Theory of Fractional Differential Equations Using Inequalities and Comparison Theorems: A Survey

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1 Introduction

In this chapter, we present a survey of the qualitative theory pertaining to fractional differential equations (FDEs) developed using differential inequalities and comparison theorems. Differential inequalities help in finding bounds for the solution of the nonlinear fractional differential equation, and once the bounds are known the constructive techniques of Quasilinearization and Monotone Iterative Technique provide the solution.

In Sect. 2, the basic concepts of lower and upper solutions are introduced and the fundamental lemma needed in the comparison theorems is given. Next, the concept of dominating component solution is introduced and existence results pertaining to these solutions are stated.

Section 3 begins with a result relating the solutions of the Caputo and the Riemann–Liouville differential equations. This is followed first by a result relating the solutions of fractional differential equations to those of ordinary differential equations and then by a variation of parameters formula for solutions of perturbed fractional differential equations in terms of solutions of ordinary differential equations. Next, a stability result using Dini derivatives is presented.

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Section 4 covers the concept of fractional trigonometric functions developed using fractional differential equations. It also covers the generalization of these results to fractional trigonometric-like functions.

Section 5 deals with impulsive fractional differential equations of two types, impulsive fractional differential equations with fixed moments of impulse and impulsive fractional differential equations with variable moments of impulse. For each type of equation, an existence and uniqueness result is given. In the case of fixed moments of impulse, the result presented was obtained using the Generalized Quasilinearization (GQL) method. Note that the Quasilinearization (QL) method is a special case of the GQL method. See [8] for an existence and uniqueness result obtained using this method. In the case of variable moments of impulse, the result was obtained using the method of lower and upper solutions and the Monotone Iterative Technique (MIT).

Results pertaining to periodic boundary value problem of Caputo fractional integro-differential equations form the content of Sect. 6.

2 Comparison Theorems, Existence Results, and Component Dominating Solution

2.1 Basic Concepts

The comparison theorems in the fractional differential equations setup require Holder continuity [22–24]. Although this requirement was used to develop iterative techniques such as the monotone iterative technique and the method of quasilinearization, there is no feasible way to check whether the functions involved are Holder continuous. However, the comparison results can be obtained using the weaker condition of continuity. In a subsequent paper [38], it was shown that the same results hold under the less restrictive condition of continuity. Similarly, in [11], differential inequalities, comparison theorems, and existence results were established under a continuity condition for impulsive fractional differential equations. Since Lemma 2.3.1 in [24] is essential in establishing the comparison theorems, we provide a sketch of the proof of this result under this weaker hypothesis. The basic differential inequality theorems and required comparison theorems are also stated.

We begin with the definition of the class $C_p[[t_0, T], \mathbb{R}]$.

Definition 2.1 $m \in C_p[[t_0, T], \mathbb{R}]$ means that $m \in C[(t_0, T], \mathbb{R}]$ and $(t - t_0)^p m(t) \in C[[t_0, T], \mathbb{R}]$ with $p + q = 1$

Definition 2.2 For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann–Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1} m(s) ds \tag{2.1}$$

Lemma 2.3 *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in [t_0, T]$ we have $m(t_1) = 0$ and, $m(t) < 0$ for $t_0 \leq t < t_1$, then it follows that*

$$D^q m(t_1) \geq 0.$$

Proof Consider $m \in C_p[[t_0, T], \mathbb{R}]$, such that $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \leq t < t_1$.

Since $m(t)$ is continuous on $(t_0, T]$, given any t_1 such that $t_0 < t_1 < T$, there exists a $k(t_1) > 0$ and $h > 0$ such that

$$-k(t_1)(t_1 - s) \leq m(t_1) - m(s) \leq k(t_1)(t_1 - s) \tag{2.2}$$

for $t_0 < t_1 - h \leq s \leq t_1 + h < T$. Set $H(t) = \int_{t_0}^{t} (t-s)^{p-1} m(s)ds$ and consider $H(t_1) - H(t_1 - h) = \int_{t_0}^{t_1-h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}]m(s)ds + \int_{t_1-h}^{t_s} (t_1 - s)^{p-1}$ $m(s)ds$.

Let $I_1 = \int_{t_0}^{t_1-h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}]m(s)ds$ and $I_2 = \int_{t_1-h}^{t_1} (t_1 - s)^{p-1}$ $m(s)ds$. Since $t_1 - s > t_1 - h - s$ and $p - 1 < 0$, we have $(t_1 - s)^{p-1} < (t_1 - h - s)^{p-1}$.

This coupled with the fact that $m(t) \leq 0$, $t_0 < t \leq t_1$, implies that $I_1 \geq 0$. Now, consider $I_2 = \int_{t_1-h}^{t_0} (t_1 - s)^{p-1} m(s)ds$. Using (2.2) and the fact that $m(t_1) = 0$, we obtain

$$m(s) \geq -k(t_1)(t_1 - s),$$

and $I_2 \geq -k(t_1) \int_{t_1-h}^{t_1} (t_1 - s)^p ds = -k(t_1)\dfrac{h^{p+1}}{p+1}$, for $s \in (t_1 - h, t_1 + h)$. Thus we have

$$H(t_1) - H(t_1 - h) \geq -\frac{k(t_1)(h^{p+1})}{p+1}$$

and

$$\lim_{h \to 0} \left[\frac{H(t_1) - H(t_1 - h)}{h} + \frac{k(t_1)h^{p+1}}{h(p+1)} \right] \geq 0.$$

Since $p \in (0, 1)$, we conclude that $\dfrac{dH(t_1)}{dt} \geq 0$, which implies that $D^q m(t_1)$ ≥ 0. □

We next state the fundamental differential inequality result in the set up of fractional derivative, which is Theorem 2.3.2 in [24] with a weaker hypothesis of continuity.

Theorem 2.4 *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$D^q v(t) \le f(t, v(t)),$$

$$D^q w(t) \ge f(t, w(t)),$$

$t_0 < t \le T$. Assume f satisfies the Lipschitz condition

$$f(t, x) - f(t, y) \le L(x - y),\ x \ge y,\ L > 0. \tag{2.3}$$

Then $v^0 \le w^0$, where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$, implies $v(t) \le w(t)$, $t \in [t_0, T]$.

Now, we define the Caputo fractional derivative, which we need in Sect. 3.

Definition 2.5 The Caputo derivative, denoted by ${}^cD^q u$, is defined as

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t - s)^{-q} u'(s) ds. \tag{2.4}$$

If $u(t)$ is Caputo differentiable, then we write $u \in C^q[[t_0, T], \mathbb{R}]$.

We now state the comparison theorem in terms of the Caputo derivative.

Theorem 2.6 *Assume that $m \in C^q[[t_0, T], \mathbb{R}]$ and*

$${}^cD^q m(t) \le g(t, m(t)),\ t_0 \le t \le T,$$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. Let $r(t)$ be the maximal solution of the initial value problem (IVP)

$${}^cD^q u = g(t, u),\ u(t_0) = u_0, \tag{2.5}$$

existing on $[t_0, T]$ such that $m(t_0) \le u_0$. Then, we have $m(t) \le r(t)$, $t_0 \le t \le T$.

The results in Sects. 2.2 and 2.3 are taken from [39].

2.2 Dominating Component Solutions of Fractional Differential Equations

Consider the IVP for the Caputo differential equation given by

$${}^cD^q x = f(t, x), \tag{2.6}$$

$$x(t_0) = x_0, \tag{2.7}$$

for $0 < q < 1$, $f \in C[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$.

If $x \in C^q[[t_0, T], \mathbb{R}^n]$ satisfies (2.6) and (2.7) then it also satisfies the Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \tag{2.8}$$

for $t_0 \le t \le T$.

