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Varsha Daftardar-Gejji
Editor

Fractional Calculus and Fractional Differential Equations

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Fractional Calculus and Fractional Differential Equations

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Foreword

It gives me immense pleasure to write the foreword for the volume edited by Prof. Varsha Daftardar-Gejji with contributions from eminent researchers in the fields of fractional calculus (FC) and fractional differential equations (FDEs). These are the most important and prominent areas for research which have emerged as interdisciplinary branch of mathematical, physical, biological sciences and engineering. This book provides a systematic, logical development of modern topics through the articles by eminent scientists and active researchers working in this area all over the globe.

Fractional calculus has a history of more than 300 years, while modelling of various phenomena in terms of fractional differential equations has gained momentum since the last two decades or so. There is an upsurge of research articles in the areas of FC and FDEs. This book is appealing and unique in this context as it encompasses numerical analysis of fractional differential equations, dynamics and stability analysis of fractional differential equations involving delay, variable-order fractional operators along with chapters on engineering applications. Moreover, the fractional analogues of classical Poisson processes, analysis of fractional differential equations using inequalities and comparison theorems are dealt with in a concise manner in this book.

Bologna, Italy

Francesco Mainardi
University of Bologna

Preface

Fractional calculus (FC) and fractional differential equations (FDEs) have emerged as the most important and prominent areas of interdisciplinary interest in recent years. FC has a history of more than 300 years, yet its applicability in different domains has been realised only recently. In the last three decades, the subject witnessed exponential growth and a number of researchers around the globe are actively working on this topic. The Department of Mathematics at Savitribai Phule Pune University (SPPU) organised a national workshop on fractional calculus in 2012, which was the first workshop in India that exclusively focussed on fractional calculus. This workshop attracted researchers of pure and applied mathematics, statisticians, physicists and engineers from all over India, working in fractional calculus and related areas. Deliberations in that workshop have been appeared earlier as a book titled *Fractional Calculus: Theory and Applications* which was very well received.

As a continuation of this, in 2017, we organised a national conference on fractional differential equations bringing together researchers in FDEs for academic exchange of ideas through discussions. Many active scientists from all parts of the country participated in this conference. It covered a significant range of topics motivating us to take up this endeavour. The present book comprises excellent contributions by the resource persons in this conference besides invited contributions from experts abroad, who willingly contributed. This book gives a panoramic overview of the latest developments and is expected to help new researchers entering this vast field.

The book comprises eight chapters which cover numerical analysis of FDEs, fractional Poisson processes, variable-order fractional operators, fractional-order delay differential equations and related phenomena including chaos, impulsive FDEs, inequalities and comparison theorems in FDEs. Moreover, artificial neural network and FDEs are also discussed by a group of engineers. New transform methods such as Sumudu transform methods are presented, and their utility for solving fractional partial differential equations (PDEs) is discussed.

The book is written keeping young researchers in mind who are planning to embark upon the research problems in FC and FDEs and related topics. There are many aspects that are still open for pursuing further research. If this book motivates some readers to venture into these areas, the aim of the endeavour will be fulfilled.

I am very grateful to all the researchers who have made wonderful contributions to this volume. My sincere thanks to Springer India Pvt. Ltd. for publishing this beautiful book. I also take this opportunity to thank the authorities of SPPU and my colleagues at the Department of Mathematics. Last but not least, my sincere thanks to my parents, husband and children for their unfailing support throughout.

Pune, India Varsha Daftardar-Gejji

Contents

About the Editor

Varsha Daftardar-Gejji is Professor at the Department of Mathematics, Savitribai Phule Pune University, India. She completed her Ph.D. at Pune University, India. She has developed original methods for solving fractional differential equations that have become widely popular. Her noteworthy contributions include analysis of fractional differential equations and developing theories of fractional-ordered dynamical systems and related phenomena such as chaos. She is the editor of the book *Fractional Calculus: Theory and Applications* and has co-authored the book *Differential Equations* (Schaum's Outline Series). She has published more than 65 papers in reputed international journals in areas of fractional calculus, fractional differential equations and general relativity.

Numerics of Fractional Differential Equations

Varsha Daftardar-Gejji

Abstract Fractional calculus has become a basic tool for modeling phenomena involving memory. However, due to the non-local nature of fractional derivatives, the computations involved in solving a fractional differential equations (FDEs) are tedious and time consuming. Developing numerical and analytical methods for solving nonlinear FDEs has been a subject of intense research at present. In the present article, we review some of the existing numerical methods for solving FDEs and some new methods developed by our group recently. We also perform their comparative study.

1 Introduction

Fractional calculus (FC) is emerging as an unavoidable tool to model many phenomena in Science and Engineering [1, 2]. Fractional differential equations (FDEs) play a pivotal role in formulating processes involving memory effects. This realization is rather recent and during the past 3–4 decades there is an upsurge of intense activity exploring various aspects of FC and FDEs. Compared to integer-order differential equations, the FDEs open up great opportunities for modeling and simulations of multi-physics phenomena, e.g., seamless transition from wave propagation to diffusion, or from local to non-local dynamics. Due to the extra free parameter order, fractional-order based methods provide an additional degree of freedom in optimization performance. Not surprisingly, many fractional-order based methods have been used in image processing [3], image denoising, cryptography, controls, and many engineering applications very successfully.

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There exist many inequivalent definitions of fractional derivatives albeit Riemann–Liouville and Caputo derivatives are most popular. Analysis of FDEs involving these derivatives has been studied extensively in the literature [4–6]. Various analytical methods have been developed for solving nonlinear fractional differential equations such as Adomian Decomposition Method [7], New Iterative Method [8], Homotopy perturbation method, and so on. In these decomposition methods, solutions are obtained without discretizing the equations or approximating the operators. As these decomposition methods yield local solutions around initial conditions, for studying long-time behavior of the solutions of FDEs one has to resort to numerical methods. An important objective for developing new numerical methods is to study fractional-ordered dynamical systems and related phenomena such as bifurcations and chaos. For simulation work in fractional-ordered dynamical systems, accurate and time-efficient numerical methods are required. Due to the nonlocal nature of fractional derivatives, FDEs are computationally expensive to solve. So developing time-efficient, accurate, and stable numerical methods for FDEs is currently an active area of research.

Present article intends to give an overview of the numerical methods that are currently used in the literature. Section 2 gives basics and preliminaries. In Sect. 3, fractional Adams predictor–corrector method (FAM) has been presented. Section 4 deals with the new predictor–corrector method developed by Daftardar-Gejji et al. [9]. In Sect. 5, predictor–corrector method introduced by Jhinga and Daftardar-Gejji has been introduced along with its error estimate [10]. In Sect. 6, some illustrative examples have been presented which are solved by all the three methods, and a comparative study is made in the context of time taken, accuracy, and performance of the method for very small values of the order of the derivative. Finally, conclusions are drawn in the last section.

2 Fractional Calculus: Preliminaries

2.1 Definitions

Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f(t) \in C[a,b]$ is defined as

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \tag{1}$$

Caputo fractional derivative of order $\alpha > 0$ of a function $f \in C^m[a,b],\ m \in \mathbb{N}$ is defined as

$$^cD_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left[\int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds \right] = I_a^{m-\alpha} D^m f(t), \quad m-1 < \alpha < m, \tag{2}$$

where $D^m f(t) = \frac{d^m f(t)}{dt^m},\ ^cD_a^m f(t) = D^m f(t)$.

2.2 Properties of the Fractional Derivatives and Integrals

1. Let $f \in C^m[a,b],\ m-1<\beta\leq m,\ m\in\mathbb{N}$ and $\alpha>0$. Then
   a. $I_a^\alpha({}^cD_a^\beta f(t)) = {}^cD_a^{\beta-\alpha} f(t),$ if $\alpha<\beta$.
   b. $I_a^\beta({}^cD_a^\beta f(t)) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^k.$
2. For $\alpha, \beta > 0$ and $f(t)$ sufficiently smooth,
   a. if $\alpha\in\mathbb{N}$, then
$$ {}^cD_a^\beta(I_a^\alpha f(t)) = I_a^{(\alpha-\beta)} f(t). \tag{3}$$
   b. For $\alpha<\beta, m-1\leq\alpha<m, n-1\leq\beta<n$,
$$ {}^cD_a^\beta(I_a^\alpha f(t)) = {}^cD_a^{\beta-\alpha} f(t) + \sum_{k=0}^{n-m} \frac{f^{(k)}(a)}{\Gamma(k+1+\alpha-\beta)}(t-a)^{k+\alpha-\beta}. \tag{4}$$
3. For $\alpha>0, n\in\mathbb{N}$,
$$ {}^cD_a^\alpha(D_a^n f(t)) = {}^cD_a^{n+\alpha} f(t). \tag{5}$$

2.3 DGJ Method

Daftardar-Gejji and Jafari [8] introduced a new decomposition method (DGJ method) for solving functional equations of the form

$$ y = f + N(y) \tag{6}$$

where f is a known function and $N(y)$ is a nonlinear operator from a Banach space $B \rightarrow B$.

