Chapter 5
Reference Dynamics Based Motion Planning for Robotic Systems with Flexible Components

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Abstract Analysis of robot dynamics based motion planning is presented in the chapter. Motion of a robot is task based and is formulated upon work dedicated to it. The focus of motion planning is positioning and velocity of the robot end-effector, which are programmed by position and kinematic constraint equations. The constraints are incorporated into the system dynamics, referred to as reference dynamics, whose outputs deliver position and velocity time histories of the end-effector and joints. A special computational procedure for the constrained dynamics generation enables development of the reference dynamics for rigid and flexible system models such that vibration, allowable velocity profiles for robot joints and other programmed motion kinematic properties can be analyzed. This analysis enables planning feasible tasks for robots and design controllers for vibration compensation.

5.1 Introduction

Analysis of robot motion planning, which is task based and dedicated to work and services delivery is presented in the chapter. The focus of motion planning is positioning and velocity of the robot end-effector, which are programmed by position and kinematic constraint equations. The constraints are incorporated into the system dynamics, referred to as reference dynamics, whose outputs deliver position and velocity time histories of the end-effector and joints. The key contribution of the paper is in three aspects of the analysis. The first one is the possibility of kinematic constraints incorporation into the system dynamics, the second one is the special computational procedure for the constrained dynamics generation, which

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provides reference dynamic models satisfying all constraints upon them. The third contribution, comparing to the results reported in the literature, see e.g. [1, 2], is in modelling flexibility of the system parts and supports. The constraints put on a system are referred to as programmed and they are imposed as control goals on system performance or service task requirements. The procedure of generating reference dynamics offers automated derivation of equations and it was successfully developed and implemented to rigid system models [5, 6]. The advantage of this procedure is that it serves both reference and control oriented dynamics derivation and the final dynamic models are obtained in the reduced state form, i.e. constraint reaction forces are eliminated. The procedure is extended on flexible subsystems of lightweight and fast machine parts and servicing equipment, which are prone to vibration in some work regimes. Vibrations may significantly affect system performance and disable effective controller designs. Flexibility of links is modelled using the rigid finite element method [3, 9, 10], which advantage relies in its ability of application of the rigid body approach to modelling flexible link elements. The novelty of the presented method is its ability to analyze any system reference motion, including flexible link vibration, in the presence of kinematic task-based constraints. The results of this analysis may contribute to verification of a system behavior when it is subjected to given kinematic constraints, help to specify desired task-based constraints properly, e.g. put tighter velocity limits, to exclude some work regimes or to design controllers correctly. The possibility of reference motion analysis, e.g. desired positions of the robot end-effector [7], in these aspects enables simulation of various work regimes for system models and selection of required and safe task based motion parameters, accordingly. The special interest of this chapter is paid to plan a robot end-effector desired velocity. Usually, the robot velocity is controlled through the joint velocities at the kinematics or dynamic levels, see e.g. [8] and references there. In [8], traditional velocity control of robot manipulators in joint space, assuming Lagrangian non-linear dynamics, by three control schemes resulting as extensions of proportional-integral velocity regulators of direct current motors is examined. In this chapter a method of desired position and velocity specification through the programmed constraints is provided. It is followed by subsequent derivation of the reference dynamics for the constrained motion and its analysis. The outcomes of the reference dynamics can be used for motion controller design. The theoretical development presented in the chapter is illustrated by simulation studies of an example of a flexible link and support manipulator model, whose service tasks are predefined by the programmed constraints, specifically its end-effector is to move according to some velocity profile. Special interest of this study is to analyze flexible link vibrations when the manipulator performs the desired tasks.

The chapter is organized as follows. After introduction, Sect. 5.2 reports the generalized programmed motion equations derivation procedure briefly. Section 5.3 presents formulation of the programmed constraints for the manipulator end-effector. In Sect. 5.4 the GPME (generalized programmed motion equations) for the manipulator are derived and simulation studies demonstrating programmed motion executions are presented in Sect. 5.5. The chapter closes with conclusions and the list of references.

5.2 Reference Dynamics Model of a Flexible Supported Manipulator

A model of the flexible supported manipulator is presented in Fig. 5.1. The manipulator is driven by a driving torque $\mathbf{t}_{dr}^{(2)}$. It is assumed that the third link of the manipulator can be treated as rigid or flexible. If the flexibility of this link is taken into account, the rigid finite element method is used for discretization.

