Chapter 11
Complex Fractional Moments for the Characterization of the Probabilistic Response of Non-linear Systems Subjected to White Noises

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Abstract In this chapter the solution of Fokker-Planck-Kolmogorov type equations is pursued with the aid of Complex Fractional Moments (CFMs). These quantities are the generalization of the well-known integer-order moments and are obtained as Mellin transform of the Probability Density Function (PDF). From this point of view, the PDF can be seen as inverse Mellin transform of the CFMs, and it can be obtained through a limited number of CFMs. These CFMs' capability allows to solve the Fokker-Planck-Kolmogorov equation governing the evolutionary PDF of non-linear systems forced by white noise with an elegant and efficient strategy. The main difference between this new approach and the other one based on integer moments lies in the fact that CFMs do not require the closure scheme because a limited number of them is sufficient to accurately describe the evolutionary PDF and no hierarchy problem occurs.

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11.1 Introduction

The dawns of stochastic differential calculus is dated back to the last century. Thanks to the pioneering papers of Itô, Wong and Zakaj, Kolmogorov and other authors new horizons were opened giving rise to the modern stochastic mechanics [5, 17, 23, 31]. In this context, a relevant problem is represented by the study of nonlinear system forced by normal white noise. Excitations such as ground motion, wind turbulence, sea waves, surface roughness, blasts and impacts loads being stochastic processes induce that structural responses are stochastic processes too. Thus, the analysis is concerned with the problem of the response statistical characterization. An approach to describe this kind of problems, that is typical of several physical applications [16], is based on the study of the Fokker-Planck equation (FPK) which represents a partial differential equation that describes the evolution of the response conditional probability density function (PDF).

Nowadays, the resolution of FPK equation or its generalized form for different kind of forced white noise (Poissonian, α-stable, etc.) still represents an open problem. Indeed, the FPK equation admits analytical solution in very few cases, for this reason we resort to numerical methods. A possible way to treat such partial differential equation problem is related to the evaluation of the moments of the PDF. This method consists in writing differential equations for the response statistical moments of any order. However, when dealing with nonlinear systems, a serious problem arises in the Moment Equation (ME) approach, the entire system is hierarchic in the sense that the equations for the moments of a fixed order, say K, contain moments of order higher than K. In this way, the ME form an infinite hierarchy. Then, due to the hierarchical nature of the forcing processes, this approach needs a truncation of the involved higher-orders moments in the solution.

Although other strategies, based upon the Hermite polynomials, and cumulants, provide some solutions for a certain few cases, these approaches show some particular limits [13, 16, 25, 26, 34]. Certainly, a meaningful limit of such classical methods is the inability to well describe the tails of the PDF that leads to serious problem in reliability analysis.

Other more complex approaches are available in literature but they are not discussed here for sake of brevity [11, 14, 20, 22, 24, 27, 29, 32, 35, 37]. Instead, in this chapter we focus on a recent development in the resolution of the FPK based on the moment approach [1, 12, 14]. Such recent improvement is obtained thanks to the introduction of the complex-order moments. It has been shown that these complex quantities, known as Complex Fractional Moments (CFMs), are related to the Mellin transform and to the Riesz integral at the origin of the PDF [30, 33]. Moreover, the link between CFMs, Mellin transform and Riesz integrals has provided several important relations and properties [7, 9, 15]. Undoubtedly, one of important properties of CFMs is the capability to reconstruct both PDF and characteristic function. Therefore, the knowledge of the CFMs represents another way to characterize random variables. As will be shown later, this property is fundamental for the resolution of FPK by this new approach. Further information on the applications of these complex quantities can be found in [2, 8, 11, 21, 28, 36].

11.2 Basic Concepts on Mellin Transform Operator

The Mellin transform operator is a very interesting tool of fractional calculus. It proves to be very useful in solving some problems of engineering interest [1, 2, 4, 7–12, 14, 15, 18, 28, 36]. Let $f(x)$ be any real function defined in $0 \leq x < \infty$. The Mellin transform, labeled as $M_f(\gamma - 1)$, is defined as

$$\mathcal{M}\{f(x);\gamma\} = M_f(\gamma - 1) = \int_0^\infty f(x)x^{\gamma-1}dx; \quad \gamma = \rho + i\,\eta \tag{11.1}$$

where $i = \sqrt{-1}$ and $\rho, \eta \in \mathbb{R}$.

If the Mellin transform exists, then the function $f(x)$ may be rewritten in the form

$$f(x) = \mathcal{M}^{-1}\left\{M_f(\gamma - 1); x\right\} = \frac{1}{2\pi}\int_{\eta=-\infty}^{\infty} M_f(\gamma - 1)x^{-\gamma}d\eta\,;\, x > 0 \tag{11.2}$$

It is noted that the integration is performed along the imaginary axis and the value of ρ remains fixed. The condition for the existence of the Mellin transform is that $-p < \rho < -q$, being p and q the order of zero at $x = 0$ and $x = \infty$, respectively. Namely

$$\lim_{x\to 0} f(x) = O(x^p); \quad \lim_{x\to\infty} f(x) = O(x^q) \tag{11.3}$$

where $O(\cdot)$ stands for the order of the term in parenthesis.

For example, let us assume that $f(x) = (1 + x)^{-1}$, since $\lim_{x\to 0} f(x) = 1\left[O(x^0)\right]$ then $p = 0$, and $\lim_{x\to\infty} f(x) = x^{-1}\left[O(x^0)\right]$, then $q = -1$; it follows that in this case the existence condition of the Mellin transform is $0 < \rho < 1$.. The strip in the complex plane such that $-p < \rho < -q$ is commonly known as *Fundamental Strip* (FS) of the Mellin transform. If $-q$ is lesser than $-p$ the Mellin transform and its inverse do not exist.

Equation (11.2) may be used in a discretized form as

$$f(x) \cong \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} M_f(\gamma_k - 1)x^{-\gamma_k} \quad ; \quad \gamma_k = \rho + i\,k\,\Delta\eta \tag{11.4}$$

where $\Delta\eta$ is the discretization step along to the imaginary axis, $m\Delta\eta = \overline{\eta}$ is a cut-off value chosen in such a way that the contribution of terms of higher order than m do not produce sensible variations on $f(x)$. It is to be remarked that $M_f(\gamma - 1)$ is analytic onto the fundamental strip, and is such that

$$M_f(\rho + i\,\eta - 1) = M_f^*(\rho - i\,\eta - 1) \tag{11.5}$$

where the star means complex conjugate. It follows that with simple manipulations the summation in (11.4) may be rewritten in a summation from 0 to m.

The Riesz fractional integral of a certain function $f(x)$ that is zero for $x < 0$, denoted as $(I^\gamma f)(x)$, is defined as

$$(I^\gamma f)(x) = \frac{1}{2\nu_c(\gamma)} \int_0^\infty f(\xi)|x - \xi|^{\gamma-1} d\xi; \quad \rho > 0, \rho \neq 1, 3, .. \tag{11.6}$$

where $\nu_c(\gamma) = \Gamma(\gamma)\cos\left(\gamma\frac{\pi}{2}\right)$ and $\Gamma(\cdot)$ is the Euler Gamma function. By comparing (11.1) and (11.6) it may be stated that the Mellin transform is related to Riesz fractional integral in $x = 0$, that is

$$2\nu_c(\gamma)(I^\gamma f)(0) = M_f(\gamma - 1) \tag{11.7}$$

Under this perspective the representation in (11.4) looks like a Taylor expansion because it involves an operator in zero and a (complex) power series on x; for more details see [33]. The main difference is that when a truncation on the classical Taylor series is performed, always the Taylor series diverges as x diverges, while no divergence problem occur using (11.4) since summation is performed along the imaginary axis and ρ remains fixed. Moreover, unless $f(x)$ belongs to the class C_∞ in zero, the various derivatives in zero may be divergent quantities and the Taylor expansion in such cases is meaningless. On the contrary the series expressed in (11.4) never diverges provided ρ belongs to the FS of the Mellin transform and then $f(x)$ is reproduced in the whole domain with the exception of the value in zero. With these simple information we can now solve the FPK equation by using Mellin transform theorem.