Next, we present a class of functions that are possible solutions of the IVP of FDEs, and which under certain conditions satisfy the relations

$${}^c D^{q^+} |x(t)| \le |{}^c D^q x(t)|$$

and

$$D^{q^+} |x(t)| \le |D^q x(t)|,$$

where ${}^c D^{q^+}$ is the Caputo Dini derivative and D^{q^+} is the Riemann–Louiville (RL) fractional Dini derivative, which are defined as follows.

Definition 2.7 The Caputo fractional Dini derivative of a function $x(t)$ is defined as

$${}^c D^{q^+} x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} D^+ x(s) ds$$

where D^+ is the usual Dini derivative defined in [25]. For more details on fractional Dini derivatives, see [21, 24].

Definition 2.8 The RL fractional Dini derivative is defined as

$$D^{q^+} x(t) = \frac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q} x(s) ds.$$

Definition 2.9 A continuous function $x : I \to \mathbb{R}^n$ is said to be a dominating component function (DCF) if there exists $i \in \{1, 2, \ldots, n\}$ such that $|x_j(s)| \le x_i(t)$ and $|x_j'(t)| \le x_i'(t)$ for all $t \in I = [t_0, T], \; j = 1, 2, \ldots, n$.

Definition 2.10 A continuous function $x : I \to \mathbb{R}^n$ is said to be a weakly dominating component function (WDCF) if there exists $i \in \{1, 2, \ldots, n\}$ such that $|x_j'|(t)| \le x_i'(t)$ for all $t \in I, \; j = 1, 2, \ldots, n$.

Remark 2.11 Every DCF is a WDCF. For example, $x(t) = (\sqrt{t}, t)), \; t \in [1, 2]$ is a DCF and a WDCF whereas $x(t) = (\frac{1}{2}t^2, \frac{-1}{2}t, \frac{-1}{3}t, t), \; t \in [1, 2]$, is a WDCF.

Definition 2.12 By a weakly dominating component solution of the IVP (2.6) and (2.7), we mean a weakly dominating component function which satisfies the IVP (2.6) and (2.7).

We now state a comparison theorem in terms of the Caputo fractional Dini derivative. Note that it is essential to use Dini derivatives when we use an absolute value function or a norm function.

Theorem 2.13 *Assume that* $f \in C[[I \times \mathbb{R}^n, \mathbb{R}^n]$ *and satisfies the relation*

$$|f(t, x)| \leq g(t, |x|), \tag{2.9}$$

where $g \in C[[I \times \mathbb{R}_+, \mathbb{R}_+]$. *Let* $r(t)$ *be the maximal solution of the scalar Caputo FDE*

$$^cD^q u = g(t, u), \;\; u(t_0) = u_0. \tag{2.10}$$

If $x(t)$ *is the weakly dominating solution of the IVP (2.6) and (2.7), then*

$$|x(t, t_0, x_0)| \leq r(t, t_0, u_0),$$

$t \in I$ *provided* $|x_0| \leq u_0$.

Proof Set $m(t) = |x(t)|$ for $t \in I$. Then, using the definition of the Caputo fractional Dini derivative and the fact that $x(t)$ is WDCF of (2.6) and (2.7), we get

$$\begin{aligned}
^cD^{q^+} m(t) &= {}^cD^{q^+}|x(t)| \\
&= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} D^+|x(s)| ds, \\
&\leq \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} |x'(s)| ds, \\
&= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \max_j |x_j'(s)| ds, \\
&\leq \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x_i'(s) ds, \\
&\leq \max_j \left| \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x_j'(s) ds \right|
\end{aligned}$$

$$= |^cD^q x(t)| = |f(t, x(t))| \leq g((t, |x(t)|) = g((t, m(t)).$$

Now, with $m(t_0) = |x_0|$, the conclusion follows from the hypothesis and the application of Theorem 2.6, which yields

$$|x(t, t_0, x_0)| \leq r(t, t_0, x_0), \ \ t \in I.$$

Thus, the proof is complete. □

Remark 2.14 If $n = 1$, the above theorem states that the result holds if the solution belongs to the set of all increasing functions. In this case, one can observe that the Caputo FDE

$$^{c}D^{q}x = Lx, \qquad x(t_0) = x_0$$

has a solution, the Mittag-Leffler function, which is also a weakly dominating component solution.

2.3 Dominating Component Solutions for Riemann–Liouville FDE

Consider the IVP given by

$$D^{q}x = f(t, x) \tag{2.11}$$

$$x(t_0) = x^0 = x(t)(t - t_0)^{1-q}|_{t=t_0} \tag{2.12}$$

where $f \in C[I \times \mathbb{R}^n, \mathbb{R}^n]$.

For the sake of completeness we give the following definitions from [24].

Definition 2.15 Let $0 < q < 1$ and $p = 1 - q$. The function space $C_p[[t_0, T], \mathbb{R}^n] = \{u \in C[[(t_0, T], \mathbb{R}^n]$ and $(t - t_0)^p u(t) \in C[[t_0, T], \mathbb{R}^n]\}$

Definition 2.16 A function $x(t)$ is said to be a solution of the IVP (2.11) and (2.12) if and only if $x \in C_p[[(t_0, T], \mathbb{R}^n]$, $D^q x(t)$ exists and $x(t)$ is continuous on $[t_0, T]$ and satisfies the relations (2.11) and (2.12).

Definition 2.17 A function $x(t)$ is said to be dominating component solution of the IVP (2.11) and (2.12) if $x(t)$ is a dominating component function and further satisfies the IVP (2.11) and (2.12).

Theorem 2.18 *Assume that f in (2.11) satisfies*

$$|f(t, x(t)| \leq g((t, |x(t)|), \tag{2.13}$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$. Let $r(t)$ be the maximal solution of the scalar Riemann–Liouville FDE

$$D^{q}u = g((t, u), \ \ u(t_0) = u^0 = u(t)(t - t_0)^{1-q}|_{t=t_0} \tag{2.14}$$

Further, if $x(t)$ is the dominating component solution of (2.11) and (2.12) then,

$$|x(t, t_0, x^0| \le r(t, t_0, u^0),$$

$t \in I$, provided $|x^0| \le u^0$.

Proof Set $m(t) = |x(t)|,\ t \in I$. Using the definition of the RL fractional derivative and the fact that $x(t)$ is a dominating component function, we get

$$\begin{aligned}
D^{q^+}m(t) &= \tfrac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q}|x(s)|ds \\
&= \tfrac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q} x_i(s)ds \\
&= \tfrac{1}{\Gamma(1-q)} \tfrac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} x_i(s)ds \\
&= f_i(t, x(t)) \\
&= |f(t, x(t))| \\
&\le g(t, |x(t)|) = g(t, m(t)).
\end{aligned}$$

Then,with $m(t_0) = |x^0|$, a result for Riemann–Liouville FDEs, parallel to Theorem 2.6, yields

$$|x(t, t_0, x^0)| \le r(t, t_0, x^0),$$

$t \in I$. □

Remark 2.19 Note that, in case of Riemann–Liouville FDE, for $n = 1$, we need the solutions to be positive and also increasing. Thus, it is clear that Riemann–Liouville FDEs are more complex than Caputo FDEs.

Next, we give criteria that will guarantee the existence of a dominating component solution for Riemann–Liouville FDE (2.11) and (2.12). Since, as will be shown in Sect. 3, any result that holds for solutions of Riemann–Liouville FDE also holds for solutions of the corresponding Caputo FDE, we obtain a sufficiency condition for the Riemann–Liouville FDEs to have a dominating component solution.