A vector of generalized coordinates (joint coordinates) which describes motion of the manipulator has the following form:

$$\mathbf{q} = (q_i)_{i=1,\dots,n_{dof}} = \left[\, \tilde{\mathbf{q}}^{(1)^T} \; \tilde{\mathbf{q}}^{(2)^T} \; \tilde{\mathbf{q}}^{(3)^T} \; \tilde{\mathbf{q}}^{(4)^T} \,\right]^T, \quad (5.1)$$

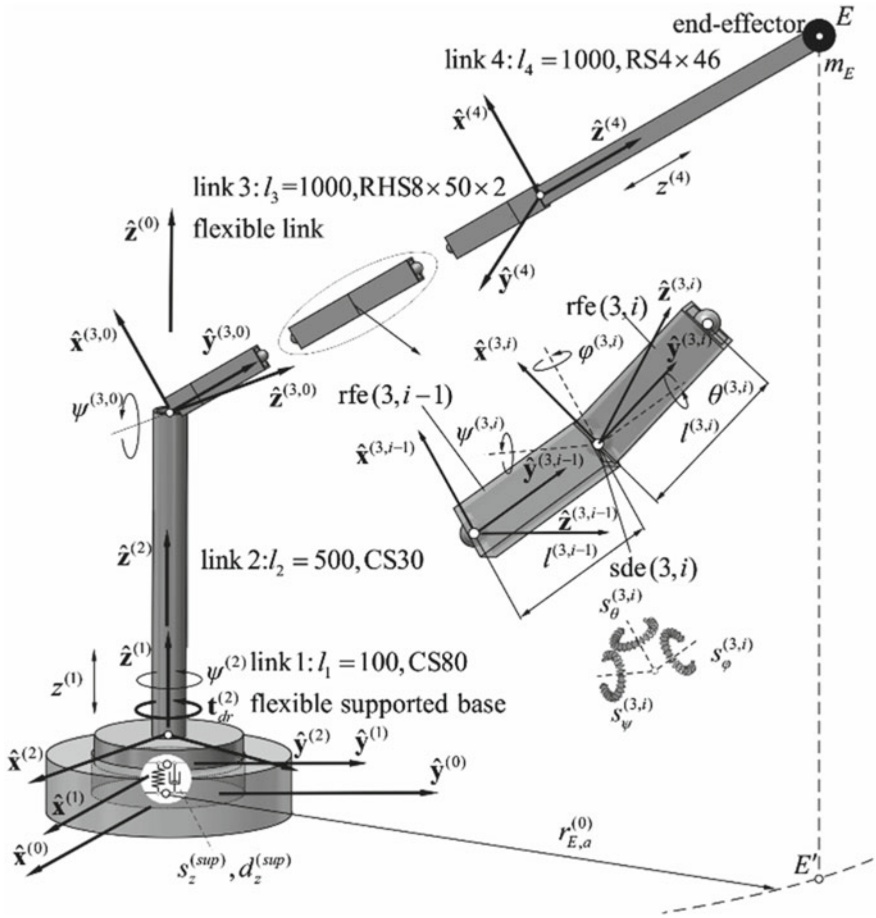

Fig. 5.1 Model of a manipulator

where: $\tilde{\mathbf{q}}^{(1)} = \left[z^{(1)}\right]$, $\tilde{\mathbf{q}}^{(2)} = \left[\psi^{(2)}\right]$, $\tilde{\mathbf{q}}^{(3)} = \left[\begin{array}{cc}\tilde{\mathbf{q}}^{(3,0)^T} & \tilde{\mathbf{q}}_f^{(3)^T}\end{array}\right]^T$

$$\tilde{\mathbf{q}}_f^{(3)} = \begin{cases} \varnothing & -\text{rigid link} \\ \left[\begin{array}{ccccc}\tilde{\mathbf{q}}^{(3,1)^T} & \ldots & \tilde{\mathbf{q}}^{(3,r)^T} & \ldots & \tilde{\mathbf{q}}^{(3,n_{rfe}^{(3)}-1)^T}\end{array}\right]^T & -\text{flexible link} \end{cases}$$

$$\tilde{\mathbf{q}}^{(3,0)} = \left[\psi^{(3,0)}\right], \tilde{\mathbf{q}}^{(3,r)} = \left[\begin{array}{ccc}\psi^{(3,r)} & \theta^{(3,r)} & \varphi^{(3,r)}\end{array}\right]^T, \tilde{\mathbf{q}}^{(4)} = \left[z^{(4)}\right].$$

The generalized coordinates describing motion of the link l with respect to the global reference frame can be presented in the following form:

$$\mathbf{q}^{(l)}\big|_{l=1,\ldots,n_l} = \left(q_i^{(l)}\right)_{i=1,\ldots,n_{dof}^{(l)}} = \left[\begin{array}{cc}\mathbf{q}^{(l-1)^T} & \tilde{\mathbf{q}}^{(l)^T}\end{array}\right]^T, \tag{5.2}$$

where $\mathbf{q}^{(0)} = \varnothing,\ \mathbf{q}^{(l-1)}\big|_{l=4} = \begin{cases}\mathbf{q}^{(3,0)} & -\text{rigid link} \\ \mathbf{q}^{(3,n_{rfe}^{(3)}-1)} & -\text{flexible link}\end{cases}.$