Equation (6) represents a variety of problems such as nonlinear ordinary differential equations, integral equations, fractional differential equations, partial differential equations, and systems of them.

In this method, we assume that solution y of Eq. (6) is of the form:

$$ y = \sum_{i=0}^{\infty} y_i. \tag{7}$$

The nonlinear operator is decomposed as

$$ N\left(\sum_{i=0}^{\infty} y_i\right) = N(y_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i} y_k\right) - N\left(\sum_{k=0}^{i-1} y_k\right)\right\} \tag{8}$$

$$= \sum_{i=0}^{\infty} G_i, \tag{9}$$

where $G_0 = N(y_0)$ and $G_i = \left\{N\left(\sum_{k=0}^{i} y_k\right) - N\left(\sum_{k=0}^{i-1} y_k\right)\right\},\ i \geq 1.$

Equation (6) takes the form

$$\sum_{i=0}^{\infty} y_i = f + \sum_{i=0}^{\infty} G_i. \tag{10}$$

y_i, $i = 0, 1, \ldots$ are then obtained by the following recurrence relation:

$$\begin{aligned} y_0 &= f, \\ y_1 &= G_0, \\ y_2 &= G_1, \\ &\vdots \\ y_i &= G_{i-1}, \\ &\vdots \end{aligned} \tag{11}$$

Then

$$(y_1 + y_2 + \cdots + y_i) = N(y_0 + y_1 + \cdots + y_{i-1}),\ i = 1, 2, \ldots,$$

and

$$y = f + \sum_{i=1}^{\infty} y_i = f + N\left(\sum_{i=0}^{\infty} y_i\right).$$

The k-term approximation is obtained by summing up first k-terms of (11) and is defined as

$$y = \sum_{i=0}^{k-1} y_i. \tag{12}$$

Bhalekar and Daftardar-Gejji [11] have done the convergence analysis of this method. Theorems regarding convergence of DGJ method are stated below [11].

Theorem 1 *If N is C^{∞} in a neighborhood of y_0 and*

$$||N^{(n)}(y_0)|| = \sup\left\{N^{(n)}(y_0)(h_1, h_2, \ldots, h_n) : ||h_i|| \leq 1, 1 \leq i \leq n\right\} \leq L,$$

for some real number $L > 0$ and for any n and $||y_i|| \leq M < \frac{1}{e}, i = 1, 2, \ldots,$ then $\sum_{i=0}^{\infty} G_i$ is absolutely convergent and moreover,

$$||G_n|| \leq L M^n e^{n-1}(e-1),\ n = 1, 2, \ldots. \tag{13}$$

Theorem 2 *If N is C^∞ and $||N^{(n)}(y_0)|| \leq L < 1/e$, $\forall n$, then the series $\sum_{i=0}^{\infty} G_i$ is absolutely convergent.*

3 Fractional Adams Method (FAM)

Consider the initial value problem (IVP) for $0 < \alpha < 1$:

$$ {}^cD_0^\alpha x(t) = f(t, x(t)),\ x(0) = x_0, \tag{14}$$

where ${}^cD_0^\alpha$, denotes Caputo derivative and $f : [0, T] \times \mathbb{D} \longrightarrow \mathbb{R},\ \mathbb{D} \subseteq \mathbb{R}$. For solving Eq. (14) on $[0, T]$, the interval is divided into l subintervals.

Let $h = \frac{T}{l}$, $t_n = nh$, $n = 0, 1, 2, \ldots, l \in \mathbb{Z}^+$. Then

$$x(t_n) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - \tau)^{\alpha-1} f(\tau, x(\tau))d\tau. \tag{15}$$

Consider an equispaced grid $t_j = t_0 + jh$, with step length h. Let x_j denote the approximate solution at t_j and $x(t_j)$ denotes the exact solution of the IVP (14) at t_j. Further denote $f_j = f(t_j, x_j)$.

$$\begin{aligned} I^\alpha f(t_n, x(t_n)) &= \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(s, x(s))ds \\ &\approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{\alpha-1} f(t_k, x(t_k))ds \\ &= h^\alpha \sum_{k=0}^{n-1} b_{n-k-1} f(t_k, x(t_k)), \\ &\text{where } b_k = \frac{1}{\Gamma(\alpha+1)}[(k+1)^\alpha - k^\alpha]. \end{aligned}$$

$$\text{Hence } x_n = x_0 + h^\alpha \sum_{k=0}^{n-1} b_{n-k-1} f_k. \tag{16}$$

Equation (16) is referred as fractional rectangle rule.

Implicit Adams quadrature method (using trapezoidal rule) gives the following formula. On each subinterval $[t_k, t_{k+1}]$, the function $f(t)$ is approximated by straight line

$$\begin{aligned} \tilde{f}(t, x(t)) \mid_{[t_k, t_{k+1}]} = \frac{t_{k+1} - t}{t_{k+1} - t_k} f(t_k, x(t_k)) + \\ \frac{t - t_k}{t_{k+1} - t_k} f(t_{k+1}, x(t_{k+1})). \end{aligned}$$

In view of this approximation

$$I_0^\alpha f(t_n, x(t_n)) \approx \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - t)^{\alpha-1} \tilde{f}(t, x(t)) \,|_{[t_k, t_{k+1}]}\, dt$$

$$= h^\alpha \sum_{k=0}^{n} a_{n-k} f(t_k, x(t_k)), \text{ where}$$

$$a_j = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)} & \text{if} \quad j = 0, \\ \dfrac{(j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if } j = 1, \ldots, n-1, \\ \dfrac{(n-1)^{\alpha+1} - n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)} & \text{if} \quad j = n. \end{cases}$$

$$\text{Hence } x_n = x_0 + h^\alpha a_n f_0 + h^\alpha \sum_{j=1}^{n} a_{n-j} f(t_j, x_j). \tag{17}$$

Equation (17) is referred as fractional trapezoidal rule.

Thus, fractional rectangle rule and fractional trapezoidal rule form a predictor–corrector algorithm. A preliminary approximation x_n^p (predictor) is made using Eq. (16), which is substituted in Eq. (17) to give a corrector. This method is also known as fractional Adams method [12], and used for simulations of FDEs extensively.

$$x_n^p = x_0 + h^\alpha \sum_{j=0}^{n-1} b_{n-j-1} f(t_j, x_j), \tag{18}$$

$$x_n^c = x_0 + h^\alpha a_n f_0 + h^\alpha \sum_{j=1}^{n-1} a_{n-j} f(t_j, x_j) + h^\alpha a_0 f(t_n, x_n^p). \tag{19}$$

Order of the method is said to be p when the error can be shown to have $O(h^p)$ as $h \to 0$ for step length $h > 0$. Order of the method is often regarded as a benchmark for comparing methods.

The error in the FAM [12] behaves as $\text{Max}_{j=0,1,\ldots,n} \left| x(t_j) - x_j \right| = O(h^p)$, where $p = \min\{2, 1+\alpha\}$.

4 New Predictor–Corrector Method (NPCM)

Though FAM is extensively used in the literature, for carrying out simulations pertaining to fractional-ordered dynamical systems, one needs more time-efficient numerical methods as solving FDEs involves memory effects. In pursuance to this, Daftardar-Gejji et al. [9] have proposed a new predictor–corrector method (NPCM). This method is developed as a combination of fractional trapezoidal rule and DGJ decomposition. We describe this method below.

Consider the initial value problem given in Eq. (14)

$$ {}^{c}D_0^{\alpha}x(t) = f(t, x(t)), x(0) = x_0, 0 < \alpha < 1. $$

Equation (15) can be discretized as follows.

$$ \begin{aligned} x(t_n) &= x(0) + h^{\alpha}\sum_{j=0}^{n} a_{n-j} f(t_j, x_j) \\ &= x(0) + h^{\alpha}\sum_{j=0}^{n-1} a_{n-j} f(t_j, x_j) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_n, x_n). \end{aligned} \tag{20} $$

The solution of Eq. (20) can be approximated by DGJ method, where

$$ N(x(t_n)) = \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_n, x_n). \tag{21} $$

We apply DGJ method to get approximate value of x_1, as follows:

$$ x(t_1) = x_1 = x_0 + h^{\alpha} a_1 f(t_0, x_0) + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, x_1), \tag{22} $$

$$ x_{1,0} = x_0 + h^{\alpha} a_1 f(t_0, x_0), $$

$$ x_{1,1} = N(x_{1,0}) = \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, x_{1,0}), $$

$$ x_{1,2} = N(x_{1,0} + x_{1,1}) - N(x_{1,0}). $$

The three-term approximation of $x_1 \approx x_{1,0} + x_{1,1} + x_{1,2} = x_{1,0} + N(x_{1,0} + x_{1,1})$. This gives a new predictor–corrector formula as follows:

$$ \begin{aligned} y_1^p &= x_{1,0}, \qquad z_1^p = N(x_{1,0}), \\ x_1^c &= y_1^p + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f(t_1, y_1^p + z_1^p). \end{aligned} $$

$x(t_2), x(t_3), \ldots$ can be obtained similarly.

