
\frac{\mathrm{d}^n[\lambda F_\lambda(t)]}{\mathrm{d}\,t^n} &= \sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\frac{e^u-1}{u}\right)\mathrm{B}_{n,k}\left(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda},\right.\\
&\qquad\left.\dots, \frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda}\right)\\
&= \sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\sum_{\ell=1}^{\infty}\frac{u^{\ell-1}}{\ell!}\right)(1+t)^{k\lambda-n}\,\mathrm{B}_{n,k}(1, \lambda-1, \dots, (\lambda-1)\cdots[\lambda-(n-k)])\\
&= \sum_{k=0}^{n}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\left(\sum_{\ell=0}^{\infty}\frac{u^{\ell}}{(\ell+1)!}\right)(1+t)^{k\lambda-n}\\
&\quad\times \mathrm{B}_{n,k}\left(1, \lambda\left(1-\frac{1}{\lambda}\right), \dots, \lambda^{n-k}\left(1-\frac{1}{\lambda}\right)\cdots\left(1-\frac{n-k}{\lambda}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}\Bigg[\sum_{\ell=0}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\bigg(\frac{u^{\ell}}{(\ell+1)!}\bigg)\Bigg](1+t)^{k\lambda-n}\\
&\quad\times \lambda^{n-k}\,\mathrm{B}_{n,k}\bigg(1, 1-\frac{1}{\lambda}, \dots, \bigg(1-\frac{1}{\lambda}\bigg)\cdots\bigg(1-\frac{n-k}{\lambda}\bigg)\bigg)\\
&= \sum_{k=0}^{n}\Bigg[\sum_{\ell=k}^{\infty}\frac{\ell!}{(\ell-k)!}\frac{u^{\ell-k}}{(\ell+1)!}\Bigg](1+t)^{k\lambda-n}\lambda^{n-k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}\Big(\ell-\frac{q}{\lambda}\Big)\\
&= \sum_{k=0}^{n}\Bigg[\sum_{\ell=k}^{\infty}\frac{1}{(\ell-k)!}\frac{u^{\ell-k}}{\ell+1}\Bigg](1+t)^{k\lambda-n}\frac{1}{\lambda^k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q)\\
&\to \sum_{k=0}^{n}\frac{1}{k+1}\frac{1}{\lambda^k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q)\\
&= \sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!\lambda^k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\ell\lambda\rangle_n
\end{aligned}
$$

as $t \to 0$ and, consequently, $u \to 0$. This implies that

$$
C_n(\lambda) = \lambda\lim_{t\to 0}F_{\lambda}^{(n)}(t) = \sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!\lambda^k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\ell\lambda\rangle_n.
$$

The explicit formula (4) is thus proved.

It is well known [1, Theorem A] that the Stirling numbers of the first kind $s(n,k)$ can be generated by

$$
\langle x\rangle_n = \sum_{k=0}^{n}s(n,k)x^k.
$$

Hence, by the L'Hôspital rule, we have

$$
\begin{aligned}
\lim_{\lambda\to 0}\frac{1}{\lambda^k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\ell\lambda\rangle_n &= \lim_{\lambda\to 0}\frac{1}{\lambda^k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{m=0}^{n}s(n,m)(\ell\lambda)^m\\
&= \frac{1}{k!}\lim_{\lambda\to 0}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\sum_{m=0}^{n}s(n,m)\langle m\rangle_k\ell^k(\ell\lambda)^{m-k}\\
&= \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}s(n,k)\langle k\rangle_k\ell^k = s(n,k)\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\ell^k = (-1)^k k!s(n,k).
\end{aligned}
$$

Combining this with (1) and (4) gives

$$
C_n = \lim_{\lambda\to 0}C_n(\lambda) = \sum_{k=0}^{n}\frac{(-1)^k}{(k+1)!}\lim_{\lambda\to 0}\frac{1}{\lambda^k}\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\langle\ell\lambda\rangle_n = \sum_{k=0}^{n}\frac{s(n,k)}{k+1}.
$$

The explicit formula (5) follows. The proof of Theorem 1 is complete. □

Proof of Theorem 2 For $n \in \mathbb{N}$, applying $u = h(t) = \frac{(1+t)^\lambda - 1}{\lambda}$ and $f(u) = \frac{e^u - 1}{u}$ to (9) and making use of (10) and (11) in sequence arrive at

$$\begin{aligned}
\frac{\mathrm{d}^n[\lambda F_\lambda(t)]}{\mathrm{d}\,t^n} &= \sum_{k=1}^{n} \frac{\mathrm{d}^k}{\mathrm{d}\,u^k}\biggl(\frac{e^u-1}{u}\biggr) \mathrm{B}_{n,k}\biggl(\frac{\lambda(1+t)^{\lambda-1}}{\lambda}, \frac{\lambda(\lambda-1)(1+t)^{\lambda-2}}{\lambda}, \\
&\quad \dots, \frac{\lambda(\lambda-1)\cdots[\lambda-(n-k)](1+t)^{\lambda-(n-k+1)}}{\lambda}\biggr) \\
&= \sum_{k=1}^{n}\biggl[\sum_{\ell=0}^{k}\binom{k}{\ell}(e^u-1)^{(\ell)}\biggl(\frac{1}{u}\biggr)^{(k-\ell)}\biggr](1+t)^{k\lambda-n} \\
&\quad \times \mathrm{B}_{n,k}(1, \lambda-1, \dots, (\lambda-1)\cdots[\lambda-(n-k)]) \\
&= \sum_{k=1}^{n}\biggl[\frac{(-1)^k k!(e^u-1)}{u^{k+1}} + \sum_{\ell=1}^{k}\binom{k}{\ell}e^u\frac{(-1)^{k-\ell}(k-\ell)!}{u^{k-\ell+1}}\biggr](1+t)^{k\lambda-n} \\
&\quad \times \mathrm{B}_{n,k}\biggl(1, \lambda\biggl(1-\frac{1}{\lambda}\biggr), \dots, \lambda^{n-k}\biggl(1-\frac{1}{\lambda}\biggr)\cdots\biggl(1-\frac{n-k}{\lambda}\biggr)\biggr) \\
&= \sum_{k=1}^{n}\biggl[\lambda F_\lambda(t)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{k-\ell}(k-\ell)!}{u^{k-\ell}} + \sum_{\ell=1}^{k}\binom{k}{\ell}\frac{(-1)^{k-\ell}(k-\ell)!}{u^{k-\ell+1}}\biggr](1+t)^{k\lambda-n} \\
&\quad \times \lambda^{n-k}\,\mathrm{B}_{n,k}\biggl(1, 1-\frac{1}{\lambda}, \dots, \biggl(1-\frac{1}{\lambda}\biggr)\cdots\biggl(1-\frac{n-k}{\lambda}\biggr)\biggr) \\
&= \sum_{k=1}^{n}\biggl[\lambda F_\lambda(t)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^{k-\ell}\lambda^{k-\ell}(k-\ell)!}{[(1+t)^\lambda-1]^{k-\ell}} + \sum_{\ell=1}^{k}\binom{k}{\ell}\frac{(-1)^{k-\ell}\lambda^{k-\ell+1}(k-\ell)!}{[(1+t)^\lambda-1]^{k-\ell+1}}\biggr] \\
&\quad \times\ (1+t)^{k\lambda-n}\lambda^{n-k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}\biggl(\ell-\frac{q}{\lambda}\biggr) \\
&= \frac{1}{(1+t)^n}\sum_{k=1}^{n}\biggl[\lambda F_\lambda(t)\sum_{\ell=0}^{k}\binom{k}{\ell}\frac{(-1)^\ell\lambda^\ell \ell!}{[(1+t)^\lambda-1]^\ell} + \sum_{\ell=0}^{k-1}\binom{k}{\ell}\frac{(-1)^\ell\lambda^{\ell+1}\ell!}{[(1+t)^\lambda-1]^{\ell+1}}\biggr] \\
&\quad \times \frac{(1+t)^{k\lambda}}{\lambda^k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q).
\end{aligned}$$

Accordingly, we have

$$\begin{aligned}
&(1+t)^n\bigl[(1+t)^\lambda-1\bigr]^n F_\lambda^{(n)}(t) = \sum_{k=1}^{n}\biggl[\frac{(1+t)^{k\lambda}}{\lambda^k}\frac{(-1)^k}{k!}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q)\biggr] \\
&\quad \times \biggl[F_\lambda(t)\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^\ell\lambda^\ell\ell![(1+t)^\lambda-1]^{n-\ell} + \sum_{\ell=0}^{k-1}\binom{k}{\ell}(-1)^\ell\lambda^\ell\ell![(1+t)^\lambda-1]^{n-\ell-1}\biggr] \\
&= \sum_{k=1}^{n}\biggl[\frac{(-1)^k}{\lambda^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q)\biggr](1+t)^{k\lambda} \\
&\quad \times \biggl[F_\lambda(t)\sum_{\ell=0}^{k}\frac{(-1)^\ell\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell}\binom{n-\ell}{m}(1+t)^{m\lambda}(-1)^{n-\ell-m}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=0}^{k-1}\frac{(-1)^\ell\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell-1}\binom{n-\ell-1}{m}(1+t)^{m\lambda}(-1)^{n-\ell-1-m}\Bigg]\\
&=(-1)^n\sum_{k=1}^{n}\Bigg[\frac{(-1)^k}{\lambda^k}\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell\lambda-q)\Bigg](1+t)^{k\lambda}\\
&\quad\times\Bigg[F_\lambda(t)\sum_{\ell=0}^{k}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell}(-1)^m\binom{n-\ell}{m}(1+t)^{m\lambda}\\
&\quad+\sum_{\ell=0}^{k-1}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell-1}(-1)^{m+1}\binom{n-\ell-1}{m}(1+t)^{m\lambda}\Bigg]\\
&=(-1)^nF_\lambda(t)\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell}(-1)^m\binom{n-\ell}{m}(1+t)^{(k+m)\lambda}\\
&\quad+(-1)^n\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k-1}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell-1}(-1)^{m+1}\binom{n-\ell-1}{m}(1+t)^{(k+m)\lambda}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell}(-1)^m\binom{n-\ell}{m}(1+t)^{(k+m)\lambda}\\
&\quad=\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n}(-1)^m\binom{n-\ell}{m}(1+t)^{(k+m)\lambda}\\
&\quad=\sum_{k=1}^{n}A_k(n,\lambda)(1+t)^{k\lambda}\sum_{m=0}^{n}(-1)^m\Bigg[\sum_{\ell=0}^{k}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m}\Bigg](1+t)^{m\lambda}\\
&\quad=\sum_{k=1}^{n}A_k(n,\lambda)(1+t)^{k\lambda}\sum_{m=0}^{n}(-1)^m\Bigg[\sum_{\ell=0}^{\min\{n-m,k\}}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m}\Bigg](1+t)^{m\lambda}\\
&\quad=\sum_{k=1}^{n}A_k(n,\lambda)\sum_{m=0}^{n}(-1)^m\Bigg[\sum_{\ell=0}^{\min\{n-m,k\}}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m}\Bigg](1+t)^{(k+m)\lambda}\\
&\quad=\sum_{i=1}^{2n}\Bigg[\sum_{\substack{k+m=i\\1\le k\le n\\0\le m\le n}}(-1)^mA_k(n,\lambda)\sum_{\ell=0}^{\min\{n-m,k\}}\frac{\lambda^\ell}{(k-\ell)!}\binom{n-\ell}{m}\Bigg](1+t)^{i\lambda}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k-1}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-\ell-1}(-1)^{m+1}\binom{n-\ell-1}{m}(1+t)^{(k+m)\lambda}\\
&\quad=\sum_{k=1}^{n}A_k(n,\lambda)\sum_{\ell=0}^{k-1}\frac{\lambda^\ell}{(k-\ell)!}\sum_{m=0}^{n-1}(-1)^{m+1}\binom{n-\ell-1}{m}(1+t)^{(k+m)\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} A_k(n,\lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[\sum_{\ell=0}^{k-1} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{k=1}^{n} A_k(n,\lambda) \sum_{m=0}^{n-1} (-1)^{m+1} \left[\sum_{\ell=0}^{\min\{k-1,n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{(k+m)\lambda} \\
&= \sum_{i=1}^{2n-1} \left[\sum_{\substack{k+m=i \\ 1\le k\le n \\ 0\le m\le n-1}} (-1)^{m+1} A_k(n,\lambda) \sum_{\ell=0}^{\min\{k-1,n-m-1\}} \frac{\lambda^\ell}{(k-\ell)!} \binom{n-\ell-1}{m} \right] (1+t)^{i\lambda},
\end{aligned}
$$

where an empty sum is understood to be 0 and $\binom{p}{q} = 0$ for $q > p \ge 0$, the Eq. (6) and the formulas (7) and 8 are thus proved. The proof of Theorem 2 is complete. $\square$

4 Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 1 Comparing (6) with (2) reveals that

$$
a_i(n,\lambda) = (-1)^n \alpha_i(n,\lambda) \quad \text{and} \quad b_i(n,\lambda) = (-1)^n \beta_i(n,\lambda).
$$

Remark 2 It is easy to see that explicit expressions (7) and (8) for $\alpha_i(n,\lambda)$ and $\beta_i(n,\lambda)$ are more meaningful and more significant than those in [2, Theorem 2.1] for $a_i(n,\lambda)$ and $b_i(n,\lambda)$ mentioned above.

Remark 3 The formula (5) was derived in [3] and mentioned in [4, 8].

Remark 4 Per requests of anonymous referees, the preprint [17] is split into and simplified as two formally published papers [18, 29].

Remark 5 The motivations in the papers [5, 6, 9–16, 20, 21, 24–27, 30–32, 34, 36–38, 40–42] are same as the one in this paper.

Remark 6 This paper is a slightly revised version of the preprint [28].

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Statistical Deferred Riesz Summability Mean and Associated Approximation Theorems for Trigonometric Functions

M. Patro, S. K. Paikray, B. B. Jena and Hemen Dutta

Abstract The notion of deferred weighted statistical convergence was introduced by Srivastava et al. (Math Methods Appl Sci 41:671–683, 2018) [20]. In the present investigation, we have used (presumably new) the notion of approximation via statistical deferred weighted (Riesz) summability mean for trigonometrical periodic functions defined over a Banach space $C_{2\pi}(\mathbb{R})$ and accordingly established a new approximation theorem (Korovkin-type). Furthermore, we have introduced the idea of the rate of statistical deferred weighted summability and also established another result for the same set of functions by using the modulus of continuity. Finally, We have also considered a number of fascinating special cases and examples in relevance to our results and definitions provided in this paper.

Keywords Banach space · Positive linear operators · Statistical convergence · Deferred weighted statistical convergence · Statistical deferred weighted summability · Rate of convergence · Periodic function · Korovkin-type approximation theorems

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1 Introduction, Preliminaries and Motivation

The theory of statistical convergence was initially studied by Fast [8] and Steinhaus [24]. Gradually, this theory became an active research area due basically to the cause that it is more extensive than that of usual convergence. Furthermore, such theory has been fairly discussed in the study in the areas of (for example) Number Theory, Fourier Analysis and Approximation Theory. For details, see the current research works [3, 6, 9, 10, 12, 14, 17–21], etc.

Let $K \subseteq \mathbb{N}$ (set of Naturals) and let

$$K_n = \{k : k \leqq n \quad \text{and} \quad k \in K\}.$$

The natural (asymptotic) density of K is given by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

provided the limit exists.

Recall that, a sequence (x_n) is statistically convergent (or stat-convergent) to L if, for every $\epsilon > 0$,

$$K_\epsilon = \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\}$$

has natural density zero (see [8, 24]). That, for each $\epsilon > 0$,

$$d(K_\epsilon) = \lim_{n\to\infty} \frac{|K_\epsilon|}{n} = 0.$$

Here, we write

$$\text{stat} \lim_{n\to\infty} x_n = L.$$

Consider the following example:

Example 1 Let us consider a sequence $x = (x_n)$ by

$$x_n = \begin{cases} \frac{1}{2} & (n = m^2, m \in \mathbb{N}) \\ \frac{n^3}{n^3+1} & (\text{otherwise}). \end{cases}$$

Observe that, the sequence (x_n) is statistically convergent to 1 but it is not usually classical convergent. Also, every convergent sequence is statistically convergent in the sense that, the subset to be discarded has natural density zero. Thus, statistical convergence is more general than usual convergence.

The basic concept of weighted statistical convergence was initially studied by Karakaya and Chishti [11]. Gradually, it was improved by Mursaleen et al. (see [16]) and accordingly some important approximation results were proved. For more results in this direction one may refers to the following works (see [4, 20]).

Suppose that (p_k) be a sequence of nonnegative numbers such that $P_n = \sum_{k=0}^{n} p_k$ with $p_0 > 0$ $(n \to \infty)$. Setting

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k x_k \quad (n = 0, 1, 2, \ldots),$$

we say (x_n) is weighted statistically convergent to a number L if, for each $\epsilon > 0$, the following set:

$$\{k : k \leqq P_n \quad \text{and} \quad p_k|x_k - L| \geqq \epsilon\}$$

has zero weighted density (see [7]). This means that, for each $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{P_n} |\{k : k \leqq P_n \quad \text{and} \quad p_k|x_k - L| \geqq \epsilon\}| = 0.$$

Similarly a sequence (x_n) is said to be statistical weighted summable to L if, for each $\epsilon > 0$, the following set:

$$\{k : k \leqq n \quad \text{and} \quad |t_k - L| \geqq \epsilon\}$$

has zero weighted density, that is,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad |t_k - L| \geqq \epsilon\}| = 0.$$

Motivated essentially by the above-mentioned works, here we wish to present the (presumably new) notion of statistical deferred weighted summability to establish certain new approximation results.

Let (a_n) and (b_n) be sequences of non-negative integers and we recall the regularity conditions of the deferred weighted mean due to Agnew [1] as $a_n < b_n (n \in \mathbb{N})$ and $\lim\limits_{n\to\infty} b_n = \infty$.

Furthermore, let (p_n) and (q_n) be the sequences of real numbers (non-negative) such that

$$P_n = \sum_{m=a_n+1}^{b_n} p_m \quad \text{and} \quad Q_n = \sum_{m=a_n+1}^{b_n} q_m.$$

Let

$$R_{a_n+1}^{b_n} = \sum_{v=a_n+1}^{b_n} p_v q_v,$$

setting

$$\sigma_n = \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m x_m,$$

it is said that (x_n) is summable to L under the deferred weighted (Riesz) summability mean defined by the associated sequences (p_n) and (q_n) or, briefly, summable $D_a^b(\overline{N}, p, q)$, if

$$\lim_{n\to\infty} \sigma_n = L.$$

Next, for our proposed method consider a definition as follows.

Definition 1 A sequence (x_n) is statistical deferred weighted summable (or stat$D(\overline{N})$-summable) to L if, for each $\epsilon > 0$,

$$\{m : m \leqq n \quad \text{and} \quad |\sigma_m - L| \geqq \epsilon\}$$

has deferred weighted density zero, that means,

$$\lim_{n\to\infty} \frac{1}{n} |\{m : m \leqq n \quad \text{and} \quad |\sigma_m - L| \geqq \epsilon\}| = 0.$$

Here, we write

$$\text{stat}D(\overline{N}) \lim x_n = L.$$

Remark 1 If, $q_n = 1(\forall\ n)$, then $D_a^b(\overline{N}, p, q)$ mean is same as $D_a^b(\overline{N}, p)$ mean (see [5]) and if $a_n = 0$, $b_n = n(\forall\ n)$ and $q_n = 1$, then $D_a^b(\overline{N}, p, q)$ mean becomes $(\overline{N}, p_n)$ mean (see [15]). Finally, if $a_n = 0$, $b_n = n(\forall\ n)$, $p_n = 1$ and $q_n = 1$, then $D_a^b(\overline{N}, p, q)$ mean is same as $(C, 1)$ mean (see [14]).

The following example illustrates that, a sequence (x_n) is statistical deferred weighted summable to L, but not deferred weighted statistical convergent to L.

Example 2 For $a_n = 2n$ and $b_n = 4n$, choose a sequence $x = (x_n)$ as,

$$x_n = \begin{cases} 0 & (n \text{ is odd}) \\ 1 & (n \text{ is even}). \end{cases} \tag{1.1}$$

Clearly, (x_n) is neither convergent nor deferred weighted statistically convergent. But, (x_n) is statistical deferred weighted summable to 1 with

$$p_n = 1 \quad \text{and} \quad q_n = 1.$$

2 A Korovkin-Type Approximation Theorem Based on Deferred Riesz Mean

In this section, by using the idea of deferred Riesz statistical summability mean for periodic functions $1, \cos x, \sin x$ over $C_{2\pi}(\mathbb{R})$, we have proved an approximation theorem (Korovkin-type). Also, our theorem effectively extends most of the results established earlier. In this direction, one may refer to the recent works [7, 19–22].

Let $F(\mathbb{R})$ be the linear space of all functions (real-valued) f on the set of real numbers $\mathbb{R}$ and $C(\mathbb{R})$ be the space of all continuous functions defined on $\mathbb{R}$. Recall that, $C(\mathbb{R})$ is a Banach space with norm

$$\|f\|_{\infty} = \sup_{x\in\mathbb{R}} |f(x)| \quad (f \in C(\mathbb{R})).$$

Suppose $C_{2\pi}(\mathbb{R})$ be the space of all continuous 2π-periodic functions (real valued) defined over $\mathbb{R}$ and $\mathfrak{L} : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be a linear operator. That means

$$\mathfrak{L}(f; x) \geqq 0 \quad \text{for} \quad x \in \mathbb{R}.$$

Also $C_{2\pi}(\mathbb{R})$ is a Banach space and for $f \in C_{2\pi}(\mathbb{R})$, the norm of f is given by

$$\|f\|_{2\pi} = \sup_{x\in\mathbb{R}} |f(x)|.$$

Theorem 1 *Let $\mathfrak{L}_m (m \in \mathbb{N})$ be a sequence of linear operators (positive) from $C_{2\pi}(\mathbb{R})$ into itself and let $f \in C_{2\pi}(\mathbb{R})$. Then*

$$stat\, D(\overline{N}) \lim_{m\to\infty} \|\mathfrak{L}_m(f; x) - f(x)\|_{2\pi} = 0 \tag{2.1}$$

$$if \ and \ only \ if$$

$$stat\, D(\overline{N}) \lim_{m\to\infty} \|\mathfrak{L}_m(1; x) - 1\|_{2\pi} = 0, \tag{2.2}$$

$$stat\, D(\overline{N}) \lim_{m\to\infty} \|\mathfrak{L}_m(\cos x; x) - \cos x\|_{2\pi} = 0, \tag{2.3}$$

$$stat\, D(\overline{N}) \lim_{m\to\infty} \|\mathfrak{L}_m(\sin x; x) - \sin x\|_{2\pi} = 0. \tag{2.4}$$

Proof. Since the functions

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad \text{and} \quad f_2(x) = \sin x$$

are in $C_{2\pi}(\mathbb{R})$, the following implication:

$$(2.1) \Longrightarrow (2.2)-(2.4)$$

is trivial. Now, for the completion of the proof of our Theorem, we have to assume that the assentations (2.2)–(2.4) are true. Suppose $f \in C_{2\pi}(\mathbb{R})$ and let $I = [-\pi, \pi] \subset \mathbb{R}$. Then, there is a constant $k > 0$ such that

$$|f(x)| \leqq k \quad (\forall\, x \in I),$$

It, thus implies that,

$$|f(t) - f(x)| \leqq 2k \quad (t, x \in I). \tag{2.5}$$

Clearly, for a given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \epsilon \quad \text{whenever} \quad |t - x| < \delta \tag{2.6}$$

for all $t, x \in I$.

Also, f is bounded, so it follows that

$$|f(t) - f(x)| \leqq 2\|f\|_{2\pi} \quad (\forall\, t, x \in I). \tag{2.7}$$

From equation (2.6) and (2.7), we get

$$|f(t) - f(x)| < \epsilon + \frac{2\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\varphi(t) \quad (t \in (x - \delta, 2\pi + x - \delta]), \tag{2.8}$$

where

$$\varphi(t) = \sin^2\left(\frac{t - x}{2}\right).$$

Since $f \in C_{2\pi}(\mathbb{R})$ is a periodic function with period 2π, the inequality (2.8) satisfied for $t \in \mathbb{R}$.

Moreover, the operator $\mathfrak{L}_m(1; x)$ being linear and monotone, so the inequality in (2.8) follows that

$$\begin{aligned}|\mathfrak{L}_m(f; x) - f(x)| &\leqq (\epsilon + |f(x)|)|\mathfrak{L}_m(1; x) - 1| + \epsilon + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\{|\mathfrak{L}_m(1; x) - 1| \\ &\quad + |\cos x||\mathfrak{L}_m(\cos t; x) - \cos x| + |\sin x||\mathfrak{L}_m(\sin t; x) - \sin x|\}. \\ &\leqq \epsilon + \left(\epsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})}\right)\{|\mathfrak{L}_m(1; x) - 1| + |\mathfrak{L}_m(\cos t; x) \\ &\quad - \cos x| + |\mathfrak{L}_m(\sin t; x) - \sin x|\}.\end{aligned} \tag{2.9}$$

Next, taking $\sup_{x \in I}$, in both side of (2.9), we get

$$\begin{aligned}\|\mathfrak{L}_m(f; x) - f(x)\|_{2\pi} \leqq \epsilon &+ B\{\|\mathfrak{L}_m(1; x) - 1\|_{2\pi} + \|\mathfrak{L}_m(\cos t; x) - \cos x\|_{2\pi} \\ &+ \|\mathfrak{L}_m(\sin t; x) - \sin x\|_{2\pi}\},\end{aligned} \tag{2.10}$$

where

$$B = \left\{ \epsilon + |f(x)|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2(\frac{\delta}{2})} \right\}.$$

Now replacing $\mathfrak{L}_m(f;x)$ by

$$\frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(f;x)$$

and then by $\Psi_m(f;x)$ in (2.10), we have for a given $r > 0$, we choose $\epsilon' > 0$, such that $0 < \epsilon' < r$. Then, by setting

$$\Psi_m(x,r) = \left| \left\{ m : m \leqq n \quad \text{and} \quad \left| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(f;x) - f(x) \right| \geqq r \right\} \right|$$

and

$$\Psi_{0,m}(x,r) = \left| \left\{ m : m \leqq n \quad \text{and} \quad \left| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(1;x) - 1 \right| \geqq \frac{r-\epsilon'}{3B} \right\} \right|,$$

$$\Psi_{1,m}(x,r) = \left| \left\{ m : m \leqq n \quad \text{and} \quad \left| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(\cos t;x) - \cos x \right| \geqq \frac{r-\epsilon'}{3B} \right\} \right|,$$

$$\Psi_{2,n}(x,r) = \left| \left\{ m : m \leqq n \quad \text{and} \quad \left| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(\sin t;x) - \sin t \right| \geqq \frac{r-\epsilon'}{3B} \right\} \right|,$$

we clearly find from (2.10) that

$$\Psi_m(x,r) \leqq \sum_{i=0}^{2} \Psi_{i,m}(x,r).$$

Thus, we get

$$\frac{\|\Psi_m(x,r)\|_{2\pi}}{n} \leqq \sum_{i=0}^{2} \frac{\|\Psi_{i,m}(x,r)\|_{2\pi}}{n}. \tag{2.11}$$

Finally, under the above assumption for the implication in (2.2)–(2.4) and also by Definition 1, the right-hand side of (2.11) tend to zero as $n \to \infty$. It clearly follows that,

$$\mathrm{stat}D(\overline{N}) \lim_{n\to\infty} \|\mathfrak{L}_n(f;x) - f(x)\|_{2\pi} = 0.$$

Therefore, the implication (2.1) is true. Which completes the proof of Theorem 1. □

Next, to recall the *Fejér convolution operators*, we consider the Fourier series of f at $t = x$ of the form,

$$f_m(x) = \frac{a_0}{2} + \sum_{m=0}^{\infty}(a_m \cos mx + b_m \sin mx).$$

Let the nth partial sum of the Fourier series of $f_m(x)$ be

$$S_n(f; x) = \frac{a_0}{2} + \sum_{m=0}^{n}(a_m \cos mx + b_m \sin mx) \quad (\forall\, n \in \mathbb{N})$$

and we write by Cesàro mean of $f_m(x)$,

$$\mathfrak{F}_n(f; x) = \frac{1}{n+1}\sum_{m=0}^{n} S_m(f; x).$$

Further, by simple calculation, we obtain:

$$\begin{aligned}\mathfrak{F}_n(f; x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{n+1}\sum_{m=0}^{n}\frac{\sin^2\left[\frac{(n+1)(x-t)}{2}\right]}{\sin^2\left[\frac{(x-t)}{2}\right]}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\phi_n(x-t)dt,\end{aligned}$$

where

$$\phi_n(x) = \begin{cases} \dfrac{\sin^2\left[\frac{(n+1)(x-t)}{2}\right]}{(n+1)\sin^2\left[\frac{(x-t)}{2}\right]} & (\text{x is an even multiple of } \pi) \\ \\ n+1 & (\text{if x is not an even multiple of } \pi). \end{cases}$$

Note that, the sequence $\{\phi_n(x) : n \in \mathbb{N}\}$ is the *Fejér kernel* and the operators $\mathfrak{F}_n(f; x)$ are the *Fejér convolution operators*.

Furthermore, consider the operator $x(1 + xD)$, where D is a differential operators. This operator was earlier used by Al-Salam [2] (Also, see [23, 25]).

Now for the validity of the operators $\mathfrak{L}_m(f; x)$ for our Theorem 1, we present the following example.

Example 3 Let $\mathfrak{L}_m : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be defined by,

$$\mathfrak{L}_m(f; x) = [1 + x_m]x(1 + xD)\mathfrak{F}_m(f) \quad (f \in C_{2\pi}(\mathbb{R}), \tag{2.12}$$

where (x_m) is the same sequence as defined in Example 2. Then, we obtain

$$\begin{aligned}\mathfrak{L}_m(1; x) &= [1 + x_m]x(1 + xD)1 = x,\\ \mathfrak{L}_m(\cos t; x) &= [1 + x_m]x(1 + xD)\frac{m-1}{m}\cos x\\ &= [1 + x_m]\frac{m-1}{m}(x\cos x - x^2\sin x)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{L}_m(\sin t; x) &= [1 + x_m]x(1 + xD)\frac{m-1}{m}\sin x\\ &= [1 + x_m]\frac{m-1}{m}(x\sin x + x^2\cos x).\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}&\mathrm{stat}D(\overline{N})\lim_{m\to\infty}\|\mathfrak{L}_m(1; x) - 1\|_{2\pi} = 0,\\ &\mathrm{stat}D(\overline{N})\lim_{m\to\infty}\|\mathfrak{L}_m(\cos x; x) - \cos x\|_{2\pi} = 0,\\ &\mathrm{stat}D(\overline{N})\lim_{m\to\infty}\|\mathfrak{L}_m(\sin x; x) - \sin x\|_{2\pi} = 0.\end{aligned}$$

It now implies, the operators $\mathfrak{L}_m(f; x)$ fairly satisfy the conditions (2.2)–(2.4). Hence, by Theorem 1, we certainly have

$$\mathrm{stat}D(\overline{N})\lim_{m\to\infty}\|\mathfrak{L}_m(f; x) - f\|_{2\pi} = 0.$$

However, since (x_n) is not deferred weighted statistically convergent, so the result of Srivastava et al. ([20], p. 5, Theorem 1) is not true for our operators defined by (2.12). Moreover, since (x_n) is statistical deferred weighted summable, therefore we conclude that our Theorem 1 works for the same operators.

3 Rate of the Statistical Deferred Weighted (Riesz) Summability

In the present section of our investigation under the consideration of the modulus of continuity, we study the rate of the statistical deferred weighted summability for a sequence of linear operators (positive) defined over $C_{2\pi}(\mathbb{R})$.

Definition 2 Let (u_n) be a positive non-increasing sequence. A sequence $x = (x_n)$ is said to be statistical deferred weighted summable to a number L with rate $o(u_n)$ if, for each $\epsilon > 0$,

$$\lim_{n\to\infty}\frac{1}{u_n n}\left|\left\{m : m \leqq n \text{ and } |\sigma_m - L| \geqq \epsilon\right\}\right| = 0.$$

Here, in this case, we may write

$$x_n - L = \text{stat}D(\overline{N}) - o(u_n).$$

We now state and prove a Lemma as follows.

Lemma 1 *Suppose (u_n) and (v_n) be two non-increasing positive sequences and let $x = (x_m)$ and $y = (y_m)$ be two sequences such that*

$$x_m - L_1 = statD(\overline{N}) - o(u_n)$$

and

$$y_m - L_2 = statD(\overline{N}) - o(v_n).$$

Then each of the following assertions hold true:

(i) $(x_m + y_m) - (L_1 + L_2) = statD(\overline{N}) - o(w_n)$*;*
(ii) $(x_m - L_1)(y_m - L_2) = statD(\overline{N}) - o(u_n v_n)$*;*
(iii) $\beta(x_m - L_1) = statD(\overline{N}) - o(u_n)$ *(for any scalar* β*);*
(iv) $\sqrt{|x_m - L_1|} = statD(\overline{N}) - o(u_n)$*,*
where $w_n = \max\{u_n, v_n\}$.

Proof. To prove the assertion (i) of Lemma 1, we consider the following sets for $\epsilon > 0$ and $x \in [0, 2\pi]$:

$$A_n(x;\epsilon) = \left|\left\{m : m \leqq n \text{ and } |(\sigma_m(x) + \sigma_m(y)) - (L_1 + L_2)| \geqq \epsilon\right\}\right|,$$
$$A_{0;n}(x;\epsilon) = \left|\left\{m : m \leqq n \text{ and } |\sigma_m(x) - L_1| \geqq \frac{\epsilon}{2}\right\}\right|$$

and

$$A_{1,n}(x;\epsilon) = \left|\left\{m : m \leqq n \text{ and } |\sigma_m(y) - L_2| \geqq \frac{\epsilon}{2}\right\}\right|.$$

Clearly, we have

$$A_n(x;\epsilon) \subseteq A_{0,n}(x;\epsilon) \cup A_{1n}(x;\epsilon).$$

Moreover, since

$$w_n = \max\{u_n, v_n\}, \tag{3.1}$$

by applying the assertion (2.1) of Theorem 1, we obtain

$$\frac{\|A_m(x;\epsilon)\|_{2\pi}}{w_n n} \leqq \frac{\|A_{0,n}(x;\epsilon)\|_{2\pi}}{u_n n} + \frac{\|A_{1,n}(x;\epsilon)\|_{\pi}}{v_n n}. \tag{3.2}$$

Also, by using the assertion (2.2)–(2.4) of Theorem 1, we obtain

$$\frac{\|A_n(x;\epsilon)\|_{2\pi}}{w_n n} = 0, \tag{3.3}$$

which proves the assertion (i) of Lemma 1.

The other assertion (ii) to (iv) of Lemma 1 being similar to (i), so in the similar lines, it can be proved. Which completes the proof Lemma 1. □

We recall here the modulus of continuity of a function $f \in C_{2\pi}(\mathbb{R})$ of the form

$$\omega(f,\delta) = \sup_{|t-x| \leqq \delta \ (t,x \in \mathbb{R})} |f(t) - f(x)| \quad (\delta > 0),$$

which implies

$$|f(t) - f(x)| \leqq \omega(f,\delta)\left(\frac{|x-t|}{\delta} + 1\right). \tag{3.4}$$

Theorem 2 *Let $\mathfrak{L}_m : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be sequences of linear operators (positive). Suppose that the following conditions:*

(i) $\|\mathfrak{L}_m(1;x) - 1\|_{2\pi} = statD(\overline{N}) - o(u_n)$;
(ii) $\omega(f,\lambda_m) = statD(\overline{N}) - o(v_n)$,
where

$$\lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2, x)} \quad \text{and} \quad \varphi(t) = \sin^2\left(\frac{t-x}{2}\right)$$

are satisfied. Then, for all $f \in C_{2\pi}(\mathbb{R})$, the following assertion holds true:

$$\|\mathfrak{L}_m(f;x) - f\|_{2\pi} = statD(\overline{N}) - o(w_n), \tag{3.5}$$

where (w_n) is given by (3.1).

Proof. Let $f \in C_{2\pi}(\mathbb{R})$ and $x \in [-\pi,\pi]$. Using (3.4), we have

$$\begin{aligned}
|\mathfrak{L}_m(f;x) - f(x)| &\leqq \mathfrak{L}_m(|f(t) - f(x)|;x) + |f(x)||\mathfrak{L}_m(1;x) - 1|,\\
&\leqq \mathfrak{L}_m\left(\frac{|x-t|}{\delta} + 1; x\right)\omega(f,\delta) + |f(x)||\mathfrak{L}_m(1;x) - 1|,\\
&\leqq \mathfrak{L}_m\left(1 + \frac{\pi^2}{\delta^2}\sin^2\left(\frac{t-x}{2}\right); x\right)\omega(f,\delta) + |f(x)||\mathfrak{L}_m(1;x) - 1|\\
&\leqq \left(\mathfrak{L}_m(1;x) + \frac{\pi^2}{\delta^2}\mathfrak{L}_m(\varphi(t);x)\right)\omega(f,\delta) + |f(x)||\mathfrak{L}_m(1;x) - 1|.
\end{aligned}$$

Now, putting $\delta = \lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2; x)}$, we get

$$\begin{aligned}
&\|\mathfrak{L}_m(f; x) - f(x)\|_{2\pi} \\
&\quad \leqq (1+\pi^2)\omega(f, \lambda_m) + \omega(f, \lambda_m)\|\mathfrak{L}_m(1; x) - 1\|_{2\pi} + \|f(x)\|_{2\pi}\|\mathfrak{L}_m(1; x) - 1\|_{2\pi} \\
&\quad \leqq \mu\{\omega(f, \lambda_m) + \omega(f, \lambda_m)\|\mathfrak{L}_m(1; x) - 1\|_{2\pi} + \|\mathfrak{L}_m(1; x) - 1\|_{2\pi}\},
\end{aligned}$$

where

$$\mu = \{\|f\|_{C_{2\pi}(\mathbb{R})}, 1+\pi^2\}.$$

This yields

$$\begin{aligned}
&\left\| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(f; x) - f(x) \right\|_{2\pi} \\
&\quad \leqq \mu \left\{ \omega(f, \lambda_m) + \omega(f, \lambda_m) \left\| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(f_0; x) - f_0(x) \right\|_{2\pi} \right. \\
&\quad \left. + \left\| \frac{1}{R_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m q_m T_m(f_0; x) - f_0(x) \right\|_{2\pi} \right\}. \qquad (3.6)
\end{aligned}$$

Finally, in view of the conditions (i) and (ii) of Theorem 2 in association with Lemma 1, this last inequality (3.6) motivates us to the assertion (3.5) of Theorem 2. Which completes the proof of Theorem 2. □

4 Conclusion

In the last section of our study, we present some further remarks and observations concerning the different results which we have provided here.

Remark 2 Let $(x_n)_{n\in\mathbb{N}}$ be a given sequence as in Example 2. As

$$\text{stat}D(\overline{N}) \lim_{n\to\infty} x_n \to 1 \text{ on } [0, 2\pi],$$

we have

$$\text{stat}D(\overline{N}) \lim_{n\to\infty} \|\mathfrak{L}_n(f_i; x) - f_i(x)\|_{2\pi} = 0 \quad (i = 0, 1, 2). \qquad (4.1)$$

Hence, by using Theorem 1, we have

$$\text{stat}D(\overline{N}) \lim_{n\to\infty} \|\mathfrak{L}_n(f; x) - f(x)\|_{2\pi} = 0, \quad (f \in C_{2\pi}\mathbb{R}), \qquad (4.2)$$

where

$$f_0(x) = 1, \quad f_1(x) = \cos x \;\text{ and }\; f_2(x) = \sin x.$$

However, since (x_n) is neither ordinarily convergent nor uniformly convergent. So, the usual Korovkin Theorem will not work here for the operators defined by (2.12). Hence, this application clearly leads that our Theorem 1 is a non-trivial extension of the classical Korovkin-type theorem (see [12, 13]).

Remark 3 Let $(x_n)_{n\in\mathbb{N}}$ be a given sequence as in Example 2. As

$$\text{stat}D(\overline{N}) \lim_{n\to\infty} x_n \to 1 \text{ on } [0, 2\pi],$$

so (4.1) holds. Now by applying (4.1) and our Theorem 1, condition (4.2) holds. However, since (x_n) does not weighted statistically convergent, so we can say that the result of Srivastava et al. ([20], p. 5, Theorem 1) does not hold true for our operator defined in (2.12). Thus, our Theorem 1 is also a non-trivial extension of [13, 20]. Moreover, based on the above results, it is infered here that our proposed method has truly worked for the operators defined in (2.12) and hence it is stronger than the earlier established classical and statistical versions of the Korovkin-type approximation theorem (see [12, 13, 20]).

Remark 4 If, we replace the conditions (i) and (ii) in our Theorem 2 by the condition,

$$|\mathfrak{L}_m(f_i; x) - f_i(x)|_{2\pi} = \text{stat}D(\overline{N}) - o(u_{n_i}) \;\; (i = 0, 1, 2), \tag{4.3}$$

then, since

$$\mathfrak{L}_m(\varphi^2; x) = |\mathfrak{L}_m(1; x) - 1| + |\cos x||\mathfrak{L}_m(\cos t; x) - \cos x| \\ + |\sin x||\mathfrak{L}_m(\sin x; x) - \sin x|,$$

we can clearly write

$$\mathfrak{L}_m(\varphi^2; x) \leq M \sum_{i=0}^{2} |\mathfrak{L}_m(f_i; x) - f_i(x)|_{2\pi}, \tag{4.4}$$

where

$$M = 1 + \|f_1\|_{2\pi} + \|f_2\|_{2\pi}.$$

It now follows from (4.3), (4.5) and Lemma 1 that

$$\lambda_m = \sqrt{\mathfrak{L}_m(\varphi^2)} = S_{D(\overline{N})} - o(d_n) \text{ on } [0, 2\pi], \tag{4.5}$$

where

$$o(d_n) = \max\{u_{n_0}, u_{n_1}, u_{n_2}\}.$$

Hence, clearly, we get

$$\omega(f, \delta) = \mathrm{stat}D(\overline{N}) - o(d_n) \text{ on } [0, 2\pi].$$

By using (4.6) in Theorem 2, we subsequently see for all $f \in C_{2\pi}(\mathbb{R})$ that

$$\mathfrak{L}_m(f; x) - f(x) = \mathrm{stat}D(\overline{N}) - o(d_n) \text{ on } [0, 2\pi]. \tag{4.6}$$

Hence, if we use the condition (4.3) in Theorem 2 in place of conditions (i) and (ii), then we fairly obtain the rates of the statistical deferred weighted summability of the sequence $(\mathfrak{L}_m)$ of linear operators (positive) in Theorem 1.

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On Pointwise Convergence of a Family of Nonlinear Integral Operators

Gumrah Uysal and Hemen Dutta

Abstract Let Λ be a non-empty index set consisting of σ indices and σ_0 is allowed to be either accumulation point of Λ or infinity. We assume that the function K_σ, $K_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and for any $\sigma \in \Lambda$ and satisfies some conditions. The main purpose of this work is to investigate the conditions under which Fatou type pointwise convergence is obtained for the operators in the following setting:

$$(T_\sigma f)(x) = \int_{-\infty}^{\infty} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right) \right) dt, \ \ x \in \mathbb{R},$$

where $P_{k,\sigma}$ and $\alpha_{k,\sigma}$ are real numbers satisfying certain conditions, at $p - \mu$-Lebesgue point of function f. The obtained results are used for presenting some theorems for the rate of convergences.

Keywords $p - \mu$-Lebesgue point · Rate of convergence · Lipschitz condition · Unified approach · Nonlinear integral operator

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1 Introduction

In the year 1962, Gadjiev et al. [5] investigated the asymptotic value of the approximation of measurable functions by integral operators of the form:

$$L_{\sigma}\left(f,x\right)=\int_{\mathbb{R}}\sum_{k=1}^{\infty}P_{k,\sigma}f\left(x+\alpha_{k,\sigma}t\right)K_{\sigma}\left(t\right)dt,\ x\in\mathbb{R},\ \sigma\in\Lambda, \tag{1.1}$$

where Λ is a non-empty set of a non-negative real parameters σ, $\alpha_{k,\sigma}$ are assumed to be non-negative real numbers for all values of k and σ with $\sup\limits_{k,\sigma}\left\{\alpha_{k,\sigma}\right\}=\alpha^{*}<\infty$ and $P_{k,\sigma}$ are real numbers satisfying $\sum_{k=1}^{\infty}\left|P_{k,\sigma}\right|\leq M$ (M is independent of σ) and $\sum_{k=1}^{\infty}P_{k,\sigma}=1$ for all $\sigma\in\Lambda$. Also, the kernel function $K_{\sigma}:\mathbb{R}\to\mathbb{R}$ satisfies some certain conditions. The operators of type (1.1) were considered later in the works [16, 17] presenting some theorems concerning convergence in the norms of $L_{1}\left(\mathbb{R}\right)$ and $L_{p}\left(\mathbb{R}\right)$ $(1<p<\infty)$, respectively. In these works, in order to obtain the desired convergence, the new modulus of continuity definitions are given.

In the year 1983, Musielak [11] built a bridge between linear and nonlinear integral operators of convolution type by considering the following setting of integral operators

$$T_{w}f(y)=\int_{G}K_{w}(x-y;f(x))dx,\ y\in G,\ w\in\Lambda, \tag{1.2}$$

where Λ is a non-empty set of indices and K_{w}, $K_{w}:G\times\mathbb{R}\to\mathbb{R}$, for any $w\in\Lambda$, is a kernel function satisfying some conditions including Lipschitz property with respect to its second variable. For some advanced studies concerning approximation by nonlinear integral operators, we refer the reader to [1, 6, 12, 20]. Also, for some works, related to linear integral operators of convolution type, we refer the reader to [2, 3, 7, 14, 21].

In [9], Mamedov handled the following m-singular integral operators

$$L_{\lambda}^{[m]}(f;x)=(-1)^{m+1}\int_{\mathbb{R}}\left[\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt)\right]K_{\lambda}(t)dt, \tag{1.3}$$

where $x\in\mathbb{R}$, $m\geq1$ is a finite natural number and $\lambda\in\Lambda$ which is a non-empty set of non-negative indices, by harnessing m-th finite difference formulas. Under certain conditions, the operators of type (1.1) may be reduced to the operators of type (1.3). Fatou type convergence of nonlinear counterparts of the operators of type (1.3) were studied in [8]. Also, for some studies concerning convergence of m-singular integral operators in different function spaces, we refer the reader to [15, 22]. In 1965, Mamedov [10] studied the saturation classes of linear operators by considering further generalization of the operators of type (1.1), that is, the summation inside the operators runs from $k=-\infty$ to $+\infty$.

Let Λ be a non-empty index set consisting of σ indices. Here, σ_0 is allowed to be either accumulation point of Λ or infinity. We assume that the function $K_\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and satisfies some conditions. The main purpose of this work is to investigate the conditions under which Fatou type pointwise convergence is obtained for the operators in the following setting:

$$(T_\sigma f)(x) = \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right) \right) dt, \ x \in \mathbb{R}, \tag{1.4}$$

where $\alpha_{k,\sigma}$ are positive real numbers with finite supremum value for all values of k and σ, that is, $\sup\limits_{k,\sigma} \left\{\alpha_{k,\sigma}\right\} = \alpha^* < \infty$, $P_{k,\sigma}$ are real numbers with $\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \leq M$ (M is independent of σ) and $\sum_{k=1}^{\infty} P_{k,\sigma} = 1$ for all $\sigma \in \Lambda$, at $p-\mu$-Lebesgue point of function f as $(x, \sigma) \to (x_0, \sigma_0)$. As in [5, 17], we also suppose that f has a majorant function, that is, there exists a function φ satisfying $|f(x)| \leq \varphi(x) < \infty$ for all $x \in \mathbb{R}$. Here, $L_p(\mathbb{R})$ $(1 \leq p < \infty)$ will denote the space of all measurable functions f for which the Lebesgue integral of $|f|^p$ has finite value on $\mathbb{R}$. The obtained results are used for presenting some theorems for the rate of convergences. Here, the operators of type (1.4) are obtained by incorporating the operators of type (1.1) and (1.2).

The paper is organized as follows: In Sect. 2, we introduce fundamental notions. In Sect. 3, we give some auxiliary theorems concerning existence and pointwise convergence of the operators of type (1.4). In Sect. 4, we present a Fatou type convergence theorem for these operators. In Sect. 5, we establish the rates of both pointwise and Fatou type convergences by using the results obtained in the previous two sections.

2 Preliminaries

The following definition is obtained by incorporating the characterization of function μ given by Gadjiev [7] which helps to generalize well-known Lebesgue point definition and the idea used in [16, 17] in order to create new modulus of continuity definitions. For some other μ-Lebesgue point characterizations, we refer the reader to [3, 8, 14, 15, 22].

Definition 1 Let $\delta_0 \in \mathbb{R}^+$ be a fixed number. A point $x \in \mathbb{R}$ satisfying the following relations:

$$\lim_{h \to 0^+} \left(\frac{1}{\mu(h)} \int_0^h \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x + \alpha_{k,\sigma} t\right) - f(x) \right] \right|^p dt \right)^{\frac{1}{p}} = 0, \tag{2.1}$$

$$\lim_{h\to 0^+}\left(\frac{1}{\mu(h)}\int_{-h}^{0}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x+\alpha_{k,\sigma}t\right)-f\left(x\right)\right]\right|^{p}dt\right)^{\frac{1}{p}}=0, \tag{2.2}$$

where $0 < h \leq \delta_0$ and relations (2.1) and (2.2) are independent of the choice of $\sigma \in \Lambda$, is called $p-\mu$-Lebesgue point of f $(1 \leq p < \infty)$. Here, $\mu : \mathbb{R} \to \mathbb{R}$ is an increasing and absolutely continuous function on $[0, \delta_0]$ with $\mu(0) = 0$.

The following definition is obtained by incorporating kernel properties used in the works [5, 17, 20]. Also, usage of Lipschitz condition is due by Musielak [11].

Definition 2 Let $1 \leq p < \infty$ and σ_0 be an accumulation point of non-empty index set Λ or infinity. A family K consisting of the functions $K_\sigma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, where $K_\sigma(\vartheta, u)$ has finite Lebesgue integral value on $\mathbb{R}$ for all values of its second variable and for any $\sigma \in \Lambda$ and the following conditions hold:

(a) $K_\sigma(\vartheta, 0) = 0$, for every $\vartheta \in \mathbb{R}$ and $\sigma \in \Lambda$.
(b) There exists a function $L_\sigma : \mathbb{R} \to \mathbb{R}_0^+$ whose Lebesgue integral has finite value on $\mathbb{R}$ for any $\sigma \in \Lambda$ such that the following inequality:

$$|K_\sigma(t,u) - K_\sigma(t,v)| \leq L_\sigma(t)\,|u-v|$$

holds for every $t \in \mathbb{R}$, $u, v \in \mathbb{R}$ and $\sigma \in \Lambda$.
(c) For every $u \in \mathbb{R}$, we have $\lim\limits_{\sigma\to\sigma_0}\left|\int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty}P_{k,\sigma}u\right)dt - u\right| = 0$.
(d) $\lim\limits_{\sigma\to\sigma_0}\left[\int_{|t|>\xi} L_\sigma(t)\,dt\right] = 0$ for every $\xi > 0$.
(e) $\lim\limits_{\sigma\to\sigma_0}\left[\int_{|t|>\xi} \eta^p(\alpha^* t)\,L_\sigma(t)\,dt\right] = 0$ for every $\xi > 0$.
(f) For a certain real number $\delta_1 > 0$, the function $L_\sigma(t)$ is non-decreasing on $(-\delta_1, 0]$ and non-increasing on $[0, \delta_1)$ with respect to t, for any $\sigma \in \Lambda$.
(g) $\int_{\mathbb{R}} \eta(\alpha^* t)\,L_\sigma(t)\,dt \leq N_1 < \infty$ and $\int_{\mathbb{R}} L_\sigma(t)\,dt \leq N_2 < \infty$ for all $\sigma \in \Lambda$ (N_1 and N_2 are independent of $\sigma \in \Lambda$).

Here,

$$\eta(t) = \sup_{\substack{x\in\mathbb{R}\\ |y|\leq t}} \frac{\varphi(x+y)}{\varphi(x)} < \infty,$$

where $\varphi : \mathbb{R} \to \mathbb{R}^+$ and φ is an aforementioned majorant function. Throughout this manuscript, we assume that K_σ satisfies above conditions.

3 Existence of the Operators and Pointwise Convergence

Theorem 1 *Let $1 \leq p < \infty$ such that there exists a positive function $\varphi \in L_p(\mathbb{R})$ satisfying $|f(x)| \leq \varphi(x) < \infty$ for all $x \in \mathbb{R}$. Then, the functions $T_\sigma f \in L_p(\mathbb{R})$ and the inequality*

$$\|T_\sigma f\|_{L_p(\mathbb{R})} \leq M \|\varphi\|_{L_p(\mathbb{R})} \int_{\mathbb{R}} \eta\left(\alpha^* t\right) L_\sigma(t)\, dt,$$

holds for every $\sigma \in \Lambda$.

Proof First, under the hypotheses, the convergence of the series

$$\sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)$$

is guaranteed for all fixed $t \in \mathbb{R}$ (for details, see [5, 17]), that is,

$$\begin{aligned}\left|\sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right| &\leq \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left|f\left(x + \alpha_{k,\sigma} t\right)\right| \\ &\leq M\varphi(x)\, \eta\left(\alpha^* t\right).\end{aligned}$$

Let $p = 1$. By conditions (a) and (b), we may write

$$\begin{aligned}\|T_\sigma f\|_{L_1(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt \right| dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} L_\sigma(t) \left| M\varphi(x)\, \eta\left(\alpha^* t\right)\right| dt dx.\end{aligned}$$

In view of Fubini theorem (see, e.g., [2]), we obtain the desired result, that is,

$$\|T_\sigma f\|_{L_1(\mathbb{R})} \leq M \|\varphi\|_{L_1(\mathbb{R})} \int_{\mathbb{R}} \eta\left(\alpha^* t\right) L_\sigma(t)\, dt.$$

Now, we prove the theorem for the case $1 < p < \infty$. By conditions (a) and (b), we may write

$$\begin{aligned}\|T_\sigma f\|_{L_p(\mathbb{R})} &= \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} L_\sigma(t) \left| M\varphi(x)\, \eta\left(\alpha^* t\right)\right| dt \right)^p dx \right)^{\frac{1}{p}}.\end{aligned}$$

Now, applying generalized Minkowski inequality to the last inequality above (see, e.g., [19]), we have

$$\begin{aligned}\|T_\sigma f\|_{L_p(\mathbb{R})} &\leq \int\limits_{\mathbb{R}} \left(\int\limits_{\mathbb{R}} L_\sigma^p (t) \left| M\varphi (x)\, \eta \left(\alpha^* t\right)\right|^p dx \right)^{\frac{1}{p}} dt \\ &= M \int\limits_{\mathbb{R}} L_\sigma (t)\, \eta \left(\alpha^* t\right) dt \left(\int\limits_{\mathbb{R}} |\varphi (x)|^p dx \right)^{\frac{1}{p}} \\ &= M \|\varphi\|_{L_p(\mathbb{R})} \int\limits_{\mathbb{R}} \eta \left(\alpha^* t\right) L_\sigma (t)\, dt.\end{aligned}$$

The desired result follows from condition (g). Thus the proof is completed. □

Now, we give a theorem concerning pointwise convergence of the operators of type (1.4).

Theorem 2 *Suppose that there exists a positive function φ satisfying $|f(x)| \leq \varphi(x) < \infty$ for all $x \in \mathbb{R}$. If $x_0 \in \mathbb{R}$ is a $p - \mu$-Lebesgue point of the function f $(1 \leq p < \infty)$, then*

$$\lim_{\sigma \to \sigma_0} |(T_\sigma f)(x_0) - f(x_0)| = 0$$

provided that $\sigma \in \Lambda_1 \subseteq \Lambda$ on which the function

$$\int\limits_{-\delta}^{\delta} \left| \{\mu(|t|)\}_t' \right| L_\sigma(t) dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, is bounded as σ tends to σ_0.

Proof We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $|I_\sigma(x_0)| = |(T_\sigma f)(x_0) - f(x_0)|$.

From (c), we can write

$$\begin{aligned}|I_\sigma(x_0)| = \Bigg| &\int\limits_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \\ &+ \int\limits_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt - \int\limits_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt \Bigg|.\end{aligned}$$

Using (b), we may easily get

$$|I_\sigma(x_0)| \le \int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| L_\sigma(t)\, dt + \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0) \right) dt - f(x_0) \right|.$$

Since whenever A and B being positive numbers the inequality $(A+B)^p \le 2^p(A^p + B^p)$ holds (see, e.g., [13]), we have

$$\begin{aligned}
|I_\sigma(x_0)|^p &\le 2^p \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| L_\sigma(t)\, dt \right)^p \\
&\quad + 2^p \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0) \right) dt - f(x_0) \right|^p \\
&= 2^p (I_1 + I_2).
\end{aligned}$$

By (c), I_2 tends to zero as σ tends to σ_0. Next, applying Hölder's inequality (see [13]) to the integral I_1 and condition (g), we obtain

$$\begin{aligned}
I_1 &= \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right| (L_\sigma(t))^{\frac{1}{p}} (L_\sigma(t))^{\frac{1}{q}}\, dt \right)^p \\
&\le \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma(t)\, dt \right) \left(\int_{\mathbb{R}} L_\sigma(t)\, dt \right)^{\frac{p}{q}} \\
&\le (N_2)^{\frac{p}{q}} \int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p L_\sigma(t)\, dt \\
&= (N_2)^{\frac{p}{q}} I_{11},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Since $x_0 \in \mathbb{R}$ is a $p - \mu$-Lebesgue point of the function f in view of relations (2.1) and (2.2) for all $\varepsilon > 0$, there exists a number $\delta > 0$ such that the following inequalities hold there:

$$\int_0^h \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} t\right) - f(x_0) \right] \right|^p dt \le \varepsilon^p \mu(h), \tag{3.1}$$

$$\int_{-h}^{0}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x_0+\alpha_{k,\sigma}t\right)-f\left(x_0\right)\right]\right|^{p}dt\leq\varepsilon^{p}\mu\left(h\right), \tag{3.2}$$

where $0<h\leq\delta$ provided that $0<\delta<\min\{\delta_0,\delta_1\}$.

Now, we consider I_{11}. It is easy to see that the following equality holds:

$$\begin{aligned}I_{11}&=\left\{\int_{|t|>\delta}+\int_{-\delta}^{\delta}\right\}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x_0+\alpha_{k,\sigma}t\right)-f(x_0)\right]\right|^{p}L_{\sigma}\left(t\right)dt\\&=I_{111}+I_{112}.\end{aligned}$$

For the integral I_{111}, we can write

$$\begin{aligned}I_{111}&=\int_{|t|>\delta}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x_0+\alpha_{k,\sigma}t\right)-f(x_0)\right]\right|^{p}L_{\sigma}\left(t\right)dt\\&\leq 2^{p}\int_{|t|>\delta}\left|\sum_{k=1}^{\infty}P_{k,\sigma}f\left(x_0+\alpha_{k,\sigma}t\right)\right|^{p}L_{\sigma}\left(t\right)dt\\&\quad+2^{p}\left|f\left(x_0\right)\right|^{p}\int_{|t|>\delta}L_{\sigma}\left(t\right)dt\\&=2^{p}\left(I_{1111}+I_{1112}\right).\end{aligned}$$

Under the hypotheses, we observe that

$$\begin{aligned}I_{1111}&=\int_{|t|>\delta}\left|\sum_{k=1}^{\infty}P_{k,\sigma}f\left(x_0+\alpha_{k,\sigma}t\right)\right|^{p}L_{\sigma}\left(t\right)dt\\&=\int_{|t|>\delta}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\frac{f\left(x_0+\alpha_{k,\sigma}t\right)}{\varphi\left(x_0+\alpha_{k,\sigma}t\right)}\frac{\varphi\left(x_0+\alpha_{k,\sigma}t\right)}{\varphi\left(x_0\right)}\varphi\left(x_0\right)\right|^{p}L_{\sigma}\left(t\right)dt\\&\leq M^{p}\varphi^{p}\left(x_0\right)\int_{|t|>\delta}\eta^{p}\left(\alpha^{*}t\right)L_{\sigma}\left(t\right)dt.\end{aligned}$$

Using (e) and (d), I_{1111} tends to 0 and I_{1112} tends to 0 as σ tends to σ_0, respectively.

Lastly, we have to show that I_{112} tends to 0 as σ tends to σ_0.

Obviously, I_{112} may be written in the form

$$I_{112}=\left\{\int_{-\delta}^{0}+\int_{0}^{\delta}\right\}\left|\sum_{k=1}^{\infty}P_{k,\sigma}\left[f\left(x_0+\alpha_{k,\sigma}t\right)-f(x_0)\right]\right|^{p}L_{\sigma}\left(t\right)dt$$

$$= I_{1121} + I_{1122}.$$

For I_{1121}, let us define

$$F(t) := \int_{t}^{0} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x_0 + \alpha_{k,\sigma} v\right) - f(x_0) \right] \right|^p dv.$$

By (3.2) the inequality

$$F(t) \le \varepsilon^p \mu(-t) \tag{3.3}$$

holds for every δ satisfying $0 < \delta < \min\{\delta_0, \delta_1\}$. In view of (3.3) and following similar strategy as in [7, 14], we have

$$|I_{1121}| \le \varepsilon^p \int_{-\delta}^{0} \mu'(-t) L_\sigma(t)\, dt.$$

Similarly,

$$|I_{1122}| \le \varepsilon^p \int_{0}^{\delta} \mu'(t) L_\sigma(t)\, dt.$$

Incorporating above results, we have

$$|I_{112}| \le \varepsilon^p \int_{-\delta}^{\delta} \left| \{\mu(|t|)\}'_t \right| L_\sigma(t)\, dt.$$

The remaining part follows from the arbitrariness of ε and boundedness of $\int_{-\delta}^{\delta} \left| \{\mu(|t|)\}'_t \right| L_\sigma(t)\, dt$ as σ tends to σ_0. This completes the proof. □

4 Main Theorem

In this section we will prove the Fatou type pointwise convergence of the operators of type (1.4). For the original description, we refer the reader to Fatou [4]. Some related works may be found in [5, 8, 14, 18]. For this purpose, we suppose that for a sufficiently small number $\delta > 0$ such that the function Ω_δ given as

$$\Omega_\delta(x, \sigma) = \int_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left| P_{k,\sigma} \right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_\sigma(t)\, dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, is bounded on the set defined as

$$Z_{C,\delta} = \{(x, \sigma) \in \mathbb{R} \times \Lambda_1 : \Omega_\delta(x, \sigma) < C\},$$

where C is positive constant which can be made arbitrarily small, as (x, σ) tends to (x_0, σ_0). Here, this set is given before the theorem, but it can be given inside the theorem up to desire.

Theorem 3 *Suppose such that there exists a positive function φ satisfying $|f(x)| \leq \varphi(x) < \infty$ for all $x \in \mathbb{R}$. If $x_0 \in \mathbb{R}$ is a $p-\mu$-Lebesgue point of the function f $(1 \leq p < \infty)$, then*

$$\lim_{(x,\sigma)\to(x_0,\sigma_0)} |(T_\sigma f)(x) - f(x_0)| = 0$$

provided that $(x, \sigma) \in Z_{C,\delta}$.

Proof We prove the theorem for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar. Let $0 < |x_0 - x| < \frac{\delta}{2}$ for a given $0 < \delta < \min\{\delta_0, \delta_1\}$.

Now, set $I_\sigma(x) = |(T_\sigma f)(x) - f(x_0)|$. Let us write

$$\begin{aligned}
|I_\sigma(x)| &= \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right| \\
&= \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x + \alpha_{k,\sigma} t\right)\right) dt - K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt \right. \\
&\quad \left. + \ K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|I_\sigma(x)|^p &\leq 2^p \left(\int_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right] \right| L_\sigma(t)\, dt \right)^p \\
&\quad + 2^p \left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|^p \\
&= 2^p \{I_1 + I_2\}.
\end{aligned}$$

Clearly, by Theorem 2, $I_2 \to 0$ as σ tends to σ_0.

The following inequality holds for I_1:

$$
\begin{aligned}
I_1 &= \left(\int\limits_{\mathbb{R}} \left| \sum_{k=1}^{\infty} P_{k,\sigma} \left[f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right] \right| L_\sigma(t)\, dt \right)^p \\
&\le \left(\left\{ \int\limits_{|t|>\delta} + \int\limits_{-\delta}^{\delta} \right\} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left| f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right| L_\sigma(t)\, dt \right)^p \\
&\le 2^p \left(\int\limits_{|t|>\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left| f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right| L_\sigma(t)\, dt \right)^p \\
&\quad + 2^p \left(\int\limits_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left| f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right| L_\sigma(t)\, dt \right)^p \\
&= 2^p \left(I_{11} + I_{12}\right).
\end{aligned}
$$

Applying Hölder's inequality to I_{11} and using condition (g), we have

$$
\begin{aligned}
I_{11} &\le \left(\int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \right)^p \left| f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right|^p L_\sigma(t)\, dt \right) \left(\int\limits_{\mathbb{R}} L_\sigma(t)\, dt \right)^{\frac{p}{q}} \\
&\le \left(\int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \right)^p \left| f\left(x+\alpha_{k,\sigma}t\right) - f\left(x_0+\alpha_{k,\sigma}t\right)\right|^p L_\sigma(t)\, dt \right) \left(N_2\right)^{\frac{p}{q}} \\
&= I_{111} \left(N_2\right)^{\frac{p}{q}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{q} = 1$. It is easy to see that

$$
\begin{aligned}
I_{111} \le 2^p &\int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \right)^p \left| f\left(x+\alpha_{k,\sigma}t\right)\right|^p L_\sigma(t)\, dt \\
&+ 2^p \int\limits_{|t|>\delta} \left(\sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \right)^p \left| f\left(x_0+\alpha_{k,\sigma}t\right)\right|^p L_\sigma(t)\, dt.
\end{aligned}
$$

Following same strategy as in Theorem 2, we have

$$
\begin{aligned}
I_{111} &\le 2^p M^p \varphi^p(x) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\, dt \\
&\quad + 2^p M^p \varphi^p(x_0) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\, dt \\
&= I_{1111} + I_{1112}.
\end{aligned}
$$

Using condition (e), $I_{1111} \to 0$ and $I_{1112} \to 0$ as $(x, \sigma) \to (x_0, \sigma_0)$. The result follows from the hypothesis on the integral $I_{12.}$ Thus the proof is completed. □

5 Rate of Convergence

Theorem 4 *Suppose that the hypotheses of Theorem 2 are satisfied. Let*

$$\Delta(\sigma, \delta) = \int_{-\delta}^{\delta} \left| \{\mu\,(|t|)\}'_t \right| L_\sigma(t)dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, *and the following conditions are satisfied:*

(i) $\Delta(\sigma, \delta)$ tends to 0 as $\sigma \to \sigma_0$ for some $\delta > 0$.

(ii) For every $\xi > 0$, we have

$$\int_{|t|>\xi} L_\sigma\,(t)\,dt = \mathbf{o}(\Delta(\sigma, \delta))$$

as $\sigma \to \sigma_0$.

(iii) For every $\xi > 0$ and $1 \leq p < \infty$, we have

$$\int_{|t|>\xi} \eta^p\left(\alpha^* t\right) L_\sigma\,(t)\,dt = \mathbf{o}(\Delta(\sigma, \delta))$$

as $\sigma \to \sigma_0$.

(iv) Letting $\sigma \to \sigma_0$, we have

$$\left| \int_{\mathbb{R}} K_\sigma \left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\,(x_0) \right) dt - f\,(x_0) \right|^p = \mathbf{o}(\Delta(\sigma, \delta)).$$

Then, at each $p - \mu$-Lebesgue point of f $(1 \leq p < \infty)$, we have

$$|(T_\sigma f)\,(x_0) - f\,(x_0)|^p = \mathbf{o}(\Delta(\sigma, \delta))$$

as σ tends to σ_0.

Proof By the hypotheses of Theorem 2, we have

$$\begin{aligned}|(T_\sigma f)(x_0) - f(x_0)|^p \le{}& \varepsilon^p 2^p (N_2)^{\frac{p}{q}} \int_{-\delta}^{\delta} \left|\{\mu(|t|)\}'_t\right| L_\sigma(t)\,dt.\\ &+ 2^{2p}\varphi^p(x_0)(N_2)^{\frac{p}{q}} M^p \int_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt.\\ &+ 2^{2p}(N_2)^{\frac{p}{q}} |f(x_0)|^p \int_{|t|>\delta} L_\sigma(t)\,dt\\ &+ 2^p \left|\int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f(x_0)\right) dt - f(x_0)\right|^p.\end{aligned}$$

The proof follows from $(i)-(iv)$. □

Theorem 5 *Suppose that the hypotheses of Theorem 3 are satisfied. Let*

$$\Omega_\delta(x,\sigma) = \int_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left|f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right)\right| L_\sigma(t)\,dt,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$, *and the following conditions are satisfied:*

(i) $\Omega_\delta(x,\sigma)$ tends to 0 as (x,σ) tends to (x_0,σ_0) for some $\delta > 0$.
(ii) For every $\xi > 0$ and $1 \le p < \infty$, we have

$$\int_{|t|>\xi} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt = \mathbf{o}(\Omega_\delta(x,\sigma))$$

as (x,σ) tends to (x_0,σ_0).
(iii) Letting $(x,\sigma) \to (x_0,\sigma_0)$,

$$\left|\int_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0)\right|^p = \mathbf{o}\left(\Omega_\delta(x,\sigma)\right).$$

Then, at each $p-\mu$-Lebesgue point of f $(1 \le p < \infty)$, we have

$$|(T_\sigma f)(x) - f(x_0)|^p = \mathbf{o}\left(\Omega_\delta(x,\sigma)\right)$$

as (x,σ) tends to (x_0,σ_0).