In further considerations the generalized coordinates vector $\mathbf{q}$ is divided into independent and dependent coordinates as follows:

$$\mathbf{q} = \begin{bmatrix}\mathbf{q}_{i_c} \\ \mathbf{q}_{d_c}\end{bmatrix}, \tag{5.3}$$

where $\mathbf{q}_{i_c} = \left(q_j\right)_{j\in i_{i_c}}$, $\mathbf{q}_{d_c} = \left(q_j\right)_{j\in i_{d_c}}$, and i_{i_c} and i_{d_c} stand for subscripts for describing dependent and independent coordinates. For the analyzed robotic system the following division into dependent and independent coordinates is assumed:

$$i_{d_c} \in \left\{3, n_{dof}\right\}, \tag{5.4.1}$$

$$i_{i_c} \in \left\{1, 2, \ldots, n_{dof}\right\} - i_{d_c}, \tag{5.4.2}$$

The generalized programmed motions equations (GPME) with first order programmed constraints are derived according to the procedure described in [5, 6]. This procedure requires formulation of some function R_1 which components depend on the kinetic and potential energy of the system, i.e.

$$R_1 = \sum_{l=1}^{n_l} \dot{E}_k^{(l)} + \sum_{l=1}^{n_l}\sum_{i=1}^{n_{dof}^{(l)}}\left(\frac{\partial E_{p,g}^{(l)}}{\partial q_i^{(l)}}\dot{q}_i^{(l)} - 2\frac{\partial E_k^{(l)}}{\partial q_i^{(l)}}\dot{q}_i^{(l)}\right) + \sum_{r=1}^{n_{rfe}^{(3)}-1}\sum_{i=1}^{n_{dof}^{(3,r)}}\frac{\partial E_{p,f_l}^{(3,r)}}{\partial q_i^{(3,r)}}\dot{q}_i^{(3,r)}$$
$$+ \frac{\partial R_{sup}^{(1)}}{\partial \dot{q}_1^{(1)}}\dot{q}_1^{(1)} + \frac{\partial E_{p,sup}^{(1)}}{\partial q_1^{(1)}}\dot{q}_1^{(1)} - \sum_{i=1}^{n_{dof}^{(2)}} t_i \dot{q}_i^{(2)}, \tag{5.5}$$

where:

$$E_k^{(l)} = \begin{cases} \frac{1}{2}\mathrm{tr}\left\{\dot{\mathbf{T}}^{(l)}\mathbf{H}^{(l)}\left(\dot{\mathbf{T}}^{(l)}\right)^T\right\}, & l \neq 3 \\ \frac{1}{2}\sum_{r=0}^{n_{rfe}^{(l)}-1} \mathrm{tr}\left\{\dot{\mathbf{T}}^{(l,r)}\mathbf{H}^{(l,r)}\left(\dot{\mathbf{T}}^{(l,r)}\right)^T\right\}, & l = 3 \end{cases}$$ – the kinetic energy of the link l,

$$E_{p,g}^{(l)} = \begin{cases} m^{(l)} g \mathbf{J}_3 \mathbf{T}^{(l)} \mathbf{r}_{C^{(l)}}^{(l)}, & l \neq 3 \\ \sum_{r=0}^{n_{rfe}^{(l)}-1} m^{(l,r)} g \mathbf{J}_3 \mathbf{T}^{(l,r)} \mathbf{r}_{C^{(l,r)}}^{(l,r)}, & l \neq 3 \end{cases}$$ – the potential energy of gravity forces of the link l,

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 \\ \mathbf{J}_2 \\ \mathbf{J}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$E_{p,sup}^{(1)} = \frac{1}{2} s_z^{(sup)} \left(z^{(1)}\right)^2$, $R_{sup}^{(1)} = \frac{1}{2} d_z^{(sup)} \left(\dot{z}^{(1)}\right)^2$—the spring deformation energy and the Rayleigh dissipation function of the flexible supported base,

$E_{p,fl}^{(3,r)} = \frac{1}{2}\left(\tilde{\mathbf{q}}^{(3,r)}\right)^T \mathbf{S}^{(3,r)} \tilde{\mathbf{q}}^{(3,r)}$, $\mathbf{S}^{(3,r)} = \mathrm{diag}\left\{s_\psi^{(3,r)}, s_\theta^{(3,r)}, s_\varphi^{(3,r)}\right\}$—the spring deformation energy of the flexible link,

$\mathbf{t} = \left[\, 0 \;\; t_{dr}^{(2)} - t_{res}^{(2)} \,\right]^T$—vector containing the driving torques.

Finally, the GPME dynamic equations of motion of the robotic system subjected to programmed constraints can be obtained as:

$$\left.\frac{\partial R_1}{\partial \dot{q}_i}\right|_{i \in i_{i_c}} + \sum_{j \in i_{d_c}} \frac{\partial R_1}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_i} = 0, \tag{5.6}$$

Notice, that the number of dynamic equations of motion (5.6) is equal to the number of the independent coordinates n_{i_c}. Therefore, the resulting equations have to be supplemented by the programmed constraints equations, in this case of the first order.