Daftardar-Gejji et al. [9] have proposed this new predictor–corrector method (NPCM), which is derived by combining fractional trapezoidal formula and DGJ method [8] and it leads to the following formula:

$$y_n^p = x_0 + h^\alpha \sum_{j=0}^{n-1} a_{n-j} f(t_j, x_j),$$

$$z_n^p = \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_n, y_n^p),$$

$$x_n^c = y_n^p + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_n, y_n^p + z_n^p),$$

where

$$a_j = \begin{cases} \dfrac{1}{\Gamma(\alpha+2)} & \text{if} \quad j=0, \\ \dfrac{(j-1)^{\alpha+1} - 2j^{\alpha+1} + (j+1)^{\alpha+1}}{\Gamma(\alpha+2)} & \text{if} \ j = 1, \ldots, n-1, \\ \dfrac{(n-1)^{\alpha+1} - n^\alpha(n-\alpha-1)}{\Gamma(\alpha+2)} & \text{if} \quad j = n. \end{cases}$$

Here y_n^p and z_n^p are called as predictors and x_n^c is the corrector. Here x_j denotes the approximate value of solution of Eq. (20) at $t = t_j$. This is called three-step iterative method for solving nonlinear equation (20).

Error Estimation in NPCM

Let ${}^cD^\alpha x(t) \in C^2[0, T],\ T > 0$, then $\max_{0\le j\le l} \mid x(t_j) - x_j \mid = O(h^2)$.
Comment: For $0 < \alpha < 1$, the error estimate for the case ${}^cD^\alpha x(t) \in C^2[0, T]$ in the FAM is of the order $O(h^{1+\alpha})$, whereas for NPCM $O(h^2)$. Hence NPCM in this case gives more accuracy.

4.1 Stability Regions

It is further noted that both FAM and three-term NPCM are strongly stable methods. Comparison of the stability regions of NPCM and FAM is given below [13]. It should be noted that the NPCM is more stable than FAM (Figs. 1, 2, 3 and 4).

S_1: Stability region of FAM, S_2: Stability region of NPCM

4.2 NPCM for System of FDEs

The NPCM can be generalized for solving following system of fractional differential equations. Consider the following system of FDEs, for $\alpha_i > 0,\ 1 \le i \le r$:

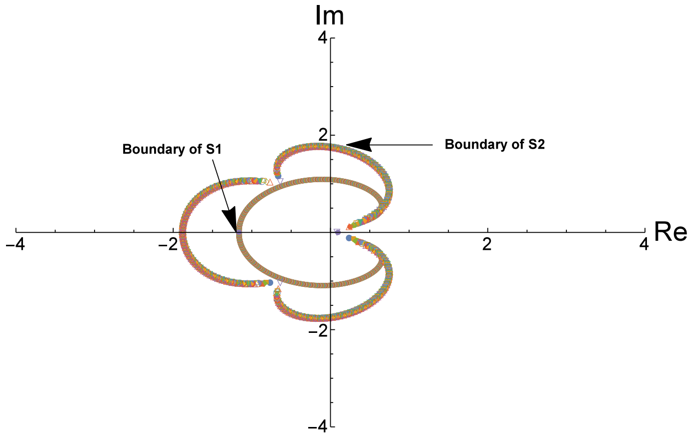

Fig. 1 $\alpha = 0.3$

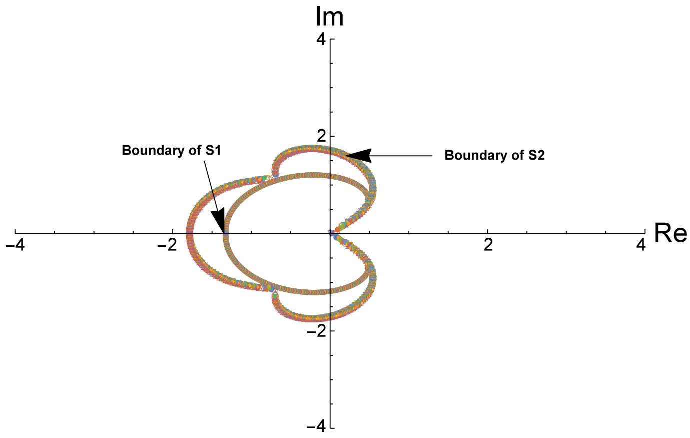

Fig. 2 $\alpha = 0.5$

$$
\begin{aligned}
{}^{c}D_0^{\alpha_1}u_1(t) &= f_1(t,\bar{u}(t)),\ u_1^{(k_1)}(0) = u_{10}^{(k_1)},\ k_1 = 0,1,2,\ldots,\lceil\alpha_1\rceil - 1,\\
{}^{c}D_0^{\alpha_2}u_2(t) &= f_2(t,\bar{u}(t)),\ u_2^{(k_2)}(0) = u_{20}^{(k_2)},\ k_2 = 0,1,2,\ldots,\lceil\alpha_2\rceil - 1,\\
&\vdots\\
{}^{c}D_0^{\alpha_r}u_r(t) &= f_r(t,\bar{u}(t)),\ u_r^{(k_r)}(0) = u_{r0}^{(k_r)},\ k_r = 0,1,2,\ldots,\lceil\alpha_r\rceil - 1.
\end{aligned}
\tag{23}
$$

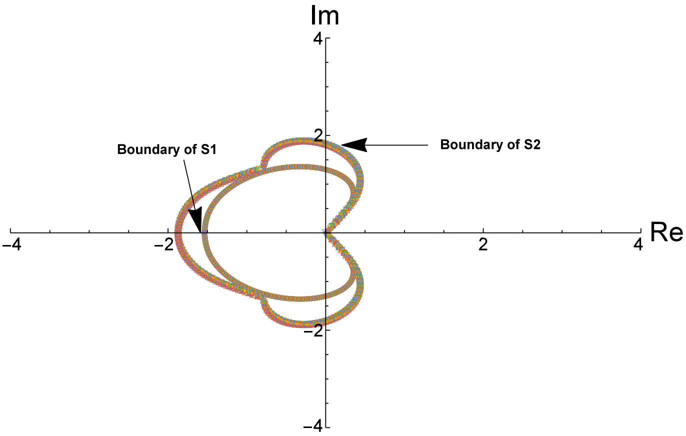

Fig. 3 $\alpha = 0.7$

Applying trapezoidal quadrature formula to the equivalent system of Volterra integral equations, it follows that

$$
\begin{aligned}
u_1(t_{n+1}) &= \sum_{k_1=0}^{\lceil\alpha_1\rceil-1} \frac{u_{10}^{(k_1)} t^{k_1}}{k_1!} + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \sum_{k=0}^{n} a_{1,k,n+1} f_1(t_k, \bar{u}(t_k)) \\
&\quad + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} f_1(t_{n+1}, \bar{u}(t_{n+1})), \\
u_2(t_{n+1}) &= \sum_{k_2=0}^{\lceil\alpha_2\rceil-1} \frac{u_{20}^{(k_2)} t^{k_2}}{k_2!} + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \sum_{k=0}^{n} a_{2,k,n+1} f_2(t_k, \bar{u}(t_k)) \\
&\quad + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} f_2(t_{n+1}, \bar{u}(t_{n+1})), \\
&\ \ \vdots \\
u_r(t_{n+1}) &= \sum_{k_r=0}^{\lceil\alpha_r\rceil-1} \frac{u_{r0}^{(k_r)} t^{k_r}}{k_r!} + \frac{h^{\alpha_r}}{\Gamma(\alpha_r+2)} \sum_{k=0}^{n} a_{r,k,n+1} f_r(t_k, \bar{u}(t_k)) \\
&\quad + \frac{h^{\alpha_r}}{\Gamma(\alpha_r+2)} f_r(t_{n+1}, \bar{u}(t_{n+1})),
\end{aligned}
$$

where

$$
a_{i,k,n+1} = \begin{cases} n^{\alpha_i+1} - (n-\alpha_i)(n+1)^{\alpha_i} & \text{if } k=0, \\ (n-k+2)^{\alpha_i+1} + (n-k)^{\alpha_i+1} - 2(n-k+1)^{\alpha_i+1} & \text{if } 1 \le k \le n, \\ 1 & \text{if } k = n+1 \end{cases}
$$