Theorem 2.20 *Suppose that $f \in C^1[I \times \mathbb{R}^n, \mathbb{R}^n]$ in (2.11) is a dominating component-bounded function, that is, there exists an $i \in \{1, 2, 3 \ldots, n\}$ such that*

$$|f_j(t, x)| \le f_i(t, x) < M \tag{2.15}$$

$$|\frac{d}{dt} f_j(t, x)| \le \frac{d}{dt} f_i(t, x) \tag{2.16}$$

where $(t, x) \in I \times \mathbb{R}^n,\ \ j = 1, 2, 3 \ldots, n$. Further, for the above fixed i assume the following criteria hold

(i) $$x_i^0 = \max\{x_1^0, x_2^0, x_3^0, \ldots, x_n^0\} \text{ and } |x_j^0| < x_i^0,\ j = 1, 2, 3, \ldots, n. \tag{2.17}$$

(ii) For every neighborhood of t_0, the following relation holds

$$(t - t_0) f_i(t, x^0) > x_i^0(1 - q) \tag{2.18}$$

(iii) For all $j \neq i$, the following relations hold in every neighborhood of t_0,

$$f_i(t_0, x^0) + f_j(t_0, x^0) \geq \frac{(1-q)}{(t-t_0)}(x_i^0 + x_j^0), \tag{2.19}$$

$$f_j(t_0, x^0) - f_i(t_0, x^0) \leq \frac{(1-q)}{(t-t_0)}(x_i^0 - x_j^0). \tag{2.20}$$

Then, there exists a dominating component solution for the IVP of Riemann–Liouville FDE (2.11) and (2.12).

3 The Variational Lyapunov Method and Stability Results

Next, we give a relation between ordinary differential equations (ODEs) and fractional differential equations (FDEs), then present the variation of parameters formula for FDEs in terms of ODEs. This is an important result, as obtaining the variation of parameters formula for FDEs in terms of fractional derivatives is still an open problem. Then, we present a stability result using the variational Lyapunov method. In order to establish the above results, a relation between the solutions of Caputo and Riemann–Liouville fractional differential equations is needed, which we give in the next section.

3.1 Relation Between the Solutions of Caputo and Riemann–Liouville Fractional Differential Equations

In this section, we begin with a relation between the solutions of Caputo FDEs and those of Riemann–Liouville FDEs. This relation leads to the observation that the solutions of Caputo FDEs have the same properties as the solutions of the Riemann–Liouville FDEs [11].

Consider the Caputo fractional differential equation and the corresponding Volterra integral differential equation given by

$$^cD^q x(t) = F(t, x), \; x(t_0) = x_0 \tag{3.1}$$

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds. \tag{3.2}$$

The aforementioned relation is established by observing that

$$^cD^q x(t) = D^q[x(t) - x(t_0)]. \tag{3.3}$$

Setting $y = x - x_0$, we have

$$^cD^q y = D^q x = F(t, x) = F(t, y + x_0)$$

which gives

$$^cD^q y = \hat{F}(t, y) \tag{3.4}$$

and

$$y^0 = [x(t) - x_0](t - t_0)^{1-q}|_{t=t_0} = 0, \tag{3.5}$$

The integral equation corresponding to (3.4) and (3.5) is given by

$$y(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \hat{F}(s, y(s))ds. \tag{3.6}$$

Suppose $y(t)$ is a solution of the Volterra fractional integral equation (3.6). Then $y(t)$ also satisfies the corresponding Riemann–Liouville fractional differential equation (3.4). Letting $y(t) = x(t) - x_0$, we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} F(s, x(s))ds,$$

which implies that $x(t)$ satisfies the integral equation (3.2) and hence is a solution of both the Caputo fractional differential equation and its corresponding Volterra integral equation.

Thus, a given Caputo FDE can be transformed into a Riemann–Liouville FDE, and hence solutions of Caputo fractional differential equations have properties similar to the properties of solutions of Riemann–Liouville fractional differential equations.

3.2 Relation Between Ordinary Differential Equations and Fractional Differential Equations

The method of variation of parameters provides a link between unknown solutions of a nonlinear system and the known solutions of another nonlinear system, and, as such, is a useful tool for the study of the qualitative behavior of the unknown solutions.

We now present a relation between ODEs and FDEs which was developed in [24]. Then, we use the variation of parameters formula to link the solutions of the two systems. Using this relation and the properties of the solutions of ODEs, which are relatively easy to find, the properties of the solutions of the corresponding FDEs can be investigated.

Consider the IVP

$$D^q x = f(t, x),\ x^0 = x(t)(t - t_0)^q|_{t=t_0}, \tag{3.7}$$

where $f \in C[([t_0, T] \times \mathbb{R}^n, \mathbb{R}^n), x \in C_p([t_0, T], \mathbb{R}^n)$, $D^q x$ is the Riemann–Liouville fractional differential operator of order $q, 0 < q < 1,\ 1 - q = p$, and assume the existence and uniqueness of the solution $x(t, t_0, x^0)$ of (3.7).

To obtain a relation between fractional and ordinary differential equations, we tentatively write

$$x(t) = x(s) + \phi(t - s), t_0 \le s \le T, \tag{3.8}$$

with the function $\phi(t - s)$ to be determined. Substituting this expression in the Riemann–Liouville fractional differential equation, we get

$$\begin{aligned} D^q x(t) &= \frac{1}{\Gamma(p+1)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1}[x(t) - \phi(t - s)]ds \\ &= \frac{1}{\Gamma(p+1)} \frac{d}{dt}[x(t)(t - t_0)^p] - \eta(t, p, \phi). \end{aligned} \tag{3.9}$$

where

$$\eta(t, p, \phi) = \frac{1}{\Gamma(p+1)} \frac{d}{dt}[\int_{t_0}^{t} (t - s)^{p-1}\phi(t - s)ds]. \tag{3.10}$$

Setting $y(t) = \dfrac{x(t)(t - t_0)^p}{\Gamma(1+p)}$, where $x(t)$ is any solution of IVP (3.7), we arrive at the IVP for ordinary differential equation, namely,

$$y'(t) = \frac{dy}{dt} = F(t, y(t)) + \eta(t, p, \phi),\ y(t_0) = x^0 \tag{3.11}$$

where

$$F(t, y) = f(t, \Gamma(1 + p)y(t)(t - t_0)^{-p}). \tag{3.12}$$

We consider the unperturbed system

$$y'(t) = F(t, y(t)),\ y(t_0) = x^0, \tag{3.13}$$

and the perturbed system (3.11) and use perturbation theory to obtain the estimates of $|y(t)|$. The nonlinear variation of parameters formula is also a very useful tool to study perturbation theory. It was developed for fractional differential equation in terms of ordinary differential equations in [24] and is presented below.

Suppose $F_y(t, y)$ exists and is continuous on $[t_0, T] \times \mathbb{R}^n$. It is known, (*see Theorem 2.5.3 in* [25]), that the solution $y(t, t_0, x^0)$ of IVP (3.13) satisfies the identity

$$\frac{\partial}{\partial t_0} y(t, t_0, x^0) + \frac{\partial}{\partial x_0} y(t, t_0, x^0) F(t_0, x_0) \equiv 0, \tag{3.14}$$

where $\frac{\partial}{\partial t_0} y(t, t_0, x^0)$ and $\frac{\partial}{\partial x_0} y(t, t_0, x^0) F(t_0, x^0)$ are solutions of the linear system

$$z' = F_y(t, y(t, t_0, x^0))z,$$

with the corresponding initial conditions $z(t_0) = -F(t_0, x^0)$ and $z(t_0) = I$, the identity matrix. Using this information, the nonlinear variation of parameters formula for the solutions of IVP (3.11) was obtained. Setting $p(s) = y(t, s, z(s))$, where $z(t)$ is the solution of the perturbed IVP (3.11), and using (3.13) we have

$$\begin{aligned} \frac{d}{ds} p(s) &= \frac{\partial}{\partial t_0} y(t, s, z(s)) + \frac{\partial}{\partial x^0} y(t, s, z(s))[F(s, z(s)) + \eta(s, t_0, \phi_0)] \\ &= \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_o). \end{aligned}$$

Integrating from t_0 to t yields the desired nonlinear variation of parameters formula, which links the solutions of the fractional differential equation to the solutions of the generated ordinary differential equation:

$$z(t, t_0, x^0) = y(t, t_0, x^0) + \int_{t_0}^{t} \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_0) ds.$$

3.3 Variational Lyapunov Method and Stability

In order to present the stability results, the Caputo fractional Dini derivative of the Lyapunov function is defined using the Grunwald–Letnikov fractional derivative, taking advantage of the series in its definition.