5.3 Programmed Constraint Equations Formulation

A programmed motion of the manipulator is defined as follows: the end-effector moves along an elliptical trajectory designed in a plane parallel to the $\hat{\mathbf{x}}^{(0)}\hat{\mathbf{y}}^{(0)}$ plane and the velocity of the end-effector in $z^{(0)}$ direction has to change according to time-dependent function $\dot{z}_{E,a}^{(0)}(t)$ shown in Fig. 5.2.

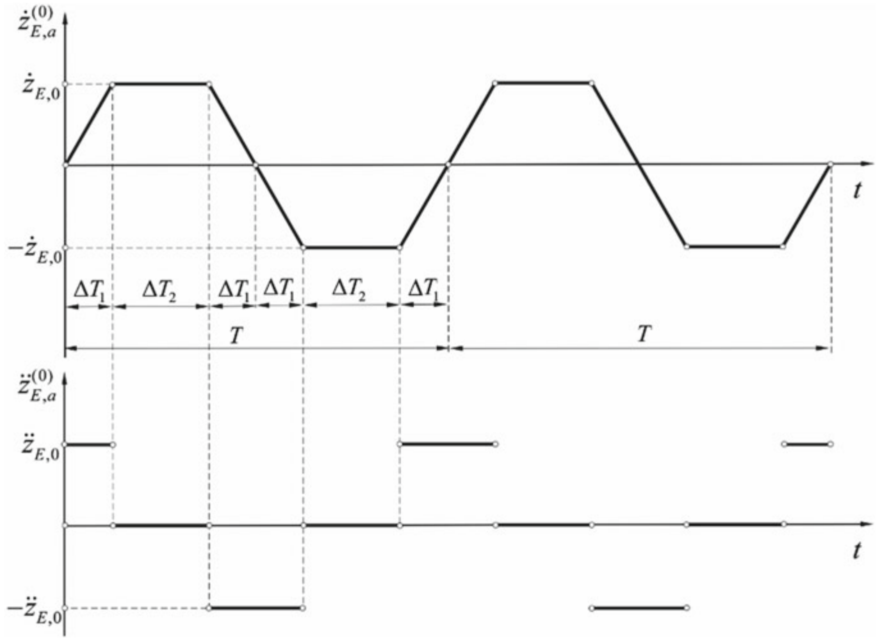

Fig. 5.2 Assumed time courses of the velocity and acceleration of the end-effector

The constraint equations corresponding to the proposed programmed motion can be presented as:

$$\Phi_1 \equiv 0 \Rightarrow \left(\frac{x_E^{(0)}}{a_{E,a}^{(0)}}\right)^2 + \left(\frac{y_E^{(0)}}{b_{E,a}^{(0)}}\right)^2 - 1 = 0 \tag{5.7.1}$$

$$\dot{\Phi}_2 \equiv 0 \Rightarrow \dot{z}_E^{(0)} - \dot{z}_{E,a}^{(0)}(t) = 0 \tag{5.7.2}$$

where $x_E^{(0)} = \mathbf{J}_1\mathbf{T}^{(4)}\mathbf{r}_E^{(4)}$, $y_E^{(0)} = \mathbf{J}_2\mathbf{T}^{(4)}\mathbf{r}_E^{(4)}$, $z_E^{(0)} = \mathbf{J}_3\mathbf{T}^{(4)}\mathbf{r}_E^{(4)}$. It can be noticed that the first constraint equation is defined at the position level and the second one is kinematic, i.e. formulated at the velocity level.

The GPME algorithm [5] requires the constraint equations in differentiated form which can be calculated as follows:

$$\dot{\Phi}_1 \equiv 0 \Rightarrow \mathbf{u}\dot{\mathbf{q}} = \mathbf{0}, \tag{5.8.1}$$

$$\dot{\Phi}_2 \equiv 0 \Rightarrow \mathbf{C}_3\dot{\mathbf{q}} - \dot{z}^{(0)}_{E,a}(t) = 0, \tag{5.8.2}$$

$$\ddot{\Phi}_1 \equiv 0 \Rightarrow \mathbf{u}\ddot{\mathbf{q}} + v = 0, \tag{5.8.3}$$

$$\ddot{\Phi}_2 \equiv 0 \Rightarrow \mathbf{C}_3\ddot{\mathbf{q}} + d_3 - \ddot{z}^{(0)}_{E,a}(t) = 0, \tag{5.8.4}$$

with:

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{bmatrix} = \left(c_{ij}\right)_{\substack{i=1,2,3 \\ j=1,\ldots,4}} = \mathbf{J}\left[\, \mathbf{T}^{(4)}_1\mathbf{r}^{(4)}_E \; \cdots \; \mathbf{T}^{(4)}_4\mathbf{r}^{(4)}_E \,\right],$$

$$\mathbf{d} = (d_i)_{i=1,\ldots,3} = \mathbf{J}\left(\left(\sum_{i=1}^{n_{dof}}\sum_{j=1}^{n_{dof}} \mathbf{T}^{(4)}_{ij}\dot{q}_i\dot{q}_j\right)\mathbf{r}^{(4)}_E\right),$$