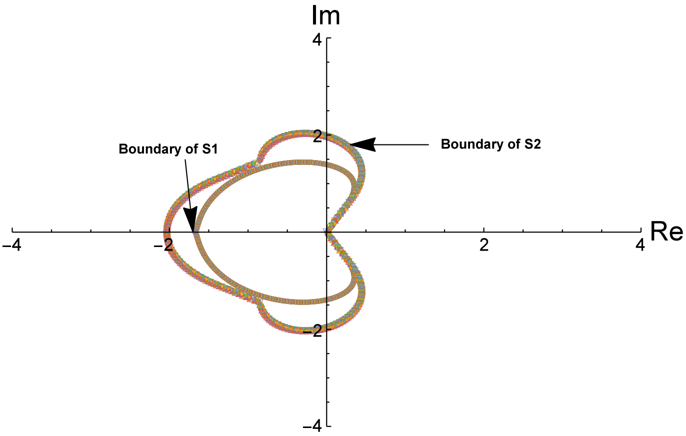

Fig. 4 $\alpha = 0.8$

for $1 \le i \le r$. In view of the NPCM algorithm, it follows that

$$\begin{aligned} u_{1,0}^{p}(t_{n+1}) &= \sum_{k_1=0}^{\lceil\alpha_1\rceil-1} \frac{u_{10}^{(k_1)} t^{k_1}}{k_1!} + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} \sum_{k=0}^{n} a_{1,k,n+1} f_1(t_k, \bar{u}(t_k)), \\ u_{2,0}^{p}(t_{n+1}) &= \sum_{k_2=0}^{\lceil\alpha_2\rceil-1} \frac{u_{20}^{(k_2)} t^{k_2}}{k_2!} + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} \sum_{k=0}^{n} a_{2,k,n+1} f_2(t_k, \bar{u}(t_k)), \\ &\vdots \\ u_{r,0}^{p}(t_{n+1}) &= \sum_{k_r=0}^{\lceil\alpha_r\rceil-1} \frac{u_{r0}^{(k_r)} t^{k_r}}{k_r!} + \frac{h^{\alpha_r}}{\Gamma(\alpha_r+2)} \sum_{k=0}^{n} a_{r,k,n+1} f_r(t_k, \bar{u}(t_k)), \end{aligned} \tag{24}$$

and

$$\begin{aligned} u_{1,1}^{p}(t_{n+1}) &= N_1(u_{1,0}^{p}, u_{2,0}^{p}, \ldots, u_{r,0}^{p}), \\ u_{2,1}^{p}(t_{n+1}) &= N_2(u_{1,0}^{p}, u_{2,0}^{p}, \ldots, u_{r,0}^{p}), \\ &\vdots \\ u_{r,1}^{p}(t_{n+1}) &= N_r(u_{1,0}^{p}, u_{2,0}^{p}, \ldots, u_{r,0}^{p}), \end{aligned} \tag{25}$$

where

$$N_i[\bar{u}(t_{n+1})] = \frac{h^{\alpha_i}}{\Gamma(\alpha_i+2)} f_i(t_{n+1}, \bar{u}(t_{n+1})), \;\; 1 \le i \le r.$$

Hence, the final NPCM algorithm reads as

$$
\begin{aligned}
u_1^c(t_{n+1}) &= u_{1,0}^p(t_{n+1}) + \frac{h^{\alpha_1}}{\Gamma(\alpha_1+2)} f_1(t_{n+1}, u_{1,0}^p(t_{n+1}) \\
&\quad + u_{1,1}^p(t_{n+1}), u_{2,0}^p(t_{n+1}) + u_{2,1}^p(t_{n+1}), \ldots, u_{r,0}^p(t_{n+1}) + u_{r,1}^p(t_{n+1})), \\
u_2^c(t_{n+1}) &= u_{2,0}^p(t_{n+1}) + \frac{h^{\alpha_2}}{\Gamma(\alpha_2+2)} f_2(t_{n+1}, u_{1,0}^p(t_{n+1}) \\
&\quad + u_{1,1}^p(t_{n+1}), u_{2,0}^p(t_{n+1}) + u_{2,1}^p(t_{n+1}), \ldots, u_{r,0}^p(t_{n+1}) + u_{r,1}^p(t_{n+1})), \\
&\quad \vdots \\
u_r^c(t_{n+1}) &= u_{r,0}^p(t_{n+1}) + \frac{h^{\alpha_r}}{\Gamma(\alpha_r+2)} f_r(t_{n+1}, u_{1,0}^p(t_{n+1}) \\
&\quad + u_{1,1}^p(t_{n+1}), u_{2,0}^p(t_{n+1}) + u_{2,1}^p(t_{n+1}), \ldots, u_{r,0}^p(t_{n+1}) + u_{r,1}^p(t_{n+1})),
\end{aligned}
$$

where $u_{i,0}^p(t_{n+1})$ and $u_{i,1}^p(t_{n+1})$, $1 \leq i \leq r$ are given in Eqs. (24) and (25).

5 L1-Predictor Corrector Method (L1-PCM)

L1 method [1, 14] is used for the numerical evaluation of the fractional derivatives of order α, $0 < \alpha < 1$. In this method, fractional derivative is numerically evaluated as follows.

$$
\begin{aligned}
\left[{}^cD_0^\alpha u(t)\right]_{t=t_n} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} (t_n - s)^{-\alpha} u'(s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} u'(s) ds \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \frac{u(t_{k+1}) - u(t_k)}{h} ds \\
&= \sum_{k=0}^{n-1} b_{n-k-1}(u(t_{k+1}) - u(t_k)),
\end{aligned} \tag{26}
$$

where

$$
b_k = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}[(k+1)^{1-\alpha} - k^{1-\alpha}].
$$

Recently, Jhinga and Daftardar-Gejji [10] have introduced yet another accurate and time-efficient predictor–corrector method by combining L1 and DGJ method [8] which is abbreviated as L1-PCM. This is given by the following formula:

$$
\begin{aligned}
u_n^p &= a_{n-1}u_0 + \sum_{k=1}^{n-1}(a_{n-1-k} - a_{n-k})u_k, \\
z_n^p &= N(u_n^p) = \Gamma(2-\alpha)h^\alpha f(t_n, u_n^p), \\
u_n^c &= u_n^p + \Gamma(2-\alpha)h^\alpha f(t_n, u_n^p + z_n^p),
\end{aligned} \tag{27}
$$

where

$$a_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \; k = 1, 2 \ldots, n-1.$$

Here u_n^p and z_n^p are predictors and u_n^c is the corrector.

Error Estimation in L1-PCM [10]

Let $u(t)$ be the exact solution of the IVP (14), $f(t, u)$ satisfy the Lipschitz condition with respect to the second argument u with a Lipschitz constant L, and $f(t, u(t)), u(t) \in C^1[0, T]$. Further u_k^c denotes the approximate solutions at $t = t_k$ obtained by the L1-PCM. Then we have for $0 < \alpha < 1$,

$$| u(t_k) - u_k^c | \leq CT^\alpha h^{2-\alpha}, \; k = 0, 1, \ldots, N, \tag{28}$$

where $C = d/(1-\alpha)$ and d is a constant.
Comment 1: L1-PCM is applicable only when $0 < \alpha < 1$.
Comment 2: For $0 < \alpha < 1$, the error estimate for L1-PCM is of the order $O(h^{2-\alpha})$, whereas the error estimate for FAM is of the order $O(h^{1+\alpha})$. Hence, L1-PCM gives more accuracy than FAM for $0 < \alpha < 0.5$.
Comment 3: Formulation of this method, for a system of FDEs is given in [10].

6 Illustrations

Example 1.

$${}^cD_0^\alpha y(x) + y^4(x) = \frac{\Gamma(2\alpha+1)x^\alpha}{\Gamma(\alpha+1)} - \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + (x^{2\alpha} - x^2)^4; \; y(0) = 0. \tag{29}$$

Exact solution of the IVP (29) is $x^{2\alpha} - x^2$. This example is solved by FAM, NPCM, and L1-PCM. Table 1 shows errors in each method and they are compared for different values of h and $\alpha = 0.6, 0.7, 0.8$. Table 2 presents CPU time required by FAM (T_1), NPCM (T_2) and L1-PCM (T_3). It may be noted that $T_3 < T_2 < T_1$. So the L1-PCM takes least time. Further we solve this equation for very small values of α, ($\alpha = 0.001$), for $x = 1$ and $x = 0.8$. We observe that both FAM and NPCM diverge, only L1-PCM gives the answer. (cf. Tables 3 and 4).