Proof Under the hypotheses of Theorem 3, we may write

$$
\begin{aligned}
|(T_\sigma f)(x) - f(x_0)|^p \le{}& (N_2)^{\frac{p}{q}} 2^{3p} M^p \varphi^p(x) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt \\
&+ (N_2)^{\frac{p}{q}} 2^{3p} M^p \varphi^p(x_0) \int\limits_{|t|>\delta} \eta^p\left(\alpha^* t\right) L_\sigma(t)\,dt \\
&+ 2^{2p} \left(\int\limits_{-\delta}^{\delta} \sum_{k=1}^{\infty} \left|P_{k,\sigma}\right| \left| f\left(x + \alpha_{k,\sigma} t\right) - f\left(x_0 + \alpha_{k,\sigma} t\right) \right| L_\sigma(t)\,dt \right)^p \\
&+ 2^p \left| \int\limits_{\mathbb{R}} K_\sigma\left(t, \sum_{k=1}^{\infty} P_{k,\sigma} f\left(x_0 + \alpha_{k,\sigma} t\right)\right) dt - f(x_0) \right|^p .
\end{aligned}
$$

By conditions $(i)-(iii)$, we obtain the desired result. This completes the proof. □

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Existence and Ulam's Type Stability of Integro Differential Equation with Non-instantaneous Impulses and Periodic Boundary Condition on Time Scales

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Abstract The present manuscript is dedicated to the study of existence and stability of integro differential equation with periodic boundary condition and non-instantaneous impulses on time scales. Banach contraction theorem and non-linear functional analysis have been used to established these results. Moreover, to outline the utilization of these outcomes an example is given.

Keywords Existence · Stability · Time scales · Non-instantaneous impulses

AMS Subject Classification 34A12 · 35F30 · 34A37 · 34N05

1 Introduction

There are many physical models which are subject to sudden changes in its states, such rapid changes are known as impulsive response. In the current hypothesis, there are two types of impulsive system, one is instantaneous and another one is known as non-instantaneous impulsive system. In the instantaneous impulsive system, the duration of these abrupt changes is very little correlation to the duration of the whole process, for example pulses, stuns and cataclysmic events [7, 16], while in the non-instantaneous impulses, the duration of these changes continues over a finite time interval. For the initial studies related with the existence, uniqueness, and controllability of non-instantaneous impulsive systems of integer and fractional order, we refer to [10, 15, 18, 21] and the references cited therein. Further, stability analysis of dynamical systems becomes an important research area and various form of

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stabilities have been developed including Lyapunov stability, Mittag-Leffler function and exponential for dynamical equations. Moreover, an interesting type of stability was introduced by Ulam and Hyers is known as Ulam-Hyers stability which is highly useful in numerical analysis and optimization for dynamical equations. The Ulam-Hyper's stability for many dynamical equations of integer and fractional order has been studied in lots of articles [4, 5, 25, 26].

In 1988, Hilger presented the time scales calculus. The investigation of analytics on time scales incorporates the continuous and discrete analysis, therefore the investigation of dynamical system on time scales has picked up an awesome consideration and numerous scientists have discovered the uses of time scales in heat transfer system [19], population dynamics [28] and economics [11, 12]. For more details about time scales one can refer the book [8, 9] and papers [2, 3, 17]. Further over the most recent couple of years, many authors talked about the existence, uniqueness and stability of dynamical system on time scales [1, 6, 13, 14, 20, 22–24, 27]. Particularly, Geng [13], presented the concepts of lower and upper solutions for a PBVP on time scales.

According as far as anyone is concerned, there is no manuscript which examined the existence, uniqueness and stability investigation of integro differential equations with non-instantaneous impulses on time scales. Spurred by the above actualities, we take the differential equations with periodic boundary condition and non-instantaneous impulses on time scale of the form:

$$\begin{aligned} v^{\Delta}(\theta) &= \mathcal{C}\left(\theta, v(\theta), \int_0^{\theta} h(\theta,\tau,v(\tau))\Delta\tau\right), \quad \theta \in \cup_{k=0}^{l}(\lambda_k,\theta_{k+1}]_{\mathbb{T}}, \\ v(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \theta\in(\theta_k,\lambda_k]_{\mathbb{T}},\ k=1,2,\ldots,l,\ q\in(0,1) \\ v(0) &= v(T) \end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\theta_k, \lambda_k \in \mathbb{T}$ are right dense points with $0 = \lambda_0 = \theta_0 < \theta_1 < \lambda_1 < \theta_2 < \cdots \lambda_l < \theta_{l+1} = T$, $v(\theta_k^-) = \lim_{h\to 0^+} v(\theta_k - h)$, $v(\theta_k^+) = \lim_{h\to 0^+} v(\theta_k + h)$, represent the left and right limits of $v(\theta)$ at $\theta = \theta_k$. The functions $\boldsymbol{g}_k(\theta, v(\theta_k^-)) \in C(I,\mathbb{R})$ represent non-instantaneous impulses during the intervals $(\theta_k,\lambda_k]_{\mathbb{T}},\ k = 1,2,\ldots,l$, so impulses at θ_k have some duration, namely on intervals $(\theta_k,\lambda_k]_{\mathbb{T}}$. $\mathcal{C} : I = [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathcal{Q}\times\mathbb{R}\to\mathbb{R}$ are given functions, where $\mathcal{Q} = \{(\theta,\tau)\in I\times I : 0\le\tau\le\theta\le T\}$.

Throughout the manuscript, we impose

$$\mathcal{M}(v(\theta)) = \int_0^{\theta} h(\theta,\tau,v(\tau))\Delta\tau.$$

The structure of the manuscript is as: In second section, we give preliminaries, fundamental definitions, useful lemmas and some important results. In the subsequent sections, the main results of the manuscript are discussed. Finally, an example is given to outline the utilization of these outcomes.

2 Preliminaries

Below, we give basic notations, fundamental definitions and useful lemmas. Let $(X, \|.\|)$ be a Banach space. $C(I, \mathbb{R})$ be the set of all continuous functions. In order to define the solution of the Eq. (1.1), we define the space $PC(I, \mathbb{R})$ of piecewise continuous functions defined as $PC(I, \mathbb{R}) = \{v : I \to \mathbb{R} : v \in C(\theta_k, \theta_{k+1}]_{\mathbb{T}}, \mathbb{R}), k = 0, 1, \ldots, l$ and there exists $v(\theta_k^-)$ and $v(\theta_k^+)$, $k = 1, 2, \ldots, l$ with $v(\theta_k^-) = v(\theta_k)\}$. It can be seen easily that $PC(I, \mathbb{R})$ is a Banach space with the TZ-norm

$$\|v\|_\Omega = \sup_{\theta \in [a,b]} \frac{\|v(\theta)\|}{e_\Omega(\theta, a)}, \text{ for some } \Omega \in \mathcal{R}^+.$$

A closed non-empty subset of real number is called time scales $\mathbb{T}$. A time scale interval is defined as $[i, m]_{\mathbb{T}} = \{\theta \in \mathbb{T} : i \le \theta \le m\}$, accordingly, we define $(i, m)_{\mathbb{T}}$, $[i, m)_{\mathbb{T}}$ and so on. Now onwards, we used a time scale interval $[i, m]$ instead of $[i, m]_{\mathbb{T}}$. Also, now onward if max $\mathbb{T}$ exists, then we take $\mathbb{T}^k = \mathbb{T}\backslash\{\max \mathbb{T}\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. The forward jump operator $\sigma : \mathbb{T}^k \to \mathbb{T}$ is defined by $\sigma(\theta) := \inf\{r \in \mathbb{T} : r > \theta\}$ with the substitution $\inf\{\phi\} = \sup \mathbb{T}$ and the graininess function $\mu : \mathbb{T}^k \to [0, \infty)$ is define as $\mu(\theta) := \sigma(\theta) - \theta, \forall \theta \in \mathbb{T}^k$.

Definition 2.1 Let $z : \mathbb{T} \to \mathbb{R}$ and $\theta \in \mathbb{T}^k$. The delta derivative $z^\Delta(\theta)$ is the number (when it exists) such that given any $\epsilon > 0$, there is a neighbourhood U of θ such that

$$|[z(\sigma(\theta)) - z(\tau)] - z^\Delta(\theta)[\sigma(\theta) - \tau]| \le \epsilon|\sigma(\theta) - \tau|, \quad \forall \tau \in U.$$

Definition 2.2 Function Z is said to be antiderivative of $z : \mathbb{T} \to \mathbb{R}$ provided $Z^\Delta(\theta) = z(\theta)$ for each $\theta \in \mathbb{T}^k$, then the delta integral is defined by

$$\int_{\theta_0}^{\theta} z(\zeta)\Delta\zeta = Z(\theta) - Z(\theta_0).$$

A function $z : \mathbb{T} \to \mathbb{R}$ is called rd-continuous on $\mathbb{T}$, if z has finite left-sided limits at points $\theta \in \mathbb{T}$ with $\sup\{r \in \mathbb{T} : r < \theta\} = \theta$ and z is continuous at points $\theta \in \mathbb{T}$ with $\sigma(\theta) = \theta$. The collection of all rd-continuous functions $z : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive (positive regressive) if $1 + \mu(\theta)p(\theta) \neq 0(> 0), \forall \theta \in \mathbb{T}$ and the set of all regressive (positive regressive) functions are denoted by $\mathcal{R}(\mathcal{R}^+)$.

Definition 2.4 The generalized exponential function is defined as

$$e_p(\theta, r) = \exp\left(\int_r^\theta \xi_{\mu(\zeta)}(p(\zeta))\Delta\zeta\right), \quad \theta, r \in \mathbb{T}, \ p \in \mathcal{R},$$

where $\xi_{\mu(\beta)}(p(\beta))$ is given by

$$\xi_{\mu(\beta)}(\varkappa) = \begin{cases} \frac{1}{\mu(\beta)} \boldsymbol{Log}(1 + \mu(\beta)\varkappa), & \text{if } \mu(\beta) \neq 0. \\ \varkappa, & \text{if } \mu(\beta) = 0. \end{cases}$$

Lemma 2.5 ([17]) *Let $\theta_1, \theta_2 \in \mathbb{T}$, such that $\theta_1 \leq \theta_2$ and $z : \mathbb{R} \to \mathbb{R}$ be a non-decreasing continuous function. Then,*

$$\int_{\theta_1}^{\theta_2} z(\zeta)\Delta\zeta \leq \int_{\theta_1}^{\theta_2} z(\zeta)d\zeta. \tag{2.1}$$

Lemma 2.6 *Let $\boldsymbol{g} : I \to \mathbb{R}$ be a right dense continuous function. Then, for any $k = 1, 2, \ldots, l$, the solution of the following problem*

$$\begin{aligned} &v^\Delta(\theta) = \boldsymbol{g}(\theta), \quad \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}], \\ &v(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \theta \in (\theta_k, \lambda_k], \ k = 1, 2, \ldots, l, \\ &v(0) = v(T), \end{aligned}$$

is given by the following integral equation

$$\begin{aligned} v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, v(\theta_l^-))\Delta\zeta + \int_{\lambda_l}^{T} \boldsymbol{g}(\zeta)\Delta\zeta + \int_0^\theta \boldsymbol{g}(\zeta)\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1], \\ v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \forall\, \theta \in (\theta_k, \lambda_k], \ k = 1, 2, \ldots, l, \\ v(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta + \int_{\lambda_k}^{\theta} \boldsymbol{g}(\zeta)\Delta\zeta, \quad \forall\, \theta \in (\lambda_k, \theta_{k+1}], \ k = 1, 2, \ldots, l. \end{aligned}$$

(H1): The non-linear function $\mathcal{C} : J_1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $J_1 = \cup_{k=0}^{l}[\lambda_k, \theta_{k+1}]$ is continuous and $\exists$ positive constants $L_{\mathcal{C}_1}$, $L_{\mathcal{C}_2}$ such that

$$|\mathcal{C}(\theta, v_1, v_2) - \mathcal{C}(\theta, w_1, w_2)| \leq L_{\mathcal{C}_1}|v_1 - w_1| + L_{\mathcal{C}_2}|v_2 - w_2|, \quad \forall\, \theta \in I,\ v_j, w_j \in \mathbb{R},\ j = 1, 2.$$

Also, $\exists$ positive constants $C_{\mathcal{C}},\ M_{\mathcal{C}}$ and $N_{\mathcal{C}}$ such that

$$|\mathcal{C}(\theta, v, w)| \leq C_{\mathcal{C}} + M_{\mathcal{C}}|v| + N_{\mathcal{C}}|w|, \quad \forall\, \theta \in I,\ v, w \in \mathbb{R}.$$

(H2): $h : \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\exists$ positive constant L_h such that

$$|h(\theta, \tau, v) - h(\theta, \tau, w)| \leq L_h|v - w|, \quad \forall\, \theta, \tau \in \mathcal{Q},\ v, w \in \mathbb{R}.$$

Also, $\exists$ positive constants $C_h,\ M_h$ such that

$$|h(\theta, \tau, v)| \leq C_h + M_h|v|, \quad \forall\, \theta, \tau \in \mathcal{Q},\ v \in \mathbb{R}.$$

(H3): The functions $g_k : I_k \times \mathbb{R} \to \mathbb{R},\ I_k = [\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$ are continuous and $\exists$ a positive constant L_g such that

$$|g_k(\theta, v) - g_k(\theta, w)| \leq L_g|v - w|, \quad \forall\, v, w \in \mathbb{R}, \theta \in I_k,\ k = 1, 2, \ldots, l.$$

Also, $\exists$ a positive constant M_g such that $|g_k(\theta, v)| \leq M_g, \quad \forall\, \theta \in I_k$ and $v \in \mathbb{R}$.

(H4): $\max_{1\leq k\leq l}\left(e_\Omega(T, \lambda_k)\left(\frac{M_{\mathcal{C}}}{\Omega} + \frac{N_{\mathcal{C}} M_h}{\Omega^2}\right)\right) < 1.$

3 Existence and Uniqueness

Theorem 3.1 *Let the assumptions* ***(H1)–(H4)*** *are holds, then Eq. (1.1) has a unique solution provided,*

$$e_\Omega(T, \lambda_l)\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right) < 1.$$

Proof Consider a subset $\mathcal{D} \subseteq PC(I, \mathbb{R})$ such that

$$\mathcal{D} = \{v \in PC(I, \mathbb{R}) : \|v\|_\Omega \leq \beta\},$$

where

$$\beta = \max_{1\leq k\leq l}\left(\frac{\frac{M_g T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + N_{\mathcal{C}} C_h (T^2 + \theta_1^2)}{1 - (1 + e_\Omega(T, \lambda_k))\left(\frac{M_{\mathcal{C}}}{\Omega} + \frac{N_{\mathcal{C}} M_h}{\Omega^2}\right)}\right).$$

Now, define an operator $\Pi : \mathcal{D} \to \mathcal{D}$ given by

$$(\Pi v)(\theta) = \int_0^\theta \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, v(\theta_l^-))\Delta\zeta$$

$$+ \int_{\lambda_l}^{T} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1],$$

$$(\Pi v)(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \forall\, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l,$$

$$(\Pi v)(\theta) = \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta + \int_{\lambda_k}^{\theta} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta,$$

$$\forall\, \theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l.$$

The proof of this theorem are divided into two steps.
Step 1: To use the Banach contraction theorem, we have to show that $\Pi : \mathcal{D} \to \mathcal{D}$.
For this, we are taking three cases as follows:
Case 1: For $\theta \in (\lambda_k, \theta_{k+1}]$, $k = 1, 2, \ldots, l$ and $v \in \mathcal{D}$, we have:

$$\begin{aligned}
|(\Pi v)(\theta)| &\le \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} |\boldsymbol{g}_k(\zeta, v(\theta_k^-))|\Delta\zeta + \int_{\lambda_k}^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta \\
&\le \frac{M_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1}\Delta\zeta + \int_{\lambda_k}^{\theta} (C_{\mathcal{C}} + M_{\mathcal{C}}|v(\zeta)| + N_{\mathcal{C}}|\mathcal{M}(v(\zeta))|)\Delta\zeta \\
&\le \frac{M_{\boldsymbol{g}}(\lambda_k - \theta_k)^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h \theta_{k+1})(\theta_{k+1} - \lambda_k) \\
&\quad + \left(M_{\mathcal{C}}\beta + \frac{N_{\mathcal{C}} M_h \beta}{\Omega} \right) \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k)\Delta\zeta \\
&\le \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h T)T + \frac{M_{\mathcal{C}}\beta e_\Omega(\theta, \lambda_k)}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta e_\Omega(\theta, \lambda_k)}{\Omega^2}.
\end{aligned}$$

Hence,

$$\|\Pi v\|_\Omega \le \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + (C_{\mathcal{C}} + N_{\mathcal{C}} C_h T)T + \frac{M_{\mathcal{C}}\beta}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta}{\Omega^2}. \tag{3.1}$$

Case 2: For $\theta \in [0, \theta_1]$ and $v \in \mathcal{D}$, we have:

$$
\begin{aligned}
|(\Pi v)(\theta)| \leq & \int_0^\theta |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} |\boldsymbol{g}_l(\zeta, v(\theta_l^-))|\Delta\zeta \\
& + \int_{\lambda_l}^{T} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))|\Delta\zeta \\
\leq & \frac{M_{\boldsymbol{g}}(\lambda_l - \theta_l)^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T - \lambda_l) + M_{\mathcal{C}}\beta \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta + N_{\mathcal{C}} C_h T(T - \lambda_l) \\
& + \frac{N_{\mathcal{C}} M_h \beta}{\Omega} \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta + C_{\mathcal{C}}\theta_1 + N_{\mathcal{C}} C_h \theta_1^2 \\
& + \left(M_{\mathcal{C}}\beta + \frac{N_{\mathcal{C}} M_h \beta}{\Omega} \right) \int_0^\theta e_\Omega(\zeta, 0)\Delta\zeta \\
\leq & \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + \frac{M_{\mathcal{C}}\beta e_\Omega(T, \lambda_l)}{\Omega} + N_{\mathcal{C}} C_h (T^2 + \theta_1^2) \\
& + \frac{N_{\mathcal{C}} M_h \beta e_\Omega(T, \lambda_l)}{\Omega^2} + \frac{M_{\mathcal{C}}\beta e_\Omega(\theta, 0)}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta e_\Omega(\theta, 0)}{\Omega^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\Pi v\|_\Omega \leq & \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)} + C_{\mathcal{C}}(T + \theta_1) + \frac{M_{\mathcal{C}}\beta e_\Omega(T, \lambda_l)}{\Omega} + N_{\mathcal{C}} C_h (T^2 + \theta_1^2) \\
& + \frac{N_{\mathcal{C}} M_h \beta e_\Omega(T, \lambda_l)}{\Omega^2} + \frac{M_{\mathcal{C}}\beta}{\Omega} + \frac{N_{\mathcal{C}} M_h \beta}{\Omega^2}.
\end{aligned} \tag{3.2}
$$

Case 3: For $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$ and $v \in \mathcal{D}$, we can easily get:

$$
\|\Pi v\|_\Omega = \frac{M_{\boldsymbol{g}} T^q}{\Gamma(q+1)}. \tag{3.3}
$$

After summarizing the above inequalities (3.1)–(3.3), we get:

$$
\|\Pi v\|_\Omega \leq \beta.
$$

Therefore, $\Pi : \mathcal{D} \to \mathcal{D}$.

Step 2: In this step, we will show that the operator Π is a contracting operator. Here also, we are taking three cases as follows:

Case 1: For any $v, w \in \mathcal{D},\ \theta \in (\lambda_k, \theta_{k+1}], k = 1, 2, \ldots, l$, we have:

$$
\begin{aligned}
|(\Pi v)(\theta) - (\Pi w)(\theta)| &\leq \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} |\boldsymbol{g}_k(\zeta, v(\theta_k^-)) - \boldsymbol{g}_k(\zeta, w(\theta_k^-))| \Delta\zeta \\
&+ \int_{\lambda_k}^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(w(\zeta)))| \Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} \frac{(\lambda_k - \zeta)^{q-1} |v(\theta_k^-) - w(\theta_k^-)| e_\Omega(\theta_k^-, \theta_k)}{e_\Omega(\theta_k^-, \theta_k)} \Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_{\lambda_k}^{\theta} \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, \lambda_k)}{e_\Omega(\zeta, \lambda_k)} \Delta\zeta \\
&+ L_{\mathcal{C}_2} \int_{\lambda_k}^{\theta} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))| \Delta\zeta \\
&\leq \frac{\|v - w\|_\Omega L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k)(\lambda_k - \theta_k)^q}{\Gamma(q+1)} \\
&+ L_{\mathcal{C}_1} \|v - w\|_\Omega \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k) \Delta\zeta \\
&+ \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_{\lambda_k}^{\theta} e_\Omega(\zeta, \lambda_k) \Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k)(\lambda_k - \theta_k)^q \|v - w\|_\Omega}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1} e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Thus, we have:

$$
\|\Pi v - \Pi w\|_\Omega \leq \left[\frac{L_{\boldsymbol{g}} e_\Omega(\theta_k^-, \theta_k) T^q}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2} \right] \|v - w\|_\Omega. \tag{3.4}
$$

Case 2: For any $v, w \in \mathcal{D}, \theta \in [0, \theta_1]$, we have:

$$
\begin{aligned}
|(\Pi v)(\theta) - (\Pi w)(\theta)| &\leq \int_0^{\theta} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))|\Delta\zeta \\
&+ \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} |\boldsymbol{g}_l(\zeta, v(\theta_l^-)) - \boldsymbol{g}_l(\zeta, w(\theta_l^-))|\Delta\zeta \\
&+ \int_{\lambda_l}^{T} |\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) - \mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))|\Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}}}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} \frac{(\lambda_l - \zeta)^{q-1} |v(\theta_l^-) - w(\theta_l^-)| e_\Omega(\theta_l^-, \theta_l)}{e_\Omega(\theta_l^-, \theta_l)} \Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_{\lambda_l}^{T} \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, \lambda_l)}{e_\Omega(\zeta, \lambda_l)} \Delta\zeta \\
&+ L_{\mathcal{C}_2} \int_{\lambda_l}^{T} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))|\Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_{0}^{\theta} \frac{|v(\zeta) - w(\zeta)| e_\Omega(\zeta, 0)}{e_\Omega(\zeta, 0)} \Delta\zeta \\
&+ L_{\mathcal{C}_2} \int_{0}^{\theta} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))|\Delta\zeta \\
&\leq \frac{L_{\boldsymbol{g}} e_\Omega(\theta_l^-, \theta_l)(\lambda_l - \theta_l)^q \|v - w\|_\Omega}{\Gamma(q+1)} + L_{\mathcal{C}_1} \|v - w\|_\Omega \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta \\
&+ \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_{\lambda_l}^{T} e_\Omega(\zeta, \lambda_l)\Delta\zeta + L_{\mathcal{C}_1} \|v - w\|_\Omega \int_0^{\theta} e_\Omega(\zeta, 0)\Delta\zeta \\
&+ \frac{L_{\mathcal{C}_2} L_h \|v - w\|_\Omega}{\Omega} \int_0^{\theta} e_\Omega(\zeta, 0)\Delta\zeta \\
&\leq \frac{L_{\mathcal{C}_2} L_h e_\Omega(T, \lambda_l) \|v - w\|_\Omega}{\Omega^2} + \frac{L_{\boldsymbol{g}} e_\Omega(\theta_l^-, \theta_l)(\lambda_l - \theta_l)^q \|v - w\|_\Omega}{\Gamma(q+1)} \\
&+ \frac{\|v - w\|_\Omega L_{\mathcal{C}_1} e_\Omega(T, \lambda_l)}{\Omega} + \frac{L_{\mathcal{C}_1} e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Therefore,

$$\|\Pi v - \Pi w\|_\Omega \leq \left[\frac{L_g e_\Omega(\theta_l^-, \theta_l)T^q}{\Gamma(q+1)} + (1 + e_\Omega(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right]\|v - w\|_\Omega. \tag{3.5}$$

Case 3: Similarly, for $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$, we get:

$$|(\Pi v)(\theta) - (\Pi w)(\theta)| \leq \frac{L_g e_\Omega(\theta_k^-, \theta_k)T^q}{\Gamma(q+1)}\|v - w\|_\Omega.$$

Therefore,

$$\|\Pi v - \Pi w\|_\Omega \leq \frac{L_g T^q}{e_\Omega(\theta_k, \theta_k^-)\Gamma(q+1)}\|v - w\|_\Omega. \tag{3.6}$$

After summarizing the inequalities (3.4)–(3.6), we get:

$$\|\Pi v - \Pi w\|_\Omega \leq L_\Pi \|v - w\|_\Omega,$$

where

$$L_\Pi = \max_{1 \leq k \leq l}\left[\frac{L_g T^q e_\Omega(\theta_k^-, \theta_k)}{\Gamma(q+1)} + (1 + e_\Omega(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right].$$

Hence, for sufficiently large Ω, Π is a strict contraction mapping. Therefore, Π has a unique fixed point and that fixed point is the solution of the taken Eq. (1.1). □

Let us consider a special case when $\mathcal{C}\left(\theta, v(\theta), \int_0^\theta h(\theta, \tau, v(\tau))\Delta\tau\right) = \mathcal{P}(\theta, v) + \int_0^\theta h(\theta, \tau, v(\tau))\Delta\tau$ then (1.1) becomes:

$$\begin{aligned}
v^\Delta(\theta) &= \mathcal{P}(\theta, v) + \int_0^\theta h(\theta, \tau, v(\tau))\Delta\tau, \quad \theta \in \cup_{k=0}^l (\lambda_k, \theta_{k+1}],\\
v(\theta) &= \frac{1}{\Gamma(q)}\int_{\theta_k}^\theta (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l,\\
v(0) &= v(T).
\end{aligned} \tag{3.7}$$

(H5): $\mathcal{P} : J_1 \times \mathbb{R} \to \mathbb{R}$ is a non-linear continuous function and $\exists$ a positive constant $L_\mathcal{P}$ such that

$$|\mathcal{P}(\theta, v) - \mathcal{P}(\theta, w)| \leq L_{\mathcal{P}}|v - w|, \quad \forall\, \theta \in I,\ v, w \in \mathbb{R}.$$

Also, ∃ positive constants $C_{\mathcal{P}}$ and $M_{\mathcal{P}}$ such that

$$|\mathcal{P}(\theta, v)| \leq C_{\mathcal{P}} + M_{\mathcal{P}}|v|, \quad \forall\, \theta \in I,\ v \in \mathbb{R}.$$

(H6): $\max_{1\leq k\leq l}\left(e_{\Omega}(T, \lambda_k)\left(\frac{M_{\mathcal{P}}}{\Omega} + \frac{M_h}{\Omega^2}\right)\right) < 1.$

Corollary 3.2 *If the assumptions (**H2**)–(**H3**) and (**H5**)–(**H6**) are holds, then the Eq. (3.7) has a unique solution, provided*

$$e_{\Omega}(T, \lambda_l)\left(\frac{L_{\mathcal{P}}}{\Omega} + \frac{L_h}{\Omega^2}\right) < 1.$$

4 Hyer-Ulam's Stability

For $\epsilon > 0$, $\psi \geq 0$, and nondecreasing $\varphi \in PC(I, \mathbb{R}^+)$, consider the below inequalities

$$\begin{cases} |w^{\Delta}(\theta) - \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta)))| \leq \epsilon, \quad \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]. \\ \left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta - \zeta)^{q-1}g_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases} \tag{4.1}$$

$$\begin{cases} |w^{\Delta}(\theta) - \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta)))| \leq \epsilon\varphi(\theta), \quad \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]. \\ \left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta - \zeta)^{q-1}g_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon\psi, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases} \tag{4.2}$$

Definition 4.1 ([25]) Equation (1.1) is called Hyer's-Ulam stable if there exists a positive constant $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}$ such that for $\epsilon > 0$ and for each solution w of inequality (4.1), there exist a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \leq H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}\epsilon, \quad \forall\, \theta \in I.$$

Definition 4.2 ([25]) Equation (1.1) is said to be generalized Hyer's-Ulam stable if there exists $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(0) = 0$ such that for each solution w of inequalities (4.1), there exists a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \le \mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(\epsilon), \quad \forall\, \theta \in I.$$

Remark 4.3 Definition (4.1) $\implies$ Definition (4.2).

Definition 4.4 ([25]) Equation (1.1) is said to be Hyers-Ulam-Rassias stable w.r.t (φ, ψ), if there exists $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g, \varphi)}$ such that for $\epsilon > 0$ and for each solution w of inequality (4.2), there exist a unique solution v of Eq. (1.1) satisfies the following inequality

$$|w(\theta) - v(\theta)| \le H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g, \varphi)}\epsilon(\varphi(\theta), \psi), \quad \forall\, \theta \in I.$$

Remark 4.5 A function $w \in PC(I, \mathbb{R})$ is a solution of inequality (4.1) if and only if there is $\mathsf{G} \in PC(I, \mathbb{R})$ and a sequence $\mathsf{G}_k,\ k = 1, 2, \ldots, l$, such that

(a) $|\mathsf{G}(\theta)| \le \epsilon, \forall\, \theta \in \cup_{k=0}^{l}(\lambda_k, \theta_{k+1}]$ and $|\mathsf{G}_k| \le \epsilon,\ \forall\, \theta \in (\theta_k, \lambda_k], k = 1, 2, \ldots, l.$
(b) $w^{\Delta}(\theta) = \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta))) + \mathsf{G}(\theta),\ \theta \in (\lambda_k, \theta_{k+1}],\ k = 0, 1, \ldots, l.$
(c) $w(\theta) = \dfrac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l.$

Now, by the above Remark 4.5, we have:

$$\begin{cases} w^{\Delta}(\theta) = \mathcal{C}(\theta, w(\theta), \mathcal{M}(w(\theta))) + \mathsf{G}(\theta),\ \theta \in (\lambda_k, \theta_{k+1}],\ k = 0, 1, \ldots, l, \\ w(\theta) = \dfrac{1}{\Gamma(q)} \displaystyle\int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \quad \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l. \end{cases}$$

From Lemma 2.6, one can find that the solution w with $w(0) = w(T)$ of the above equation is given by

$$\begin{aligned} w(\theta) &= \int_0^{\theta} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta + \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, w(\theta_l^-))\Delta\zeta + \mathsf{G}_l \\ &\quad + \int_{\lambda_l}^{T} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta, \quad \forall\, \theta \in [0, \theta_1], \\ w(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k, \quad \forall\, \theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l, \\ w(\theta) &= \frac{1}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} \boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta + \mathsf{G}_k + \int_{\lambda_k}^{\theta} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) + \mathsf{G}(\zeta))\Delta\zeta, \\ &\qquad \forall\, \theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l. \end{aligned}$$

Therefore, for $\theta \in (\lambda_k, \theta_{k+1}],\ k = 1, 2, \ldots, l$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta - \int_{\lambda_k}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right|$$
$$\leq |\mathsf{G}_k| + \int_{\lambda_k}^{\theta}|\mathsf{G}(\zeta)|\Delta\zeta \leq \epsilon(1+T).$$

Also, for $\theta \in [0, \theta_1]$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_l}^{\lambda_l}(\lambda_l-\zeta)^{q-1}\boldsymbol{g}_l(\zeta, w(\theta_l^-))\Delta\zeta - \int_{\lambda_l}^{T}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right.$$
$$\left. - \int_{0}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right| \leq |\mathsf{G}_l| + \int_{\lambda_l}^{T}|\mathsf{G}(\zeta)|\Delta\zeta + \int_{0}^{\theta}|\mathsf{G}(\zeta)|\Delta\zeta$$
$$\leq \epsilon(1+2T).$$

Similarly, for $\theta \in (\theta_k, \lambda_k],\ k = 1, 2, \ldots, l$, we have:

$$\left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\theta}(\theta-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right| \leq \epsilon.$$

We have similar remark for the inequality (4.2).

Theorem 4.6 *If the assumptions of Theorem 3.1 are holds, then the Eq. (1.1) is Hyer-Ulam stable.*

Proof Let $w \in PC(I, \mathbb{R})$ be the solution of inequality (4.1) and $v \in PC(I, \mathbb{R})$ be a unique solution of the Eq. (1.1). Therefore, for $\theta \in (\lambda_k, \theta_{k+1}]$, $k = 1, 2, \ldots, l$, we have:

$$|w(\theta) - v(\theta)| \leq \left|w(\theta) - \int_{\lambda_k}^{\theta}\mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta)))\Delta\zeta\right| - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta$$
$$\leq \left|w(\theta) - \frac{1}{\Gamma(q)}\int_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}\boldsymbol{g}_k(\zeta, w(\theta_k^-))\Delta\zeta\right.$$
$$\left. - \int_{\lambda_k}^{\theta}\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta)))\Delta\zeta\right|$$
$$+ \left|\frac{1}{\Gamma(q)}\int_{\theta_k}^{\lambda_k}(\lambda_k-\zeta)^{q-1}(\boldsymbol{g}_k(\zeta, w(\theta_k^-)) - \boldsymbol{g}_k(\zeta, v(\theta_k^-)))\Delta\zeta\right|$$

$$
\begin{aligned}
&+ \left| \int_{\lambda_k}^{\theta} (\mathcal{C}(\zeta, w(\zeta), \mathcal{M}(w(\zeta))) - \mathcal{C}(\zeta, w(\zeta), \mathcal{M}(v(\zeta)))) \Delta\zeta \right| \\
&\leq \epsilon(1+T) + \frac{L_g}{\Gamma(q)} \int_{\theta_k}^{\lambda_k} (\lambda_k - \zeta)^{q-1} |w(\theta_k^-) - v(\theta_k^-)| \Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_{\lambda_k}^{\theta} |w(\zeta) - v(\zeta)| \Delta\zeta + L_{\mathcal{C}_2} \int_{\lambda_k}^{\theta} |\mathcal{M}(w(\zeta)) - \mathcal{M}(v(\zeta))| \Delta\zeta \\
&\leq \epsilon(1+T) + \frac{L_g e_\Omega(\theta_k^-, \theta_k)(\lambda_k - \theta_k)^q \|v - w\|_\Omega}{\Gamma(q+1)} \\
&+ \frac{L_{\mathcal{C}_1} e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega} + \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, \lambda_k) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Hence,

$$
\|w - v\|_\Omega \leq \epsilon(1+T) + \left[\frac{L_g e_\Omega(\theta_k^-, \theta_k) T^q}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2} \right] \|v - w\|_\Omega. \quad (4.3)
$$

Also, for $\theta \in [0, \theta_1]$, we have:

$$
\begin{aligned}
|w(\theta) - v(\theta)| &\leq \left| w(\theta) - \int_0^{\theta} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) \Delta\zeta \right| \\
&- \frac{1}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} \boldsymbol{g}_l(\zeta, v(\theta_l^-)) \Delta\zeta - \int_{\lambda_l}^{T} \mathcal{C}(\zeta, v(\zeta), \mathcal{M}(v(\zeta))) \Delta\zeta \\
&\leq \epsilon(1+2T) + \frac{L_g}{\Gamma(q)} \int_{\theta_l}^{\lambda_l} (\lambda_l - \zeta)^{q-1} |v(\theta_l^-) - w(\theta_l^-)| \Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_{\lambda_l}^{T} |v(\zeta) - w(\zeta)| \Delta\zeta + L_{\mathcal{C}_2} \int_{\lambda_l}^{T} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))| \Delta\zeta \\
&+ L_{\mathcal{C}_1} \int_0^{\theta} |v(\zeta) - w(\zeta)| \Delta\zeta + L_{\mathcal{C}_2} \int_0^{\theta} |\mathcal{M}(v(\zeta)) - \mathcal{M}(w(\zeta))| \Delta\zeta \\
&\leq \epsilon(1+2T) + \frac{L_g e_\Omega(\theta_l^-, \theta_l)(\lambda_l - \theta_l)^q \|v - w\|_\Omega}{\Gamma(q+1)} + \frac{L_{\mathcal{C}_1} e_\Omega(T, \lambda_l) \|v - w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2} L_h e_\Omega(T, \lambda_l) \|v - w\|_\Omega}{\Omega^2} + \frac{L_{\mathcal{C}_1} e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega} \\
&+ \frac{L_{\mathcal{C}_2} L_h e_\Omega(\theta, 0) \|v - w\|_\Omega}{\Omega^2}.
\end{aligned}
$$

Thus,

$$\|w-v\|_{\Omega} \leq \epsilon(1+2T) + \left[\frac{L_{\boldsymbol{g}} e_{\Omega}(\theta_l^-, \theta_l) T^q}{\Gamma(q+1)} + (1+e_{\Omega}(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right]\|v-w\|_{\Omega}. \tag{4.4}$$

Similarly, for $\theta \in (\theta_k, \lambda_k],\ k=1,2,\ldots,l$, we can easily find that

$$\begin{aligned} |w(\theta)-v(\theta)| &\leq \left|w(\theta) - \frac{1}{\Gamma(q)} \int_{\theta_k}^{\theta} (\theta-\zeta)^{q-1} \boldsymbol{g}_k(\zeta, v(\theta_k^-))\Delta\zeta\right| \\ &\leq \epsilon + \frac{L_{\boldsymbol{g}}(\lambda_k-\theta_k)^q e_{\Omega}(\theta_k^-, \theta_k)\|w-v\|_{\Omega}}{\Gamma(q+1)}. \end{aligned}$$

Therefore,

$$\|w-v\|_{\Omega} \leq \epsilon + \frac{L_{\boldsymbol{g}} T^q}{e_{\Omega}(\theta_k, \theta_k^-)\Gamma(q+1)}\|v-w\|_{\Omega}. \tag{4.5}$$

After summarizing the above inequalities (4.3)–(4.5), we get:

$$\begin{aligned} \|w-v\|_{\Omega} \leq \epsilon(1+2T) + &\left[\frac{L_{\boldsymbol{g}} e_{\Omega}(\theta_l^-, \theta_l) T^q}{\Gamma(q+1)} + (1+e_{\Omega}(T, \lambda_l))\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right)\right] \\ &\times \|v-w\|_{\Omega},\ \forall\, \theta \in I. \end{aligned}$$

Hence,

$$\|w-v\|_{\Omega} \leq H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}\epsilon, \quad \theta \in I,$$

where $H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)} = \dfrac{1+2T}{1-L_{\Pi}} > 0$. Thus, the Eq. (1.1) is Ulam-Hyer's stable. Moreover, if we put $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(\epsilon) = H_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}\epsilon$, $\mathcal{H}_{(L_{\mathcal{C}_1}, L_{\mathcal{C}_2}, L_h, L_g)}(0) = 0$, then the Eq. (1.1) is generalized Ulam-Hyer's stable. □

(H7): There exists a $\delta_{\varphi} > 0$ such that $\int_0^{\theta} \varphi(\zeta)\Delta\zeta \leq \delta_{\varphi}\varphi(\theta),\ \forall\, \theta \in I$.

The following theorem is the consequence of the Theorem 4.6.

Theorem 4.7 *If the conditions of Theorem 3.1 and* ***(H7)*** *are holds, then the Eq. (1.1) is Hyer's-Ulam-Rassias stable.*

5 Example

Consider the following equation with impulses on $\mathbb{T}$, $(0, 3/5, 4/5, 1 \in \mathbb{T})$

$$v^{\Delta}(\theta) = \frac{5 + |v(\theta)|}{20e^{\theta+3}(1 + |v(\theta)|)} + \frac{1}{10}\int_{0}^{\theta} \frac{\theta\tau^2 \sin(v(\tau))}{e^{\tau+5}}\Delta\tau, \quad \theta \in I' = [0, 1]_{\mathbb{T}} \setminus (\theta_1, \lambda_1]_{\mathbb{T}},$$

$$v(\theta) = \frac{1}{\Gamma(q)}\int_{\theta_1}^{\theta} \frac{(\theta - \zeta)^{q-1}(1 + \zeta^2 \sin(v(\theta_1^-)))}{15}\Delta\zeta, \quad \theta \in (\theta_1, \lambda_1]_{\mathbb{T}}, \tag{5.1}$$

$$v(0) = v(1).$$

Set,

$$\mathcal{C}(\theta, v, w) = \frac{5 + |v(\theta)|}{20e^{\theta+3}(1 + |v(\theta)|)} + \frac{1}{10}w, \ \theta \in I', \ v, w \in \mathbb{R},$$

$$h(\theta, \tau, v) = \frac{\theta\tau^2 \sin(v(\tau))}{e^{\tau+5}}, \ \forall\, \theta, \tau \in I', v \in \mathbb{R},$$

and

$$\boldsymbol{g}_1(\theta, v) = \frac{1 + \theta^2 \sin(v(\theta_1^-))}{15}, \ \theta \in (\theta_1, \lambda 1], \ v \in \mathbb{R}.$$

Then, $\forall\, \theta, \tau \in I = [0, 1], \ v, w, x, y \in \mathbb{R}$, we have:

$$\begin{aligned}
|f(\theta, v, w) - f(\theta, x, y)| &\le \frac{1}{20e^3}|v - x| + \frac{1}{10}|w - y|, \\
|f(\theta, v, w)| &\le \frac{5 + |v|}{20e^3} + \frac{1}{10}|w|, \\
|\boldsymbol{g}_1(\theta, v) - \boldsymbol{g}_1(\theta, w)| &\le \frac{1}{15}|v - w|, \ |h(\theta, \tau, v)| \le \frac{1}{e^5} + \frac{1}{e^5}|v|, \\
|h(\theta, \tau, v) - h(\theta, \tau, w)| &\le \frac{1}{e^5}|v - w|.
\end{aligned}$$

Hence, the assumptions (H1)–(H4) are holds with $L_{\mathcal{C}_1} = \frac{1}{20e^3}$, $L_{\mathcal{C}_2} = \frac{1}{10}$, $C_{\mathcal{C}} = \frac{5}{20e^3}$, $M_{\mathcal{C}} = \frac{1}{20e^3}$, $N_{\mathcal{C}} = \frac{1}{10}$, $L_h = \frac{1}{e^5}$, $C_h = \frac{1}{e^5}$, $M_h = \frac{1}{e^5}$, $L_g = \frac{1}{15}$, $M_g = \frac{2}{15}$. Also, for $l = 1, \ \theta_1 = 3/5, \ \lambda_1 = 4/5, \ T = 1, \ \Omega = 10$, the condition

$$e_{\Omega}(T, \lambda_1)\left(\frac{L_{\mathcal{C}_1}}{\Omega} + \frac{L_{\mathcal{C}_2} L_h}{\Omega^2}\right) = 0.0039\ (<1)$$

holds. Thus, from Theorems 3.1 and 4.6, Eq. (5.1) has a Ulam Hyer's stable solution which is unique.

6 Conclusion

In this manuscript, we have successfully established the existence of a unique solution for the system (1.1) by using the Banach contraction theorem and nonlinear functional analysis. Also, we established the Ulam-Hyer's stability of the taken problem (1.1). To illustrate the application of obtained results, we have given an example.

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Introduction to Class of Uniformly Fractional Differentiable Functions

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Abstract In this paper, authors introduced new concept of uniformly fractional differentiable functions on an arbitrary interval I of R by using Caputo-type fractional derivative instead of the commonly used first-order derivative. Their interesting properties with few illustrations have been discussed in this paper.

Keywords Uniformly differentiable functions · Uniformly continuous functions · Uniformly fractional differentiable functions · Caputo fractional derivative

Mathematics Subject Classification (2000) 26A33 · 34A08 · 34A12

1 Introduction

The fractional calculus is a theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n-fold integration. We shall explain the result connected to classical analysis, namely uniformly differential functions given by Patel [1], can be extended to fractional calculus, i.e they can be generalized by replacing the first order the first derivatives and integrals, respectively, by derivatives and integrals of non-integer. The uniformly differentiable function can be defined as:

Definition 1 Let I be an interval in R. A differentiable function $f : I \to R$ is uniformly differentiable, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in I$ satisfying $|x - y| < \delta$,

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$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \epsilon \tag{1}$$

and

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \epsilon \tag{2}$$

The collection of all uniformly differentiable functions on I will be denoted by $UD(I)$. The class of uniformly differentiable function has connection with class of uniformly continuous functions which are well-known class of functions in classical analysis. The uniform continuous defined by Apostol [2] as:

Definition 2 A function $f : I \to R$ is uniformly continuous function on interval I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$|f(x) - f(y)| < \epsilon \tag{3}$$

Definition 3 The Caputo fractional derivative of order α defined by Caputo [3] as

$$^{C}D^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha-1}}\, d\tau \;(n-1 < \alpha < n) \tag{4}$$

The following theorem is given by Diethelm [4].

Theorem 4 *Let* $0 < \alpha \leq 1$, $a < b$ *and* $f \in C[a, b]$ *be such that* $^{C}D^{\alpha}(f) \in C[a, b]$. *Then there exist* $\xi \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{(b-a)^{\alpha}} = \frac{1}{\Gamma(\alpha)} {}^{C}D^{\alpha}(f(\xi)) \tag{5}$$

Also some properties of Local fractional calculus was studied by Yang [5] and Yang and Gao [6]. Kachhia and Prajapati [7] introduced concept of functions of bounded fractional differential variation using the Caputo-type fractional derivative.

Definition 5 A Caputo fractional differentiable function f is absolutely fractional differentiable function on interval I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for an collection of pairwise disjoint intervals $\{(a_i, b_i)\}$ in I satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$,

$$\sum_{i=1}^{n} \left| \Gamma(\alpha) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(a_i)) \right| < \epsilon \tag{6}$$

and

$$\sum_{i=1}^{n} \left| \Gamma(\alpha) \left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(b_i)) \right| < \epsilon \tag{7}$$

where $0 < \alpha \leq 1$.

The Hölder continuous function defined by Gilberg and Trudinger [8] as:

Definition 6 A function $f : R \to C$ is said to be Hölder continuous if for all $x, y \in R$, there are non-negative real constants M, α such that

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}$$

2 Uniformly Fractional Differentiable Functions

In this section, authors introduced the new concept of uniformly factional differentiable functions as:

Definition 7 Let I be an interval in R. A Caputo fractional differentiable function f is uniformly fractional differentiable function on I, if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in I$ satisfying $|x - y| < \delta$,

$$\left| \Gamma(\alpha) \left(\frac{f(x) - f(y)}{(x - y)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(x)) \right| < \epsilon \tag{8}$$

and

$$\left| \Gamma(\alpha) \left(\frac{f(x) - f(y)}{(x - y)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(y)) \right| < \epsilon \tag{9}$$

where $0 < \alpha \leq 1$.

If we take $\alpha = 1$, then Eqs. (8) and (9) reduces to Eqs. (1) and (2) respectively. The collection of all uniformly fractional differentiable functions on I will be denoted by $UFD(I)$.

Theorem 8 *A function f is uniformly fractional differentiable function on an interval I if and only if ${}^{C}D^{\alpha}(f)$ is uniformly continuous on I.*

Proof Let $f : I \to R$ be uniformly fractional differentiable. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left| \Gamma(\alpha) \left(\frac{f(x) - f(y)}{(x - y)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(x)) \right| < \frac{\epsilon}{2} \tag{10}$$

and

$$\left| \Gamma(\alpha) \left(\frac{f(x) - f(y)}{(x - y)^{\alpha}} \right) - {}^{C}D^{\alpha}(f(y)) \right| < \frac{\epsilon}{2} \tag{11}$$

Now for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left|{}^C D^\alpha(f(x)) - {}^C D^\alpha(f(y))\right| =$$
$$\left|{}^C D^\alpha(f(x)) - \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) + \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(y))\right| \tag{12}$$

We get

$$\begin{aligned}\left|{}^C D^\alpha(f(x)) - {}^C D^\alpha(f(y))\right| \le &\left|{}^C D^\alpha(f(x)) - \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right)\right| \\ &+ \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(y))\right|\end{aligned} \tag{13}$$

By using Eqs. (10) and (11), we obtain

$$|{}^C D^\alpha(f(b_i)) - {}^C D^\alpha(f(a_i))| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{14}$$

Hence ${}^C D^\alpha(f)$ is a uniformly continuous on I.
Conversely suppose that ${}^C D^\alpha(f)$ is uniformly continuous on I. Let $\epsilon > 0$ be given. Then there exist a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left|{}^C D^\alpha(f(x)) - {}^C D^\alpha(f(y))\right| < \epsilon \tag{15}$$

Then from Theorem 4, there exist $c \in (y, x)$ such that

$$f(x) - f(y) = \frac{{}^C D^\alpha(f(x))(x-y)^\alpha}{\Gamma(\alpha)} \tag{16}$$

Since $|c - y| < \delta$, for any $\epsilon > 0$, there exist a $\delta > 0$ such that for any x, y in I

$$|{}^C D^\alpha(f(c)) - {}^C D^\alpha(f(y))| < \epsilon \tag{17}$$

By using Eq. (16)

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^\alpha}\right) - {}^C D^\alpha(f(y))\right| < \epsilon \tag{18}$$

Again $|x - c| < \delta$, then for any $\epsilon > 0$, there exist a $\delta > 0$ such that for any x, y in I

$$\sum_{i=1}^{n}\left|{}^C D^\alpha(f(x)) - {}^C D^\alpha(f(c))\right| < \epsilon \tag{19}$$

By using Eq. (16)

$$\sum_{i=1}^{n} \left| {}^{C}D^{\alpha}(f(x)) - \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) \right| < \epsilon \tag{20}$$

Therefore f is uniformly fractional differentiable on I.

Example 9 The $\frac{1}{2}$ order Caputo derivative of function $f(t) = t$ is $2\sqrt{\frac{t}{\pi}}$ which is uniformly continuous on $[0, c]$. Then by Theorem 8 uniformly fractional differentiable functions on $[0, c]$ of order $\frac{1}{2}$.

In fact, using Theorem 8, several examples of uniformly fractional differentiable functions can be constructed.
The following is motivated by the principle that differentiability implies continuity.

Theorem 10 *If f is uniformly fractional differentiable function on an interval I, then f is uniformly continuous on I.*

Proof Since a function $f : I \to R$ is uniformly fractional differentiable, then if for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x - y| < \delta$,

$$\left| \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x)) \right| < \epsilon \tag{21}$$

and

$$\left| \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(y)) \right| < \epsilon \tag{22}$$

Since ${}^{C}D^{\alpha}(f)$ is bounded on I, so there exit $M > 0$ such that

$$|{}^{C}D^{\alpha}(f(t))| \leq M \ (\forall\, t \in I) \tag{23}$$

Take $\delta_0 = \min\{(\delta)^{\frac{1}{\alpha}}, (\frac{\epsilon}{\epsilon+M})^{\frac{1}{\alpha}}\}$. Let $x, y \in I$ satisfying $|x - y| < \delta_0$.
Now

$$\begin{aligned} &|f(x) - f(y)| \leq \\ &\left| \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)(x-y)^{\alpha} - {}^{C}D^{\alpha}(f(x))(x-y)^{\alpha} + {}^{C}D^{\alpha}(f(x))(x-y)^{\alpha} \right| \end{aligned} \tag{24}$$

Therefore

$$\begin{aligned} &|f(x) - f(y)| \leq \\ &\left| \Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x)) \right| |x-y|^{\alpha} + |{}^{C}D^{\alpha}(f(x))||x-y|^{\alpha} \end{aligned} \tag{25}$$

Finally

$$|f(x) - f(y)| < \delta_0\epsilon + M\delta_0 = \delta_0(\epsilon + M) < \epsilon \tag{26}$$

Hence f is an uniformly continuous on I.

Theorem 11 *Every absolutely fractional differentiable function on I is uniformly fractional differentiable on I.*

Proof Since $f : I \to R$ is an absolutely fractional differentiable. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any finite collection of pairwise disjoint intervals $\{(a_i, b_i)\}$ in I satisfying $\sum_{i=1}^{n}(b_i - a_i) < \delta$,

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^\alpha}\right) - {}^C D^\alpha(f(a_i))\right| < \epsilon \tag{27}$$

and

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{f(b_i) - f(a_i)}{(b_i - a_i)^\alpha}\right) - {}^C D^\alpha(f(b_i))\right| < \epsilon \tag{28}$$

In particular

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^\alpha}\right) - {}^C D^\alpha(f(x))\right| < \epsilon \tag{29}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^\alpha}\right) - {}^C D^\alpha(f(y))\right| < \epsilon \tag{30}$$

Hence f is uniformly fractional differentiable function on I.

Proposition 12 *If f is uniformly fractional differential function on I and if ${}^C D_a^\alpha(f)$ is bounded on I, then f is Hölder continuous on I.*

Proof Let f is uniformly fractional differential function. Then for x, y in I satisfying $|x - y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^\alpha}\right) - {}^C D^\alpha(f(x))\right| < \epsilon \tag{31}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x) - f(y)}{(x - y)^\alpha}\right) - {}^C D^\alpha(f(y))\right| < \epsilon \tag{32}$$

Since ${}^C D^\alpha(f)$ is bounded on I, so there exit $M > 0$ such that

$$|{}^C D^\alpha(f(t))| \leq M \ (\forall\, t \in I) \tag{33}$$

Now

$$
\begin{aligned}
|f(x)-f(y)| &= \left|\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)(x-y)^{\alpha} - {}^{C}D_a^{\alpha}(f(y))(x-y)^{\alpha} + {}^{C}D_a^{\alpha}(f(y))(x-y)^{\alpha}\right| \\
&\leq \left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D_a^{\alpha}(f(y))\right| |x-y|^{\alpha} + |{}^{C}D_a^{\alpha}(f(y))||x-y|^{\alpha} \\
&\leq (\epsilon + M)|x-y|^{\alpha}
\end{aligned}
$$

Hence f is Hölder continuous function on R.

Theorem 13 *The space $UFD(I)$ of uniformly fractional differentiable functions on interval I is a vector space with pointwise operations.*

Proof Let $f, g \in UFD(I)$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x-y| < \delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \frac{\epsilon}{2} \tag{34}$$

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| < \frac{\epsilon}{2} \tag{35}$$

$$\left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(g(x))\right| < \frac{\epsilon}{2} \tag{36}$$

and

$$\left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(g(x))\right| < \frac{\epsilon}{2} \tag{37}$$

Now for any $\epsilon > 0$, there is a $\delta > 0$ such that for any x, y in I satisfying $|x-y| < \delta$,

$$
\begin{aligned}
&\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}((f+g)(x))\right| = \\
&\quad \left|\Gamma(\alpha)\left(\frac{(f(x)+g(x))-(f(y)+g(y))}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}((f(x)+(g(x))\right|
\end{aligned} \tag{38}
$$

Then

$$
\begin{aligned}
&\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}((f+g)(x))\right| \leq \\
&\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(f(x))\right| + \left|\Gamma(\alpha)\left(\frac{g(x)-g(y)}{(x-y)^{\alpha}}\right) - {}^{C}D^{\alpha}(g(x))\right|
\end{aligned} \tag{39}
$$

By using Eqs. (34) and (35) the Eq. (39) reduces to

$$\sum_{i=1}^{n}\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(x)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((f+g)(x))\right|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon \tag{40}$$

Similarly by using Eqs. (36) and (37) we obtain

$$\left|\Gamma(\alpha)\left(\frac{(f+g)(x)-(f+g)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((f+g)(y))\right|<\epsilon \tag{41}$$

Hence $f+g\in UFD(I)$. Now let $f\in UFD(I)$ and $k\in C$. Then for any $\epsilon>0$, there is a $\delta>0$ such that for any x, y in I satisfying $|x-y|<\delta$,

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(x))\right|<\frac{\epsilon}{k}, \tag{42}$$

and

$$\left|\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}(f(y))\right|<\frac{\epsilon}{k}, \tag{43}$$

Now for any $\epsilon>0$, there is a $\delta>0$ such that for any x, y in I satisfying $|x-y|<\delta$,

$$\begin{aligned}\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(x))\right| = \\ \left|k\Gamma(\alpha)\left(\frac{f(x)-f(y)}{(x-y)^{\alpha}}\right)-k\,{}^{C}D^{\alpha}((f(x))\right|\end{aligned} \tag{44}$$

By using Eq. (42) the above equation reduces to

$$\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(x))\right|<k\frac{\epsilon}{k}=\epsilon \tag{45}$$

Similarly by using Eq. (43) we obtain

$$\left|\Gamma(\alpha)\left(\frac{(kf)(x)-(kf)(y)}{(x-y)^{\alpha}}\right)-{}^{C}D^{\alpha}((kf)(y))\right|<\epsilon \tag{46}$$

Thus $kf\in UFD(I)$.
Therefore the space $UFD(I)$ of uniformly fractional differentiable functions on I is a vector space with pointwise operations.

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Asymptotically Almost Automorphic Solution for Neutral Functional Integro Evolution Equations on Time Scales

Soniya Dhama and Syed Abbas

Abstract The script is dedicated to look at the existence, uniqueness with stability consequence of asymptotically almost automorphic ($\mathcal{AAA}$) solution for integro neutral evolution equation on time scales by applying fixed point hypothesis. We give the time scale adaptation of ($\mathcal{AAA}$) functions. Toward the end, a precedent is given for the adequacy of the hypothetical outcomes.

Keywords Asymptotically almost automorphic function · Evolution system · Neutral · Integro · Time scales

1 Introduction

Generally, one study the continuous and discrete cases differently and there are many different sets which are very utilizable. Ergo, this an arduous task that we study differently for all cases. So for evading this type quandary, Hilger, in 1988, [1] present time scales hypothesis which cumulates discrete and continuous investigation. This hypothesis present a robust actualize for applications to populace models, financial matters and quantum material science among others. Thus, managing issues of differential conditions on time scales turns out to be extremely noteworthy and deliberate in the examination field of dynamic frameworks. For more subtle elements of this theme, we allude to the papers [2–4] and the books [5, 6]. These give a glorious portrayal of time scale hypothesis and its apparatus.

Almost automorphy, which is a natural generalization of almost periodicity introduced by Bochner [7]. In [8, 9], the literature of almost automorphy and its applications to differential equations are describe. Recently, the existence of almost automorphic ($\mathcal{AA}$) type solutions for evolution equations has attracted more and

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more attention. The literature of the concept of asymptotically almost automorphy, as a natural extension of almost automorphy, was introduced by N'Guérékata [10]. Now a days, these type of functions have made lots of developments and applications in real life, we refer for more details [11–15].

There are numerous marvels, for example, in the investigation of oscillatory frameworks and in the displaying of a few physical issues, where the theory of neutral differential equations arises [16]. There are many papers on existence of $\mathcal{AAA}$ solution for continuous cases. As per our knowledge, there is no paper on time scale where these type of solution is discussed with neutral functional term in abstract space. The rationale of the present article is discover the existence and uniqueness with stability of $\mathcal{AAA}$ solution for the neutral integro evolution equation on periodic time scale $\mathbb{T}$,

$$\begin{aligned}[y(r) - g(r, y(\varkappa(r)))]^{\Delta} &= A(r)[y(r) - g(r, y(\varkappa(r)))] + \mathcal{P}(r, y(r)) \\ &\quad + \int_{-\infty}^{r} k(r, \sigma(s))h(r, y(s))\Delta s,\end{aligned} \tag{1.1}$$

$r \in \mathbb{T}$. $A(r) : \mathcal{D}(A(r)) \subset Y \to Y$ is a family of linear operators, where Y is Banach space. $|k(r,s)| \leq ce_{\ominus\lambda}(r,s)$, c and λ are positive constant and $\varkappa : \mathbb{T} \to \mathbb{T}$ satisfying $\varkappa(r) \leq r$ for all $r \in \mathbb{T}$. The functions $\mathcal{P} : \mathbb{T} \times Y \to Y$, $g, h : \mathbb{T} \times Y \to Y$ are defined later with specified conditions in next section.

Whatever is left of this article as follows. In Sect. 2, we give basic definitions, results and lemmas. In Sect. 3, using Banach contraction principle, existence and uniqueness of $\mathcal{AAA}$ solution of system (1.1) is discussed. In Sect. 4, some conditions for stability are obtained . In last Sect. 5 a numerical example is shown for potency of hypothetical outcomes.

2 Preliminaries

In this segment, some essential hypothesis and facts for time scales is given which is required for further work.

A time scale, $\mathbb{T}$, is a non empty closed subset of real line. The backward and forward operator is define by $\rho(\zeta) = \sup\{s \in \mathbb{T} : s < \zeta\}$ and $\sigma(\zeta) = \inf\{s \in \mathbb{T} : s > \zeta\}$ respectively. A point ζ is a left dense point and left scattered point when $\rho(\zeta) = \zeta$ and $\rho(\zeta) < \zeta$ respectively with $\zeta > \inf \mathbb{T}$. Also, ζ is right scattered point and right dense when $\sigma(\zeta) > \zeta$ and $\sigma(\zeta) = \zeta$ respectively with $\zeta < \sup \mathbb{T}$. A function $\mu : \mathbb{T} \to [0, \infty)$ is given by $\mu(\zeta) = \sigma(\zeta) - \zeta$, $\forall \zeta \in \mathbb{T}$, is known as the graininess operator. We will mean the interval $[c, d]_{\mathbb{T}} = \{\zeta \in \mathbb{T} : c \leq \zeta \leq d\}$.

Definition 2.1 If $\Lambda : \mathbb{T} \to \mathbb{R}$ is a function and at left dense points, its left-side limits exist and continuous at right dense points of $\mathbb{T}$ then it is known as rd-continuous. The collection of all rd-continuous functions $\Lambda : \mathbb{T} \to \mathbb{R}$ will be mean by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 Reference [5] A function $q : \mathbb{T} \to \mathbb{R}$ is said to be regressive (positive regressive) if $1 + \mu(\zeta)q(\zeta) \neq 0 (> 0)$, $\forall \zeta \in \mathbb{T}$. The collection of regressive (positive regressive) functions is represented by $\mathcal{R}(\mathcal{R}^+)$.

Definition 2.3 Reference [5] Let $\Lambda : \mathbb{T} \to \mathbb{R}$ and $\zeta \in \mathbb{T}$. Δ-derivative, $\Lambda^{\Delta}(\zeta)$ is the number if exist, such that given any $\varepsilon > 0$, $\exists$ a neighbourhood U of ζ such that

$$\left|[\Lambda(\sigma(\zeta)) - \Lambda(s)] - \Lambda^{\Delta}(\zeta)[\sigma(\zeta) - s]\right| \leq \varepsilon|\sigma(\zeta) - s|, \ \forall s \in U.$$

Let Λ is rd-continuous; if $\Lambda_*^{\Delta}(\zeta) = \Lambda(\zeta)$, the delta integral is defined by,

$$\int_r^s \Lambda(\zeta)\Delta\zeta = \Lambda_*(s) - \Lambda_*(r), \quad s, r \in \mathbb{T}.$$

Definition 2.4 The exp function on $\mathbb{T}$ is defined as

$$e_q(\tau, \zeta) = \exp\left(\int_\zeta^\tau \xi_{\mu(t)}(q(t))\Delta t\right), \quad \tau, \zeta \in \mathbb{T}, q \in \mathcal{R}.$$

For $b > 0$,

$$\xi_b(Z) = \frac{1}{b} log(1 + Zb).$$

For $b = 0$, $\xi_0(Z) = Z$.

Definition 2.5 Reference [6] Let $q, p \in \mathcal{R}$, define

$$\ominus q = \frac{-q}{1 + \mu q}, \quad q \oplus p = q + p + \mu qp, \quad q \ominus p = q \oplus (\ominus p).$$

Lemma 2.6 *Reference [6] Let us suppose that $p, q \in \mathcal{R}$, then*

1. $e_0(\zeta, r) = 1, \quad e_p(\zeta, \zeta) = 1;$
2. $e_p(\sigma(\zeta), r) = (1 + \mu(\zeta)p)e_p(\zeta, r);$
3. $e_p(\zeta, r) = 1/e_p(r, \zeta) = e_{\ominus p}(r, \zeta);$
4. $e_p(\zeta, r)e_p(r, s) = e_p(\zeta, s);$
5. $e_p(\zeta, r)e_q(\zeta, r) = e_{p \oplus q}(\zeta, r);$
6. $(1/e_p(\zeta, r))^{\Delta} = -p(\zeta)/e_p(\sigma(\zeta), r).$

Lemma 2.7 *Reference [6] Let $q \in \mathcal{R}$ and $b, c, d \in \mathbb{T}$, then*

$$\int_b^c q(\zeta)e_q(d, \sigma(\zeta))\Delta\zeta = e_q(d, b) - e_q(d, c).$$

Lemma 2.8 *Reference [17] For* $0 < \lambda$, $e_{\ominus\lambda}(\zeta, \eta) \leq 1$, $\forall \eta, \zeta \in \mathbb{T}$, *where* $\eta \leq \zeta$.

Definition 2.9 Reference [17] $\mathbb{T}$ is called periodic time scale, if

$$\Pi := \{w \in \mathbb{R} : \zeta \pm w \in \mathbb{T}, \forall \zeta \in \mathbb{T}\} \neq \{0\}.$$

The notations in this section follow as: Y is Banach space with sup norm $\|y\|_\infty = \sup_{r\in\mathbb{T}} \|y(r)\|$. $C(\mathbb{T}, Y)$ contains the collection of continuous functions from $\mathbb{T}$ to Y. $C_0(\mathbb{T}, Y)$ is proper subset of $C(\mathbb{T}, Y)$ containing functions $g : \mathbb{T} \to Y$ which vanish at infinity i.e., $\lim_{|r|\to\infty} \|g(r)\| = 0$ and $C_0(\mathbb{T} \times Y, Y)$ denotes the collection of functions $g : \mathbb{T} \times Y \to Y$ such that $\lim_{|r|\to\infty} \|g(r, y)\| = 0$ uniformly for y in any compact subset of Y.

Definition 2.10 A function $g(r) \in C(\mathbb{T}, Y)$ is called almost automorphic ($\mathcal{AA}$) if for every sequence $(\tau'_n) \subset \Pi$, we can extract a subsequence (τ_n) such that

$$g^*(r) := \lim_{n\to\infty} g(r + \tau_n),$$

and

$$\lim_{n\to\infty} g^*(r - \tau_n) = g(r),$$

for each $r \in \mathbb{T}$. We note that the convergence is pointwise. Then, the function g^* not necessarily continuous, but measurable. Moreover, we note if we consider that convergence is uniform on $\mathbb{T}$ instead of pointwise convergence, we get that the function g is almost periodic.

We set $\mathcal{AA}(\mathbb{T}, Y)$ for the collection of all almost automophic functions from $\mathbb{T}$ into Y.

Example 2.11 Let $G : \mathbb{T} \to X$ be a function defined by

$$G(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right).$$

It is $\mathcal{AA}$. However, it not almost periodic because this function is not uniformly continuous on $\mathbb{T}$.

Definition 2.12 A continuous function $g : \mathbb{T} \times Y \to Y$ is called $\mathcal{AA}$ if $g(r, y)$ is $\mathcal{AA}$ in $r \in \mathbb{T}$ uniformly $\forall y$ in any bounded subset of Y.

$\mathcal{AA}(\mathbb{T} \times Y, Y)$ is the collection of all such functions.

Definition 2.13 A continuous function $g : \mathbb{T} \to Y$ is said to be $\mathcal{AAA}$ if $g(r)$ can be decomposed into two parts like that $g(r) = g_1(r) + g_2(r)$, where $g_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$ and $g_2(r) \in C_0(\mathbb{T}, Y)$.

$\mathcal{AAA}(\mathbb{T}, Y)$ is the collection of all such functions.

Definition 2.14 A continuous function $g : \mathbb{T} \times Y \to Y$ is said to be $\mathcal{AAA}$ in r uniformly for y in any compact subset of Y if $g(r, y)$ can be decomposed into two parts like that $g(r, y) = g_1(r, y) + g_2(r, y)$, where $g_1 \in \mathcal{AA}(\mathbb{T} \times Y, Y)$ and $g_2(r) \in C_0(\mathbb{T} \times Y, Y)$.

We set $\mathcal{AAA}(\mathbb{T} \times Y, Y)$ is the collection of all such functions.

Example 2.15 Let $\chi : \mathbb{T} \to Y$ be a function such that

$$\chi(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right) + e^{-|r|}.$$

This function is $\mathcal{AAA}$ as first part belongs to $\mathcal{AA}(\mathbb{T}, Y)$ and second part belongs to $C_0(\mathbb{T}, Y)$.

Example 2.16 Let $\wp : \mathbb{T} \times Y \to Y$ be a function such that

$$\wp(r) = \sin\left(\frac{1}{2 + \sin(r) + \sin(\sqrt{2}r)}\right)\cos y + \frac{1}{1 + r^2}\sin y.$$

This function is $\mathcal{AAA}$ in $r \in \mathbb{T}$ for each $y \in Y$ because first part belongs to $\mathcal{AA}(\mathbb{T} \times Y, Y)$ and second part belongs to $C_0(\mathbb{T} \times Y, Y)$.

Lemma 2.17 *If $g_1, g_2, g \in \mathcal{AAA}(\mathbb{T}, Y)$, then:*

- $g_1 + g_2 \in \mathcal{AAA}(\mathbb{T}, Y)$;
- $\lambda g \in \mathcal{AAA}(\mathbb{T}, Y)$, *for any scalar* λ;
- *If* $\alpha \in \mathbb{R}$ *is a constant then,* $g_\alpha \in \mathcal{AAA}(\mathbb{T}, Y)$, *where* $g_\alpha : \mathbb{T} \to Y$ *define as* $g_\alpha(\cdot) = g(\cdot + \alpha)$;
- *The range* $R_g = \{g(r) : r \in \mathbb{T}\}$ *is relatively compact of* Y, *thus* g *is bounded with respect to norm.*

Definition 2.18 A function $g(r, s)$ is said to be bi-$\mathcal{AA}$ if for every sequence $\tau_n' \subset \Pi$, there is a subsequence τ_n and a function $g^*(r, s)$ such that the translation of g converge to g^*, that is $\|g(r + \tau_n, s + \tau_n) - g^*(r, s)\| \to 0$ as $n \to \infty$ and $\|g^*(r - \tau_n, s - \tau_n) - g(r, s)\| \to 0$ as $n \to \infty$, $\forall\, r, s \in \mathbb{T}$.

We set $bi\mathcal{AA}(\mathbb{T} \times \mathbb{T}, Y)$ is the collection of all such functions.

Remark 2.19 Exponential function on time scale is bi $\mathcal{AA}$ function.