11.2.1 Use of CFMs to Construct Probability Density Functions

In the ensuing derivations, for simplicity sake's, we suppose that the PDF of a stochastic process $X(t)$, in the following denoted as $p_X(x,t)$, is symmetric, namely $p_X(x,t) = p_X(-x,t)$.

The Mellin transform of $p_X(x,t)$, denoted as $M_{p_X}(\gamma - 1)$, is given in the form

$$M_{p_X}(\gamma - 1, t) = \int_0^\infty p_X(x,t)x^{\gamma-1}dx = \frac{1}{2}E\left[|X(t)|^{\gamma-1}\right] \tag{11.8}$$

where $E[\cdot]$ means ensemble average. From this equation it may be stated that the Mellin transform of the PDF is strictly related to moments of the type $E\left[|X(t)|^{\gamma-1}\right]$.

According to (11.4) the discretized version of the inverse Mellin Transform is written for $x > 0$ in the equivalent forms

$$p_X(x,t) = \frac{1}{4b}\sum_{k=-m}^{m} E\left[|X(t)|^{\gamma_k-1}\right]x^{-\gamma_k} = \frac{1}{2b}x^{-\rho}\sum_{k=-m}^{m} M_{p_X}(\gamma_k-1,t)x^{-i\frac{k\pi}{b}};$$
$$\gamma_k = \rho + i\frac{k\pi}{b} \tag{11.9}$$

where $b = \pi/\Delta\eta$ and ρ belongs to the FS of $p_X(x,t)$. Since $p_X(x,t) \geq 0$ and the area of the PDF in $0 \div \infty$ is 1/2 then $\lim_{x\to\infty} p_X(x,t) = 0$. It follows that the fundamental strip of $p_X(x,t)$ always exists and, for $p_X(0,t) \neq 0$, it is $0 < \rho < u$. The value of u depends of the order of zero of the PDF at $x = \infty$. As an example for α-stable random variable the moments $E[|X|^{\beta}](\beta \in \Re)$ do not diverge only in the range $-1 < \beta < \alpha$ [33]. Then for such random variable the FS is $0 < \rho < \alpha + 1$. In general if for a given stochastic process the integer moments diverge starting from a certain value, say r, then the strictest FS is $0 < \rho < r + 1$.

An important issue of this representation of the PDF is the discretization of the inverse Mellin transform, more specifically the number m that define the number of CFMs to be used in order to efficiently represent the PDF. In order to properly define the parameter m, some considerations are necessary: (i) the choice of m strictly depends of $\Delta\eta$ since $m\Delta\eta = \bar{\eta}$ is the truncation of $M_p(\gamma - 1)$ that in turns depends of the value of ρ selected; (ii) higher value of ρ, at a parity of the PDF at hands produces oscillations in $M_p(\gamma - 1)$ as shown in Fig. 11.1 in which CFM are reported for different values of ρ ($\rho = 0.5$; $\rho = 10$). It follows that in order to properly discretize the inverse Mellin transform it is necessary of a smaller value of $\Delta\eta$ as ρ increase.

In the case of α-stable Lévy white noise the selection of ρ is obligated by the limitations of the FS of the Mellin transform and on the non-linearity. So because m and consequently $\Delta\eta$ depends on many parameters we can proceed with trial and error (two or three attempts are enough) or if we have a crude estimation on the

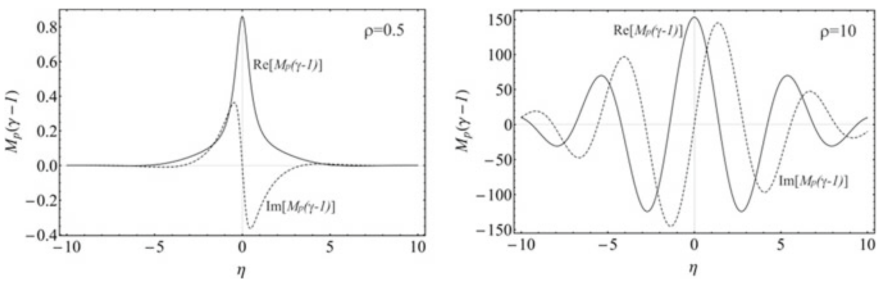

Fig. 11.1 CFMs of a Gaussian distribution with unitary variance and different values of ρ

PDF of the response at steady state by using approximate techniques (like stochastic linearization) then a preliminary choice of m may be readily performed. In quiescent systems, since in t = 0 all CFMs are zero and the tails of the PDF increase then the worst situation will remain the PDF at the steady state or when the scale attains the maximum value. It follows that as the PDF is well reproduced for the steady state or in correspondence of the maximum scale (or of the variance if it exists) then all the parameters (m and $\bar{\eta}$) may be used also in the transient zone.

11.3 Applications of CFMs for the Solution of FPK-Type Equation

In this section we will show how to solve the equations ruling the evolution of the PDF describing the motion of a spring-dashpot system (first-order differential equation) subjected to a Gaussian white noise (Fokker-Planck equation), to a-stable white noise (Fractional Fokker-Planck equation) and to Poissonian white noise (Kolomogorov-Feller equation). For all the three cases some numerical applications are also presented in order to show the accuracy of this approach.

11.3.1 Fokker-Planck Equation (Gaussian White Noise)

Let us suppose that the equation of motion of a (mass-less) non-linear system is given in the form

$$\begin{cases} \dot{X} = f(X,t) + W(t) \\ X(0) = X_0 \end{cases} \tag{11.10}$$

$W(t)$ is a normal zero mean white noise, formal derivative of the Brownian motion $B(t)$, $(dB(t)/dt = W(t))$ characterized by $E\left[dB^2(t)\right] = q\,dt$, being q the intensity of the white noise. In (11.10) it is assumed that $f(X,t) = -f(-X,t)$ is a deterministic non-linear function of the stochastic output process $X(t)$. X_0 is a random variable with assigned distribution $(p_X(x,0) = p_X(-x,0))$. Under these assumptions the output stochastic process has a symmetric distribution $p_X(x,t)$.

The Fokker-Planck equation, ruling the transition probability of $X(t)$, is written in the form

$$\begin{cases} \frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(f(x,t)p_X(x,t)\right) + \frac{q}{2}\frac{\partial^2 p_X(x,t)}{\partial x^2} \\ p_X(x,0) = \overline{p_X}(x) \end{cases} \tag{11.11}$$

where the overbar means assigned PDF in $t = 0$.

The differential equations of integer moments may be obtained multiplying (11.11) by $x^k dx (k = 0, 1, \ldots, n)$, and integrating over $-\infty \div \infty$. The solution of the FPK equation in terms of integer moments cannot be obtained, unless $f(X, t) = cX$, since the set of differential equations is hierarchical. That is, the moment equation of an assigned order, say s, involves moments of higher order than s. Since fractional moments are able to return the PDF in the whole range (excluding the value in zero), one may wonder what happens when CFM are used instead of integer moments. In order to answer this question, we multiply (11.11) by $x^{\gamma-1}dx$ and integrating over the range $0 \div \infty$ yields

$$\begin{aligned}\frac{\partial M_{p_X}(\gamma-1,t)}{\partial t} &= -\left[f(x,t)x^{\gamma-1}p_X(x,t)\right]_0^\infty + \\ (\gamma-1)\int_0^\infty x^{\gamma-2}f(x,t)p_X(x,t)dx &+ \frac{q}{2}\left[\frac{\partial p_X(x,t)}{\partial x}x^{\gamma-1}\right]_0^\infty + \\ -\frac{q}{2}(\gamma-1)\left[x^{\gamma-2}p_X(x,t)\right]_0^\infty &+ \frac{q}{2}(\gamma-1)(\gamma-2)\int_0^\infty x^{\gamma-3}p_X(x,t)dx \end{aligned} \tag{11.12}$$

where the first, third and fourth term at the right-hand side of (11.12) come out from integration by parts.

Under the hypothesis that $X(t)$ is stable in distribution and moments up to the m-order are stable, by properly selecting $\rho > 2$, it may be easily demonstrated that the first, the third and the fourth term in (11.12) vanish. For more details see [14].