Table 1 Example 1: Error in FAM (E1), NPCM (E2), L1-PCM (E3), $x = 1$

Step length (h)		$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
0.1	E1	–0.00487443	–0.00396479	–0.00287403
	E2	–0.00487407	–0.0039647	–0.00287399
	E3	–0.01614330	–0.01621376	–0.01419765
0.01	E1	–0.00010385	–0.0000715422	–0.0000456169
	E2	–0.000103852	–0.0000715422	–0.0000456169
	E3	–0.00064236	–0.00078227	–0.00084073
0.001	E1	-2.40227×10^{-6}	-1.33008×10^{-6}	-7.04757×10^{-7}
	E2	-2.4023×10^{-6}	-1.33008×10^{-6}	-7.04757×10^{-7}
	E3	-2.47×10^{-5}	-3.798×10^{-6}	-5.185×10^{-6}

Table 2 Example 1: CPU time by FAM (T1), NPCM (T2), and L1-PCM (T3)

No. of iterations	T1 (s)	T2 (s)	T3 (s)
100	0.02256	0.012677	0.003541
500	0.523676	0.27807	0.077884
1000	2.078472	1.086413	0.289886
10000	206.413373	107.445599	22.684047

Table 3 Example 1 for $\alpha = 0.001$ and $x = 1$

Step length (h)	FAM	3-term NPCM	L1-PCM	Exact
h = 0.01	Diverges	Diverges	–0.004355	0
h = 0.001	Diverges	Diverges	–0.004205	0

Example 2. Consider the following fractional initial value problem:

$$^{c}D_0^{\alpha} y(x) = \frac{\Gamma(3)}{\Gamma(3-\alpha)} x^{2-\alpha} - \frac{2\Gamma(2)}{\Gamma(2-\alpha)} x^{1-\alpha} + (x^2 - 2x)^3 - y^3(x); \;\; y(0) = 0. \tag{30}$$

The exact solution of the IVP is $x^2 - 2x$. We apply FAM, NPCM, and L1-PCM to solve the IVP (30) and compare with exact solution. Errors in FAM, NPCM, and L1-PCM are tabulated in Table 5. CPU time required by FAM (T_1), NPCM (T_2) and L1-PCM (T_3) is given in Table 6. It may be noted that $T_3 < T_2 < T_1$. Further we solve this example numerically for small values of α such as $\alpha = 0.005$ and $\alpha = 0.001$ and observe that FAM and NPCM do not converge for these small values of α, whereas L1-PCM converges. These observations are presented in Tables 7 and 8.

Table 4 Example 1 for $\alpha = 0.001$ and $x = 0.8$

Step length (h)	FAM	3-term NPCM	L1-PCM	Exact
h = 0.01	Diverges	Diverges	0.351546	0.359553
h = 0.001	Diverges	Diverges	0.35171574	0.359553

Table 5 Example 2: error in FAM (E1), NPCM (E2), L1-PCM (E3), $x = 2$

Step length (h)		$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$
0.1	E1	–0.00576076	–0.00255451	–0.00011668
	E2	–0.00438643	0.00172467	0.00411813
	E3	0.05435552	0.05297444	0.06904374
0.01	E1	–0.00003499	0.00007466	0.00019599
	E2	–0.0000147	0.0001458	0.0002493
	E3	0.00193404	0.00172231	0.00289783
0.001	E1	7.4×10^{-7}	6.21×10^{-6}	1.478×10^{-5}
	E2	2.17×10^{-6}	7.66×10^{-6}	1.559×10^{-5}
	E3	9.641×10^{-5}	-7.757×10^{-5}	1.6428×10^{-4}

Table 6 Example 2: CPU time required by FAM (T1), NPCM (T2) and L1-PCM (T3)

No. of iterations	T1 (s)	T2 (s)	T3 (s)
100	0.017772	0.010457	0.00335
500	0.405403	0.217934	0.05734
1000	1.612247	0.861459	0.224005
10000	159.338239	84.556418	20.778786

Table 7 Example 2 for $\alpha = 0.001$ and $x = 2$

Steplength	FAM	3-term NPCM	L1-PCM	Exact
h = 0.01	Diverges	Diverges	0.004811	0
h = 0.001	Diverges	Diverges	0.004702	0

Table 8 Example 2 for $\alpha = 0.005$ and $x = 1.7$

Steplength	FAM	3-term NPCM	L1-PCM	Exact
h = 0.01	Diverges	Diverges	–0.388979	–0.51
h = 0.001	Diverges	Diverges	–0.391465	–0.51

7 Conclusion

Numerous problems in Physics, Chemistry, Biology, and Engineering are modeled mathematically by fractional differential equations. Hence, developing methods to solve FDEs is of paramount interest.

In pursuance to developing accurate and reliable numerical methods, Daftardar-Gejji and co-workers have developed new predictor–corrector method (NPCM) and L1-predictor corrector method (L1-PCM). NPCM and L1-PCM are more time efficient than fractional Adams method (FAM) as they deal with half the weights used in FAM. The time taken by FAM (T1), NPCM (T2), L1-PCM (T3) follows: $T1 > T2 > T3$, so L1-PCM is most time efficient. It should be noted that L1-PCM is applicable when $0 < \alpha < 1$, whereas FAM and NPCM work even for $\alpha > 1$. The order of accuracy in case of NPCM is $O(h^2)$, for L1-PCM $O(h^{2-\alpha})$, where as $O(h^{1+\alpha})$ in case of FAM. So NPCM is most accurate. For $0 < \alpha < 0.5$, L1-PCM gives more accurate results than FAM. Further it is noted that L1-PCM converges even for very small values of α, while FAM and NPCM diverge.

References

1. Oldham, K., Spanier, J.: The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order. Elsevier, Amsterdam (1974)
2. Daftardar-Gejji, V.: An introduction to fractional calculus. In: Daftardar-Gejji, V. (ed.) Fractional Calculus: Theory and Applications. Narosa, Delhi (2014)
3. Yang, Q., Chen, D., Zhao, T., Chen, Y.: Fractional calculus in image processing: a review. Fract. Calc. Appl. Anal. **19**(5), 1222–1249 (2016)
4. Samko, S.G., Kilbas, A.A., Mariuchev, O.I.: Fractional Integrals and Derivatives. Taylor and Francis, Milton Park (1993)
5. Daftardar-Gejji, V., Babakhani, A.: Analysis of a system of fractional differential equations. J. Math. Anal. Appl. **293**(2), 511–522 (2004)
6. Daftardar-Gejji, V., Jafari, H.: Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives. J. Math. Anal. Appl. **328**, 1026–1033 (2007)
7. Adomian, G.: Solving Frontier Problems in Physics: The Decomposition Method. Kluwer Academic, Boston (1994)
8. Daftardar-Gejji, V., Jafari, H.: An iterative method for solving nonlinear functional equations differential equations. J. Math. Anal. Appl. **316**(2), 321–354 (2006)
9. Daftardar-Gejji, V., Sukale, Y., Bhalekar, S.: A new predictor-corrector method for fractional differential equations. Appl. Math. Comput. **244**, 158–182 (2014)
10. Jhinga, A., Daftardar-Gejji, V.: A new finite difference predictor-corrector method for fractional differential equations. Appl. Math. Comput. **336**, 418–432 (2018)
11. Bhalekar, S., Daftardar-Gejji, V.: Convergence of the new iterative method. Int. J. Differ. Equ. (2011)
12. Diethelm, K., Ford, N., Freed, A.: Detailed error analysis for a fractional Adams method. Numer. Algorithms **36**(1), 31–52 (2004)
13. Sukale, Y.: PhD thesis, Savitribai Phule Pune University (2016)
14. Li, C., Zheng, F.: Numerical methods for Fractional Calculus. CRC Press, New York (2015)

Adomian Decomposition Method and Fractional Poisson Processes: A Survey

K. K. Kataria and P. Vellaisamy

Abstract This paper gives a survey of recent results related to the applications of the Adomian decomposition method (ADM) to certain fractional generalizations of the homogeneous Poisson process. First, we briefly discuss the ADM and its advantages over existing methods. As applications, this method is employed to obtain the state probabilities of the time fractional Poisson process (TFPP), space fractional Poisson process (SFPP) and Saigo space–time fractional Poisson process (SSTFPP). Usually, the Laplace transform technique is used to obtain the state probabilities of fractional processes. However, for certain state-dependent fractional Poisson processes, the Laplace transform method is difficult to apply, but the ADM method could be effectively used to obtain the state probabilities of such processes.