Lemma 2.20 *The decomposition of $\mathcal{AAA}$ function $g = g_1 + g_2$, where $g_1 \in \mathcal{AA}(\mathbb{T}, Y)$ and $g_2 \in C_0(\mathbb{T}, Y)$ is unique i.e., $g = g_1 \oplus g_2$.*

Proof From the definition, we can easily observe $g_1(\mathbb{T}) \subset \overline{g(\mathbb{T})}$. Assume that $g = g_1 + g_2$ and $g = h_1 + h_2$ then $0 = (g_1 - h_1) + (g_2 - h_2) \in \mathcal{AAA}(\mathbb{T}, Y)$, where $(g_1 - h_1) \in \mathcal{AA}(\mathbb{T}, Y)$ and $(g_2 - h_2) \in C_0(\mathbb{T}, Y)$. In view of above result $g_1 - h_1 = 0$. Consequently, $g_2 - h_2 = 0$, i.e., $g_1 = h_1$ and $g_2 = h_2$.

Lemma 2.21 *The space $\mathcal{AAA}(\mathbb{T}, Y)$ is a Banach space with sup norm*

$$\|g\|_\infty = \sup_{r\in\mathbb{T}} \|g(r)\|.$$

Proof Consider $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{AAA}(\mathbb{T}, Y)$. We can express uniquely $g_n = f_n + h_n$, where f_n is a sequence in $\mathcal{AA}(\mathbb{T}, Y)$ and h_n is in $C_0(\mathbb{T}, Y)$. From Lemma 2.20, we see $\|f_n - f_m\|_\infty \le \|g_n - g_m\|_\infty$. We deduce from here that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy sequence in $AA(\mathbb{T}, Y)$. So, $h_n = g_n - f_n$ is Cauchy sequence in $C_0(\mathbb{T}, Y)$. We conclude that $f_n \to f \in \mathcal{AA}(\mathbb{T}, Y)$ and $h_n \to h \in C_0(\mathbb{T}, Y)$ and finally $g_n \to f + h \in \mathcal{AAA}(\mathbb{T}, Y)$.

Lemma 2.22 *Let $g : \mathbb{T} \times Y \to Y,\ (r, y) \to g(r, y) \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ in $r \in \mathbb{T}$, for each $y \in Y$ and assume that g satisfies Lipschitz condition i.e.,*

$$\|g(r, y) - g(r, y^*)\| \le L\|y - y^*\|,$$

for all $y, y^* \in Y$ and for every $r \in \mathbb{T}$, where $L > 0$ is constant. Then $G : \mathbb{T} \to Y$ given by $G(\cdot) = g(\cdot, y(\cdot))$ is $\mathcal{AAA}$ provided $y : \mathbb{T} \to Y$ is $\mathcal{AAA}$.

Proof Since $g, y \in \mathcal{AAA}$, then we can decompose as

$$g = g_1 + g_2, \quad y = y_1 + y_2,$$

where $g_1 \in \mathcal{AA}(\mathbb{T} \times Y, Y),\ g_2 \in C_0(\mathbb{T} \times Y, Y),\ y_1 \in \mathcal{AA}(\mathbb{T}, Y),\ y_2 \in C_0(\mathbb{T}, Y)$. We can write

$$g(r, y(r)) = g_1(r, y_1(r)) + g(r, y(r)) - g(r, y_1(r)) + g_2(r, y_1(r))$$

By *Lemma* 3.3 in [18] $g_1(r, y_1(r)) \in \mathcal{AA}(\mathbb{T}, Y)$. Noticing that $\|g(r, y(r)) - g(r, y_1(r))\| \le L\|y_2(r)\| \to 0$ as $|r| \to \infty$. Hence $g(r, y(r)) - g(r, y_1(r)) \in C_0(\mathbb{T}, Y)$. Now, since $\overline{\{y_1(r),\ r \in \mathbb{T}\}}$ is compact set of Y, $g_2(r, y_1(r)) \in C_0(\mathbb{T}, Y)$. In conclusion, $g(r, y(r)) \in \mathcal{AAA}(\mathbb{T}, Y)$.

Definition 2.23 A continuous function $y : \mathbb{T} \to Y$ is called $\mathcal{AAA}$ solution of system (1.1) on $\mathbb{T}$ if $y(r)$ is and satisfies $\mathcal{AAA}$

$$y(r) = S(r, a)[y(a) - g(a, y(\varkappa(a)))] + g(r, y(\varkappa(r))) + \int_a^r S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s$$
$$+ \int_a^r S(r, \sigma(s)) \int_{-\infty}^s k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s, \qquad \forall\, r \ge a \in \mathbb{T}. \tag{2.1}$$

3 Main Result

To prove main result of this manuscript, we assume the following assumptions which are further mandatory:
A1. The system

$$y^{\Delta}(r) = A(r)y(r), \quad s \leq r, \quad r, s \in \mathbb{T},$$

has an evolution family of operators $\{S(r, s) : s \leq s,\ s, r \in \mathbb{T}\}$. $S(r, s)$ is asymptotically stable i.e., $\exists$ constants $R_0, \omega > 0$ satisfying

$$\|S(r, s)\| \leq R_0 e_{\ominus\omega}(r, s)$$

for all $r, s \in \mathbb{T}$ with $r \geq s$.
A2. For any sequence $\{\tau_n'\}_{n\in\mathbb{N}} \subset \Pi$, we can find a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that for any $\varepsilon > 0$, $\exists N \in \mathbb{N}$,

$$||S(r + \tau_n, s + \tau_n) - S(r, s)|| \leq \varepsilon e_{\ominus\omega(r,s)} \quad \text{and} \quad ||S(r - \tau_n, s - \tau_n) - S(r, s)|| \leq \varepsilon e_{\ominus\omega(r,s)},$$

$\forall n > N,\ \forall r, s \in \mathbb{T}, r \geq s$.
A3. $g \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist constant $L_g > 0$ such that

$$\|g(r, y) - g(r, x)\| \leq L_g \|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

A4. $h \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist a constant $L_h > 0$ such that

$$\|h(r, y) - h(r, x)\| \leq L_h \|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

A5. $\mathcal{P} \in \mathcal{AAA}(\mathbb{T} \times Y, Y)$ and there exist a constant $L_{\mathcal{P}} > 0$ such that

$$\|\mathcal{P}(r, y) - \mathcal{P}(r, x)\| \leq L_{\mathcal{P}} \|y - x\|,\ r \in \mathbb{T},\ x, y \in Y.$$

Lemma 3.1 *Suppose* $\xi \in \mathcal{AAA}(\mathbb{T}, Y)$ *holds,* $\Xi(\eta) : \mathbb{T} \to Y$ *defined by*

$$\Xi(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi(\zeta)\Delta\zeta, \quad \eta \in \mathbb{T},$$

is $\mathcal{AAA}(\mathbb{T}, Y)$.

Proof Since $\xi \in \mathcal{AAA}(\mathbb{T}, Y)$. So, we can decompose it as $\xi(\eta) = \xi_1(\eta) + \xi_2(\eta)$, where $\xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$ and $\xi_2(\eta) \in C_0(\mathbb{T}, Y)$. Now,

$$\Xi(\eta) = \Xi_1(\eta) + \Xi_2(\eta),$$

where $\Xi_1(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1(\zeta)\Delta\zeta$ and $\Xi_2(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta$. To proof complete, we have to prove $\Xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$, $\Xi_2(\eta) \in C_0(\mathbb{T}, Y)$. Since $\xi_1 \in \mathcal{AA}(\mathbb{T}, Y)$, there exists ξ_1^* and a subsequence $\{\tau_n\} \subset \Pi$ for each sequence $\{\tau_n'\}$ such that

$$\lim_{n\to\infty} \|\xi_1(\eta + \tau_n) - \xi_1^*(\eta)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\xi_1^*(\eta - \tau_n) - \xi_1(\eta)\| = 0. \tag{3.1}$$

Now, corresponding to ξ_1^*, let us define $\Xi_1^*(\eta) = \int_{-\infty}^{\eta} k(\eta, \sigma(s))\xi_1^*(s)ds$. Now, we compute

$$\begin{aligned}
\|\Xi_1(\eta + \tau_n) - \Xi_1^*(\eta)\| &= \Big\| \int_{-\infty}^{\eta+\tau_n} k(\eta + \tau_n, \sigma(\zeta))\xi_1(\zeta)\Delta\zeta - \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1^*(\zeta)\Delta\zeta \Big\| \\
&= \Big\| \int_{-\infty}^{\eta} k(\eta + \tau_n, \sigma(\zeta) + \tau_n)\xi_1(\zeta + \tau_n)\Delta\zeta - \int_{-\infty}^{\eta} k(\eta, \sigma(\zeta))\xi_1^*(\zeta)\Delta\zeta \Big\| \\
&\leq c \int_{-\infty}^{\eta} \Big\| e_{\ominus\lambda}(\eta + \tau_n, \sigma(\zeta) + \tau_n) - e_{\ominus\lambda}(\eta, \sigma(\zeta)) \Big\| \big\| \xi_1(\zeta + \tau_n) \big\| \Delta\zeta \\
&\quad + c \int_{-\infty}^{\eta} e_{\ominus\lambda}(\eta, \sigma(\zeta)) \big\| \xi_1(\zeta + \tau_n) - \xi_1^*(\zeta) \big\| \Delta\zeta \\
&\leq c\|\xi_1\|_\infty \int_{-\infty}^{\eta} \Big\| e_{\ominus\lambda}(\eta + \tau_n, \sigma(\zeta) + \tau_n) - e_{\ominus\lambda}(\eta, \sigma(\zeta)) \Big\| \\
&\quad + \frac{c(1 + \bar{\mu}\lambda)}{\lambda} \sup_{\eta\in\mathbb{T}} \big\| \xi_1(\eta + \tau_n) - \xi_1^*(\eta) \big\|,
\end{aligned}$$

where $\bar{\mu} = \sup_{\eta\in\mathbb{T}} \mu(\eta)$. From Remark 2.19 and Eq. 3.1, we have $\lim_{n\to\infty} \|\Xi_1(\eta + \tau_n) - \Xi_1^*(\eta)\| = 0$. Using the similar arguments, we get $\lim_{n\to\infty} \|\Xi_1^*(\eta - \tau_n) - \Xi_1(\eta)\| = 0$. Hence $\Xi_1(\eta) \in \mathcal{AA}(\mathbb{T}, Y)$.

Now, since $\xi_2(\eta) \in C_0(\mathbb{T}, Y)$ then $\forall\varepsilon > 0$, $\exists$ a constant $R > 0$ such that

$$\|\xi_2(\eta)\| < \varepsilon, \quad |\eta| > R. \tag{3.2}$$

which yields that

$$\begin{aligned}
\|\Xi_2(\eta)\| &= \Big\| \int_{-\infty}^{R} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta + \int_{R}^{\eta} k(\eta, \sigma(\zeta))\xi_2(\zeta)\Delta\zeta \Big\| \\
&\leq c\|\xi_2\|_\infty \int_{-\infty}^{R} e_{\ominus\lambda}(\eta, \sigma(\zeta))\Delta\zeta + \varepsilon c \int_{R}^{\eta} e_{\ominus\lambda}(\eta, \sigma(\zeta))\Delta\zeta
\end{aligned}$$

$$\leq c\|\xi_2\|_\infty \frac{(1+\bar{\mu}\lambda)}{\lambda} e_\lambda(R, |\eta|) + \varepsilon c \frac{(1+\bar{\mu}\lambda)}{\lambda}[1 - e_{\ominus\lambda}(\eta, R)]$$
$$\leq c\|\xi_2\|_\infty \frac{(1+\bar{\mu}\lambda)}{\lambda} e^{\lambda(R-|\eta|)} + \varepsilon c \frac{(1+\bar{\mu}\lambda)}{\lambda}$$
$$\lim_{|\eta|\to\infty} \|\Xi_2(\eta)\| = 0.$$

Therefore, $\Xi_2(\eta) \in C_0(\mathbb{T}, Y)$. □

Lemma 3.2 *Let $P \in \mathcal{AAA}(\mathbb{T}, Y)$ and suppose $(A1)$–$(A2)$ is satisfied. If $\mathfrak{P} : \mathbb{T} \to Y$ is defined by*

$$\mathfrak{P}(r) = \int_{-\infty}^{r} S(r, \sigma(s))P(s)\Delta s, \quad r \in \mathbb{T},$$

then $\mathfrak{P}(\cdot) \in \mathcal{AAA}(\mathbb{T}, Y)$.

Proof Since $P \in \mathcal{AAA}(\mathbb{T}, Y)$. So, we can decompose it as $P(r) = P_1(r) + P_2(r)$, where $P_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$ and $P_2(r) \in C_0(\mathbb{T}, Y)$. Now,

$$\mathfrak{P}(r) = \mathfrak{P}_1(r) + \mathfrak{P}_2(r)$$

where $\mathfrak{P}_1(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_1(s)\Delta s$ and $\mathfrak{P}_2(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_2(s)\Delta\zeta$. To proof complete, we have to prove $\mathfrak{P}_1(r) \in \mathcal{AA}(\mathbb{T}, Y)$, $\mathfrak{P}_2(r) \in C_0(\mathbb{T}, Y)$. Since $P_1 \in \mathcal{AA}(\mathbb{T}, Y)$ there exists P_1^* and a subsequence $\{\tau_n\} \subset \Pi$ for each sequence $\{\tau_n'\}$ such that

$$\lim_{n\to\infty} \|P_1(r+\tau_n) - P_1^*(r)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|P_1^*(r-\tau_n) - P_1(r)\| = 0 \qquad (3.3)$$

Now, corresponding to P_1^*, let us define $\mathfrak{P}_1^*(r) = \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)ds$. Now, we compute

$$\|\mathfrak{P}_1(r+\tau_n) - \mathfrak{P}_1^*(r)\| = \Big\| \int_{-\infty}^{r+\tau_n} S(r+\tau_n, \sigma(s))P_1(s)\Delta s - \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)\Delta s \Big\|$$
$$= \Big\| \int_{-\infty}^{r} S(r+\tau_n, \sigma(s)+\tau_n)P_1(s+\tau_n)\Delta s - \int_{-\infty}^{r} S(r, \sigma(s))P_1^*(s)\Delta s \Big\|$$
$$\leq \int_{-\infty}^{r} \Big\| S(r+\tau_n, \sigma(s)+\tau_n) - S(r, \sigma(s)) \Big\| \Big\| P_1(s+\tau_n) \Big\| \Delta s$$
$$+ \int_{-\infty}^{r} \|S(r, \sigma(s))\| \Big\| P_1(s+\tau_n) - P_1^*(s) \Big\| \Delta s$$
$$\leq \|P_1\|_\infty \frac{\varepsilon(1+\bar{\mu}\omega)}{\omega} + \frac{R_0(1+\bar{\mu}\omega)}{\omega} \sup_{r\in\mathbb{T}} \Big\| P_1(r+\tau_n) - P_1^*(r) \Big\|.$$

From Eq. 3.3, we have $\lim_{n\to\infty} \|\mathfrak{P}_1(r+\tau_n) - \mathfrak{P}_1^*(r)\| = 0$. Using the similar arguments, we get $\lim_{n\to\infty} \|\mathfrak{P}_1^*(r-\tau_n) - \mathfrak{P}_1(r)\| = 0$. Hence $\mathfrak{P}_1(r) \in AA(\mathbb{T}, Y)$.

Now, analogously to the previous lemma proof we can easily find $\lim_{|r|\to\infty} \|\mathfrak{P}_2(r)\| = 0$. Hence $\mathfrak{P}_2(\cdot) \in C_0(\mathbb{T}, Y)$. □

Now we are prepare for our main result which gives the unique $\mathcal{AAA}$ solution of system (1.1).

Theorem 3.3 *Let us assumptions* $(A1)$–$(A5)$ *hold, the system (1.1) has a unique* $\mathcal{AAA}$ *solution* $y : \mathbb{T} \to Y$ *provided*

$$\Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) < 1. \tag{3.4}$$

Proof Firstly, let us define a nonlinear operator

$$(\mathcal{G}y)(r) = g(r, y(\varkappa(r))) + \int_{-\infty}^{r} S(r,\sigma(s))\mathcal{P}(s, y(s))\Delta s + \int_{-\infty}^{r} S(r,\sigma(s)) \int_{-\infty}^{s} k(s,\sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s.$$

From the assumptions, Lemmas 2.17, 2.22, 3.1 and 3.2, we conclude that the operator $\mathcal{G}$ is from $AAA(\mathbb{T}, Y)$ into $AAA(\mathbb{T}, Y)$ which is Banach space from Lemma 2.21. To prove the remaining part, suppose $y, x \in \mathcal{AAA}(\mathbb{T}, Y)$, then

$$\begin{aligned}
&\|(\mathcal{G}y)(r) - (\mathcal{G}x)(r)\| \\
&\le \|g(r, y(\varkappa(r))) - g(r, x(\varkappa(r))\| + \Big\|\int_{-\infty}^{r} S(r,\sigma(s))[\mathcal{P}(s, y(s)) - \mathcal{P}(s, x(s))]\Delta s\Big\| \\
&+ \Big\|\int_{-\infty}^{r} S(r,\sigma(s)) \int_{-\infty}^{s} k(s,\sigma(\zeta))[h(\zeta, y(\zeta)) - h(\zeta, x(\zeta))]\Delta\zeta\Delta s\Big\| \\
&\le L_g\|y(\varkappa(r)) - x(\varkappa(r))\| + K_0 L_{\mathcal{P}} \int_{-\infty}^{r} e_{\ominus\omega}(r,\sigma(s))\|y(s) - x(s)\|\Delta s \\
&+ R_0 c L_h \int_{-\infty}^{r} e_{\ominus\omega}(r,\sigma(s)) \int_{-\infty}^{s} e_{\ominus\lambda}(s,\sigma(\zeta))\|y(\zeta) - x(\zeta)\|\Delta\zeta\Delta s \\
&\le \Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) \sup_{r\in\mathbb{T}} \|y(r) - x(r)\|
\end{aligned}$$

where $\bar{\mu} = \sup_{r\in\mathbb{T}} \mu(r)$.

$$\|(\mathcal{G}y) - (\mathcal{G}x)\|_\infty = M\|y - x\|_\infty,$$

where $M = \left(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\right)$. According to condition (3.4), $M < 1$ which implies $\mathcal{G}$ is a contraction mapping. Therefore using the Banach contraction theorem, we get a unique fixed point $y(r)$ in $\mathcal{AAA}(\mathbb{T}, Y)$ such that $\mathcal{G}y = y$ that is

$$
\begin{aligned}
y(r) = g(r, y(\varkappa(r))) + \int_{-\infty}^{r} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s \\
+ \int_{-\infty}^{r} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s
\end{aligned}
$$

for all $r \in \mathbb{T}$. If we let $a \in \mathbb{T}$, then

$$
\begin{aligned}
y(a) = g(a, y(\varkappa(a))) + \int_{-\infty}^{a} S(a, \sigma(s))\mathcal{P}(s, y(s))\Delta s \\
+ \int_{-\infty}^{a} S(a, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s
\end{aligned}
$$

using evolution operator property $S(r, t)S(t, s) = S(r, s),\ s \le t \le r$.

$$
\begin{aligned}
S(r, a)y(a) = S(r, a)g(a, y(\varkappa(a))) + \int_{-\infty}^{a} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s \\
+ \int_{-\infty}^{a} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s \\
S(r, a)[y(a) - g(a, y(\varkappa(a)))] = y(r) - g(r, y(\varkappa(r))) - \int_{a}^{r} S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s \\
- \int_{a}^{r} S(r, \sigma(s)) \int_{-\infty}^{s} k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s.
\end{aligned}
$$

From last equality, we find that system (1.1) has a unique $\mathcal{AAA}$ solution, given by (2.1). □

4 Stability Result

Definition 4.1 A solution y is called stable, if for any arbitrary $0 < \varepsilon$, there exists $0 < \delta$ such that

$$\|y(r) - \overline{y}(r)\| < \varepsilon, \quad \forall r \geq a, \quad r, a \in \mathbb{T}$$

whenever $\|y(a) - \overline{y}(a)\| < \delta$, where $\overline{y}$ is the solution of System (1.1) with initial condition $\overline{y}(a) \in Y$.

Theorem 4.2 *If the conditions of Theorem 3.3 satisfies, system (1.1) has a unique stable $\mathcal{AAA}$ mild solution.*

Proof By Theorem 3.3, we get that problem (1.1) has a unique $\mathcal{AAA}$ mild solution whose integral form is given by,

$$y(r) = S(r,a)[y(a) - g(a, y(\varkappa(a)))] + g(r, y(\varkappa(r))) + \int_a^t S(r, \sigma(s))\mathcal{P}(s, y(s))\Delta s$$
$$+ \int_a^t S(r, \sigma(s)) \int_{-\infty}^s k(s, \sigma(\zeta))h(\zeta, y(\zeta))\Delta\zeta\Delta s,$$

for $\forall r > a \in \mathbb{T}$. Now, let us suppose that $y(r)$ is $\mathcal{AAA}$ solution of the system (1.1) and $\overline{y}(r)$ is another solution of the system (1.1).

$$\begin{aligned}
&\|y(r) - \overline{y}(r)\| \\
&\quad \leq \left\|S(r,a)[y(a) - \overline{y}(a)]\right\| + \left\|S(r,a)[g(a, y(\varkappa(a))) - g(a, \overline{y}(\varkappa(a))]\right\| \\
&+ \left\|\int_a^r S(r, \sigma(s))[\mathcal{P}(s, y(s)) - \mathcal{P}(s, \overline{y}(s))]\Delta s\right\| + \left\|g(r, y(\varkappa(r))) - g(r, \overline{y}(\varkappa(r))\right\| \\
&\qquad + \left\|\int_a^r S(r, \sigma(s)) \int_{-\infty}^s k(s, \sigma(\zeta))[h(\zeta, y(\zeta)) - h(\zeta, \overline{y}(\zeta))]\Delta\zeta\Delta s\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq R_0(1 + L_g)e_{\ominus\omega}(r,a)\|y(a) - \overline{y}(a)\| + L_g\|y(\varkappa(r)) - \overline{y}(\varkappa(r))\| \\
&\quad + \left(K_0 L_{\mathcal{P}} + \frac{K_0 c L_h(1 + \bar{\mu}\lambda)}{\lambda}\right) \int_a^r e_{\ominus\omega}(r, \sigma(s)) \sup_{s\in\mathbb{T}} \|y(s) - \overline{y}(s)\|\Delta s \\
&\leq R_0(1 + L_g)\|y(a) - \overline{y}(a)\| + \\
&\quad \left(L_g + \frac{K_0 L_{\mathcal{P}}(1 + \bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h(1 + \bar{\mu}\lambda)(1 + \bar{\mu}\omega)}{\lambda\omega}\right) \sup_{r\in\mathbb{T}} \|y(r) - \overline{y}(r)\|
\end{aligned}$$

$$\|y - \overline{y}\|_\infty - M\|y - \overline{y}\|_\infty \leq R_0(1 + L_g)\|y(a) - \overline{y}(a)\|$$

$$\|y - \overline{y}\|_\infty \leq \frac{R_0(1 + L_g)\|y(a) - \overline{y}(a)\|}{1 - M}$$

where $\frac{R_0(1+L_g)}{1-M} > 0$, choose a $\delta > 0$ such that $\delta < \frac{\varepsilon(1-M)}{R_0(1+L_g)}$, then

$$\|y - \overline{y}\| < \varepsilon.$$

From Definition 4.1, the system (1.1) is stable. □

5 Example

Here, we give an example on different different time scale which shows the fruitfulness of results obtained in previous sections.

Consider the PDE on general periodic time scales $\mathbb{T}$,

$$\begin{aligned}
&\frac{\partial}{\Delta_1 r} U(r, y) \\
&= \frac{\partial^2}{\Delta_2 x^2} U(r, y) + \frac{\partial}{\Delta_1 r}\Big[\frac{1}{250}\sin\Big(\frac{1}{1 + \sin r + \sin\sqrt{2}r}\Big)\sin U(r, y) + \frac{1}{250}e^{-|r|}\cos U(r, y)\Big] \\
&\quad + \frac{1}{250}\cos\Big(\frac{1}{1 + \sin r + \cos\sqrt{2}r}\Big)\cos U(r, y) + \frac{1}{250}\frac{1}{1 + r^2}\sin U(r, y) \\
&\quad + \int_{-\infty}^{r} e_{-\frac{1}{4}}(r, \sigma(s))\Big[\cos\sqrt{2}s \sin U(s, y) + \frac{1}{1 + s^2 + s^4}\cos U(s, y)\Big]\Delta s, \qquad y \in [0, \pi]_{\mathbb{T}}
\end{aligned} \tag{5.1}$$

$$U(r, 0) = U(r, \pi) = 0, \quad r \in \mathbb{T},$$

Let $\vartheta(r) = U(r, \cdot)$, we consider the operator A by

$$A\vartheta = \frac{\partial^2}{\Delta_2 y^2}\vartheta, \quad \vartheta \in \mathcal{D}(A) = \{\mathbb{H}_0^1[0, \pi]_{\mathbb{T}} \cap \mathbb{H}_0^2[0, \pi]_{\mathbb{T}}\}.$$

As the similar argument of Sect. 3.1 in [19] and in [20], any one can simply find that the evolution system $\{S(r, s) : r \geq s\}$ satisfies $||S(r, s)|| \leq e_{\ominus\frac{1}{2}}(r, s), r \geq s,$ with $R_0 = 1$ and $\omega = \frac{1}{2}$. On based of above things, system (5.1) can be converted in form as (1.1) and satisfied all assumptions with $L_g, L_h, L_{\mathcal{P}} = \frac{1}{125}, \ c = 1, \ \lambda = \frac{1}{4}$. Now, it remains to check one condition for different different time scales.

Case1: If $\mathbb{T} = \mathbb{R}$, then $\bar{\mu} = 0$, hence

$$\Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) = 0.104 < 1.$$

Case2: If $\mathbb{T} = \mathbb{Z}$, then $\bar{\mu} = 1$, hence

$$\Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) = 0.176 < 1.$$

Case3: If $\mathbb{T} = 2\mathbb{Z}$, then $\bar{\mu} = 2$, hence

$$\Big(L_g + \frac{K_0 L_{\mathcal{P}}(1+\bar{\mu}\omega)}{\omega} + \frac{K_0 c L_h (1+\bar{\mu}\lambda)(1+\bar{\mu}\omega)}{\lambda\omega}\Big) = 0.264 < 1.$$

In all of cases, we find that all conditions of Theorems 3.3 and 4.2 satisfy, so we derive that problem (5.1) has a unique stable $\mathcal{AAA}$ solution.

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An Integral Relation Associated with a General Class of Polynomials and the Aleph Function

Monika Jain and Sapna Tyagi

Abstract A new finite integral involving two general class of polynomials with the Aleph function has been obtained in the present paper. This integral is supposed to be one of the most universal integral evaluated until now and act as a key component from which we can obtain as its different special cases, integrals relating a large number of simpler special functions and polynomials. Some useful unique cases of the main outcome have also been discussed in the paper.

Keywords The general class of polynomials · Aleph-function

2000 Mathematics Subject Classifications 26A33 · 33C60

1 Introduction

The Aleph-function is a new generalization of the well-known H-function [1] and the I-function [2, 3].

The Aleph-function is defined and represented as follows [4, 5].

$$\aleph[z] = \aleph^{M,N}_{P_i,Q_i,\tau_i;r}[z] = \aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z \left| \begin{matrix} (a_j, A_j)_{1,N}, \ldots, [\tau_i(a_j, A_j)]_{N+1,P_i} \\ (b_j, B_j)_{1,M}, \ldots, [\tau_i(b_j, B_j)]_{M+1,Q_i} \end{matrix} \right.\right]$$
$$= \frac{1}{2\pi\omega}\int_L \Phi(\xi) z^{-\xi} d\xi \tag{1.1}$$

for all $z \neq 0$, where $\omega = \sqrt{-1}$ and

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$$\Phi(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j + B_j\xi) \prod_{j=1}^{N} \Gamma(1 - a_j - A_j\xi)}{\sum_{i=1}^{r} \tau_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} + A_{ji}\xi) \prod_{j=M+1}^{Q_i} \Gamma(1 - b_{ji} - B_{ji}\xi)} \tag{1.2}$$

The path of integration $L = L_{i\Upsilon\infty}, \Upsilon \in R$ extends from $\Upsilon - i\infty$ to $\Upsilon + i\infty$. The poles of $\Gamma(b_j + B_j\xi)$, $j = \overline{1, M}$. which do not coincide to the poles of $\Gamma(1 - a_j - A_j\xi), j = \overline{1, N}$ are taken as simple poles. The parameters p_i, q_i are non-negative integers $0 \le N \le P_i$, $1 \le M \le Q_i$, $\tau_i > 0$ for $i = \overline{1, r}$. The parameters $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji} \in C$. The product in (1.2) is interpreted as unity. The existence conditions for the described integral (1.1) are given beneath:

$$\theta_\ell > 0, \ |\arg(z)| < \frac{\pi}{2}\theta_\ell, \ell = \overline{1, r}; \tag{1.3}$$

$$\theta_\ell > 0, |\arg(z)| < \frac{\pi}{2}\theta_\ell \text{ and } \mathrm{Re}\{\zeta_\ell\} + 1 < 0, \tag{1.4}$$

where

$$\theta_\ell = \sum_{j=1}^{N} A_j + \sum_{j=1}^{M} B_j - \tau_\ell \left(\sum_{j=N+1}^{P_\ell} A_{j\ell} + \sum_{j=M+1}^{Q_\ell} B_{j\ell} \right) \tag{1.5}$$

$$\zeta_\ell = \sum_{j=1}^{M} b_j - \sum_{j=1}^{N} a_j + \tau_\ell \left(\sum_{j=M+1}^{Q_\ell} b_{j\ell} - \sum_{j=N+1}^{P_\ell} a_{j\ell} \right) + \frac{1}{2}(P_\ell - Q_\ell), \ell = \overline{1, r}, \tag{1.6}$$

Note 1 The simplification of the sum in the denominator of (1.2) in terms of a polynomial in ξ, the factor of this polynomial can be uttered by a fraction of Euler's Gamma function leading to H-function, see [6], p. 325.

Note 2 It might be seen that there is no recorded name given to (1.1), compared to [5]. The Mellin transform of this function is coefficient of $z^{-\zeta}$ in the integrand of (1.1).

Note 3 Taking $\tau_i = 1, i = 1, \ldots, r$, in (1.1), the $\aleph$-function lessens to the notable I-function [3].

Note 4 Putting $r = 1$ and $\tau_1 = \tau_2 = \ldots = \tau_3 = 1$, then $\aleph$-function reduces to the known H-function [7].

Following definition of general class of polynomials is required which was introduced by Srivastava [8, Eq. (1)].

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \qquad n = 0, 1, 2, \ldots \tag{1.7}$$

Here the coefficients $A_{n,k}(n, k \ge 0)$ are subjective real or complex constants, whereas M_1 is an arbitrarily chosen positive integer.

On suitably specializing the coefficients $A_{n,k}$ occurring in (1.7), the general class polynomials $S_n^m[x]$ can be reduced to the known traditional orthogonal polynomials and the generalized hypergeometric polynomials as its particular cases. These incorporate, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials and a couple of others.

2 Main Result

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$
$$.\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s\left(\frac{b-x}{x-c}\right)^t \middle| \begin{matrix}(a_j,A_j)_{1,N},\ldots,[\tau_i(a_j,A_j)]_{N+1,P_i}\\(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}\end{matrix}\right]$$
$$.S^{m_1}_{n_1}\left[z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}\right]S^{m_2}_{n_2}\left[z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}\right]dx$$
$$=\sum_{k_1=0}^{[n_1/m_1]}\sum_{k_2=0}^{[n_2/m_2]}\frac{(-n_1)_{m_1k_1}(-n_2)_{m_2k_2}}{k_1!k_2!}A_{n_1,k_1}\ A_{n_2,k_2}z_1^{k_1}z_2^{k_2}$$
$$.(b-a)^{u+v+(\lambda+\mu)k_1+(\lambda'+\mu')k_2-1}(b-c)^{-u-\lambda k_1-\lambda' k_2}(a-c)^{-v-\mu k_1-\mu' k_2}$$
$$.\aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left[z\left(\frac{b-a}{b-c}\right)^s\left(\frac{b-a}{a-c}\right)^t \middle| \begin{matrix}(1-u-\lambda k_1-\lambda' k_2,s),(1-v-\mu k_1-\mu' k_2,t)\\(b_j,B_j)_{1,M},\ldots,[\tau_i(b_j,B_j)]_{M+1,Q_i}\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,\alpha_j)_{1,N},\ \tau_j(a_j,\alpha_j)_{N+1,P_i;r}\\(1-u-v-\lambda k_1-\mu k_1-\lambda' k_2-\mu' k_2,s+t)\end{matrix}\right], \tag{2.1}$$

where $s>0$, $t>0$, $\mathrm{Re}\,(u+s\,b_j/\beta_j)>0$, $\mathrm{Re}\,(v+t\,b_j/\beta_j)>0$,

$j=1,\ldots,M$, λ, λ', μ and μ' are positive integers. A_{n_1,k_1} and A_{n_2,k_2} $(n_1,\ k_1, n_2,\ k_2 \geq 0)$ are arbitrary constants, real or complex.

Proof To establish (2.1), expressing the $\aleph$-function by (1.2) and general class of polynomials by (1.7), then the order of summations and integration are interchanged (which is justified due to the absolute convergence of the integral in the process), we calculate the integral with the help of a result ([7], p. 287 (3.119)), and get the desired outcome.

3 Special Cases

(A) Taking $S_n^2[y] = y^{n/2}H_n\left[\frac{1}{2\sqrt{y}}\right]$ in the result obtained in (2.1) to the case of Hermite polynomials ([9], Eq. (5.5.4), p. 106 and [3], p. 158)

in which case $m_1 = 2$, $A_{n_1,k_1} = (-1)^{k_1}$ and also letting $m_2 = 2$, $A_{n_2,k_2} = (-1)^{k_2}$, we have

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$
$$.\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s\left(\frac{b-x}{x-c}\right)^t \middle| \begin{matrix}(a_j,\ A_j)_{1,N},\ldots,[\tau_i(a_j,\ A_j)]_{N+1,P_i}\\(b_j,\ B_j)_{1,M},\ldots,[\tau_i(b_j,\ B_j)]_{M+1,Q_i}\end{matrix}\right]$$
$$.\left[z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}\right]^{n_1/2} H_{n_1}\left[\frac{1}{2\sqrt{z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}}}\right]$$
$$.\left[z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}\right]^{n_2/2} H_{n_2}\left[\frac{1}{2\sqrt{z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}}}\right]dx$$
$$= \sum_{k_1=0}^{[n_1/m_1]}\sum_{k_2=0}^{n_2/m_2}\frac{(-n_1)_{2k_1}(-n_2)_{2k_2}}{k_1!k_2!}(-1)^{k_1}(-1)^{k_2}z_1^{k_1}z_2^{k_2}$$
$$.(b-a)^{u+v+(\lambda+\mu)k_1+(\lambda'+\mu')k_2-1}(b-c)^{-u-\lambda k_1-\lambda' k_2}(a-c)^{-v-\mu k_1-\mu' k_2}$$
$$.\aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left[z\left(\frac{b-a}{b-c}\right)^s\left(\frac{b-a}{a-c}\right)^s \middle| \begin{matrix}(1-u-\lambda k_1-\lambda' k_2, s),(1-v-\mu k_1-\mu' k_2, t)\\(b_j,\ B_j)_{1,M},\ldots,[\tau_i(b_j,\ B_j)]_{M+1,Q_i}\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j,\ \alpha_j)_{1,N},\tau_j(a_j,\ \alpha_j)_{N+1,P_i;r}\\(1-u-v-\lambda k_1-\mu k_1-\lambda' k_2-\mu' k_2, s+t)\end{matrix}\right], \tag{3.1}$$

applicable under the conditions as available from (2.1).

(B) For the Jacobi polynomials ([9], Eq. (4.3.2), p. 68 and [3], p. 158), our result (2.1) yields the following result by setting

$$S_n^1[x] = P_n^{(\alpha',\beta')}(1-2x) \text{ in which case}$$

$$m_1 = 1,\quad A_{n_1,k_1} = \binom{n_1+k_1}{n_1}\frac{(\alpha'+\beta'+n_1+1)_{k_1}}{(\alpha'+1)_{k_1}}$$

and also taking

$$m_2 = 1, \quad A_{n_2,k_2} = \binom{n_2 + k_2}{n_2} \frac{\left(\alpha'' + \beta'' + n_2 + 1\right)_{k_2}}{(\alpha'' + 1)_{k_2}},$$

we obtain

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$
$$.\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s\left(\frac{b-x}{x-c}\right)^t \middle| \begin{matrix}(a_j, A_j)_{1,N}, \dots, [\tau_i(a_j, A_j)]_{N+1,P_i} \\ (b_j, B_j)_{1,M}, \dots, [\tau_i(b_j, B_j)]_{M+1,Q_i}\end{matrix}\right]$$
$$.P^{(\alpha',\beta')}_{n_1}\left[1 - 2z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}\right]$$
$$.P^{(\alpha'',\beta'')}_{n_2}\left[1 - 2z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}\right]dx$$
$$= \sum_{k_1=0}^{[n_1]}\sum_{k_2=0}^{[n2]}\binom{n_1+\alpha'}{n_1-k_1}\binom{n_2+\alpha''}{n_2-k_2}(-z_1)^{k_1}(-z_2)^{k_2}$$
$$.\binom{\alpha'+\beta'+n_1+k_1}{k_1}\binom{\alpha''+\beta''+n_2+k_2}{k_2}$$
$$.(b-a)^{u+v+(\lambda+\mu)k_1+(\lambda'+\mu')k_2-1}(b-c)^{-u-\lambda k_1-\lambda' k_2}(a-c)^{-v-\mu k_1-\mu' k_2}$$
$$.\aleph^{M,\,N+2}_{P_i+2,\,Q_i+1,\,\tau_i;r}\left[z\left(\frac{b-a}{b-c}\right)^s\left(\frac{b-a}{a-c}\right)^t \middle| \begin{matrix}\left(1-u-\lambda k_1-\lambda' k_2, s\right), \left(1-v-\mu k_1-\mu' k_2, t\right) \\ (b_j, B_j)_{1,M}, \dots, [\tau_i(b_j, B_j)]_{M+1,Q_i}\end{matrix}\right.$$
$$\left.\begin{matrix}\left(a_j,\ \alpha_j\right)_{1,N}, \tau_j\left(a_j,\ \alpha_j\right)_{N+1,P_i;r} \\ \left(1-u-v-\lambda k_1-\mu k_1-\lambda' k_2-\mu' k_2,\ s+t\right)\end{matrix}\right], \tag{3.2}$$

valid under the conditions as obtainable from (2.1).

(C) For the Laguerre polynomials ([9], Eq. (5.1.6), p. 10 and [3], p. 158), we have the following interesting consequence of our result (2.1), by setting

$$S^1_n[x] \to L^{(\alpha')}_n(x) \text{ in which case}$$

$$m_1 = 1, \quad A_{n_1,k_1} = \binom{n_1+\alpha'}{n_1}\frac{1}{(\alpha'+1)_{k_1}}$$

and also taking

$$m_2 = 1, \quad A_{n_2,k_2} = \binom{n_2 + \alpha''}{n_2} \frac{1}{(\alpha'' + 1)_{k_2}},$$

we get

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$

$$.\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s \left(\frac{b-x}{x-c}\right)^t \middle| \begin{matrix} (a_j, A_j)_{1,N}, \ldots, [\tau_i(a_j, A_j)]_{N+1,P_i} \\ (b_j, B_j)_{1,M}, \ldots, [\tau_i(b_j, B_j)]_{M+1,Q_i} \end{matrix} \right]$$

$$.L^{(\alpha')}_{n_1}\left[z_1 \left(\frac{x-a}{x-c}\right)^{\lambda} \left(\frac{b-x}{x-c}\right)^{\mu} \right] \; .L^{(\alpha'')}_{n_2}\left[z_2 \left(\frac{x-a}{x-c}\right)^{\lambda'} \left(\frac{b-x}{x-c}\right)^{\mu'} \right] dx$$

$$= \sum_{k_1=0}^{[n_1]} \sum_{k_2=0}^{[n_2]} \frac{(-n_1)_{k_1}(-n_2)_{k_2}}{k_1! k_2!} \binom{n_1 + \alpha'}{n_1} \frac{1}{(\alpha'+1)k_1} \binom{n_2 + \alpha''}{n_2} \frac{1}{(\alpha''+1)K_2} z_1^{k_1} z_2^{k_2}$$

$$.(b-a)^{u+v+(\lambda+\mu)k_1 + (\lambda'+\mu')k_2 - 1} (b-c)^{-u-\lambda k_1 - \lambda' k_2} (a-c)^{-v-\mu k_1 - \mu' k_2}$$

$$.\aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left[z\left(\tfrac{b-a}{b-c}\right)^s \left(\tfrac{b-a}{a-c}\right)^t \middle| \begin{matrix} (1-u-\lambda k_1 - \lambda' k_2, s),\ (1-v-\mu k_1 - \mu' k_2, t) \\ (b_j, B_j)_{1,M}, \ldots, [\tau_i(b_j, B_j)]_{M+1,Q_i} \end{matrix} \right.$$
$$\left. \begin{matrix} (a_j,\ \alpha_j)_{1,N},\ \tau_j(a_j,\ \alpha_j)_{N+1,P_i;r} \\ (1-u-v-\lambda k_1 - \mu k_1 - \lambda' k_2 - \mu' k_2,\ s+t) \end{matrix} \right], \tag{3.3}$$

suitable under the conditions as required sufficiently for (2.1).

(D) Letting $n_2 \to 0$ in (2.1), we have

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$

$$.\aleph^{M,N}_{P_i,Q_i,\tau_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s \left(\frac{b-x}{x-c}\right)^t \middle| \begin{matrix} (a_j, A_j)_{1,N}, \ldots, [\tau_i(a_j, A_j)]_{N+1,P_i} \\ (b_j, B_j)_{1,M}, \ldots, [\tau_i(b_j, B_j)]_{M+1,Q_i} \end{matrix} \right]$$

$$.S^{m_1}_{n_1}\left[z_1 \left(\frac{x-a}{x-c}\right)^{\lambda} \left(\frac{b-x}{x-c}\right)^{\mu} \right] dx$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1!} A_{n_1,k_1} z_1^{k_1} .(b-a)^{u+v+(\lambda+\mu)k_1 - 1} (b-c)^{-u-\lambda k_1} (a-c)^{-v-\mu k_1}$$

$$.\aleph^{M,N+2}_{P_i+2,Q_i+1,\tau_i;r}\left[z\left(\frac{b-a}{b-c}\right)^s\left(\frac{b-a}{a-c}\right)^t\middle|\begin{array}{l}\left(1-u-\lambda k_1-\lambda' k_2, s\right),\left(1-v-\mu k_1-\mu' k_2, t\right)\\(b_j,\ B_j)_{1,M},\ldots,[\tau_i(b_j,\ B_j)]_{M+1,Q_i}\end{array}\right.$$
$$\left.\begin{array}{c}\left(a_j,\alpha_j\right)_{1,N},\tau_j\left(a_j,\alpha_j\right)_{N+1,P_i;r}\\\left(1-u-v-\lambda k_1-\mu k_1-\lambda' k_2-\mu' k_2,\ s+t\right)\end{array}\right], \tag{3.4}$$

valid under the conditions as essential for (2.1).

(E) Taking $\tau_i \to 1$ in (2.1), the I-function given by Saxena [2, 3] is obtained from Aleph function and the main integral (2.1) converts in the following form:

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}$$
$$.I^{M,N}_{P_i,Q_i;r}\left[z\left(\frac{x-a}{x-c}\right)^s\left(\frac{b-x}{x-c}\right)^t\middle|\begin{array}{l}(a_j,A_j)_{1,N},(a_j,A_j)_{N+1,P_i}\\(b_j,B_j)_{1,M},(b_j,B_j)_{M+1,Q_i}\end{array}\right]$$
$$.S^{m_1}_{n_1}\left[z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}\right]S^{m_2}_{n_2}\left[z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}\right]dx$$
$$=\sum_{k_1=0}^{[n_1/m_1]}\sum_{k_2=0}^{[n_2/m_2]}\frac{(-n_1)_{m_1k_2}(-n_2)_{m_2k_2}}{k_1!k_2!}A_{n_1,k_1}A_{n_2,k_2}z_1^{k_1}z_2^{k_2}$$
$$.(b-a)^{u+v+(\lambda+\mu)k_1+\left(\lambda'+\mu'\right)k_2-1}(b-c)^{-u-\lambda k_1-\lambda' k_2}(a-c)^{-v-\mu k_1-\mu' k_2}$$
$$.I^{M,N+2}_{P_i+2,Q_i+1;r}\left[z\left(\frac{b-a}{b-c}\right)^s\left(\frac{b-a}{a-c}\right)^t\middle|\begin{array}{l}\left(1-u-\lambda k_1-\lambda' k_2, s\right),\left(1-v-\mu K_1-\mu' K_2, t\right)\\(b_j,B_j)_{1,M},(b_j,B_j)]_{M+1,Q_i}\end{array}\right.$$
$$\left.\begin{array}{c}\left(a_j,\alpha_j\right)_{1,N},\left(a_j,\alpha_j\right)_{N+1,P_i;r}\\\left(1-u-v-\lambda k_1-\mu k_1-\lambda' k_2-\mu' k_2,\ s+t\right)\end{array}\right], \tag{3.5}$$

valid under the conditions as required sufficiently for (2.1).

(F) If we take $\tau_i \to 1$ and $r=1$ in (2.1), the Aleph function reduces to Fox's H-function [1] and the main integral takes the following form:

$$\int_a^b (x-a)^{u-1}(b-x)^{v-1}(x-c)^{-u-v}\quad .H^{M,N}_{P,Q}\left[z\left(\frac{x-a}{x-c}\right)^s\left(\frac{b-x}{x-c}\right)^t\middle|\begin{array}{l}(a_j,A_j)\\(b_j,B_j)\end{array}\right]$$
$$.S^{m_1}_{n_1}\left[z_1\left(\frac{x-a}{x-c}\right)^{\lambda}\left(\frac{b-x}{x-c}\right)^{\mu}\right]S^{m_2}_{n_2}\left[z_2\left(\frac{x-a}{x-c}\right)^{\lambda'}\left(\frac{b-x}{x-c}\right)^{\mu'}\right]dx$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \sum_{k_2=0}^{[n_2/m_2]} \frac{(-n_1)_{m_1 k_2}(-n_2)_{m_2 k_2}}{k_1! k_2!} A_{n_1,k_1} A_{n_2,k_2} z_1^{k_1} z_2^{k_2}$$

$$.(b-a)^{u+v+(\lambda+\mu)k_1+(\lambda'+\mu')k_2-1}(b-c)^{-u-\lambda k_1-\lambda' k_2}(a-c)^{-v-\mu k_1-\mu' k_2}$$

$$.H^{M,N+2}_{P+2,Q+1,}\left[z\left(\frac{b-a}{b-c}\right)^s \left(\frac{b-a}{a-c}\right)^t \middle| \begin{matrix} (1-u-\lambda k_1-\lambda' k_2, s), (1-v-\mu k_1-\mu' k_2, t)(a_1,\alpha_1), (a_p,\alpha_p) \\ (b_1,B_1), (b_q,B_q)(1-u-v-\lambda k_1-\mu K_1-\lambda' k_2-\mu' k_2, s+t) \end{matrix} \right], \tag{3.6}$$

valid under the conditions as required sufficiently for (2.1).

The significance of outcomes lies in its various generalizations. In perspective of the generality of the function and polynomials of very broad nature involved in the results, our results encompass several particular cases of interest scattered hitherto in the literature.

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On the New Fractional Operator and Application to Nonlinear Bloch System

J. F. Gómez-Aguilar, Behzad Ghanbari and Ebenezer Bonyah

Abstract In this chapter, we analyze the nonlinear Bloch system with a new fractional operator without singular kernel proposed by Michele Caputo and Mauro Fabrizio. The commensurate and non-commensurate order nonlinear Bloch system is considered. Special solutions using a numerical scheme based in Lagrange interpolations were obtained. We studied the uniqueness and existence of the solutions employing the fixed point theorem. Novel chaotic attractors with total order less than 3 are obtained.

Keywords Fractional calculus · Bloch system · Exponential-decay law · Lagrange interpolation

1 Introduction

The nonlinear Bloch system is a system consisting of three nonlinear ODEs which can be used to model time-dependent nuclear magnetization. These equations are efficient tool to describe the Nuclear Magnetic Resonance (NMR). The dynamic balance between externally applied magnetic fields and also internal sample relaxation times [1] is explained by the Bloch system. Taking advantage of fractional-order differential equations, we model this relaxation as a multiexponential process.

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Fractional order systems are suitable for describing the memory properties of several materials, because it has a stronger memory function. It is well known that, fractional-order derivatives have made great progress in mathematical modeling of the dynamics of complex systems, multi-scale phenomena and viscoelastic materials [2–11]. The fractional derivative with power-law singular kernel imposes an artifical singularity to mathematical models and the memory effects cannot describe accurately. Due to this inconvenience, a new fractional derivative considering the exponential function as non-singular kernel was proposed by Caputo and Fabrizio [12]. This new operator allows to describe more efficiently the memory effect and do not impose artificial singularities as in the old Liouville-Caputo derivative. Several problems in chemical reactions, luminescence, heat transfer, geophysics, physical optics, radioactivity, thermoelectricity, vibrations and electromagnetism are naturally governed by the exponential decay law. These natural phenomena can be studied considered the exponential kernel suggested by Caputo and Fabrizio. Furthermore, this new operator has supplementary properties, it can portray substance heterogeneities and configurations with different scales, which noticeably cannot be managed with the other representations [13–15]. Losada and Nieto in [16] studied the further properties.

Atangana and Baleanu generalized the exponential function and proposed the Mittag-Leffler law as kernel of differentiation [17] arising the Atangana-Baleanu fractional derivative. The fractional-order derivatives with non-singular kernel allows to describe two different waiting times distribution, which is an ideal waiting time distribution as such is observed in nuclear magnetization. The crossover behavior of both operators is due to their capacity of not obeying the classical index-law imposed in fractional calculus. This apparent limitation allows to permits describe more appropriate real world problems [18–21]. In [22], several examples of non-commutative and non-associative problems were presented. The authors justify why the fractional derivatives with non-singular kernel are needed to describe real-world problems. The authors conclude that the commutativity or index-law and semi-group principle are irrelevant in fractional calculus, ending the controversy generated for the use of these fractional-order derivatives.

In recent years, the generalized nonlinear Bloch equation with fractional-order derivatives has attracted great interest of many researchers and scholars in literature [23–30]. A predictor-corrector approach to solve the multi-term time-fractional Bloch equations has been developed in [31]. Also, for some other variants of the equation including Bloch equations with Riemann-Liouville fractional derivative [32–35] or the delay-dependent fractional Bloch equations [36–38].

In this chapter, we apply the new fractional operator with exponential-decay law to the nonlinear Bloch model. We studied the uniqueness and existence of the solutions employing the fixed point theorem. The manuscript is organized as follows. The paper is structured as follows. In Sect. 2, we recall the fractional operators of type Liouville-Caputo sense. In Sect. 3, we formulate the fractional order nonlinear Bloch model, and then the existence and uniqueness of the coupled solutions is proved. We consider numerical simulations in Sect. 4. Finally, we summarize and conclude in Sect. 5.

2 Fractional Operators

Based in the exponential-decay law, the Caputo-Fabrizio fractional operator without singular kernel in Liouville-Caputo sense (CFC) is given by [12]

$$
{}_{0}^{CFC}\mathcal{D}_{t}^{\gamma}\{f(t)\} = \frac{M(\gamma)}{n-\gamma}\int_{0}^{t}\frac{d^{n}}{dt^{n}}f(\theta)\exp\Big[-\frac{\gamma}{n-\gamma}(t-\theta)\Big]d\theta, \quad n-1<\gamma\leq n, \tag{1}
$$

where $M(\gamma)$ is a normalization function such that $M(0) = M(1) = 1$.

The Caputo-Fabrizio fractional integral is defined below [16]

$$
{}_{0}^{CF}I_{t}^{\gamma} f(t) = \frac{2(1-\gamma)}{M(\gamma)(2-\gamma)}f(t) + \frac{2\gamma}{M(\gamma)(2-\gamma)}\int_{0}^{t} f(s)ds. \qquad t \geq 0.
$$

where,

$$
M(\gamma) = \frac{2}{2-\gamma}, \qquad 0<\gamma<1. \tag{2}
$$

Losada and Nieto [16] analyzed more properties of this newly presented fractional operator.

3 Bloch System with Non-singular Kernel

The nonlinear Bloch system [36] in Caputo-Fabrizio-Caputo sense is given by

$$
\begin{aligned}
{}_{0}^{CFC}\mathcal{D}_{t}^{\gamma_1}x(t) &= \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t),\\
{}_{0}^{CFC}\mathcal{D}_{t}^{\gamma_2}y(t) &= -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t),\\
{}_{0}^{CFC}\mathcal{D}_{t}^{\gamma_3}z(t) &= y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t)-1),
\end{aligned} \tag{3}
$$

with initial conditions

$$
x(t) = x(0), \qquad y(t) = y(0), \qquad z(t) = z(0). \tag{4}
$$

System (3) can be made more realistic as the nuclear magnetization as a function of time should not follow the same fractional order dynamics. For this reason, we introducing three different orders of the fractional-differential operators $\gamma_i \in (0, 1]$

for $i = 1, 2, 3$. The system (3) is called commensurate when $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, otherwise is called non-commensurate (for this case, the total order of the system is then changed from 3 to the sum of each particular order).

Existence of the coupled solutions.
We investigate the numerical results predicted by the fractional model given by the system (3). Firstly start to investigate the existence and uniqueness of the solutions. By using the fixed-point theorem, we define the existence of the solution. First, transform system (3) into an integral equation as follows

$$x(t) - x(0) = {}_0^{CF} I_t^{\gamma_1}\Big[\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\Big],$$

$$y(t) - y(0) = {}_0^{CF} I_t^{\gamma_2}\Big[-\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\Big],$$

$$z(t) - z(0) = {}_0^{CF} I_t^{\gamma_3}\Big[y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\Big]. \quad (5)$$

On using the definition (2), we get

$$x(t) = x_0 + \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big\{\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\Big\}$$
$$+ \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t \Big[\zeta y(s) + \varrho z(s)(x(s)\sin(\varphi) - y(s)\cos(\varphi)) - \frac{1}{\Psi_2}x(s)\Big]ds, \quad (6)$$

$$y(t) = y_0 + \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big\{-\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\Big\}$$
$$+ \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t \Big[-\zeta x(s) - z(s) + \varrho z(s)(y(s)\sin(\varphi) + x(s)\cos(\varphi)) - \frac{1}{\Psi_2}y(s)\Big]ds, \quad (7)$$

$$z(t) = z_0 + \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big\{y(t) - \varrho\sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\Big\}$$
$$+ \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t \Big[y(s) - \varrho\sin(\varphi)(x(s)^2 + y(s)^2) - \frac{1}{\Psi_1}(z(s) - 1)\Big]ds. \quad (8)$$

Now, we consider the following kernels

$$G_1(t, x(t)) = \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t),$$

$$G_2(t, y(t)) = -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t), \quad (9)$$

$$G_3(t, z(t)) = y(t) - \varrho \sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1).$$

Now, we prove that the kernels G_1, G_2 and G_3 satisfy the Lipschitz condition. To achieve we first prove this condition for each kernel proposed. We start with the kernel 1. Let x and X be two functions, using the Cauchy's inequality, then we assess the following

$$||G_1(t, x(t)) - G_1(t, X(t))|| \leq \left|\left|\zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2}x(t)\right|\right|. \quad (10)$$

Similarly for the second and third cases, we have

$$\begin{aligned} ||G_2(t, y(t)) - G_2(t, Y(t))|| \leq \Big|\Big| &- \zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) \\ &+ x(t)\cos(\varphi)) - \frac{1}{\Psi_2}y(t)\Big|\Big|, \end{aligned}$$

$$||G_3(t, z(t)) - G_3(t, Z(t))|| \leq \left|\left|y(t) - \varrho \sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1)\right|\right|, \quad (11)$$

consider the following recursive formula, we have

$$x_{(n)}(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}G_1(t, x_{(n-1)}) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t G_1(s, x_{(n-1)})ds,$$

$$y_{(n)}(t) = \frac{2(1-\gamma_2)}{(M(\gamma_2)(2-\gamma_2)}G_2(t, y_{(n-1)}) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t G_2(s, y_{(n-1)})ds,$$

$$z_{(n)} = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}G_3(t, z_{(n-1)}) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t G_3(s, z_{(n-1)})ds. \quad (12)$$

Applying the norm and the triangular inequality, we get

$$
\begin{aligned}
||a_{(n)}(t)|| &= ||x_{(n)}(t) - X_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}||G_1(t, x_{(n-1)}(t)) - G_1(t, X_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Big|\Big|\int_0^t \Big[G_1(s, x_{(n-1)}(s)) - G_1(s, X_{(n-2)}(s))\Big]\Big|\Big|ds, \\
||b_{(n)}(t)|| &= ||y_{(n)}(t) - Y_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}||G_2(t, y_{(n-1)}(t)) - G_2(t, Y_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Big|\Big|\int_0^t \Big[G_2(s, y_{(n-1)}(s)) - G_2(s, Y_{(n-2)}(s))\Big]\Big|\Big|ds, \\
||c_{(n)}(t)|| &= ||z_{(n)}(t) - Z_{(n-1)}(t)|| \\
&\leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}||G_3(t, z_{(n-1)}(t)) - G_3(t, Z_{(n-2)}(t))|| \\
&\quad + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Big|\Big|\int_0^t \Big[G_3(s, z_{(n-1)}(s)) - G_3(s, Z_{(n-2)}(s))\Big]\Big|\Big|ds,
\end{aligned}
\tag{13}
$$

where,

$$
x_{(n)}(t) = \sum_{m=0}^{\infty} a_m(t); \qquad y_{(n)}(t) = \sum_{m=0}^{\infty} b_m(t); \qquad z_{(n)}(t) = \sum_{m=0}^{\infty} c_m(t). \tag{14}
$$

Since the kernels satisfies the Lipschitz condition, we have

$$
\begin{aligned}
||a_{(n)}(t)|| = ||x_{(n)}(t) - X_{(n-1)}(t)|| &\leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1||x_{(n-1)}(t) - X_{(n-2)}(t)|| \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\int_0^t ||x_{(n-1)}(s) - X_{(n-2)}(s)||ds,
\end{aligned}
$$

$$
\begin{aligned}
||b_{(n)}(t)|| = ||y_{(n)}(t) - Y_{(n-1)}(t)|| &\leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3||y_{(n-1)}(t) - Y_{(n-2)}(t)|| \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4\int_0^t ||y_{(n-1)}(s) - Y_{(n-2)}(s)||ds,
\end{aligned}
$$

$$||c_{(n)}(t)|| = ||z_{(n)}(t) - Z_{(n-1)}(t)|| \le \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5||z_{(n-1)}(t) - Z_{(n-2)}(t)|| + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6\int_0^t ||z_{(n-1)}(s) - Z_{(n-2)}(s)||ds. \tag{15}$$

Considering system (13) bounded, we have proven that the kernels satisfy Lipschitz condition, therefore following the results obtained in (13) using the recursive technique, we get the following relation

$$||a_{(n)}(t)|| \le ||x(0)|| + \left\{\left\{\frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\right\}^n + \left\{\frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2 t\right\}^n\right\},$$

$$||b_{(n)}(t)|| \le ||y(0)|| + \left\{\left\{\frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3\right\}^n + \left\{\frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4 t\right\}^n\right\},$$

$$||c_{(n)}(t)|| \le ||z(0)|| + \left\{\left\{\frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5\right\}^n + \left\{\frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6 t\right\}^n\right\}. \tag{16}$$

Therefore, Eq. (16) exists and is continuous. Nonetheless, to show that the above functions are a system of solutions of Eq. (3), we assume

$$x(t) = x_{(n)}(t) - \Xi_{1(n)}(t); \quad y(t) = y_{(n)}(t) - \Xi_{2(n)}(t); \quad z(t) = z_{(n)}(t) - \Xi_{3(n)}(t), \tag{17}$$

where $\Xi_{1(n)}$, $\Xi_{2(n)}$ and $\Xi_{3(n)}$ are reminder terms of series solution. Thus

$$x(t) - X_{(n)}(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}G_1(t, x - \Xi_{1(n)}(t)) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\int_0^t G_1(s, x - \Xi_{1(n)}(s))ds,$$

$$y(t) - Y_{(n)}(t) = \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}G_2(t, y - \Xi_{2(n)}(t)) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\int_0^t G_2(s, y - \Xi_{2(n)}(s))ds,$$

$$z(t) - Z_{(n)}(t) = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}G_3(t, z - \Xi_{3(n)}(t)) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\int_0^t G_3(s, z - \Xi_{3(n)}(s))ds. \tag{18}$$

Applying the norm on both sides and using the Lipschitz condition, we get

$$\left\|x(t) - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)} G_1(t, x(t)) - x(0) - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \int_0^t G_1(s, x(s))ds\right\|$$

$$\leq ||\Xi_{1(n)}(t))|| + \left\{\frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1 + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2 t\right\}||\Xi_{1(n)}(t)||,$$

$$\left\|y(t) - \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)} G_2(t, y(t)) - y(0) - \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \int_0^t G_2(s, y(s))ds\right\|$$

$$\leq ||\Xi_{2(n)}(t))|| + \left\{\frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3 + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4 t\right\}||\Xi_{2(n)}(t)||,$$

$$\left\|z(t) - \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)} G_3(t, z(t)) - z(0) - \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \int_0^t G_3(s, z(s))ds\right\|$$

$$\leq ||\Xi_{3(n)}(t))|| + \left\{\frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5 + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6 t\right\}||\Xi_{3(n)}(t)||. \quad (19)$$

On taking the limit $n \to \infty$ of Eq. (19), we get

$$x(t) = \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)} G_1(t, x(t)) + x(0) + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \int_0^t G_1(s, x(s))ds,$$

$$y(t) = \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)} G_2(t, y(t)) + y(0) + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \int_0^t G_2(s, y(s))ds,$$

$$z(t) = \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)} G_3(t, z(t)) + z(0) + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \int_0^t G_3(s, z(s))ds. \quad (20)$$

Uniqueness of the solutions.

We assume that we can find another solutions for Eq. (3); say $x(t)$, $y(t)$ and $z(t)$; then

$$\begin{aligned}
x(t) - X(t) &= \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big[G_1(t, x(t)) - G_1(t, X(t))\Big] \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \int_0^t \Big[G_1(s, x(s)) - G_1(s, X(s))\Big] ds, \\
y(t) - Y(t) &= \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big[G_2(t, y(t)) - G_2(t, Y(t))\Big] \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \int_0^t \Big[G_2(s, y(s)) - G_2(s, Y(s))\Big] ds, \\
z(t) - Z(t) &= \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big[G_3(t, z(t)) - G_3(t, Z(t))\Big] \\
&\quad + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \int_0^t \Big[G_3(s, z(s)) - G_3(s, Z(s))\Big] ds.
\end{aligned} \tag{21}$$

Apply the norm both sides of Eq. (21), we have

$$\begin{aligned}
||x(t) - X(t)|| &\leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Big[\Big|\Big|G_1(t, x(t)) - G_1(t, X(t))\Big|\Big|\Big] \\
&\quad + \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)} \int_0^t \Big[\Big|\Big|G_1(s, x(s)) - G_1(s, X(s))\Big|\Big|\Big] ds, \\
||y(t) - Y(t)|| &\leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Big[\Big|\Big|G_2(t, y(t)) - G_2(t, Y(t))\Big|\Big|\Big] \\
&\quad + \frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)} \int_0^t \Big[\Big|\Big|G_2(s, y(s)) - G_2(s, Y(s))\Big|\Big|\Big] ds, \\
||z(t) - Z(t)|| &\leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Big[\Big|\Big|G_3(t, z(t)) - G_3(t, Z(t))\Big|\Big|\Big] \\
&\quad + \frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)} \int_0^t \Big[\Big|\Big|G_3(s, z(s)) - G_3(s, Z(s))\Big|\Big|\Big] ds,
\end{aligned} \tag{22}$$

considering the Lipschitz condition, having the fact in mind that the solutions are bounded, we get

$$||x(t) - X(t)|| \leq \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 + \left\{\frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right\}^n,$$

$$||y(t) - Y(t)|| \leq \frac{2(1-\gamma_2)}{M(\gamma_2)(2-\gamma_2)}\Delta_3\xi_3 + \left\{\frac{2\gamma_2}{M(\gamma_2)(2-\gamma_2)}\Delta_4\xi_4 t\right\}^n,$$

$$||z(t) - Z(t)|| \leq \frac{2(1-\gamma_3)}{M(\gamma_3)(2-\gamma_3)}\Delta_5\xi_5 + \left\{\frac{2\gamma_3}{M(\gamma_3)(2-\gamma_3)}\Delta_6\xi_6 t\right\}^n, \tag{23}$$

this is true for any n.

The system given by Eq. (3) has a unique solution if the below condition holds.

$$\left(1 - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right) \geq 0. \tag{24}$$

If the condition (24) holds, then

$$||x(t) - X(t)||\left(1 - \frac{2(1-\gamma_1)}{M(\gamma_1)(2-\gamma_1)}\Delta_1\xi_1 - \frac{2\gamma_1}{M(\gamma_1)(2-\gamma_1)}\Delta_2\xi_2 t\right) \leq 0, \tag{25}$$

implies that $||x(t) - X(t)|| = 0$. Then we get, $x(t) = X(t)$.

Employing the same way, we have

$$x(t) = X(t); \qquad y(t) = Y(t); \qquad z(t) = Z(t). \tag{26}$$

Therefore, we verified the uniqueness of coupled-solutions.

Now we propose a numerical solution of the nonlinear Bloch system considering the fractional derivative of Caputo-Fabrizio in Liouville-Caputo sense using the numerical scheme proposed by Atangana and Toufik in [39].

First we consider the following fractional differential equation with fading memory

$${}_0^{CFC}\mathscr{D}_t^{\alpha} y(t) = f(t, y(t)), \tag{27}$$

using the fundamental theorem of fractional calculus we obtain the solution of the above equation [39]

$$y_{n+1} = y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f(t_n, y_n) - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f(t_{n-1}, y_{n-1}) \tag{28}$$

Again, we apply the numerical scheme (28) to have a numerical solution to Eq. (3) in Caputo-Fabrizio-Caputo sense

$$
\begin{aligned}
x_{n+1}(t) &= x_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_1(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_1(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), \\
y_{n+1} &= y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_2(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_2(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}), \\
z_{n+1} &= z_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right) f_3(t_n, x_n, y_n, z_n) \\
&\quad - \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right) f_3(t_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}),
\end{aligned} \tag{29}
$$

where,

$$
\begin{aligned}
f_1(t, x(t), y(t), z(t)) &:= \zeta y(t) + \varrho z(t)(x(t)\sin(\varphi) - y(t)\cos(\varphi)) - \frac{1}{\Psi_2} x(t), \\
f_2(t, x(t), y(t), z(t)) &:= -\zeta x(t) - z(t) + \varrho z(t)(y(t)\sin(\varphi) + x(t)\cos(\varphi)) - \frac{1}{\Psi_2} y(t), \\
f_3(t, x(t), y(t), z(t)) &:= y(t) - \varrho \sin(\varphi)(x(t)^2 + y(t)^2) - \frac{1}{\Psi_1}(z(t) - 1).
\end{aligned} \tag{30}
$$

In the next section, we consider Eq. (29) for obtain several numerical solutions considering different values of the fractional order γ arbitrarily chosen.

4 Numerical Simulations

Numerical solutions of the system (3) have been depicted in Fig. 1a–f and Fig. 2a–f for the commensurate and non-commensurate order system, respectively. The parameter values used in the simulations are $\zeta = 1.26$, $\varrho = 10$, $\varphi = 0.7764$, $\Psi_1 = 0.5$, $\Psi_2 = 0.25$ and the initial conditions are $x(t) = 0.1$, $y(t) = 0.1$ and $z(t) = 0.1$. The step size used in evaluating the approximate solution was $h = 0.0001$.

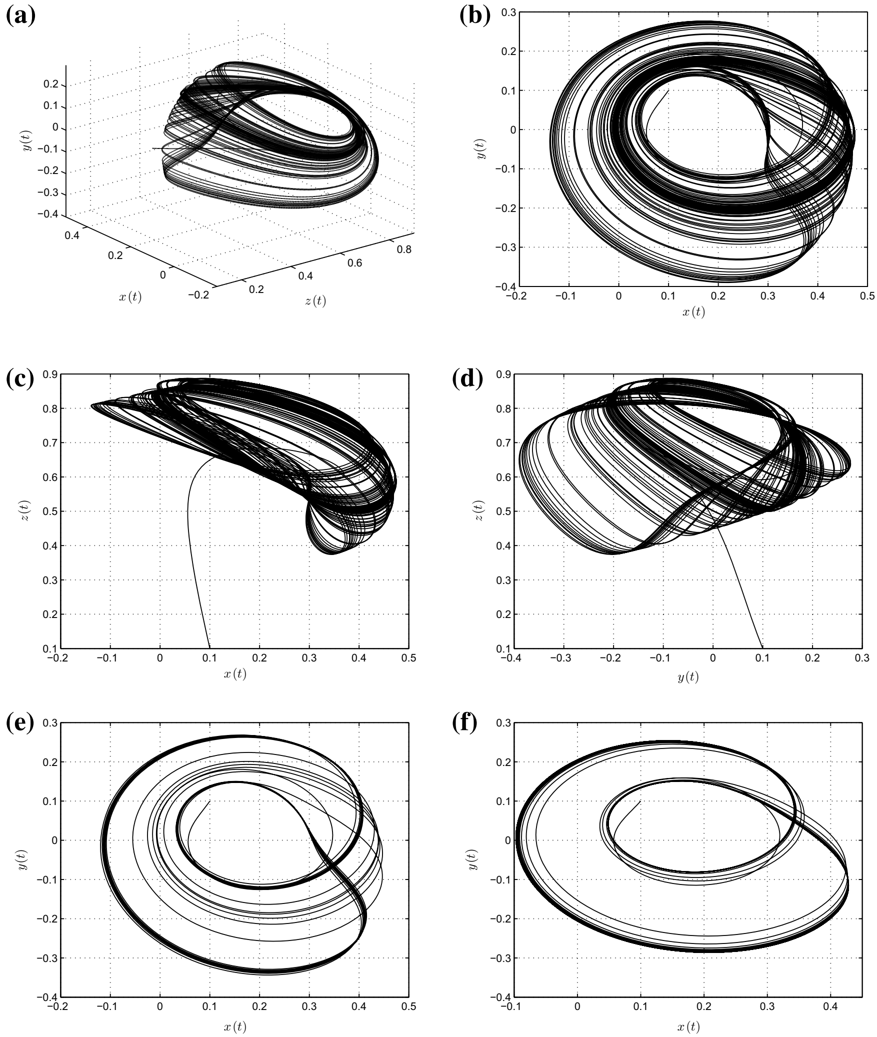

Fig. 1 Numerical simulation for the commensurate nonlinear Bloch system with non-singular kernel. In **a**–**d** projections of chaos for $\gamma = 0.95$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma = 0.92$ and $\gamma = 0.87$, respectively

Numerical solutions of the system (3) have been depicted in Fig. 3a–f and Fig. 4a–f for the commensurate and non-commensurate order system, respectively. The parameter values used in the simulations are $\zeta = -1.26$, $\varrho = 35$, $\varphi = 0.173$, $\Psi_1 = 5$, $\Psi_2 = 2.5$ and the initial conditions are $x(t) = 0.1$, $y(t) = 0.1$ and $z(t) = 0.1$. The step size used in evaluating the approximate solution was $h = 0.0001$.