$$\mathbf{u} = \left(u_j\right)_{j=1,\ldots,4} = \frac{1}{\left(a^{(0)}_{E,a}\right)^2}\mathbf{J}_1\mathbf{T}^{(4)}\mathbf{r}^{(4)}_E\mathbf{C}_1 + \frac{1}{\left(b^{(0)}_{E,a}\right)^2}\mathbf{J}_2\mathbf{T}^{(4)}\mathbf{r}^{(4)}_E\mathbf{C}_2,$$

$$v = \frac{1}{\left(a^{(0)}_{E,a}\right)^2}\left((\mathbf{C}_1\dot{\mathbf{q}})^2 + \mathbf{J}_1\mathbf{T}^{(4)}\mathbf{r}^{(4)}_E d_1\right) + \frac{1}{\left(b^{(0)}_{E,a}\right)^2}\left((\mathbf{C}_2\dot{\mathbf{q}})^2 + \mathbf{J}_2\mathbf{T}^{(4)}\mathbf{r}^{(4)}_E d_2\right),$$

$a^{(0)}_{E,a}$, $b^{(0)}_{E,a}$ are elliptical trajectory semi-major and semi-minor axes, respectively.

Relations between dependent and independent velocities can be found by proper transformations of (5.8). As the result, these relations can be written as follows:

$$\dot{\mathbf{q}}_{d_c} = -\mathbf{K}^{-1}_{d_c}\mathbf{K}_{i_c}\dot{\mathbf{q}}_{i_c}, \tag{5.9}$$

where:

$$\mathbf{K}_{d_c} = \begin{bmatrix} u_2 & u_4 \\ c_{12} & c_{14} \end{bmatrix}, \mathbf{K}_{i_c} = \begin{bmatrix} u_1 & u_3 \\ c_{11} & c_{13} \end{bmatrix}.$$

5.4 The GPME Generation for the Constrained Manipulator Model

The GPME (6) supplemented by the equations of the programmed constraints (5.8.3) and (5.8.4) in the stabilized form can be written as follows:

$$\begin{bmatrix} \mathbf{M}_{1\cdot} + \sum\limits_{j\in i_{dc}} \mathbf{M}_{j\cdot} \frac{\partial \dot{q}_j}{\partial \dot{q}_1} \\ \vdots \\ \mathbf{M}_{n_{ic}\cdot} + \sum\limits_{j\in i_{dc}} \mathbf{M}_{j\cdot} \frac{\partial \dot{q}_j}{\partial \dot{q}_{n_{ic}}} \\ \mathbf{u} \\ \mathbf{C}_3 \end{bmatrix} \ddot{\mathbf{q}}$$

$$= \begin{bmatrix} h_1 + Q_1 + \sum\limits_{j=1}^{n_{dof}} \dot{q}_j \frac{\partial Q_j}{\partial \dot{q}_1} + \sum\limits_{k\in i_{dc}} \left(h_k + Q_k + \sum\limits_{j=1}^{n_{dof}} \dot{q}_j \frac{\partial Q_j}{\partial \dot{q}_k} \right) \frac{\partial \dot{q}_k}{\partial \dot{q}_1} \\ \vdots \\ h_{n_{ic}} + Q_{n_{ic}} + \sum\limits_{j=1}^{n_{dof}} \dot{q}_j \frac{\partial Q_j}{\partial \dot{q}_{n_{ic}}} + \sum\limits_{k\in i_{dc}} \left(h_k + Q_k + \sum\limits_{j=1}^{n_{dof}} \dot{q}_j \frac{\partial Q_j}{\partial \dot{q}_k} \right) \frac{\partial \dot{q}_k}{\partial \dot{q}_{n_{ic}}} \\ -v - 2\alpha\, \dot{\Phi}_1 - \beta^2\, \Phi_1 \\ -d_3 + \ddot{z}_{E,a}^{(0)}(t) - 2\alpha\, \dot{\Phi}_2 \end{bmatrix}, \quad (5.10)$$

where:

$$\mathbf{M} = \sum_{\substack{l=1 \\ l\neq 3}}^{n_l} \mathbf{M}^{(l)} + \sum_{r=0}^{n_{rfe}^{(3)}-1} \mathbf{M}^{(3,r)}, \mathbf{M}^{(\bullet)} = \left(m_{ij}^{(\bullet)} \right)_{i,j=1,\ldots,n_{dof}^{(\bullet)}}, m_{ij}^{(\bullet)} = \mathrm{tr}\left\{ \mathbf{T}_i^{(\bullet)} \mathbf{H}^{(\bullet)} \left(\mathbf{T}_j^{(\bullet)} \right)^T \right\},$$

$$\mathbf{M}_{i\cdot}|_{i\in i_{ic}} = \mathrm{row}_i(\mathbf{M}),\ \mathbf{M}_{j\cdot}\big|_{j\in i_{dc}} = \mathrm{row}_j(\mathbf{M}),$$