Next, let us suppose that $f(X,t) = -\sum_{j=1}^{n} c_j |X(t)|^{\beta_j} sgn(X(t)) \left(c_j > 0, \beta_j > 0\right)$, then the equation in terms of fractional moments is written as

$$\begin{cases}\frac{\partial M_{p_X}(\gamma-1,t)}{\partial t} = -(\gamma-1)\sum_{j=1}^{n} c_j M_{p_X}(\gamma+\beta_j-2,t) + \\ \qquad \frac{q}{2}(\gamma-1)(\gamma-2)M_{p_X}(\gamma-3,t); \quad \rho > 2 \\ M_{p_X}(\gamma-1,0) = \int_0^\infty x^{\gamma-1}\overline{p}_X(x)dx \quad \text{assigned}\end{cases} \tag{11.13}$$

This equation may be discretized for $\gamma_k = \rho + i\,k\frac{\pi}{b}$ so obtaining a set of $(2m+1)$ ordinary (linear) differential equations, being m the truncation of the discretized inverse Mellin transform of the PDF.

The main difficulty in solving such a set of differential equations is that the fractional moments are evaluated for different values of ρ. This problem is the analogue of the infinite hierarchy problem. Then at first glance it seems that the use of complex fractional moments does not open new breaks for the solution of the FPK equation. However, to overcome this drawback the following strategy can be adopted.

Since (11.9) remains valid for every value of ρ, provided it belongs to the FS, we equate (11.9) for two different values of ρ say $\rho_1 = \rho$ and $\rho_2 = \rho + \Delta\rho$, denoting as $M_{p_X}(\gamma_k^{(1)} - 1, t)$ and $M_{p_X}(\gamma_k^{(2)} - 1, t)$ the CFM evaluated in $\gamma_k^{(j)} = \rho_j + i\,k\Delta\eta\ (j = 1, 2)$. Then multiplying such equation for $x^{-1/2}$ gives

$$x^{-1/2}\sum_{k=-m}^{m} M_{px}(\gamma_k^{(1)}-1,t)e^{-ik\frac{\pi}{b}\ln x} = x^{-(\Delta\rho+1/2)}\sum_{k=-m}^{m} M_{px}(\gamma_k^{(2)}-1,t)e^{-ik\frac{\pi}{b}\ln x}; \quad x>0 \tag{11.14}$$

It is to be emphasized that equality in (11.14) strictly holds for $x > 0$, since zero singularities appear. Now it is assumed that $M_{px}(\gamma_k^{(2)}-1,t)$ are already known and thus it is possible to evaluate $M_{px}(\gamma_k^{(1)}-1,t)$, i.e., to evaluate $M_{px}(\gamma-1,t)$ for different values of ρ. Because (11.9) is an approximation then (11.14) is to be satisfied in a weak sense in the interval $x_1 > 0$, $x_2 \gg x_1$, i.e.,

$$\begin{aligned}&\int_{x_1}^{x_2}\frac{1}{x}\left\{\left[\sum_{k=-m}^{m} M_{px}(\gamma_k^{(1)}-1,t)e^{-ik\frac{\pi}{b}\ln x} - x^{-\Delta\rho}\sum_{k=-m}^{m} M_{px}(\gamma_k^{(2)}-1,t)e^{-ik\frac{\pi}{b}\ln x}\right]\times\right.\\ &\left.\times\left[\sum_{k=-m}^{m} M^*_{px}(\gamma_k^{(1)}-1,t)e^{ik\frac{\pi}{b}\ln x} - x^{-\Delta\rho}\sum_{k=-m}^{m} M^*_{px}(\gamma_k^{(2)}-1,t)e^{ik\frac{\pi}{b}\ln x}\right]\right\}dx =\\ &= min(M_{px}(\gamma_k^{(1)}-1,t))\end{aligned} \tag{11.15}$$

Now performing the following change of variable

$$\xi = \ln x, \quad d\xi = \frac{dx}{x}; \xi_j = \ln x_j, \quad j = 1,2 \tag{11.16}$$

In order to find $M_{px}(\gamma_s^{(1)}-1,t)$ as a linear combination of $M_{px}(\gamma_k^{(2)}-1,t)$ we perform variations and instead of putting $x_1 = 0, x_2 = \infty$, we put $x_1 = e^{-b}$ and $x_2 = e^b$: In this way three goals are achieved: (i) the interval $e^{-b} \div e^b$ is very large since $b = \pi/\Delta\eta$ and $\Delta\eta$ is of order $0.3 \div 0.5$ then the interval $e^{-b} \div e^b$ is of order $e^{-10} \div e^{10}$ (for $\Delta\eta = 0.314$) or $e^{-6.28} \div e^{6.28}$ (for $\Delta\eta = 0.5$); (ii) the value $x_1 = 0$ is excluded, that is the main problem to perform variations in (11.15) since in zero a divergence occurs; and (iii) with the choice $e^{-b} \div e^b$ the integral (11.15), taking into account the position of (11.16), is in the range $-b \div b$.

It follows that with the choice of the interval $e^{-b} \div e^b$, (11.15), with the positions in (11.16), is written as

$$\begin{aligned}&\int_{-b}^{b}\left\{\left[\sum_{k=-m}^{m} M_{px}(\gamma_k^{(1)}-1,t)e^{-ik\frac{\pi}{b}\xi} - e^{-\Delta\rho\xi}\sum_{k=-m}^{m} M_{px}(\gamma_k^{(2)}-1,t)e^{-ik\frac{\pi}{b}\xi}\right]\times\right.\\ &\left.\left[\sum_{k=-m}^{m} M^*_{px}(\gamma_k^{(1)}-1,t)e^{ik\frac{\pi}{b}\xi} - e^{-\Delta\rho\xi}\sum_{k=-m}^{m} M^*_{px}(\gamma_k^{(2)}-1,t)e^{ik\frac{\pi}{b}\xi}\right]\right\}d\xi =\\ &= min(M_{px}(\gamma_k^{(1)}-1,t))\end{aligned} \tag{11.17}$$

with the orthogonality condition of $e^{ik\frac{\pi}{b}\xi}$ and after minimization we get

$$2bM_{p_X}(\gamma_s^{(1)}-1,t)=\sum_{k=-m}^{m}M_{p_X}(\gamma_k^{(2)}-1,t)a_{ks}(\Delta\rho) \tag{11.18}$$

where

$$a_{ks}(\Delta\rho)=\int_{-b}^{b}e^{-\Delta\rho\xi}e^{-i(k-s)\frac{\pi}{b}\xi}d\xi=\frac{2b\sin[\pi(k-s)-ib\Delta\rho]}{\pi(k-s)-ib\Delta\rho} \tag{11.19}$$

From (11.18) we recognize that $M_{p_X}(\gamma_s^{(1)}-1,t)$ may be obtained as a linear combination of $M_{p_X}(\gamma_k^{(2)}-1,t)$, i.e., it is possible to solve FPK equation by using Mellin transform.

Since in (11.13) we have $M_{p_X}(\gamma_s-1,t)$, $M_{p_X}(\gamma_s+\beta-2,t)$, and $M_{p_X}(\gamma_s-3,t)$, then we select the initial value of $\rho>2$. In this manner we are sure that $Re(\gamma_s-2)>0$ is inside the FS. Thus, taking into account (11.13) and (11.18), yields

$$M_{p_X}(\gamma_s+\beta-2,t)=\frac{1}{2b}\sum_{k=-m}^{m}M_{p_X}(\gamma_k-1,t)a_{ks}(1-\beta)$$
$$M_{p_X}(\gamma_s-3,t)=\frac{1}{2b}\sum_{k=-m}^{m}M_{p_X}(\gamma_k-1,t)a_{ks}(2) \tag{11.20}$$

By inserting these equations in (11.13) for $\gamma=\gamma_s$ $(s=-m,..,0,..m)$ we get a set of complex ordinary differential equations in the unknowns $M_{p_X}(\gamma_s-1,t)$.