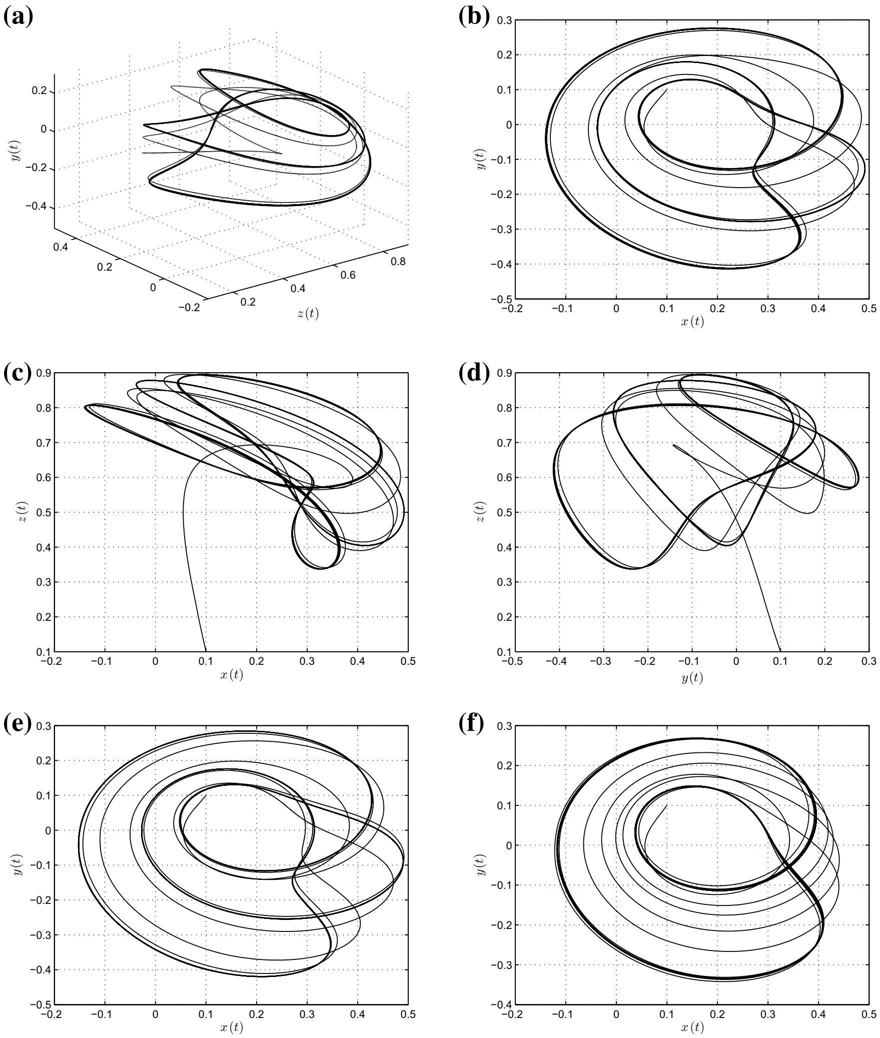

Fig. 2 Numerical simulation for the non-commensurate nonlinear Bloch system with non-singular kernel. In **a**–**d** projections of chaos for $\gamma_1 = 1$, $\gamma_2 = 0.95$ and $\gamma_3 = 1$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma_1 = 0.94$, $\gamma_2 = 1$ and $\gamma_3 = 1$ and $\gamma_1 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 0.92$, respectively

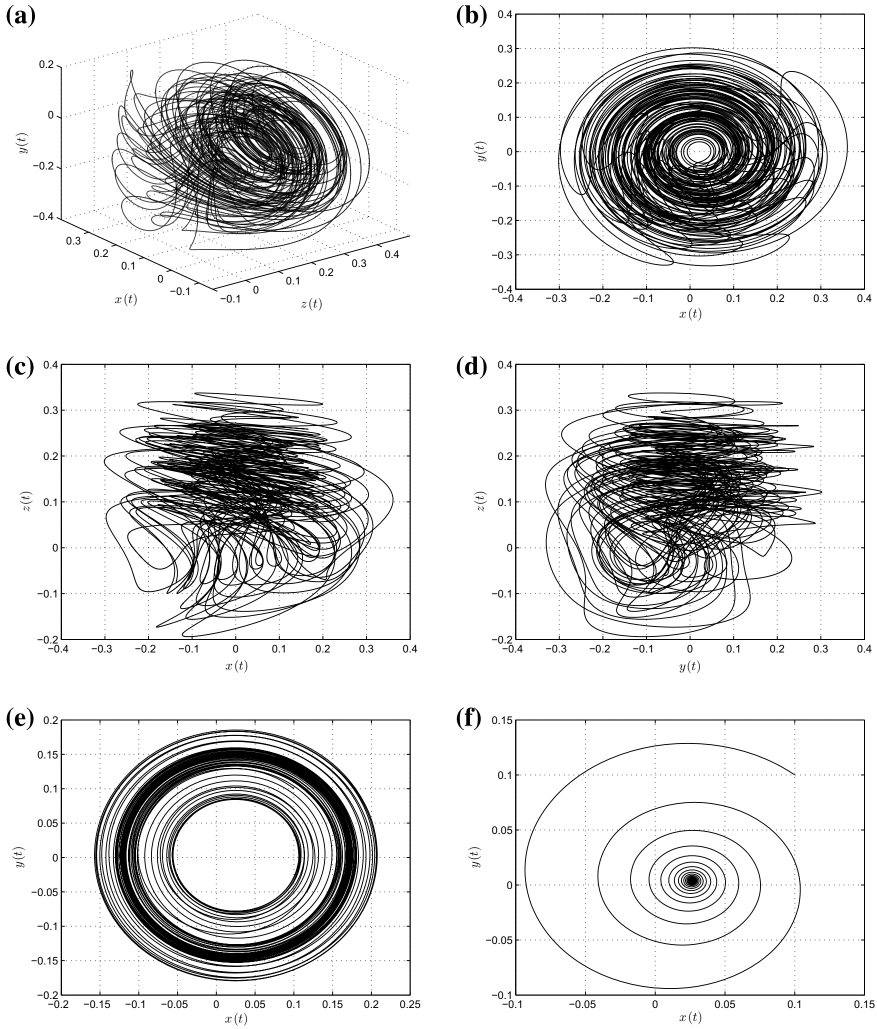

Fig. 3 Numerical simulation for the commensurate nonlinear Bloch system with with non-singular kernel. In **a**–**d** projections of chaos for $\gamma = 0.95$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma = 0.92$ and $\gamma = 0.87$, respectively

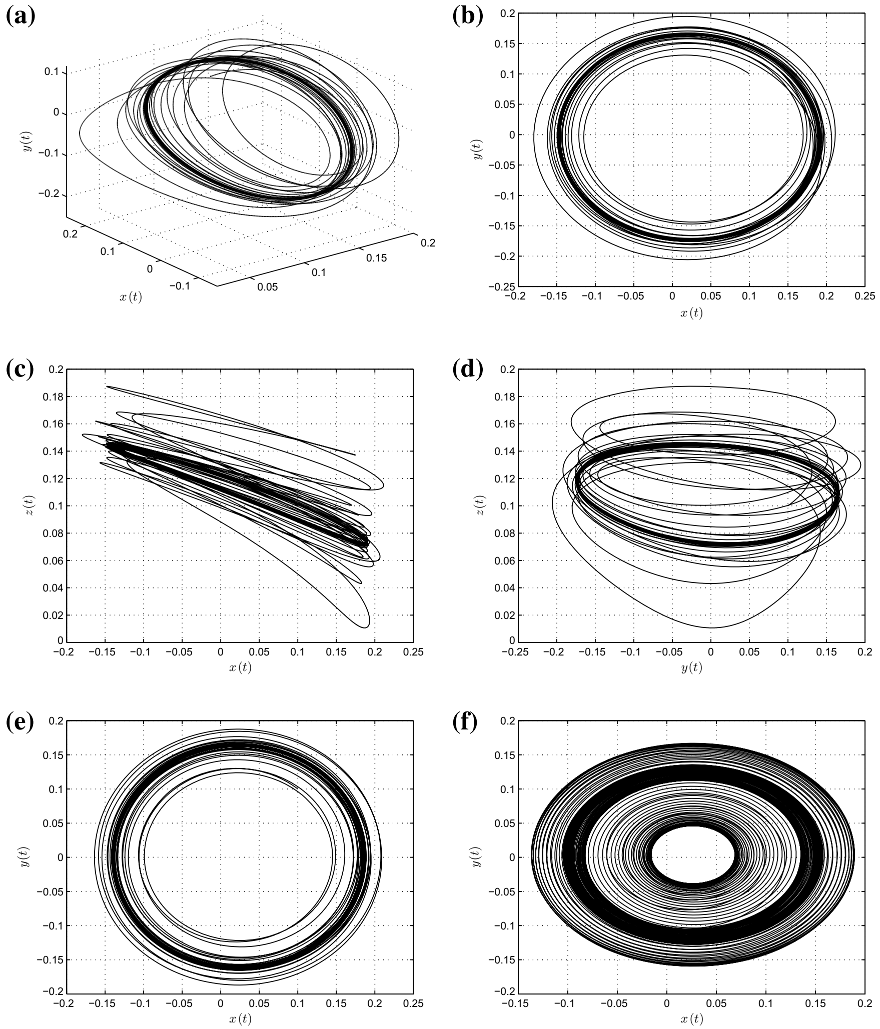

Fig. 4 Numerical simulation for the non-commensurate nonlinear Bloch system with with non-singular kernel. In **a**–**d** projections of chaos for $\gamma_1 = 1$, $\gamma_2 = 0.95$ and $\gamma_3 = 1$. In **e**–**f** chaotic phase trajectory $x(t) - y(t)$, for $\gamma_1 = 0.94$, $\gamma_2 = 1$ and $\gamma_3 = 1$ and $\gamma_1 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 0.92$, respectively

5 Conclusions

In this chapter, we used the new definition of fractional operator with an exponential kernel proposed by Caputo and Fabrizio. This new operator can describe material heterogeneities and structures with different scales, which cannot be handling with the classical theories. To further apply this operator, we have modified the nonlinear Bloch equation with feedback. We prove the existence and uniqueness of the coupled-solutions. The numerical results for nonlinear Bloch with non-singular kernel shows that with decreases the order of time-fractional operator ($\gamma \to 0$), several irregular attractors are formed and the model exhibit transient chaos. The characteristics of the alternative model, in contrast with the classical model, memory properties, the nuclear magnetization or other independent quantities are considered.

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Fractional Order Integration and Certain Integrals of Generalized Multiindex Bessel Function

K. S. Nisar, S. D. Purohit, D. L. Suthar and Jagdev Singh

Abstract We aim to introduce the generalized multiindex Bessel function $J_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$ and to present some formulas of the Riemann-Liouville fractional integration and differentiation operators. Further, we also derive certain integral formulas involving the newly defined generalized multiindex Bessel function $J_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$. We prove that such integrals are expressed in terms of the Fox-Wright function ${}_p\Psi_q(z)$. The results presented here are of general in nature and easily reducible to new and known results.

Keywords Generalized (Wright) hypergeometric functions · Generalized multiindex Bessel function · Fractional calculus · Integral formulas

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1 Introduction and Preliminaries

Fractional calculus, which has a long history, is an important branch of mathematical analysis (calculus) where differentiations and integrations can be of arbitrary non-integer order. The operators of Riemann-Liouville fractional integrals and derivatives are defined, for $\alpha \in \mathbb{C}$ $(\Re(\lambda) > 0)$ and $x > 0$ (see, for details, [8, 18])

$$\left(I_{0+}^{\lambda} f\right)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x \frac{f(t)}{(x-t)^{1-\lambda}} dt, \tag{1.1}$$

$$\left(I_{-}^{\lambda} f\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty \frac{f(t)}{(t-x)^{1-\lambda}} dt, \tag{1.2}$$

$$\begin{aligned}\left(D_{0+}^{\lambda} f\right)(x) &= \left(\frac{d}{dx}\right)^{[\Re(\lambda)]+1} \left(I_{0+}^{1-\lambda+[\Re(\lambda)]} f\right)(x) \\ &= \left(\frac{d}{dx}\right)^{[\Re(\lambda)]+1} \frac{1}{\Gamma(1-\lambda+\Re[\lambda])} \int_0^x \frac{f(t)}{(x-t)^{\lambda-[\Re(\lambda)]}}\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}\left(D_{-}^{\lambda} f\right)(x) &= \left(-\frac{d}{dx}\right)^{[\Re(\lambda)]+1} \left(I_{-}^{1-\lambda+[\Re(\lambda)]} f\right)(x) \\ &= \left(-\frac{d}{dx}\right)^{[\Re(\lambda)+1]} \frac{1}{\Gamma(1-\lambda+[\Re(\lambda)])} \int_x^\infty \frac{f(t)}{(t-x)^{\lambda-[\Re(\lambda)]}} dt\end{aligned} \tag{1.4}$$

respectively, where $[\Re(\lambda)]$ is the integral part of $\Re(\lambda)$. The following lemma is needed in sequel [18, (2.44)],

Lemma 1.1 *Let* $\lambda \in \mathbb{C}$ $(\Re(\lambda) > 0)$ *and* $\delta \in \mathbb{C}$ *then*

(a) If $\Re(\delta) > 0$ *then*

$$\left(I_{0+}^{\lambda} t^{\delta-1}\right)(x) = \frac{\Gamma(\delta)}{\Gamma(\lambda+\delta)} x^{\lambda+\delta-1}. \tag{1.5}$$

(b) If $\Re(\delta) > \Re(\lambda) > 0$ *then*

$$\left(I_{-}^{\lambda} t^{-\delta}\right)(x) = \frac{\Gamma(\delta-\lambda)}{\Gamma(\delta)} x^{\lambda-\delta}. \tag{1.6}$$

In this paper, we aim to introduce a new generalized multiindex Bessel function and to study its compositions with the classical Riemann-Liouville fractional integration and differentiation operators. Further, we derive certain integral formulas involving the newly defined generalized multiindex Bessel function $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$. We prove that such integrals are expressed in terms of the Fox-Wright function ${}_p\Psi_q(z)$.

2 Fractional Calculus Approach of $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$

In this section, we introduce a generalized multiindex Bessel function $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$ as follows:

For $\alpha_j, \beta_j, \gamma, b, c \in \mathbb{C}$ $(j = 1, 2, \ldots, m)$ be such that $\sum_{j=1}^{m} \Re\left(\alpha_j\right) > \max 0;$ $\{\Re(\kappa) - 1\}$; $\kappa > 0, \Re\left(\beta_j\right) > 0$ and $\Re(\gamma) > 0$, then

$$\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z] = \sum_{n=0}^{\infty} \frac{c^n\,(\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{z^n}{n!} \quad (m \in \mathbb{N}). \tag{2.1}$$

Here and in the following, $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the Gamma function Γ (see [19, Section 1.1]), by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \ \lambda \in \mathbb{C}). \end{cases} \tag{2.2}$$

2.1 Fractional Integration

We first recall the definition of the Fox-Wright function ${}_p\Psi_q(z)$ $(p,\ q \in \mathbb{N}_0)$ (see, for details, [6, 22]):

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix}\ z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n)\cdots\Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n)\cdots\Gamma(\beta_q + B_q n)} \frac{z^n}{n!} \tag{2.3}$$

$$\left(A_j \in \mathbb{R}^+\ (j = 1, \ldots, p);\ \ B_j \in \mathbb{R}^+\ (j = 1, \ldots, q);\ \ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0\right),$$

where the equality in the convergence condition holds true for

$$|z| < \nabla := \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \cdot \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

Now we present the Riemann-Liouville fractional integration of the generalized multiindex Bessel function $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$ in the following theorems.

Theorem 1 *Let* $\lambda, \delta \in \mathbb{C}$ *be such that* $\Re(\lambda) > 0$, $\Re(\delta) > 0$ *and the conditions given in (2.1) is satisfied, then for* $x > 0$, *the following integral formula holds true*

$$\left(I_{0+}^{\lambda} \left\{ t^{\delta-1} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}(t) \right\} \right)(x) = \frac{x^{\lambda+\delta-1}}{\Gamma(\gamma)} {}_2\Psi_{m+1} \left[\begin{array}{c} (\gamma, k), (\delta, 1) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^{m}, (\lambda+\delta, 1) \end{array} \Big| cx \right]. \tag{2.4}$$

Proof Let us denote the left-hand side of (2.4) by $\mathcal{I}_1$. Using the definition (2.1), we have

$$\begin{aligned} \mathcal{I}_1 &= \left(I_{0+}^{\lambda} \left\{ t^{\delta-1} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}(t) \right\} \right)(x) \\ &= \left(I_{0+}^{\lambda} \left\{ t^{\delta-1} \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{t^n}{n!} \right\} \right)(x). \end{aligned} \tag{2.5}$$

Interchanging the integration and the summation in (2.5) and using the definition of Pochhammer symbol (2.2), we get

$$\mathcal{I}_1 = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\gamma + \kappa n)}{\Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right) n!} \left(I_{0+}^{\lambda} t^{\delta+n-1} \right)(x).$$

Applying the relation (1.5) in Lemma 1.1, we get

$$\mathcal{I}_1 = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\gamma + \kappa n)}{\Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right) n!} \frac{\Gamma(\delta + n)}{\Gamma(\lambda + \delta + n)} x^{\lambda+\delta+n-1}.$$

In view of the definition of the Fox-Wright function (2.3), we arrived at the desired result. □

Theorem 2 *Let* $\lambda, \delta \in \mathbb{C}$ *such that* $\Re(\delta) > \Re(\lambda) > 0$ *and the conditions given in (2.1) is satisfied, then for* $x > 0$, *the following integral formula holds true*

$$\left(I_{-}^{\lambda}\left\{t^{-\delta}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(\frac{1}{t}\right)\right\}\right)(x)=\frac{x^{\lambda-\delta}}{\Gamma(\gamma)}\,{}_2\Psi_{m+1}\left[\begin{matrix}(\gamma,k),(\delta-\lambda,1)\\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{j=1}^{m},(\delta,1)\end{matrix}\middle|\frac{c}{x}\right]. \tag{2.6}$$

Proof Denoting the left-hand side of (2.5) by $\mathcal{I}_2$. Using (2.1), we have

$$\begin{aligned}\mathcal{I}_2&=\left(I_{-}^{\lambda}\left\{t^{-\delta}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(\frac{1}{t}\right)\right\}\right)(x)\\&=\left(I_{-}^{\lambda}\left\{t^{-\delta}\sum_{n=0}^{\infty}\frac{c^n\,(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+\frac{b+1}{2}\right)}\frac{t^{-n}}{n!}\right\}\right)(x)\,.\end{aligned} \tag{2.7}$$

Interchanging the integration and the summation in (2.7) and using the definition of Pochhammer symbol (2.2), we get

$$\mathcal{I}_2=\sum_{n=0}^{\infty}\frac{c^n\Gamma(\gamma+\kappa n)}{\Gamma(\gamma)\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+\frac{b+1}{2}\right)n!}\left(I_{-}^{\lambda}t^{-\delta-n}\right)(x)\,.$$

Applying the relation (1.6) in Lemma 1.1, we get

$$\mathcal{I}_1=\sum_{n=0}^{\infty}\frac{c^n\Gamma(\gamma+\kappa n)}{\Gamma(\gamma)\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+\frac{b+1}{2}\right)n!}\frac{\Gamma(\delta+n-\lambda)}{\Gamma(\delta+n)}x^{\lambda-\delta-n}.$$

In view of the definition of the Fox-Wright function (2.3), we arrived at the desired result. □

2.2 Fractional Differentiation

In this subsection, we establish the fractional differentiation of generalized multiindex Bessel function given in (2.1).

Theorem 3 *Let* $\lambda,\delta\in\mathbb{C}$ *such that* $\Re(\lambda)>0,\Re(\delta)>0$ *and the conditions given in (2.1) is satified, then for* $x>0$*, the following fractional differentiation formula holds true*

$$\left(D_{0+}^{\lambda}\left\{t^{\delta-1}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}(t)\right\}\right)(x)=\frac{x^{\delta-\lambda-1}}{\Gamma(\gamma)}\,{}_2\Psi_{m+1}\left[\begin{matrix}(\gamma,k),(\delta,1)\\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{i=1}^{m},(\delta-\lambda,1)\end{matrix}\middle|cx\right]. \tag{2.8}$$

Proof Let $\mathcal{I}_3$ denote the left-hand side of (2.8). Using the definition (2.1), we have

$$\begin{aligned}\mathcal{I}_3 &= \left(D_{0+}^{\lambda}\left\{t^{\delta-1}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}(t)\right\}\right)(x)\\ &= \left(\frac{d}{dx}\right)^n\left(I_{0+}^{n-\lambda}\left\{t^{\delta-1}\sum_{r=0}^{\infty}\frac{c^r(\gamma)_{\kappa r}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j+\frac{b+1}{2}\right)}\frac{t^r}{r!}\right\}\right)(x)\ ,\\ &= \left(\frac{d}{dx}\right)^n\sum_{r=0}^{\infty}\frac{c^r(\gamma)_{\kappa r}}{\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j+\frac{b+1}{2}\right)r!}\left(I_{0+}^{n-\lambda}t^{\delta+r-1}\right)(x)\,.\end{aligned}$$

Using the relation (1.5) and the definition of the Pochhammer symbol (2.2), we get

$$\mathcal{I}_3 = \left(\frac{d}{dx}\right)^n\sum_{r=0}^{\infty}\frac{c^r\Gamma(\gamma+\kappa r)}{\Gamma(\gamma)\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j+\frac{b+1}{2}\right)r!}\frac{\Gamma(\delta+r)}{\Gamma(n-\lambda+\delta+r)}x^{n-\lambda+\delta+r-1}.$$

By interchanging the differentiation and the summation, we get

$$\begin{aligned}\mathcal{I}_3 &= \sum_{r=0}^{\infty}\frac{c^r\Gamma(\gamma+\kappa r)}{\Gamma(\gamma)\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j+\frac{b+1}{2}\right)r!}\frac{\Gamma(\delta+r)}{\Gamma(n-\lambda+\delta+r)}\left(\frac{d}{dx}\right)^n x^{n-\lambda+\delta+r-1}\\ &= \frac{1}{\Gamma(\gamma)}\sum_{r=0}^{\infty}\frac{c^r\Gamma(\gamma+\kappa r)}{\prod_{j=1}^{m}\Gamma\left(\alpha_j r+\beta_j+\frac{b+1}{2}\right)r!}\frac{\Gamma(\delta+r)\Gamma(n-\lambda+\delta+r)}{\Gamma(n-\lambda+\delta+r)\Gamma(\delta-\lambda+r)}x^{\delta-\lambda+\delta+r-1}.\end{aligned}$$

In view of the definition of the Fox-Wright function (2.3), we arrived at the desired result. □

Theorem 4 *Let $\lambda, \delta \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\delta) > [\Re(\lambda)] + 1 - \Re(\lambda)$ and the conditions given in (2.1) is satisfied, then the fractional differentiation D_{-}^{λ} of generalized multiindex Bessel function is given by*

$$\left(D_{-}^{\lambda}\left\{t^{-\delta}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(\frac{1}{t}\right)\right\}\right)(x) = \frac{x^{1-\lambda-\delta}}{\Gamma(\gamma)}\,{}_2\Psi_{m+1}\left[\begin{matrix}(\gamma,k),(\lambda+\delta,1)\\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{j=1}^{m},(\delta,1)\end{matrix}\,\middle|\,\frac{c}{x}\right]. \tag{2.9}$$

Proof Let $\mathcal{I}_4$ denote the left-hand side of (2.9). Applying the definition (2.1), we have

$$\mathcal{I}_4 = \left(D_{-}^{\lambda} \left\{ t^{-\delta} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c} \left(\frac{1}{t} \right) \right\} \right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{n-\lambda} \left(t^{-\delta} \sum_{r=0}^{\infty} \frac{c^r (\gamma)_{\kappa r}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right)} \frac{t^{-r}}{r!} \right) \right)(x) \ ,$$

$$= \left(-\frac{d}{dx}\right)^n \sum_{r=0}^{\infty} \frac{c^r (\gamma)_{\kappa r}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right) r!} \left(I_{-}^{n-\lambda} t^{-\delta-r} \right)(x) .$$

Using the relation (1.6) and the definition of the Pochhammer symbol (2.2), we get

$$\mathcal{I}_4 = \left(-\frac{d}{dx}\right)^n \sum_{r=0}^{\infty} \frac{c^r \Gamma(\gamma+\kappa r)}{\Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right) r!} \frac{\Gamma(\delta+r-n+\lambda)}{\Gamma(\delta+r)} x^{n-\lambda-\delta-r},$$

By interchanging the derivatives and the summation, we get

$$\mathcal{I}_4 = \sum_{r=0}^{\infty} \frac{c^r \Gamma(\gamma+\kappa r)}{\Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right) r!} \frac{\Gamma(\delta+r-n+\lambda)}{\Gamma(\delta+r)} \left(-\frac{d}{dx}\right)^n x^{n-\lambda-\delta-r}$$

$$= \frac{1}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \frac{c^r \Gamma(\gamma+\kappa r)}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right) r!} \frac{\Gamma(\delta+r-n+\lambda)}{\Gamma(\delta+r)} \frac{(-1)^n \Gamma(n-\lambda-\delta-r+1)}{\Gamma(-\lambda-\delta-r+1)} x^{1-\delta-\lambda-r}$$

$$= \frac{x^{1-\delta-\lambda}}{\Gamma(\gamma)} \sum_{r=0}^{\infty} \frac{c^r \Gamma(\gamma+\kappa r)}{\prod_{j=1}^{m} \Gamma\left(\alpha_j r + \beta_j + \frac{b+1}{2}\right) r!} \frac{\Gamma(\lambda+\delta+r)}{\Gamma(\delta+r)}.$$

In view of the definition of the Fox-Wright function (2.3), we arrived at the desired result. □

3 Certain Integrals of the $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$

Recently many researchers are developing a large number of integral formulas involving a variety of special functions [1, 2, 4, 5, 7, 10–15, 17]. In this section, four integral formulas involving generalized multi-index Bessel function $\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}[z]$ are established, which are expressed in terms of the Fox-Wright function. For the present investigation, we need the following result of Oberhettinger [16]

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx=2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \quad (3.1)$$

provided $0<\Re(\mu)<\Re(\lambda)$ and the following integral formula due to Lavoie [9]

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (3.2)$$

with $\Re(\alpha)>0, \Re(\beta)>0$.

Theorem 5 *Let* $\alpha_j, \beta_j, \gamma, , b, c \in \mathbb{C}$ $(j=1,2,\ldots,m)$ *be such that* $\sum_{j=1}^{m}\Re\left(\alpha_j\right)>\max\{0;\Re(\kappa)-1\}$ *with* $\kappa>0, \Re(\beta)>-1, \Re(\gamma)>0, 0<\Re(\mu)<\Re(\lambda+n)$ *and* $x>0$ *, then*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=\frac{2^{1-\mu}a^{-\lambda+\mu}\Gamma(2\mu)}{\Gamma(\gamma)}\,{}_3\Psi_{m+2}\left[\begin{array}{c}(\gamma,k),(\lambda+1,1),(\lambda-\mu,1)\\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{j=1}^{m},(\lambda,1),(1+\lambda+\mu,1)\end{array}\middle|\frac{-cy}{a}\right]. \quad (3.3)$$

Proof Let us denote the right-hand side of (3.3) by $\mathcal{I}_5$ and using the definition (2.1), we have

$$\begin{aligned}\mathcal{I}_1&=\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\mathcal{J}_{(\beta_j)m,\kappa,b}^{(\alpha_j)m,\gamma,c}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx\\&=\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}\\&\qquad\times\sum_{n=0}^{\infty}\frac{(-c)^n(\gamma)_{\kappa n}}{n!\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+\frac{b+1}{2}\right)}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)^n.\end{aligned}$$

Interchanging the integration and summation under the suitable convergence condition gives

$$\mathcal{I}_1=\sum_{n=0}^{\infty}\frac{(-c)^n(\gamma)_{\kappa n}\,y^n}{n!\prod_{j=1}^{m}\Gamma\left(\alpha_j n+\beta_j+\frac{b+1}{2}\right)}\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+n)}dx, \quad (3.4)$$

Applying (3.1) in (3.4), we get

$$\mathcal{I}_1 = \sum_{n=0}^{\infty} \frac{(-c)^n (\gamma)_{\kappa n}\, y^n}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} 2(\lambda+n)\, a^{-(\lambda+n)} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\,\Gamma(\lambda+n-\mu)}{\Gamma(1+\lambda+\mu+n)},$$

provided $\Re(\lambda+n) > \Re(\mu) > 0$. Now using the definition of Pochhammer symbol, we get

$$\mathcal{I}_1 = \frac{2^{1-\mu} a^{-\lambda+\mu}\Gamma(2\mu)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\kappa n)}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{\Gamma(\lambda+n+1)\,\Gamma(\lambda-\mu+n)}{\Gamma(\lambda+n)\,\Gamma(1+\lambda+\mu+n)} \frac{\left(-\frac{cy}{a}\right)^n}{n!}.$$

In view of the definition of Fox-Wright function (2.3), we arrived the desired result. □

Theorem 6 *Let* $\alpha_j, \beta_j, \gamma, b, c \in \mathbb{C}$ $(j = 1, 2, \ldots, m)$ *be such that* $\sum_{j=1}^{m} \Re\left(\alpha_j\right) >$ $\max\{0; \Re(\kappa) - 1\}$ *with* $\kappa > 0$, $\Re(\beta) > -1$, $\Re(\gamma) > 0$, $0 < \Re(\mu+n) < \Re(\lambda+n)$ *and* $x > 0$ *, then*

$$\int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2+2ax}\right)^{-\lambda} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \frac{2^{1-\mu}a^{-\lambda+\mu}\Gamma(2\mu)}{\Gamma(\gamma)}\, {}_3\Psi_{m+2}\left[\begin{matrix} (\gamma, k), (\lambda+1, 1), (2\mu, 2) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^{m}, (\lambda, 1), (1+\lambda+\mu, 2) \end{matrix} \middle| \frac{-cy}{a}\right]. \tag{3.5}$$

Proof Let us denote the right-hand side of (3.5) by $\mathcal{I}_6$ and using the definition (2.1), we have

$$\mathcal{I}_6 = \int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2+2ax}\right)^{-\lambda} \mathcal{J}_{(\beta_j)m,\kappa,b}^{(\alpha_j)m,\gamma,c}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$
$$= \int_0^{\infty} x^{\mu-1} \left(x + a + \sqrt{x^2+2ax}\right)^{-\lambda}$$
$$\times \sum_{n=0}^{\infty} \frac{(-c)^n (\gamma)_{\kappa n}}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)^n.$$

Interchanging the integration and summation under the given condition, yields

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} \frac{(-c)^n (\gamma)_{\kappa n} y^n}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \int_0^{\infty} x^{\mu+n-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-(\lambda+n)} dx. \tag{3.6}$$

Applying (3.1) on (3.6), we get

$$\mathcal{I}_6 = \sum_{n=0}^{\infty} \frac{(-c)^n (\gamma)_{\kappa n} y^n}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} 2(\lambda+n) a^{-(\lambda+n)} \left(\frac{a}{2}\right)^{\mu+n} \frac{\Gamma(2\mu+2n)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu+2n)},$$

provided $\Re(\lambda+n) > \Re(\mu+n) > 0$.

In view of definition of Pochhammer symbol (2.2), we get

$$\mathcal{I}_6 = \frac{2^{1-\mu} a^{-\lambda+\mu} \Gamma(\lambda-\mu)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+\kappa n)}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{\Gamma(\lambda+n+1)\Gamma(2\mu+2n)}{\Gamma(\lambda+n)\Gamma(1+\lambda+\mu+2n)} \frac{\left(-\frac{cy}{2}\right)^n}{n!}.$$

Using the definition of Fox-Wright function (2.3), we arrived the desired result. □

Theorem 7 *For* $\xi, \sigma \in \mathbb{C}$ *with* $\Re(\xi+\sigma) > 0, \Re(\xi+n) > 0$ *and then for* $x > 0$,

$$\int_0^1 x^{\xi+\sigma-1} (1-x)^{2\xi-1} \left(1-\frac{x}{3}\right)^{2(\xi+\sigma)-1} \left(1-\frac{x}{4}\right)^{\xi-1} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c} \left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx$$
$$= \frac{\Gamma(\xi+\sigma)}{\Gamma(\gamma)} \left(\frac{2}{3}\right)^{2(\xi+\sigma)} {}_2\Psi_{m+1}\left[\begin{matrix} (\gamma, k), (\xi, 1) \\ \left(\beta_j + \frac{b+1}{2}, \alpha_j\right)_{j=1}^{m}, (2\xi+\sigma, 1) \end{matrix} \middle| cy\right].$$

Proof Denoting the left-hand side of theorem by $\mathcal{I}_7$ and using (2.1) ,we get

$$\begin{aligned}\mathcal{I}_7 &= \int_0^1 x^{\xi+\sigma-1} (1-x)^{2\xi-1} \left(1-\frac{x}{3}\right)^{2(\xi+\sigma)-1} \left(1-\frac{x}{4}\right)^{\xi-1} \\ &\quad \times \mathcal{J}_{(\beta_j)m,\kappa,b}^{(\alpha_j)m,\gamma,c} \left(y\left(1-\frac{x}{4}\right)(1-x)^2\right) dx, \\ &= \int_0^1 x^{\xi+\sigma-1} (1-x)^{2\xi-1} \left(1-\frac{x}{3}\right)^{2(\xi+\sigma)-1} \left(1-\frac{x}{4}\right)^{\xi-1} \\ &\quad \times \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{y^n \left(1-\frac{x}{4}\right)^n (1-x)^{2n}}{n!} dx.\end{aligned}$$

Interchanging the integration and summation gives,

$$\mathcal{I}_7 = \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{\kappa n} y^n}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \times \int_0^1 x^{\xi+\sigma-1} (1-x)^{2(\xi+n)-1} \left(1-\frac{x}{3}\right)^{2(\xi+\sigma)-1} \left(1-\frac{x}{4}\right)^{\xi+n-1} dx.$$

Now using (3.2) and the definition of Pochhammer symbol,

$$\mathcal{I}_7 = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\kappa+\gamma n) y^n}{n! \Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \left(\frac{2}{3}\right)^{2(\xi+\sigma)} \frac{\Gamma(\xi+\sigma)\Gamma(\xi+n)}{\Gamma(2\xi+\sigma+n)}.$$

Using the definition of Fox-Wright function (2.3), we obtained the required result. □

Theorem 8 *For* $\xi, \sigma \in \mathbb{C}$ *with* $\Re(\xi+\sigma) > 0, \Re(\xi+n) > 0$ *then for* $x > 0$

$$\int_0^1 x^{\xi-1} (1-x)^{2(\xi+\sigma)-1} \left(1-\frac{x}{3}\right)^{2\xi-1} \left(1-\frac{x}{4}\right)^{(\xi+\sigma)-1} \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(yx\left(1-\frac{x}{3}\right)^2\right) dx$$
$$= \frac{\Gamma(\xi+\sigma)}{\Gamma(\gamma)} \left(\frac{2}{3}\right)^{2\xi} {}_2\Psi_{m+1}\left[\begin{matrix}(\gamma,k),(\xi,1)\\ \left(\beta_j+\frac{b+1}{2},\alpha_j\right)_{j=1}^m,(2\xi+\sigma,1)\end{matrix}\,\middle|\,\frac{4cy}{9}\right].$$

Proof Taking left-hand side of theorem by $\mathcal{I}_8$ and using (2.1), we get

$$\mathcal{I}_8 = \int_0^1 x^{\xi-1} (1-x)^{2(\xi+\sigma)-1} \left(1-\frac{x}{3}\right)^{2\xi-1} \left(1-\frac{x}{4}\right)^{(\xi+\sigma)-1} \times \mathcal{J}_{(\beta_j)_m,\kappa,b}^{(\alpha_j)_m,\gamma,c}\left(yx\left(1-\frac{x}{3}\right)^2\right) dx,$$
$$= \int_0^1 x^{\xi-1} (1-x)^{2(\xi+\sigma)-1} \left(1-\frac{x}{3}\right)^{2\xi-1} \left(1-\frac{x}{4}\right)^{(\xi+\sigma)-1} \times \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \frac{x^n y^n \left(1-\frac{x}{3}\right)^{2n}}{n!} dx.$$

Interchanging the integration and summation gives,

$$\mathcal{I}_8 = \sum_{n=0}^{\infty} \frac{c^n (\gamma)_{\kappa n} \, y^n}{n! \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)}$$
$$\times \int_0^1 x^{\xi+n-1} (1-x)^{2(\xi+\sigma)-1} \left(1 - \frac{x}{3}\right)^{2(\xi+n)-1} \left(1 - \frac{x}{4}\right)^{\xi+\sigma-1} dx.$$

Now using (3.2) and the definition of Pochhammer symbol (2.2),

$$\mathcal{I}_8 = \sum_{n=0}^{\infty} \frac{c^n \Gamma(\kappa + \gamma n) \, y^n}{n! \Gamma(\gamma) \prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + \frac{b+1}{2}\right)} \left(\frac{2}{3}\right)^{2\xi} \frac{\Gamma(\xi+n) \Gamma(\xi+\sigma)}{\Gamma(2\xi+\sigma+n)}.$$

Using the definition of Fox-Wright function (2.3), we obtained the desired result. □

4 Concluding Remark and Discussion

The fractional calculus and the integral formulae of the newly defined generalized multiindex Bessel function are investigated here. Various special cases of the derived results in the paper can be evaluate by taking suitable values of parameters involved. For example, if we set $c = -1$ and $b = 1$ in (2.1), we immediately obtain the result due to Choi and Agarwal [3]:

$$J_{(\beta_j)_m, \kappa, 1}^{(\alpha_j)_m, \gamma, -1}[z] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m} \Gamma\left(\alpha_j n + \beta_j + 1\right)} \frac{(-z)^n}{n!} \quad (m \in \mathbb{N}). \tag{4.1}$$

For various other special cases we refer [3, 20, 21] and we left results for the interested readers.

Conflict of Interests The authors declare that there is no conflict of interests regarding the publication of this paper.

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Fractional Variational Iteration Method for Time Fractional Fourth-Order Diffusion-Wave Equation

Amit Prakash and Manoj Kumar

Abstract In the present article, Fractional variational iteration method (FVIM) is used to solve numerically time-fractional diffusion wave equation of order four. By using FVIM we obtain a sequence converging rapidly to the exact solution of the fourth order fractional diffusion wave equation. Two test problem are presented to prove the merit of the proposed technique. Plotted graph shows that the numerical solution acquired by employed technique is similar to the exact solution.

Keywords Mittag-Leffler function · Fractional variational iteration method · Diffusion wave equation of order four · Fractional derivative in the sense of Caputo

2010 Mathematics Subject Classification 44A99 · 35Q99

1 Introduction

In few last years, popular progress has been presumed in the area of newly branch of calculus named fractional calculus. In fractional calculus, numerous fractional differential equations are used to mold a sort of projects as in field of ion-acoustic wave, bio-informatics, nanotechnology, heat conduction, electromagnetic waves, diffusion equations, chemical engineering, mechanical engineering and almost every part of science and technology. Due to its eerie range and praxis in numerous fields, a great consideration is taken in the numerical approach as analytic solution does not exist always. Many researchers have taken interest in the use of modeling and controlling in numerous dynamical systems with the help of fractional partial differential equations and it is also key thing to find the solution technique of these type of models.

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Caputo fractional derivatives are defined only for differentiable functions and in this paper, we considered u as a differentiable function. Also, we have used Caputo fractional derivative because it has an advantage that with these derivatives, initial conditions for fractional differential equations undertake the similar form as for the integer order differential equations.

It is also of same importance to strut critical points which produce casual divergence, branching and convergence of the numerical solutions of the given model. In order to find the exact and numerical solution of the fractional order ordinary and partial differential equations, numerous approaches have been employed in past time.

In (1997) He [1–4] established a novel method, called, Variational iteration method (VIM) to find numerical and exact solution of models generated by linear and nonlinear fractional differential equations. After this Odibat and Momani [5] and Yulita Molliq et al. [6] employed VIM to find the numerical solution of nonlinear fractional Zakharov–Kuznetsov equations. Lu [7] and Sakar et al. [8, 9] employed FVIM and AVIM to solve numerically Fornberg–Whitham equation. Prakash et al. applied FVIM [10–14] and HPTM [15] to find numerically solution of various nonlinear partial differential equation of fractional order and many others by different technique [16–21]. By using the FVIM technique, numerical as well as exact solutions can be obtained as a convergent sequence and series rapidly. We can get extremely correct numerical results and exact solution in the form of a convergent sequence for fractional differential equations with the help of proposed technique.

There is an epochal role of time-fractional diffusion-wave equations in the field of mathematical physics. Agarwal [22] obtained the time-fractional diffusion wave equation of order four with the help of standard diffusion wave equation by changing the time dependent derivative using fractional order derivative $\boldsymbol{\alpha}, \mathbf{0} < \alpha < 1$ or $\mathbf{1} < \alpha < 2$. It can be concluded that as α changes between 0 and 2, the procedure gets changed starting from low dispersal to standard wave process. From the past many authors like Mainardi [23] and El-Sayed [24] have studied the diffusion wave equation of fractional order and its characters. These type of equations have prominent uses in the area of mathematical physics. Fractional diffusion wave equation has been used to define diffusion in resources with fractal geometry by Nigmatullin [25]. Fractional diffusion equation has been employed to describe phenomena of relaxation in complex viscoelastic materials by Ginoa et al. [26]. In many models, we have to use a space dependent fourth order derivative term. For instance, when we make a model in beams of wave propagation during the construction of grooves on a flat surface as grain involve fourth-order space derivative terms in its modelling. In this article we have taken the time-fractional fourth order diffusion wave equation with the given conditions as

$$D_t^{\alpha} u(x,t) = \beta \frac{\partial^4 u(x,t)}{\partial x^4}, u(0,t) = u(L,t) = \frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = c, t \geq 0,$$
$$u(x,0) = g(x), u_t(x,0) = 0,\ 0 < x < L$$

Recently fractional diffusion wave equation of order four has been solved by Adomian decomposition method by Dehghan et al. [27]. But fractional model of fourth order diffusion wave equation has not been solved by FVIM. The core motive of the present article is to apply fractional Variational iteration method to solve time-fractional fourth order diffusion wave equation for different values of fractional order α.

2 Preliminaries

Definition 2.1 A real valued function $g(p)$, $p > 0$ is in C_α, $\alpha \in R$ if there is $m > \alpha$, where m is a real number such that $g(p) = p^m g_1(p)$ where $g_1 \in C[0, \infty]$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$ [28–31].

Definition 2.2 A function $g(p)$, $p > 0$ is in C_α^m, $m \in N \cup \{0\}$ if $g^{(m)} \in C_\alpha$ [28–31].

Definition 2.3 Fractional integral in the sense of Riemann-Liouville of order $\mu > 0$, [28–31] of a $g \in C_\alpha, \alpha \geq -1$ is as:

$$I^\mu g(p) = \frac{1}{\Gamma(\mu)} \int_0^p \frac{g(\tau)}{(p-\tau)^{1-\mu}} \mathrm{d}\tau = \frac{1}{\Gamma(\mu+1)} \int_0^p g(\tau)(\mathrm{d}\tau)^\mu,$$

$$I^0 g(p) = g(p).$$

Definition 2.4 Fractional derivative in the sense of Caputo of g, $g \in C_{-1}^m, m \in \mathrm{N} \cup \{0\}$ [28–31],

$$D_p^\mu g(p) = \begin{cases} \left[I^{m-\mu} g^{(m)}(p)\right], & m-1 < \mu < m, m \in \mathrm{N}, \\ \frac{d^m}{\mathrm{d}p^m} g(p), & \mu = m. \end{cases}$$

a. $I_p^\alpha g(x,p) = \frac{1}{\Gamma(\alpha)} \int_0^p (p-s)^{\alpha-1} g(x,s)\mathrm{d}s, \alpha, p > 0.$

b. $D_p^\alpha u(x,p) = I_p^{m-\alpha} \frac{\partial^m u(x,p)}{\partial p^m} g(p), m-1 < \alpha < m.$

c. $I^\mu p^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)} p^{\mu+\gamma}.$

Definition 2.5 The Mittag-Leffler function denoted by $E_\beta(z)$ with $\beta > 0$, in the form of a series valid in domain of complex plane [22–25] is given as $E_\beta(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\beta n+1)}, \beta > 0, z \in C.$

3 Description of Fractional Variational Iteration Method (FVIM)

To apply the proposed technique on the time-fractional diffusion wave equation of order four, we take the general equation as

$$D_t^{\alpha} v(r,s,t) = \beta \frac{\partial^4 v(r,s,t)}{\partial r^4} + \gamma \frac{\partial^4 v(r,s,t)}{\partial s^4}; r,s \in R, 0 < \alpha \le 1, t \ge 0. \quad (1)$$

Using FVIM, we construct a correction functional [3] for this as

$$v_{n+1}(r,s,t) = v_n(r,s,t) + \int_0^t \lambda \left(\frac{\partial^{\alpha} v_n(\mathrm{r,s},\tau)}{\partial \tau^{\alpha}} - \beta \frac{\partial^4 \tilde{v}_n(\mathrm{r,s},\tau)}{\partial r^4} - \gamma \frac{\partial^4 \tilde{v}_n(r,s,\tau)}{\partial s^4} \right) (\mathrm{d}\tau)^{\alpha}. \quad (2)$$

Now by the variational theory λ must satisfy $\frac{\partial^{\alpha}\lambda}{\partial\tau^{\alpha}} = 0$ and $1 + \lambda|_{\tau=t} = 0$. From these equations, we obtain $\lambda = -1$ and by using $\lambda = -1$ in (2), we get the result as

$$v_{n+1}(r,s,t) = v_n(r,s,t) - \int_0^t \left(\frac{\partial^{\alpha} v_n(\mathrm{r,s},\tau)}{\partial \tau^{\alpha}} - \beta \frac{\partial^4 \mathrm{v}_n(\mathrm{r,s},\tau)}{\partial r^4} - \gamma \frac{\partial^4 \mathrm{v}_n(\mathrm{r,s},\tau)}{\partial s^4} \right) (\mathrm{d}\tau)^{\alpha}. \quad (3)$$

Now, we can construct a sequence of approximations v_n, $n \ge 0$ by evaluating λ, a general Lagrange's multiplier, that can be find out with the help of variational theory. The function $\tilde{v}_n$ is a restricted variation that imply $\delta\tilde{v}_n = 0$. Thus we first find λ with the help of integration and then construct succeeding iterations $\mathrm{v}_{n+1}(\mathrm{x,t})$, $\mathrm{n} \ge 0$ and then exact solution can be find out as $\mathrm{v(r,s,t)} = \lim_{n\to\infty} \mathrm{v_n(r,s,t)}$.

4 Test Examples

In present segment, we employed suggested technique on two test problems.

Example 4.1 Consider the first test problems as [23, 28]

$$D_t^{\alpha} u(x,t) = -\frac{\partial^4 u(x,t)}{\partial x^4}, t > 0, u(x,0) = e^{-x}, 0 < \alpha \le 1 \quad (4)$$

Comparing Eq. (4) with Eq. (1), by FVIM, we get Lagrangian multiplier $\lambda = -1$. So, we can assume

$$u_0(x,t) = u(x,0) = e^{-x},$$

Then by using Fractional variational iteration method (FVIM), we get

$$u_1(x,t) = e^{-x}\left(1 - \tfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right),$$
$$u_2(x,t) = e^{-x}\left(1 - \tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right),$$
$$u_3(x,t) = e^{-x}\left(1 - \tfrac{t^{\alpha}}{\Gamma(1+\alpha)} + \tfrac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \tfrac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right),$$
$$\vdots$$
$$u_n(x,t) = e^{-x}\left[1 - \tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \cdots + (-1)^n \tfrac{t^{n\alpha}}{\Gamma(n\alpha+1)}\right].$$

Then we obtain solution as

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = e^{-x} E_{\alpha}(-t),$$

which is the exact solution of Eq. (4), where $E_{\alpha}(t)$ is Mittag-Leffler function.

Example 4.2 We consider the second test problems as [23, 28]

$$D_t^{\alpha} u(x,y,t) = -2\left(\frac{\partial^4 u(x,y,t)}{\partial x^4} + \frac{\partial^4 u(x,y,t)}{\partial y^4}\right), u(x,y,0)$$
$$= \cos x \cos y, \frac{\partial(x,y,0)}{\partial t} = 0, \;\; 1 < \alpha \le 2. \tag{5}$$

Comparing Eq. (5) with Eq. (1), by FVIM, we get Lagrangian multiplier $\lambda = -1$. So, we can assume

$$u_0(x,y,t) = u(x,y,0) = \cos x \cos y.$$

Then by using Fractional variational iteration method (FVIM), we get

$$u_1(x,y,t) = \cos x \cos y\left(1 - \tfrac{4t^{\alpha}}{\Gamma(1+\alpha)}\right),$$
$$u_2(x,y,t) = \cos x \cos y\left(1 - \tfrac{4t^{\alpha}}{\Gamma(1+\alpha)} + \tfrac{4^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right),$$
$$\vdots$$
$$u_n(x,y,t) = \cos x \cos y\left(1 - \tfrac{4t^{\alpha}}{\Gamma(1+\alpha)} + \tfrac{4^2 t^{2\alpha}}{\Gamma(1+2\alpha)} - \cdots + \tfrac{(-1)^n 4^n t^{n\alpha}}{\Gamma(1+n\alpha)}\right),$$

Then, we obtain solution as

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t) = \cos x \cos y E_{\alpha}(-4t^{\alpha}),$$

which is the exact solution of Eq. (5), where $E_{\alpha}(t)$ is Mittag-Leffler function.

5 Numerical Results and Discussions

Figure 1 depicts comparison between exact solution and approximate solution when $\alpha = 1$ at $t = 1$. Figure 2 demonstrate the comparison between exact solution and approximate solution when $\alpha = 1$ acquired with the help of proposed technique FVIM. Figure 2a represent exact solution and Fig. 2b represent the numerical solution. It can be observed from Fig. 2 that the solution attained by FVIM is same as exact solution. It can be observed that only the tenth order term of FVIM was employed for finding the numerical solution. Table 1 shows that the absolute error

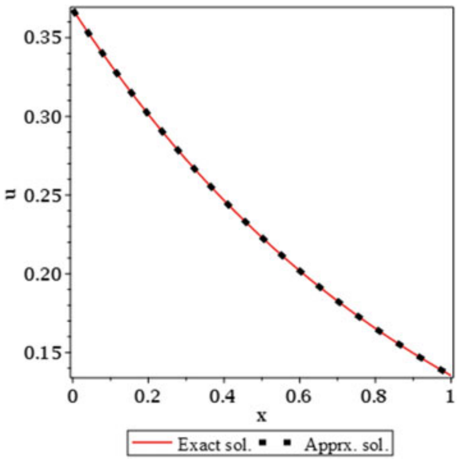

Fig. 1 Comparison between approximate solution and exact solution for $\alpha = 1$ at $t = 1$ for Example 4.1

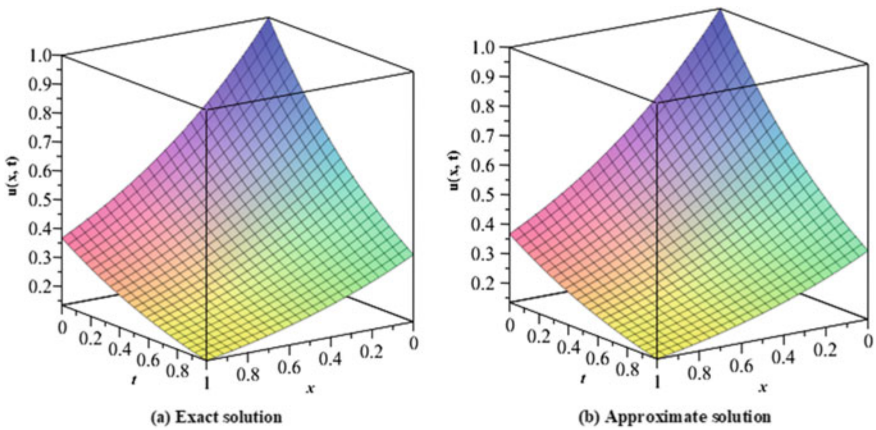

Fig. 2 Comparison of approximate solution and exact solution for $\alpha = 1$ for Example 4.1

Table 1 Absolute error $|u(x,t) - u_{10}(x,t)|$ for Example 4.1

x	t	$\alpha = 1$	$\alpha = 0.9$	x	t	$\alpha = 1$	$\alpha = 0.9$
0.1	0.2	5.6×10^{-16}	2.56×10^{-2}	0.1	0.6	7.0×10^{-11}	1.47×10^{-2}
0.3	0.2	4.4×10^{-16}	2.09×10^{-2}	0.3	0.6	6.4×10^{-11}	1.21×10^{-2}
0.5	0.2	3.3×10^{-16}	1.71×10^{-2}	0.5	0.6	5.2×10^{-11}	9.91×10^{-3}
0.7	0.2	2.8×10^{-16}	1.40×10^{-2}	0.7	0.6	4.3×10^{-11}	8.81×10^{-3}
0.9	0.2	2.2×10^{-16}	1.15×10^{-2}	0.9	0.6	3.5×10^{-11}	6.64×10^{-3}
0.1	0.4	9.2×10^{-13}	2.24×10^{-2}	0.1	0.8	1.8×10^{-9}	6.87×10^{-3}
0.3	0.4	7.5×10^{-13}	1.83×10^{-2}	0.3	0.8	1.5×10^{-9}	5.56×10^{-3}
0.5	0.4	6.1×10^{-13}	1.50×10^{-2}	0.5	0.8	1.2×10^{-9}	4.60×10^{-3}
0.7	0.4	5.1×10^{-13}	1.23×10^{-2}	0.7	0.8	1.1×10^{-9}	3.77×10^{-3}
0.9	0.4	4.1×10^{-13}	1.00×10^{-2}	0.9	0.8	8.2×10^{-10}	3.08×10^{-3}

between the approximate solution and exact solution is very small for $\alpha = 0.9$ and $\alpha = 1$. Figure 3 depicts comparison between numerical solution and exact solution for $\alpha = 2$ at $y = 0.5$, t = 1. Figure 3 demonstrate the comparison between the exact and the numerical solution for $\alpha = 2$ acquired with the help of proposed technique FVIM. Figure 4a represent the exact solution and Fig. 4b represent the approximate solution. It can be observed from Fig. 4 that the solution attained by FVIM is same as the exact solution. It can be observed that only the tenth order term of FVIM was employed for finding the numerical solution. Table 2 shows that the absolute error between the approximate and the exact solution is very small for $\alpha = 1.7$, 1.8 and $\alpha = 1.9$.

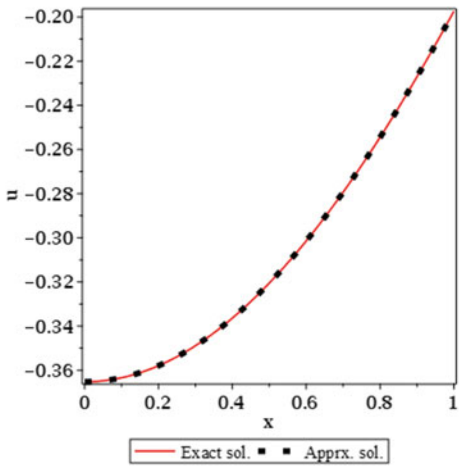

Fig. 3 Comparison between approximate solution and exact solution for $\alpha = 2$ at $y = 0.5$, $t = 1$ for Example 4.2

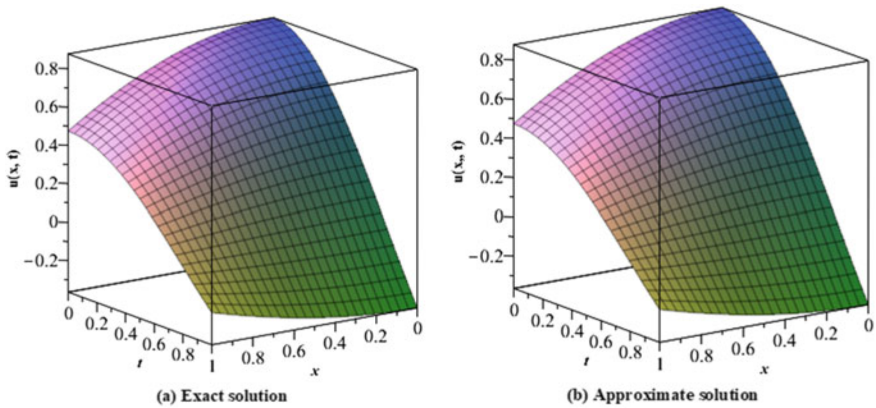

Fig. 4 Comparison between approximate solution and exact solution for $\alpha = 2$ for Example 4.2

Table 2 Absolute error $|u(x, t) - u_{10}(x, t)|$ for Example 4.2

x	t	$\alpha = 1.7$	$\alpha = 1.8$	$\alpha = 1.9$
0	0	0	0	0
0	1	5.76×10^{-11}	2.19×10^{-12}	7.92×10^{-14}
0	2	2.33×10^{-5}	1.92×10^{-6}	1.48×10^{-7}
1	0	0	0	0
1	1	3.11×10^{-11}	1.18×10^{-12}	4.27×10^{-14}
1	2	1.261×10^{-5}	1.04×10^{-6}	8.01×10^{-8}
2	0	0	0	0
2	1	2.39×10^{-11}	9.13×10^{-13}	3.29×10^{-14}
2	2	9.71×10^{-6}	8×10^{-7}	6.16×10^{-8}
3	0	0	0	0
3	1	5.70×10^{-11}	2.17×10^{-12}	7.83×10^{-14}
3	2	2.31×10^{-5}	1.9×10^{-6}	1.46×10^{-7}

6 Conclusion

In present article, Fractional Variational iteration method (FVIM) is employed to solve numerically time-fractional diffusion wave equation of fourth order. It can be clearly seen that fractional variation iteration method (FVIM) is an efficient and powerful numerical tool to find the numerical analytic solution. The advantage of this method over other methods is that it can be used directly without the use of polynomial used in adomian method, linearization, perturbation or restrictive assumptions. So we can conclude that FVIM is easier and more suitable than any other numerical methods.

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Analytical Approach to Fractional Navier–Stokes Equations by Iterative Laplace Transform Method

Rajendra K. Bairwa and Jagdev Singh

Abstract In this paper, we have presented iterative Laplace transform scheme to examine fractional Navier–Stokes equations in cylindrical coordinates with initial conditions. The arbitrary ordered derivatives are described in terms of Caputo. By utilizing only the initial conditions, the analytical expressions are derived in the closed form. The results achieved with the aid of the proposed technique are graphically presented.

Keywords Laplace transform · Navier–Stokes equations · Iterative method · Caputo fractional derivative

1 Introduction

The fractional calculus has become a strong mechanism for finding the solutions of many problems pertaining to control engineering, physics, signal processing, mathematical biology, viscoelasticity, electromagnetism, and mathematical physics and other areas of sciences as well as technology. Several methods can be found in the literature to derive the solution of fractional order differential equation such as ADM [12], HAM [14], HPM [5], Homotopy perturbation transform method (HPTM) [9, 10, 19] and fractional Laplace Adomian decomposition method (FLADM) [7], LPM [20], LHAM [21] and so on. The above mentioned techniques provide immediate and easily seen symbolic terms of numerical approximate solutions as well as of analytical solutions to both linear and nonlinear fractional differential equations.

In 2006, Daftardar-Gejji and Jafari introduced the iterative technique for examining numerically to non-linear functional equations [4, 6, 7]. Since then the iterative approach is being used to find the solution of several non-linear differential equations

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of arbitrary order [1] and viewing fractional BVP [3]. Recently, Jafari et al. has made elegant use of Laplace transform in this iterative method and it became a popular method known as iterative Laplace transform method (ILTM) [8] to examine a system of partial differential equations of fractional order, Fokker–Plank equation [18] as well. In recent, time-fractional Schrödinger equations [15], fractional heat and wave-like equation [16] and fractional Telegraph equations [17] are solved successfully by the use of ILTM.

In the present study, we consider the time-fractional Navier–Stokes equation having initial condition in cylindrical coordinate and are expressed in operator form as

$$D_t^{\alpha} u(r,t) = P + v\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right), u(r,o) = f(r), \quad 0 < \alpha \leq 1 \tag{1.1}$$

where $D_t^{\alpha} u(r,t)$ indicates the Caputo fractional derivative of order α , $P = -\frac{\partial p}{\rho \partial z}$, u indicates the velocity, ρ is the pressure, v is the kinematics viscosity, t is the time and α is a parameter representing the order of the time–fractional derivatives. In particular for $\alpha = 1$, the fractional Navier–Stokes Eq. (1.1) reduces to the standard Navier–Stokes equation.

The main object of this paper, we shall extend the application of Iterative Laplace transform algorithm to derive the solution of the time-fractional Navier–Stokes equations.

2 Some Basic Definitions

In this portion, we list certain basic definitions of fractional calculus along with elegant properties of Laplace transform.

Definition 1 The Caputo derivative of arbitrary order [2] of function $u(r,t)$ is presented as

$$\begin{aligned} D_t^{\alpha} u(r,t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\eta)^{m-\alpha-1} u^{(m)}(r,\eta)\, d\eta, \quad m-1 < \alpha \leq m, m \in N, \\ &= J_t^{m-\alpha} D^m u(r,t). \end{aligned} \tag{2.1}$$

Here $D^m \equiv \frac{d^m}{dt^m}$ and J_t^{α} indicates the Riemann-Liouville integral operator of fractional order $\alpha > 0$, presented as [11]

$$J_t^{\alpha} u(r,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} u(r,\eta)\, d\eta, \ \ \eta > 0, (m-1 < \alpha \leq m), m \in N. \tag{2.2}$$

Definition 2 The Laplace transform of $f(t)$, $t > 0$ is expressed as [11, 13]

$$L[f(t)] = F(t) = \int_0^\infty e^{-st} f(t)dt. \tag{2.3}$$

Definition 3 The Laplace transform of $D_t^\alpha u(r,t)$ is presented in following manner [11, 13]

$$L[D_t^\alpha u\ (r,t)] = L[u(r,t)] - \sum_{k=o}^{m-1} u^k(r,0)\, s^{\alpha-k-1},\ m-1 < \alpha \le m, m \in N, \tag{2.4}$$

3 Basic Idea of ILTM

To explain the basic idea of iterative Laplace transform approach [8], we take the subsequent fractional non-linear partial differential equation having the prescribed initial conditions can be expressed in the form of an operator as

$$D_t^\alpha\, u(r,t) + R\, u(r,t) + N\, u(r,t) = g(r,t), \quad m-1 < \alpha \le m, \quad m \in N, \tag{3.1}$$

$$u^{(k)}(r,0) = h_k(r), \quad k = 0, 1, 2, \ldots, m-1, \tag{3.2}$$

where $D_t^\alpha u(r,t)$ is the Caputo derivative of arbitrary order α, $m-1 < \alpha \le m$, presented by Eq. (2.1), R is a linear operator and may contain rest of fractional derivatives of order less than α, N indicates a non-linear operator which may contain other derivatives of fractional order less than α and $g(r,t)$ is a known analytic function.

Applying the Laplace transform on Eq. (3.1), we have

$$L\,[D_t^\alpha\, u(r,t)] + L\,[R\,u\ (r,t) + Nu(r,t)] = L[g(r,t)]. \tag{3.3}$$

Making use of the differentiation property of the Laplace transform, we find

$$L\,[u(r,t)] = \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-1-k} u^k(r,0) + \frac{1}{s^\alpha} L\,[g(r,t)] - \frac{1}{s^\alpha} L\,[R\,u(r,t) + N\,u(r,t)]. \tag{3.4}$$

On taking inverse Laplace transform on Eq. (3.4), we have

$$u(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^{k}(r,0) + L[g(r,t)]\right)\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L[Ru(r,t) + Nu(r,t)]\right]. \tag{3.5}$$

Now, applying the iterative method,

$$u(r,t) = \sum_{i=0}^{\infty} u_i(r,t). \tag{3.6}$$

As R is a linear operator, so we have

$$R\left(\sum_{i=0}^{\infty} u_i(r,t)\right) = \sum_{i=0}^{\infty} R[u_i(r,t)], \tag{3.7}$$

whereas the non-linear operator N is splitted as

$$N\left(\sum_{i=0}^{\infty} u_i(r,t)\right) = N[u_0(r,t)] + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i} u_k(r,t)\right) - N\left(\sum_{k=0}^{i-1} u_k(r,t)\right)\right\}. \tag{3.8}$$

Putting the results given by Eqs. from (3.6) to (3.8) in the Eq. (3.5), we obtain

$$\sum_{i=0}^{\infty} u_i(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^{k}(r,0) + L\,[g(r,t)]\right)\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\sum_{i=0}^{\infty} R[u_i(r,t)] \ + N\,[u_0(r,t)] + \sum_{i=1}^{\infty} \left\{N\left(\sum_{k=0}^{i} u_k(r,t)\right) - N\left(\sum_{k=0}^{i-1} u_k(r,t)\right)\right\}\right]\right]. \tag{3.9}$$

We have defined the recurrence formulae as

$$u_0(r,t) = L^{-1}\left[\frac{1}{s^{\alpha}}\left(\sum_{k=0}^{m-1} s^{\alpha-1-k} u^{k}(r,0) + L\,(g(r,t))\right)\right] \tag{3.10}$$

$$u_1(r,t) = -L^{-1}\left[\frac{1}{s^{\alpha}} L\,[R\,(u_0(r,t)) + N(u_0(r,t))]\right], \tag{3.11}$$

$$u_{m+1}(r,t) = -L^{-1}\left[\frac{1}{s^{\alpha}}L\left[R\,(u_m(r,t)) - \left\{N\left(\sum_{k=0}^{m} u_k(r,t)\right)\right.\right.\right.$$
$$\left.\left.\left. - N\left(\sum_{k=0}^{m-1} u_k(r,t)\right)\right\}\right]\right], m \geq 1 \tag{3.12}$$

Therefore the m-term approximate solution of Eqs. (3.1) and (3.2) in series form is given by

$$u(r,t) \cong u_0(r,t) + u_1(r,t) + u_2(r,t)+, \ldots, +u_m(r,t), \quad m = 1, 2, \ldots. \tag{3.13}$$

4 Solutions of the Time-Fractional Navier–Stokes Equations

In this part, we have made an attempt to solve the time-fractional Navier–Stokes equations by the application of iterative Laplace transform scheme.

Example 1 Consider the subsequent Navier–Stokes equation involving time–fractional derivative written by

$$D_t^{\alpha} u = P + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, 0 < \alpha \leq 1, \tag{4.1}$$

Surrounding the initial condition

$$u(r,t) = 1 - r^2 \tag{4.2}$$

Taking the Laplace transform of the Eq. (4.1), and making use of the result given by (4.2), we get,

$$L\,[u(r,t)] = \frac{1}{s}\left(1 - r^2\right) + \frac{P}{s^{\alpha+1}} + \frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right). \tag{4.4}$$

Applying inverse Laplace transform to the Eq. (4.4), we arrive at the subsequent result

$$u(r,t) = (1 - r^2) + P\frac{t^{\alpha}}{\Gamma(\alpha+1)} + L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)\right]. \tag{4.5}$$

Now, making use of the iterative method, substituting the results of the Eqs. from (3.6) to (3.8) in the Eq. (4.5) and making use of the results given by the Eqs. (3.10) to (3.12), we determine the components of the ILTM solution as follows

$$u_0(r,t) = (1-r^2) + P\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{4.6}$$

$$u_1(r,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u_0}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial u_0}{\partial \mathrm{r}}\right]\right]$$
$$= -\frac{4t^\alpha}{\Gamma(\alpha+1)}, \tag{4.7}$$

$$u_{n+1}(r,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left(\frac{\partial^2 u_n}{\partial r^2} + \frac{1}{r}\frac{\partial u_n}{\partial r}\right)\right] = 0. \quad \forall\, n \geq 1. \tag{4.8}$$

The other components may be obtained accordingly.

Thus, the closed form solution in the series form is can be obtained as

$$u(r,\ t) = u_0(r,t) + u_1(r,t) + u_2(r,t) + u_3(r,t) +, \ldots,$$
$$= \left(1-r^2\right) + (P-4)\frac{t^\alpha}{\Gamma(\alpha+1)}\ . \tag{4.9}$$

Special Cases

(i) The result in (4.9) was derived by Momani and Odibat [12] with the aid of the different scheme that is ADM.

(ii) The result in (4.9) deduced by Ragab et al. [14] by the application of HAM.

(iii) A result in (4.9) has an analogy with the result of Ganji et al. [5] has been obtained by using HPM.

(iv) For $\alpha = 1$, the result in (4.9) reduces to the following simple form

$$u(r,t) = \left(1-r^2\right) + (P-4)t. \tag{4.10}$$

This result was obtained earlier by Kumar et al. [10] by using the method of HPTM.