Definition 3.1 The Grunwald–Letnikov (GL) fractional derivative is defined as

$$D_0^q x(t) = \lim_{\substack{h \to 0_+ \\ nh = t - t_0}} \frac{1}{h^q} \sum_{r=0}^{n} (-1)^r \, {}_qC_r x(t - rh) \tag{3.15}$$

or

$$D_0^q x(t) = \lim_{h \to 0_+} \frac{1}{h^q} x_h^q(t),$$

where

$$\begin{aligned} x_h^q(t) &= \frac{1}{h^q} \sum_{r=0}^{n} (-1)^r \,{}_qC_r x(t - rh) \\ &= \frac{1}{h^q}[x(t) - S(x, h, r, q)] \end{aligned} \tag{3.16}$$

with

$$S(x, h, r, q) = \sum_{r=1}^{n} (-1)^{r+1} \,{}_qC_r x(t - rh). \tag{3.17}$$

Now, using (3.15) we define the GL fractional Dini derivative by

$$D_{0_+}^q x(t) = \limsup_{h \to 0_+} \frac{1}{h^q} \sum_{r=0}^{n} (-1)^r \,{}_qC_r x(t - rh). \tag{3.18}$$

Since the Caputo fractional derivative and GL fractional derivative are related by the equation

$${}^cD^q x(t) = D_0^q [x(t) - x(t_0)],$$

we define the Caputo fractional Dini derivative by

$${}^cD_+^q x(t) = D_{0_+}^q [x(t) - x(t_0)]. \tag{3.19}$$

Consider the Caputo differential equation

$${}^cD^q x = f(t, x), \;\; x(t_0) = x_0. \tag{3.20}$$

Then, using relations (3.19) and (3.20), we get

$$\begin{aligned} f(t, x) &= \limsup_{h \to 0_+} \frac{1}{h^q} \sum_{r=0}^{n} (-1)^r \,{}_qC_r [x(t - rh) - x_0] \\ &= \limsup_{h \to 0_+} \frac{1}{h^q} [x(t) - x_0 - S(x, h, r, q)] \end{aligned}$$

where $S(x, h, r, q) = \sum_{r=1}^{n} (-1)^{r+1} \,{}_qC_r [x(-rh) - x_0]$. This yields

$$S(x, h, r, q) = x(t) - x(t_0) - h^q f(t, x) - \epsilon(h^q)], \tag{3.21}$$

where $\frac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$. The definition of the Caputo fractional Dini derivative for the Lyapunov function is given below.

Definition 3.2 Let $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ where $S(\rho) = \{x : |x| < \rho\}$. Let $V(t, x)$ be locally Lipschitzian in x. The Grunwald–Letnikov fractional Dini derivative of $V(t, x)$ is defined by

$$D^q_{0_+} V(t, x) = \limsup_{h \to 0_+} \frac{1}{h^q} [V(t, x) - \sum_{r=1}^{n} (-1)^{r+1} {}_qC_r V(t - rh, S(x, h, r, q))]$$

where $S(x, h, r, q) = x(t) - h^q f(t, x) - \epsilon(h^q)$ with $\frac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$. Then, the Caputo fractional Dini derivative of $V(t, x)$ is defined as

$${}^cD^q_+ V(t, x) = \limsup_{h \to 0_+} \frac{1}{h^q} [V(t, x) - V(t - h, x - h^q f(t, x)) - V(t_0, x_0)].$$

Definition 3.3 The zero solution of (3.1) is said to be

(i) *stable* if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \epsilon$ for $t \geq t_0$;
(ii) *uniformly stable* if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$, the inequality $|x(t; t_0, x_0)| < \epsilon$ holds for $t \geq t_0$;
(iii) *uniformly attractive* if for $\beta > 0$ and for every $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that for any $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ with $|x_0| < \beta$, the inequality $|x(t; t_0, x_0)| < \epsilon$ holds for $t \geq t_0 + T$;
(iv) *uniformly asymptotically stable* if the zero solution is uniformly stable and uniformly attractive.

Now we present a comparison theorem, which uses the variation of parameters formula and relate the solutions of a perturbed system to the known solutions of an unperturbed system in terms of the solution of a comparison scalar fractional differential equation.

Consider the two fractional differential systems given by

$${}^cD^q y = f(t, y), \;\; y(t_0) = y_0, \tag{3.22}$$

$${}^cD^q x = F(t, x), \;\; x(t_0) = x_0 \tag{3.23}$$

where $f, F \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}^n]$, and assume the following assumption relative to system (3.22).

(H) The solutions $y(t, t_0, x_0)$ of (3.22) exist for all $t \geq t_0$, are unique and continuous with respect to the initial data, and $|y(t, t_0, x_0|$ is locally Lipschitzian in x_0.

Let $|x_0| < \rho$ and suppose that $|y(t, t_0, x_0)| < \rho$ for $t_0 \leq t \leq T$. For any $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ and for any fixed $t \in [t_0, T]$, we define the Grunwald–Letnikov fractional Dini derivative of V by

$$
\begin{aligned}
&D^q_{0_+} V(s, y(t, s, x)) \\
&\quad = \limsup_{h \to 0_+} \frac{1}{h^q} \{V(s, y(t, s, x)) - \sum_{r=1}^{n} (-1)^{r+1} \, {}_qC_r V(s - rh, s - h^q F(s, x))\}.
\end{aligned}
$$

The Caputo fractional Dini derivative of the Lyapunov function $V(s, y(t, s, x))$, for any fixed $t \in [t_0, T]$, any arbitrary point $s \in [t_0, T]$ and $x \in \mathbb{R}^n$, is given by

$$
\begin{aligned}
&{}^c D^q_+ V(s, y(t, s, x)) \\
&\quad = \limsup_{h \to 0_+} \frac{1}{h^q} \{V(s, y(t, s, x)) - V(s - h, y(t, s - h, x - h^q F(s, x)))\},
\end{aligned}
$$

where

$$
\begin{aligned}
&V(s - h, y(t, s - h, x - h^q F(s, x))) \\
&\quad = \sum_{r=1}^{n} (-1)^{r+1} \, {}_qC_r \; V(s - rh, y(t, s - rh, x - h^q F(s, x))).
\end{aligned}
$$

Theorem 3.4 *Assume that assumption (H) holds. Suppose that*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $V(t, x)$ *is locally Lipschitzian in x with Lipschitz constant* $L > 0$*, and for* $t_0 \le s \le t$ *and* $x \in S(\rho)$,

$$ {}^c D^q_+ V(s, y(t, s, x)) \le g(s, V(s, y(t, s, x)); \tag{3.24} $$

(ii) $g \in C[\mathbb{R}^2_+, \mathbb{R}]$ *and the maximal solution* $r(t, t_0, u_0)$ *of*

$$ {}^c D^q u = g(t, u), \;\; u(t_0) = u_0 \ge 0 \tag{3.25} $$

exists for $t_0 \le t \le T$.
Then, if $x(t) = x(t, t_0, x_0)$ *is any solution of (3.23), we have* $V(t, x(t, t_0, x_0)) \le r(t, t_0, u_0), \;\; t_0 \le t \le T$*, provided* $V(t_0, y(t, t_0, x_0)) \le u_0$.

The following stability result is an application of Theorem 3.4.