If the system of differential equations is directly implemented using a computer program the solution is not correct because we need of another information, i.e., the area of the PDF into the interval $e^{-b}\div e^{b}$ will be 1/2. This constraint may be enforced very easily. Taking into account (11.9), we get

$$\frac{1}{2b}\sum_{k=-m}^{m}M_{p_X}(\gamma_k-1,t)\int_{e^{-b}}^{e^{b}}x^{-\gamma_k}dx=\frac{1}{2} \tag{11.21}$$

This equation gives the following information in the Mellin transform domain

$$M_{p_X}(\gamma_0-1,t)=\frac{1-\rho}{e_0}\left[b-\sum_{\substack{k=-m\\k\neq 0}}^{m}\left(\frac{e_k}{1-\gamma_k}\right)M_{p_X}(\gamma_k-1,t)\right] \tag{11.22}$$

where

$$e_0=\left(e^{b}\right)^{1-\rho}-\left(e^{-b}\right)^{1-\rho} \tag{11.23}$$

and

$$e_k = \left(e^b\right)^{1-\gamma_k} - \left(e^{-b}\right)^{1-\gamma_k} \tag{11.24}$$

In this manner, a set of 2 m linear (complex) differential equations is obtained, which involves only CFMs evaluated in the same value of ρ ruling the evolution of the CFMs. The *s-th* equation is

$$\begin{aligned}\frac{\partial M_{p_X}(\gamma_s-1,t)}{\partial t} &= -(\gamma_s-1)\sum_{j=1}^{n} c_j\left[\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)a_{ks}(1-\beta_j)+\right.\\&\left.+\frac{1-\rho}{e_0}a_{0s}(1-\beta_j)\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right)\right]+\\&+\frac{q}{2}(\gamma_s-1)(\gamma_s-2)\left[\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)a_{ks}(2)+\right.\\&\left.+\frac{1-\rho}{e_0}a_{0s}(2)\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right)\right] \quad s=-m,\ldots,-1,1,\ldots,m\end{aligned} \tag{11.25}$$

Equation (11.25) constitute a set of linear coupled ordinary differential equations in the unknown $M_{p_X}(\gamma_s-1,t)$ that may be easily solved by inserting the initial conditions given in (11.13). Moreover $M_{p_X}(\gamma_0-1,t)$ in (11.25) is given in (11.22). Thus (11.25) is not homogeneous and the steady state solution may be readily found. If the system is quiescent at $t=0$, that is $\bar{p}_X(x)=\delta(x)$, then all $M_{p_X}(\gamma_s-1,0)$ are zeros.

System of (11.25) may be reduced to only m equations by taking into account that $M_p(\gamma_s-1,t)=M_p^*(\gamma_{-s}-1,t)$.

In order to show the capability of the method, we suppose that the nonlinear function in (11.10) is $f(X,t)=-c|X|^{\beta}\text{sgn}(X)$, with $c>0$ and $\beta\geq 0$. Moreover, let us suppose that $\bar{p}_X(x)=\delta(x)$, that is the system is quiescent at $t=0$. For this system for the case $\beta=1$ (linear system) the transient response is already known and is given in the form

$$p_X(x,t)=\frac{1}{\sqrt{2\pi}\sigma(t)}exp\left(-\frac{x^2}{2\sigma^2(t)}\right) \tag{11.26}$$

where

$$\sigma^2(t)=\frac{q}{2c}(1-e^{-2ct}) \tag{11.27}$$

while if $\beta\neq 1$ the stationary solution is known in analytical form and it reads

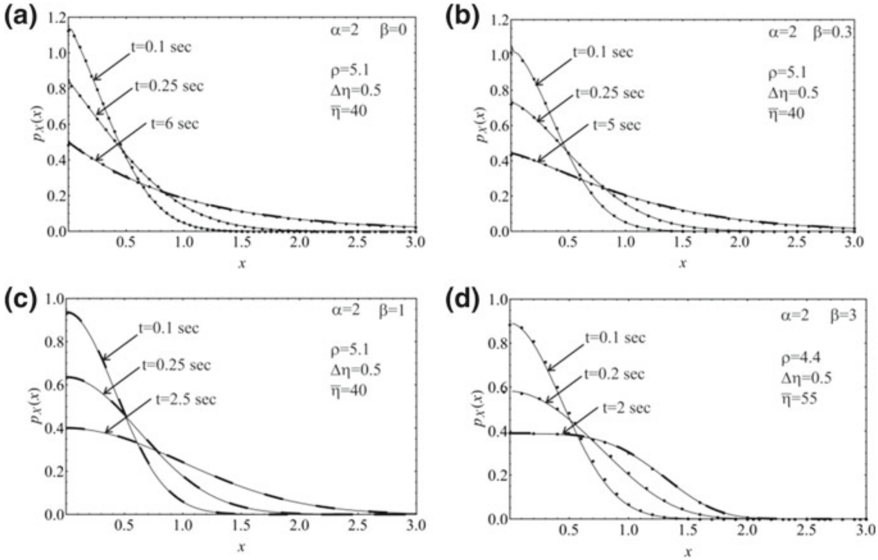

Fig. 11.2 Probability density function for different values of β; continuous line solution in terms of CFM, dotted line Monte Carlo simulation; dashed line exact solution

$$p_X(x,\infty) = \nu \exp\left(-\frac{2c|x|^{\beta+1}}{(\beta+1)q}\right) \tag{11.28}$$

where ν is a normalization constant such that $\int_0^\infty p_X(x,\infty)dx = 1/2$.

In Fig. 11.2 the PDF of the nonlinear system given by the procedure outlined above is contrasted with the solution obtained by the Monte Carlo simulation with 10^6 samples, for different values of β ($\beta = 0,\ \beta = 0.3,\ \beta = 1,\ \beta = 3$). In particular for $\beta = 1$ also the exact solution given in (11.26) is plotted (dashed line) at various time instants. Further in Fig. 11.2a, b and d also the steady state solution is plotted in dashed line and contrasted with the results obtained by CFMs. During the transitory phase the comparison is made with the PDF obtained by Monte Carlo simulation. The value of c selected for these applications is $c = 1$ and $q = 2$. The various parameters ($\rho,\ \Delta\eta,\ \bar{\eta}$) are given in the figures.

11.3.2 Fractional Fokker-Planck Equation (α-Stable White Noise)

Let us now suppose that the same mechanical system of the previous section (spring-dashpot system) is subjected to an α-stable white noise $W_\alpha(t)$. Without loss of generality we assume that $W_\alpha(t)$, formal derivative of the Lévy α-stable process $L_\alpha(t)$, is a symmetric α-stable ($S_\alpha S$) process.

The corresponding non-linear Langevin equation may be written as follows

$$\begin{cases} \dot{X} = f(X,t) + W_\alpha(t) \\ X(0) = X_0 \end{cases} \tag{11.29}$$

The Itô equation associated to (11.29) may be written in the form

$$dX(t) = f(X,t)dt + dL_\alpha(t) \tag{11.30}$$

where the characteristic function (CF) of $dL_\alpha(t)$ is in the form

$$\phi_{dL_\alpha}(t) = \exp(-dt\sigma|\theta|^\alpha) \tag{11.31}$$

where σ is the scale factor (not the standard deviation) and α is the stability index. The equation ruling the evolution of the PDF of the output process is known as Fractional Fokker-Planck (FPP) equation and is given in the form

$$\begin{cases} \frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}(f(x,t)p_X(x,t)) + \sigma^\alpha D_X^\alpha(p_X(x,t)) \\ p_X(x,0) = \overline{p_X}(x) \end{cases} \tag{11.32}$$

where the symbol $D_X^\alpha(\cdot)$ denotes the Riesz fractional derivative defined as

$$D_x^\alpha(u(x,t)) = \begin{cases} -\frac{1}{2\cos(\pi\alpha/2)}\left[D_{x^+}^\alpha(u(x,t)) + D_{x^-}^\alpha(u(x,t))\right]; & \alpha \neq 1 \\ -\frac{d}{dx}\mathcal{H}[u(x,t)]; & \alpha = 1 \end{cases} \tag{11.33}$$

In (11.33) $D_{x^+}^\alpha$ and $D_{x^-}^\alpha$ are the left and right Riemann-Liouville fractional derivatives that may be written in the form

$$\begin{aligned} D_{x^+}^\alpha(u(x,t)) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha-n+1}}d\xi \\ D_{x^-}^\alpha(u(x,t)) &= \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_{x}^{\infty}\frac{u(\xi,t)}{(\xi-x)^{\alpha-n+1}}d\xi \end{aligned} \tag{11.34}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α and $\mathcal{H}[\cdot]$ is the Hilbert transform operator defined as

$$\mathcal{H}[u(x,t)] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{u(\xi,t)}{|x-\xi|}d\xi \tag{11.35}$$

being $\mathcal{P}$ the Cauchy principal value.