Example 2 Next, consider the subsequent Navier–Stokes equation concerning to time–fractional derivative given by

$$D_t^\alpha u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \quad 0 < \alpha \leq 1, \tag{4.11}$$

with the initial condition

$$u(r,0) = r\ , \tag{4.12}$$

Taking the Laplace transform of the Eq. (4.11), and making use of the result given by (4.12), we have,

$$L\,[u(r,t)] = \frac{r}{s} + \frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right). \tag{4.14}$$

Applying inverse Laplace transform to the Eq. (4.14), we get

$$u(r,t) = r + L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right) \right]. \tag{4.15}$$

Now, making use of the iterative method, substituting the results of the Eqs. from (3.6) to (3.8) in the Eq. (4.15) and making use of the results given by the Eqs. (3.10) to (3.12), we determine the components of the ILTM solution as follows

$$u_0(r,t) = \ r\,, \tag{4.16}$$

$$\begin{aligned} u_1(r,t) &= L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r}\frac{\partial u_0}{\partial r} \right) \right] \\ &= \frac{1}{r}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{4.17}$$

$$\begin{aligned} u_2(r,t) &= L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_1}{\partial r^2} + \frac{1}{r}\frac{\partial u_1}{\partial r} \right) \right] \\ &= \frac{1}{r^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} u_3(r,t) &= L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} \right) \right] \\ &= \frac{9}{r^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} u_n(r,t) &= L^{-1}\left[\frac{1}{s^{\alpha}} L\left(\frac{\partial^2 u_{n-1}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{n-1}}{\partial r} \right) \right] \\ &= \frac{1^2 \times 3^2 \cdots (2n-1)^2}{r^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \end{aligned} \tag{4.20}$$

and so on. The other components may be obtained accordingly.

Thus, the closed form solution in the series form is can be obtained as

$$\begin{aligned} u(r,\ t) &= u_0(r,t) + u_1(r,t) + u_2(r,t) + u_3(r,t) +, \ldots, \\ &= r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \cdots (2n-1)^2}{r^{2n-1}}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{aligned} \tag{4.21}$$

Special Cases

(i) The result in (4.21) was obtained by Ragab et al. [14] using the different method known as HAM.
(ii) The result in (4.21) was given by Ganji et al. [5] using the different technique known as HPM technique.
(iii) The result in (4.21) deduced by Momani and Odibat [12] by the application of ADM.
(iv) For $\alpha = 1$, the result in (4.21) reduces to the following simple form

$$u(r,t) = r + \sum_{n=1}^{\infty} \frac{1^2 \times 3^2 \cdots (2n-1)^2}{r^{2n-1}} \frac{t^n}{n!}. \tag{4.22}$$

This result was obtained earlier by Kumar et al. [10] by using the method of HTPM.

5 Numerical Results and Discussions

In this part, we present some numerical results for Navier–Stokes equation concerning to time–fractional derivative. Figures 1 and 2 present the ILTM solution of Navier–Stokes equation concerning to time–fractional derivative for $\alpha = 1$ and 2 respectively. Figure 3 presents the ILTM solution of Navier–Stokes equation concerning to time–fractional derivative with respect to r for distinct values of α.

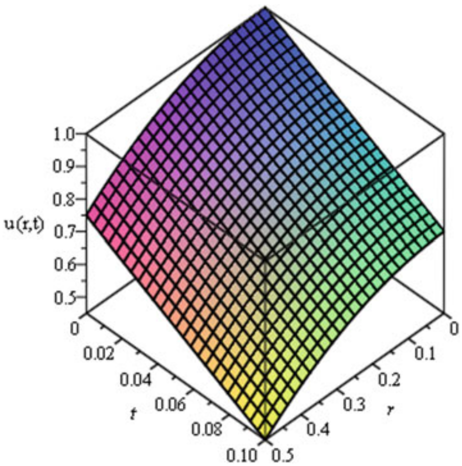

Fig. 1 The surface of solution $u(r,t)$, when $\alpha = 1$, $P = 1$ for Eq. (4.9)

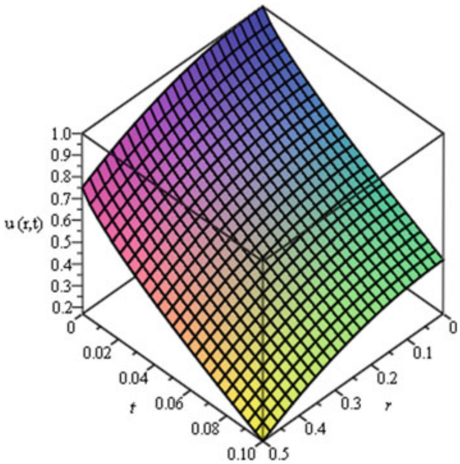

Fig. 2 The surface of solution $u(r, t)$, when $\alpha = 0.5$, $P = 1$ for Eq. (4.9)

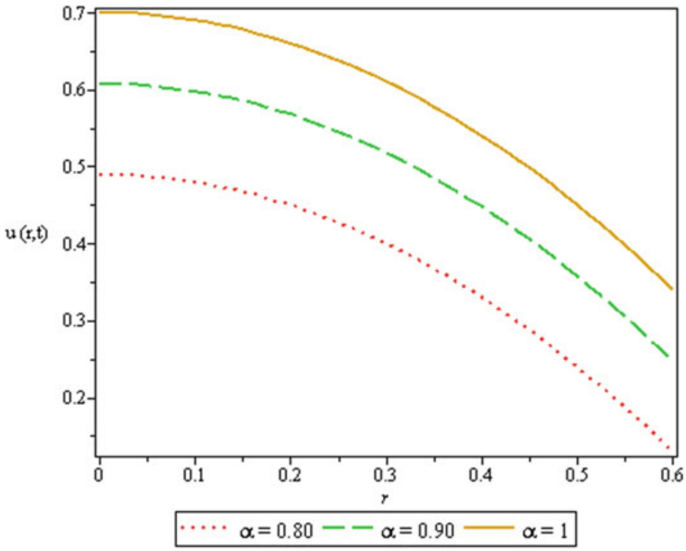

Fig. 3 The nature of the solution $u(r, t)$ w.r.t. r, when $P = 1$ for diverse values of α for Eq. (4.9)

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Biological Model of Dengue Spread with Non-Markovian Properties

Sonal Jain and Abdon Atangana

Abstract A fatal and infectious called Dengue found in the tropical zone of the world is a mosquito-borne and caused by four viruses namely Den 1-Den 4. The transmission is achieved from one person to another via a bite of female adult Aedes mosquitoes. The dynamic of spread does not really follow the Markovian process therefore does have memory effect, thus can well be described by using nonlocal differential operators with non-singular and non-local kernel as these operators have a crossover from exponential decay law to power law as waiting time distribution. In this chapter, we reverted the classical model to fractional model by using the concept of recently established fractional differential operators known as the Caputo-Fabrizio derivative. To include into mathematical system the memory and the crossover effects. The new model was subjected to analysis of existence and uniqueness of the system solution to insure the well poseness of the modified system. Due to the complexity of the new system, a newly introduced numerical scheme was used to solve the system and some numerical simulations where performed to see the effect of the Mittag-Leffler law that brings the crossover effect.

Keywords Caputo-Fabrizio derivative · Dengue model · Fractional differential equations · Existence and uniqueness · Fixed point theorem

1 Introduction

Dengue disease is a common arboviral disease in tropical regions of the world. It is transferral to humans by the bite of Aedes mosquitoes. There are four types of virus which is denoted by one, two, three, and four. The bites of the Aedes mosquitoes

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is the reason of the viruses that transferral to humans. If a person infected once in the life by these one of the four serotypes of viruses will never get infected by that serotype again but loses immunity to other three stereotypes of the viruses [8]. There are lots of Bio-mathematical models have been proposed to recognize the transferral dynamics of these type of infectious diseases. In recent years, modeling has become a valuable tool in the analysis of dengue disease transferral dynamics and to determine the factors that influence the spread of disease to support control measures. Many researchers have proposed [5–8, 13, 14, 16, 18] epidemic model [10] to study the transferral dynamics of dengue disease.

There is no specific medicine to cure dengue disease. Awareness programs can be helpful in reducing the prevalence of the disease. Different Bio-mathematical models have been proposed to study the impact of awareness in controlling dengue and these type diseases. Prevention of mosquitoes bites is one of the ways to prevent dengue disease. The mosquitoes bite humans during day and night when lights are on. So, to get rid of mosquitoes bite, people can use mosquito repellents and nets. If infected hosts feel they have symptoms of the disease and approach the doctor in time for the supportive treatment, they can recover fast. This type of awareness can help controlling the disease. Another way of controlling dengue is destroying larval breeding sites of mosquitoes and killing them. Spray of insecticides may be applied to control larvae or adult mosquitoes which can transmit dengue viruses. This type of biological model have two properties as we observed Markovian and Non-Markovian. In dengue spread model does not really follow the Markovian process therefore does have memory effect, thus can well be described using the concept of nonlocal differential operators with non local and non singular kernel as these operators have a crossover from exponential decay law to power law as waiting time distribution.

Fc is applied in various directions of Bio-mathematics, physics, signal-processing, fluid-mechanics, visco-elasticity, finance, electro-chemistry and in many more. In the branch of fc, we study fractional integral and fractional derivative as an important aspects. Recently, many researcher and scientists have studied various type of issues in this special branch [1–3, 9]. The Caputo-Fabrizo derivative brought new weapons into applied mathematics to model complex real-world problems more accurately. Caputo-Fabrizio derivative is give the result of non-Markovian process. In the RL derivative the kernal inside it is gives the result for power law but Caputo-Fabrizio shows the result for exponential decay.

The main objective of this chapter is to discuss fractional Caputo-Fabrizio derivative for the mathematical system to finding the crossover effects and memory effect Also by using fixed point theorem we are finding the details of the uniqueness and exactness and of the solution. The development of this article is as follows. In Sect. 2, we discuss the Caputo-Fabrizio and AB derivative. In Sect. 3, the mathematical portion of fractional dengue spread model and also by applying CF derivative we find the approximate solution. In Sect. 4, by using fixed point theorem, we proved the uniqueness and existence of system of solutions in Sect. 6, Numerical Solution are discuss and in the last Sect. 7 we presented concluding remarks.

2 Preliminaries

Some definitions and properties of the fractional derivative are presented here.

Definition 2.1 Let f be a function not necessarily differentiable, and κ be a real number such that $0 < \kappa \leq 1$, then the Riemann-Liouville derivative with κ order with power law is given as [15]

$$ {}^{RL}D_t^{\kappa}[f(t)] = \frac{1}{\Gamma(1-\kappa)} \frac{d}{dt} \int_0^t (t-y)^{-\kappa} f(y) dy. \tag{2.1} $$

Definition 2.2 Let $f \in H^1(a, b), b > a, \kappa \in [0, 1]$ then the new Caputo derivative of fractional order is given by:

$$ D_t^{\kappa}(f(t)) = \frac{M(\kappa)}{(1-\kappa)} \int_a^t f^{'}(x) \exp\left[-\kappa \frac{t-x}{1-\kappa}\right] dx. \tag{2.2} $$

where $M(\kappa)$ is a normalization function such that $M(0) = M(1) = 1$ [4]. But, if the function does not belong to $H_1(a, b)$ then, the derivative can be reformulated as

$$ D_t^{\kappa}(f(t)) = \frac{M(\kappa)}{(1-\kappa)} \int_a^t (f(t) - f(x)) \exp\left[-\kappa \frac{t-x}{1-\kappa}\right] dx. \tag{2.3} $$

Remark 2.1 The authors remarked that, if $\sigma = \frac{1-\kappa}{\kappa} \in [0, \infty)$, $\kappa = \frac{1}{1+\kappa} \in [0, 1]$, then Eq. (2.1) assumes the form

$$ D_t^{\kappa}(f(t)) = \frac{N(\sigma)}{(\sigma)} \int_a^t f^{'}(x) \exp\left[-\frac{t-x}{\sigma}\right] dx, \quad N(0) = N(\infty) = 1 \tag{2.4} $$

In Addition,

$$ \lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t) \tag{2.5} $$

Now after the introduction of a new derivative, the associate anti-derivative becomes important, the associated integral of the new Caputo derivative with fractional order was proposed by Losada and Nieto [11].

Definition 2.3 [11] Let $0 < \kappa < 1$. The fractional integral of order κ of a function f is defined by

$$I_{\kappa}^{t}(f(t)) = \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}f(t) + \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_{0}^{t} f(s)ds, t \geq 0. \tag{2.6}$$

Remark 2.2 Note that, according to above definition, the fractional integral of Caputo type of function of order $0 < \kappa < 1$ is an average between function f and its integral of order one. This therefore imposes

$$\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}f(t) + \frac{2\kappa}{(2-\kappa)M(\kappa)} = 1 \tag{2.7}$$

The above expression yields an explicit formula for

$$M(\kappa) = \frac{2}{2-\kappa}, 0 \leq \kappa \leq 1 \tag{2.8}$$

Because of the above, Losada and Nieto proposed that the new Caputo derivative of order $0 < \kappa < 1$ can be reformulated as

$$D_{t}^{\kappa}(f(t)) = \frac{1}{1-\kappa}\int_{a}^{t} f^{'}(x)\exp\left[-\kappa\frac{t-x}{1-\kappa}\right]dx. \tag{2.9}$$

3 Model Description

In the given model, total host human population, N_h. We divided this human population into four parts: R_h(recovered), I_h(infectious), E_h(exposed), S_h(susceptible) and total vector (mosquito) population, also we divide N_v into three parts: I_v (infectious), E_v (exposed), S_v (susceptible). We assume that the fraction u_1 of susceptible hosts use mosquito repellents to avoid mosquitoes bite. So, the fraction $(1-u_1)$ of susceptible hosts interact with infectious mosquitoes. The fraction u_2 of infectious hosts seek for the timely supportive treatment and recover fast by the rate $r_h(r > 1)$. The fraction r_1u_2 (r_1 is the proportionality constant) of infectious hosts use mosquito repellents to avoid mosquitoes bite. u_3 is a control variable that represents the eradication effort of insecticide spraying. That follows that morality rate of mosquito population increases at a rate r_2u_3 (r_2 is the proportionality constant) and also it is assume that recruitment rate of this is reduced by a factor of $1-u_3$.

In this section, we describes the geometry of dengue disease together with control measures. The system of differential equations which shows the present SEIR-SEI vector host model is given in [13].

$$\begin{aligned}
\frac{dS_h}{dt} &= \mu_h N_h - (1-u_1)\frac{b\beta_h}{N_h} S_h I_\nu - \mu_h S_h \\
\frac{dE_h}{dt} &= (1-u_1)\frac{b\beta_h}{N_h} S_h I_\nu - (\nu_h + \mu_h) E_h \\
\frac{dI_h}{dt} &= \nu_h E_h - [ru_2\gamma h + (1-u_2)\gamma h + \mu_h] I_h \\
\frac{dR_h}{dt} &= [ru_2\gamma_h + (1-u_2)\gamma_h] I_h - \mu_h R_h \\
\frac{dS_\nu}{dt} &= (1-u_3)\pi_\nu - (1-r_1u_2)\frac{b\beta_h\nu}{N_h} S_\nu I_h - (r_2u_3 + \mu_h\nu) S_\nu \\
\frac{dE_\nu}{dt} &= (1-r_1u_2)\frac{b\beta_h\nu}{N_h} S_\nu I_h - (r_2u_3 + \nu_\nu + \mu_h\nu) E_\nu \\
\frac{dI_\nu}{dt} &= \nu_\nu E_\nu - (r_2u_3 + \mu\nu) I_\nu
\end{aligned} \tag{3.1}$$

The parameters of the model are given in the following table.

Symbols	Description
μ_h	Death rate of host population
ν_h	Host's incubation rate
γ_h	Recovery rate of host population
β_h	Transmission probability from vector to host
π_ν	Vector population recruitment rate
μ_ν	Vector population death rate
ν_ν	Vector's incubation rate
β_ν	Host to vector the transmission probability
b	Rate (biting) of vector

Total host population, $N_h = R_h + I_h + E_h + S_h$, total vector population, $N_\nu = I_\nu + E_\nu + S_\nu$.

$$\frac{dN_h}{dt} = 0 \text{ and } \frac{dN_\nu}{dt} = (1-u_3)\pi_\nu - (r_2u_3 + \mu_\nu)N_\nu.$$

So, N_h remains constant and N_ν approaches the equilibrium $(1-u_3)\pi_\nu(r_2u_3 + \mu_\nu\nu)$ as $t \to \infty$. Introducing the proportions

$$s_\nu = \frac{S_\nu}{(1-u_3)\pi\nu/(r_2u_3+\mu_\nu)}, s_h = \frac{S_h}{N_h}, \quad e_h = \frac{E_h}{N_h}, \quad i_h = \frac{I_h}{N_h}, \quad r_h = \frac{R_h}{N_h},$$
$$e_\nu = \frac{E_\nu}{(1-u_3)\pi_\nu\nu/(r_2u_3+\mu_\nu\nu)}, \quad i_\nu = \frac{I_\nu}{(1-u_3)\pi_\nu/(r_2u_3+\mu_\nu\nu)}$$

Since $s_\nu = 1 - e_\nu - i_\nu$ and $r_h = 1 - s_h - e_h - i_h$ the system of Eq. (3.1) is the equivalent written by five dimensional non-linear system of ODEs:

$$\begin{aligned}
\frac{ds_h}{dt} &= \mu_h(1 - s_h) - \alpha s_h i_\nu \\
\frac{de_h}{dt} &= \alpha s_h i_\nu - \beta e_h \\
\frac{di_h}{dt} &= \nu_h e_h - \gamma i_h \\
\frac{de_\nu}{dt} &= \delta s_\nu i_h - (\epsilon + \nu_\nu) e_\nu \\
\frac{di_\nu}{dt} &= \nu_\nu e_\nu - \epsilon i_\nu
\end{aligned} \tag{3.2}$$

Here,

$$\alpha = \frac{b\beta_h \pi_\nu (1 - u_1)(1 - u_3)}{N_h(r_2 u_3 + \mu\nu)}, \quad \beta = \nu_h + \mu_h, \quad \gamma = r u_2 \gamma_h + (1 - u_2)\gamma_h + \mu_h,$$
$$\delta = (1 - r_1 u_2) b\beta_\nu, \quad \epsilon = r_2 u_3 + \mu_\nu.$$

Due to Markovian process, this system is exponentially stable with no memory. Thus, to include the memory effect into this bio-mathematical model, we introduced Caputo-Fabrizio arbitrarily ordered derivative to moderate this system by non Markovian process as given by

$$\begin{aligned}
{}_0^{CF}D_t^\kappa s_h &= \mu_h(1 - s_h) - \alpha s_h i_\nu \\
{}_0^{CF}D_t^\kappa e_h &= \alpha s_h i_\nu - \beta e_h \\
{}_0^{CF}D_t^\kappa i_h &= \nu_h e_h - \gamma i_h \\
{}_0^{CF}D_t^\kappa e_\nu &= \delta s_\nu i_h - (\epsilon + \nu_\nu) e_\nu \\
{}_0^{CF}D_t^\kappa i_\nu &= \nu_\nu e_\nu - \epsilon i_\nu
\end{aligned} \tag{3.3}$$

These come with the initial conditions

$$i_\nu(0) = \delta_5, \quad e_\nu(0) = \delta_4, \quad i_h(0) = \delta_3, \quad e_h(0) = \delta_2, \quad s_h(0) = \delta_1. \tag{3.4}$$

4 Uniqueness and Existence of a System of Solutions of Dengue Models with Non-Markovian Properties

In this section investigate numerical result of fractional model based on CF derivative. We discuss the uniqueness and existence of the solutions by fixed point theorem. For this we apply the fractional integral operator due to Nieto and Losada [11] on Eq. (3.3), to examine the existence of the system of solutions. We obtain

$$
\begin{aligned}
s_h(t) - s_h(0) &= {}_0^{CF}I_t^{\kappa}\,\{\mu_h(1-s_h) - \alpha s_h i_\nu\}\\
e_h(t) - e_h(0) &= {}_0^{CF}I_t^{\kappa}\,\{\alpha s_h i_\nu - \beta e_h\}\\
i_h(t) - i_h(0) &= {}_0^{CF}I_t^{\kappa}\,\{\nu_h e_h - \gamma i_h\}\\
e_\nu(t) - e_\nu(0) &= {}_0^{CF}I_t^{\kappa}\,\{\delta s_\nu i_h - (\epsilon + \nu_\nu) e_\nu\}\\
i_\nu(t) - i_\nu(0) &= {}_0^{CF}I_t^{\kappa}\,\{\nu_\nu e_\nu - \epsilon i_\nu\}
\end{aligned}
\tag{4.1}
$$

By using the equation discussed by Nieto and Losada [11], we have

$$
\begin{aligned}
i_\nu(t) - i_\nu(0) &= \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \{\nu_\nu(y)e_\nu(y) - \epsilon i_\nu(y)\}\,dy\\
&\quad + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\{\nu_\nu(t)e_\nu(t) - \epsilon i_\nu(t)\}\\
e_\nu(t) - e_\nu(0) &= \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \{\delta s_\nu(y)i_h(y) - (\epsilon + \nu_\nu(y))e_\nu(y)\}\,dy\\
&\quad + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\{\delta s_\nu(t)i_h(t) - (\epsilon + \nu_\nu(t))e_\nu(t)\}\\
i_h(t) - i_h(0) &= +\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \{\nu_h(y)e_h(y) - \gamma i_h(y)\}\,dy\\
&\quad + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\{\nu_h(t)e_h(t) - \gamma i_h(t)\}\\
e_h(t) - e_h(0) &= \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \{\alpha s_h(y)i_\nu(y) - \beta e_h(y)\}\,dy\\
&\quad + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\{\alpha s_h(t)i_\nu(t) - \beta e_h(t)\}\\
s_h(t) - s_h(0) &= \frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \{\mu_h(1 - s_h(y)) - \alpha s_h(y)i_\nu(y)\}\,dy\\
&\quad + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\{\mu_h(1 - s_h(t)) - \alpha s_h(t)i_\nu(t)\}
\end{aligned}
\tag{4.2}
$$

So we can write for clarity

$$Z_1(t, s_h) = \mu_h(1 - s_h(t)) - \alpha s_h(t)i_\nu(t),$$

$$Z_2(t, e_h) = \mu_h(1 - s_h(y)) - \alpha s_h(y)i_\nu(y)$$

$$Z_3(t, i_h) = \nu_h(t)e_h(t) - \gamma i_h(t)$$

$$
\begin{aligned}
Z_4(t, e_\nu) &= \delta s_\nu(t) i_h(t) - (\epsilon + \nu_\nu(t)) e_\nu(t) \\
Z_5(t, i_\nu) &= \nu_\nu(t) e_\nu(t) - \epsilon i_\nu(t)
\end{aligned} \tag{4.3}
$$

Theorem 4.1 *If the following inequality holds then The kernels Z_1, Z_2, Z_3, Z_4 and Z_5 satisfy the Lipschitz condition and contraction.*

$$0 < a_1 + \beta b_1 \leq 1.$$

Proof Starting with the kernel Z_1. Let two function is s_{h_1} and s_{h_2} then we get the following:

$$\|Z_1(t, s_h) - Z_1(t, s_{h_1})\| = \| - \mu_h \left\{ s_h(t) - s_{h_1}(t) \right\} - \alpha \left\{ s_h(t) - s_{h_1}(t) \right\} i_\nu(t) \|. \tag{4.4}$$

Now using the triangular inequality (4.4), we have

$$
\begin{aligned}
\|Z_1(t, s_h) - Z_1(t, s_{h_1})\| &\leq \|\alpha \left\{ s_h(t) - s_{h_1}(t) \right\} i_\nu(t)\| + \|\mu_h \left\{ s_h(t) - s_{h_1}(t) \right\} \| \\
&\leq \|s_h(t) - s_{h_1}(t)\| \left\{ a_1 + b_1 \|i_\nu(t)\| \right\} \\
&\leq \left\{ a_1 + b_1 \beta \right\} \|s_h(t) - s_{h_1}(t)\| \leq \gamma_1 \|s_h(t) - s_{h_1}(t)\|
\end{aligned} \tag{4.5}
$$

Taking $\gamma_1 = a_1 + \beta b_1$ here the $\beta = i_\nu(t)$ are bounded functions, then we have

$$\|Z_5(t, i_\nu) - Z_1(t, i_{\nu_1})\| = \gamma_5 \|i_\nu(t) - i_{\nu_1}(t)\| \tag{4.6}$$

Hence, the Lipschitz condition is satisfied for Z_1, and if additionally $0 < (a_1 + \beta b_1 \leq 1)$, this condition is satisfy then it gives us a contraction for Z_1.
Similarly all the cases II, II, III and IV satisfy the Lipschitz condition as follows:

$$
\begin{aligned}
\|Z_4(t, e_\nu) - Z_1(t, e_{\nu_1})\| &= \gamma_4 \|e_\nu(t) - e_{\nu_1}(t)\|, \\
\|Z_3(t, i_h) - Z_1(t, i_{h_1})\| &= \gamma_3 \|i_h(t) - i_{h_1}(t)\|, \\
\|Z_2(t, e_h) - Z_1(t, e_{h_1})\| &= \gamma_2 \|e_h(t) - e_{h_1}(t)\|, \\
\|Z_1(t, s_h) - Z_1(t, s_{h_1})\| &= \gamma_1 \|s_h(t) - s_{h_1}(t)\|.
\end{aligned} \tag{4.7}
$$

when we consider the kernels, the Eq. (4.2) becomes

$$
\begin{aligned}
i_\nu(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t (Z_5(y, i_\nu))\, dy + i_\nu(0) + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_5(t, i_\nu) \\
e_\nu(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t (Z_4(y, e_\nu))\, dy + e_\nu(0) + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_4(t, e_\nu),
\end{aligned}
$$

$$i_h(t) = \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t (Z_3(y, i_h))\, dy + i_h(0) + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_3(t, i_h),$$

$$e_h(t) = \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t (Z_2(y, e_h))\, dy + e_h(0) + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_2(t, e_h),$$

$$s_h(t) = \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t (Z_1(y, s_h))\, dy + s_h(0) + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_1(t, s_h). \quad (4.8)$$

Now, presenting the following recursive formula:

$$\begin{aligned}
i_{v_n}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_5(y, i_{v_{n-1}})\big)\, dy + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_5(t, i_{v_{n-1}}) \\
e_{v_n}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_4(y, e_{v_{n-1}})\big)\, dy, + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_4(t, e_{v_{n-1}}) \\
i_{h_n}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_3(y, i_{h_{n-1}})\big)\, dy + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_3(t, i_{h_{n-1}}), \qquad (4.9) \\
e_{h_n}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_2(y, e_{h_{n-1}})\big)\, dy + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_2(t, e_{h_{n-1}}), \\
s_{h_n}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_1(y, s_{h_{n-1}})\big)\, dy + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} Z_1(t, s_{h_{n-1}}),
\end{aligned}$$

and the initial conditions are gives as below:

$$i_{v_0}(t) = i_v(0), \quad e_{v_0}(t) = e_v(0), \quad i_{h_0}(t) = i_h(0), \quad e_{h_0}(t) = e_h(0), \quad s_{h_0}(t) = s_h(0). \quad (4.10)$$

Now, difference between the successive terms are presented as follow:

$$\begin{aligned}
\varsigma_n(t) = i_{v_n}(t) - i_{v_{n-1}}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_5(y, i_{v_{n-1}}) - Z_5(y, i_{v_{n-2}})\big)\, dy \\
&+ \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} \big(Z_5(t, i_{v_{n-1}}) - Z_5(t, i_{v_{n-2}})\big) \\
\chi_n(t) = e_{v_n}(t) - e_{v_{n-1}}(t) &= \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \big(Z_4(y, e_{v_{n-1}}) - Z_4(y, e_{v_{n-2}})\big)\, dy
\end{aligned}$$

$$+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z_4(t,e_{v_{n-1}})-Z_4(t,e_{v_{n-2}})\right),$$

$$\xi_n(t)=i_{h_n}(t)-i_{h_{n-1}}(t)=\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t\left(Z_3(y,i_{h_{n-1}})-Z_3(y,i_{h_{n-2}})\right)dy$$
$$+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z_3(t,i_{h_{n-1}})-Z_3(t,i_{h_{n-2}})\right),$$

$$\psi_n(t)=e_{h_n}(t)-e_{h_{n-1}}(t)=\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t\left(Z_2(y,e_{h_{n-1}})-Z_2(y,e_{h_{n-2}})\right)dy$$
$$+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z_2(t,e_{h_{n-1}})-Z_2(t,e_{h_{n-2}})\right)$$

$$\phi_n(t)=s_{h_n}(t)-s_{h_{n-1}}(t)=\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t\left(Z_1(y,s_{h_{n-1}})-Z_1(y,s_{h_{n-2}})\right)dy$$
$$+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z_1(t,s_{h_{n-1}})-Z_1(t,s_{h_{n-2}})\right) \quad (4.11)$$

Noticing that

$$\begin{aligned}
s_{h_n}(t)&=\sum_{i=0}^{n}\phi_i(t),\\
e_{h_n}(t)&=\sum_{i=0}^{n}\psi_i(t),\\
i_{h_n}(t)&=\sum_{i=0}^{n}\xi_i(t),\\
e_{v_n}(t)&=\sum_{i=0}^{n}\chi_i(t),\\
i_{v_n}(t)&=\sum_{i=0}^{n}\varsigma_n(t).
\end{aligned} \quad (4.12)$$

Step by step we get

$$\begin{aligned}
\|\phi_n(t)\|&=\|s_{h_n}(t)-s_{h_{n-1}}(t)\|\\
&=\left\|\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t\left(Z_1(y,s_{h_{n-1}})-Z_1(y,s_{h_{n-2}})\right)dy\right.\\
&\quad\left.+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z_1(t,s_{h_{n-1}})-Z_1(t,s_{h_{n-2}})\right)\right\|
\end{aligned} \quad (4.13)$$

Employing the triangular inequality, Eq. (4.13) reduces to

$$\begin{aligned}\|s_{h_n}(t)-s_{h_{n-1}}(t)\| \leq{}& \frac{2\kappa}{(2-\kappa)M(\kappa)}\left\|\int_0^t \left(Z_1(y,s_{h_{n-1}})-Z_1(y,s_{h_{n-2}})\right)dy\right\| \\ &+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left\|\left(Z_1(t,s_{h_{n-1}})-Z_1(t,s_{h_{n-2}})\right)\right\|.\end{aligned} \tag{4.14}$$

The Lipschitz condition is satisfy with the kernel, we have

$$\begin{aligned}\|s_{h_n}(t)-s_{h_{n-1}}(t)\| \leq{}& \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1\int_0^t \left\|s_{h_{n-1}}-s_{h_{n-2}}dy\right\| \\ &+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1\left\|s_{h_{n-1}}-s_{h_{n-2}}\right\|,\end{aligned} \tag{4.15}$$

then we get

$$\|\phi_n(t)\| \leq \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1\int_0^t \|\phi_{n-1}(y)\|\,dy+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1\|\phi_{n-1}(t)\|. \tag{4.16}$$

Similarly, the following results are obtained by us:

$$\begin{aligned}\|\varsigma_n(t)\| &\leq \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_5\int_0^t \|\varsigma_{n-1}(y)\|\,dy+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_5\|\varsigma_{n-1}(t)\|,\\ \|\chi_n(t)\| &\leq \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_4\int_0^t \|\chi_{n-1}(y)\|\,dy+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_4\|\chi_{n-1}(t)\|,\\ \|\xi_n(t)\| &\leq \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_3\int_0^t \|\xi_{n-1}(y)\|\,dy+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_3\|\xi_{n-1}(t)\|,\\ \|\psi_n(t)\| &\leq \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_2\int_0^t \|\psi_{n-1}(y)\|\,dy+\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_2\|\psi_{n-1}(t)\|.\end{aligned} \tag{4.17}$$

Now we are presenting the subsequent theorem by consideration of the above results, □

Theorem 4.2 *The fractional dengue Models* (3.3) *with Non-Markovian Properties has a system of solutions under the conditions that we can find* t_0 *such that*

$$\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1 t_0 + \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1 \le 1$$

Proof Here first we considered that the functions $i_\nu(t), e_\nu(t), i_h(t), e_h(t), s_h(t)$ are bounded and Also, we prove that Lipschitz condition is satisfy with the kernels and hence on consideration of the results of Eqs. (4.16) and (4.17) and by employing the recursive method, we derive the relation as follows:

$$\begin{aligned}
\|\phi_n(t)\| &\le \|s_h(0)\|\left[\left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1\right)+\left(\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1 t\right)\right]^n,\\
\|\psi_n(t)\| &\le \|e_h(0)\|\left[\left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_2\right)+\left(\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_2 t\right)\right]^n,\\
\|\xi_n(t)\| &\le \|i_h(0)\|\left[\left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_3\right)+\left(\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_3 t\right)\right]^n,\\
\|\chi_n(t)\| &\le \|e_\nu(0)\|\left[\left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_4\right)+\left(\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_4 t\right)\right]^n,\\
\|\varsigma_n(t)\| &\le \|i_\nu(0)\|\left[\left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_5\right)+\left(\frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_5 t\right)\right]^n.
\end{aligned}\tag{4.18}$$

Therefore, the system of functions (4.12) is smooth and exists. However, to show that the above functions are the system of solutions of the given system of Eq. (3.3), we assume that

$$\begin{aligned}
i_\nu(t)-i_\nu(0) &= i_\nu(t)-F_{\nu_n}(t)\\
e_\nu(t)-e_\nu(0) &= e_{\nu_n}(t)-E_{\nu_n}(t),\\
i_h(t)-i_h(0) &= i_{h_n}(t)-D_{h_n}(t),\\
e_h(t)-e_h(0) &= e_{h_n}(t)-C_{h_n}(t),\\
s_h(t)-s_h(0) &= s_{h_n}(t)-B_{h_n}(t).
\end{aligned}\tag{4.19}$$

So, we have

$$\begin{aligned}
\left\|B_{h_n}(t)\right\| &= \Bigg\|\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left(Z(t,s_h)-Z(t,s_{h_{n-1}})\right)\\
&\quad+\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \left(Z(y,s_h)-Z(y,s_{h_{n-1}})\right)dy\Bigg\|\\
&\le \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left\|\left(Z(t,s_h)-Z(t,s_{h_{n-1}})\right)\right\|\\
&\quad+\frac{2\kappa}{(2-\kappa)M(\kappa)}\int_0^t \left\|\left(Z(y,s_h)-Z(y,s_{h_{n-1}})\right)\right\|dy\\
&\le \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1\left\|s_h-s_{h_{n-1}}\right\|
\end{aligned}$$

$$+ \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \gamma_1 \left\| s_h - s_{h_{n-1}} \right\| t. \tag{4.20}$$

On using this process recursively, it yields

$$\left\| B_{h_n}(t) \right\| \leq \left(\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} + \frac{2\kappa}{(2-\kappa)M(\kappa)} t \right)^{n+1} \gamma_1^{n+1} \alpha. \tag{4.21}$$

On taking the limit on Eq. (4.21) as $n \to \infty$, we get

$$\left\| B_{h_n}(t) \right\| \to 0.$$

Similarly, we get
$\left\| F_{\nu_n}(t) \to 0 \right\|$, $\left\| E_{\nu_n}(t) \to 0 \right\|$, $\left\| D_{h_n}(t) \to 0 \right\|$, and $\left\| C_{h_n}(t) \right\| \to 0$.
Hence existence is verified. □

Now, On proving the uniqueness of a system of solutions of Eq. (3.3)
Let there exist another system of solutions of (3.3) $s_{h_1}(t), e_{h_1}(t), i_{h_1}(t), e_{\nu_1}(t)$ and $i_{\nu_1}(t)$ then

$$\begin{aligned} s_h(t) - s_{h_1}(t) &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} \left(Z_1(t, s_h) - Z_1(t, s_{h_1}) \right) \\ &+ \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \left(Z_1(y, s_h) - Z_1(y, s_{h_1}) \right) dy. \end{aligned} \tag{4.22}$$

On Eq. (4.22), if we applying norm then we get,

$$\begin{aligned} \left\| s_h(t) - s_{h_1}(t) \right\| &\leq \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} \left\| \left(Z_1(t, s_h) - Z_1(t, s_{h_1}) \right) \right\| \\ &+ \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t \left\| \left(Z_1(y, s_h) - Z_1(y, s_{h_1}) \right) \right\| dy. \end{aligned} \tag{4.23}$$

From employing the Lipschitz conditions of the kernel, we have

$$\left\| s_h(t) - s_{h_1}(t) \right\| \leq \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} \gamma_1 \left\| s_h - s_{h_1} \right\| + \frac{2\kappa}{(2-\kappa)M(\kappa)} \gamma_1 t \left\| \left(s_h - s_{h_1} \right) \right\|. \tag{4.24}$$

It gives

$$\left\| s_h(t) - s_{h_1}(t) \right\| \left(1 - \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} \gamma_1 - \frac{2\kappa}{(2-\kappa)M(\kappa)} \gamma_1 t \right) \leq 0. \tag{4.25}$$

Theorem 4.3 *The system of Eq. (3.3) has a unique system of solutions if the following condition holds:*

$$\left\| s_h(t) - s_{h_1}(t) \right\| \left(1 - \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1 - \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1 t \right) \geq 0. \tag{4.26}$$

Proof If the condition holds (4.26), then

$$\left\| s_h(t) - s_{h_1}(t) \right\| \left(1 - \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\gamma_1 - \frac{2\kappa}{(2-\kappa)M(\kappa)}\gamma_1 t \right) \leq 0, \tag{4.27}$$

then we have

$$\left\| s_h(t) - s_{h_1}(t) \right\| = 0.$$

Then we get

$$s_h(t) = s_{h_1}(t) \tag{4.28}$$

Similarly, we have

$$\begin{aligned} i_\nu(t) &= i_{\nu_1}(t), \\ e_\nu(t) &= e_{\nu_1}(t), \\ i_h(t) &= i_{h_1}(t), \\ e_{h_1}(t) &= e_{h_1}(t). \end{aligned} \tag{4.29}$$

Therefore, this verified the uniqueness of the system of solutions of Eq. (3.3). □

5 Numerical Solution

In this section, we construct a numerical scheme for fractional model based on the CF derivative. On applying this scheme we first consider the following non-linear fractional ODE:

$$\begin{cases} {}_0^{CF}D_t^\kappa u(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases} \tag{5.1}$$

On applying the fundamental theorem of fc The above eq can be converted to a fractional integral equation:

$$u(t) - u(0) = \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)} f(t, u(t)) + \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_0^t f(\tau, u(\tau))d\tau, \tag{5.2}$$

At a given point t_{n+1}, $n = 0, 1, 2, \ldots$ we reformulated the above equation as

$$u(t_{n+1}) - u(t_0) = \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}[f(t_{n+1}) - f(t_n)] + \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_{t_n}^{t_{n+1}} f(\tau, u(\tau))d\tau \tag{5.3}$$

The second step is approximation of our numerical scheme of the function $f(t, u(t))$. Thus we approximate $f(t, u(t))$ by using the well-known Lagrange interpolation polynomial to obtain following result for the interval $[t_n, t_{n+1}]$,

$$P(\tau)(\approx f(\tau, u(\tau))) = \left\{\frac{(\tau - t_{n-1})}{(t_n - t_{n-1})}\right\} f(t_n, u_n) + \left\{\frac{(\tau - t_n)}{(t_{n-1} - t_n)}\right\} f(t_{n-1}, u_{n-1}) \tag{5.4}$$

$$P(\tau)(\approx f(\tau, u(\tau))) = \left\{\frac{(\tau - t_{n-1})}{(t_n - t_{n-1})}\right\} f_n + \left\{\frac{(\tau - t_n)}{(t_{n-1} - t_n)}\right\} f_{n-1} \tag{5.5}$$

The above approximation can included in Eq. (5.3) to produce

$$\begin{aligned} u(t_{n+1}) - u(t_0) &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}[f(t_{n+1}) - f(t_n)] \\ &\quad + \frac{2\kappa}{(2-\kappa)M(\kappa)} \int_{t_n}^{t_{n+1}} \left[\left\{\frac{(\tau - t_{n-1})}{(t_n - t_{n-1})}\right\} f_n + \left\{\frac{(\tau - t_n)}{(t_{n-1} - t_n)}\right\} f_{n-1}\right] d\tau \end{aligned} \tag{5.6}$$

thus, after some simplifications and integrating, the following equation is obtained:

$$u_{n+1} - u_n = \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}[f_{n+1} - f_n] + \frac{2\kappa}{(2-\kappa)M(\kappa)} h \left[\frac{3}{2} f_n - \frac{1}{2} f_{n-1}\right] \tag{5.7}$$

Now for finding the numerical solution of fractional model based on the CF derivative. For the Eq. (3.3) we get the solution

$$\begin{aligned}
s_{h_{n+1}} - s_{h_n} &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left[\mu_h(1 - s_{h_{n+1}}) - \alpha s_{h_{n+1}} i_{v_{n+1}} - \mu_h(1 - s_{h_n}) + \alpha s_{h_n} i_{v_n}\right] \\
&\quad + \frac{2\kappa h}{(2-\kappa)M(\kappa)}\left[\frac{3}{2}[\mu_h(1 - s_{h_n}) - \alpha s_{h_n} i_{v_n}] - \frac{1}{2}[\mu_h(1 - s_{h_{n-1}}) - \alpha s_{h_{n-1}} i_{v_{n-1}}]\right] \\
e_{h_{n+1}} - e_{h_n} &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left[\alpha s_{h_{n+1}} i_{v_{n+1}} - \beta e_{h_{n+1}} - \alpha s_{h_n} i_{v_n} + \beta e_{h_n}\right] \\
&\quad + \frac{2\kappa h}{(2-\kappa)M(\kappa)}\left[\frac{3}{2}\left[\alpha s_{h_n} i_{v_n} - \beta e_{h_n}\right] - \frac{1}{2}\left[\alpha s_{h_{n-1}} i_{v_{n-1}} - \beta e_{h_{n-1}}\right]\right] \\
i_{h_{n+1}} - i_{h_n} &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left[\nu_{h_{n+1}} e_{h_{n+1}} - \gamma i_{h_{n+1}} - \nu_{h_n} e_{h_n} + \gamma i_{h_n}\right] \\
&\quad + \frac{2\kappa h}{(2-\kappa)M(\kappa)}\left[\frac{3}{2}\left[\nu_{h_n} e_{h_n} - \gamma i_{h_n}\right] - \frac{1}{2}[\nu_{h_{n-1}} e_{h_{n-1}} - \gamma i_{h_{n-1}}]\right] \\
e_{v_{n+1}} - e_{v_n} &= \frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left[\delta s_{v_{n+1}} i_{h_{n+1}} - (\epsilon + \nu_{v_{n+1}}) e_{v_{n+1}} - \delta s_{v_n} i_{h_n} + (\epsilon + \nu_{v_n}) e_{v_n}\right]
\end{aligned}$$

$$+\frac{2\kappa h}{(2-\kappa)M(\kappa)}\left[\frac{3}{2}\left[\delta s_{\nu_n} i_{h_n}-(\epsilon+\nu_{\nu_n})e_{\nu_n}\right]-\frac{1}{2}\left[\delta s_{\nu_{n-1}} i_{h_{n-1}}-(\epsilon+\nu_{\nu_{n-1}})e_{\nu_{n-1}}\right]\right]$$

$$i_{\nu_{n+1}}-i_{\nu_n}=\frac{2(1-\kappa)}{(2-\kappa)M(\kappa)}\left[\nu_{\nu_{n+1}} e_{\nu_{n+1}}-\epsilon i_{\nu_{n+1}}-\nu_{\nu_n} e_{\nu_n}+\epsilon i_{\nu_n}\right]$$
$$+\frac{2\kappa h}{(2-\kappa)M(\kappa)}\left[\frac{3}{2}\left[\nu_{\nu_n} e_{\nu_n}-\epsilon i_{\nu_n}\right]-\frac{1}{2}\left[\nu_{\nu_{n-1}} e_{\nu_{n-1}}-\epsilon i_{\nu_{n-1}}\right]\right] \tag{5.8}$$

6 Numerical Simulation

In this part, By using the proposed numerical scheme of the model for different values of fractional order we present the numerical simulation. The numerical simulations are shown in Figs. 1, 2, 3, 4 and 5. Figure 1 is considered κ to be 1, Fig. 2 is considered κ to be 0.75, Fig. 3 is considered κ to be 0.55, in Fig. 4 is considered κ to be 0.35 and finally Fig. 5 is considered κ to be 0.15.

To achieve our numerical simulation the following initial conditions and parameters were used [17].

$N_h = 5{,}071{,}126$, $\pi_\nu = 2{,}500{,}000$, $\nu_h = 0.1667$, $\mu_h = 0.0045$, $\mu_\nu = 0.02941$, $\gamma_h = 0.328833$, $b\beta_h = 0.75$, $b\beta_\nu = 0.375$, $\nu_\nu = 0.1428$.

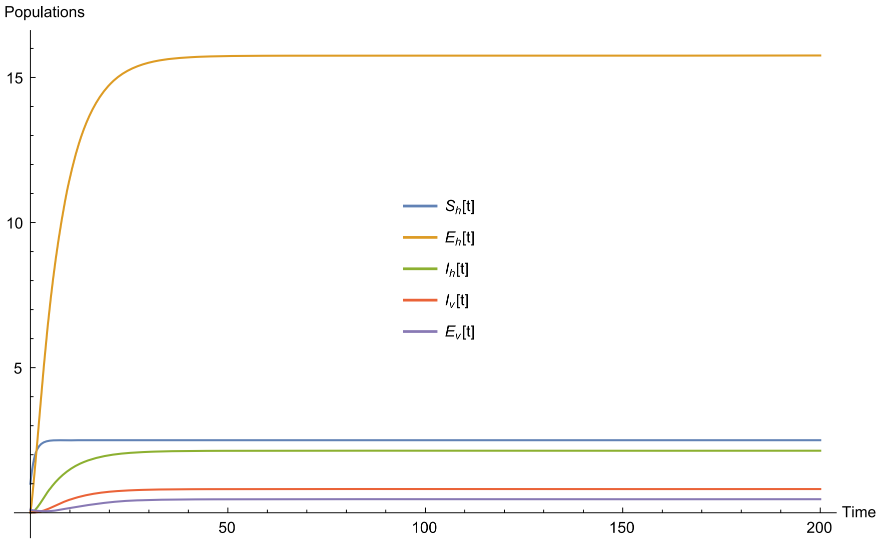

Fig. 1 For $\kappa = 1$

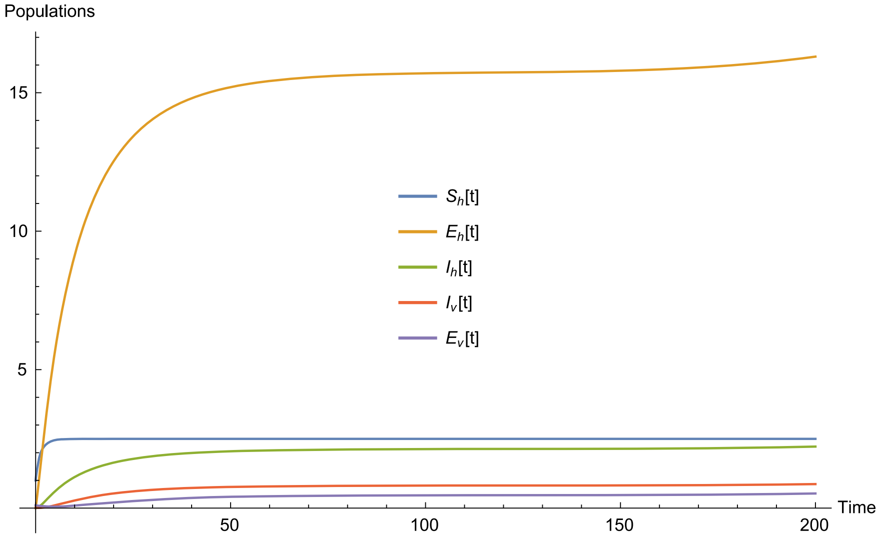

Fig. 2 For $\kappa = 0.75$

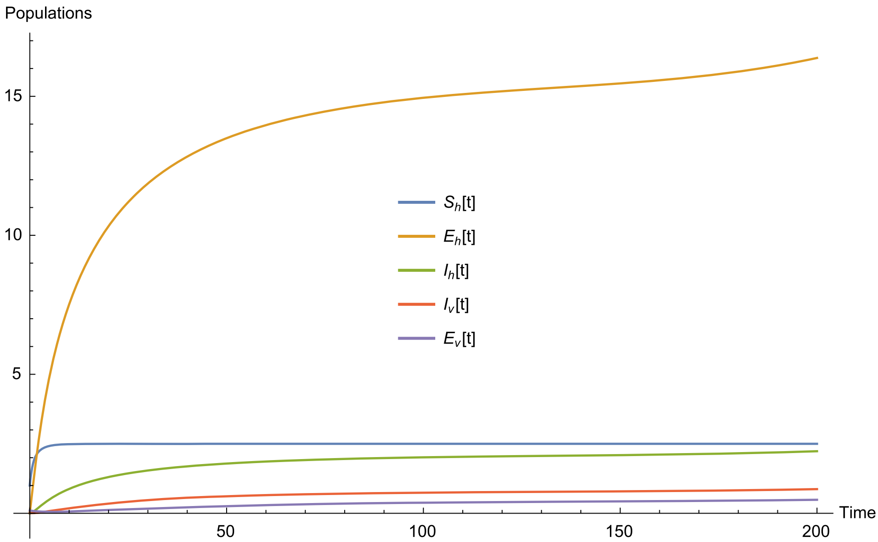

Fig. 3 For $\kappa = 0.55$

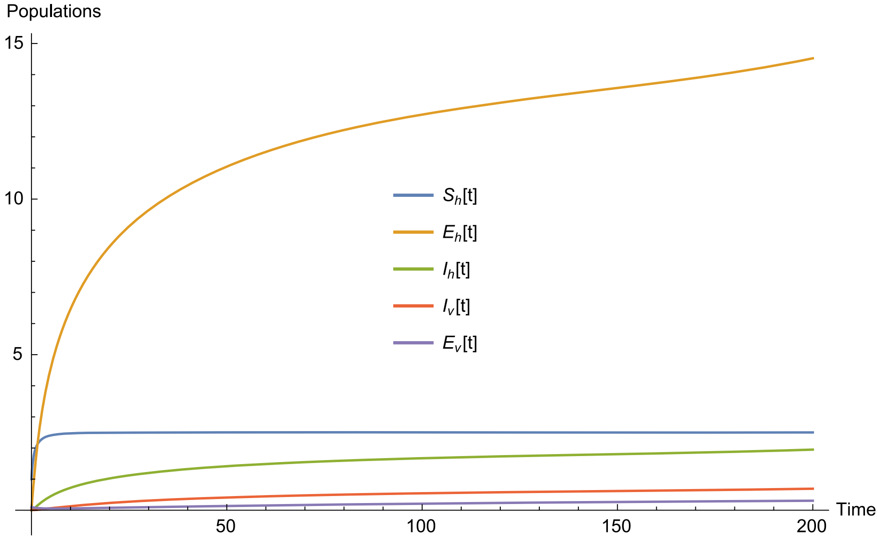

Fig. 4 For $\kappa = 0.35$

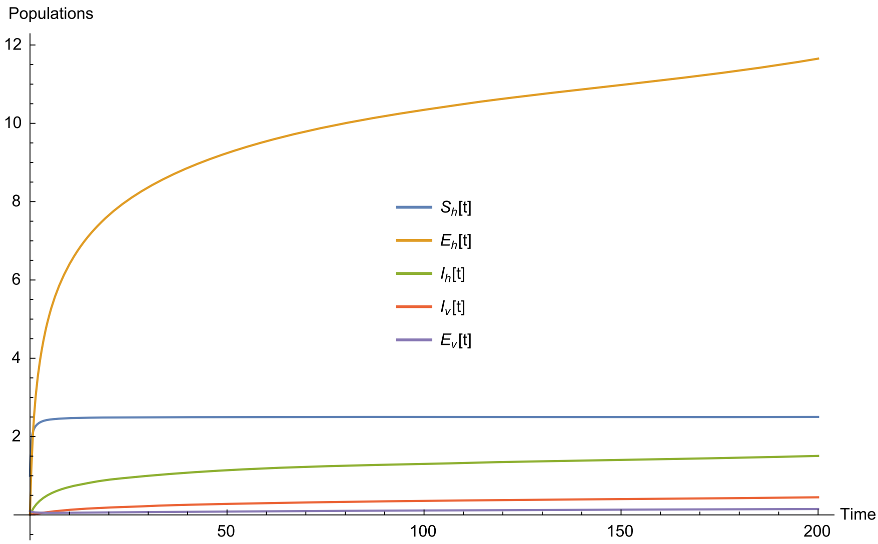

Fig. 5 For $\kappa = 0.15$

7 Conclusion

Although the dynamics spread of Dengue fever has in the attention of many researchers in the same field of applied mathematics in biology, it is worth noting that, still there is no attention has been given to modeling the spread with a differential operator having non-Markovian properties but the associated evolution equation having Markovian properties. If we consider the recent development in fractional differentiation and integration, a derivative with non-local kernel and non-singular was suggested by Caputo and Fabrizio and posses several properties that one observed in many problems occurring in biological modeling. We these properties, we devoted our paper to the discussion and analysis underpinning the dynamical spread of Dengue in given population. We provided a motivation to underpin why this operator is used for this model, then, we presented a detailed analysis of uniqueness and existence and the exact solution using the fixed-point theorem in Banach space. With the aim of improving the accuracy of numerical scheme, a new method was suggested by Toufit and Atangana [19] and was found to be highly accurate and very easier to implement. We used this numerical scheme to solve the new model with fading memory induces by the exponential kernel and presented numerical simulation.

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Approximate Solution of Higher Order Two Point Boundary Value Problems Using Uniform Haar Wavelet Collocation Method

Akmal Raza and Arshad Khan

Abstract An efficient collocation method is proposed for the numerical solution of second and fourth order two-point boundary value problems (B.V.P.) based on uniform Haar wavelet. We have converted higher order differential equations into a system of differential equations of lower order and then solve it by uniform Haar wavelet, which reduces the time and complexity of the system. The technique introduced here is easy to apply. The performance of the present method yield more accurate results on increasing the resolution level. To demonstrate the robustness and accuracy of the Haar wavelet collocation method, five problems have been solved and compared with the existing methods present in the literature [1–6].

Keywords Haar wavelet · Collocation points

2000 Mathematics Subject Classification: 65M99 · 65N35 · 65N55 · 65L10

1 Introduction

Wavelet Analysis is a new development in the field of Mathematics. Wavelets were introduced in seismology to provide a time localisation to seismic analysis. Wavelet theory involves representing square integrable functions in terms of simple wavelet functions at different scale and positions. The fundamental idea of wavelet is translation and scaling according to the need [7–10]. The best property of wavelet is compact support, which is boom for the numerical solution of differential equations. Meanwhile in numerical analysis, wavelet methods have become an important tool for solution of differential and integral equations that has been discussed in many research papers with different approaches such as Galerkin method, finite

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element method, finite difference method, filter-bank method, adaptive method etc. [7, 8, 11–15]. One of the best and easiest wavelet in wavelet theory is Haar wavelet. Gaussian and Legendre wavelet is also applied in treatment of Numerical solutions of differential equations but lots of numerical difficulties appeared on using these wavelets. The detection of singularities, local high frequencies, irregular structures and transient phenomena exhibited by analyzed function is possible on using wavelets. Use of orthogonal functions to construct the solution of differential equations was initially established in 1995 by Chen and Hsiao [14]. During the last two decades different types of functions have been applied to find the approximate solution of differential equations. But Haar wavelet gives the desirable results for such types of problems due to its simplicity, orthogonality and compact support.

2 Multiresolution Analysis and Haar Wavelet

Definition: A multiresolution analysis consists of a sequence $\{V_j : j \in Z\}$ of embedded closed subspace of $L^2(R)$ that satisfy the following properties:

1. **Increasing**: $V_j \subset V_{j+1} : j \in Z$
2. **Density**: $\overline{\bigcup_{j\in Z} V_j} = L^2(R)$
3. **Separation**: $\bigcap_{j\in Z} V_j = \{0\}$
4. **Scaling**: $f(t) \in V_j$ if and only if $f(2t) \in V_{j+1}$
5. **Orthonormal basis**: $\exists$ a scaling function $\phi \in V_0$ such that $\{\phi_{0,k}(t) = \phi(t-k) : k \in Z\}$ is an orthonormal basis for V_0.

Haar Wavelet: Haar function was discovered long before the wavelet was introduced by Hungarian Mathematician Alfred Haar in 1909. Haar is the simplest orthonormal wavelet with compact support [16].
The Haar wavelet family for $t \in [0, 1]$ is defined as follows:

$$h_u(t) = \begin{cases} 1, & \xi_1(u) \le t < \xi_2(u) \\ -1, & \xi_2(u) \le t < \xi_3(u) \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where u indicates the wavelet number and

$\xi_1(u) = \frac{k}{m}$, $\xi_2(u) = \frac{k+0.5}{m}$, $\xi_3(u) = \frac{k+1}{m}$
$m = 2^j$, $j = 0, 1, 2..., J$, and integer $k = 0, 1..., m-1$.

Also J indicates the level of resolution and k represents the translation parameter. Index u is calculated as $u = m + k + 1$ which is true for $u \ge 2$.
For $u = 1$ the Haar wavelet is given by

$$h_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Because of constant and piecewise nature of Haar wavelet, derivative vanishes. Due to lack of differentiability authors move towards integration approach instead of differentiation [14].
The integration of Haar wavelet has been obtained from [13] and given as follows:

$$I_1 h_u(t) = \begin{cases} t - \xi_1(u), & \xi_1(u) \le t < \xi_2(u) \\ \xi_3(u) - t, & \xi_2(u) \le t < \xi_3(u) \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The double integration of Haar wavelet can be given as follows:

$$I_2 h_u(t) = \begin{cases} \frac{1}{2}(t - \xi_1(u))^2, & \xi_1(u) \le t < \xi_2(u) \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3(u) - t)^2, & \xi_2(u) \le t < \xi_3(u) \\ \frac{1}{4m^2}, & \xi_3(u) \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

Proceeding in similar manner the nth integration of Haar wavelet can be written as:

$$I_n h_u(t) = \begin{cases} 0, & if \quad t < \xi_1(u) \\ \frac{1}{n!}[t - \xi_1(u)]^n, & \xi_1(u) \le t < \xi_2(u) \\ \frac{1}{n!}[(t - \xi_1(u))^n - 2(t - \xi_2(u))^n], & \xi_2(u) \le t < \xi_3(u) \\ \frac{1}{n!}[(t - \xi_1(u))^n - 2(t - \xi_2(u))^n + (t - \xi_3(u))^n], & \xi_3(u) \le t \end{cases} \tag{2.5}$$

Now consider, any square integrable function $f(t) \in L^2[0, 1]$, can be approximated by the dialation and translation of Haar wavelet [12, 13]

$$f(t) = \sum_{u=1}^{N} a_u h_u(t) \tag{2.6}$$

The Haar wavelet coefficients a_u are calculated as

$$a_u =< y(t), h_u(t) >= \int_0^1 y(t) \cdot \overline{h_u(t)} dt. \tag{2.7}$$

The collocation points are given as

$$X(u) = \frac{2u - 1}{m}, u = 1, 2, ..., m. \tag{2.8}$$

The matrix of Haar wavelet with respect to the collocation points is given as

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The matrix of integral and double integral of Haar wavelet with respect to the collocation points are given as:

$$I_1H = \frac{1}{16}\begin{pmatrix} 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 1 & 3 & 5 & 7 & 7 & 5 & 3 & 1 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, I_2H = \frac{1}{512}\begin{pmatrix} 1 & 9 & 25 & 49 & 81 & 121 & 169 & 225 \\ 1 & 9 & 25 & 49 & 79 & 103 & 119 & 127 \\ 1 & 9 & 23 & 31 & 32 & 32 & 32 & 32 \\ 0 & 0 & 0 & 0 & 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 0 & 1 & 7 & 8 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 1 & 7 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

3 Methods of Solution

3.1 Method for Solving Second Order Differential Equations

Consider a second order differential equation

$$y'' = \phi(t, y, y') \tag{3.1}$$

with boundary conditions

$$y(0) = \alpha, y(1) = \beta. \tag{3.2}$$

Let us suppose that

$$y'(t) = z(t) \Rightarrow y''(t) = z'(t) \tag{3.3}$$

$$y'(t) = \sum_{u=1}^{N} a_u h_u(t) \tag{3.4}$$

$$z'(t) = \sum_{u=1}^{N} b_u h_u(t). \tag{3.5}$$

Now integrating Eqs. (3.4) and (3.5) with respect to t from 0 to t we get

$$y(t) = \sum_{u=1}^{N} a_u I_1 h_u(t) + y(0) \tag{3.6}$$

and

$$z(t) = \sum_{u=1}^{N} b_u I_1 h_u(t) + z(0). \tag{3.7}$$

Substituting the values from Eqs. (3.3–3.7) in (3.1), we get the following system of equations

$$\sum_{u=1}^{N} b_u h_u(t) = \phi(t, \sum_{u=1}^{N} a_u I_1 h_u(t) + y(0), \sum_{u=1}^{N} b_u I_1 h_u(t) + z(0)) \tag{3.8}$$

Solving the above system of equation and find out the unknown Haar wavelet coefficient a_u and b_u with the help of Eq. (3.3) and then put in Eq. (3.6) to get the approximate solution of the differential equation.

3.2 Method for Solving Fourth Order Differential Equations

Consider the fourth order ordinary linear differential equation of the form.

$$y'''' = \phi(t, y, y', y'', y''') \tag{3.9}$$

with boundary conditions

$$y(0) = a, y(1) = b, y''(0) = c, y''(1) = d.$$

Let us suppose that

$$y''(t) = z(t), \tag{3.10}$$

$$y'''(t) = z'(t), \tag{3.11}$$

$$y''''(t) = z''(t). \tag{3.12}$$

$$\text{and} \quad y''(t) = \sum_{u=1}^{N} a_u h_u(t). \tag{3.13}$$

On integrating Eq. (3.13) from 0 to t with respect to t we get

$$y'(t) = \sum_{u=1}^{N} a_u P_{u,1}(t) + y'(0) \tag{3.14}$$

Again integrating Eq. (3.14) from 0 to t, with respect to t we get

$$y(t) = \sum_{u=1}^{N} a_u P_{u,2}(t) + t y'(0) + y(0) \tag{3.15}$$

Also we assume that

$$z'' = \sum_{u=1}^{N} b_u h_u(t) \tag{3.16}$$

On integrating Eq. (3.16) from 0 to t we get,

$$z'(t) = \sum_{u=1}^{N} b_u P_{u,1}(t) + z'(0) \tag{3.17}$$

Again integrating Eq. (3.17) from 0 to t we get

$$z(t) = \sum_{u=1}^{N} b_u P_{u,2}(t) + t z'(0) + z(0) \tag{3.18}$$

We can find the values of $y'(0)$, $y''(0)$, $y'''(0)$ and $y''''(0)$ from the boundary conditions. Now put Eqs. (3.10–3.18) in (3.9), we get the following system of equation

$$\begin{aligned}\sum_{u=1}^{N} b_u h_u(t) = \phi(t, \sum_{u=1}^{N} a_u P_{u,2}(t) + t.y'(0) + y(0), \sum_{u=1}^{N} a_u P_{u,1}(t) + y'(0),\\ \sum_{u=1}^{N} b_u P_{u,2}(t) + t.z'(0) + z(0), \sum_{u=1}^{N} b_u P_{u,1}(t) + z'(0))\end{aligned} \tag{3.19}$$

Find the value of the vector a_u and then put these values in the Eq. (3.15) to get the Haar approximate solution of the required differential equation.

4 Numerical Examples

In this section we have tested five problems to demonstrate the accuracy and effectiveness of proposed method.

Problem 1 Consider the second order differential equations [1]

$$y'' = 100y, \tag{4.1}$$

with boundary conditions

$$y(0) = y(1) = 1. \tag{4.2}$$

Exact solution of the problem is

$$y = \frac{\cos h(10t - 5)}{\cos h5} \tag{4.3}$$

Obtained maximum absolute errors for different resolutions are given in Table 1 and graph for J = 4 is given in Fig. 1.

Problem 2 Consider Dirichlet problem given in [4]:

$$-y'' = (\frac{537}{10}\pi)^2 \sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi)^2 \sin(\frac{23}{10}\pi t), \tag{4.4}$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0, \qquad t \in [0, 1]. \tag{4.5}$$

Exact solution is

$$y = (\frac{537}{10}\pi) \sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi) \sin(\frac{23}{10}\pi t) \qquad . \tag{4.6}$$

Table 1 Maximum absolute error for Problem 1

Level of resolution J	Our method	[1]
3	1.6719e−04	1.2800e−03
4	2.2854e−05	3.0700e−04
5	2.9131e−06	–
10	8.4470e−11	–

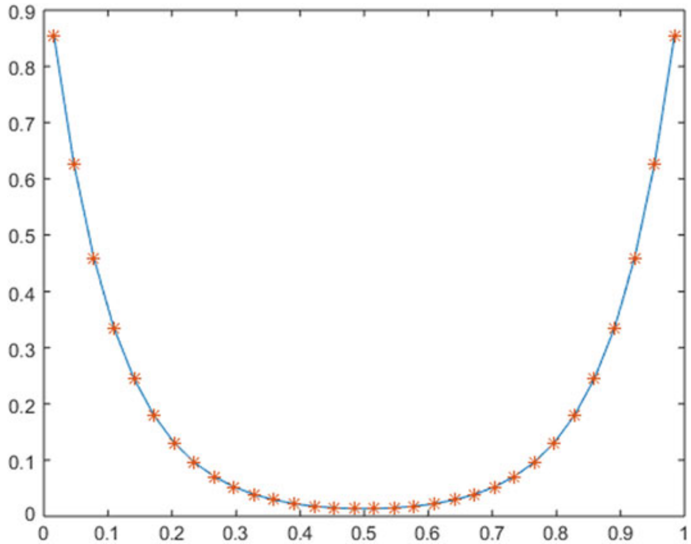

Fig. 1 Exact and Haar solution of Problem 1 for J = 4

Table 2 Maximum absolute error for Problem 2

Level of resolution	J = 6	7	8	9
Our method	9.5472e−04	1.2490e−04	1.5790e−05	1.9750e−06
[4]	1.2100e−02	1.3260e−02	1.0820e−04	7.3580e−06

Obtained maximum absolute errors for different resolutions are given in Table 2 and graph for $J = 6$ is given in Fig. 2.

Problem 3 Consider Dirichlet problem given in [4]:

$$-y'' + y = [1 + (\frac{537}{10}\pi)^2]\sin(\frac{537}{10}\pi t) + [1 + (\frac{23}{10}\pi)^2]\sin(\frac{23}{10}\pi t), \quad (4.7)$$

with boundary conditions

$$y(0) = 0, \quad y(1) = 0. \quad t \in [0, 1] \quad (4.8)$$

Exact solution is

$$y = (\frac{537}{10}\pi)\sin(\frac{537}{10}\pi t) + (\frac{23}{10}\pi)\sin(\frac{23}{10}\pi t) \qquad . \quad (4.9)$$

Obtained maximum absolute errors for different resolutions are given in Table 3 and graph is given in Fig. 3.

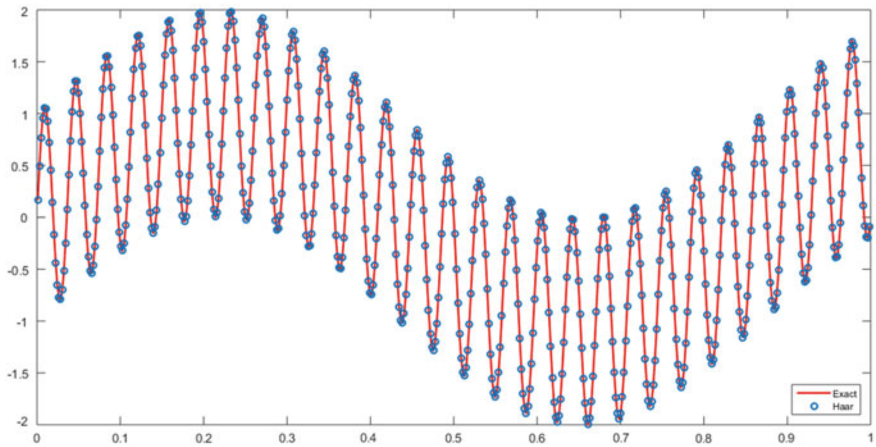

Fig. 2 Exact and Haar solution of Problem 2

Table 3 Maximum absolute error for Problem 3

Level of resolution	J = 6	7	8	9
Our method	9.4584e−04	1.2457e−04	1.5741e−05	1.9690e−06
[4]	1.2100e−2	1.3260e−3	1.0820e−4	7.3590e−6

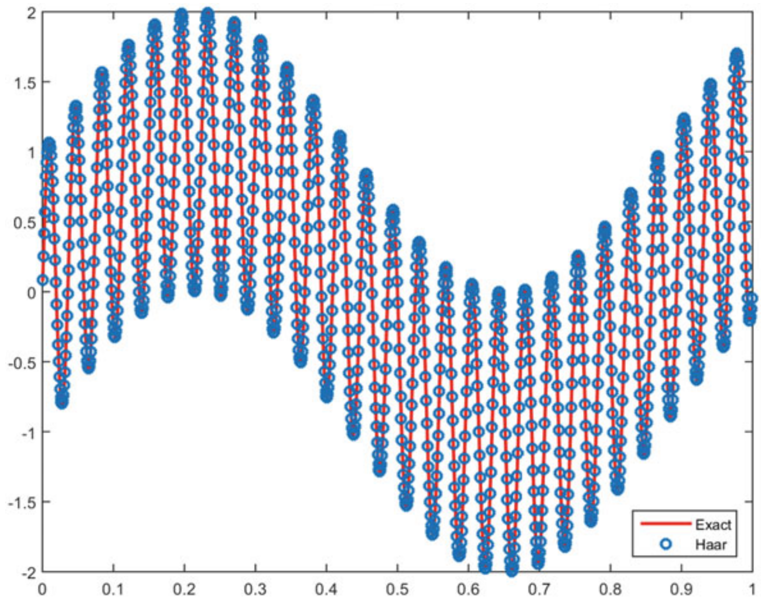

Fig. 3 Exact and Haar solution of Problem 3 for J = 9

Problem 4 Let us assume the fourth order B.V.P. given in [3]

$$y'''' + ty = -(t^3 + 7t + 8)e^t, \quad t \in [0, 1] \tag{4.10}$$

with boundary conditions

$$y(0) = y(1) = 0, \tag{4.11}$$

$$y''(t) = 1 \quad when \quad t = 0, \tag{4.12}$$

$$y''(t) = -4e. \quad when \quad t = 1 \tag{4.13}$$

Exact solution of the problem is

$$y(t) = t(1-t)e^t. \tag{4.14}$$

Obtained maximum absolute errors for different resolutions are given in Table 4 and graph is given in Fig. 4.