Keywords Adomian decomposition method · Fractional derivatives · Fractional point processes

Classifications Primary: 60G22 · Secondary: 60G55

1 Introduction

The Poisson process is a commonly used model for count data. Several characterizations of the homogeneous Poisson process $\{N(t, \lambda)\}_{t \geq 0}$, $\lambda > 0$, are available in the literature. It can be defined as a pure birth process with rate λ, a process with iid interarrival times distributed exponentially or a process with independent and stationary increments, where $N(t, \lambda)$ follows Poisson distribution with parameter λt. Here, $N(t, \lambda)$ denotes the number of events in $(0, t]$ with $N(0, \lambda) = 0$. A

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martingale characterization of the Poisson process is also available in the literature (see [27]), *viz.*, a point process is a homogeneous Poisson point process if and only if $\{N(t,\lambda)-\lambda t\}_{t\geq 0}$ is a martingale. All these characterizations are equivalent. The Poisson process can also be defined in terms of the Kolmogorov equations as follows:

A stochastic process $\{N(t,\lambda)\}_{t\geq 0}$ with independent and stationary increments is said to be a Poisson process with intensity parameter $\lambda > 0$ if its state probabilities $p(n,t) = \Pr\{N(t,\lambda)=n\}$ satisfy

$$\frac{\mathrm{d}}{\mathrm{dt}}p(n,t) = -\lambda(p(n,t)-p(n-1,t)) = -\lambda(1-B)p(n,t), \quad n \geq 0, \tag{1.1}$$

with initial condition $p(0,0)=1$. In the above Kolmogorov equations, B is the backward shift operator acting on the state space, i.e., $B(p(n,t)) = p(n-1,t)$. The conditions $p(0,0)=1$, $N(0,\lambda)=0$ *a.s.*, and $p(n,0)=0$ for all $n\geq 1$ are essentially equivalent.

In the usual definition of the Poisson process, the state probabilities are given by

$$p(n,t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,1,2,\ldots. \tag{1.2}$$

The Poisson process is a Lévy process used to model the counting phenomenon. However, it has certain limitations. The data traffic of bursty nature, especially on multiple time scales, cannot be modeled using the Poisson process. It is known that the waiting times of a Poisson process are independent and identically distributed (iid) exponential random variables. In certain empirical studies, it has been shown that the power law decay offers a better model than an exponential decay, for example, in the case of network connection sessions. In order to overcome such limitations, several authors have tried to improve the Poisson model. This leads to many generalizations of the Poisson process such as the non-homogeneous Poisson process, Cox point process, higher dimensional Poisson process, etc.

The fractional generalizations of the Poisson process known as the fractional Poisson processes (FPP) have drawn the interest of several researchers. The FPP gives some interesting connections between fractional calculus, stochastic subordination, and the renewal theory. Laskin [13] used time fractional Poisson process (TFPP) to define a new family of quantum coherent states. He also introduced the fractional generalizations of the Bell polynomials, Bell numbers, and Stirling's numbers of the second kind. Biard and Saussereau [4] have shown that the TFPP is a nonstationary process and it has the long-range dependence property.

2 A Brief Survey on FPP

In this section, we briefly discuss the three main approaches to the FPP.

2.1 FPP as a Stochastic Subordination

The time-changed stochastic processes have found applications in several areas of physical science such as telecommunications, turbulence, image processing, bioengineering, hydrology, and finance. The TFPP has the long-range dependence property (see [14]) and many authors time-changed the TFPP to define new processes. Meerschaert et al. [15] give a time-changed characterization of the TFPP $N^{\alpha}(t, \lambda)$, $0 < \alpha < 1$. They have shown that

$$N^{\alpha}(t, \lambda) \stackrel{d}{=} N(E_{\alpha}(t), \lambda), \tag{2.1}$$

where $\stackrel{d}{=}$ means equal in distribution and $\{E_{\alpha}(t)\}_{t\geq 0}$ is the inverse α-stable subordinator (see [17]) independent of the Poisson process $\{N(t, \lambda)\}_{t\geq 0}$. Similar characterization for the space fractional Poisson process (SFPP), $N_{\nu}(t, \lambda)$, $0 < \nu < 1$, is given by Orsingher and Polito [19], where the homogeneous Poisson process $\{N(t, \lambda)\}_{t\geq 0}$ is subordinated by an independent ν-stable subordinator $\{D_{\nu}(t)\}_{t\geq 0}$, i.e.,

$$N_{\nu}(t, \lambda) \stackrel{d}{=} N(D_{\nu}(t), \lambda),\ t \geq 0. \tag{2.2}$$

2.2 FPP as a Renewal Process

The Poisson process is a renewal process (see [16]) with iid waiting times W_i's such that $W_i \sim \mathrm{Exp}(\lambda),\ \lambda > 0,\ i \geq 1$. Let, $N(t, \lambda) := \max\{n \geq 0 : S_n \leq t\},\ t \geq 0$, where $S_n = W_1 + W_2 + \cdots + W_n$. Then, $N(t, \lambda)$ is a Poisson process with intensity parameter $\lambda > 0$. Suppose now the waiting times W_i^{β} follow iid Mittag-Leffler distribution, i.e., $\Pr\{W_i^{\beta} > t\} = E_{\beta}(-\lambda t^{\beta}),\ 0 < \beta < 1, i \geq 1$, where $E_{\beta}(x)$ is the Mittag-Leffler function defined by

$$E_{\beta}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\beta + 1)},\ \ \beta > 0,\ x \in \mathbb{R}. \tag{2.3}$$

Then, the Mittag-Leffler renewal process (see [15])

$$N_{\beta}(t, \lambda) := \max\{n \geq 0 : S_n^{\beta} \leq t\},\ t \geq 0,\ 0 < \beta < 1,$$

is the TFPP with fractional index β. Here, $S_n^{\beta} = W_1^{\beta} + W_2^{\beta} + \cdots + W_n^{\beta}$. The interarrival times of the FPP follow Mittag-Leffler distribution which exhibits power law asymptotic decay, which may find applications in network connection problems. However, we mention that the SFPP or its generalizations are not renewal processes.

2.3 FPP as a Solution of Fractional Difference-Differential Equations

There is an interesting connection between the fractional-order derivatives, diffusion process, and stochastic subordination. The state probabilities of the TFPP $\{N^{\alpha}(t,\lambda)\}_{t\geq 0}$, $0<\alpha\leq 1$, satisfy (see [3, 12])

$$\partial_t^{\alpha} p^{\alpha}(n,t) = -\lambda(1-B)p^{\alpha}(n,t), \quad n\geq 0,$$

with $p^{\alpha}(-1,t)=0$, $t\geq 0$ and the initial conditions $p^{\alpha}(0,0)=1$ and $p^{\alpha}(n,0)=0$, $n\geq 1$. Here, $p^{\alpha}(n,t)=\Pr\{N^{\alpha}(t,\lambda)=n\}$ denotes the probability mass function (pmf) and ∂_t^{α} denotes the Dzhrbashyan–Caputo fractional derivative defined as

$$\partial_t^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s, & 0<\alpha<1, \\ f'(t), & \alpha=1. \end{cases} \tag{2.4}$$

Also, it is known that the state probabilities $p_{\nu}(n,t)=\Pr\{N_{\nu}(t,\lambda)=n\}$ of the SFPP $\{N_{\nu}(t,\lambda)\}_{t\geq 0}$, $0<\nu\leq 1$, satisfy (see [19])

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\nu}(n,t) = -\lambda^{\nu}(1-B)^{\nu} p_{\nu}(n,t), \quad n\geq 0,$$

with $p_{\nu}(0,0)=1$ and $p_{\nu}(n,0)=0$, $n\geq 1$. Here, $(1-B)^{\nu}=\sum_{r=0}^{\infty}\frac{(\nu)_r}{r!}(-1)^r B^r$ is the fractional difference operator and $(\nu)_r$ denotes the falling factorials, i.e., $(\nu)_r=\nu(\nu-1)\ldots(\nu-r+1)$.

Further, the space–time fractional Poisson process (STFPP) $\{N_{\nu}^{\alpha}(t,\lambda)\}_{t\geq 0}$, $0<\alpha\leq 1$, $0<\nu\leq 1$, is a counting process whose pmf $p_{\nu}^{\alpha}(n,t)=\Pr\{N_{\nu}^{\alpha}(t,\lambda)=n\}$, satisfies

$$\partial_t^{\alpha} p_{\nu}^{\alpha}(n,t) = -\lambda^{\nu}(1-B)^{\nu} p_{\nu}^{\alpha}(n,t), \quad n\geq 0,$$

with initial conditions $p_{\nu}^{\alpha}(0,0)=1$ and $p_{\nu}^{\alpha}(n,0)=0$, $n\geq 1$. Also, $p_{\nu}^{\alpha}(-n,t)=0$, $t\geq 0$, $n\geq 1$.