In order to solve the FFP equation the same approach of the previous section is adopted. We suppose that $f(X,t) = -\sum_{j=1}^{r} c_j |X(t)|^{\beta_j} sgn(X(t))$ $(c_j > 0,\ \beta_j > 0)$ and we perform Mellin transform of (11.32):

$$\begin{aligned}\frac{\partial M_p(\gamma-1,t)}{\partial t} &= \sum_{j=1}^{r} c_j \left[x^{\gamma-1+\beta_j} p_X(x,t)\right]_0^{\infty} - (\gamma-1)\sum_{j=1}^{r} c_j M_p(\gamma-2+\beta_j,t) + \\ &-\sum_{k=0}^{n-1}\frac{\Gamma(\gamma-1+k)}{\Gamma(\gamma-1)}\left[\frac{d^{n-k-1}}{dx^{n-k-1}}\left(\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(x-\xi)^{\alpha-n+1}}d\xi + \right.\right.\\ &\left.\left.+(-1)^n\int_{x}^{\infty}\frac{p_X(\xi,t)}{(\xi-x)^{\alpha-n+1}}d\xi\right)x^{\gamma-k-1}\right]_0^{\infty} - \sigma^{\alpha}\frac{\nu_c(\gamma)}{\nu_c(\gamma-\alpha)}M_p(\gamma-1-\alpha,t)\end{aligned} \tag{11.36}$$

The terms in square brackets, coming from integration by parts, vanish by properly selecting the value of ρ and (11.36) reduces to

$$\frac{\partial M_p(\gamma-1,t)}{\partial t} = -(\gamma-1)\sum_{j=1}^{r} c_j M_p(\gamma-2+\beta_j,t) - \sigma^{\alpha}\frac{\nu_c(\gamma)}{\nu_c(\gamma-\alpha)}M_p(\gamma-1-\alpha,t) \tag{11.37}$$

By evaluating (11.36) for different values in $2m+1$ values $\gamma_k = \rho + ik\Delta\eta$, a set of ordinary linear differential equations is obtained. In order to solve this system, it is necessary to write all CFMs in terms of CFMs of one order. Then, following the results of the previous section, we may write:

$$M_{p_X}(\gamma_s+\beta_j-2,t) = \frac{1}{2b}\sum_{k=-m}^{m} M_{p_X}(\gamma_k-1,t)a_{ks}(1-\beta_j) \tag{11.38}$$

$$M_{p_X}(\gamma_s-1-\alpha,t) = \frac{1}{2b}\sum_{k=-m}^{m} M_{p_X}(\gamma_k-1,t)a_{ks}(\alpha) \tag{11.39}$$

By inserting (11.38) and (11.39) into (11.37) and enforcing the normalization condition (11.21)–(11.22) a solvable set of 2 m linear differential equations ruling the time evolution of CFMs is obtained. The *s-th* equation is written as

$$\begin{aligned}\frac{\partial M_p(\gamma_s-1,t)}{\partial t} &= -(\gamma-1)\sum_{j=1}^{r} c_j\left(\sum_{\substack{k=-m\\k\neq m}}^{m} M_p(\gamma_k-1,t)\,a_{ks}(1-\beta_j) + \right.\\ &\left.+\frac{1-\rho}{e_0}a_{0s}(1-\beta_j)\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right)\right) + \\ &-\sigma^{\alpha}\frac{\nu_c(\gamma)}{\nu_c(\gamma-\alpha)}\left(\sum_{\substack{k=-m\\k\neq m}}^{m} M_p(\gamma_k-1,t)a_{ks}(\alpha) + \frac{1-\rho}{e_0}a_{0s}(\alpha)\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m} M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right)\right)\end{aligned} \tag{11.40}$$

The solution of FFP in terms of CFMs has been tested with different values of the stability index α and with different order of non-linearity β. The case $\alpha = 2$ corresponds to the case of Gaussian white noise that has been treated in the previous section, hence for the sake of brevity it is not repeated in the following. The solution in terms of CFMs have been contrasted with analytical solutions, when available, and with results of Monte Carlo simulations with 10^6 samples.

Consider

- $\alpha = 1.5$

This value of α has been investigated as a general case in the range $1 \div 2$. When stability index is less than 2 the fundamental strip depends on the values of α and β, because of the decay of the PDF for $x \to \infty$. In particular, it has been demonstrated by Chechkin et al. [6] that for α-stable input, the tails of the PDF of the output decay as a power law x^{-u}, being $u = \alpha + 1$ for the linear dashpot-system ($\beta = 1$) and $u = \alpha + 3$ for the quartic system ($\beta = 3$), so in both cases $u = \alpha + \beta$. This allows us to do some considerations on the FS that is unknown. These considerations resulted in the choice of ρ in the range $0 \div 1 + \alpha$. From this descends that, since in the Mellin transform of FFP equation there are CFMs evaluated for different value of ρ, we cannot solve the system with some values of $\beta > 1.7$ because CFMs from the drift term and from diffusive term are evaluated in value of ρ outside the FS, for more details see [1]. In the following, results for $\alpha = 1.5$ are reported in Fig. 11.3.

- $\alpha = 1$

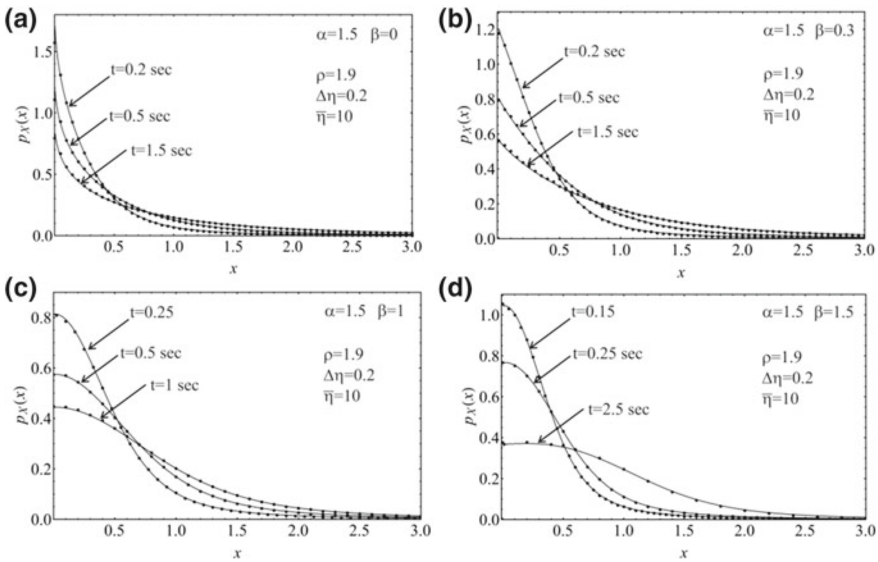

Fig. 11.3 Probability density function for different values of β; continuous line solution in terms of CFM, dotted line Monte Carlo simulation

This is the case when the input is a Cauchy process. In this case the steady state solution for $\beta = 1$ is known as

$$p_X(x,\infty) = \frac{\sigma c}{\pi(\sigma^2 + c^2x^2)} \tag{11.41}$$

Figure 11.4 shows the response pdf for various value for β at different time instant.

- $\alpha = 0.8$

This case is taken as general case in the range $0 \leq \alpha \leq 1$. In the following result for various values for β at different instants are shown in Fig. 11.5.