Theorem 3.5 *Assume that (H) holds and condition (i) of Theorem 3.4 is satisfied. Suppose that* $g \in C[\mathbb{R}^2, \mathbb{R}], \;\; g(t, 0) = 0, \;\; f(t, 0) = 0, \;\; F(t, 0) = 0$ *and for* $(t, x) \in \mathbb{R}_+ \times S(\rho)$,

$$ b(|x|) \le V(t, x) \le a(|x|) $$

$a, b \in \mathbb{K} = \{c \in C[[0, \rho), \mathbb{R}_+] : c(0) = 0$ *and c is monotonically increasing\}. Further suppose that the trivial solution of (3.22) is uniformly stable and* $u \equiv 0$ *of (3.25) is asymptotically stable. Then, the trivial solution of (3.23) is uniformly asymptotically stable.*

4 Fractional Trigonometric Functions

It is well known that trigonometric functions play a vital role in understanding physical phenomena that exhibit oscillatory behavior. The generalization of trigonometric functions has been made through differential equations. In this section, we give a brief summary of the work done in order to introduce fractional trigonometric functions and their generalizations through fractional differential equations of a specific type [35]. Fractional hyperbolic functions and their generalizations are also described in a similar fashion in [36].

Consider the following αth order homogeneous fractional initial value with Caputo derivative

$$ {}^cD^{\alpha}x(t) + x(t) = 0,\ 1 < \alpha < 2,\ t \geq 0, \tag{4.1}$$

$$ x(0) = 1,\ {}^cD^{q}x(0) = 0,\ \text{where } \alpha = 2q, 0 < q < 1. \tag{4.2}$$

The general solution of (4.1) and (4.2) is given by $c_1x(t) + c_2y(t)$, where c_1 and c_2 are arbitrary constants, and where $x(t)$ and $y(t)$ are infinite series solutions of the form

$$x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)},\ y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)},\ t \geq 0,\ 0 < q < 1.$$

We designate these series by $\cos_q t$ and $\cos_q t$, respectively. Then,

$$\cos_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)}, \tag{4.3}$$

$$\sin_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)}, \tag{4.4}$$

which we denote $M^q_{2,0}(t)$ and $M^q_{2,1}(t)$, respectively, for future convenience. Observe that if $q = 1,\ \cos_q t = \cos t$ and $\sin_q t = \sin t$. Using the FDE (4.1) and the initial condition (4.2), one can prove the following properties of $x(t)$ and $y(t)$:

(1) $x^2(t) + y^2(t) = 1,\ t \geq 0$
(2) $x(t)$ and $y(t)$ have at least on zero in $\mathbb{R}_+$.
(3) The zeros of $x(t)$ and $y(t)$ interlace each other, i.e., between any two consecutive zeros of $y(t)$ there exists one and only one zero of $x(t)$.
(4) For $t \geq 0$ and $\eta \geq 0$

$$y(t+\eta) = y(t)x(\eta) + y(\eta)x(t)$$

$$x(t+\eta) = x(t)x(\eta) + y(\eta)y(t)$$

(5) $x(t)$ is an even function, but for $q \neq 1$, $y(t)$ is not an odd function
(6) Euler's Formulae:
The solutions of FDE (4.1) can also be expressed as $E_q(it^q)$ and $E_q(-it^q)$ where $\pm i$ are the roots of $\lambda^2 + 1 = 0$. $E_q(-it^q)$ can be expressed in terms of $M^q_{2,0}(t)$ *and* $M^q_{2,1}(t)$ as

$$(i)\ E_q(it^q) = 1 - \frac{t^{2q}}{\sum(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots + i\left(\frac{t^q}{\sum(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \ldots\right),$$

$$= M^q_{2,0}(t) + iM^q_{2,1}(t)$$

$$(ii)\ E_q(-it^q) = 1 - \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots - i\left(\frac{t^q}{\Gamma(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \ldots\right).$$
$$= M^q_{2,0}(t) - iM^q_{2,1}(t)$$

$$\text{Thus,}\quad M^q_{2,0}(t) = \frac{1}{2}(E_q(it^q) + E_q(-it^q)), \text{ and}$$

$$M^q_{2,1}(t) = \frac{1}{2i}(E_q(it^q) - E_q(-it^q)), t \in \mathbb{R}^+.$$

The following theorem generalizes the notion of fractional trigonometric functions using an αth order fractional differential equation of the type considered in (4.1).

Theorem 4.1 *Consider the* αth *order fractional IVP of the form*

$$^cD^\alpha x(t) + x(t) = 0,\ x(0) = 1,\ ^cD^q x(0) = 0, \ldots,\ ^cD^{(n-1)_q} x(0) = 0 \qquad (4.7)$$

where $n < \alpha < n+1$, *with* $\alpha = nq$, $0 < q < 1$, *n fixed.*

The general solution of this equation is given by $c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$ *where* $c_1, c_2, \ldots, c_n$ *are arbitrary constants and* $x_1(t), x_2(t), \ldots, x_n(t)$ *are infinite series of the form*

$$x_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{nkq}}{\Gamma(1+nkq)}$$
$$x_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+1)q}}{\Gamma(1+(nk+1)q)}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$x_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)},$$

which are denoted by $M^q_{n,0}(t),\ M^q_{n,1}(t), \ldots,\ M^q_{n,n-1}(t)$, *respectively.*

More generally, let

$$M^q_{n,r}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+r)q}}{\Gamma(1+(nk+r)q)}, \qquad n \in N, t \geq 0.$$

These are the n linearly independent solutions of the Caputo FDE (4.7).

Let $x_1(t),\ x_2(t), \ldots, x_n(t)$ be n solutions of the nth order Caputo FDE for $t \in \mathbb{R}^+$. Then, the Wronskian $W(t)$ of the n solutions is defined as

$$W(t) = \begin{vmatrix} x_1 & x_2 \cdots & x_n \\ -x_n & x_1 \cdots & x_{n-1} \\ -x_{n-1} & -x_n & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ -x_2 & -x_3 & \cdots & x_1 \end{vmatrix} (t).$$

Theorem 4.2 *Let $x_1(t),\ x_2(t), \ldots, x_n(t)$ be n solutions of (4.7). Then, these solutions are linearly independent on $\mathbb{R}_+$ if and only if $W(t) \neq 0$ for every $t \in \mathbb{R}_+$.*

Finally, we give the addition formula of the solutions of (4.7) for $\eta \geq 0$ and $t \geq 0$,

$$M^q_{n,r}(t+\eta) = \sum_{k=0}^{r} M^q_{n,k}(t) M^q_{n,r-k}(\eta) - \sum_{k=r+1}^{n-1} M^q_{n,k}(t) M^q_{n,n+r-k}(\eta).$$

5 Impulsive Differential Equations

It is well established that many evolutionary processes exhibit impulses, which are perturbations whose duration is negligible compared to the duration of the process. Thus, differential equations with impulses are appropriate mathematical models for the study of physical phenomena exhibiting sudden change. As fractional differential equations are considered better models of processes that have memory and hereditary properties, it is natural to use FDEs with impulses to study perturbations or sudden changes in these systems.

In this section, we present known existence and uniqueness results for impulsive fractional differential equation with both fixed and variable moments of impulse.

In both cases, we use the theory of inequalities and comparison theorems, the method of lower and upper solutions and the iterative methods of quasilinearization (QL) and monotone iterative technique (MIT). In order to illustrate this approach, we present an existence and uniqueness result for impulsive FDEs using the generalized QL method for fixed moments of impulse and using the method lower and upper solutions and the MIT for variable moments of impulse.

5.1 FDE with Fixed Moments of Impulse

We begin with the basic notation and a definition of the solution of a FDE with fixed moments of impulse and then proceed to the generalized QL method.

Definition 5.1 Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC_p[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C_p-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t, y) = h(t_k^+, x)$$

exists for $k = 1, 2, \ldots, n-1$.