$$\mathbf{h} = \sum_{\substack{l=1 \\ l\neq 3}}^{n_l} \mathbf{h}^{(l)} + \sum_{r=0}^{n_{rfe}^{(3)}-1} \mathbf{h}^{(3,r)}, \mathbf{h}^{(\bullet)} = \left(h_i^{(\bullet)} \right)_{i=1,\ldots,n_{dof}^{(\bullet)}},$$

$$h_i^{(\bullet)} = \sum_{m=1}^{n_{dof}^{(\bullet)}} \sum_{n=1}^{n_{dof}^{(\bullet)}} \mathrm{tr}\left\{ \mathbf{T}_m^{(\bullet)} \mathbf{H}^{(\bullet)} \left(\mathbf{T}_{m,n}^{(\bullet)} \right)^T \right\} \dot{q}_m^{(\bullet)} \dot{q}_n^{(\bullet)} + 2 \sum_{m=1}^{n_{dof}^{(\bullet)}} \sum_{n=1}^{n_{dof}^{(\bullet)}} \mathrm{tr}\left\{ \mathbf{T}_m^{(\bullet)} \mathbf{H}^{(\bullet)} \left(\mathbf{T}_{i,n}^{(\bullet)} \right)^T \right\} \dot{q}_m^{(\bullet)} \dot{q}_n^{(\bullet)},$$

$$\mathbf{T}_i^{(\bullet)} = \frac{\partial \mathbf{T}^{(\bullet)}}{\partial q_i^{(\bullet)}}, \mathbf{T}_{i,j}^{(\bullet)} = \frac{\partial^2 \mathbf{T}^{(\bullet)}}{\partial q_i^{(\bullet)} \partial q_j^{(\bullet)}},$$

$$\mathbf{Q} = -\left(\mathbf{g} + \mathbf{d}_{sup} + \mathbf{f}_{sup} + \mathbf{f}_{f_l} \right) + \mathbf{t}_{dr},$$

$$\mathbf{g} = \sum_{\substack{l=1 \\ l\neq 3}}^{n_l} \mathbf{g}^{(l)} + \sum_{r=0}^{n_{rfe}^{(1)}-1} \mathbf{g}^{(3,r)}, \mathbf{g}^{(\bullet)} = \left(g_i^{(\bullet)} \right)_{i=1,\ldots,n_{dof}^{(\bullet)}}, g_i^{(\bullet)} = m^{(\bullet)} g \mathbf{J}_3 \mathbf{T}_i^{(\bullet)} \mathbf{r}_{C^{(\bullet)}}^{(\bullet)},$$

$$\mathbf{f}_{sup} = \left(f_{sup,i} \right)_{i=1,\ldots,n_{dof}} = \left[s_z^{(sup)} z^{(1)}\ \mathbf{0} \right]^T, \mathbf{d}_{sup} = \left(d_{sup,i} \right)_{i=1,\ldots,n_{dof}} = \left[d_z^{(sup)} \dot{z}^{(1)}\ \mathbf{0} \right]^T,$$

$$\mathbf{f}_{f_l} = \left(f_{f_l,i} \right)_{i=1,\ldots,n_{dof}} = \left[\mathbf{0}\ \mathbf{S}^{(3,1)} \tilde{\mathbf{q}}^{(3,1)}\ \ldots\ \mathbf{S}^{(3,n_{rfe}^{(3)}-1)} \tilde{\mathbf{q}}^{(3,n_{rfe}^{(3)}-1)} \right],$$

$$\mathbf{t}_{dr} = \left(t_{dr,i}\right)_{i=1,\dots,n_{dof}} = \left[\mathbf{t}^T \ \mathbf{0}\right]^T,$$

in which **H** is the pseudo-inertia matrix, g is the acceleration of gravity, and α, β are coefficients for the Baumgarte method.

For simulations, the Baumgarte stabilization method [4] is applied to eliminate constraint violation at position and velocity levels. The Baumgarte method is simple in implementations and provides satisfactory stabilization results to constrained motion equations solutions. Unfortunately, the parameters α, β have to be selected by the trial and error method. However, the method returns good adjustment of the parameters, such which secure the solution convergence, after a couple of simulation runs.

Finally, the dynamic equations of motion (5.10) form $(n_{dof} - 2)$ ordinary differential equations, which need to be supplemented by 2 equations corresponding to the programmed constraints (5.8.3) and (5.8.4). Notice that the GPME derivation procedure eliminates the constraint reaction forces from the equations. Thus (5.10) are the smallest set of motion equations.

5.5 Numerical Simulation Studies—End-Effector Programmed Motion

The procedure of generation of reference dynamics (5.10) enables simulations of the programmed motion of the flexible supported three link manipulator. Parameters of the analyzed robotic system applied in simulations are gathered in Table 5.1.