A more generalized space fractional Poisson process (GSFPP) is recently introduced and studied by Polito and Scalas [22]. The GSFPP $\{N_{\nu,\eta}^{\delta}(t,\lambda)\}_{t\geq 0}$ is defined as the stochastic process whose pmf $p_{\nu,\eta}^{\delta}(n,t)=\Pr\{N_{\nu,\eta}^{\delta}(t,\lambda)=n\}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{\nu,\eta}^{\delta}(n,t) = -\left((\eta+\lambda^{\nu}(1-B)^{\nu})^{\delta}-\eta^{\delta}\right) p_{\nu,\eta}^{\delta}(n,t), \quad n\geq 0,$$

with initial conditions $p_{\nu,\eta}^{\delta}(0,0)=1$ and $p_{\nu,\eta}^{\delta}(n,0)=0$, $n\geq 1$. Also, $p_{\nu,\eta}^{\delta}(-n,t)=0$, $t\geq 0$, $n\geq 1$. Here, $m=\lceil\delta\rceil$, $\nu m\in(0,1)$ and $\delta,\eta,\lambda>0$.

These three approaches to the fractional generalization of the homogeneous Poisson process reveal the connections among the Mittag-Leffler function, fractional derivatives, and stochastic subordination. The fractional generalizations of the Poisson process are obtained by replacing the derivative involved in the difference-differential equations (1.1) by certain fractional derivatives. For instance, Laskin [12], Meerschaert et al. [15] used the Riemann–Liouville fractional derivative whereas Beghin and Orsingher [3] used the Caputo fractional derivative. The other generalized fractional derivatives such as Prabhakar derivative and Saigo fractional derivative are used by Polito and Scalas [22], Kataria and Vellaisamy [8]. The state probabilities of these generalized Poisson processes are generally obtained by evaluating the respective Laplace transforms, and then inverting them. However, in certain cases, the inversion may becomes too cumbersome and involved. Recently, Garra et al. [6] introduced and studied certain state-dependent versions of the TFPP. They obtained the Laplace transforms of those state-dependent models but did not derive the corresponding state probabilities, as the associated Laplace transforms are difficult to invert. The state probabilities of these processes are obtained by Kataria and Vellaisamy [10] by applying the Adomian decomposition method (ADM). These results are discussed and are stated without proofs in Sect. 6.

3 Adomian Decomposition Method

In this section, we briefly explain the ADM (for details we refer the reader to Adomian [1, 2]). In ADM, the solution $u(x,t)$ of the functional equation

$$L(u) + H(u) + f = u, \tag{3.1}$$

where L and H are linear and nonlinear operators and f is a known function, is expressed in the form of an infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{3.2}$$

Note that $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$, where α and β are scalars. The nonlinear term $H(u)$ is assumed to satisfy

$$H(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n). \tag{3.3}$$

Here, A_n denotes the nth Adomian polynomial in $u_0, u_1, \ldots, u_n$. Also, the series (3.2) and (3.3) are assumed to be absolutely convergent. So, (3.1) can be rewritten as

$$\sum_{n=0}^{\infty} L(u_n) + \sum_{n=0}^{\infty} A_n + f = \sum_{n=0}^{\infty} u_n. \tag{3.4}$$

Thus u_n's are obtained by the following recursive relation:

$$u_0 = f \quad \text{and} \quad u_n = L(u_{n-1}) + A_{n-1}, n \geq 1.$$

The important step involved in ADM is the computation of Adomian polynomials. A method for determining these polynomials was given by Adomian [1]. By parametrizing u as $u_\lambda = \sum_{n=0}^{\infty} u_n \lambda^n$ and assuming $H(u_\lambda)$ to be analytic in λ which decomposes as

$$H(u_\lambda) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n)\lambda^n,$$

the Adomian polynomials can be obtained as

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n H(u_\lambda)}{\partial \lambda^n}\bigg|_{\lambda=0}, \ \forall\, n \in \mathbb{N}_0, \tag{3.5}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

A slightly refined version of the above result was given by Zhu et al. [28] as

$$A_n(u_0, u_1, \ldots, u_n) = \frac{1}{n!} \frac{\partial^n H(\sum_{k=0}^{n} u_k \lambda^k)}{\partial \lambda^n}\bigg|_{\lambda=0}, \ \forall\, n \in \mathbb{N}_0. \tag{3.6}$$

The following formula for these polynomials was given by Rach [23]: $A_0(u_0) = H(u_0)$,

$$A_n(u_0, u_1, \ldots, u_n) = \sum_{k=1}^{n} C(k, n) H^{(k)}(u_0), \ \forall\, n \geq 1, \tag{3.7}$$

where

$$C(k, n) = \sum_{\Theta_n^k} \prod_{j=1}^{n} \frac{u_j^{k_j}}{k_j!}, \tag{3.8}$$

and the sum is taken over

$$\Theta_n^k = \left\{ (k_1, k_2, \ldots, k_n) : \sum_{j=1}^{n} k_j = k, \sum_{j=1}^{n} j k_j = n, k_j \in \mathbb{N}_0 \right\}.$$

Note that $H^{(k)}(.)$ denotes the kth derivative of $H(.)$. The equivalence between (3.5) and (3.7) can be established using Faà di Bruno's formula. Two parametrization methods for generating these Adomian polynomials were obtained by Kataria and Vellaisamy [7]. It is also established that the nth Adomian polynomial for any non-linear operator $H(.)$ can be expressed explicitly in terms of the partial exponential Bell polynomials; for details, we refer the reader to Kataria and Vellaisamy [9].

An important observation is that the functional equations corresponding to the difference-differential equations governing the state probabilities of various fractional Poisson processes do not involve any nonlinear term. Thus, the ADM can be used conveniently to obtain these state probabilities as the series solutions of the corresponding difference-differential equations.

Next, we demonstrate the use of ADM to various fractional Poisson processes.

4 Application of ADM to Fractional Poisson Processes

In this section, we discuss the applications of the ADM to two fractional versions of the Poisson process. We start with the time fractional version.

4.1 Time Fractional Poisson Process (TFPP)

The TFPP $\{N^{\alpha}(t,\lambda)\}_{t\geq 0}$, $0<\alpha\leq 1$, is defined as the stochastic process, whose probability mass function (pmf) $p^{\alpha}(n,t)=\Pr\{N^{\alpha}(t,\lambda)=n\}$ satisfies

$$\partial_t^{\alpha} p^{\alpha}(n,t) = -\lambda(1-B)p^{\alpha}(n,t), \quad n\geq 0, \tag{4.1}$$

with $p^{\alpha}(-1,t)=0$, $t\geq 0$ and the initial conditions $p^{\alpha}(0,0)=1$ and $p^{\alpha}(n,0)=0$, $n\geq 1$. Note that (4.1) is obtained by replacing $\dfrac{\mathrm{d}}{\mathrm{dt}}$ in (1.1) by the Caputo fractional derivative ∂_t^{α}, defined in (2.4). The functional equation corresponding to (4.1) is

$$p^{\alpha}(n,t) = p^{\alpha}(n,0) - \lambda I_t^{\alpha}(1-B)p^{\alpha}(n,t), \quad n\geq 0, \tag{4.2}$$

where I_t^{α} denotes the Riemann–Liouville (RL) fractional integral defined by

$$I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad \alpha>0. \tag{4.3}$$

We have $I_t^{\alpha}\partial_t^{\alpha} f(t) = f(t)-f(0)$, $0<\alpha\leq 1$ (see Eq. 2.4.44, [11]). The following result for power functions of the RL integral holds (see Eq. 2.1.16, [11]): For $\alpha,\rho>0$,

$$I_t^{\alpha} t^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+\alpha)} t^{\rho+\alpha-1}.$$

Laskin [12] obtained the pmf of TFPP by using the method of generating function and is given by

$$p^{\alpha}(n,t) = \frac{(\lambda t^{\alpha})^n}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\lambda t^{\alpha})^k}{\Gamma((k+n)\alpha+1)}, \quad n\geq 0. \tag{4.4}$$

Moreover, for $0 < \alpha < 1$, Meerschaert et al. [15] showed that

$$N^{\alpha}(t, \lambda) \stackrel{d}{=} N(E_{\alpha}(t), \lambda), \tag{4.5}$$

where $\{E_{\alpha}(t)\}_{t\geq 0}$ is the inverse α-stable subordinator independent of the Poisson process $\{N(t, \lambda)\}_{t\geq 0}$. We next show that how ADM could be employed to obtain the solutions $p^{\alpha}(n, t)$ of (4.2) for each $n \geq 0$.