Definition 5.2 Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC^q[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C^q-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t, y) = h(t_k^+, x)$$

exists for $k = 1, 2, \ldots, n-1$.

Consider the impulsive Caputo fractional differential system defined by

$$\begin{cases} {}^cD^q x = f(t, x), \ t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), \ k = 1, 2, \ldots, n-1, \\ x(t_0) = x_0, \end{cases} \tag{5.1}$$

where $f \in PC[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, n-1$.

Definition 5.3 By a solution $x(t, t_0, x_0)$ of system (5.1), we mean a PC^q continuous function $x \in PC^q[[t_0, T], \mathbb{R}^n]$, such that

$$x(t) = \begin{cases} x_0(t, t_0, x_0), & t_0 \leq t \leq t_1, \\ x_1(t, t_1, x_1^+), & t_1 \leq t \leq t_2, \\ \cdot \\ \cdot \\ \cdot \\ x_k(t, t_k, x_k^+), & t_k < t \leq t_{k+1}, \\ \cdot \\ \cdot \\ x_{n-1}(t, t_{n-1}, x_{n-1}^+), & t_{n-1} < t \leq T, \end{cases} \tag{5.2}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_{n-1} \leq T$ and $x_k(t, t_k, x_k^+)$ is the solution of the following fractional initial value problem

$$\begin{aligned} {}^cD_x^q &= f(t, x), \\ x_k^+ = x(t_k^+) &= I_k(x(t_k)) \end{aligned}.$$

Definition 5.4 $\alpha, \beta \in PC^q[[t_0, T], \mathbb{R}]$ are said to be lower and upper solutions of equation (5.1), if and only if they satisfy the following inequalities:

$$\begin{cases} {}^cD^q\alpha \le f(t,\alpha) + g(t,\alpha), t \ne t_k, \\ \alpha(t_k^+) \le I_k(\alpha(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \alpha(t_0) \le x_0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^cD^q\beta \ge f(t,\beta) + g(t,\beta), t \ne t_k, \\ \beta(t_k^+) \ge I_k(\beta(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \beta(t_0) \ge x_0, \end{cases} \tag{5.4}$$

respectively.

We first state two lemmas [9] that are needed to prove the main theorem.

Lemma 5.5 *The linear, nonhomogeneous impulsive Caputo initial value problem*

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y), \ t \ne k, \\ x(t_k^+) = (I_k(x(t_k)), \ k = 1, 2, \ldots, n-1, \\ x(t_0) = x_0, \end{cases}$$

has a unique solution on the interval $[t_0, T]$.

Lemma 5.6 *Suppose that*

(i) $\alpha_0(t)$ *and* $\beta_0(t)$ *are lower and upper solutions of the hybrid Caputo fractional differential equation (5.1).*
(ii) $\alpha_1(t)$ *and* $\beta_1(t)$ *are the unique solutions of the following linear, impulsive Caputo initial value problems,*

$$\begin{cases} {}^cD^q\alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha-\alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0), t \ne t_k, \\ \alpha_1(t_k^+) = I_k(\alpha_0(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \alpha_1(t_0) = x_0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} {}^cD^q\beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g(t,\beta_0) + g_x(t,\beta_0)(\beta_1-\beta_0), t \ne t_k, \\ \beta_1(t_k^+) = I_k(\beta_0(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \beta_1(t_0) = x_0, \end{cases} \tag{5.6}$$

respectively;
(iii) $I_k(x)$ *is nondecreasing function in* x *for each* $k = 1, 2, 3, \ldots, n-1$*;*
(iv) f_x, g_x *are continuous and Lipschitz in* x *on* $[t_0, T]$.

Then, $\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t)$ *on* $[t_0, T]$.

We now state the main result.

Theorem 5.7 *Assume that*

(i) $f, g \in PC[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $\alpha_0, \beta_0 \in PC^q[[t_0, T], \mathbb{R}]$ *are lower and upper solutions of (5.1) such that* $\alpha_0(t) \le \beta_0(t), t \in [t_0, T]$*;*

(ii) $f_x(t,x)$ *exists, is increasing in* x *for each* t, $f(t,x) \geq f(t,y) + f_x(t,y)(x-y), x \geq y$ *and* $|f_x(t,x) - f_x(t,y| \leq L_1|x-y|$, *and further suppose that* $g_x(t,x)$ *exists, is decreasing in* x *for each* t, $g(t,x) \geq g(t,y) + g_x(t,y)(x-y), x \geq y$ *and* $|g_x(t,x) - g_x(t,y| \leq L_2|x-y|$;

(iii) I_k *is increasing and Lipschitz in* x *for each* $k = 1, 2, 3, \ldots, n-1$.

Then, there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ *such that* $\alpha_n \to \rho, \beta_n \to r$, *as* $n \to \infty$, *uniformly and monotonically to the unique solution* $\rho = r = x$ *of (5.1) on* $[t_0, T]$, *and the convergence is quadratic.*

Remark 5.8 Observe that if we set $I_k \equiv 0$ for all k, then (5.1) reduces to a Caputo fractional differential equation, for which the generalized quasilinearization for this type of equations has been studied in [24], under the assumption of a Holder continuity. However, Theorem 5.7, with $I_k \equiv 0$, shows that those results also hold with the weakened hypothesis of C^q-continuity.

5.2 *Impulsive Differential Equation with Variable Moments of Impulse*

Consider a sequence of surfaces $\{S_k\}$ given by $S_k : t = \tau_k(x),\ k = 1, 2, 3, \ldots;\ \tau_k : \mathbb{R} \to \mathbb{R}$ such that $\tau_k(x) < \tau_{k+1}(x)$ and $\lim\limits_{k\to\infty} \tau_k(x) = \infty$. Then, the impulsive Caputo FDE with variable moments of impulse is given by

$$\left.\begin{aligned} {}^cD^q x &= f(t,x),\ t \neq \tau_k(x) \\ x(t^+) &= x(t) + I_k(x(t)),\ t = \tau_k(x). \end{aligned}\right\} \tag{5.7}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}$ is an open set, $\tau_k \in C[\Omega, (0, \infty)],\ I_k(x(t)) = \triangle(x(t)) = x(t^+) - x(t^-)$, and $I_k \in C[\Omega, \mathbb{R}], k = 1, 2, 3, \ldots$.

In this case, the moments of impulse depend on the solutions satisfying $t_k = \tau_k(x(t_k))$, for each k. Thus, solutions starting at different points will have different points of discontinuity. Also, a solution may hit the same surface $S_k : t = \tau_k(x)$ several times and we shall call such a behavior "*pulse phenomenon*". In addition, different solutions may coincide after some time and behave as a single solution thereafter. This phenomenon is called "*confluence*".

In order to construct the method of lower and upper solutions in the given interval, we have to ensure that the solution does not exhibit a pulse phenomenon. The following theorem gives a simple set of sufficient conditions for any solution to meet each surface exactly once and shows the interplay between the functions f, τ_k, and I_k [14]. In the rest of the section,we shall assume that the solution of (5.7) exists for $t \geq t_0$ and is C_p continuous.