Table 4 Maximum absolute errors for Problem 4

Level of resolution J	Our method	[3]
2	5.2806e−04	4.5900e−04
3	9.4372e−05	1.9000e−4
4	1.6728e−05	5.2300e−05
5	2.9591e−06	–
6	5.2319e−07	–

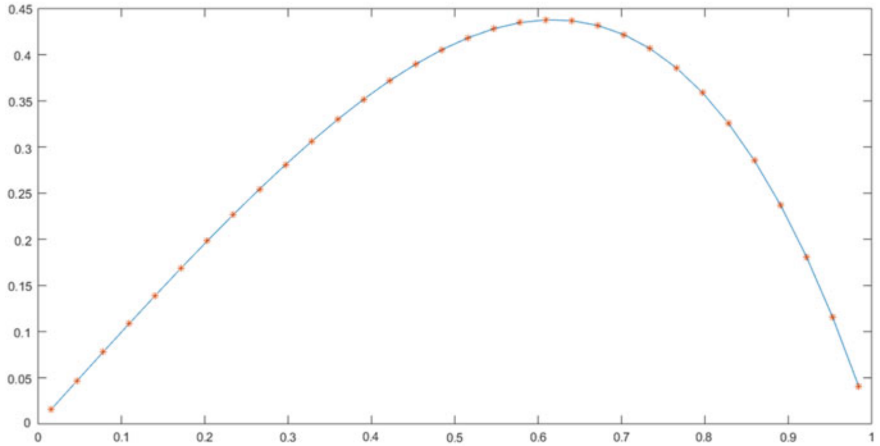

Fig. 4 Exact and Haar solution of Problem 4 for J = 4

Problem 5 Let us assume the fourth order B.V.P. given in [3]

$$y'''' - y = -4(2t\cos t + 3\sin t), \quad t \in [0, 1] \tag{4.15}$$

with boundary conditions

$$y(0) = y(1) = 0, \quad y''(t) = 0, \quad when \quad t = 0, y''(1) = 4\cos 1 + 2\sin 1. \tag{4.16}$$

Exact solution of the problem is

$$y(t) = (t^2 - 1)\sin t. \tag{4.17}$$

Obtained maximum absolute errors for different resolutions are given in Table 5 and graph is given in Fig. 5.

Table 5 Maximum absolute errors for Problem 5

Level of resolution J	Our method	[3]
2	5.6800e−04	6.6600e−04
3	9.7369e−05	1.6500e−04
4	1.4350e−05	4.1200e−05
5	1.5638e−06	–

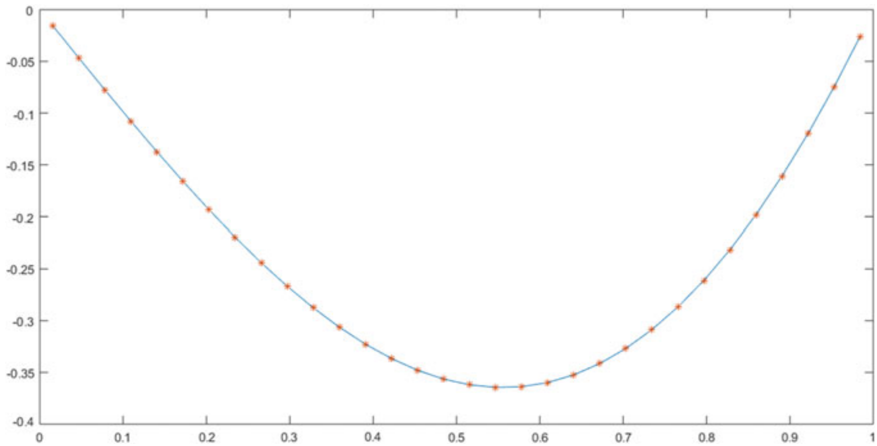

Fig. 5 Exact and Haar solution of Problem 5 for J = 4

5 Conclusion

We have converted second order differential equation into system of first order and fourth order differential equations into system of second order of differential equation, which is easy to solve to get the approximations of higher order two point boundary value problems. Haar wavelet collocation method has been applied on second and fourth order two point B.V.P. We have compared our results with the existing method given in [1, 3–6] which shows that our results are better.

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Solving Multi-objective Fractional Transportation Problem

Vishwas Deep Joshi and Rachana Saini

Abstract In classical fractional transportation programming problem, we want to optimize the objective function in the form of one or several ratios subject to some linear constraints. If in multi-objective transportation problem, objective function is in ration of two linear function under some linear restrictions, then the problem is called multi-objective linear fractional transportation problem (MOLFTP). In this paper we propose a new method to solve multi-objective linear fractional transportation problem which is extension of Nomani et al. (Int J Manag Sci Eng Manag (2016) [9]). Two numerical problems are presented to validate the proposed algorithm.

Keywords Fractional transportation programming · Multi-objective programming

1 Introduction

Transportation-distribution planning problems play an important role in management science. In recent scenario a distribution company often faced problems related multivehicle routing problem. To fulfill the multi respective demand, problem objective must be in the divided in multi-objective form. This paper studies fractional transportation problem with several objectives.

The transportation problem is to transfer goods from various origins to several destinations in a minimum cost. Bit et al. [1] solve multi-objective solid transportation problem using fuzzy programming approach and find both efficient and compromise optimal solution. Li and Lie [7] developed a fuzzy approach to solve the multi-objective transportation problem and obtain a non-dominated compromise solution at which the synthetic membership degree of the global evaluation for all objectives

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is maximum. Ivan et al. [3] a method providing the efficient way of construction of weighted coefficients for linear weighted sum method. Kumar and Pandey [6] applied fuzzy programming approach to solve multi-objective transportation problem.

Joshi and Gupta [5] proposed method for identification of more-for-less paradox in the linear fractional transportation problem. Gupta et al. [2] solve a compromise solution for multi-objective chance constraint capacitated transportation problem. Ota and Ojha [10] developed weighted sum method for solving multi-objective geometric programming problem (MOGPP) and compared the result with fuzzy programming method. Jadhav and Doke [4] developed fuzziness in the objective function is handled with fuzzy programming techniques in the sense of multi-objective approach. Cost and profit coefficients are trapezoidal fuzzy numbers and for each set of crisp part the fuzzy number a single fractional objective is considered.

Nomani et al. [9] developed a different approach for solving multi-objective transportation problems and compared solution with weighted sum approach. Maruti [8] solve each of the transportation problem as single objective and then using Taylor series approach expand each of the problem about its optimal solution and ignoring second and higher order error terms each of the objective and converted objectives into linear one. Then the problem reduces to MOLTPP. Evaluate each of the objectives at every optimal solution.

A new method is introduced to solve the problem studied in this paper. The paper is organized as follows. Section 2 outlines the MOLFTP and necessary definitions; In Sect. 3 the weighted sum method for fractional transportation problem discussed; Sect. 4 explain the proposed method for MOLFTP problem; Sect. 5 analyzes the computational performance of the algorithm proposed on randomly generated examples; Conclusions and remarks are discussed in Sect. 6.

2 Problem Description and Definitions

2.1 Mathematical Formulation of Multi-objective Linear Fractional Transportation Problem

The MOLFTP is formulated as follows:

$$\text{Min } Z_k\left(x_{ij}\right) = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^{k} x_{ij}}, k = 1, 2, 3, \ldots, K$$

Subject to:

$$\sum_{i=1}^{m} x_{ij} = a_{i,} \quad j = 1, 2, 3, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = b_j, \quad i = 1, 2, 3, \ldots, m$$

$$x_{ij} \geq 0 \qquad i = 1, 2, 3, \ldots, m \text{ and } j = 1, 2, 3, \ldots, n \tag{1}$$

Supply points $a_i, i = 1, 2, 3, \ldots, m$, there are m supply points which goods transported. Demand points b_j, $j = 1, 2, 3, \ldots, n$, there are n demand centers where goods required. The profit of c_{ij}^k is transporting one unit from i origin to j destination. The cost of d_{ij}^k is transporting one unit from i origin to j destination. Suppose variable x_{ij} denotes number units to be transported from jth origin to jth destination. Problem must be in balanced form (sum of supply = sum of demand).

2.2 *Pareto Optimal Solution*

A solution is called Pareto optimal solution if none of the objective functions can be improved in value without degrading one or more of the other objective values. Without additional subjective preference information, all Pareto optimal solutions are considered equally good [3].

3 Weighted Sum Method

Weighted sum method is single-objective optimization problem. It is following as:

$$\text{Min } Z = \sum_{k=1}^{K} w_k Z_k = \sum_{k=1}^{K} w_k \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^k x_{ij}}$$

Subject to:

$$\sum_{i=1}^{m} x_{ij} = a_{i,} \quad j = 1, 2, 3, \ldots, n$$

$$\sum_{j=1}^{n} x_{ij} = b_j, \quad i = 1, 2, 3, \ldots, m$$

$$x_{ij} \geq 0 \qquad i = 1, 2, 3, \ldots, m \text{ and } j = 1, 2, 3, \ldots, n \tag{2}$$

where the weights $w_k, k = 1, 2, 3, \ldots, K$, corresponding to the objective functions satisfy the following conditions:

$$\sum_{k=1}^{K} w_k = 1, w_k \geq 0, k = 1, 2, 3, \ldots, K.$$

4 Proposed Method

In proposed method we convert the MOLFTP into a non-linear problem where the objective is to Min $\rho' = \sum \rho(1 - w_k)$, where all objectives have common deviational variable ρ and w_k the weight for the kth objective function. The multi-objective linear fractional transportation problem (1) convert into the single objective problem as follows:

$$\begin{aligned}
&\text{Min } \rho' = \sum \rho(1 - w_k) \\
&\text{Subject to } \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}^{k} x_{ij}} \le Z_k^* + \rho(1 - w_k), \forall k = 1, 2, 3, \ldots, K \\
&\qquad \sum_{i=1}^{m} x_{ij} = a_{i,} \quad j = 1, 2, 3, \ldots, n \\
&\qquad \sum_{j=1}^{n} x_{ij} = b_j, \quad i = 1, 2, 3, \ldots, m \\
&\qquad 0 \le w_k \le 1, \quad k = 1, 2, 3, \ldots, K \\
&\qquad x_{ij} \ge 0 \qquad i = 1, 2, 3, \ldots, m \text{ and } j = 1, 2, 3, \ldots, n
\end{aligned} \tag{3}$$

In this method, we have introduced a deviational function $\rho(1 - w_k)$ instead of using a deviational variable.

5 Numerical Examples

Let us consider two examples with two and three objectives respectively. Both examples solved by proposed method.

Example 1 Let us consider two fractional objectives ration of $\left(c_{ij}^{k}\right)$ and $\left(d_{ij}^{k}\right)$ in Table 1.

We obtain optimal solution as follows:

$X^1 = (5, 5, 5, 0, 10, 0, 0, 15, 0, 20, 0, 0)$, $Z_1(X^1) = 0.60377$, $Z_1(X^2) = 1.13913$ and $X^2 = (0, 0, 0, 15, 0, 20, 5, 0, 15, 5, 0, 0)$, $Z_2(X^1) = 1.02963$, $Z_2(X^2) = 0.64$.

Then the proposed model generates result for numerical example 1 as shown in Table 2.

Example 2 Let us consider three objectives ratio of $\left(c_{ij}^{k}\right)$ and $\left(d_{ij}^{k}\right)$ in Table 3.

We obtain optimal solution as follows:

$X^1 = (0, 0, 7, 0, 0, 8, 0, 1, 5, 0, 0, 13)$, $Z_1(X^1) = 1.105802$, $Z_2(X^1) = 1.054688$, $Z_3(X^1) = 0.994898$;

Table 1 Objective matrix for numerical Example 1

	Destination 1	Destination 2	Destination 3	Destination 4	Supply
Origin 1	10, 14 $\left(c_{ij}^{k}\right)$ 15, 12 $\left(d_{ij}^{k}\right)$	14, 9 12, 14	8, 11 16, 7	12, 9 8, 17	15
Origin 2	8, 12 10, 6	12, 9 6, 11	14, 6 13, 13	8, 15 12, 10	25
Origin 3	9, 6 13, 9	6, 9 15, 15	15, 12 12, 12	9, 10 10, 16	20
Demand	15	25	5	15	60

Table 2 Solutions for different weights for numerical example 1

	Weight assigned to objective function (w_1, w_2)	Z_1, Z_2
1	$w_1 = 0.1, w_2 = 0.9$	0.99909, 0.68392
2	$w_1 = 0.2, w_2 = 0.8$	0.90532, 0.71572
3	$w_1 = 0.3, w_2 = 0.7$	0.83869, 0.74068
4	$w_1 = 0.4, w_2 = 0.6$	0.79704, 0.76885
5	$w_1 = 0.5, w_2 = 0.5$	0.76249, 0.79872
6	$w_1 = 0.6, w_2 = 0.4$	0.73039, 0.82993
7	$w_1 = 0.7, w_2 = 0.3$	0.69926, 0.86279
8	$w_1 = 0.8, w_2 = 0.2$	0.66828, 0.89803
9	$w_1 = 0.9, w_2 = 0.1$	0.63762, 0.94464
10	Without preference	1.13913, 0.64

Table 3 Objective matrix for numerical Example 2

	Destination 1	Destination 2	Destination 3	Destination 4	Supply
Origin 1	5, 6, 8 $\left(c_{ij}^{k}\right)$ 9, 12, 7 $\left(d_{ij}^{k}\right)$	2, 3, 7 5, 7, 2	3, 9, 5 9, 8, 7	7, 9, 12 2, 15, 6	7
Origin 2	16, 2, 9 8, 6, 9	8, 9, 5 13, 8, 5	9, 2, 3 7, 2, 5	10, 6, 13 3, 5, 9	9
Origin 3	12, 5, 11 9, 8, 2	9, 12, 13 10, 11, 7	14, 8, 8 6, 7, 12	13, 8, 4 6, 7, 8	18
Demand	5	8	7	14	34

$X^2 = (0, 0, 0, 7, 2, 0, 7, 0, 3, 8, 0, 7)$; $Z_1(X^2) = 1.504386$, $Z_2(X^2) = 0.849315$, $Z_3(X^2) = 1.352113$;
$X^3 = (5, 0, 2, 0, 0, 8, 1, 0, 0, 0, 4, 14)$, $Z_1(X^3) = 1.212766$, $Z_2(X^3) = 0.992537$, $Z_3(X^3) = 0.712598$.

Table 4 Solutions for different weights for numerical example 2

	Weights assigned (w_1, w_2, w_3)	Z_1, Z_2, Z_3
1	$w_1 = 0.1, w_2 = 0.9, w_3 = 0.0$	1.383647, 0.880187, 1.021315
2	$w_1 = 0.2, w_2 = 0.8, w_3 = 0.0$	1.323226, 0.903671, 0.984379
3	$w_1 = 0.3, w_2 = 0.7, w_3 = 0.0$	1.275302, 0.921958, 0.952431
4	$w_1 = 0.4, w_2 = 0.0, w_3 = 0.6$	1.191198, 0.991643, 0.769529
5	$w_1 = 0.5, w_2 = 0.0, w_3 = 0.5$	1.179208, 0.996127, 0.786005
6	$w_1 = 0.6, w_2 = 0.0, w_3 = 0.4$	1.16653, 1.001136, 0.803691
7	$w_1 = 0.0, w_2 = 0.3, w_3 = 0.7$	1.262712, 0.959152, 0.759671
8	$w_1 = 0.0, w_2 = 0.2, w_3 = 0.8$	1.253422, 0.967411, 0.742122
9	$w_1 = 0.0, w_2 = 0.1, w_3 = 0.9$	1.245205, 0.974778, 0.726539
10	$w_1 = 0.3, w_2 = 0.3, w_3 = 0.4$	1.221772, 0.965285, 0.812001
11	$w_1 = 0.3, w_2 = 0.4, w_3 = 0.3$	1.230194, 0.955937, 0.836991
12	$w_1 = 0.4, w_2 = 0.3, w_3 = 0.3$	1.207223, 0.967639, 0.830922
13	Without preference	1.212766, 0.992537, 0.712598

Then the proposed model generates result for numerical example 2 as shown in Table 4.

Comparison in Tables 5 and 6, it can be seen that for different weights objective values do not change consistently for the weighted sum method. Examples shows that our method gives better results as compare to weighted sum method.

The objective value increases and decreases by proposed method in Figs. 1 and 2.

Table 5 Compare the result obtained by proposed method and weighted sum method for Example 1

	Weights assigned (w_1, w_2)	Z_1, Z_2	
		Proposed method	Weighted sum method
1	$w_1 = 0.1, w_2 = 0.9$	0.99909, 0.68392	1.13913, 0.64
2	$w_1 = 0.2, w_2 = 0.8$	0.90532, 0.71572	1.13913, 0.64
3	$w_1 = 0.3, w_2 = 0.7$	0.83869, 0.74068	0.827068, 0.745098
4	$w_1 = 0.4, w_2 = 0.6$	0.79704, 0.76885	0.827068, 0.745098
5	$w_1 = 0.5, w_2 = 0.5$	0.76249, 0.79872	0.73768, 0.822585
6	$w_1 = 0.6, w_2 = 0.4$	0.73039, 0.82993	0.645963, 0.92517
7	$w_1 = 0.7, w_2 = 0.3$	0.69926, 0.86279	0.641968, 0.93443
8	$w_1 = 0.8, w_2 = 0.2$	0.66828, 0.89803	0.603774, 1.02963
9	$w_1 = 0.9, w_2 = 0.1$	0.63762, 0.94464	0.603774, 1.02963
10	Without preference	1.13913, 0.64	–

Table 6 Compare the result obtained by proposed method and weighted sum method for Example 2

	Weights assigned (w_1, w_2, w_3)	Z_1, Z_2, Z_3	
		Proposed method	Weighted sum method
1	$w_1 = 0.1$, $w_2 = 0.9$, $w_3 = 0.0$	1.383647, 0.880187, 1.021315	1.688372, 0.861751, 0.850427
2	$w_1 = 0.2$, $w_2 = 0.8$, $w_3 = 0.0$	1.323226, 0.903671, 0.984379	1.691244, 0.86758, 0.820084
3	$w_1 = 0.3$, $w_2 = 0.7$, $w_3 = 0.0$	1.275302, 0.921958, 0.952431	1.691244, 0.86758, 0.820084
4	$w_1 = 0.4$, $w_2 = 0.0$, $w_3 = 0.6$	1.191198, 0.991643, 0.769529	1.212766, 0.992537, 0.712598
5	$w_1 = 0.5$, $w_2 = 0.0$, $w_3 = 0.5$	1.179208, 0.996127, 0.786005	1.305344, 0.944882, 0.741803
6	$w_1 = 0.6$, $w_2 = 0.0$, $w_3 = 0.4$	1.16653, 1.001136, 0.803691	1.305344, 0.944882, 0.741803
7	$w_1 = 0.0$, $w_2 = 0.3$, $w_3 = 0.7$	1.262712, 0.959152, 0.759671	1.212766, 0.992537, 0.712598
8	$w_1 = 0.0$, $w_2 = 0.2$, $w_3 = 0.8$	1.253422, 0.967411, 0.742122	1.212766, 0.992537, 0.712598
9	$w_1 = 0.0$, $w_2 = 0.1$, $w_3 = 0.9$	1.245205, 0.974778, 0.726539	1.212766, 0.992537, 0.712598
10	$w_1 = 0.3$, $w_2 = 0.3$, $w_3 = 0.4$	1.221772, 0.965285, 0.812001	1.305344, 0.944882, 0.741803
11	$w_1 = 0.3$, $w_2 = 0.4$, $w_3 = 0.3$	1.230194, 0.955937, 0.836991	1.691244, 0.86758, 0.820084
12	$w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.3$	1.207223, 0.967639, 0.830922	1.691244, 0.86758, 0.820084
13	Without preference	1.212766, 0.992537, 0.712598	–

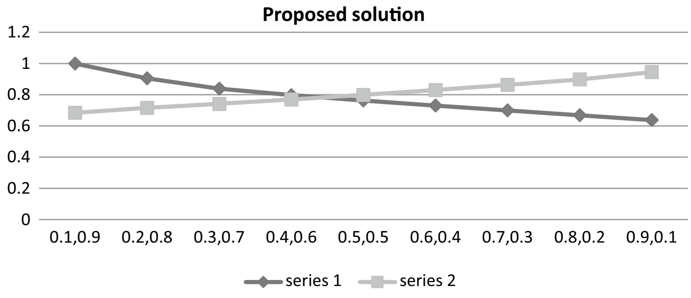

Fig. 1 Solution with different weights for Example 1

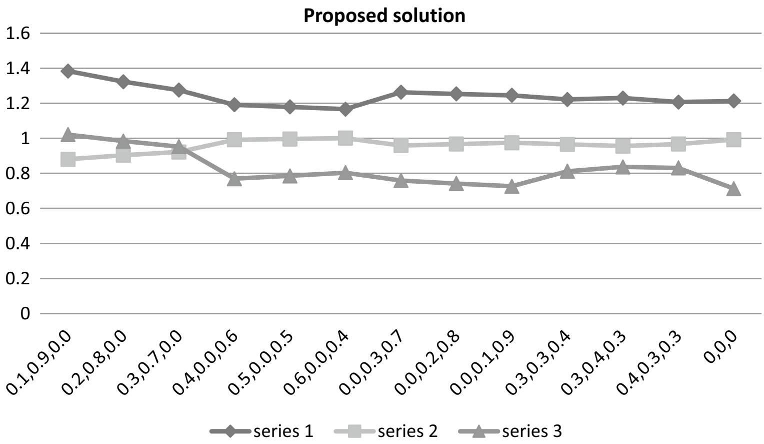

Fig. 2 Solution with different weights for Example 2

6 Conclusion

In this paper, we proposed a new technique for solving multi-objective linear fractional transportation problem. The proposed method is able to given Pareto optimal solutions without preference as well as solutions based on preferences. Than comparison weighted sum method and proposed solution. For solving all mathematical models in Sect. 5, LINGO* 17.0 software was used.

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On the Dark and Bright Solitons to the Negative-Order Breaking Soliton Model with (2+1)-Dimensional

Haci Mehmet Baskonus

Abstract This paper deal with the complex the dynamic of cnoidal waves via the negative-order breaking soliton model with (2+1)-dimensional. This model is arisen in the (2+1)-dimensional interaction of the Riemann wave propagated between y-axis and x-axis. The Improved bernoulli sub-equation function method is used in obtaining some complex and dark solutions with hyperbolic function structure. We present the interesting contour surfaces along with 2D and 3D graphics of the obtained analytical solutions in this study, plotted by using several computational programmes such as Matlap, Mathematica and so on. We finally present a comprehensive conclusion.

Keywords Nonlinear negative-order breaking soliton model · Improved bernoulli sub-equation function method · Complex hyperbolic solutions

PACS 02.30.Jr · 02.30.Hq · 04.20.Jb · 04.20.Cv · 52.35.Bj

1 Introduction

Today, the works carried on the solutions of mathematical models are of an outstanding area among scientists because solitons provides more information into the relevant from nonlinear sciences to engineering applications [1–54]. The first soliton model proposed by Korteweg and de Vries was KdV equation in 1895. Afterwards, Zabusky and Kruskal have presented an important paper on the interaction of "solitons" in a collisionless plasma in 1965 [26]. More recently, many scientific and engineering applications including vital real world problems on solitons have been presented to the literature. Bogoyavlenskii has presented some important models, which are entirely integrable solitons and N-solitons [27]. He has derived the connection with the Kadomtsev–Petviashvili equation with the help of the Painlevé

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method. One of the most important properties of integrable models is that these models produce many soliton solutions. Therefore, many experts have focused on the investigations of solitons arising in real world problems. Moreover, they have changed general structures of models for getting more and clear understanding of the models. This plays a major role in solitary waves theory and soliton theory. In this sense, Wazwaz has investigated the negative-order breaking soliton equations by using simplified Hirota's method [28]. Fei and Cao have observed explicit soliton-cnoidal wave interaction solutions for the (2+1)-dimensional negative-order breaking soliton equation (NOBSE) [29] defined as

$$u_t - v_x = 0, \quad u_y + v_{xxx} - 4uv_x - 2u_x v = 0. \tag{1.1}$$

This model was used to symbolize the (2+1)-dimensional interaction of the Riemann wave propagated along the y-axis with a long wave propagated along the x-axis [28–32]. Fei et al. [29] have derived the explicit soliton-cnoidal wave interaction solutions to the Eq. (1.1) by using an analytic method. The paper is organized as follows. In Sect. 2, we present the Improved bernoulli sub-equation function method (IBSEFM) in a comprehensive manner. Section 3 is devoted to obtain new complex travelling wave soliton solutions to the NOBSE. A conclusion and discussion is given in the last section.

2 General Properties of IBSEFM

The general properties of IBSEFM are given as follows:
Step 1. It can be considered that the following nonlinear model in two variables and a dependent variable v;

$$P(u, u_x, u_y, u_t, \ldots) = 0. \tag{2.1}$$

and take the wave transformation;

$$u(x, y, t) = U(\eta), \eta = \mu(x + \alpha y - kt). \tag{2.2}$$

where μ, α, k are constants and can be determined later. By substituting Eq. (2.2), Eq. (2.1) converts a nonlinear ordinary differential equation (NODE) as following;

$$N(U, U', U'', U''', \ldots) = 0. \tag{2.3}$$

Step 2. Considering trial equation of solution in Eq. (2.3), it can be written as following;

$$U(\eta) = \frac{\sum_{i=0}^{n} a_i F^i(\eta)}{\sum_{j=0}^{m} b_i F^j(\eta)} = \frac{a_0 + a_1 F(\eta) + a_2 F^2(\eta) + \cdots + a_n F^n(\eta)}{b_0 + b_1 F(\eta) + b_2 F^2(\eta) + \cdots + b_m F^m(\eta)}. \tag{2.4}$$

According to the Bernoulli theory, we can consider the general form of Bernoulli differential equation for F' as following;

$$F' = wF + \lambda F^M, w \neq 0, \lambda \neq 0, \ M \in \mathbb{R} - \{0, 1, 2\}. \tag{2.5}$$

where $F = F(\eta)$ is Bernoulli differential polynomial. Substituting above relations in Eq. (2.3), it yields us an equation of polynomial $\Omega(F)$ of F as following;

$$\Omega(F) = \rho_s F^s + \cdots + \rho_1 F + \rho_0 = 0. \tag{2.6}$$

According to the balance principle, we can determine the relationship between n, m and M.
Step 3. The coefficients of $\Omega(F)$ all be zero will yield us an algebraic system of equations;

$$\rho_i = 0, i = 0, \ldots, s. \tag{2.7}$$

Solving this system, we will specify the values of $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$.
Step 4. When we solve nonlinear Bernoulli differential equation Eq. (2.6), we obtain the following two situations according to b and d,

$$F(\eta) = \left[\frac{-\lambda}{w} + \frac{E}{e^{w(M-1)\eta}}\right]^{\frac{1}{1-M}}, w \neq \lambda. \tag{2.8}$$

$$F(\eta) = \left[\frac{(E-1) + (E+1)tanh(w(1-M)\frac{\eta}{2})}{1 - tanh(w(1-M)\frac{\eta}{2})}\right], w = \lambda, E \in \mathbb{R}. \tag{2.9}$$

Using a complete discrimination system for polynomial of F, we solve this system with the help of computer programming and classify the exact solutions to Eq. (2.3).

3 Application of the IBSEFM

In this section, IBSEFM has been successfully considered to the NOBSE to obtain more and novel complex solutions.

Example Taking the travelling wave transformation as

$$u(x, y, t) = U(\xi), \xi = kx + wy - ct, \quad v(x, y, t) = V(\xi), \xi = kx + wy - ct, \tag{3.1}$$

which k, w, c are real constants and non-zero in Eq. (1.1), we get the following nonlinear ordinary differential equation;

$$wU' - ck^2U''' + 6cUU' = 0. \tag{3.2}$$

with

$$V = \frac{-c}{k} U. \tag{3.3}$$

Integrating once and getting to the zero of integration constants, Eq. (3.2) can be rewritten as

$$wU - ck^2 U'' + 3cU^2 = 0. \tag{3.4}$$

With the help of balance principle for U'' and U^2, relationship between M, m and n can be obtained as follows;

$$2M + m = n + 2. \tag{3.5}$$

Case 1: Choosing $M = 3$, $n = 5$ and $m = 1$, we can find and its derivatives from Eq. (3.5) as follows:

$$U = \frac{a_0 + a_1 F + a_2 F^2 + a_3 F^3 + a_4 F^4 + a_5 F^5}{b_0 + b_1 F} = \frac{\Upsilon}{\Psi}, \tag{3.6}$$

$$U' = \frac{\Upsilon' \Psi - \Upsilon \Psi'}{\Psi^2}, \tag{3.7}$$

$$U'' = \ldots, \tag{3.8}$$

where $F' = pF + dF^3$, $a_5 \neq 0$, $b_1 \neq 0$, $p \neq 0$, $d \neq 0$. Substituting Eq. (3.6) with Eq. (3.8) into Eq. (3.4), a system of algebraic equations including various power of F can be found. Solving the system by using different computer programming such as Mathematica, Maple, and Matlap gives the complex structures;

Case-1a: For $p \neq d$ the following coefficients;

$$a_0 = \frac{-wb_0}{3c}, a_1 = \frac{-wb_1}{3c}, a_2 = \frac{i\sqrt{2}\sqrt{w}\sqrt{a_4}\sqrt{b_0}}{\sqrt{c}}, a_3 = \frac{i\sqrt{2}\sqrt{w}\sqrt{a_4} b_1}{\sqrt{b_0}\sqrt{c}}, a_5 = \frac{b_1 a_4}{b_0},$$
$$p = \frac{i\sqrt{2} d\sqrt{w}\sqrt{b_0}}{\sqrt{a_4}\sqrt{c}}, k = \frac{\sqrt{a_4}}{2\sqrt{2} d\sqrt{b_0}}, \tag{3.9}$$

we have the following new complex travelling wave solution

$$u_1 = \frac{-w}{3c} + 4wa_4 \left(i\sqrt{2}\sqrt{c}\sqrt{a_4} + 2e^{\frac{i\sqrt{w}}{\sqrt{2}\sqrt{c}\sqrt{a_4}}(\sqrt{2}x\sqrt{a_4} + 4d(ct - wy)\sqrt{b_0})} E\sqrt{b_0}\sqrt{w}\right)^{-2}$$
$$+ \frac{1}{\frac{c}{2w} - \frac{i\sqrt{c}E\sqrt{b_0}}{\sqrt{w}\sqrt{a_4}\sqrt{2}} e^{\frac{i\sqrt{w}}{\sqrt{2}\sqrt{c}\sqrt{a_4}}(\sqrt{2}x\sqrt{a_4} + 4d(ct - wy)\sqrt{b_0})}}, \tag{3.10}$$

$$v_1 = \frac{-2c\sqrt{2}d\sqrt{b_0}}{\sqrt{a_4}} u_1. \tag{3.11}$$

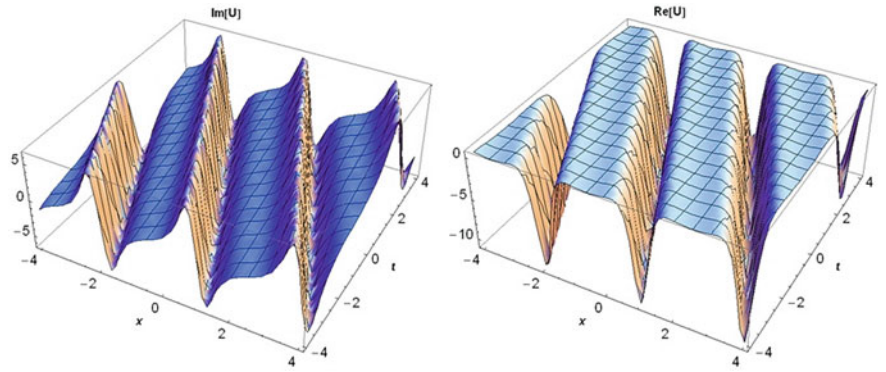

Fig. 1 The periodic wave surfaces of Eq. (3.10) for $w = 0.9$, $c = 0.2$, $a_4 = 0.3$, $b_0 = 0.5$, $d = 0.6$, $E = 0.1$, $y = 3$, $-4 < x < 4$, $-4 < t < 4$

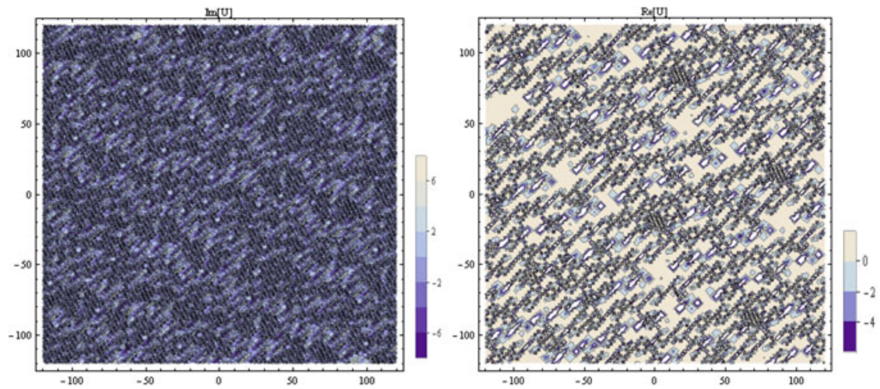

Fig. 2 The contour graphs of Eq. (3.10) for $w = 0.9$, $c = 0.2$, $a_4 = 0.3$, $b_0 = 0.5$, $d = 0.6$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

For better understanding of wave propagation meaning of via Eq. (3.10), and also, for suitable values of parameters, 2D and 3D figures along with contour graphs may be observed in Figs. 1, 2, 3 and 4.

Case-1b: When

$$a_0 = \frac{-wb_0}{3c},\ a_1 = \frac{-wb_1}{3c},\ a_2 = \frac{4idk\sqrt{w}\sqrt{b_0}}{\sqrt{c}},\ a_3 = \frac{4dki\sqrt{w}b_1}{\sqrt{c}}, \\ a_4 = 8d^2k^2b_0,\ a_5 = 8d^2k^2b_1,\ p = \frac{i\sqrt{w}}{2k\sqrt{c}}, \tag{3.12}$$

we have the following new complex bright soliton solution to the Eq. (1.1)

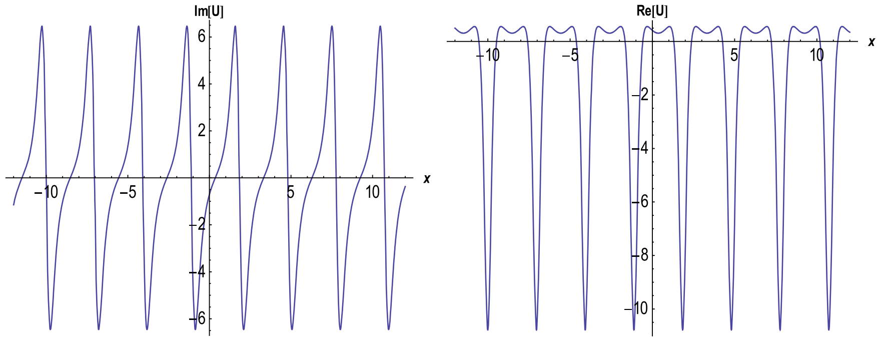

Fig. 3 The periodic wave surfaces of Eq. (3.10) for $w = 0.9, c = 0.2, a_4 = 0.3, b_0 = 0.5, d = 0.6, E = 0.1, y = 3, t = 0.85, -4 < x < 4$

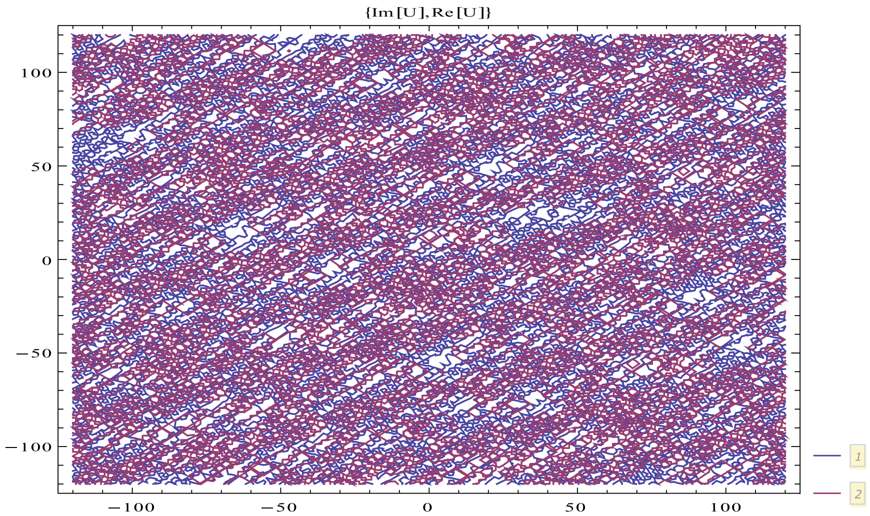

Fig. 4 The combination of contour graphs of both side of Eq. (3.10) for $w = 0.9, c = 0.2, a_4 = 0.3, b_0 = 0.5, d = 0.6, E = 0.1, y = 3, -120 < x < 120, -120 < t < 120$

$$
\begin{aligned}
u_2 =\ & \frac{8d^2k^2 sech^2(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t))}{(E + E\sqrt{1 - sech^2(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t))} + \frac{2idk\sqrt{c}}{\sqrt{w}} sech(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t)))^2} \\
& + \frac{4idk\sqrt{w} sech(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t))}{(E\sqrt{c} + \sqrt{1 - sech^2(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t))} E\sqrt{c} + \frac{2icdk}{\sqrt{w}} sech(\frac{-i\sqrt{w}}{k\sqrt{c}} f(x,y,t)))} - \frac{w}{3c}, \\
v_2 =\ & \frac{-c}{k} u_2
\end{aligned}
\tag{3.13}
$$

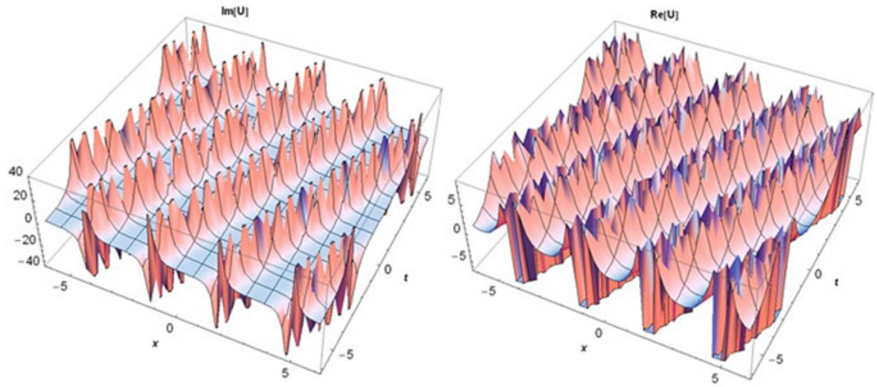

Fig. 5 The 3D graphs of Eq. (3.13) for $w = 0.9$, $c = 0.2$, $d = 0.3$, $k = 0.5$, $E = 0.1$, $y = 3$, $-6 < x < 6$, $-6 < t < 6$

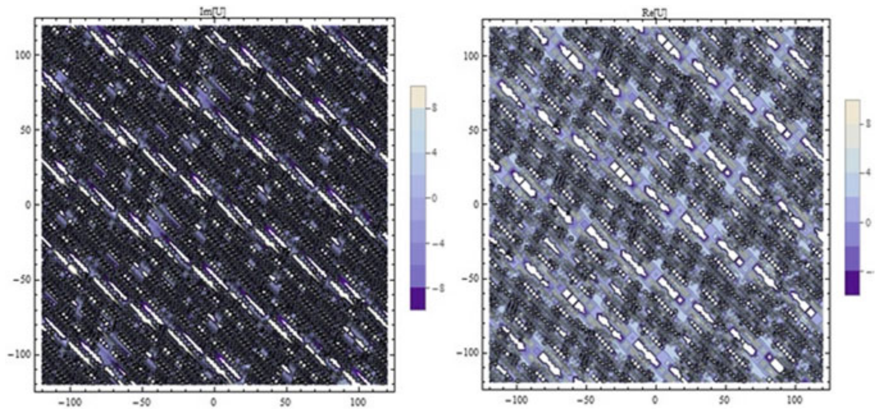

Fig. 6 The contour graphs of Eq. (3.13) for $w = 0.9$, $c = 0.2$, $d = 0.3$, $k = 0.5$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

in which $f(x, y, t) = kx + wy - ct$. With a view to the deeper investigation of complex travelling wave structure of Eq. (3.13) along with suitable values of parameters, 2D and 3D figures along with contour graphs may be seen in Figs. 5, 6, 7 and 8.

Case-1c: Once we consider as

$$\begin{aligned} b_0 =& \frac{-3ca_0}{w}, a_1 = \frac{-wb_1}{3c}, a_2 = \frac{-3ca_0a_3}{wb_1}, a_4 = \frac{3c^2a_0a_3^2}{2w^2b_1^2}, \\ a_5 =& \frac{-ca_3^2}{2wb_1}, p = \frac{i\sqrt{w}}{2k\sqrt{c}}, d = \frac{-i\sqrt{c}a_3}{4k\sqrt{w}b_1}, \end{aligned} \tag{3.14}$$

we have the following new complex dark soliton solution to the Eq. (1.1);

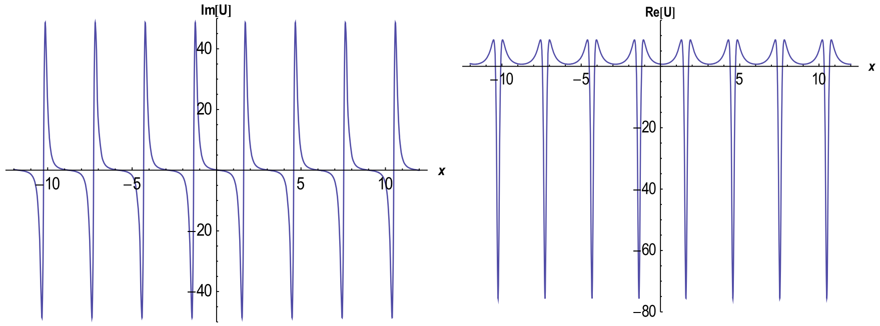

Fig. 7 The periodic wave surfaces of Eq. (3.13) for $w = 0.9$, $c = 0.2$, $d = 0.3$, $k = 0.5$, $E = 0.1$, $y = 3$, $t = 0.85$, $-6 < x < 6$

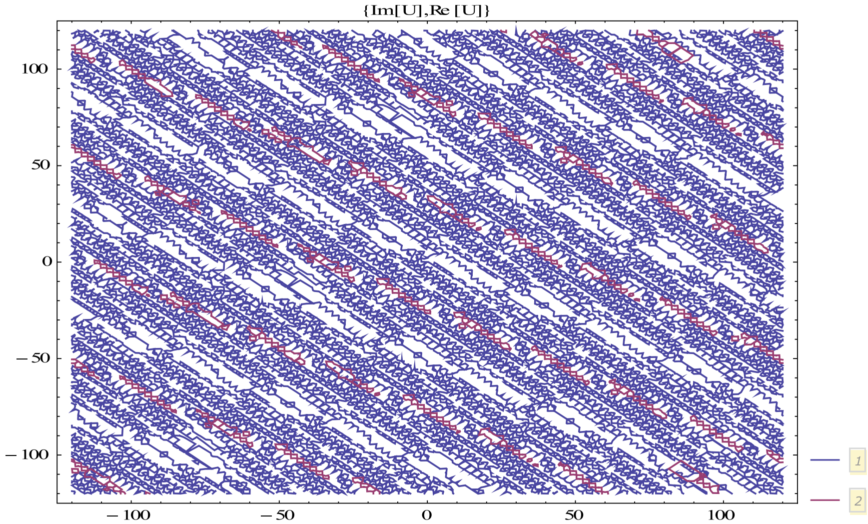

Fig. 8 The combination of contour graphs of both side of Eq. (3.13) for $w = 0.9$, $c = 0.2$, $d = 0.3$, $k = 0.5$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

$$
\begin{aligned}
u_3 &= \frac{a_3\sqrt{-1 + tanh(-if(x, y, t))}}{\frac{c}{2w}a_3\sqrt{-1 + tanh(-if(x, y, t))} + Eb_1\sqrt{-1 - tanh(-if(x, y, t))}} \\
&- \frac{ca_3^2(-1 + tanh(\frac{-i\sqrt{w}}{k\sqrt{c}}(kx + wy - ct)))}{2w(\frac{ca_3}{2w}\sqrt{-1 + tanh(-if(x, y, t))} + Eb_1\sqrt{-1 - tanh(-if(x, y, t))})^2} - \frac{w}{3c}, \\
v_3 &= \frac{-c}{k}u_3,
\end{aligned}
\tag{3.15}
$$

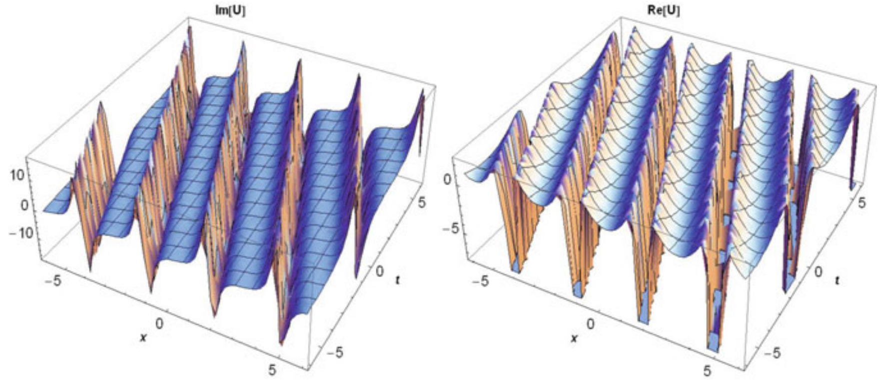

Fig. 9 The 3D graphs of Eq. (3.15) for $w = 0.9, c = 0.2, a_3 = 0.3, k = -0.5, b_1 = -0.6, E = 0.1, y = 3, -6 < x < 6, -6 < t < 6$

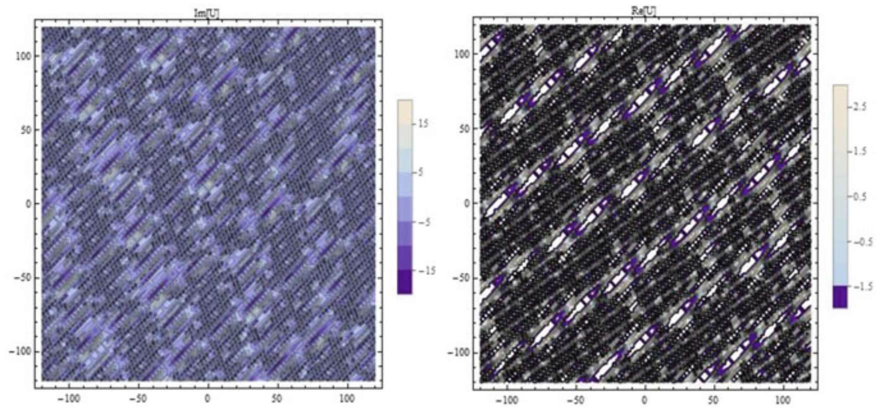

Fig. 10 The contour graphs of Eq. (3.15) for $w = 0.9$, $c = 0.2$, $a_3 = 0.3$, $k = -0.5$, $b_1 = -0.6$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

in which $f(x, y, t) = \frac{\sqrt{w}}{k\sqrt{c}}(kx + wy - ct)$. For suitable values of parameters, 2D and 3D figures along with contour graphs of Eq. (3.15) may be observed in Figs. 9, 10, 11, 12 and 13.

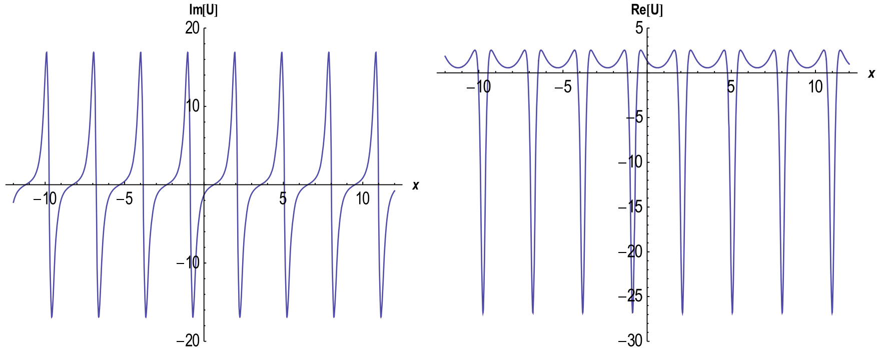

Fig. 11 The periodic wave surfaces of Eq. (3.15) for $w = 0.9$, $c = 0.2$, $a_3 = 0.3$, $k = -0.5$, $b_1 = -0.6$, $E = 0.1$, $y = 3$, $t = 0.85$, $-6 < x < 6$

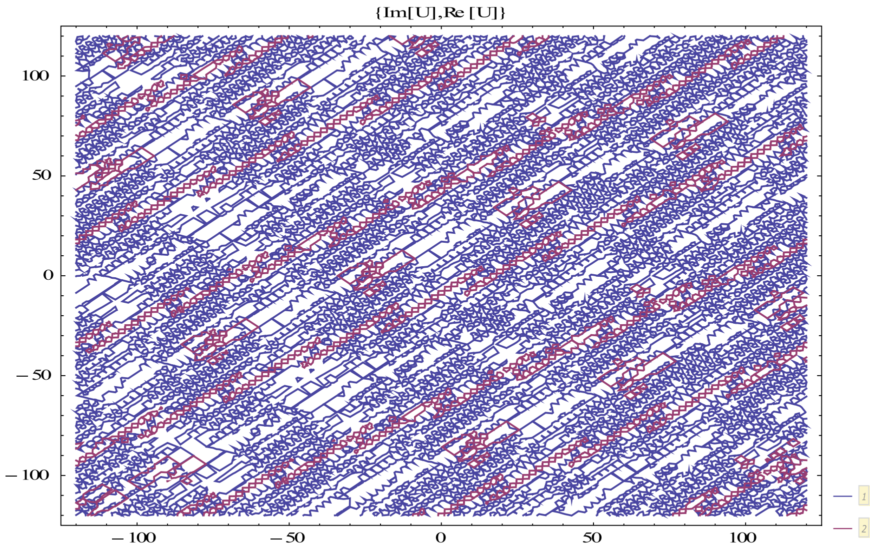

Fig. 12 The combination of contour graphs of both side of Eq. (3.15) for $w = 0.9$, $c = 0.2$, $a_3 = 0.3$, $k = -0.5$, $b_1 = -0.6$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

4 Conclusion

In this manuscript, the complex dark and bright soliton solutions to the Eq. (1.1) have been obtained by using IBSEFM. It has been observed that all solutions found in this paper have been satisfied the Eq. (1.1) considered. With the suitable values for parameters, based on the physical meanings and properties of model taken, and also, for better understanding of the physical meanings of the dark and bright soliton solutions, the three- and two-dimensional graphs and contour simulations

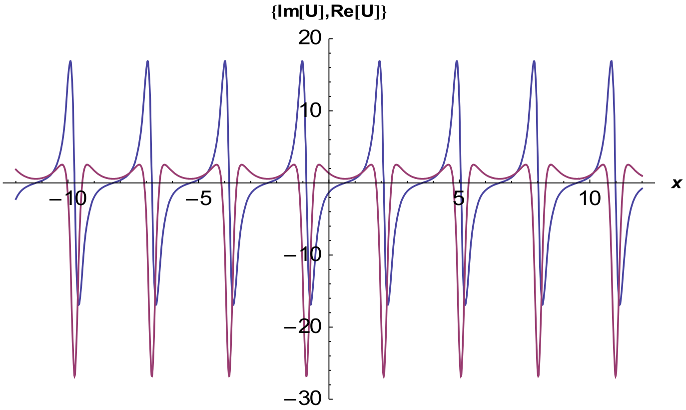

Fig. 13 Periodic wave surfaces of combination of real and imaginary part of Eq. (3.15) for $w = 0.9$, $c = 0.2$, $a_3 = 0.3$, $k = -0.5$, $b_1 = -0.6$, $E = 0.1$, $y = 3$, $-120 < x < 120$, $-120 < t < 120$

have been plotted with the help of several computer programs. The solitons of wave propagations can be observed from 3D Figs. 1, 5 and 9 along with 2D Figs. 3, 7 and 11. Moreover, high points of the mixed dark and bright soliton solutions, being Eqs. (3.10), (3.13) and (3.15), can be seen from contour surfaces of Figs. 2, 6 and 10, as an alternative and new perspective to the 3D graph. Combinations of contour graphs of real and imaginary parts of mixed dark and bright soliton solutions can be also viewed from Figs. 4, 8 and 12. Furthermore, more reality surfaces of solitons can be observed from Fig. 13 being combination of 2D graphs of real and imaginary parts of mixed dark and bright soliton solutions of Eq. (3.15). After all simulations, it can be understood that complex mixed dark and bright soliton solutions have shown the expected physical properties. Comparing some paper existing in literature [29], it can bee viewed that solutions of Eqs. (3.10), (3.13) and (3.15) are entirely new complex mixed dark and bright soliton solutions to the Eq. (1.1). To the best of our knowledge, the application of IBSEFM to the negative-order breaking soliton model with (2+1)-dimensional has been not submitted in advance.

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A Reliable Analytical Algorithm for Cubic Isothermal Auto-Catalytic Chemical System

Khaled M. Saad, H. M. Srivastava and Devendra Kumar

Abstract In this work we apply an algorithm for the q-homotopy analysis transform method (q-HATM) to solve the Cubic Isothermal Auto-catalytic Chemical System (CIACS). This technique is a combination of the Laplace decomposition method and the homotopy analysis scheme. This method gives the solution in the form of a rapidly convergent series with h-curves are employed to determine the intervals of convergent. Averaged residual errors are used to determine the optimal values of h. We show the behavior of the solutions graphically. The q-HATM solutions are compared with Numerical results by Mathematica and with finite difference method and excellent agreement is found.

Keywords Cubic isothermal auto-catalytic chemical system · Laplace transform · q-homotopy analysis transform method

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1 Introduction

Merkin et al. in [26] investigated the reaction-diffusion traveling waves that occur in isothermal auto-catalysis chemical system. The researchers proposed that the reactions take place in two regions. These regions are separated and parallel. The quadratic auto-catalysis represents the reaction in region I and is presented by

$$A + B \to 2B(rate\ k_1 ab), \tag{1.1}$$

with the step of the linear decay

$$B \to C(rate\ k_2 b), \tag{1.2}$$

where a and b are indicating the concentrations of reactant A and auto-catalyst B, the $k_i (i = 1, 2)$ are the rate constants and C is some inert product of reaction. The reaction in region II was the quadratic auto-catalytic step (1.1) only. The two regions were considered to be coupled through a linear diffusive interchange of the auto-catalytic species B. In this study we assume a similar kind of system as I, but having cubic auto-catalysis

$$A + 2B \to 3B(rate\ k_3 ab^2) \tag{1.3}$$

together with a linear decay step

$$B \to C(rate\ k_4 b). \tag{1.4}$$

This gives to the system of equations below.

The subsequent nonlinear problem on $\varsigma > 0$ and $\tau > 0$ for the dimensionless concentrations (α_1, β_1) in region I and (α_2, β_2) in region II of species A and B is considered

$$\frac{\partial \alpha_1}{\partial \tau} = \frac{\partial^2 \alpha_1}{\partial \varsigma^2} - \alpha_1 \beta_1^2, \tag{1.5}$$

$$\frac{\partial \beta_1}{\partial \tau} = \frac{\partial^2 \beta_1}{\partial \varsigma^2} + \alpha_1 \beta_1^2 - k\beta_1 + \gamma(\beta_2 - \beta_1), \tag{1.6}$$

$$\frac{\partial \alpha_2}{\partial \tau} = \frac{\partial^2 \alpha_2}{\partial \varsigma^2} - \alpha_2 \beta_2^2, \tag{1.7}$$

$$\frac{\partial \beta_2}{\partial \tau} = \frac{\partial^2 \beta_2}{\partial \varsigma^2} + \alpha_2 \beta_2^2 + \gamma(\beta_1 - \beta_2), \tag{1.8}$$

with the boundary conditions

$$\alpha_i(0,\tau) = \alpha_i(L,\tau) = 1, \quad \beta_i(0,\tau) = \beta_i(L,\tau) = 0. \tag{1.9}$$

The dimensionless constants k and γ indicates the strength of the auto-catalyst decay and the coupling between the two regions respectively.

The system of Eqs. (1.5)–(1.8) also studied by [30] for space-fractional derivative. The fractional extension of CIACS is similarly useful and gives very interesting consequences, in this regards one can refer the work on fractional calculus [5, 18, 34, 37, 40]. The main idea of this work is to apply the q-HATM [19] on the CIACS and study the effectiveness and accuracy of this method. The q-HATM is a combination of q-HAM [19] and Laplace transform. Also we modified the work [31, 32] to q-HATM [19]. The convergence of q-HAM and applications of this method on models are studied in details [7, 14–17, 27].

The present article is organized as follows. The second section describes the basic idea of the standard q-HATM. The third section is devoted to the application of q-HATM to CIACS. The forth section is devoted to the numerical results. In the last section, we summarize the results in the conclusion.

2 Basic Ideas of the *q*-HATM

Definition 2.1 If D_τ^r is linear differential operator of order r, then the Laplace transform for the fractional derivative $D_\tau^r f(\tau)$ is given as

$$\mathcal{L}(D_\tau^r f(\tau)) = s^r F(s) - \sum_{k=0}^{r-1} f^{(k)}(0^+) s^{r-k-1}, \quad \tau > 0, \tag{2.1}$$

$$F(s) = \int_0^\infty f(\tau) e^{-s\tau} d\tau.$$

In order to illustrate the basic concepts and the treatment of this method we let $\mathcal{N}[\alpha(\varsigma,\tau)] = g(\varsigma,\tau)$, where $\mathcal{N}$ represents the nonlinear partial differential operator in general. The Linear operator can be divided into two parts. The first part represents the linear operator of the highest order and indicates by L. The second part represents the reminder parts of the linear operator and indicates by R. So, it can be illustrated as

$$L\alpha(\varsigma,\tau) + R\alpha(\varsigma,\tau) + N\alpha(\varsigma,\tau) = g(\varsigma,\tau), \tag{2.2}$$

where $N\alpha(\varsigma, \tau)$ denotes the nonlinear terms. Now, if we let $L = D_\tau^r$ and apply the Laplace transform to Eq. (2.2) we obtain

$$\mathcal{L}[D_\tau^r \alpha(\varsigma, \tau)] + \mathcal{L}[R\alpha(\varsigma, \tau)] + \mathcal{L}[N\alpha(\varsigma, \tau)] = \mathcal{L}[g(\varsigma, \tau)]. \tag{2.3}$$

Making use of (2.1) we then have

$$\mathcal{L}[\alpha(\varsigma, \tau)] - \frac{1}{s}\sum_{i=0}^{r-1} \alpha^{(i)}(\varsigma, 0)s^{-i-1} + \frac{1}{s}\mathcal{L}[R\alpha(\varsigma, \tau) + N\alpha(\varsigma, \tau) - g(\varsigma, \tau)] = 0. \tag{2.4}$$

We express a nonlinear operator as

$$\begin{aligned}\mathcal{N}[\phi(\varsigma, \tau, q)] = \mathcal{L}[\phi(\varsigma, \tau; q)] - \frac{1}{s}\sum_{i=0}^{r-1} \phi^{(i)}(\varsigma, 0)s^{-i-1} \\ + \frac{1}{s}\mathcal{L}[R(\phi(\varsigma, \tau; q)) + N\phi((\varsigma, \tau; q)) - g(\varsigma, \tau))],\end{aligned} \tag{2.5}$$

In the above expression $q \in [0, 1/n]$ is denoting an embedding parameter and $\phi(\varsigma, \tau; q)$ is a real function of ς, τ and q. By modifying the well known concept of homotopy methods Liao [20–23] constructed the deformation equation of zero order written as

$$(1 - nq)\mathcal{L}[\phi(\varsigma, \tau; q) - \alpha_0(\varsigma, \tau)] = qhH(\varsigma, \tau)\mathcal{N}[\phi(\varsigma, \tau; q)], \tag{2.6}$$

Here $h \neq 0$ is an auxiliary parameter, $H(\varsigma, \tau) \neq 0$ is an auxiliary function, $\alpha_0(\varsigma, \tau)$ is an initial approximation for $\alpha(\varsigma, \tau)$ and $\phi(\varsigma, \tau; q)$ is an unknown function. It is obvious that, when $q = 0$ and $q = 1/n$, we have

$$\phi(\varsigma, \tau; 0) = \alpha_0(\varsigma, \tau), \quad \phi(\varsigma, \tau; 1) = \alpha(\varsigma, \tau), \tag{2.7}$$

respectively. Therefore, as q increases from 0 to $1/n$, then there is a variation in solution $\phi(\varsigma, \tau; q)$ from the initial approximation $\alpha_0(\varsigma, \tau)$ to the solution $\alpha(\varsigma, \tau)$. Writing $\phi(\varsigma, \tau; q)$ in series form by using Taylor theorem about q we get the following result

$$\phi(\varsigma, \tau; q) = \alpha_0(\varsigma, \tau) + \sum_{m=1}^{\infty} \alpha_m(\varsigma, \tau)q^m, \tag{2.8}$$

where

$$\alpha_m(\varsigma, \tau) = \frac{1}{m!}\frac{\partial^m \phi(\varsigma, \tau; q)}{\partial q^m}\,|_{q=0}\,. \tag{2.9}$$

If various parameters, operators and the initial approximation are properly selected, the series (2.8) converges at $q = \frac{1}{n}$ and we get

$$\alpha(\varsigma, \tau) = \alpha_0(\varsigma, \tau) + \sum_{m=1}^{\infty} \varsigma_m(\varsigma, \tau)\left(\frac{1}{n}\right)^m. \tag{2.10}$$

Let us now define the vectors

$$\vec{\alpha}_m(\varsigma, \tau) = \{\alpha_0(\varsigma, \tau), \alpha_1(\varsigma, \tau), \alpha_2(\varsigma, \tau), \ldots, \alpha_m(\varsigma, \tau)\}\,. \tag{2.11}$$

Now we differentiate the Eq. (2.6) m times with respect to q, then set $q = 0$ and finally divide them by $m!$, and we get

$$\mathcal{L}[\alpha_m(\varsigma, \tau) - \mathcal{X}_m \alpha_{m-1}(\varsigma, \tau)] = hH(\varsigma, \tau)\mathcal{R}_m(\vec{\alpha}_{m-1}(\varsigma, \tau)). \tag{2.12}$$

Here

$$\mathcal{R}_m(\vec{\alpha}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}(\mathcal{N}[\phi(\varsigma, \tau; q)])}{\partial q^{m-1}}|_{q=0} \tag{2.13}$$

and

$$\mathcal{X}_m = \begin{cases} 0 \text{ if } m \le 1, \\ n \text{ if } m > 1. \end{cases}$$

On finding the inverse of Laplace transform of (2.12) we get a power series solution $\alpha(\varsigma, \tau) = \sum_{m=0}^{\infty} \alpha_m(\varsigma, \tau)(\frac{1}{n})^m$ of the original Eq. (2.2).

To determine the interval of convergence of the q-HATM solutions, we use the h-curves. We can obtain the h-curves by plotting the derivative of the q-HATM solutions with respect to τ against h and then setting $\tau = 0$. Finally, the horizontal line in the h curve which parallels the ς axis gives the interval of convergence [21]. However, this procedure cannot determine the optimal value of h. Hence, we use the procedure which has been discussed by [3, 10, 24, 31, 32, 39]. Let

$$\triangle(h) = \int_\Omega (\mathcal{N}(\alpha_n(\varsigma, \tau)))^2\, d\Omega, \tag{2.14}$$

which denotes the exact square residual error for Eq. (2.2) integrated over the whole physical region. As $\triangle(h) \to 0$, the rate of convergence of the q-HATM solution increases. To obtain the optimal values of the convergence control parameter h, we minimize $\triangle(h)$ associated with the nonlinear algebraic equation

$$\frac{\mathrm{d}\triangle(h)}{\mathrm{d}h} = 0. \tag{2.15}$$

2.1 Convergence Analysis

To establish the convergence of the solution, we first need to give some conditions needed to prove the convergence of the series (2.10). These have been given by Odibat [29] and Elbeleze et al. [8] and Huseen and El-Tawil [14] via the following theorem:

Theorem 2.1.1 *Let the solution components $\alpha_0, \alpha_1, \alpha_2, \ldots$ be expressed as given in (2.12). The series solution $\sum_{m=0}^{\infty} \alpha_m (\frac{1}{n})^m$ written in (2.10) converges if $\exists\, 0 < r < n$ s.t. $||\alpha_{m+1}|| \leq (\frac{r}{n})||\alpha_m||$ for all $m \geq m_0$, for some $m_0 \in N$.*

Moreover, the estimated error is given by

$$||\alpha - \sum_{m=0}^{k} \alpha_m (\frac{1}{n})^m|| \leq \frac{1}{1-(\frac{r}{n})} (\frac{r}{n})^{k+1} ||\alpha_0||. \tag{2.16}$$

3 *q*-HATM solution of CIACS

In this portion, we apply the q-HATM on CIACS. We take the initial conditions to satisfy the boundary conditions, namely

$$\alpha_i(\varsigma, 0) = 1 - \sum_{n=1}^{\infty} a_{ni} \cos(0.5(L-2\varsigma)\lambda) \sin(\lambda L/2), (i = 1, 2), \tag{3.1}$$

$$\beta_i(\varsigma, 0) = \sum_{n=1}^{\infty} b_{ni} \cos(0.5(L-2\varsigma)\lambda) \sin(\lambda L/2), (i = 1, 2), \tag{3.2}$$

where $\lambda = \frac{n\pi}{L}$. As we know that HAM is based on a particular type of continuous mapping

$$\alpha_i(\varsigma, \tau) \to \phi_i(\varsigma, \tau; q), \quad \beta_i(\varsigma, \tau) \to \psi_i(\varsigma, \tau; q)$$

such that, as the embedding parameter q increases from 0 to $1/n$, $\phi_i(\varsigma, \tau; q)$, $\psi_i(\varsigma, \tau; q)$ and $i = 1, 2$ varies from the initial iteration to the exact solution.