- $\alpha = 0.5$

In this case the input is a symmetric Lévy process. For this value of α we are actually able to solve only the linear case for which the steady state solution may be obtained in the following form

$$p_X(x,\infty) = \sqrt{\frac{\bar{\sigma}}{2\pi|x|^3}}\left(\cos\left(\frac{\bar{\sigma}}{4x}\right)\left(\frac{1}{2} - F_c\left(\sqrt{\frac{\bar{\sigma}}{2\pi|x|}}\right)\right) + \sin\left(\frac{\bar{\sigma}}{4x}\right)\left(\frac{1}{2} - F_s\left(\sqrt{\frac{\bar{\sigma}}{2\pi|x|}}\right)\right)\right) \tag{11.42}$$

where $F_c(\cdot)$ and $F_s(\cdot)$ are the Fresnel integrals defined as follow

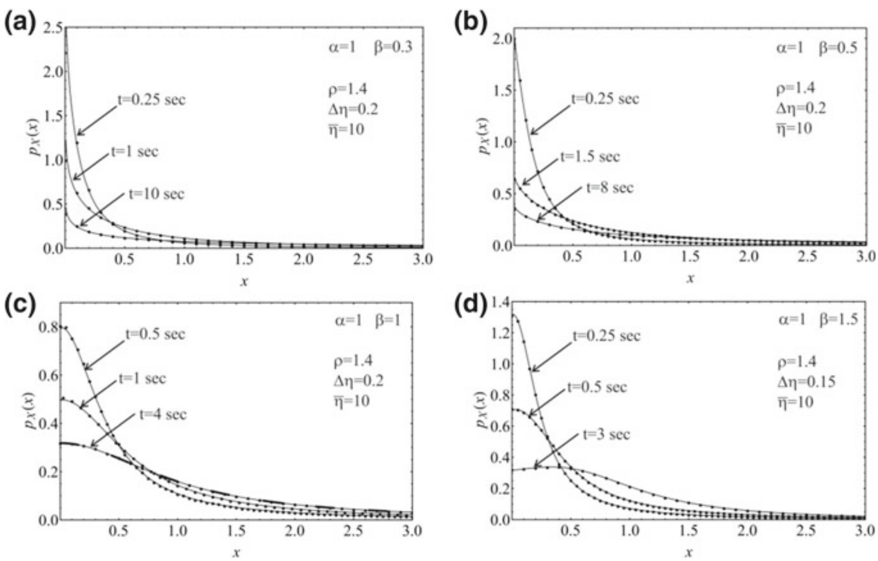

Fig. 11.4 Probability density function for different values of β; continuous line solution in terms of CFM, dotted line Monte Carlo simulation, dashed line exact solution

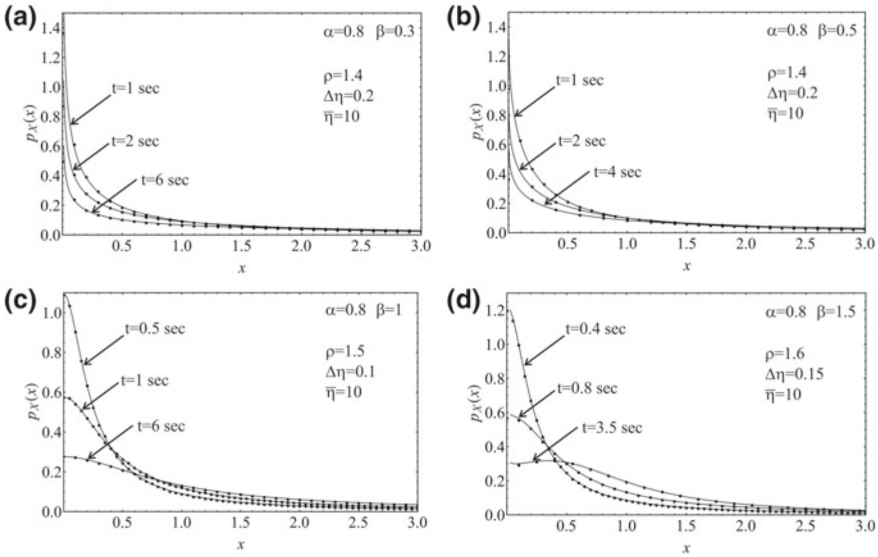

Fig. 11.5 Probability density function for different values of β; continuous line solution in terms of CFM, dotted line Monte Carlo simulation

$$F_c(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt$$
$$F_s(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \tag{11.43}$$

and $\bar{\sigma}$ is the scale factor of the output defined as

$$\bar{\sigma} = \sigma \left(\frac{c}{2}\right)^{-2} \tag{11.44}$$

The following Fig. 11.6 shows the results for $\beta = 1$.

- Trend of the PDF at ∞

Figure 11.7a, b show logarithmic plots of the stationary solution of the FFP equation for the linear case ($\beta = 1$) and for two different values of α 1 and 0.5, respectively, for which the stationary solution is known in analytical form. From these figures it is possible to observe that the solution provided by the proposed method coalesces with the exact one also for large values of x. This fact is very important because other methods of solution fail in the description of the long tails of the PDF.

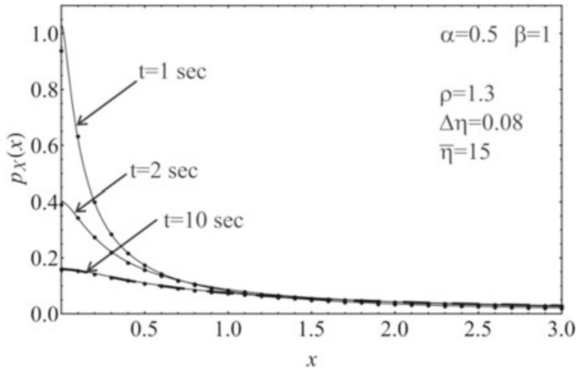

Fig. 11.6 Probability density function for $\beta = 1$ and $\alpha = 0.5$; continuous line solution in terms of CFM, dotted line Monte Carlo simulation, dashed line exact stationary solution

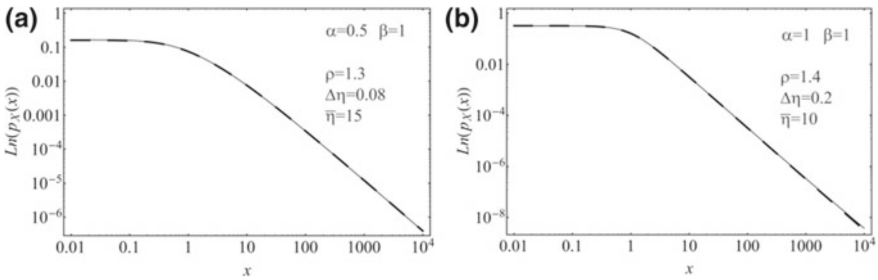

Fig. 11.7 Log-Log plot of the stationary solution for $\beta = 1$ and $\alpha = 0.5, 1$ contrasted with exact steady state solution

11.3.3 Kolmogorov-Feller Equation (Poissonian White Noise)

Let us now consider the case of a non-linear system, as in (11.10), in which, however, $W(t)$ is now a Poisson white noise process. This process can be assumed as constituted by a train of impulses of random amplitude Y, with assigned PDF $p_Y(y,t)$. The impulse occurrence is distributed in time according to a Poisson law. Thus, each impulse Y_k occurs at a time instant T_k, with random independent distribution T. Under these assumptions the Poisson white noise $W(t)$ is given by

$$W(t) = \sum_{k=1}^{N(t)} Y_k\, \delta(t - T_k) \tag{11.45}$$

where $\delta(\cdot)$ is the Dirac's delta and $N(t)$ is a Poisson counting process giving the number of impulses in $0 \div t$.

In this case, the equation governing the evolution of the transition probability of $X(t)$ is the so-called Kolmogorv-Feller (KF) equation, which can be written as

$$\begin{cases} \dfrac{\partial p_X(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}(f(x,t)p_X(x,t)) - \lambda(t)p_X(x,t) + \lambda(t)\displaystyle\int_{-\infty}^{\infty} p_Y(\xi)\; p_X(x-\xi,t)d\xi \\ p_X(x,0) = \overline{p_X}(x) \end{cases} \tag{11.46}$$

Note that, the drift term in (11.46) is identical to the corresponding one in the case of normal white noise input (FPK equation) in (11.11). On the other hand, the diffusive expression (second and third term at the right-hand side of (11.46)) contains a convolution integral instead of the second derivative of the PDF.