Theorem 5.9 *Assume that*

(i) $f \in C[[t_0, T] \times \Omega, \mathbb{R}],\ t_0 \geq 0, I_k \in C[\Omega, \mathbb{R}],\ \tau_k \in C[\Omega, (0, \infty)]$, *is linear and bounded, and* $\tau_k(x) < \tau_{k+1}(x)$ *for each* k*;*

(ii) (a) $\dfrac{\partial \tau_k(x)}{\partial x} f(t,x) < \dfrac{(t-\tilde{t})^p}{\Gamma(p+1)}$, *whenever* $t = \tau_k(x(t,\tilde{t},\tilde{x}))$,

(b) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_k(x))\right) I_k(x) < 0$, *and*

(c) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_{k-1}(x))\right) I_{k-1}(x) \geq 0, 0 \leq s \leq 1,\ \ x + I_k(x) \in \Omega$ *whenever* $x \in \Omega$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *of IVP (5.7), such that* $0 \leq t_0 < \tau_1(x_0)$, *meets each surface* S_k *exactly once.*

Next, we consider the following initial value problem:

$$\begin{aligned} &{}^cD^q x = f(t,x), t \neq \tau(x), \\ &x(t^+) = x(t) + I(x(t)),\ \ t = \tau(x) \\ &x(t_0^+) = x_0, \end{aligned} \tag{5.8}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R}, \mathbb{R}]$, and $\tau \in C^q[\mathbb{R}, (0, \infty)]$, with $J = [t_0, T]$, $t_0 \geq 0$, $\tau(x)$ is linear of the form $\lambda_0 x + \lambda_1$, $\lambda_0 \in \mathbb{R}^+$, $\lambda_1 \in \mathbb{R}$, and $\tau(x)$ is increasing.

The lower and upper solutions of (5.8) are defined as follows:

Definition 5.10 A function $v \in C_p[J, \mathbb{R}]$ is said to be a lower solution of (5.8) if it satisfies the following inequalities

$$\begin{aligned} &{}^cD^q v \leq f(t,v), t \neq \tau(v(t)), \\ &v(t^+) \leq v(t) + I(v(t)),\ \ t = \tau(v(t)) \\ &v(t_0^+) \leq x_0, \end{aligned} \tag{5.9}$$

Definition 5.11 A function $w \in C_p[J, \mathbb{R}]$ is said to be an upper solution of (5.8), if it satisfies the following inequalities

$$\begin{aligned} &{}^cD^q w \geq f(t,w), t \neq \tau(w(t)), \\ &w(t^+) \geq w(t) + I(w(t)),\ \ t = \tau(w(t)) \\ &w(t_0^+) > x_0, \end{aligned} \tag{5.10}$$

The following result is the fundamental inequality theorem in the theory of Caputo fractional differential inequalities with variable moments of impulse [13].

Theorem 5.12 *Assume that*

(i) $v, w \in C_p[J, R]$ *are lower and upper solutions of (5.8), respectively;*
(ii) $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R}, \mathbb{R}]$, $\tau \in C^q[\mathbb{R}, (0, \infty)]$, τ *is linear and increasing;*
(iii) $\tau_x(v + sI(v))I(v) < 0,\ t = \tau(v(t)),\ 0 \leq s \leq 1;$
(iv) $\tau_x(w + sI(w))I(w) > 0,\ t = \tau(w(t)),\ 0 \leq s \leq 1;$

(v) $\tau_x(v)f(t,v) < \dfrac{(t-t_1)^p}{p}$, *whenever* $t = \tau(v(t,t_1,v_1))$, *where* $v(t,t_1,v_1)$ *is the lower solution of (5.8) starting at* (t_1,v_1), $t_1, t \in J$;

(vi) $\tau_x(w)f(t,w) > \left\{\dfrac{(t-t_1)^p}{p}\right\}$, *whenever* $t = \tau(w(t,t_1,w_1))$, *where* $w(t,t_1,w_1)$ *is the upper solution of (5.8) starting at* (t_1,w_1), $t_1, t \in J$.

(vii) $f(t,x) - f(t,y) \le L(x-y), x \ge y, L > 0$.

Then, $v(t_0) \le w(t_0)$ *implies* $v(t) \le w(t)$, $t_0 \le t \le T$.

Next, we state an existence result based on the existence of upper and lower solutions.

Theorem 5.13 *Let* $v, w \in C_p[J,\mathbb{R}]$ *be lower and upper solutions of (5.8), respectively, such that* $v(t) \le w(t)$ *on* J. *Suppose that* $w(t)$ *hits the surface* $S : t = \tau(x)$ *only once at* $t = t_* \in (t_0, T]$ *and* $w(t_*) < w(t_*^+)$. *Also, assume*

(i) $f \in C[J \times \mathbb{R}, \mathbb{R}], \tau \in C^q[\mathbb{R},(0,\infty)]$, τ *is linear and increasing for* $v(t) \le x \le w(t), t \in J$;

(ii) $\tau_x(x + sI(x))I(x) < 0,\ 0 \le s \le 1,\ t = \tau(x),\ v(t) \le x \le w(t),\ t \in J$;

(iii) $\tau_x(x)f(t,x) < \dfrac{(t-t_1)^p}{p}$ *whenever* $t = \tau(x(t,t_1,x_1))$, $v(t) \le x \le w(t)$, $t, t_1 \in J$,

(iv) For any (t,x) *such that* $t = \tau(x)$, $v(t) \le x \le w(t)$ *implies* $v(t) \le x^+ \le w(t)$, $t \in J$.

Then, there exists a solution $x(t)$ *of (5.8) such that* $v(t) \le x(t) \le w(t)$ *on* J.

The method of upper and lower solutions, described previously, gives a theoretical result, namely, the existence of a solution of (5.8) in a closed sector, whereas the monotone iterative technique is a constructive method, which gives a sequence that converges to a solution of (5.8). In the case of impulsive Caputo fractional differential equations with variable moments of impulsive, this practical method involves working with sequences of solutions of a simple linear Caputo fractional differential equation of order q, $0 < q < 1$, with variable moments of impulse. This result is given in the following theorem [12].

Theorem 5.14 *Assume that*

(i) $v_0,\ w \in PC_p[J,\mathbb{R}]$ *are lower and upper solutions of (5.8) respectively, such that* $v_0(t) \le w(t)$ *on* J, *and* $w(t)$ *hits the surface* $S : t = \tau(x)$ *only once at* $t = t_* \in (t_0, T]$ *and* $w(t_*) < w(t_*^+)$, $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R},\mathbb{R}], \tau \in C^q[\mathbb{R},(0,\infty)]$ *and* $\tau(x)$ *is linear and increasing for* $v_0(t) \le x \le w(t), t \in J$;

(ii) $\tau_x(x + sI(x))I(x) < 0, 0 \le s \le 1, t = \tau(x), v_0(t) \le x \le w(t), t \in J$;

(iii) $\dfrac{\partial \tau}{\partial x} f(t,x) < \dfrac{(t-t_1)^p}{p}$, *whenever* $t = \tau(x(t,t_1,x_1)), v_0(t) \le x \le w(t)$;

(iv) $f(t,x) - f(t,y) \ge -M(x-y),\ v_0(t) \le y \le w(t),\ t \in J, M > 0$;

(v) for any (t,x) *such that* $t = \tau(x)$, $v_0(t) \le x \le w(t)$ *implies* $v_0(t) \le x^+ \le w(t), t \in J$,

Then, there exists a monotone sequence $\{v_n\}$ *such that* $v_n \to \rho$ *as* $n \to \infty$ *monotonically on* J. *Also,* ρ *is the minimal solution of (5.8).*

6 Fractional Integro-Differential Equations

It is well known that integro-differential equations are used to mathematically model physical phenomena,where past information is necessary to understand the present. On the other hand, fractional differential equations play an important role in studying processes that have memory and hereditary properties. Fractional integro-differential equations combine these two topics. In this section, we present a summary of results involving periodic-boundary value problems (PBVP) for fractional integro-differential equations using inequalities and comparison theorems [37].

Consider the following Caputo fractional integro-differential equation

$$ {}^cD^q u = f(t, u, I^q u) \tag{6.1}$$

$$ u(0) = u_0, \tag{6.2}$$

where $f \in C[J \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}],\ u \in C^1[J, \mathbb{R}], J = [0, T],$

$$ \text{and} \qquad I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds. \tag{6.3}$$

The following theorem gives the explicit solution of the linear Caputo fractional integro-differential initial value problem.