We now present the nonlinear operators

$$
\begin{aligned}
\mathcal{N}_i(\phi_i(\varsigma,\tau;q)) &= \mathcal{L}_i\left(\phi_i(\varsigma,\tau;q)\right) - \frac{1}{s}\alpha_i(\varsigma,0) \\
&\quad + \frac{1}{s}\mathcal{L}_i\left(-\phi_{i,\varsigma\varsigma}(\varsigma,\tau;q) + \phi_i(\varsigma,\tau;q)\psi_i^2(\varsigma,\tau;q)\right), \\
\mathcal{M}_i(\psi_i(\varsigma,\tau;q)) &= \mathcal{L}_i\left(\psi_i(\varsigma,\tau;q)\right) - \frac{1}{s}\beta_i(\varsigma,0) \\
&\quad + \frac{1}{s}\mathcal{L}_i\left(-\psi_{i,\varsigma\varsigma}(\varsigma,\tau;q) + (-2(i-1)k + ik)\psi_i(\varsigma,\tau;q)\right. \\
&\quad \left. + (-1)^i\gamma(\psi_1(\varsigma,\tau;q) - \psi_2(\varsigma,\tau;q)) - \phi_i(\varsigma,\tau;q)\psi_i^2(\varsigma,\tau;q)\right).
\end{aligned}
$$

Now, we develop a set of equations, using the embedding parameter q

$$(1-nq)\mathcal{L}_i(\phi_i(\varsigma,\tau;q) - \alpha_{i0}(\varsigma,\tau)) = qhH(\varsigma,\tau)\mathcal{N}_i(\phi_i(\varsigma,\tau;q)),$$

$$(1-nq)\mathcal{L}_i(\psi_i(\varsigma,\tau;q) - \beta_{i0}(\varsigma,\tau)) = qhH(\varsigma,\tau)\mathcal{M}_i(\psi_i(\varsigma,\tau;q)),$$

with the initial conditions

$$\phi_i(\varsigma,0;q) = \alpha_{i0}(\varsigma,0), \quad \psi_i(\varsigma,0;q) = \beta_{i0}(\varsigma,0), (i=1,2)$$

where $h \neq 0$ and $H(\varsigma,\tau) \neq 0$ are the auxiliary parameter and the auxiliary function, respectively. We expand $\phi_i(\varsigma,\tau;q)$ and $\psi_i(\varsigma,\tau;q)$ in series form by employing the Taylor theorem with respect to q, and get

$$\phi_i(\varsigma,\tau;q) = \alpha_{i0}(\varsigma,\tau) + \sum_{m=1}^{\infty}\alpha_{im}(\varsigma,\tau)q^m, \tag{3.3}$$

$$\psi_i(\varsigma,\tau;q) = \beta_{i0}(\varsigma,\tau) + \sum_{m=1}^{\infty}\beta_{im}(\varsigma,\tau)q^m, \tag{3.4}$$

where

$$\alpha_{im}(\varsigma,\tau) = \frac{1}{m!}\frac{\partial^m\phi_i(\varsigma,\tau;q)}{\partial q^m}|_{q=0},$$

$$\beta_{im}(\varsigma,\tau) = \frac{1}{m!}\frac{\partial^m\psi_i(\varsigma,\tau;q)}{\partial q^m}|_{q=0}.$$

If we let $q=\frac{1}{n}$ into (3.3)–(3.4), the series become

$$\alpha_i(\varsigma,\tau) = \alpha_{i0}(\varsigma,\tau) + \sum_{m=1}^{\infty}\alpha_{im}(\varsigma,\tau)\left(\frac{1}{n}\right)^m,$$

$$\beta_i(\varsigma,\tau) = \beta_{i0}(\varsigma,\tau) + \sum_{m=1}^{\infty} \beta_{im}(\varsigma,\tau)\left(\frac{1}{n}\right)^m.$$

Now, we construct the mth-order deformation equation from (2.12)–(2.13) as follows:

$$\mathcal{L}_i(\alpha_{im}(\varsigma,\tau) - \mathcal{X}_m \alpha_{i(m-1)}(\varsigma,\tau)) = hH(\varsigma,\tau)R_1((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})),$$

$$\mathcal{L}_i(\beta_{im}(\varsigma,\tau) - \mathcal{X}_m \beta_{i(m-1)}(\varsigma,\tau)) = hH(\varsigma,\tau)R_2((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})),$$

with initial conditions $\alpha_{im}(\varsigma,0) = 0, \quad \beta_{im}(\varsigma,0) = 0, m > 1$ where

$$\begin{aligned} R_1((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})) &= \mathcal{L}_i\left(\alpha_{i(m-1)}(\varsigma,\tau)\right) - \frac{1}{s}\alpha_i(\varsigma,0)(1-\frac{\mathcal{X}_m}{n}) \\ &+ \frac{1}{s}\mathcal{L}_i\left(-\alpha_{i(m-1),\varsigma\varsigma}(\varsigma,t) + \alpha_{i(m-1)}(\varsigma,\tau)\beta^2_{i(m-1)}(\varsigma,\tau)\right), \end{aligned}$$

$$\begin{aligned} R_2((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})) &= \mathcal{L}_{i(m-1)}\left(\beta_{i(m-1)}(\varsigma,\tau)\right) - \frac{1}{s}\beta_i(\varsigma,0)\left(1-\frac{\mathcal{X}_m}{n}\right) \\ &+ \frac{1}{s}\mathcal{L}_i\left(-\beta_{i(m-1),\varsigma\varsigma}(\varsigma,\tau) + (-2(i-1)k+ik)\beta_{i(m-1)}(\varsigma,\tau)\right. \\ &+ (-1)^i\gamma(\beta_{1(m-1)}(\varsigma,\tau) - \beta_{2(m-1)}(\varsigma,\tau)) \\ &- \left.\alpha_{i(m-1)}(\varsigma,\tau)\beta^2_{i(m-1)}(\varsigma,\tau;q)\right). \end{aligned}$$

If we take $\mathcal{L}_i$ = Laplace transform ($i = 1, 2$) then the right inverse of $\mathcal{L}_i$ = inverse Laplace transform will be $\mathcal{L}_i^{-1}$

$$\alpha_{im} = \mathcal{X}_m\alpha_{i(m-1)} + h\mathcal{L}_i^{-1}R_1((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})), \tag{3.5}$$

$$\beta_{im} = \mathcal{X}_m\beta_{i(m-1)} + h\mathcal{L}_i^{-1}R_2((\vec{\alpha}_{i(m-1)}, \vec{\beta}_{i(m-1)})). \tag{3.6}$$

4 Numerical Results

In this part, we compute the first approximations. We show the behavior of the solution graphically and investigate the intervals of convergence by the h-curves. Also, we will compute the average residual error. Finally, we will check the accuracy of the q-HATM solutions by comparing with another numerical method using the command NDSolve by Mathematica. We take the initial approximation

$$\alpha_{i0}(\varsigma,\tau) = \alpha_{i0}(\varsigma,0), \quad \beta_{i0}(\varsigma,\tau) = \beta_{i0}(\varsigma,0). \tag{4.1}$$

For $m = 1$, we obtain the first approximation as following:

$$\alpha_{i1} = h\mathcal{L}_i^{-1}\left(\mathcal{L}_i\left(\alpha_{i0}(\varsigma,\tau)\right) - \frac{1}{s}\alpha_i(\varsigma,0)(1-\frac{\mathcal{X}_m}{n})\right. \tag{4.2}$$

$$\left.+\frac{1}{s}\mathcal{L}_i\left(-\alpha_{i0,\varsigma\varsigma}(\varsigma,\tau) + \alpha_{i0}(\varsigma,\tau)\beta_{i0}^2(\varsigma,\tau)\right)\right), \tag{4.3}$$

$$\beta_{i1} = h\mathcal{L}_i^{-1}\left(\mathcal{L}_i\left(\beta_{i0}(\varsigma,\tau)\right) - \frac{1}{s}\beta_i(\varsigma,0)(1-\frac{\mathcal{X}_m}{n})\right. \tag{4.4}$$

$$+\frac{1}{s}\mathcal{L}_i\left(-\beta_{i0,\varsigma\varsigma}(\varsigma,\tau) + (-2(i-1)k + ik)\beta_{i0}(\varsigma,\tau)\right. \tag{4.5}$$

$$\left.\left.+(-1)^i\gamma(\beta_{10}(\varsigma,\tau) - \beta_{20}(\varsigma,\tau)) - \alpha_{i0}(\varsigma,\tau)\beta_{i0}^2(\varsigma,\tau;q)\right)\right). \tag{4.6}$$

And by the similar procedure we can evaluate the rest of the approximation.

First we show the q-HATM solutions for CIACS for different values of τ. In Fig. 1 the q-HATM solutions are displayed against ς for $n = 5$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $a_{n_1} = 0.08$, $a_{n_2} = 0.07$, $b_{n_1} = 0.0054$, $b_{n_2} = 0.0055$ with $\tau = 0.5, 15, 50$. From this figure we find that the oscillation produced by the reaction in the system of finite size. And also, we find that, beside the boundaries, the q-HATM solutions are more significant compared the q-HATM solutions far away from the boundaries. The amplitude of the oscillation decays with increasing the distance from the boundaries. These behaviors agree with [4, 6, 9]. It is clear that the symmetric pattern for CIACS with respect to $\varsigma = L/2$. The two dominant modes generated from the boundaries are travelling towards the center. Thus permanent travelling waves solution exists in systems of finite size with periodic initial conditions and these behaviors agree with [25]. For more details for the effects of other parameters on the behaviors of CIACS see [26, 33].

4.1 h-Curves

To observe the intervals of convergence of the q-HATM solutions, we draw the h-curves of 5 terms of q-HATM solutions in Figs. 2, 3 and 4 for $n = 1, 5$ and $n = 20$ respectively. In Fig. 2a, we draw $\alpha_{1\tau}(\varsigma, 0)$, $\alpha_{2\tau}(\varsigma, 0)$ and in Fig. 2b we draw $\beta_{1\tau}(\varsigma, 0)$, $\beta_{2\tau}(\varsigma, 0)$ against h respectively at $k = 0.01$, $\gamma = 0.4$, $L = 100$, $\varsigma = 20$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. From these figures, we note that the straight line that parallels the h-axis provides the valid region of the convergence [21].

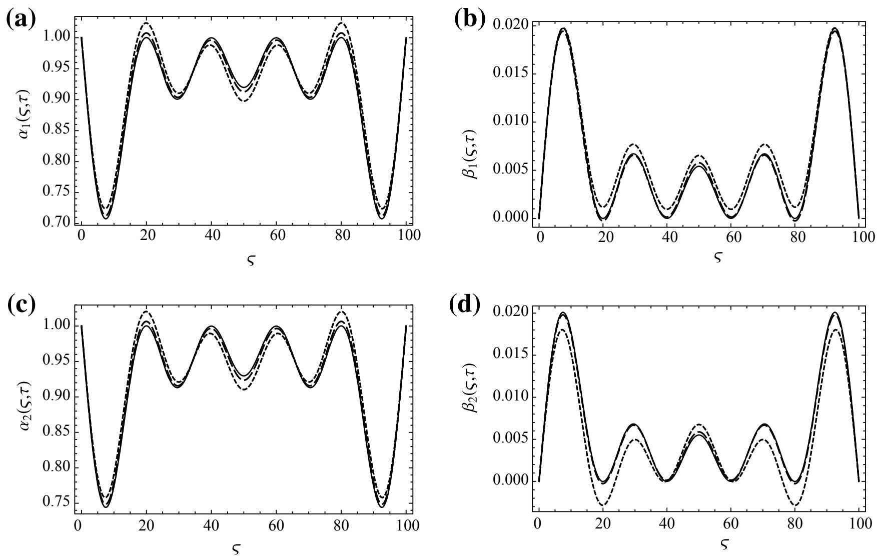

Fig. 1 The q-HATM solutions are displayed against ς for $n = 5$, $k = 0.01$, $\gamma = 0.4$, $L = 100$, $a_{n_1} = 0.08$, $a_{n_2} = 0.07$, $b_{n_1} = 0.0054$, $b_{n_2} = 0.0055$. Solid line: $\tau = 0.5$, Dash line: $\tau = 15$, and Dot line: $\tau = 50$

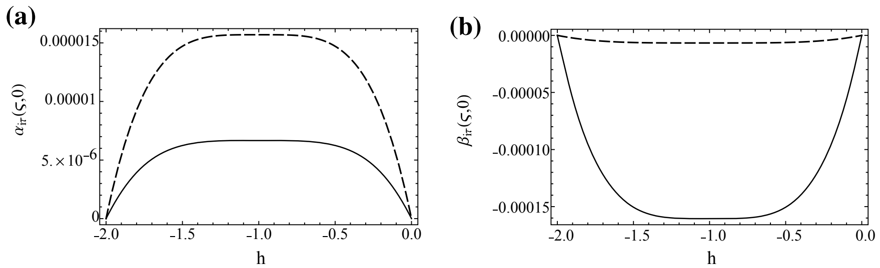

Fig. 2 The h-curve of the 5-terms of q-HATM solutions at $n = 1$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $\varsigma = 20$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. Solid line $= \alpha_{1\tau}(\varsigma, 0)$, $\beta_{1\tau}(\varsigma, 0)$, Dash line $= \alpha_{2\tau}(\varsigma, 0)$, $\beta_{2\tau}(\varsigma, 0)$

4.2 Average Residual Errors

We notice, however, that h-curve does not give the best value of the parameter h. So, we evaluate the optimal values of the convergence-control parameters by the minimum of the averaged residual errors [1–3, 11, 13, 24, 31, 32, 35, 36, 38, 39]

$$E_{\alpha_i}(h) = \frac{1}{NM}\sum_{s=0}^{N}\sum_{j=0}^{M}\left[\mathcal{N}\left(\sum_{k=0}^{m}\alpha_{ik}\left(\frac{100s}{N}, \frac{30j}{M}\right)\right)\right]^2, \tag{4.7}$$

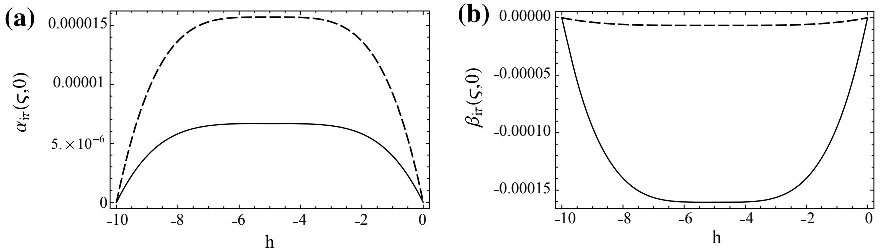

Fig. 3 The h-curve of the 5-terms of q-HATM solutions at $n = 5$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $\varsigma = 20$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. Solid line $= \alpha_{1\tau}(\varsigma, 0)$, $\beta_{1\tau}(\varsigma, 0)$, Dash line $= \alpha_{2\tau}(\varsigma, 0)$, $\beta_{2\tau}(\varsigma, 0)$

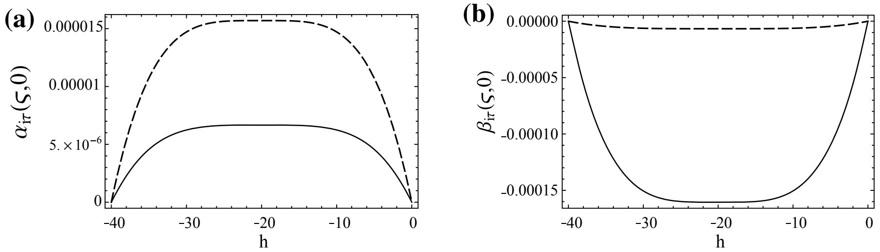

Fig. 4 The h-curve of the 5-terms of q-HATM solutions at $n = 20$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $\varsigma = 20$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. Solid line $= \beta_{1\tau}(\varsigma, 0)$, $\alpha_{1\tau}(\varsigma, 0)$, Dash line $= \alpha_{2\tau}(\varsigma, 0)$, $\beta_{2\tau}(\varsigma, 0)$

$$E_{\beta_i}(h) = \frac{1}{NM} \sum_{s=0}^{N} \sum_{j=0}^{M} \left[\mathcal{M} \left(\sum_{k=0}^{m} \beta_{ik} \left(\frac{100s}{N}, \frac{30j}{M} \right) \right) \right]^2, \tag{4.8}$$

corresponding to a nonlinear algebraic equations

$$\frac{dE_{\alpha_i}(h)}{dh} = 0, \tag{4.9}$$

$$\frac{dE_{\beta_i}(h)}{dh} = 0. \tag{4.10}$$

We show $E_{\alpha_i}(h)$ and $E_{\beta_i}(h)$ in Figs. 5, 6, 7 and 8 and in Table 1 for different values of n. Figures 3–8 and Table 2 show that the $E_{\alpha_i}(h)$ and $E_{\beta_i}(h)$ for 5 terms q-HATM solutions. We set into (4.9)–(4.10) $N = 100$ and $M = 30$ with $k = 0.1$, $\gamma = 0.2$, $L = 100, a_{n_1} = 0.001, a_{n_2} = 0.002, b_{n_1} = 0.001, b_{n_2} = 0.002$. We use the command Find Minimum and Minimize of Mathematica and the plotting of residual error against h to get the optimal values h.

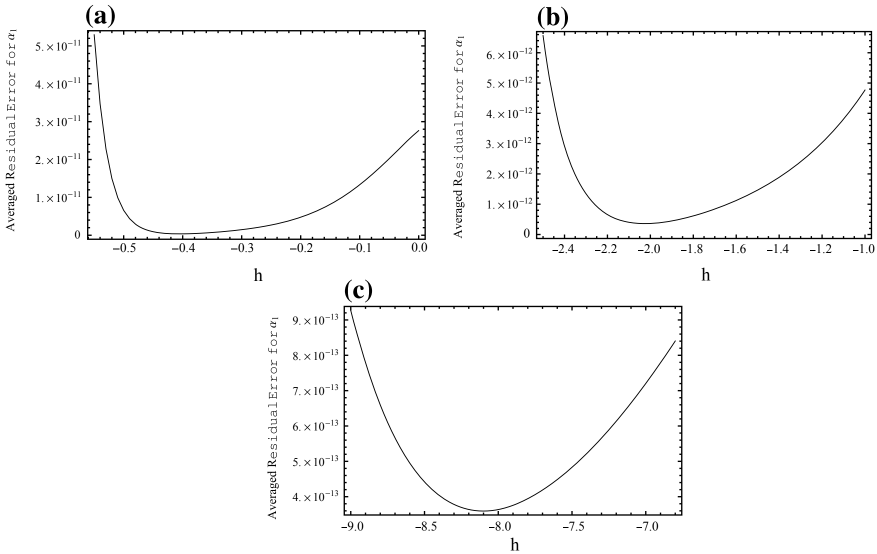

Fig. 5 The averaged residual errors at the 5-terms of the q-HATM solutions for $\alpha_1(\varsigma, \tau)$ with $0 \leq \varsigma \leq 100, 0 \leq \tau \leq 30,\ k = 0.1,\ \gamma = 0.2,\ L = 100,\ a_{n_1} = 0.001,\ a_{n_2} = 0.002,\ b_{n_1} = 0.001,$ $b_{n_2} = 0.002$. **a** $n = 1$, **b** $n = 5$, **c** $n = 20$

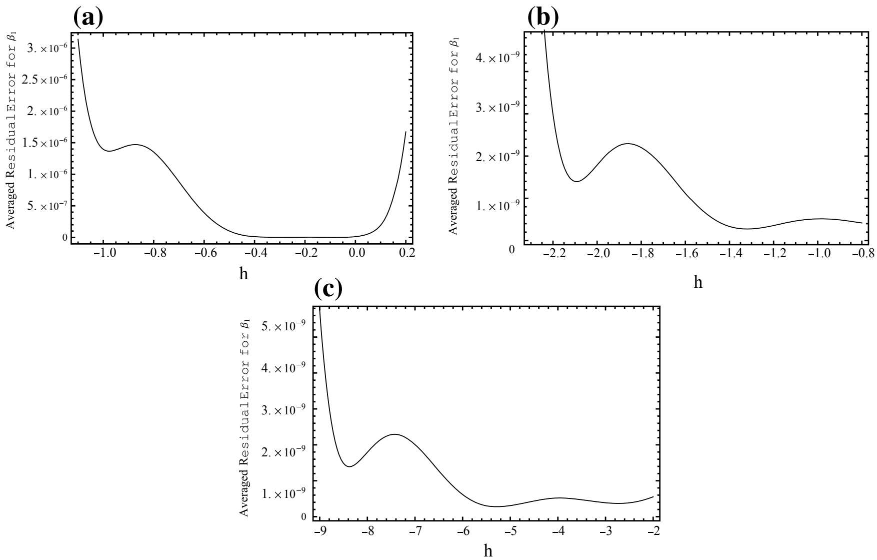

Fig. 6 The averaged residual errors at the 5-terms of the q-HATM solutions for $\beta_1(\varsigma, \tau)$ with $0 \leq \varsigma \leq 100, 0 \leq \tau \leq 30,\ k = 0.1,\ \gamma = 0.2,\ L = 100,\ a_{n_1} = 0.001,\ a_{n_2} = 0.002,\ b_{n_1} = 0.001,$ $b_{n_2} = 0.002$. **a** $n = 1$, **b** $n = 5$, **c** $n = 20$

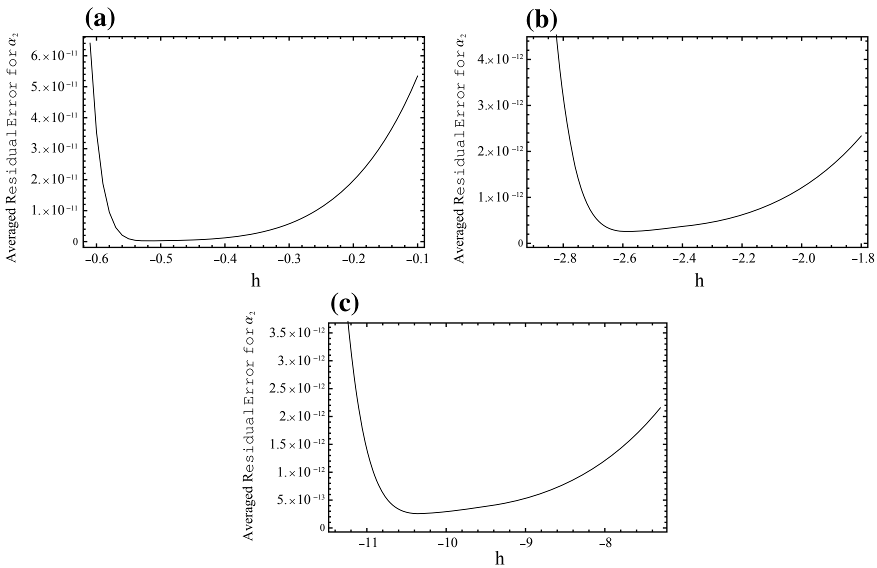

Fig. 7 The averaged residual errors at the 5-terms of the q-HATM solutions for $\alpha_2(\varsigma, \tau)$ with $0 \leq \varsigma \leq 100, 0 \leq \tau \leq 30,\ k = 0.1,\ \gamma = 0.2,\ L = 100,\ a_{n_1} = 0.001,\ a_{n_2} = 0.002,\ b_{n_1} = 0.001,$ $b_{n_2} = 0.002$. **a** $n = 1$, **b** $n = 5$, **c** $n = 20$

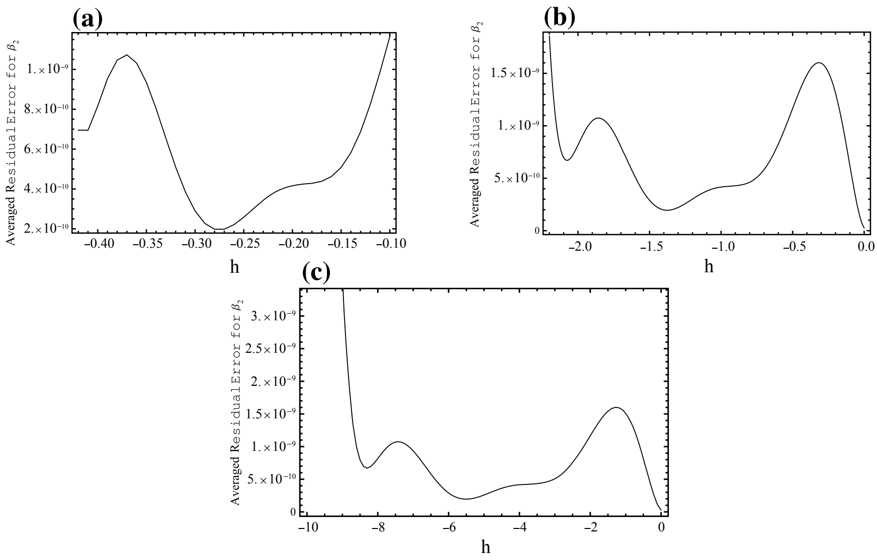

Fig. 8 The averaged residual errors at the 5-terms of the q-HATM solutions for $\beta_2(\varsigma, \tau)$ with $0 \leq \varsigma \leq 100, 0 \leq \tau \leq 30,\ k = 0.1,\ \gamma = 0.2,\ L = 100,\ a_{n_1} = 0.001,\ a_{n_2} = 0.002,\ b_{n_1} = 0.001,$ $b_{n_2} = 0.002$. **a** $n = 1$, **b** $n = 5$, **c** $n = 20$

Table 1 Optimal values of h for q-HATM solutions of $\alpha_i(\varsigma,\tau)$ at $0 \le \varsigma \le 100$, $0 \le \tau \le 30$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$

n	Optimal value of h_{α_1}	Minimum of $E_{\alpha_1}(h)$	Optimal value of h_{α_2}	Minimum of $E_{\alpha_2}(h)$
1	-0.404028	3.59782×10^{-13}	-0.520508	2.3569×10^{-13}
5	-2.02603	3.59593×10^{-13}	-2.63657	3.02373×10^{-13}
20	-8.10413	3.59593×10^{-13}	-10.3873	2.55769×10^{-13}

Table 2 Optimal values of h for q-HATM solutions of $\beta_i(\varsigma,\tau)$ at $0 \le \varsigma \le 100$, $0 \le \tau \le 30$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$

n	Optimal value of h_{β_1}	Minimum of $E_{\beta_1}(h)$	Optimal value of h_{β_2}	Minimum of $E_{\beta_2}(h)$
1	-0.137431	7.02541×10^{-10}	-0.223388	3.67024×10^{-10}
5	-1.18981	3.67977×10^{-10}	-1.38421	1.95288×10^{-10}
20	-5.34697	2.76228×10^{-10}	-5.50912	1.94929×10^{-10}

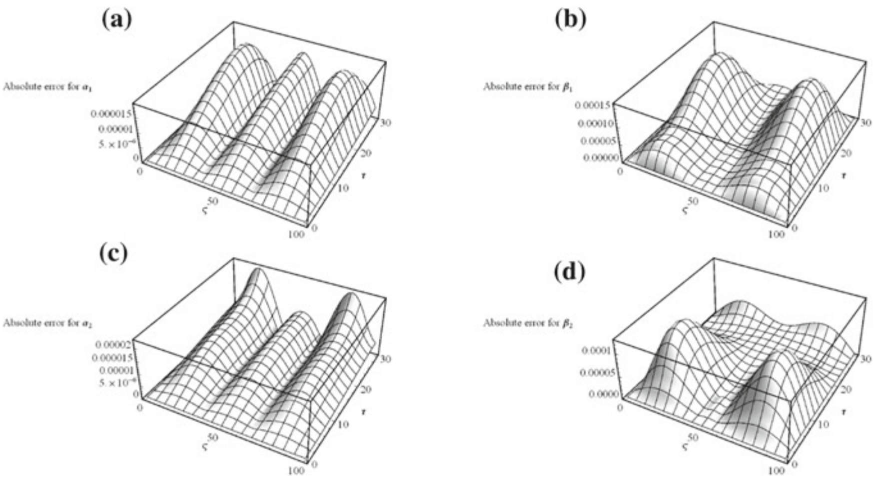

Fig. 9 The comparison of the 5-terms of the q-HATM solutions with numerical method in Mathematica for $n = 5$, $h_{\alpha_1} = -0.30$, $h_{\beta_1} = -0.18$, $h_{\alpha_2} = -0.30$, $h_{\beta_2} = -0.21$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$

4.3 Comparison Analysis

Now, we compare 5-terms of q-HATM solutions obtained with a numerical method using the commands with Mathematica 9 for solving CIACS numerically. We draw the 5-terms of HATM solutions in Fig. 9. Figure 9 shows the comparison of q-HATM solutions with numerical method for $n = 5$, $k = 0.1$, $\gamma = 0.2$, $L = 100$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. We observed from this figure that the QHATM solutions have a good agreement with the results by Mathematica.

We also compare our results also with finite differences method. We descretise with time step: $\Delta\tau = \frac{T}{N_\tau}$ and in space with grid spacing $\Delta\varsigma = \frac{L}{N_\varsigma}$, and let $\tau_j = j\Delta\tau$, where $0 \le j \le N_\tau$ and $\varsigma_n = n\Delta\varsigma$, $0 \le n \le N_\varsigma$. We put $\alpha_{1,n}^j = \alpha_1(\varsigma,\tau)$, $\beta_{1,n}^j = \beta_1(\varsigma,\tau)$, $\alpha_{2,n}^j = \alpha_2(\varsigma,\tau)$ and $\alpha_{2,n}^j = \alpha_2(\varsigma,\tau)$. Then the finite differences approximations for (1.5)–(1.8) are given by

$$\alpha_{1,n}^{j+1} = (1-2r)\alpha_{1,n}^j + r(\alpha_{1,n+1}^j + \alpha_{1,n-1}^j) - \Delta\tau(\alpha_{1,n}^j(\beta_{1,n}^j)^2), \tag{4.11}$$

$$\begin{aligned}\beta_{1,n}^{j+1} &= (1-2r)\beta_{1,n}^j + r(\beta_{1,n+1}^j + \beta_{1,n-1}^j)\\ &\quad + \Delta\tau\left(-k\beta_{1,n}^j + \gamma(\beta_{2,n}^j - \beta_{1,n}^j) - (\alpha_{1,n}^j(\beta_{1,n}^j)^2)\right),\end{aligned} \tag{4.12}$$

$$\alpha_{2,n}^{j+1} = (1-2r)\alpha_{2,n}^j + r(\alpha_{2,n+1}^j + \alpha_{2,n-1}^j) - \Delta\tau(\alpha_{2,n}^j(\beta_{2,n}^j)^2), \tag{4.13}$$

$$\begin{aligned}\beta_{1,n}^{j+1} &= (1-2r)\beta_{1,n}^j + r(\beta_{1,n+1}^j + \beta_{1,n-1}^j) - \beta_{1,n}^j)\\ &\quad + \gamma(\beta_{1,n}^j - \beta_{2,n}^j) - \Delta\tau(\alpha_{1,n}^j(\beta_{1,n}^j)^2),\end{aligned} \tag{4.14}$$

where $r = \frac{\Delta\tau}{(\Delta\varsigma)^2}$. We mention that here we use the central difference scheme for the space derivatives of second order and the forward difference scheme for the time derivative of order one [28]. The initial and boundary conditions become

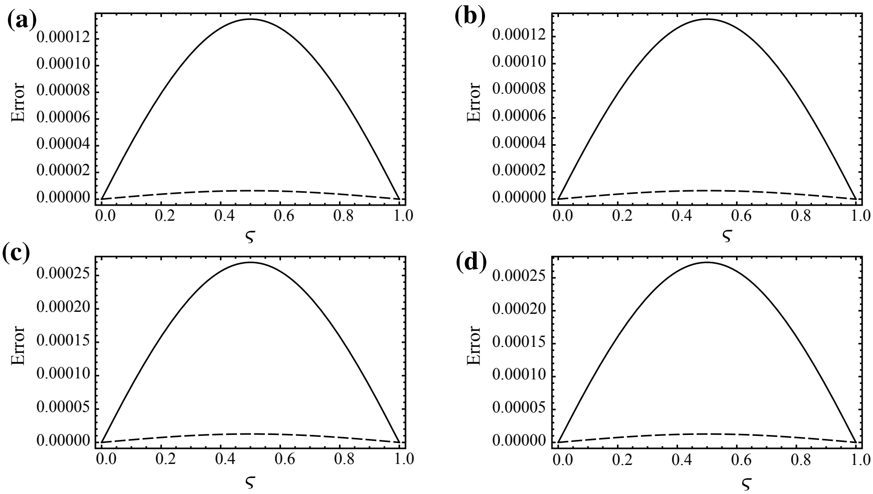

Fig. 10 The absolute error between the 6-terms of the q-HATM solutions with numerical solutions by (4.11)–(4.14) scheme for **a** α_1, **b** β_1, **c** α_2, and **d** β_2 with $h = -1.95$, $k = 0.1$, $\gamma = 0.2$, $L = 1$, $T = 1$, $\Delta\varsigma = \frac{1}{50}$, $\Delta\tau = \frac{1}{9000}$, $a_{n_1} = 0.001$, $a_{n_2} = 0.002$, $b_{n_1} = 0.001$, $b_{n_2} = 0.002$. Solid line ($n = 1$), Dashed line ($n = 5$)

$$\alpha_{i,n}^{0} = \alpha_i(\varsigma(n)) = \alpha_{i,n}, \beta_{i,n}^{0} = \beta_i(\varsigma(n)) = \beta_{i,n}, \quad i = 1, 2, \quad n = 0, 1, 2, \ldots, N_\varsigma,$$

$$\alpha_{i,0}^{j} = 1 = \alpha_{i,N}^{j}, \beta_{i,0}^{j} = 0 = \beta_{i,N}^{j}, i = 1, 2, j = 1, 2, \ldots, N_\tau.$$

Stable solutions with the (4.11)–(4.14) scheme are only obtained if $r < \frac{1}{2}$. See, e.g., [12, 28] for a proof that this condition gives the stability limit for the (4.11)–(4.14) scheme. In Fig. 10, the absolute error between the q-HATM solutions and the numerical solutions by the (4.11)–(4.14) scheme are plotted. Also, in this figure we show that the effect of the factor $\frac{1}{n}$ on the accelerate of the convergence. It is clear when n is increasing, the absolute error is decreasing.

5 Conclusion

In this paper, the q-HATM was employed to analytically compute approximate solutions of CIACS. By comparing q-HATM solutions with results by Mathimatica, the averaged residual error the residual error and finite difference method were found an excellent agreement. Also the effected on the accelerating of the convergence by the factor $\frac{1}{n}$ is shown. Mathematica was used for the computations of this article.

Competing Interests The authors declare that there is no conflict of interests regarding the publication of this paper.

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Numerical Study of Effects of Adrenal Hormones on Lymphocytes

Shikaa Samuel, Vinod Gill, Devendra Kumar and Yudhveer Singh

Abstract Lymphocytes play significant defensive role to keep the body healthy. However, there is substantial evidence that adrenal hormones such as epinephrine, norepinephrine, and cortisol generated by psychological stress suppress the activities of the immune system or alter the activation and mobilization several immune cells particularly lymphocytes during infections. Glucocorticoid receptors expressed by the immune cells makes binding those hormones possible. This work formulates a mathematical model to examine the impact of adrenal hormones on the immune system with respect to time evolution and spatial distribution cells in response to hormones concentration. The steady state of the model is studied and found to be uniformly and asymptotically stable subject to the secretion and decay rates of hormones. The numerical experiments using the free diffusion equations further investigates the dynamic behaviour of the "bound" lymphocytes secretion rate of the adrenal hormones induced by psychological stress.

Keywords Reaction–diffusion equations · Lymphocytes · Adrenal hormones · Mathematical model

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1 Introduction

The adrenal hormones are known to influence the activities of immune system in human and other animals. People exposed to life threatening issues are prone to chronic and persistent stress. For instance, a person diagnosed with any terminal disease such as HIV infection or cancer faces social and emotional challenges. Psychological stress that comes with the diagnosis of such illnesses often requires as much attention as the infection [1]. Lymphocytes are specialized white blood cells whose function is to identify and destroy invading antigens [2, 3].

The lymphocytes are vital components of the immune system alongside with macrophages, antigen receptors and antigen-presenting cells [4, 5]. Psychological stress is an unpleasant state of emotional and physiological arousal that people experience in situations that they perceive as dangerous or threatening to their well-being [6]. Psychological stress gets inside the body through the brain by the influence of the impulses via the nerve fibres that descend from the brain into the bone marrow and thymus, spleen and lymph nodes that connect with lymphoid tissues. These fibres release adrenal hormones such as epinephrine, norepinephrine, and cortisol that bind on the receptors on lymphocytes thereby changing the functionality immune system [7]. When psychological stress is excessive, prolonged and chronic, it breaks down the body's defense mechanism and leaves the body vulnerable to infections [8].

In light of the above, we propose a deterministic mathematical model to study the temporal-spatial dynamics of lymphocytes and Adrenal Hormones interaction via numerical experimentation inspired by [9, 10]. The secretions of adrenal hormones during chronic and persistent stress cases are separately examined. The rest of the paper is organized thus. In the second section, the mathematical model is proposed which is followed by tabular description of each equation. In the same section, the stability of the diffusion free system is investigated; equilibrium point obtained and studied. In addition, some estimates of the full diffusion model are also examined in appropriate Sobolev spaces. In section three, the diffusion free model is solved using classical Runge-Kutta method while the full diffusion model is solved by explicit forward in time, central in space (FTCS) method with appropriate stability condition of the scheme and the corresponding results are presented alongside. The last section is concluding remarks.

2 Formulation and Analysis of the Model Equations

2.1 Model Formulation

In these model equations, the diffusions of the respective component are modelled using Laplace operator. The zero flux boundary conditions are imposed on the system to study the phenomenon in bounded two dimensional domain Ω. $u_1(x, t)$ represents the density of normal lymphocytes at time t; $u_2(x, t)$: The concentration of adrenal

Table 1 Biological meaning of the system

Equation	Terms description	
Eq. (1)	$D_1\Delta u_1$: diffusion term for normal cells	δ: source term for normal cells
	βu_3: proportion of bound cells that revert to normal	$-\alpha_1 u_1 u_2$: Proportion of normal cells upon which adrenal hormone bound
	$-\mu_1 u_1$: proportion of dead normal cells	
Eq. (2)	$D_2\Delta u_2$: diffusion term for adrenal hormones	$\alpha_2 u_2$: natural secretion of adrenal hormones
	$-\mu_2 u_2$: decay term for adrenal hormones	$\varepsilon(x,t)$: secretion of adrenal hormones by psychological stress
Eq. (3)	$D_3\Delta u_3$: diffusion term for bound cells	
	$-(\beta+\mu_3)u_3$: sum of proportions of dead bound cells and those that revert to normal	$\alpha_1 u_1 u_2$: proportions cells bound by natural and stress induced secretions

hormones at time t; $u_3(x,t)$: The density of bound lymphocytes at time t.

$$\frac{\partial u_1}{\partial t} = D_1\Delta u_1 + \delta + \beta u_3 - \alpha_1 u_1 u_2 - \mu_1 u_1 \tag{1}$$

$$\frac{\partial u_2}{\partial t} = D_2\Delta u_2 + \alpha_2 u_2 - \mu_2 u_2 + \varepsilon(x,t) \tag{2}$$

$$\frac{\partial u_3}{\partial t} = D_3\Delta u_3 + \alpha_1 u_1 u_2 - (\beta+\mu_3)u_3 \tag{3}$$

$$u_1(.,0) = u_1^0,\, u_2(.,0) = u_2^0,\, u_3(.,0) = u_3^0, \quad \text{in } \Omega$$

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \frac{\partial u_3}{\partial n} = 0 \quad \text{on } \partial\Omega$$

where $\alpha_1, \alpha_2, \beta, \mu_1, \mu_2, \mu_3 > 0, D_1, D_2, D_3 \geq 0,\ \varepsilon(x,t) \to 0$ as $t \to \infty$.

We described system, Eqs. (1)–(3) term by term in Table 1 and parameter values given in Table 2.

2.2 Model Analysis

Here, the steady state solutions of the ODE system is obtain and the system linearize around the equilibrium point. The eigenvalues of the associated matrix of the linearized system determines stability as in [14]. In the case of the PDE, we obtained L^2 and L^∞ estimates.

Table 2 Parameter values

Symbol	Description	Values
α_1	Binding rate of adrenal hormones on lymphocytes	0.002/day
α_2	Natural secretion rate of adrenal hormones	0.04/day [11]
δ	Source term form lymphocytes	3.63E02cells [12]
β	Rate bound lymphocytes revert to normal	0.001/day
μ_1	Death rate of normal lymphocytes	0.06/day [12]
μ_2	The adrenal hormone decay rate	0.1/day [11]
μ_3	Death rate of bound lymphocytes	0.06/day
$\varepsilon(x, t)$	Secretion of adrenal hormones induced by psychological stress	
D_1	Diffusion coefficient of lymphocytes	0.0045 mm^2/day [13]
D_2	Diffusion coefficient of adrenal hormones	0.0052 mm^2/day
D_3	Diffusion coefficient of bound lymphocytes	0.0045 mm^2/day [13]

Theorem 1 *For $\mu_2 > \alpha_2$ and $\frac{\varepsilon^*}{\mu_2-\alpha_2} \geq \frac{\mu_1}{\alpha_1}$, the system, Eqs. (1)–(3) admits a spatially homogeneous steady state $\wp\left(u_1^*, u_2^*, u_3^*\right)$.*

Proof Assume that diffusion of the component decrease slowly to a negligible value, then at equilibrium state, set $\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} = \frac{\partial u_3}{\partial t} = 0$, we have

$$0 = \delta + \beta u_3 - \alpha_1 u_1 u_2 - \mu_1 u_1, \tag{4}$$

$$0 = \alpha_2 u_2 - \mu_2 u_2 + \varepsilon^*, \tag{5}$$

$$0 = \alpha_1 u_1 u_2 - (\beta + \mu_3) u_3, \tag{6}$$

Solving Eqs. (4)–(6) simultaneously, we obtain positive equilibrium values

$$u_1^* = \frac{\delta(\beta + \mu_3)}{(\beta + \mu_3)\left(\alpha_1 u_2^* - \mu_1\right) + \beta\alpha_1 u_2^*}, \tag{7}$$

$$u_2^* = \frac{\varepsilon^*}{\mu_2 - \alpha_2}, \tag{8}$$

$$u_3^* = \frac{\delta\alpha_1 u_2^*}{(\beta + \mu_3)\left(\alpha_1 u_2^* - \mu_1\right) + \beta\alpha_1 u_2^*}. \tag{9}$$

provided $\mu_2 > \alpha_2$ and $\frac{\varepsilon^*}{\mu_2-\alpha_2} \geq \frac{\mu_1}{\alpha_1}$, hence the proof.

Theorem 2 *Let $\boldsymbol{u}' = \boldsymbol{J}\boldsymbol{u}$ be a linearized system of Eqs. (11)–(13). Suppose that the Jacobian matrix $\boldsymbol{J}$ is a constant matrix, with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and $Re(\lambda_i) < 0$*

for all $i = 1, 2, 3$*, then the spatially homogeneous steady state* $\wp\big(u_1^*, u_2^*, u_3^*\big)$ *of Eqs. (1)–(3) is uniformly and asymptotically stable.*

Proof Now, let the kinetic parts of Eqs. (1)–(3) be expressed as below:

$$\varphi_1 = \delta + \beta u_3 - \alpha_1 u_1 u_2 - \mu_1 u_1, \varphi_2 = \alpha_1 u_2 - \mu_2 u_2 + \varepsilon(x, t),$$
$$\varphi_3 = \alpha_1 u_1 u_2 - (\beta + \mu_3) u_3.$$

Then, the Jacobian matrix evaluated at (u_1^*, u_2^*, u_3^*) is given by

$$J(u_1^*, u_2^*, u_3^*) = \begin{pmatrix} -\alpha_1 u_2 - \mu_1 & -\alpha_1 u_1 & \beta \\ 0 & \alpha_1 - \mu_2 & 0 \\ \alpha_1 u_2 & \alpha_1 u_1 & -(\beta + \mu_3) \end{pmatrix}.$$

Now, we solve for the eigenvalues from the characteristics equation as follows:

$$\big|J(u_1^*, u_2^*, u_3^*) - \lambda I\big| = \begin{vmatrix} -\alpha_1 u_2^* - \mu_1 - \lambda & -\alpha_1 u_1^* & \beta \\ 0 & \alpha_1 - \mu_2 - \lambda & 0 \\ \alpha_1 u_2^* & \alpha_1 u_1^* & -(\beta + \mu_3) - \lambda \end{vmatrix} = 0, \tag{10}$$

where λ is the eigenvalues while I is the 3×3 identity matrix. This leads to the characteristic equation

$$(\alpha_1 - \mu_2 - \lambda)\big(\lambda^2 - (A_1 + A_2)\lambda + A_1 A_2 + A_3\big) = 0, \tag{11}$$

where $A_1 = -\alpha_1 u_2^* - \mu_1, A_2 = -(\beta + \mu_3), A_3 = -\beta\alpha_1 u_2^*$.

Solving (11), we obtained the following eigenvalues

$$\lambda_1 = \alpha_1 - \mu_2, \lambda_2 = \frac{(A_1 + A_2) - \sqrt{(A_1 + A_2)^2 - 4(A_1 A_2 + A_3)}}{2},$$
$$\lambda_3 = \frac{(A_1 + A_2) + \sqrt{(A_1 + A_2)^2 - 4(A_1 A_2 + A_3)}}{2} \tag{12}$$

It remains to check whether the real parts of Eq. (12) are negative. Clearly, λ_1 and λ_2 are both negative but $\lambda_3 < 0$ if and only if

$$(A_1 + A_2) > \sqrt{(A_1 + A_2)^2 - 4(A_1 A_2 + A_3)}. \tag{13}$$

This suffices to show that $(A_1 A_2 + A_3) > 0$

$$A_1 A_2 + A_3 = \alpha_1(\beta + \mu_3) u_2^* + \mu_1(\beta + \mu_3) - \beta\alpha_1 u_2^* \tag{14}$$

Since $u_2^* > 0$ and β, μ_3 are positive constants, *therefore*

$$A_1A_2 + A_3 > \beta\alpha_1 u_2^* - \beta\alpha_1 u_2^* + \mu_1(\beta + \mu_3) = \mu_1(\beta + \mu_3) > 0, \Rightarrow \lambda_3 < 0. \tag{15}$$

Since all the eigenvalues Eq. (12) are negative, the system is uniformly and asymptotically stable around the equilibrium point (u_1^*, u_2^*, u_3^*) and this completes the proof.

Now, we define the time dependent Sobolev spaces to enable us obtain the estimates.

Definition 3 [15]: let X be a generic nonempty set and $1 \leq p < \infty$

$$C([0, T]; X) := \{u | u : [0, T] \to \mathrm{X} \text{ continuous }\}, \tag{16}$$

$$L^p(0, T; X) := \left\{ u | u \text{ measurable}, \int_0^T \|u(t)\|^p dt < \infty \right\}. \tag{17}$$

For an integer $m > 0$ and real p with $1 \leq p < \infty$ and $X = \Omega \subset \mathbb{R}^2$, we define the Sobolev space

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq m \right\} \tag{18}$$

equipped with the following norms

$$u_{W^{m,p}(\Omega)} := \begin{cases} \left(\sum\limits_{|\alpha| \leq m} \int\limits_\Omega |D^\alpha u|^p dx \right)^{1/p} & 1 \leq p < \infty, \\ \sum\limits_{|\alpha| \leq m} ess \sup\limits_\Omega |D^\alpha u| & p = \infty, \end{cases} \tag{19}$$

$$|u|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| = m} \int_\Omega |D^\alpha u|^p dx \right)^{1/p} \quad 1 \leq p < \infty. \tag{20}$$

Now, for $p = 2$, a Hilbert space is defined $W^{m,2}(\Omega) = H^m(\Omega)$ with the inner product

$$(u, v)_{m,\Omega} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{0,\Omega}. \tag{21}$$

$H_0^1(\Omega) = \left\{ u \in H^1 | u = 0 \text{ on } \partial\Omega \right\}$ with dual $H^{-1}(\Omega)$.

Theorem 4 *Let* $u_1^0, u_2^0, u_3^0 \in L^2(\Omega)$ *and* $(x, t) \in L^2\big(0, T; L^2(\Omega)\big)$, *then* $u_1, u_2, u_3 \in L^2\big(0, T; H_0^1(\Omega)\big)$ *with* $\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_3}{\partial t} \in L^2\big(0, T; H_0^{-1}(\Omega)\big)$, *furthermore the estimates*

$$\begin{pmatrix} \|u_1(t)\|_{L^2(H_0^1)} \\ \|u_2(t)\|_{L^2(H_0^1)} \\ \|u_3(t)\|_{L^2(H_0^1)} \end{pmatrix} \text{ and } \begin{pmatrix} \|u_1(t)\|_{L^\infty(L^2)} \\ \|u_2(t)\|_{L^\infty(L^2)} \\ \|u_3(t)\|_{L^\infty(L^2)} \end{pmatrix}$$

are bounded by the data.

Proof Multiplying Eq. (1) by u_1 and integrating over the domain, we have

$$\int_\Omega \frac{\partial u_1}{\partial t} u_1 = D_1 \int_\Omega \Delta u_1 u_1 + \int_\Omega \delta u_1 + \beta \int_\Omega u_3 u_1 - \alpha_1 \int_\Omega u_1 u_2 u_1 - \mu_1 \int_\Omega u_1 u_1 \quad (22)$$

Using the Young's inequality, integrating the first term on the right hand side by parts, applying the boundary condition and dropping the negative terms, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega u_1^2 + D_1 \int_\Omega \nabla u_1 . \nabla u_1 \leq \frac{1}{2}\int_\Omega \delta u_1^2 + \frac{\beta}{2}\int_\Omega \left(u_3^2 + u_1^2\right), \quad (23)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_1(t)\|_{L^2}^2 + D_1\|u_1(t)\|_{H_0^1}^2 \leq \frac{\delta+\beta}{2}\|u_1(t)\|_{L^2}^2 + \frac{\beta}{2}\|u_3(t)\|_{L^2}^2 \quad (24)$$

In the same manner for Eqs. (2) and (3), we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_2(t)\|_{L^2}^2 + D_2\|u_2(t)\|_{H_0^1}^2 \leq \frac{\alpha_2}{2}\|u_2(t)\|_{L^2}^2 + \|\varepsilon(.,t)\|_{L^2}\|u_2(t)\|_{L^2} \quad (25)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_3(t)\|_{L^2}^2 + D_3\|u_3(t)\|_{H_0^1}^2 \leq \frac{\alpha_1}{2}\left(\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 + \|u_3(t)\|_{L^2}^2\right) \quad (26)$$

Writing Eqs. (24)–(26) in vector form, we realize

$$\frac{1}{2}\frac{\partial}{\partial t}\begin{pmatrix} \|u_1(t)\|_{L^2}^2 \\ \|u_2(t)\|_{L^2}^2 \\ \|u_3(t)\|_{L^2}^2 \end{pmatrix} + \begin{pmatrix} \|u_1(t)\|_{H_0^1}^2 \\ \|u_2(t)\|_{H_0^1}^2 \\ \|u_3(t)\|_{H_0^1}^2 \end{pmatrix} \leq M \begin{pmatrix} \|u_1(t)\|_{L^2}^2 \\ \|u_2(t)\|_{L^2}^2 \\ \|u_3(t)\|_{L^2}^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \|\varepsilon(.,t)\|_{L^2}\|u_2(t)\|_{L^2} \\ 0 \end{pmatrix} \quad (27)$$

$$M = \max\left(\frac{\delta+\alpha_1+\beta}{2}, \frac{\alpha_1+\alpha_2}{2}, \frac{\alpha_1+\beta}{2}\right) = \frac{\delta+\alpha_1+\beta}{2}.$$

Now integrating in time and using of Cauchy-Schwarz inequality leads to

$$\frac{1}{2}\begin{pmatrix} \|u_1(T)\|_{L^2}^2 \\ \|u_2(T)\|_{L^2}^2 \\ \|u_3(T)\|_{L^2}^2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \|u_1^0\|_{L^2}^2 \\ \|u_2^0\|_{L^2}^2 \\ \|u_3^0\|_{L^2}^2 \end{pmatrix} + \begin{pmatrix} \|u_1(t)\|_{L^2(H_0^1)}^2 \\ \|u_2(t)\|_{L^2(H_0^1)}^2 \\ \|u_3(t)\|_{L^2(H_0^1)}^2 \end{pmatrix}$$

$$\leq M\begin{pmatrix}\|u_1(t)\|^2_{L^2(L^2)}\\ \|u_2(t)\|^2_{L^2(L^2)}\\ \|u_3(t)\|^2_{L^2(L^2)}\end{pmatrix}+\begin{pmatrix}0\\ \|\varepsilon(.,t)\|_{L^2(L^2)}\|u_2(t)\|_{L^2(L^2)}\\ 0\end{pmatrix} \tag{28}$$

Using Poincare's inequality on the right hand side of Eq. (28) and that

$$\begin{pmatrix}\|u_1(t)\|_{L^2(H_0^1)}\\ \|u_2(t)\|_{L^2(H_0^1)}\\ \|u_3(t)\|_{L^2(H_0^1)}\end{pmatrix}\leq C\left\{\begin{pmatrix}0\\ \|\varepsilon\|_{L^2(L^2)}\\ 0\end{pmatrix}+\begin{pmatrix}\|u_1^0\|^2_{L^2}\\ \|u_2^0\|^2_{L^2}\\ \|u_3^0\|^2_{L^2}\end{pmatrix}\right\}<\infty \tag{29}$$

$$\begin{pmatrix}\|u_1(t)\|_{L^\infty(L^2)}\\ \|u_2(t)\|_{L^\infty(L^2)}\\ \|u_3(t)\|_{L^\infty(L^2)}\end{pmatrix}\leq C\left\{\begin{pmatrix}0\\ \|\varepsilon\|_{L^2(L^2)}\\ 0\end{pmatrix}+\begin{pmatrix}\|u_1^0\|^2_{L^2}\\ \|u_2^0\|^2_{L^2}\\ \|u_3^0\|^2_{L^2}\end{pmatrix}\right\}<\infty \tag{30}$$

3 Numerical Solution

3.1 Diffusion Model

Assuming the diffusions of respective interacting components decrease to zero; we solve the resulting system of ordinary differential equations using classical Runge-Kutta method:

$$\frac{du_1}{dt}=\delta+\beta u_3-\alpha_1u_1u_2-\mu_1u_1 \tag{31}$$

$$\frac{du_2}{dt}=\alpha_2u_2-\mu_2u_2+\varepsilon(t) \tag{32}$$

$$\frac{du_3}{dt}=\alpha_1u_1u_2-(\beta+\mu_3)u_3 \tag{33}$$

$$u_1(0)=u_1^0,\ u_2(0)=u_2^0,\ u_3(0)=u_3^0,$$

Now, let $u=(u_1,u_2,u_3)$ and $t_{n+1}=t_n+h, n=0,1,2\ldots,$ the fourth order Runge-Kutta [16]

$$u_{n+1}=u_n+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1=f(t_n,u_n)$$

$$k_2 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2\right)$$

$$k_4 = f(t_n + h, u_n + hk_3)$$

3.1.1 Chronic Stress

Using the initial values $u_1(0) = 3.6E02, u_2(0) = 2.8, u_3(0) = 0$ and parameter values in Table 1. In case of chronic stress transient function

$$\varepsilon(t) = t^2 e^{-0.15t}$$

We infer from Fig. 1 below that, in the scenario where $\alpha_2 = 0.04$, $\mu_2 = 0.1$, in line the stability condition $\mu_2 > \alpha_2$, the density of normal lymphocytes (blue line) struggled initially but eventually recover from stress induced secretion of adrenal hormones. The second scenario is when $\mu_2 < \alpha_2$ which is against the stability condition. Here, the escalation of concentration of adrenal hormones leads to exponential increase of number of bound lymphocytes which spell abnormal immune response or reaction. This causes activation or inhibition depending on the particular hormone and lymphocyte involved. This will lead to chronic stress related complications such

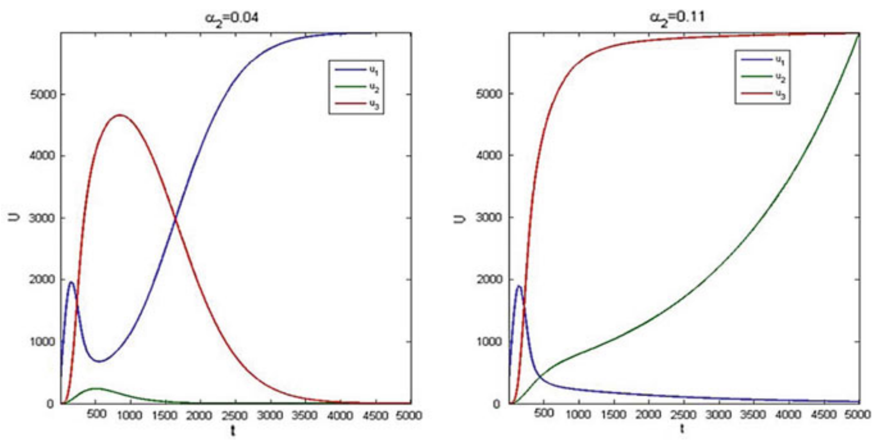

Fig. 1 Density of normal lymphocytes $u_1(x, t)$ (blue); the concentration of adrenal hormones $u_2(x, t)$ (green) during chronic stress; the density of bound lymphocytes $u_3(x, t)$ (red) for $\alpha_2 = 0.04$ and $\alpha_2 = 0.11$

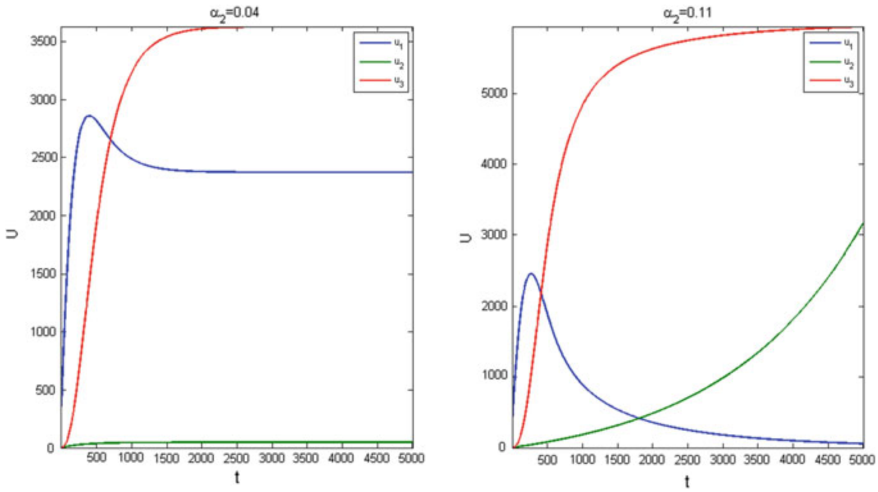

Fig. 2 Density of normal lymphocytes $u_1(x, t)$ (blue); the concentration of adrenal hormones $u_2(x, t)$ (green) during persistent stress; the density of bound lymphocytes $u_3(x, t)$ (red) for $\alpha_2 = 0.04$ and $\alpha_2 = 0.11$

as high blood pressure, hypertension and diabetes. However, the second scenario can be explored to treat hyper immune reaction related diseases.

3.1.2 Persistent Stress

Here, a constant function is used with the same value as the initial value i.e. $\varepsilon(t) = 2.8$. This is to emphasize that the fact, the initial concentration of adrenal hormones persisted for a period of time. The numerical results shown in Fig. 2, further illustrated that, even when the stability condition $\mu_2 > \alpha_2$ is satisfied with $\mu_2 = 0.1$, $\alpha_2 > 0.04$, the normal lymphocytes cannot recover back to original density. In case of $\mu_2 < \alpha_2$, the density of normal lymphocytes crashed. The two cases will lead to stress related complications.

3.2 Full Diffusion Model

We use an explicit forward in time, central in space (FTCS) method [17] to solve the system. Let the compact form Eqs. (1)–(3) be given as

$$\frac{\partial U}{\partial t} = D\Delta U + F(U), \tag{34}$$

such that the two dimensions discretize form of Eq. (34) reduces to

$$\frac{U_{ij}^{n+1} - U_{ij}^{n}}{\Delta t} = D\left(\underbrace{\frac{U_{i+1j}^{n} - 2U_{ij}^{n} + U_{i-1j}^{n}}{\Delta x^2} + \frac{U_{ij+1}^{n} - 2U_{ij}^{n} + U_{ij-1}^{n}}{\Delta y^2}}_{\widetilde{\Delta} U_{ij}^{n}}\right) + F(U^n) \tag{35}$$

This scheme has the stability condition [15]

$$\Delta t \leq \frac{\Delta x^2 \Delta y^2}{2D\big(\Delta x^2 + \Delta y^2\big)}$$

$$u_{1ij}^{n+1} = u_{1ij}^{n} + \Delta t\Big(D_1 \widetilde{\Delta} u_{1ij}^{n} + \delta + \beta u_{3ij}^{n} - \alpha_1 u_{1ij}^{n} u_{2ij}^{n} - \mu_1 u_{1ij}^{n}\Big) \tag{36}$$

$$u_{2ij}^{n+1} = u_{2ij}^{n} + \Delta t\Big(D_2 \widetilde{\Delta} u_{2ij}^{n} + (\alpha_2 - \mu_2) u_{2ij}^{n} + \varepsilon(x_n, y_n, t_n)\Big) \tag{37}$$

$$u_{3ij}^{n+1} = u_{3ij}^{n} + \Delta t\Big(D_3 \widetilde{\Delta} u_{3ij}^{n} + \alpha_1 u_{1ij}^{n} u_{2ij}^{n} - (\beta + \mu_3) u_{3ij}^{n}\Big) \tag{38}$$

In Theorem 1, this two conditions $\mu_2 > \alpha_2$ and $\frac{\varepsilon^*}{\mu_2 - \alpha_2} \geq \frac{\mu_1}{\alpha_1}$ must be satisfied for the positivity of the solution. The parameter values are taken from Table 2 and we used the initial conditions

$$u_1(x, y, 0) = e^{-0.7(x+2)^2 - 0.7(y+2)^2}, u_2(x, y, 0) = 2.8, u_3(x, y, 0) = 0 \tag{39}$$

Note that, from the initial conditions, it is assumed that, in a square domain $\Omega = [-4, 4]^2$ the initial population of normal lymphocytes is densed at $x = -2, y = -2$ and the average concentrations adrenal hormone is constant. Also, it is assume that the secretion of adrenal hormones induced by psychological stress is transient given by

$$\varepsilon(x, y, t) = \cos\Big(\frac{\pi x}{2}\Big)\cos\Big(\frac{\pi y}{2}\Big)t^2 e^{-0.15t} \tag{40}$$

It is observed from Sect. 3.1 that, the system is highly sensitive to the net secretion rate of the adrenal hormone. Here, a dynamic behaviour is also observed in Figs. 3, 4 and 5 shown at $t = 1.25$ and $t = 2.5$ for each component. Particularly, our light is beamed on the density of bound lymphocytes. It is inferred in Fig. 5 that, bound lymphocytes are more densed at areas of high concentration of adrenal hormones. Indeed, this is in consonance with previous results on cortisol association with T cell activation during HIV infection [9].

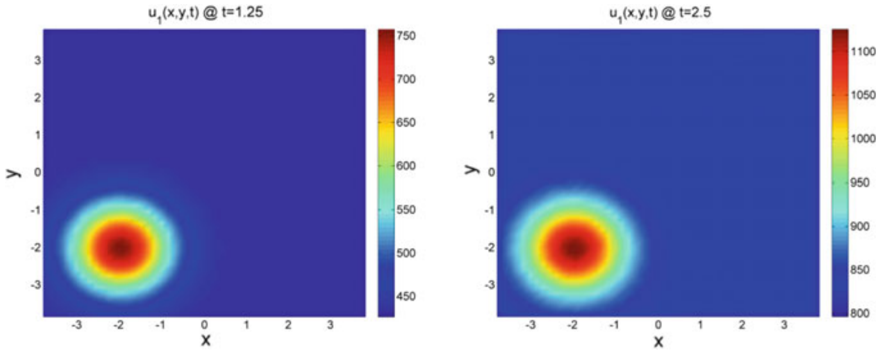

Fig. 3 Density of normal lymphocytes $u_1(x, t)$ at $t = 1.25$ and $t = 2.5$

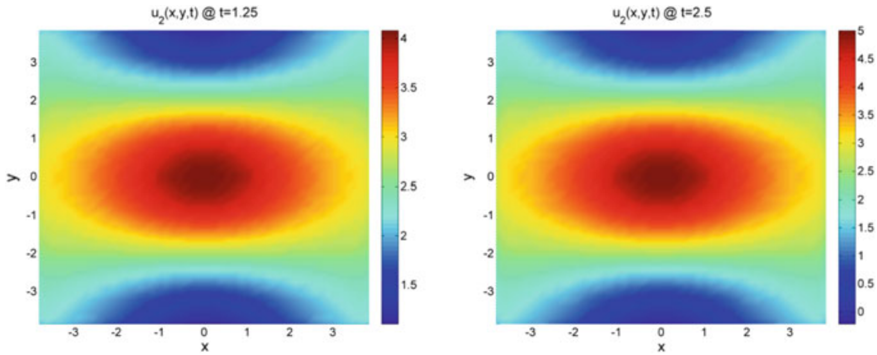

Fig. 4 The concentration of adrenal hormones $u_2(x, t)$ at $t = 1.25$ and $t = 2.5$

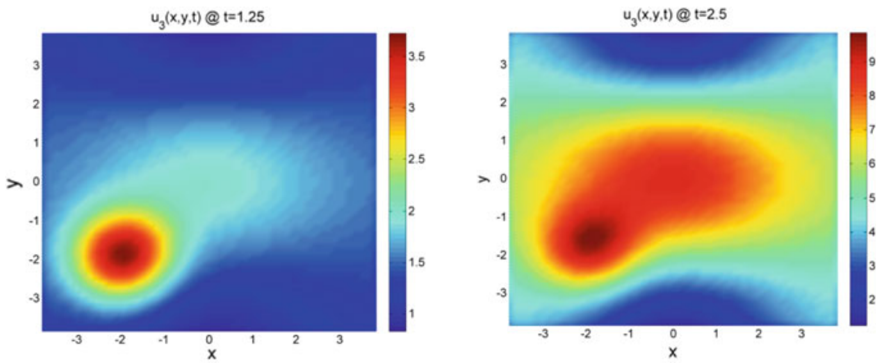

Fig. 5 The density of bound lymphocytes $u_3(x, t)$ at $t = 1.25$ and $t = 2.5$

4 Conclusion

In this paper, a coupled system of reaction-diffusion equations to study the interaction of adrenal hormones induced by psychological stress on the human immune system has been formulated. The system has only one critical point which is proved to be uniformly and asymptotically stable (UAS) under certain prescribed constrains $\mu_2 > \alpha_2$ and $\frac{\varepsilon^*}{\mu_2-\alpha_2} \geq \frac{\mu_1}{\alpha_1}$. Numerical solutions have further shown that, increase in net secretion rate of stress absorbing hormones has great negative effect on the human immune cell. Further research can be carried out in connection with other terminal disease models such as cancer and HIV.

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Mathematical Modelling of Poor Nutrition in the Human Life Cycle

Ebenezer Bonyah, Kojo Ababio and Patience Pokuaa Gambrah

Abstract Nutrition is very crucial in the survival of human race and more importantly the development of a child from the womb to adulthood. In some instances, the age of the individuals determines the kind of nutrients required. Therefore, the human cycle has something to do with the nutrients obtained. We formulate a mathematical model as a system of non-linear ordinary differential equations to investigate the effects of poor nutrition from conception to adulthood using the poor pregnant woman nutrient status. The steady states are studied and R_0 of poor nutrition in the society are calculated. To keep the society healthy and free of malnutrition, malnourished pregnant females are encouraged to eat foods that contain all the nutrients needed for development. The model is supported with numerical simulation.

Keywords Nutrition · Reproduction number · Pregnant women · Steady states · Conception

1 Introduction

Malnutrition means basically an individual who is over or under nutrition. The World Food Programme (WFP) classifies malnutrition as a condition where the physical function of the human body cannot be performed such as normal growth, pregnancy, recovery from injury and diseases [1, 2].

From conception through pregnancy, birth, childhood, adolescence and adulthood, nutrition plays a vital role in every stage which supports health and wellness and improving the quality of life. Good nutrition for pregnant women plays an important

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role in giving birth to a healthy baby and also ensuring good health status of the nursing mother [2]. The effects of poor nutrition begin in the womb, continues well into childhood, adulthood and cycles across generations [3, 4]. When a pregnant woman is malnourished in essential elements such as potassium, calcium, iodine, and others the unborn baby begins to face challenges in the proper development which could affect such individual to the adulthood. There is a positive correlation between the health status of the pregnant mother and the health status of the child to be born. In many cases, the medical health practitioners are able to detect and advice appropriately the food and other activities necessary for the health of the child [5]. A malnourished pregnant woman is provided with micro-nutrients in order to improve the nutrient status which will lead to healthy pregnancy. Anaemia is very dangerous for pregnant woman and all effort must be put in place to reduce or prevent which will bring about health growth with the right birth weights. Good birth weights depict the healthy status of the baby which will lead to the proper development of the child [3].

About one billion and nine hundred million (1.9 billion) adults worldwide are deemed overweight, while four hundred and sixty-two (462) millionaire also known to be underweight [6–10]. It has been found from studies that approximately 41 million children under the age of five (5) years are considered overweight or obese. In addition, 159 million children in the entire world are found to be stunted and 50 million also identified as wasted. Adding to this burden, are the 528 million or 29% of women of reproductive age around the world affected by anaemia, for which approximately half would be amenable to iron supplementation [11].

Nutrition has been identified to be a major factor in every human stages of development from conception to adulthood. For one to become healthy in a society, good nutrition is therefore required especially for pregnant women since the life cycle starts from conception through pregnancy. Malnutrition is the most serious and common health problem that occurs when a person's diet does not contain the proper amounts of nutrients. Mathematical modelling has become an indispensable tool in investigating many scientific processes in the world including social, health, economic etc. that addresses challenges in the absence of real data in the society.

Nita et al. [12], formulated a mathematical model in order to analyze transmission dynamics of malnutrition and underweight individuals in pregnant women in the society. They calculated the basic reproduction number R_0 at the equilibrium state of the model which decided the existence of malnutrition and underweight in the society. Local stability, global stability and numerical simulation were done for this model. Their result suggested that, to live a better and healthy life, one must consume healthy and nutritive food. They further suggested that in future work, deciding (optimum) dosage of nutrients at the early stage and incorporate different layers of the society to have more realistic analysis.

Nita et al. [13], proposed a transmission model of poor nutrition in the human life cycle to study the spread of poor nutrition at different stages of life from a malnutrited pregnant female. They modelled the sample fertile female population using the application of SEIR model constructed as a system of non-linear ordinary differential equations for the various compartments. They calculated the basic reproduction number R_0 at an endemic equilibrium point which decided the existence of poor nutrition in the society. Local stability, global stability and numerical simulation were done for this model. Their results suggested that the transmission rate of healthy pregnant female giving birth to low weight babies due to pregnancy complications contribute largely to making a poor nutritional life cycle in the society.

Senelani et al. [14], constructed a mathematical model to explore the effect of malnutrition on the spread of cholera. In their study, both nourished and malnourished individuals were included in the model as those susceptible and infected of cholera respectively. The sensitivity analysis carried out in their study revealed that an increase in the number of individuals susceptible to cholera as a result of malnutrition led to a higher number of cholera infected individuals in a community. They concluded that nutritional related matters should be attended to immediately so as to improve the nutrition status of the rural communities affected by cholera. Diana [15], developed a model hinged on the first law of thermodynamics which basically focused on controlling and managing weight variations in the human body. In this work, the authors assumed the human body as an open system which accommodates input into the system in the form of food. They partitioned the human population into resting metabolic rate, non-exercise activity thermogenesis and dietary induced thermogenesis [15]. As a result of interactions between the various compartments a set of nonlinear ordinary differential equations were obtained. Their study quantified a metabolic adaptation because of caloric restriction that seek to defend baseline body weight.

Carson et al. [16], also constructed a model that focused on a general description with respect to general body weight, a given time interval and how the body will behave. The available data suggested that there is no clear distinction between body composition and mass, and an invariant manifold. For a constant food intake rate with the corresponding physical activity level as well as the body weight all will lead to a steady state and this matches with a unique body weight.

Dumitru et al. [17], proposed a mathematical model for poor nutrition in life cycle in humans. They used the Caputo, Atangana-Baleanu and Fabrizio derivatives on the model to investigate poor pregnant women nutrient status. They calculated the basic reproduction number R_0 at an endemic equilibrium points which decided the existence of poor nutrition in the society. The proposed model was examined in fractional derivatives sense via Caputo Fabrizio, Atanagan-Baleanu and Caputo. Comparative numerical analysis of these operators was extensively carried out and showed that Caputo and Atangana-Baleanu derivative in all alpha values produced similar results. The Fabrizio Caputo operator converged quickly as compared to the other two operator and therefore more efficient.