Next, assuming that also $p_Y(y)$ has a symmetric distribution, i.e., the response PDF is symmetric, and taking into account (11.4) the discretized version of the Mellin Transform of $p_X(x-\xi,t)$ is

$$p_X(x-\xi,t) \cong \frac{1}{2b}\sum_{k=-m}^{m} \mathcal{M}_{p_X}(\gamma_k - 1, t)\,|x-\xi|^{-\gamma_k}; \quad \gamma_k = \rho + i\frac{k\pi}{b} \tag{11.47}$$

Further, following the procedure described in Sect. 11.3.1, and after some algebra, yields the equation evaluated for $\gamma_s = \rho + i\,s\,\Delta\eta$ as

$$\begin{aligned} \frac{\partial M_{p_X}(\gamma_s-1,t)}{\partial t} &= -\Big[f(x,t)p_X(x,t)x^{\gamma_s-1}\Big]_0^{\infty} + (\gamma_s-1)\int_0^{\infty} f(x,t)p_X(x,t)x^{\gamma_s-2}dx + \\ &- \lambda M_{p_X}(\gamma_s-1,t) + \frac{\lambda}{b}\sum_{k=-m}^{m} M_{p_X}\big(\gamma_k-1,t\big)\,\nu_c\big(1-\gamma_k\big)\mu_{ks}(t);\; s=-m,\ldots,0,\ldots,m \end{aligned} \tag{11.48}$$

where

$$\mu_{ks}(t) = \int_0^{\infty} \big(\mathrm{I}^{1-\gamma_k} p_Y(y,t)\big)x^{\gamma_s-1}dx \tag{11.49}$$

which represents the Mellin transform of the Riesz fractional integral of the function $p_Y(y,t)$, defined as

$$(\mathrm{I}^{\gamma} p_Y)(y,t) = \frac{1}{2\nu_c(\gamma)}\int_{-\infty}^{\infty} \frac{p_Y(\xi,t)}{|y-\xi|^{1-\gamma}}d\xi; \quad \rho \neq 1,3,\ldots \tag{11.50}$$

Now consider again the nonlinear function of the form

$$f(X,t) = -c|X|^{\beta} sgn(X),\; \beta \geq 0,\; c > 0,$$

then if $\beta+\rho-1>0$ (that is $\rho>1-\beta$), the first term at the right hand side of (11.48) vanishes leading to the equation of CFMs in the form

$$\frac{\partial M_{p_X}(\gamma_s-1,t)}{\partial t}=-c(\gamma_s-1)M_{p_X}(\gamma_s+\beta-2,t)-\lambda M_{p_X}(\gamma_s-1,t)+ \\ +\frac{\lambda}{b}\sum_{k=-m}^{m}M_{p_X}(\gamma_k-1,t)\,\nu_c(1-\gamma_k)\mu_{ks}(t);\quad s=-m,\dots,0,\dots,m \tag{11.51}$$

Further, taking into account the condition in (11.22), yields

$$\frac{\partial M_{p_X}(\gamma_s-1,t)}{\partial t}=-c(\gamma_s-1)\sum_{\substack{k=-m\\k\neq 0}}^{m}M_{p_X}(\gamma_k-1,t)\,a_{ks}(1-\beta)-\lambda M_{p_X}(\gamma_s-1,t)+ \\ +\frac{\lambda}{b}\sum_{\substack{k=-m\\k\neq 0}}^{m}M_{p_X}(\gamma_k-1,t)\,\nu_c(1-\gamma_k)\mu_{ks}(t)\;-c(\gamma_s-1)\frac{1-\rho}{e_0}a_{0s}\times \\ \times\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m}M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right)+ \\ +\frac{\lambda}{b}\frac{1-\rho}{e_0}\nu_c(1-\gamma_0)\mu_{0s}(t)\left(b-\sum_{\substack{k=-m\\k\neq 0}}^{m}M_{p_X}(\gamma_k-1,t)\frac{e_k}{1-\gamma_k}\right);\quad s=-m\div m;\; s\neq 0 \tag{11.52}$$

In this manner, a set of 2 m linear (complex) differential equations is obtained, which involves only CFMs evaluated in the same value of ρ.

In order to show the accuracy of the method, consider the non-linear system with $c=0.2$ and $\beta=0.6$. Further, let the assigned PDF at the initial time instant be given as

$$\bar{p}_X(x)=\frac{1}{\sqrt{2\pi}\sigma_0}\exp\left(-\frac{x^2}{2\sigma_0^2}\right) \tag{11.53}$$

and the PDF of the impulse amplitude be

$$p_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left(-\frac{y^2}{2\sigma_y^2}\right) \tag{11.54}$$

with $\lambda(t)=\lambda=1$, $\sigma_0=1$ and $\sigma_y=0.5$.

Figure 11.8 shows the evolution of the response PDF of the system. Specifically, the system in (11.52) is solved assuming $\rho=0.95$, $\Delta\eta=0.5$ and a cut-off value $\bar{\eta}=50$ (thus $m=100$). Solution obtained by the proposed procedure is compared with MCS data, using 40000 samples.

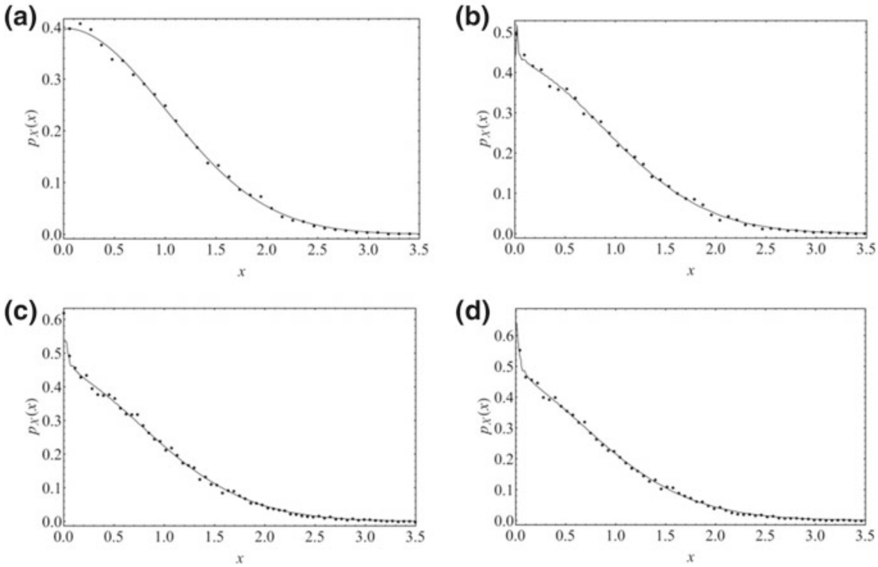

Fig. 11.8 Probability density function at various time instant; continuous line solution in terms of CFM, dots Monte Carlo simulation: **a** t = 0 s; **b** t = 0.5 s; **c** t = 1 s; **d** t = 1.5 s

11.4 PDF Representation Through CFMS Versus Integer Moments

In this section an emphasis is given to the comparison between the capability of integer order moments and CFMs to efficiently describe PDF and to solve FP-type equations. The expression of PDF in terms of CFMs reminds that one in terms of cumulants of integer order j K_j in the form at steady state condition as [19]

$$p_x(x) \cong \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\left(1+\sum_{j=3}^{m}\frac{K_j(-1)^j}{j!\sigma^j}H_j\left(\frac{x-\mu}{\sigma^2}\right)\right) \quad (11.55)$$

being σ the standard deviation, μ the mean, $H_j(x)$ the probabilistic Hermite polynomials and K_j the cumulants of order j which are related to the integer moments through the following relation

$$E[X^j] = K_j + \sum_{r=1}^{j-1}\frac{(j-1)!}{r!(j-1-r)!}K_{j-r}E[X^r] \quad (11.56)$$

With the above expression the PDF of the system response is approximated with the Gram-Charlier series. However, as it is well known, such a series can be inconsistent with probability theory, e.g., negative probabilities may result. Moreover another problem related to this expression is the *jth*-order hierarchy truncation method.

In fact, the cumulants K_j are written once all integer moments $E\left[X(t)^j\right]$ are known solving the following system of differential equation

$$\dot{E}\left[X(t)^j\right] = (j)\int_0^{\infty} x^{k-1} f(x,t) p_X(x,t) dx + \frac{q}{2}(j)(j-1)E\left[X(t)^{j-2}\right] \quad (11.57)$$

Such a strategy belongs to the moment equation (ME) approach, proposed in 1978 [3] as an alternative method to Monte Carlo approach. If on one hand the ME method requires much less computation involving the solution of a system of coupled deterministic ordinary differential equations, on the other hand the disadvantage of the ME is that, unless for linear systems or special case of nonlinear ones, the differential equations for moments of a given order will contain terms involving higher-order moments leading to an infinite hierarchy of coupled equations requiring a closure scheme-procedure. Then, the *jth*-order hierarchy truncation will require approximations for the (*j + 1*)*th*- and (*j + 2*)*th*-order moments.