Table 5.1 Parameters of the manipulator

Parameters	Value
Stiffness coefficient of the support $s_z^{(sup)}$	10^4 Nm^{-1}
Damping coefficient of the support $d_z^{(sup)}$	2.5 N s m^{-1}
Young modulus E	$2.1 \cdot 10^5$ MPa
Poisson ratio v	0.3
Density ρ	7801 kg m^3
Mass of the end-effector m_E	2 kg
Number of rigid finite elements $n_{rfe}^{(3)}$	4
Semi-major radius of the ellipse-shaped trajectory $a_{E,a}^{(0)}$	0.875 m
Semi-minor radius of the ellipse-shaped trajectory $b_{E,a}^{(0)}$	1.75 m
The Baumgarte coefficient α	10^3
The Baumgarte coefficient β	10^2

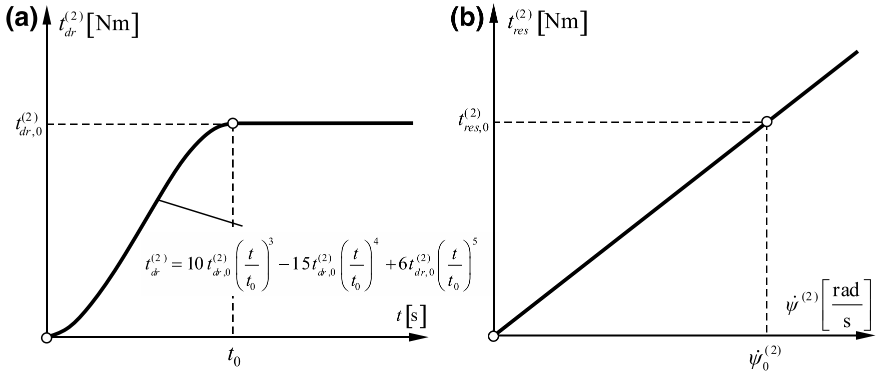

Fig. 5.3 Courses of the driving and resistance torques

Motion of the manipulator is forced by the driving torque together with the resistance torque applied to column (2) whose courses are presented Fig. 5.3.

The following is assumed: $t_{dr,0}^{(2)} = 2\,\text{Nm}$, $t_0 = 10$ and $\psi_0^{(2)} = 9\,\text{Nm}$.

At the initial configuration the flexible link is inclined to the plane $\hat{\mathbf{x}}^{(0)}\hat{\mathbf{y}}^{(0)}$ at the angle 45°. The generalized coordinates vector at time $t = 0\,\text{s}$ can be presented as follows:

$$q_i|_{\substack{t=0\text{s} \\ i=1,\ldots,n_{dof}}} = \begin{cases} 0, & i \neq 3 \\ 45°, & i = 3 \end{cases} \tag{5.11.1}$$

$$\dot{q}_i|_{\substack{t=0\text{s} \\ i=1,\ldots,n_{dof}}} = 0 \tag{5.11.2}$$

At the first step of the simulation, static analysis of the manipulator is performed. Initially, the position of the end-effector with respect to the inertial frame was $\mathbf{r}_E^{(0)} = \begin{bmatrix} 1.4142 & 0 & 2.5142 & 1 \end{bmatrix}^T$. After applying the gravity forces, the position of the end-effector changed to $\mathbf{r}_E^{(0)} = \begin{bmatrix} 1.5065 & 0 & 2.3813 & 1 \end{bmatrix}^T$. The dynamic equations of motion are integrated using 4th order Runge-Kutta scheme with the constant step size $h = 10^{-3}$ s, when all links of the manipulator are treated as rigid, and $h = 10^{-5}$ s if link's flexibility is taken into account. Time courses of values of displacements of the base and joints are shown in Fig. 5.4.

It can be noticed that flexibility of link (3) has a significant effect on dynamics of the manipulator. Vibrations due to link's flexibility are compensated by an appropriate selection of the drive function acting in joint (3, 0). The influence of link's (3) flexibility on other links and the flexible supported base motion is also clearly visible.

Programmed trajectories of the end-effector projected on planes $\hat{\mathbf{x}}^{(0)}\hat{\mathbf{y}}^{(0)}$ and $\hat{\mathbf{x}}^{(0)}\hat{\mathbf{z}}^{(0)}$ are presented in Fig. 5.5. Time courses of the end-effector vertical displacement and velocity are shown in Fig. 5.6.

Analyzing the obtained results, it can be stated that the programmed constraints resulting from the assumed elliptical trajectory and the end-effector vertical veloc-