Milinda et al. [18], investigated nutrition status which concentrated on undernourished children in a malarial formulated model. Logistic regression was employed to explore mortality rate of malaria infection. They found out that insecticide-treated bed nets given to under-nutritioned children led to fewer malaria deaths related cases. The authors suggested that free bed net can be given to the vulnerable in the communities.

Several mathematical models on epidemiology have been formulated and analyzed, however, there are few models constructed on nutrition related issues such as underweight and overweight. The formulation and analysis of nutrition related model would go a long way to provide some qualitative information on this serious health issue.

The main aim of this work is to present a modified model on poor nutrition in the human life cycle and analyze to present some useful qualitative information for decision making process.

2 Model Formulation

In this section, we assume that malnutrition is transmissible and can be treated. Based on the above assumption, ‘we formulate mathematical model from conception through pregnancy, birth, childhood and adolescence. The total population is denoted by $N(t), \forall t > 0$. The population is divided into five compartments.

2.1 Model Diagram

The proportion of babies from malnutrited pregnant female being low weight is β_1 and high weight is $(1 - \beta_1)$. When low weight babies are not given proper breast feeding or formula feeding for the first 6 months and medical care, they grow to become child undergrowth at rate of δ. With good nutrition and health care, high weight baby and under growth children grow to become healthy adolescents at rates η and γ respectively. The induced mortality rate for malnutrition pregnant female and low birth weight is denoted by α_1 and natural mortality rate is μ. The recruitment rate into malnutrited pregnant female is Λ (Table 1).

2.2 Model Equation

The nonlinear differential equations below describes lack of nutrition from one compartment to other in Fig. 1.

$$\frac{dF_{MP}}{dt} = \Lambda - \beta_1 F_{MP} B_{LW} - (1 - \beta_1) F_{MP} B_{HW} - (\alpha_1 + \mu) F_{MP}$$

$$\frac{dB_{LW}}{dt} = \beta_1 F_{MP} B_{LW} - (\delta + \alpha_1 + \mu) B_{LW}$$

$$\frac{dB_{HW}}{dt} = (1 - \beta_1) F_{MP} B_{HW} - (\eta + \mu) B_{HW} \qquad (2.2.1)$$

Table 1 The table gives detailed explanation of the model parameter and variable

Variables	Description
N	Sample size of female in fertile stage
F_{MP}	Malnutrited pregnant female
B_{LW}	Low weight baby
B_{HW}	High weight baby
C_U	Child undergrowth
A_H	Adolescent healthy growth
Parameters	Description
Λ	Recruitment rate into malnutrited pregnant female
β_1	Proportion of babies from F_{MP} being low weight baby
$(1-\beta_1)$	Proportion of babies from F_{MP} being high weight baby
α_1	Induced death rate of F_{MP} and low birth weight
δ	Rate at which individuals from B_{LW} moves to C_U compartment
η	Rate at which individuals from B_{HW} grows to A_H compartment
γ	Rate at which individuals from C_U grows to A_H compartment
μ	Natural death rate

$$\frac{dC_U}{dt} = \delta B_{LW} - (\gamma + \mu)C_U$$

$$\frac{dA_H}{dt} = \eta B_{HW} + \gamma C_U - \mu A_H$$

The total population at time t is represented by $N(t)$, $N(t) = F_{MP} + B_{LW} + B_{HW} + C_U + A_H$,

$$\text{Then, } \frac{dN}{dt} = \Lambda - \alpha_1(F_{MP} + B_{LW}) - \mu N. \tag{2.2.2}$$

2.3 *Model Analysis*

In this section, we verify some basic properties and perform stability analysis of the system (2.2.1).

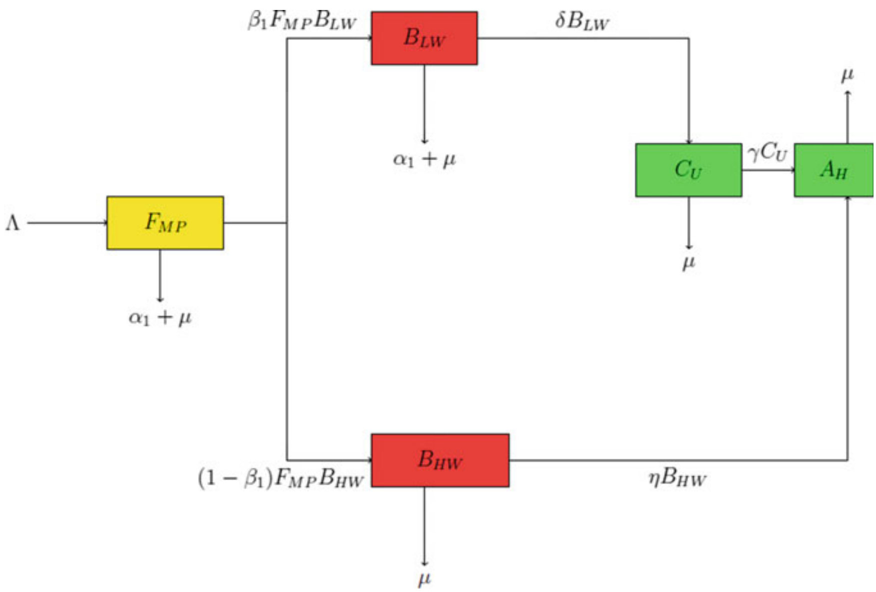


Fig. 1 Compartmental model for the transmission of poor nutrition where arrows with head means moving out or coming in of a compartment

2.3.1 Boundedness of the Solution

Lemma 2.3.1 The closed set $\phi = \left\{ (F_{MP}, B_{LW}, B_{HW}, C_U, A_H) \in \mathbb{R}^5_+ : N \leq \frac{\Lambda_H}{\mu} \right\}$ is positively invariant with respect to model (2.2.1).

Proof Assuming $(F_{MP}, B_{LW}, B_{HW}, C_U, A_H) \in \mathbb{R}^5_+$ for all $t > 0$, we want to prove that the region ϕ is positively invariant so that it becomes sufficient to look at the dynamics of the system (2.2.1). From Eq. (2.2.2), we have the rate at which the total population changes over time as:

$$\frac{dN}{dt} = \Lambda - \alpha_1(F_{MP} + B_{LW}) - \mu N$$

In the absence of malnutrition, this equation can be rewritten as

$$\frac{dN}{dt} + \mu N = \Lambda \tag{2.3.1}$$

Solving the differential equation using the integrating factor, we obtain

$$N(t) = \frac{\Lambda}{\mu} + Ke^{-\mu t}$$

Using initial conditions, $t = 0$, $N(0)$, we have

$$N(t) = \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu}\right)e^{-\mu t}$$

$$N(t) = N(0)e^{-\mu t} + \frac{\Lambda}{\mu}\left(1 - e^{-\mu t}\right) \tag{2.3.2}$$

as $t \to \infty$:

$$\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu}$$

if $N(0) \leq \frac{\Lambda}{\mu}$, then we have $N(t) = \frac{\Lambda}{\mu}$, $\forall t > 0$ as $t \to \infty$.

Also, if $N(0) > \frac{\Lambda}{\mu}$, then the solutions $(F_{MP}(t), B_{LW}(t), B_{HW}(t), C_U(t), A_H(t))$ in the region ϕ is positively invariant. We conclude from this theorem that it is sufficient to deal with the dynamics of system (2.2.1) in ϕ. Based on that, the model can be assume to be epidemiologically well-posed for mathematical analysis [19].

2.4 Analysis of Malnutrition Free Steady State

Equation 2.4.1 Steady state. At steady state, we assume the population is constant over time. We determine the steady state of malnutrition free by putting the right hand side of system (2.2.1) to zero.

$$\begin{aligned}
\Lambda - \beta_1 F_{MP} B_{LW} - (1-\beta_1) F_{MP} B_{HW} - (\alpha_1 + \mu) F_{MP} &= 0 \\
\beta_1 F_{MP} B_{LW} - (\delta + \alpha_1 + \mu) B_{LW} &= 0 \\
(1-\beta_1) F_{MP} B_{HW} - (\eta + \mu) B_{HW} &= 0 \\
\delta B_{LW} - (\gamma + \mu) C_U &= 0 \\
\eta B_{HW} + \gamma C_U - \mu A_H &= 0
\end{aligned} \tag{2.4.1}$$

Therefore the malnutrition free equilibrium (MFE) is given by

$$E_0 = \left(F_{MP}^0, 0, 0, 0, 0\right) = \left(\frac{\Lambda}{(\mu + \alpha_1)}, 0, 0, 0, 0\right)$$

Solving the equations, we obtained endemic state of the system.

2.5 The Basic Reproduction Number

The basic reproduction number R_{LW}, which provides some useful information on the spread of disease was computed in this work [20]. The next generation matrix approach was employed to drive the threshold R_{0LW} which is given by:

$$R_{LW} = \rho\left(FV^{-1}\right)$$
$$R_{LW} = \frac{\beta_1 \Lambda}{(\mu + \alpha_1)(\delta + \alpha_1 + \mu)} + \frac{(1-\beta_1)\Lambda}{(\mu + \alpha_1)(\eta + \mu)}$$

2.6 Stability Analysis of Steady States

Let

$$\phi = \left\{ (F_{MP}, B_{LW}, B_{HW}, C_U, A_H) \in \mathbb{R}^5_+ : N \leq \frac{\Lambda_H}{\mu} \right\}$$

2.6.1 Local Stability of Malnutrition Free Steady State

Theorem 2.6.2 *The malnutrition free equilibrium* (E_0) *is locally asymptotically stable if* $R_{WL} < 1$ *and unstable if* $R_{WL} > 1$.

Proof The Jacobian matrix of the system (2.2.1) is given by

$$J = \begin{bmatrix} -(\alpha_1 + \mu) - \beta_1 B_{LW} - (1-\beta_1)B_{HW} & -\beta_1 F_{MP} & -(1-\beta_1)F_{MP} & 0 & 0 \\ \beta_1 B_{LW} & p\beta_1 F_{MP} - (\delta + \alpha_1 + \mu) & 0 & 0 & 0 \\ (1-\beta_1)B_{HW} & 0 & (1-\beta_1)F_{MP} - (\eta + \mu) & 0 & 0 \\ 0 & \delta & 0 & -(\gamma + \mu) & 0 \\ 0 & 0 & \eta & \gamma & -\mu \end{bmatrix}$$

Evaluating at the malnutrition free equilibrium point gives

$$J(E_0) = \begin{bmatrix} -(\alpha_1 + \mu) & -\frac{\beta_1 \Lambda}{(\mu+\alpha_1)} & -\frac{(1-\beta_1)\Lambda}{(\mu+\alpha_1)} & 0 & 0 \\ 0 & \frac{\beta_1 \Lambda}{(\mu+\alpha_1)} - (\delta + \alpha_1 + \mu) & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\beta_1)\Lambda}{(\mu+\alpha_1)} - (\eta + \mu) & 0 & 0 \\ 0 & \delta & 0 & -(\gamma + \mu) & 0 \\ 0 & 0 & \eta & \gamma & -\mu \end{bmatrix}$$

From the Jacobian matrix we obtain the eigenvalues as follows:

$$\lambda_1 = -(\alpha_1 + \mu), \lambda_2 = \frac{(\mu + \alpha_1)(\delta + \alpha_1 + \mu) - \beta_1 \Lambda}{(\mu + \alpha_1)}, \lambda_3 = \frac{(\mu + \alpha_1)(\eta + \mu) + (1-\beta_1)\Lambda}{(\mu + \alpha_1)},$$
$$\lambda_4 = -(\gamma + \mu) \text{ and } \lambda_5 = -\mu$$

Since all the eigenvalues are negative then $R_{WL} < 1$. So we conclude that malnutrition free is locally asymptotically stable if $R_{WL} > 1$.

2.6.2 Global Stability of the Malnutrition Free Steady State

Theorem 2.6.4 *The disease steady-state free E_0 whenever it exists, is globally asymptotically stable if $R_{0WL} \leq 1$ when all solutions of system (2.2.1) in $\mathbb{R}^5$ are bounded.*

Proof The proof requires that a suitable Lyapunov function is chosen by taking into account the infective classes of the non-linear ordinary differential equations of the system (2.2.1).

$$V(t) = c_1\left(B_{LW} - B^0_{LW} - B^0_{LW}\ln\frac{B_{LW}}{B^0_{LW}}\right) + c_2\left(B_{HW} - B^0_{HW} - B^0_{HW}\ln\frac{B_{HW}}{B^0_{HW}}\right)$$

where c_1 and c_2 are non-negative constant to be determined. Then V is C^1 on the interior of ϕ, E_0 is global minimum of V on ϕ, and $V\left(B^0_{LW}, B^0_{HW}\right) = 0$. The time derivative of $V(t)$ computed along solutions of (2.2.1) is $\dot{V}(t) = c_1\frac{dB_{LW}}{dt} + c_2\frac{dB_{HW}}{dt}$

$\dot{V}(t) = c_1(\delta + \alpha_1 + \mu)(R_{WL} - 1)B_{LW} + c_2(\eta + \mu)(R_0 - 1)B_{HW} \leq 0$, if $R_0 \leq 1$

Now $\dot{V}(t)$ is negative if $R_0 < 1$ and $\dot{V}(t) = 0 \Leftrightarrow B_{LW} = B_{HW} = 0$, if $R_{WL} = 1$. Therefore, the malnutrition free equilibrium is globally asymptotically stable if $R_0 \leq 1$.

2.7 *Stability of Endemic Steady State*

$$\begin{aligned}
&\Lambda - \beta_1 F_{MP}B_{LW} - (1-\beta_1)F_{MP}B_{HW} - (\alpha_1 + \mu)F_{MP} = 0\\
&\beta_1 F_{MP}B_{LW} - (\delta + \alpha_1 + \mu)B_{LW} = 0\\
&(1-\beta_1)F_{MP}B_{HW} - (\eta + \mu)B_{HW} = 0\\
&\delta B_{LW} - (\gamma + \mu)C_U = 0\\
&\eta B_{HW} + \gamma C_U - \mu A_H = 0
\end{aligned} \tag{2.7.1}$$

Solving the Eq. (2.7.1), we obtained an endemic equilibrium point
$E^*_1 = \left(F^*_{MP}, B^*_{LW}, B^*_{HW}, C^*_U, A^*_H\right)$ and $E^*_2 = \left(F^*_{MP}, B^*_{LW}, B^*_{HW}, C^*_U, A^*_H\right)$

where

$$E_1^* = \left[\frac{(\delta+\mu+\alpha_1)}{\beta_1}, \frac{(\mu+\alpha_1)}{\beta_1}(R_{LW}-1), 0, \frac{(\mu+\alpha_1)\delta}{\beta_1(\mu+\gamma)}(R_{LW}-1), \delta\frac{(\mu+\alpha_1)}{\beta_1(\mu+\gamma)}(R_{LW}-1)\right] \text{and}$$
$$E_2^* = \left[\frac{\eta+\mu}{(1-\beta_1)}, 0, \frac{(\mu+\alpha_1)}{(1-\beta_1)}(R_{HW}-1), 0, \frac{\eta(\mu+\alpha_1)\delta}{\mu(1-\beta_1)}(R_{HW}-1)\right].$$

Theorem 2.7.1 *The malnutrition endemic equilibrium is locally asymptotically stable if $R_{WL} > 1$ and unstable if $R_{WL} \le 1$*

Proof Evaluating the Jacobian matrix at the endemic equilibrium points gives

$$J(E_1^*) = \begin{bmatrix} -(\mu+\alpha_1)+(\mu+\alpha_1)(1-R_{LW}) & -(\delta+\mu+\alpha_1) & -\frac{(1-\beta_1)(\delta+\mu+\alpha_1)}{\beta_1} & 0 & 0 \\ -(\mu+\alpha_1)(1-R_{LW}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\beta_1)(\delta+\mu+\alpha_1)}{\beta_1}-(\eta+\mu) & 0 & 0 \\ 0 & \delta & 0 & -(\gamma+\mu) & 0 \\ 0 & 0 & \eta & \gamma & -\mu \end{bmatrix}$$

and

$$J(E_2^*) = \begin{bmatrix} -(\mu+\alpha_1)+(\mu+\alpha_1)(1-R_{HW}) & -\frac{\beta_1(\eta+\mu)}{(1-\beta_1)} & -(\eta+\mu) & 0 & 0 \\ 0 & \frac{\beta_1(\eta+\mu)}{(1-\beta_1)}-(\delta+\mu+\alpha_1) & 0 & 0 & 0 \\ -(\mu+\alpha_1)(1-R_{HW}) & 0 & \frac{(1-\beta_1)(\delta+\mu+\alpha_1)}{\beta_1}-(\eta+\mu) & 0 & 0 \\ 0 & \delta & 0 & -(\gamma+\mu) & 0 \\ 0 & 0 & \eta & \gamma & -\mu \end{bmatrix}$$

where
$R_{LW} = -\frac{\beta_1\Lambda}{\mu(\delta+\mu+\alpha_1)}$ and $R_{HW} = \frac{(1-\beta_1)\Lambda}{\mu(\eta+\mu)}$
The trace of $J(E_1^*)$ is
$tr(J(E_1^*)) = -(\alpha_1+\mu)+(\alpha_1+\mu)(1-R_{LW})+\frac{(1-\beta_1)(\delta+\alpha_1+\mu)}{\beta_1}-(\eta+\mu)-(\gamma+\mu)-\mu < 0$ if
$\frac{(1-\beta_1)(\delta+\alpha_1+\mu)}{\beta_1} < (\eta+\mu)$ and determinant is given by

$$\det(J(E_1^*)) = -\left[(\delta+\alpha_1+\mu)(\alpha_1+\mu)(1-R_{LW})\left((\eta+\mu)-\frac{(1-\beta_1)(\delta+\alpha_1+\mu)}{\beta_1}\right)(\gamma+\mu)\mu\right] > 0$$

If $R_{LW} > 1$. Also the trace of $J(E_2^*)$ is
$tr(J(E_2^*)) = -(\alpha_1+\mu)+(\alpha_1+\mu)(1-R_{HW})+\frac{\beta_1(\eta+\mu)}{(1-\beta_1)}-(\delta+\alpha_1+\mu)-(\gamma+\mu)-\mu < 0$ if
$\frac{\beta_1(\eta+\mu)}{(1-\beta_1)} < (\delta+\alpha_1+\mu)$ and determinant is given by
$\det(J(E_2^*)) = -\left[(\eta+\mu)(\alpha_1+\mu)(1-R_{HW})\left(\frac{\beta_1(\eta+\mu)}{(1-\beta_1)}-(\delta+\alpha_1+\mu)\right)(\gamma+\mu)\mu\right] > 0$ if $R_{HW} > 1$.

Since the trace is negative and the determinant is positive then $R_{WL} > 1$ in both cases. We then conclude that malnutrition at endemic equilibrium is locally asymptotically stable whenever $R_{WL} > 1$.

3 Numerical Simulation

In this section, we will estimate the parameters in the model based on literature values, perform sensitivity analysis and finally we will conduct numerical simulations.

3.1 Parameter Estimation

The estimation of parameters has been a major challenge in the validation of epidemiological modelling. In this section we tried to estimate some of the parameter values of system (2.2.1).

Some estimated assumptions were considered in order make purposeful of illustrations in tracking the dynamics of malnutrition. For unavailability of data, we used literature values as indicated in the Table 2 based on model system Eq. (2.2.1). The following initial conditions were used for the purpose of numerical simulation F_{MP} $= 30$, $B_{LW} = 20$, $B_{HW} = 20$, $C_U = 10$ and $A_H = 35$. The parameter values used for the work is given in Table 2.

3.2 Sensitivity Analysis

Sensitivity analysis seeks to present to characterization of the uncertainty of parameters with regard to a given model. It offers the opportunity to have information on the effect of a particular parameter in the modeling processes [21].

We performed sensitivity analysis on R_{LW} with respect to the parameter value so that vital parameter values influence can be measured for the malnutrition model Eq. (2.2.1). For one to increase or reduce a parameter it is essential for one to have some relative information regarding the human morbidity and mortality in relation to the transmission dynamics of malnutrition. value In determining how best to reduce human mortality and morbidity due to malnutrition. According to Chintnis et al. [22], sensitivity analysis is commonly used to determine the robustness of

Table 2 The value of the parameters of the model

Parameter	Range	Sources
Λ	(0, 0.35)	[12, 13]
β_1	(0, 1)	Assumed
α_1	0.1	[12]
δ	0.006	[13]
η	0.013	Assumed
γ	(0, 1)	Assumed
μ	0.3	[12]

Table 3 Sensitivity index for malnutrition

R_0	Parameter	Sensitivity index
R_{0LW}	Λ	1
	β_1	1
	α_1	−0.46739
	δ	−0.1304347
	μ	−0.1304347
R_{0HW}	Λ	1
	β_1	−0.0012014
	α_1	−0.25
	η	−0.0415335
	μ	−1.708466

model predictions to parameter values. Thus, we are concerned with parameters that would significantly affect the model's basic reproduction number which is usually responsible for the spread of the phenomenon. Sensitivity analysis allows the measure of the relative variation in a state variable when there is a parameter variation.

Definition 3.2.1 The normalized forward sensitivity index of a variable u, which depends differentially on a parameter, p, is defined as: $r_p^u = \frac{\partial u}{\partial p} \times \frac{p}{u}$.

Since the reproduction number R_0 is a differentiable function of the parameters, the sensitivity index may alternatively be defined using partial derivatives as: $S = \frac{\partial R_0}{\partial \rho} \times \frac{\rho}{R_0}$, where ρ is the parameter of interest.

From Table 3, it depicts that R_{LW} was sensitive to β_1 and Λ. When each one of them increases making other parameters fixed, their values rose up the since they have positive indices. The most sensitive parameter observed was Λ which has an effect on malnutrited pregnant female population as well as the infected compartment.

4 Results

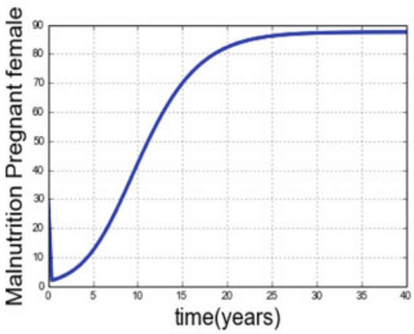
4.1 Simulation Results

From Fig. 2a, we observed that as time increases, the malnutrition pregnant female population increases to a maximum point and approaches the carrying capacity. That is the upper bound of the population.

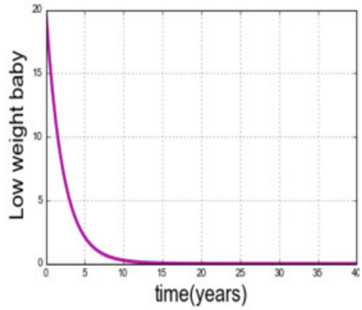
Also, Fig. 2b the low weight baby converge to the equilibrium point zero as time increases. Thus low weight baby dies out from the population with time.

In Fig. 2c, we observe that as time increases, the high weight baby converged to equilibrium point zero. That is high weight baby dies out from the population.

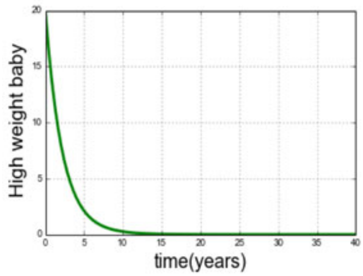
Furthermore, Fig. 2d shows that child undergrowth will converge to the equilibrium point zero with time. Thus child undergrowth dies out from the population.



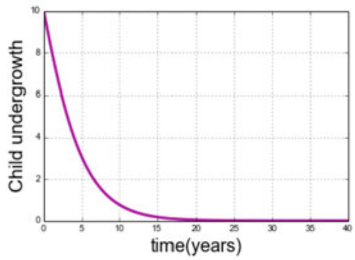
(a) Malnutrition pregnant females against time when $R_0<1$



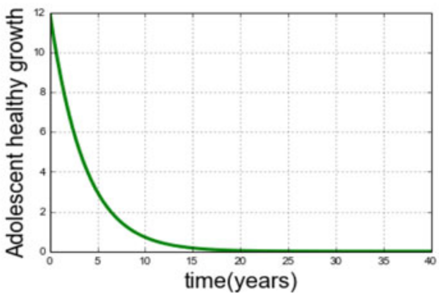
(b) Low weight baby against time when $R_0< 1$



(c) high weight baby against time when $R_0< 1$



(d) child undergrowth against time when $R_0< 1$



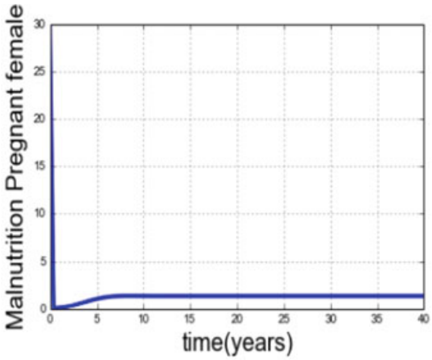
(e) Healthy adolescence growth against time when $R_0< 1$

Fig. 2 Simulation results for malnutrition free with $R_0 < 1$

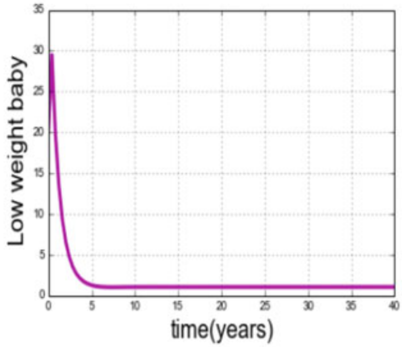
Figure 2e shows that healthy adolescent growth will converge to equilibrium point zero with time. That is healthy adolescent growth dies out from the population.

Again, we observed from Fig. 2b–e that all the populations converge to zero as time evolves. This shows that malnutrition can be minimized with time. Therefore, local stability of the malnutrition free state holds as shown in Theorem 2.6.2.

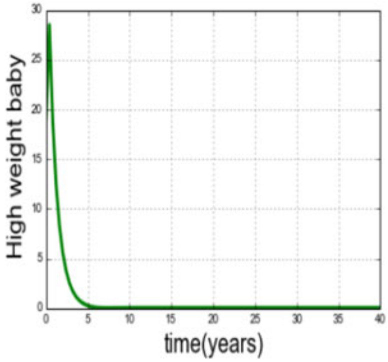
In Fig. 3a, it is observed that as time increases, the malnutrition pregnant female population decreases which shows that there is movement to another compartment. Thus, malnutrition pregnant female still exists in the population. In Fig. 3b, the low



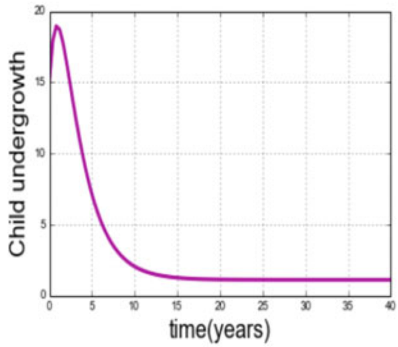
(a) malnutrition pregnant females against time



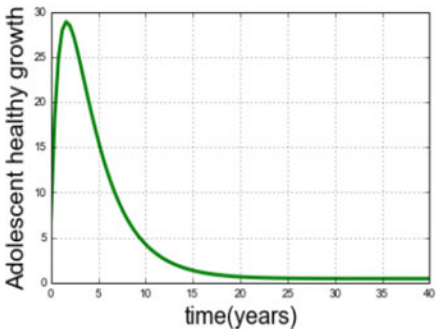
(b) A plot of low weight baby against time



(c) A plot of high weight baby against time



(d) A plot of child under growth against time



(e) A plot of healthy adolescence growth against time

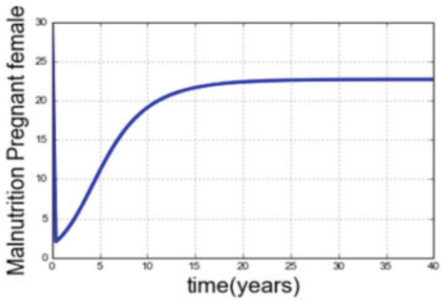
Fig. 3 Simulation results for endemic steady states E_1^*

weight baby decreases to equilibrium point as time increases. Thus, low weight baby still exists in the population.

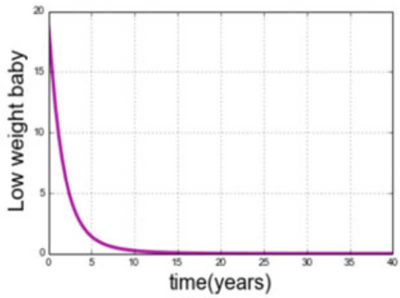
Also, Fig. 2c, we observe that as time increases, the high weight baby converged to equilibrium point zero with time. This confirms the results on the steady state E_1^*.

Furthermore, Fig. 3d shows that child undergrowth decreases to the equilibrium point as time increases. Thus, child undergrowth will still exist in the population.

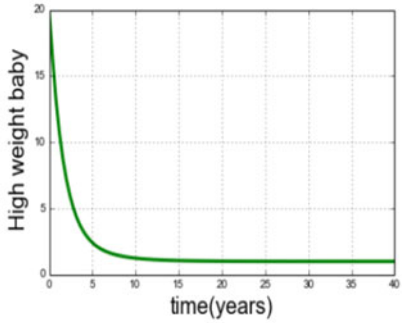
From Fig. 4a, we observe that as time increases, the malnutrition pregnant female population reduces which shows that there is movement to another compartment. Thus, malnutrition pregnant female still exists in the population.



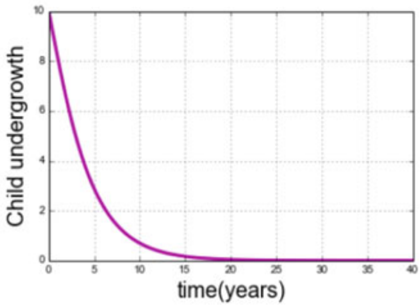
(a) A plot of malnutrition pregnant females against Time



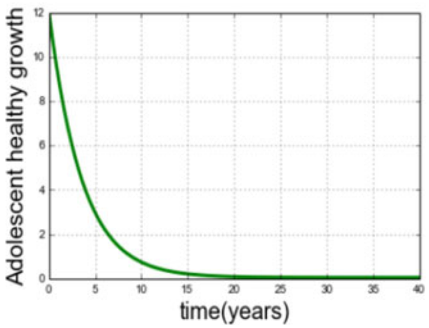
(b) A plot of low weight baby against time



(c) A plot of high weight baby against time

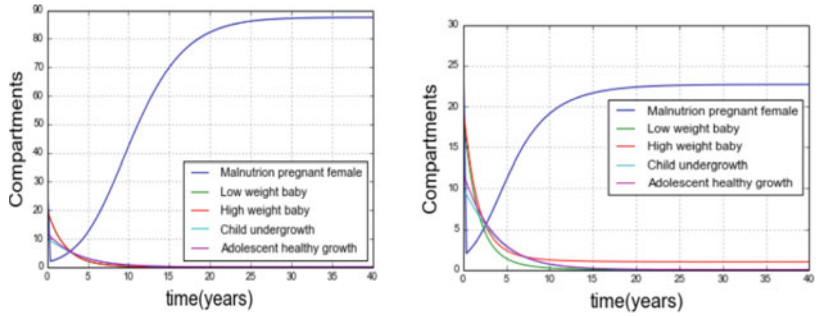


(d) A plot of child undergrowth against time



(e) A plot of healthy adolescence growth against time

Fig. 4 Simulation results for endemic steady states E_2^*



(a) A plot of all the compartments with $R_{0LW} < 1$ (b) A plot of all the compartments with $R_{0LW} > 1$

Fig. 5 Simulation results for $R_{0LW} < 1$ and $R_{0LW} > 1$

In 4b, d, we observe that as time increases, the low weight baby and child undergrowth converged to equilibrium point zero with time. This confirms the results on our steady state E_2^*. Also, Fig. 4c, we observe that as time increases, the high weight baby decreases to equilibrium point. That is high weight baby will still exist in the population with time. Furthermore, from 3e and 4e, the healthy adolescent population will converge to zero as time increases.

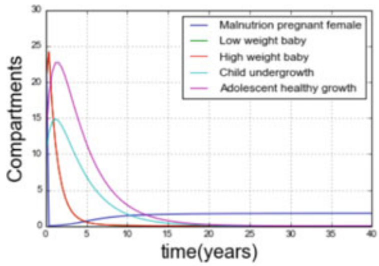
Figure 5 shows how severe malnutrition at individual compartments. When $R_{0LW} < 1$ all the infected compartments converge to zero. Thus, effect of malnutrition dies out from the population at malnutrition free.

Also, when $R_{0LW} > 1$, three of the compartments converge to zero (dies out from the population).

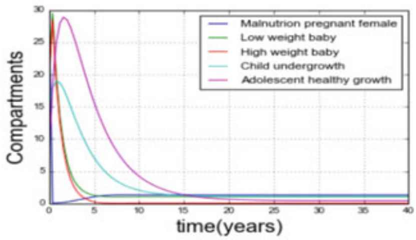
This confirms the steady states where both low weight babies and child undergrowth were found to be zero. Malnutrition is minimized but cannot be eradicated. Therefore, the existence malnutrition in the population.

From Fig. 6a, when $R_{0LW} < 1$ and $R_{0HW} < 1$, we observe malnutrition free equilibrium point E_0. That is malnutrition is stable. Also, in Fig. 6b when $R_{0HW} < 1$ and $R_{0LW} > 1$, E_0 will be unstable and the endemic equilibrium E_2^* will be stable. This means that high weigh babies and healthy adolescent will die out. However, Fig. 6c shows the case where E_0 will be unstable and the endemic equilibrium point E_1^* will be stable.

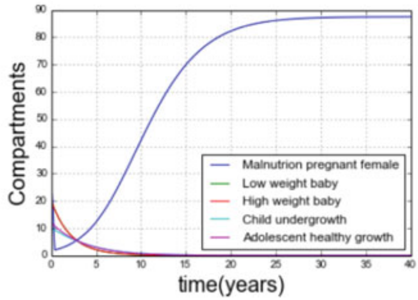
From Fig. 6b, c we observe that malnutrition cannot be eradicated but can be minimized. Hence, malnutrition at endemic equilibrium is locally stable as shown in Theorem 2.7.1.



(a) all the compartments with $R_{0LW} < 1$ and $R_{0HW} < 1$



(b) all the compartments with $R_{0HW} < 1$ and $R_{0LW} > 1$



(c) A plot of all the compartments with $R_{0LW} < 1$ and $R_{0HW} > 1$

Fig. 6 Simulation results for malnutrition free and endemic steady states

5 Conclusion

In this paper, we have successfully studied the effect of poor nutrition in the human life cycle. The effect of poor nutrition begins in the womb, continues well into childhood, adulthood and cycles across generations. We formulated a mathematical model from the conception to adulthood using the pregnant women nutrient status. The basic reproduction number R_0 was calculated. This serves as a threshold to which malnutrition will die out when $R_0 < 1$ or will persist when $R_0 > 1$. The model is supported with numerical simulation. We had a multiple steady state, which was written in terms of the reproduction number for low weight babies and high weight babies. The analysis on the steady state suggested that both malnutrition free and endemic equilibrium are both locally stable. Results from numerical simulations for all the compartments showed that malnutrition will die out locally and become unstable globally at the endemic state. Malnutrition can be minimized for a period but cannot be eradicated completely from the society. This suggests that good nutrition is very important in every stage of human development in the society. Good nutrition will reduce the rate at which fertile female becomes malnourished in the society. The pregnant women nutritional status should be improved to give birth to healthy baby which will grow eventually to become healthy adolescent. Literature values and assumed parameters were used because of the unavailability of data on malnutrition. In future work, this model can be improved to consider all stages throughout the human life cycle.

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Characteristics of Homogeneous Heterogeneous Reaction on Flow of Walters' B Liquid Under the Statistical Paradigm

Anum Shafiq, T. N. Sindhu and Z. Hammouch

Abstract In this article, significance of inclined MHD stagnant point flow of Walters B liquid because of stretched surface is investigated. Flow phenomenon is studied with Newtonian heating, homogeneous heterogeneous reactions, Joule heating and viscous dissipation. The nonlinear PDEs are converted to get nonlinear system of ODEs by invoking suitable transformations and solved by utilizing OHAM. Statistical methodology is used to check the significance and insignificance of the physical parameters via correlation coefficients and probable error. Characteristics of various sundry parameters on velocity, concentration and temperature fields are studied. Friction and Nusselt numbers are calculated and discuss in detail.

Keywords Statistical approach · Newtonian heating · Walters-B liquid · Inclined MHD · Joule heating · Homogeneous heterogeneous reaction · OHAM

1 Introduction

The investigation of magnetohydrodynamics flow with heat transfer phenomenon in non-Newtonian liquids has substantial usages in technology and science, like construction of heat exchangers, installation of nuclear accelerators, design for cooling of nuclear reactors, turbo machinery, blood flow measurement techniques. Ahmed et al. [1] investigated impact of MHD on Jeffrey liquid flow along an extended surface using power law temperature. Analytical solutions of non-linear PDEs using slip con-

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ditions are instigated flow of non-Newtonian MHD liquid in a pipe towards a porous medium was analyzed by Zeeshan and Ellahi [2]. Nejad et al. [3] reported MHD stream of electrically conducting power law liquids towards an isothermal vertical wavy sheet. 3-Dimensional MHD Jeffrey nanoliquid flow along thermally radiative surface under heat generation phenomenon was examined by Shehzad et al. [4]. Das et al. [5] reported behavior of melting phenomenon on MHD stagnant point Jeffrey liquid stream towards an extended surface with slip conditions. Venkateswarlu and Satya Narayana [6] scrutinized behavior of chemical reactant on viscoelastic liquid stream along a vertical plate with MHD. Rashidi et al. [7] explored impact of magnetohydrodynamic and heat phenomena on two dimensional liquid flows along a porous medium. Sheikholeslami et al. [8] obtained the simulation of problem of CuO-water nanoliquid stream with convective heat phenomenon. Ellahi et al. [9] reported simultaneous impacts of magnetohydrodynamic and partial slip on peristaltic stream of Jeffery liquid in a rectangular duct. Significance of Joule heating phenomenon in third-grade liquid stream towards a radiative plate was studied by Hayat et al. [10].

Both homogeneous and heterogeneous reactants are involved in numerous chemically reacting schemes. Some of them have capability to proceed slow or not, excluding catalyst. The homogeneous and heterogeneous reactants interplay is very compound including consumption and production of reactant species at different rates both within liquid and on catalytic exterior like reactions occurring in production of polymer and ceramics, hydrometallurgical industry, crops damage via freezing, dispersion and fog formation, food processing, equipment design for chemical processing, cooling towers and temperature fields and moisture over agricultural fields and groves of fruit trees. Merkin [11] considered homogeneous-heterogeneous reactions model in stream of viscous liquid towards a flat plate. He noted that outer reaction is superior mechanism near leading edge of surface. Significance of homogenous-heterogeneous reactants in stream of viscous liquid was numerically investigated by Chaudhary and Merkin [12]. Stagnant-point stream along an extended plate using homogeneous/heterogeneous reactants was analyzed by Bachok et al. [13]. Khan and Pop [14] reported significance of homogeneous-heterogeneous reactants of viscoelastic liquid stream along an extended surface. Homogeneous heterogeneous reactants in micropolar liquid flow along a permeable extended/shrinking plate was examined by Shaw et al. [15]. Khan and Pop's [14] work was extended by Kameswaran et al. [16] for nanoliquid along a porous extended plate. Importance of homogeneous-heterogeneous reactants in stagnant point carbon nanotubes flow with Newtonian heating was reported by Hayat et al. [17]. Behaviour of nanoliquid MHD flow with homogeneous heterogeneous reactants and condition for velocity slip was also examined by Hayat et al. [18]. Hayat et al. [19] reported significance of homogeneous-heterogeneous reactants in Powell-Eyring liquid flow. Hayat et al. [20] examined Oldroyd-B MHD liquid flow using homogeneous heterogeneous reactions with Cattaneo-Christov model. Significance of MHD in bi-directional stream of nanoliquid with homogeneous heterogeneous reactants and second-order velocity slip was analyzed by Hayat et al. [21].

Newtonian heating (or cooling process) is process where internal resistance is supposed to be neglected in comparing with its surface resistance. Currently this phenomenon has been used by various researchers because of its practical usages like to configuration heat exchanger, conjugate warmth exchange around fins and furthermore in convective streams setup where bounding edges absorb heat by solar radiations. The Von Kármán stream and heat phenomenon of an electrically conducting liquid was given by Sahoo [22]. Salleh et al. [23] examined heat transfer flow along an extended surface using Newtonian heating. Significance of Newtonian heating in second grade liquid flow along an extended surface was considered in [24]. Unsteady viscous liquid MHD flow towards a flat surface using Navier slip and Newtonian heating effects was reported Makinde [25]. Uddin et al. [26] is analyzed MHD flow of nanoliquid towards a flat vertical surface with Newtonian heating. Sarif et al. [27] numerically studied viscous flow induced by extended plate using Newtonian heating through Keller Box technique. 3-Dimensional couple stress magnetohydrodynamic liquid flow with Newtonian heating is studied in [28]. Impact of viscous dissipation and Newtonian heating on nanoliquids flow towards a flat surface was investigated by Makinde [29]. The flow of Walters B liquid with Newtonian heating was reported in [30].

The heterogeneous homogeneous reactants and Newtonian heating phenomenon in flow of Walters B liquid along a stretched plate is investigated. Inclined MHD, Stagnant flow and Joule heating is also considered. The non-linear ODEs are solved by OHAM [31, 32]. Statistical approach is used to check the statistical significance of physical parameters and the drag forces/local Nusselt number. Significance of various sundry parameters on velocity, temperature and concentration fields, skin friction and Nusselt numbers are examined very carefully.

2 Formulation

Walters B stagnation-point liquid flow with homogeneous and heterogeneous reactions over a stretched plate is considered here. The flow is confined to $y \geq 0$. Applied magnetic field in such a way thats its making angle ψ with axis. Surface is also subjected to Newtonian heating. Contribution due to viscous-dissipation and Joule heating is present.

Simple homogeneous heterogeneous reactant model is [20]

$$\mathbf{A} + 2\mathbf{B} \rightarrow 3\mathbf{B}, \qquad \text{rate} = ab^2 k_1. \tag{1}$$

with

$$\mathbf{A} \rightarrow \mathbf{B}, \qquad \text{rate} = ak_2, \tag{2}$$

where (a, b) are concentrations of chemical species $(\mathbf{A}, \mathbf{B})$ on the other side, rate constants are presented as (k_1, k_2). These reactants equations tells us that in external stream and at outer boundary layer edge, reaction rate is zero. The governing equations are [14, 15]

$$\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{u}}{\partial x}=0, \tag{3}$$

$$\begin{aligned}&\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}-\nu\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{k_0}{\rho}\left[\bar{u}\frac{\partial^3 \bar{u}}{\partial x\partial y^2}+\frac{\partial \bar{u}}{\partial x}\frac{\partial^2 \bar{u}}{\partial y^2}\right.\\&\left.+\bar{v}\frac{\partial^3 u}{\partial y^3}-\frac{\partial \bar{u}}{\partial y}\frac{\partial^2 \bar{u}}{\partial x\partial y}\right]-U_e\frac{dU_e}{dx}+\frac{\sigma B_0^2}{\rho}\sin^2\psi\left(\bar{u}-U_e\right)=0,\end{aligned} \tag{4}$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}-\frac{K}{\rho c_p}\frac{\partial^2 \bar{T}}{\partial y^2}-\frac{\sigma B_0^2}{\rho c_p}\sin^2\psi\left(\bar{u}-U_e\right)^2=0, \tag{5}$$

$$\bar{u}\frac{\partial \bar{a}}{\partial x}+\bar{v}\frac{\partial \bar{a}}{\partial y}+k_1\bar{a}\bar{b}^2-D_A\frac{\partial^2 \bar{a}}{\partial y^2}=0, \tag{6}$$

$$\bar{u}\frac{\partial \bar{b}}{\partial x}+\bar{v}\frac{\partial \bar{b}}{\partial y}-k_1\bar{a}\bar{b}^2-D_B\frac{\partial^2 \bar{b}}{\partial y^2}=0, \tag{7}$$

with

$$\begin{aligned}&\bar{u}\left(x,0\right)=cx,\ \ \bar{v}(x,0)=0,\ \ \left.\frac{\partial \bar{T}}{\partial y}\right|_{y=0}=-h_s\bar{T},\\&D_B\left.\frac{\partial \bar{b}}{\partial y}\right|_{y=0}+k_s\bar{a}=0,\ D_A\left.\frac{\partial \bar{a}}{\partial y}\right|_{y=0}-k_2\bar{a}=0,\end{aligned} \tag{8}$$

$$\bar{u}\to U_e(x)=dx,\ \ \bar{T}\to\bar{T}_\infty\ \bar{a}\to\bar{a}_0,\ \ \bar{b}\to 0 \text{ as } y\to\infty. \tag{9}$$

σ electrical conductivity, B_0 magnetic field, k_0 liquid material parameters, U_w stretching velocity, T temperature, ν kinematic viscosity, U_e free stream velocity, thermal conductivity denoted by K, ρ density of liquid, c_p specific heat, D_B and D_A diffusion species coefficients of B and A, T_∞ ambient liquid temperature, a_0 positive dimensional constant, h_s denotes heat transfer coefficient and c represents stretching rate.

Introducing dimensionless variables

$$\begin{aligned}&\bar{v}\left(x,y\right)=-\sqrt{c\nu}f\left(\eta\right),\ \bar{u}\left(x,y\right)=cxf'\left(\eta\right),\\&\theta=\frac{\bar{T}-\bar{T}_\infty}{\bar{T}_\infty},\ \ h(\eta)=\frac{\bar{b}}{\bar{a}_0},\ \ g(\eta)=\frac{\bar{a}}{\bar{a}_0},\ \ \eta=\sqrt{\frac{c}{\nu}}y.\end{aligned} \tag{10}$$

Thus

$$f''' - \left(f'\right)^2 - We\left[2f'f''' - \left(f''\right)^2 - ff^{(iv)}\right] + ff'' + A^2$$
$$- M^2 \sin^2\psi\left(-A + f'\right) = 0,$$

$$f'(0) = 1,\ f'(\infty) = A,\ \ f(0) = 0, \tag{11}$$

$$\theta'' + \Pr f\,\theta' + M^2 \Pr Ec \sin^2\psi\left(f' - A\right)^2 = 0,$$

$$\theta'(0) = -\gamma - \gamma\theta(0),\ \ \theta(\infty) = 0, \tag{12}$$

$$\frac{1}{Sc}g'' - Kgh^2 + fg' = 0,$$

$$g'(0) = K_2 g(0),\ \ g(\infty) \to 1, \tag{13}$$

$$\frac{\delta_1}{Sc}h'' + Kgh^2 + fh' = 0,$$

$$\delta_1 h'(0) = -K_2 g(0),\ \ h(\infty) \to 0. \tag{14}$$

Hartman number is denoted by M, ratio parameter is given by A, Weissenberg number denoted by We, Prandtl number is given by Pr, Eckert number is denoted by Ec and conjugate parameter is given by γ, strength of homogeneous reactant parameter is denoted by K, δ_1 the ratio of mass diffusion coefficient, K_2 the strength of heterogeneous reaction parameter and Sc the Schmidt number and defined as

$$M = \sqrt{\frac{\sigma B_0^2}{\rho c}},\quad A = \frac{a}{c},\ We = \frac{k_0 c}{\mu_0},\ \Pr = \frac{\mu_0 c_p}{K},$$
$$Ec = \frac{U_m^2}{c_p T_\infty},\ \gamma = h_s\sqrt{\frac{\nu}{a}},\ K = \frac{k_1 a_0^2}{c},$$
$$K_2 = \frac{k_2 l \operatorname{Re}_x^{-1/2}}{D},\ \delta_1 = \frac{D_B}{D_A},\ Sc = \frac{\nu}{D_A}. \tag{15}$$

Where coefficients of diffusion of chemical species (**B**, **A**) are of comparable size. This argument provides us to make further supposition that diffusion coefficients (D_B, D_A) are equal i.e. $\delta_1 = 1$ and therefore [12]:

$$h(\eta) + g(\eta) - 1 = 0. \tag{16}$$

Now Eqs. (13) and (14) yield

$$\frac{1}{Sc}g'' - Kg(g-1)^2 + fg' = 0, \tag{17}$$

with

$$g'(0) - K_2 g(0) = 0,\ g(\eta) \to 1 \text{ as } \eta \to \infty. \tag{18}$$

The expression of C_{fx} and Nu_x are

$$C_{fx} = \frac{\tau_w}{\rho U_w^2},\ Nu_x = \frac{x q_w}{K(T - T_\infty)}, \tag{19}$$

where

$$q_w = -K\left(\frac{\partial \bar{T}}{\partial y}\right)_{y=0},\ \tau_w = \left[\mu_0 \frac{\partial \bar{u}}{\partial y} - k_0\left(\bar{v}\frac{\partial^2 \bar{u}}{\partial y^2} + \bar{u}\frac{\partial^2 \bar{u}}{\partial x \partial y} + 2\frac{\partial \bar{u}}{\partial x}\frac{\partial \bar{u}}{\partial y}\right)\right]_{y=0}. \tag{20}$$

Thus

$$\left(\underset{x}{\mathrm{Re}}\right)^{1/2} C_f = f''(0) - 3Wef''(0), \tag{21}$$

$$\left(\underset{x}{\mathrm{Re}}\right)^{-1/2} Nu_x = \gamma\left(1 + \frac{1}{\theta(0)}\right), \tag{22}$$

where $\mathrm{Re}_x = cx^2/\nu$ denotes local Reynolds parameter.

3 OHAM

The governing system of nonlinear ODEs are solved analytically by invoking reliable methodology called optimal homotopy analysis method. Average squared residual errors (ASRE) and corresponding optimal convergence control parameters are computed. Initial guesses and linear operators $\left(f_0, \theta_0, L_f, L_\theta\right)$ are

$$f_0(\eta) = A\eta - (A-1)(1 - \exp(-\eta)),\ \theta_0(\eta) = \frac{\gamma \exp(-\eta)}{1-\gamma},\ g_0(\eta) = 1 - \frac{1}{2}\exp(-K_2\eta), \tag{23}$$

$$L_f[f(\eta)] = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta},\ L_\theta[\theta(\eta)] = \frac{d^2\theta}{d\eta^2} - \theta,\ L_g[g(\eta)] = \frac{d^2 g}{d\eta^2} - g, \tag{24}$$

satisfying the following properties

$$L_f\left[\bar{C}_1 + \bar{C}_2 e^{\eta} + \bar{C}_3 e^{-\eta}\right] = 0, \tag{25}$$

$$L_\theta\left[\bar{C}_4 e^{\eta} + \bar{C}_5 e^{-\eta}\right] = 0, \tag{26}$$

$$L_g\left[\bar{C}_6 e^{\eta} + \bar{C}_7 e^{-\eta}\right] = 0, \tag{27}$$

in which $\bar{C}_i$ $(i = 1, \ldots, 7)$ are constants.

4 Optimal Convergence Control Parameters

Convergence control parameters ($\hbar_f$, $\hbar_\theta$, $\hbar_g$) are calculated by BVPh2.0 package. These values can be obtained by minimizing total error. The average square residual error (ASRE) are used for minimizing the CPU time, at mth-order of approximation as follows

$$\varepsilon_m^f\left(\hbar_f, \hbar_\theta\right) = \frac{1}{1+N_1}\sum_{j=0}^{N_1}\left[\sum_{i=0}^{k}(f_i)_{\eta=j\pi}\right]^2,$$

$$\varepsilon_m^\theta\left(\hbar_f, \hbar_\theta\right) = \frac{1}{1+N_1}\sum_{j=0}^{N_1}\left[\sum_{i=0}^{k}(f_i)_{\eta=j\pi}, \sum_{i=0}^{k}(\theta_i)_{\eta=j\pi}\right]^2,$$

and

$$\varepsilon_m^g\left(\hbar_f, \hbar_g\right) = \frac{1}{1+N_1}\sum_{j=0}^{N_1}\left[\sum_{i=0}^{k}(f_i)_{\eta=j\pi}, \sum_{i=0}^{k}(g_i)_{\eta=j\pi}\right]^2.$$

The optimal values of $\left(\hbar_f, \hbar_\theta, \hbar_g\right)$ are $\hbar_f = -0.427635$, $\hbar_\theta = -0.8078$ and $\hbar_g = -0.93495$. when $A = 0.1$, $We = 0.2$, $\psi = \pi/3$, $\delta = 0.2$, $K_1 = 1.5$, $\Pr = 0.7$, $K_2 = 0.2$, $Ec = 0.4$ and $Sc = 0.4$. In Fig. 1, corresponding total residual error is plotted. Optimal convergence control parameters are given in Table 1. Table 1 shows individual averaged squared residual errors of momentum, energy equations at various order of approximation. By increasing order of approximation, squared residual error decreases.

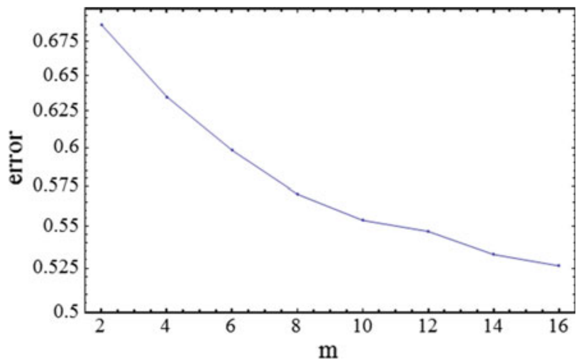

Fig. 1 Total error versus order of approximations

Table 1 ASREs at various order of approximations when $\psi = \pi/3$, $A = 0.1$, $We = 0.2$, $K_2 = 0.2$, $\delta = 0.2$, $\Pr = 0.7$, $K_1 = 1.5$, $Sc = 0.4$ and $Ec = 0.4$ by means of optimal control parameters $\hbar_f = -0.427635$, $\hbar_\theta = -0.8078$, $\hbar_g = -0.93495$

m	ε_m^f	ε_m^θ	ε_m^g	CPU time (s)
2	2.97384×10^{-2}	4.10708×10^{-3}	0.68448	1.24000
4	1.80930×10^{-2}	2.00429×10^{-3}	0.63293	5.84001
6	1.18885×10^{-2}	1.25117×10^{-3}	0.59691	18.1200
8	8.26666×10^{-3}	8.90896×10^{-4}	0.56910	42.1301
12	4.50178×10^{-3}	5.54549×10^{-4}	0.54652	142.530
14	3.46788×10^{-3}	4.63210×10^{-4}	0.53276	225.510
16	2.72896×10^{-3}	3.95808×10^{-4}	0.52610	352.510

5 Discussion

In this section, physical interpretation of the results for velocity, temperature and concentration fields are discussed.

Change in velocity field with an increment in We is plotted in Fig. 2a. When velocity of extending plate is larger than free stream velocity i.e. ($A < 1$), velocity field reduces for larger values of We. However for $A > 1$, velocity profile rises. Irrespective of A, corresponding boundary layer thins with enhancement in We. Physically, larger values of We rises tensile stresses as a result oppose momentum transport. Consequently, boundary layer width reduces. Significance of M on f' for both $A > 1$ and $A < 1$ cases are drawn in Fig. 2b. Enhancement in magnetic number corresponds the reduction in velocity field when $A < 1$ and reverse effect is observed on velocity field for $A > 1$. Since, applied transverse magnetic field creates a retardant force. It has ability to resist liquid motion and because of this reason, corresponding boundary layer width reduces for increment in magnetic field.

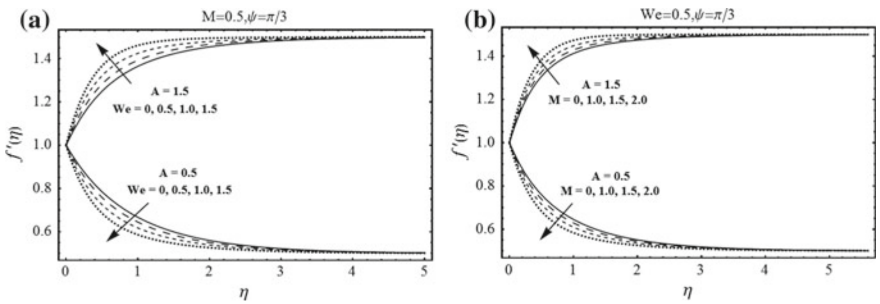

Fig. 2 **a** Impact of We on $f'(\eta)$. **b** Impact of M on $f'(\eta)$

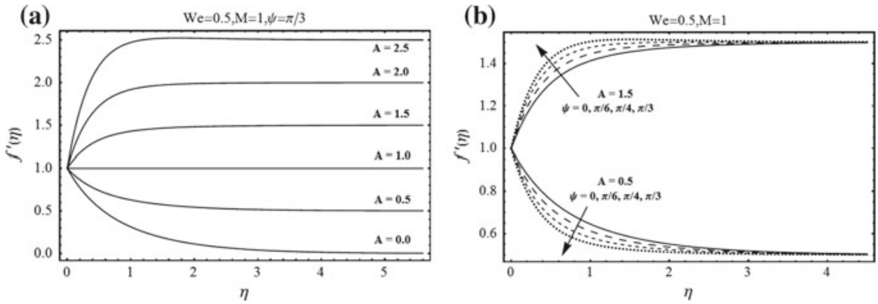

Fig. 3 **a** Impact of A on $f'(\eta)$. **b** Impact of Pr on $\theta(\eta)$

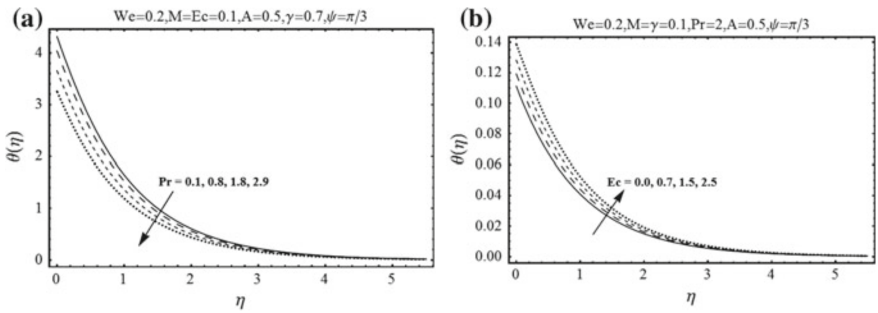

Fig. 4 **a** Impact of Pr on $\theta(\eta)$. **b** Impact of Ec on $\theta(\eta)$

f' is mounting function of A is reported in Fig. 3a. Boundary layer width rises with enhancement in A with $A < 1$, while thinner boundary layer becomes for case of enhancement in A provided $A > 1$. Additionally, no boundary layer is noted for $A = 1$.

Figure 3b is elucidated behavior of inclination angle ψ on $f'(\eta)$ for cases $A < 1$ and $A > 1$. It is noticed that velocity field is decreasing for ψ when $A < 1$ and reverse behavior is observed for $A > 1$. In fact, with augmented ψ, significance of magnetic field on liquid particles rises because of rise in Lorentz force. Therefore, velocity field reduces. It is also examined that for $\psi = 0$, magnetic field impact on velocity profile is zero while for $\psi = \pi/2$, maximum resistance is observed.

The ratio of momentum to thermal diffusivity is defined as Prandtl parameter which enhances pure convection but reduces conduction. Therefore, thermal boundary layer thins and heat transfer rate at surface rises for increment Pr (see Fig. 4a).

Higher values of Eckert number, heat up liquid near vicinity of bounding surface and therefore, corresponding boundary layer width rises (Fig. 4b).

Effect of γ, characterizing Newtonian heating strength on temperature field is sketched in Fig. 5a. It is examined that stronger convective heating permits thermal impact to penetrate deeper into quiescent liquid. Hence, corresponding thermal boundary layer width for larger γ surface heat flux, being proportional to γ is mounting function of γ.

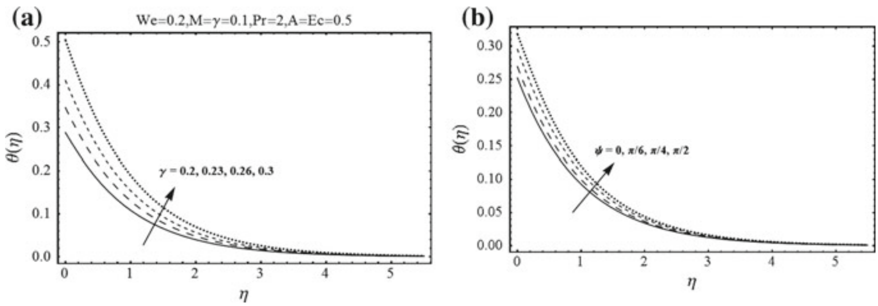

Fig. 5 **a** Impact of γ on $\theta\,(\eta)$. **b** Impact of ψ on $\theta\,(\eta)$

Figure 5b is drawn ψ on temperature distribution. Temperature field is increased for high values of ψ. Because, higher ψ corresponds to increase magnetic field which opposes liquid flow. Therefore, rise in temperature field occur.

Influence of K_1 on concentration profile is investigated in Fig. 6a. Concentration field reduces. Additionally boundary layer width increases for higher strength of homogeneous reaction number.

Significance of K_2 on concentration field is analyzed in Fig. 6b. Concentration field reduces near surface of plate and it rises away from surface for larger K_2.

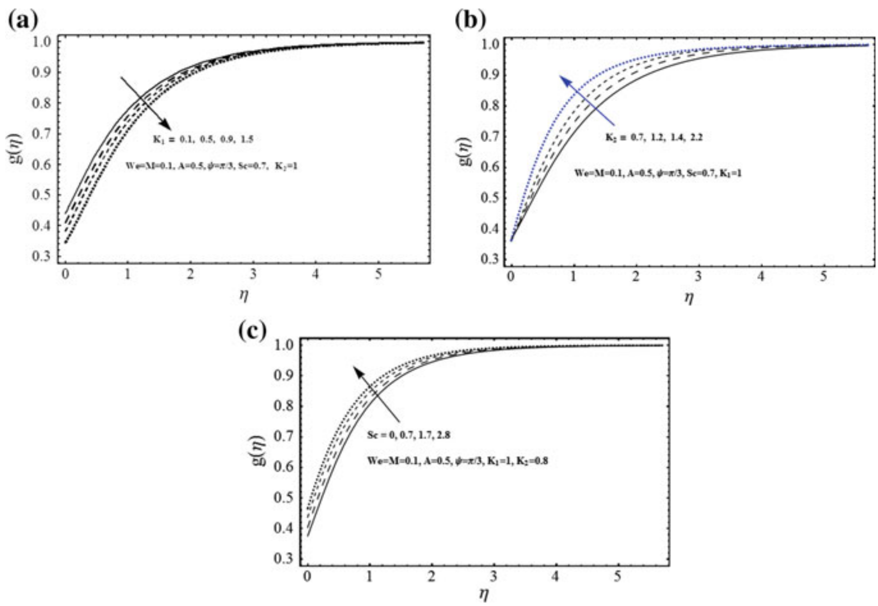

Fig. 6 **a** Impact of K_1 on $g\,(\eta)$. **b** Impact of K_2 on $g\,(\eta)$. **c** Impact of Sc on $g\,(\eta)$

Table 2 Variation of skin friction coefficient for various values of physical parameter

M	We	A	ψ	$\mathrm{Re}_x^{-1/2} C_f$
0.0	0.2	0.1	$\pi/3$	0.39670
0.1				0.39795
0.2				0.40158
0.2	0.0	0.1	$\pi/3$	0.59370
	0.1			0.62795
	0.2			0.67358
0.2	0.2	0.0	$\pi/3$	0.41420
		0.1		0.42158
		0.3		0.44247
0.2	0.2	0.1	0.0	0.39670
			$\pi/3$	0.40160
			$\pi/2$	0.40320

Effect of Sc on concentration field is illustrated in Fig. 6c. Concentration field decreases for higher Schmidt parameter. Additionally, solutal boundary layer width reduces. As Schmidt parameter is ratio of diffusivity of momentum to mass, so larger Schmidt parameter corresponds to little mass diffusivity. Consequently, concentration profile reduces.

Friction and local Nusselt numbers for different values of sundry parameters are provided in Tables 2 and 3. Friction coefficient is enhanced by increasing M, We, ψ and A. On other side, local Nusselt parameter rises for increment in M, A, We, γ and Pr while it reduces for large ψ and Ec.

6 Statistical Paradigm

We lengthen our examination for different out-turn of significant parameters on the inspected issue. Arranged by need to comprehend correlation between different sundry parameter and friction coefficient (F.C.) and furthermore for Nusselt number. We revealed estimations of F.C. in Table 2 and Nusselt number in Table 3. The estimations of correlation coefficients (c.c) are examined and recorded in Tables 4 and 5 concerning F.C. and Nusselt number. It is evident that estimation of c.c is limited between $(-1, 1)$. Moreover absolute value is limited somewhere in range of 0 and 1. The c.c is not just investigate the connection between two variates yet in addition uncovers the opposite and direct correspondence between them.

The c.c has accompanying interpretations:

- Positive perfect linear relationship of variables occurs if $r = 1$.
- Negative perfect linear relationship of variables occurs if $r = -1$.

Table 3 Numerical values of Nusselt number

M	We	A	ψ	γ	Pr	Ec	$-\mathrm{Re}_x^{1/2}\,Nu_x$
0.0	0.2	0.1	$\pi/3$	0.2	0.7	0.4	0.49750
0.1							0.49835
0.2							0.49908
0.2	0.0	0.1	$\pi/3$	0.2	0.7	0.4	0.59850
	0.1						0.69535
	0.2						0.72308
0.2	0.2	0.0	$\pi/3$	0.2	0.7	0.4	0.48190
		0.1					0.49308
		0.2					0.50571
0.2	0.2	0.1	0	0.2	0.7	0.4	0.49750
			$\pi/3$				0.49310
			$\pi/2$				0.49170
0.2	0.2	0.1	$\pi/3$	0.0	0.7	0.4	0.43570
				0.1			0.44534
				0.4			0.52885
0.2	0.2	0.1	$\pi/3$	0.2	0.1	0.4	0.33570
					0.2		0.41534
					0.4		0.52185
0.2	0.2	0.1	$\pi/3$	0.2	0.7	0.0	0.49660
						0.2	0.49308
						0.4	0.48961

Table 4 Correlation coefficient for skin friction coefficient

r	$\mathrm{Re}_x^{1/2}\,C_x$
M	0.9625690
We	0.9966344
A	0.9973672
ψ	0.9954905

- Strong +ve linear relationship of variables occurs if $0.7 \leq r \leq 1$.
- Strong −ve linear relationship of variables occurs if $-1 \leq r \leq -0.7$.
- No linear relationship of variables holds if $r = 0$.

  It is noted from Table 4, that strongly positive correlation is hold for friction coefficient according to all physical attributes under study. Whereas for the Nusselt number, we have found positive and negative correlation for all the parameters in Table 5.

Table 5 Correlation coefficient for Nusselt number

r	$\mathrm{Re}_x^{-1/2} Nu_x$
M	0.9637310
We	0.9523335
A	0.9993825
ψ	−0.9949968
Pr	0.9890194
Ec	−0.9965454

7 Probable Error (P.E.)

The P.E. of c.r. can be computed by invoking following formula

$$P.E.(r) = 0.6745\frac{\left(1 - r^2\right)}{\sqrt{n}},$$

where c.c. is denoted by r and number of observations is denoted by n. The c.c. is insignificant if r is less than $P.E$. This shows that no correlation between variables exists. The correlation is said to be certain when value of r is 6 times more than the $P.E$. and insignificant when r is less than $P.E.(r)$. This reveals that r is significant. Thus P.E. is computed to see reliability of value of c.c. Probable error of friction and local Nusselt number are given in Tables 6 and 7. It is noted that for insignificant correlation $r < P.E.(r)$, and for significant correlation $r > 6P.E.(r)$.

Table 6 P.E. for skin friction coefficient

$P.E.(r)$	$\mathrm{Re}_x^{1/2} C_x$
M	0.0028607380
We	0.0002616859
A	0.0002047810
ψ	0.0003504271

Table 7 P.E. for Nusselt number

$P.E.(r)$	$\mathrm{Re}_x^{-1/2} Nu_x$
M	0.002773569
We	0.003624006
A	4.80818×10^{-5}
ψ	0.0003886940
Pr	0.0008505241
Ec	0.0002685984

Table 8 Values of $\frac{r}{P.E.(r)}$ for skin friction coefficient

$\frac{r}{P.E.(r)}$	$\mathrm{Re}_x^{1/2} C_x$
M	336.4758
We	3808.515
A	4870.410
ψ	2840.792

Table 9 Values of $\frac{r}{P.E.(r)}$ for Nusselt number

$\frac{r}{P.E.(r)}$	$\mathrm{Re}_x^{-1/2} Nu_x$
M	347.4697
We	262.7847
A	20785.05
ψ	−2559.846
Pr	1162.835
Ec	−1505.629

7.1 Statistical Proclamation

Tables 8 and 9 are made for values of $\frac{r}{P.E.(r)}$. From these tables, it is noted that all values are satisfied abovementioned relation (see Table 8). Also, for ψ and Ec, $r < P.E.(r)$ which tells us the statistically insignificance of correlation coefficient. For $r = 1$, we obtain perfect significant correlation. Consequently, here correlation coefficients are remarkable and parameters are greatly interconnected to physical attributes (see Tables 8 and 9).

8 Conclusions

Here we studied significance of homogeneous/heterogeneous reactants and inclined MHD in stagnant point flow of Walters' B liquid. Heat transfer phenomenon using Newtonian heating is carried out. The key points are mentioned below.

- $f'(\eta)$ is decaying function of M and We for $A < 1$, while it is mounting function of M and We according to $A > 1$.
- Increment in M and We corresponds to a thinner momentum boundary layer.
- Significance rise is noted in temperature profile for higher conjugate parameter.
- Strongly positive correlation exists for friction coefficient according to all the physical attributes on the contrary the negative relation is observed for ψ and Ec with the Nusselt number.

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