At this point, some important remarks come out:

(i) Although the system (11.57) is very similar to system (11.13) (setting $(\gamma - 1) =$ j) the hierarchy problem does not in the latter case.
(ii) At first glance the required evaluation of CFMs in different values of ρ may mislead. However, if one thinks that the same requirement occurs for linear systems, it will be clear that this is not a closure scheme procedure.

11.4.1 Numerical Applications

Let the nonlinear function $f(X,t)$ in (11.10) be given in the form $f(X,t) = -c_1 X - c_2|X|^{\beta}\text{sgn}(X)$ with $\beta > 0$. Further let the assigned PDF in zero be $\bar{p}(x) = \delta(x)$, that is the system is quiescent in $t = 0$. In order to compare the accuracies of the proposed and integer moments approach, the case of a bimodal PDF is considered. Thus, let $c_1 < 0$, $c_2 > 0$ and $\beta = 3$ (quartic system). Note that in this case the steady state PDF is known in closed form as

$$p_X(x,\infty) = \nu \exp\left[\frac{1}{2q}\left(x^2 - \frac{x^4}{2}\right)\right] \quad (11.58)$$

in which ν is a normalization constant such that $\int_0^{\infty} p_X(x,\infty)\, dx = 1/2$.

As far as the Gram-Charlier series expansion in (11.55) is concerned, the equation of integers moments for the steady state case can be particularized as

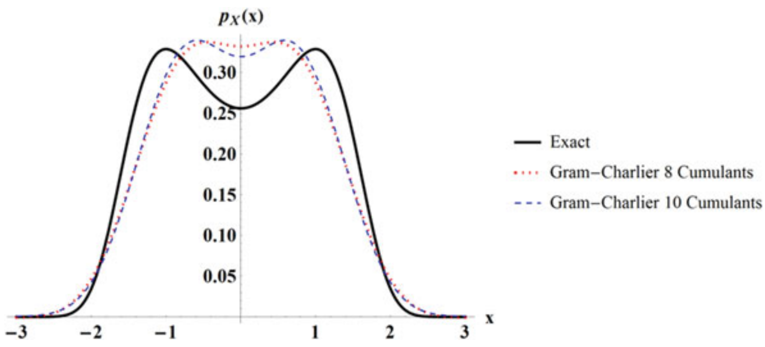

Fig. 11.9 Comparison among the Exact steady state PDF (black line) and Gram-Charlier series expansion with 8 cumulants (red dotted line) and 10 cumulants (blue dashed line)

$$-kc_1 E[X^k] - kc_2 E[X^{k-1+\beta}] + \frac{q}{2} k(k-1) E[X^{k-2}] = 0 \tag{11.59}$$

Note that this equation cannot be solved since an infinite order hierarchy problem appears. However, the aforementioned issue can be circumvented by expressing integer moments in terms of cumulants through (11.56) and considering equal to zero cumulants of order $n > \tilde{m}$ with $\tilde{m}$ arbitrary.

Figure 11.9 shows a comparison among the exact steady state PDF and the PDF obtained through (11.55), for the case $c_1 = -0.5$ and $c_2 = 0.5$, considering two different values of $\tilde{m}$.

As it can be observed from this figure, as the number of cumulants increases, the Gram-Charlier expansion does not lead to the exact solution and even considering 10 cumulants the approximated PDF is rather different from the exact steady state solution $p_X(x, \infty)$.

On the other hand, as far as the series form of the PDF through CFMs is concerned, for the system under consideration the equation ruling the evolution of the CFMs is explicitly given as

$$\begin{aligned} \dot{\mathrm{M}}_{p_X}(\gamma - 1, t) &= -c_1(\gamma - 1)\mathrm{M}_{p_X}(\gamma - 1, t) - c_2(\gamma - 1)\mathrm{M}_{p_X}(\gamma + \beta - 2, t) + \\ &+ \frac{q}{2}(\gamma - 1)(\gamma - 2)\mathrm{M}_{p_X}(\gamma - 3, t) \end{aligned} \tag{11.60}$$

in which CFMs $\mathcal{M}_{p_X}(\gamma + \beta - 2, t)$ and $\mathcal{M}_{p_X}(\gamma - 3, t)$ can be easily evaluated through the following relations

$$\mathcal{M}_{p_X}(\gamma + \beta - 2, t) = \sum_{k=-m}^{m} \mathcal{M}_p(\gamma_k - 1) a_{ks}(1 - \beta) \tag{11.61}$$

$$\mathcal{M}_{p_X}(\gamma - 3, t) = \sum_{k=-m}^{m} \mathcal{M}_p(\gamma_k - 1) a_{ks}(2) \tag{11.62}$$

Following the approach extensively discussed in Sect. 11.3 the system in (11.60) may be reduced to a set of *2* m coupled ordinary differential equations which solutions in terms of CFMs is easily found.

Figure 11.10 shows the evolution of the system response PDF for various time instants vis-à-vis the exact steady state solution. In this case the values of $\Delta\eta = 0.5$ and $m = 140$ have been chosen for solution in terms of CFMs. Note that, even if the value m of CFMs is greater than the number of cumulants chosen $\tilde{m}$, computational time is comparable for the two approaches.

Finally, in order to show the accuracy of the proposed approach with respect to the closure method, Fig. 11.11 shows the solution obtained through CFMs is contrasted with the Gram-Charlier expansion for the steady state case in (11.58).

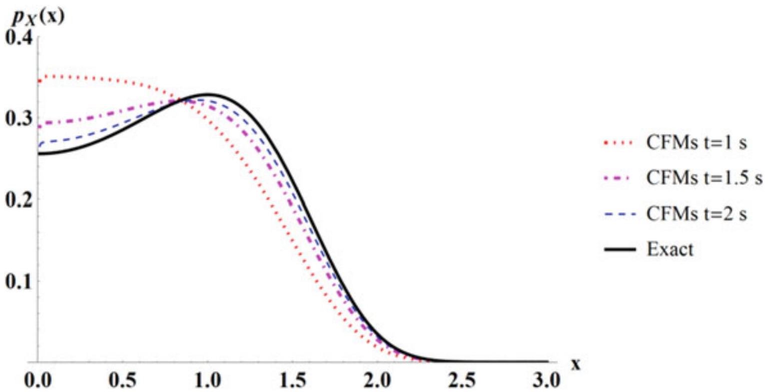

Fig. 11.10 Evolution of the response PDF

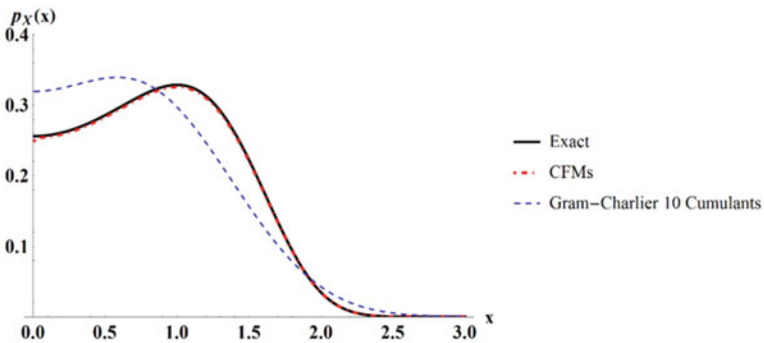

Fig. 11.11 Comparison among Exact steady state solution (Black line), CFMs (Red dashed line) and Gram-Charlier Expansion for 10 cumulants (Blue dashed line)

11.5 Conclusions

This chapter presented an efficient method to analyse the stochastic response of non-linear systems in terms of fractional moments. Instead of using moments of integer order, the FPK equation is written in terms of complex fractional order obtained as Mellin transform of the PDF. The main advantage in using CFMs instead of classical integer order ones is that thanks to the properties of Mellin transform operator the PDF may be reconstructed accurately with a limited number of terms. Moreover, CFMs of a given order may be written in terms of CFMs of a different order, thus eliminating the infinite hierarchy problem that affects the integer order moment approach. Numerical applications have extensively shown that the method is very accurate not only in describing the steady-state PDF but also its evolution for a range of non-linear systems forced by Gaussian, Lévy and Poissonian white noises.

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