Theorem 6.1 *Let* $\lambda \in C^1([0, T], \mathbb{R})$. *The solution of* ${}^cD^q \lambda(t) = L\lambda(t) + MI^q \lambda(t)$ *is given by*

$$ \lambda(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{{}_{n+k}C_k M^n L^k 2^{n+k} \lambda(0)}{\Gamma[(2n+1)q+1]} t^{(2n+1)q} $$

where $L, M > 0$.

The following comparison theorem is needed to prove the main result.

Theorem 6.2 *Let* $J = [0, T],\ f \in C[J \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}], v, w \in C^1[J, \mathbb{R}]$ *and suppose that the following inequalities hold for all* $t \in J$.

$$ {}^cD^q v(t) \le f(t, v(t), I^q v(t)), v(0) \le u_0 \tag{6.4}$$

$$ {}^cD^q w(t) \ge f(t, w(t), I^q w(t)), w(0) \ge u_0. \tag{6.5}$$

Suppose further that $f(t, u(t), I^q u(t))$ *satisfies the following Lipschitz-like condition,*

$$ f(t, x, I^q x) - f(t, y, I^q y) \le L(x-y) + M(I^q x - I^q y), \tag{6.6}$$

for $x \ge y, L, M > 0$. *Then,* $v(0) \le w(0)$ *implies that* $u(t) \le w(t), 0 \le t \le T$.

Corollary 6.3 *Let $m \in C^1[J, \mathbb{R}]$ be such that*

$$^cD^q m(t) \le L\, m(t) + MI^q m(t),\ \ m(0) = m_0 \le 1,$$

then

$$m(t) \le \lambda(t)$$

for $0 \le t \le T,\ L, M > 0;\ \lambda(0) = 1$ *and* $\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{{}_{n+k}C_k M^n L^k 2^{n+k}}{\Gamma[(2n+1)q+1]} t^{(2n+1)q}.$

Proof We have

$$\begin{aligned} {}^cD^q m(t) &\le L\, m(t) + MI^q m(t),\\ {}^cD^q \lambda(t) + 2L\, \lambda(t) + 2MI^q \lambda(t) &\\ &\ge L\lambda(t) + MI^q \lambda(t),\\ \text{for } m(0) = m_0 \le 1 = \lambda(0). \end{aligned}$$

Hence, from Theorem 6.2 we conclude that $m(t) \le \lambda(t), t \in J$. □

The result in the above corollary is true even if $L = M = 0$, which we state below.

Corollary 6.4 *Let* $^cD^q m(t) \le 0$ *on* $[0, T]$. *If* $m(0) \le 0$ *then* $m(t) \le 0, t \in J$.

Proof By definition of $^cD^q m(t)$ and by hypothesis,

$$^cD^q m(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} m'(s) ds \le 0,$$

which implies that $m'(t) \le 0$, on $[0, T]$. Therefore $m(t) \le m(0) \le 0$ on $[0, T]$. The proof is complete. □

Next, we present a result which uses the generalized monotone iterative technique in order to obtain minimal and maximal solutions of the Caputo fractional integro-differential equation

$$^cD^q u = F(t, u, I^q u) + G(t, u, I^q u), \tag{6.7}$$

with the boundary condition

$$g(u(0), u(T)) = 0, \tag{6.8}$$

where $F, G \in C[J \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}], u \in C^1[J, \mathbb{R}]$.

Definition 6.5 Let $v_0, w_0 \in C^1[J, \mathbb{R}]$. Then v_0 and w_0 are said to be coupled lower and upper solutions of Type I of (6.7) and (6.8) if

$$\begin{aligned} {}^cD^q v_0(t) &\le F(t, v_0(t), I^q v_0(t)) + G(t, w_0(t), I^q w_0(t)), \\ g(v_0(0), v_0(T)) &\le 0 \end{aligned} \tag{6.9}$$

$$\begin{aligned} {}^cD^q w_0(t) &\le F(t, w_0(t), I^q w_0(t)) + G(t, v_0(t), I^q v_0(t)), \\ g(w_0(0), w_0(T)) &\ge 0 \end{aligned} \tag{6.10}$$

The monotone iterative technique for (6.7) and (6.8) was developed using sequences of iterates which are solutions of linear fractional integro-differential initial value problems. Since the solution of a linear Caputo fractional differential equation is unique, the sequence of iterates is a unique sequence converging to a solution of (6.7) and (6.8). In this approach,it is not necessary to prove the existence of a solution of the Caputo fractional integro-differential equation as it follows from the construction of the monotone sequences.

In the following theorem, coupled lower and upper solutions of Type I are used to obtain monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (6.7) and (6.8).

Theorem 6.6 *Suppose that*

(i) v_0, w_0 *are coupled lower and upper solutions of Type I for (6.7) and (6.8) with* $v_0(t) \le w_0(t)$ *on* J*;*

(ii) the function $g(u, v) \in C[\mathbb{R}^2, \mathbb{R}]$ *is nonincreasing in* v *for each* u*, and there exists a constant* $M > 0$ *such that*

$$g(u_1, v) - g(u_2, v) \le M(u_1 - u_2),$$

for $v_0(0) \le u_2 \le u_1 \le w_0(0),\ v_0(T) \le v \le w_0(T)$*;*

(iii) $F, G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$ *and* $F(t, x_1, x_2)$ *is nondecreasing in* x_1 *for each* $(t, x_2) \in J \times \mathbb{R}_+$ *and is nondecreasing in* x_2 *for each* $(t, x_1) \in J \times \mathbb{R}$; *Further,* $G(t, y_1, y_2)$ *is nonincreasing in* y_1 *for each* $(t, y_2) \in J \times \mathbb{R}_+$ *and is nonincreasing in* y_2 *for each* $(t, y_1) \in J \times \mathbb{R}$.

Then, the iterative scheme given by

$$\begin{aligned} {}^cD^q v_{n+1} &= F(t, v_n, I^q v_n) + G(t, w_n, I^q w_n), \\ v_{n+1}(0) &= v_n(0) - \frac{1}{M} g(v_n(0), v_n(T)) \\ {}^cD^q w_{n+1} &= F(t, w_n, I^q w_n) + G(t, v_n, I^q v_n), \\ w_{n+1}(0) &= w_n(0) - \frac{1}{M} g(w_n(0), w_n(T)), \end{aligned}$$

yields two monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *such that*

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0.$$

Further, $v_n \to \rho$ and $w_n \to r$ in $C^1[J, \mathbb{R}]$, uniformly and monotonically, such that ρ and r are,respectively, the coupled minimal and maximal solutions of (6.7) and (6.8), that is, ρ and r satisfy the coupled system

$$\begin{aligned} {}^cD^q\rho &= F(t, \rho, I^q\rho) + G(t, r, I^q r), \\ & g(\rho(0), \rho(T)) = 0, \\ {}^cD^q r &= F(t, r, I^q r) + G(t, \rho, I^q\rho), \\ & g(r(0), r(T)) = 0. \end{aligned}$$

7 Conclusion

Our aim in this chapter was to give a brief survey of the qualitative theory of fractional differential equations developed using the fundamental concepts of differential inequalities and comparison theorems, as well as constructive monotone iterative methods. The results presented here constitute only a representative sample of the work done using these tools. For additional results see, for example, Abbas and Bechohra [1], Agarwal et al. [3–6], Jankowski [16–20], Lin et. al. [26], Nanware [29], Sambadham et al. [31, 32], Vatsala et al. [10, 30, 33, 34], Wang et al. [40–44], Yakar et al. [45, 46], and Zhang [46].

The main results in Sects. 2 and 3 are from Devi et al. [38, 39], Drici et al. [11] and Lakshmikantham et al. [22–24]. The main result in Sect. 4 is from Devi et al. [35]. The main results in Sect. 5 are from Giribabu et al. [12–14] and Devi and Radhika [8, 9]. The result in Sect. 6 is from Devi and Sreedhar [37].

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