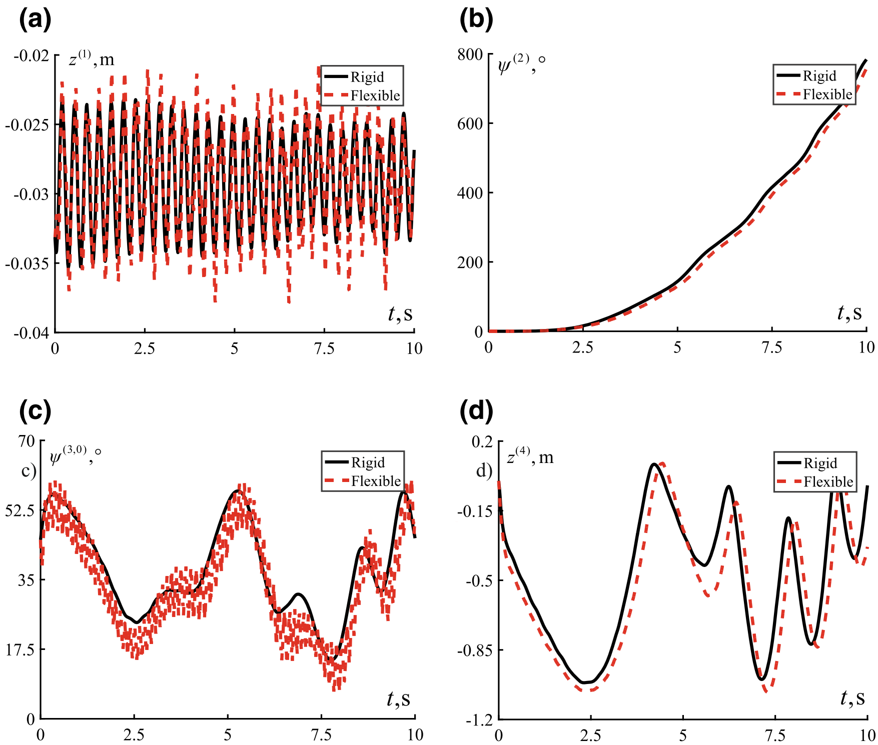

Fig. 5.4 Time course of the joint coordinates

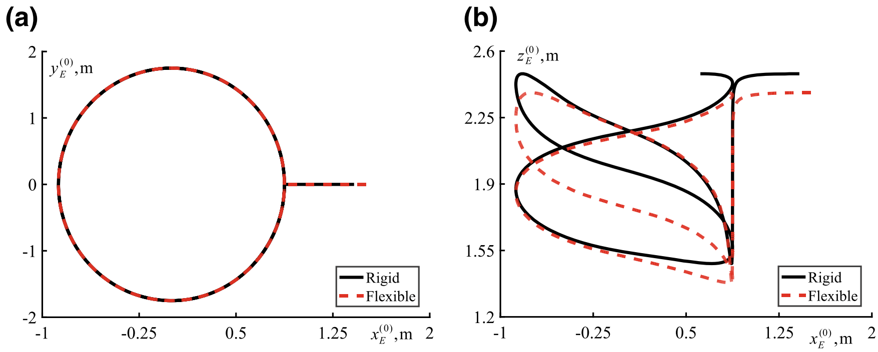

Fig. 5.5 A trajectory of the end-effector

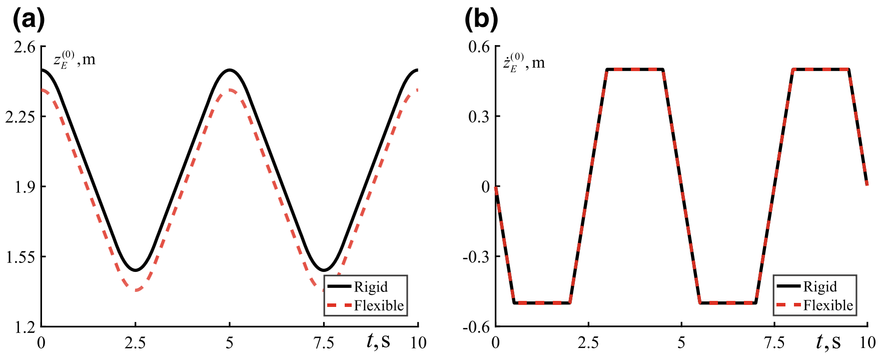

Fig. 5.6 Z-component of the end-effector displacement and the velocity vector

ity are satisfied. Thus, the reference dynamics provides motion samples along the specified program. This motion can be analyzed with respect to the manipulator capabilities and vibrations of its links. For the model with the flexible link, it can be seen that the trajectory of the end-effector in the plane $\hat{\mathbf{x}}^{(0)}\hat{\mathbf{z}}^{(0)}$ is also well executed comparing to the trajectory obtained for the model with rigid links. Moreover, this offset results from the initial deformation of the flexible link (3) due to the effects of gravity. This is also noticeable in time courses of the end-effector displacement $z_E^{(0)}$.

5.6 Conclusions

In the chapter a mathematical model of the example of a robotic system with a flexible link is analyzed. Its motion is subjected to the programmed constraints resulting from task oriented motion planning. An automated numerical procedure based on the GMPE algorithm was applied to generate dynamic equations of a multi-link robotic system subjected to programmed constraints. It enables calculating time histories of joint positions and their derivatives in the programmed motion. Joint coordinates and homogeneous transformation matrices were applied to describe kinematics of the manipulator. Links flexibility was modelled using the rigid finite element method. The presented approach can be easily generalized for robotic systems with any number of rigid or flexible links subjected to other kinematic constraint equations. The reference dynamics and the results of the programmed motion analysis can be used directly for a motion controller design what is the next step of the research plan.

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