Consider the following difference-differential equations governing the state probabilities of the TFPP:

$$\partial_t^{\alpha} p^{\alpha}(n, t) = -\lambda(p^{\alpha}(n, t) - p^{\alpha}(n-1, t)), \ \ 0 < \alpha \leq 1, \ n \geq 0, \tag{4.6}$$

with $p^{\alpha}(0, 0) = 1$ and $p^{\alpha}(n, 0) = 0$, $n \geq 1$.

Applying RL integral I_t^{α} on both sides of (4.6), we get

$$p^{\alpha}(n, t) = p^{\alpha}(n, 0) - \lambda I_t^{\alpha}(p^{\alpha}(n, t) - p^{\alpha}(n-1, t)), \ \ n \geq 0. \tag{4.7}$$

Note that $p^{\alpha}(-1, t) = 0$ for $t \geq 0$. Next we solve (4.7) for every $n \geq 0$. Consider first the case $n = 0$. Substitute $p^{\alpha}(0, t) = \sum_{k=0}^{\infty} p_k^{\alpha}(0, t)$ in (4.7) and apply ADM to get

$$\sum_{k=0}^{\infty} p_k^{\alpha}(0, t) = p^{\alpha}(0, 0) - \lambda \sum_{k=0}^{\infty} I_t^{\alpha} p_k^{\alpha}(0, t).$$

The above equation is of the form (3.4) with the known function $f = p^{\alpha}(0, 0)$, the linear term $L(p_k^{\alpha}(0, t)) = -\lambda I_t^{\alpha} p_k^{\alpha}(0, t)$ and no nonlinear term. Thus, $p_0^{\alpha}(0, t) = p^{\alpha}(0, 0) = 1$ and $p_k^{\alpha}(0, t) = -\lambda I_t^{\alpha} p_{k-1}^{\alpha}(0, t)$, $k \geq 1$. Hence,

$$p_1^{\alpha}(0, t) = -\lambda I_t^{\alpha} p_0^{\alpha}(0, t) = -\lambda I_t^{\alpha} t^0 = \frac{-\lambda t^{\alpha}}{\Gamma(\alpha + 1)},$$

and similarly

$$p_2^{\alpha}(0, t) = \frac{(-\lambda t^{\alpha})^2}{\Gamma(2\alpha + 1)}, \ \ p_3^{\alpha}(0, t) = \frac{(-\lambda t^{\alpha})^3}{\Gamma(3\alpha + 1)}.$$

Let us now assume the following suspected form for $p_{k-1}^{\alpha}(0, t)$:

$$p_{k-1}^{\alpha}(0, t) = \frac{(-\lambda t^{\alpha})^{k-1}}{\Gamma((k-1)\alpha + 1)}. \tag{4.8}$$

Now we use the method of induction to complete the argument for the case $n = 0$,

$$p_k^{\alpha}(0, t) = -\lambda I_t^{\alpha} p_{k-1}^{\alpha}(0, t) = \frac{(-\lambda)^k}{\Gamma((k-1)\alpha + 1)} I_t^{\alpha} t^{(k-1)\alpha} = \frac{(-\lambda t^{\alpha})^k}{\Gamma(k\alpha + 1)}, \ \ k \geq 0.$$

Therefore,

$$p^{\alpha}(0,t)=\sum_{k=0}^{\infty}\frac{(-\lambda t^{\alpha})^{k}}{\Gamma(k\alpha+1)}, \tag{4.9}$$

and thus the result holds for $n=0$.

For $n=1$, substituting $p^{\alpha}(1,t)=\sum_{k=0}^{\infty}p_k^{\alpha}(1,t)$ in (4.7) and applying ADM, we get

$$\sum_{k=0}^{\infty}p_k^{\alpha}(1,t)=p^{\alpha}(1,0)-\lambda\sum_{k=0}^{\infty}I_t^{\alpha}\left(p_k^{\alpha}(1,t)-p_k^{\alpha}(0,t)\right).$$

Thus, $p_0^{\alpha}(1,t)=p^{\alpha}(1,0)=0$ and $p_k^{\alpha}(1,t)=-\lambda I_t^{\alpha}\left(p_{k-1}^{\alpha}(1,t)-p_{k-1}^{\alpha}(0,t)\right)$, $k\geq 1$. Hence,

$$\begin{aligned}
p_1^{\alpha}(1,t)&=-\lambda I_t^{\alpha}\left(p_0^{\alpha}(1,t)-p_0^{\alpha}(0,t)\right)=\lambda I_t^{\alpha}t^{0}=\frac{-(-\lambda t^{\alpha})}{\Gamma(\alpha+1)},\\
p_2^{\alpha}(1,t)&=-\lambda I_t^{\alpha}\left(p_1^{\alpha}(1,t)-p_1^{\alpha}(0,t)\right)=\frac{-2\lambda^{2}}{\Gamma(\alpha+1)}I_t^{\alpha}t^{\alpha}=\frac{-2(-\lambda t^{\alpha})^{2}}{\Gamma(2\alpha+1)},\\
p_3^{\alpha}(1,t)&=-\lambda I_t^{\alpha}\left(p_2^{\alpha}(1,t)-p_2^{\alpha}(0,t)\right)=\frac{3\lambda^{3}}{\Gamma(2\alpha+1)}I_t^{\alpha}t^{2\alpha}=\frac{-3(-\lambda t^{\alpha})^{3}}{\Gamma(3\alpha+1)}.
\end{aligned}$$

Assume now

$$p_{k-1}^{\alpha}(1,t)=\frac{-(k-1)(-\lambda t^{\alpha})^{k-1}}{\Gamma((k-1)\alpha+1)},\quad k\geq 1. \tag{4.10}$$

Then

$$\begin{aligned}
p_k^{\alpha}(1,t)=-\lambda I_t^{\alpha}\left(p_{k-1}^{\alpha}(1,t)-p_{k-1}^{\alpha}(0,t)\right)&=\frac{(-1)^{k+1}k\lambda^{k}}{\Gamma((k-1)\alpha+1)}I_t^{\alpha}t^{(k-1)\alpha}\\
&=\frac{-k(-\lambda t^{\alpha})^{k}}{\Gamma(k\alpha+1)},\quad k\geq 1.
\end{aligned}$$

Therefore

$$p^{\alpha}(1,t)=-\sum_{k=1}^{\infty}\frac{k(-\lambda t^{\alpha})^{k}}{\Gamma(k\alpha+1)}=\lambda t^{\alpha}\sum_{k=0}^{\infty}\frac{(k+1)(-\lambda t^{\alpha})^{k}}{\Gamma((k+1)\alpha+1)}, \tag{4.11}$$

and thus the result holds for $n=1$.

Now, the method of induction can be successfully applied to obtain (see [8] for more details)

$$p^{\alpha}(n,t)=\frac{(\lambda t^{\alpha})^{n}}{n!}\sum_{k=0}^{\infty}\frac{(k+n)!}{k!}\frac{(-\lambda t^{\alpha})^{k}}{\Gamma\left((k+n)\alpha+1\right)},\quad n\geq 0. \tag{4.12}$$

We next discuss the application of the ADM to space fractional Poisson process.

4.2 Space Fractional Poisson Process (SFPP)

Orsingher and Polito [19] introduced a fractional difference operator in (1.1) to obtain a space fractional generalization. The SFPP $\{N_\nu(t,\lambda)\}_{t\geq 0}$, $0<\nu\leq 1$, is defined as the stochastic process whose pmf $p_\nu(n,t)=\Pr\{N_\nu(t,\lambda)=n\}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}p_\nu(n,t)=-\lambda^\nu(1-B)^\nu p_\nu(n,t),\ \ n\geq 0, \tag{4.13}$$

with initial conditions $p_\nu(0,0)=1$ and $p_\nu(n,0)=0$, $n\geq 1$. Also, $p_\nu(-n,t)=0$, $t\geq 0$, $n\geq 1$. Here, $(1-B)^\nu=\sum_{r=0}^{\infty}\frac{(\nu)_r}{r!}(-1)^r B^r$ is the fractional difference operator and hence (4.13) can be equivalently written as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_\nu(n,t)=-\lambda^\nu\sum_{r=0}^{n}\frac{(\nu)_r}{r!}(-1)^r p_\nu(n-r,t),\ \ n\geq 0, \tag{4.14}$$

where $(\nu)_r=\nu(\nu-1)\dots(\nu-r+1)$ denotes the falling factorial. They obtained the pmf of SFPP using the method of generating function as

$$p_\nu(n,t)=\frac{(-1)^n}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda^\nu t)^k}{k!}\frac{\Gamma(k\nu+1)}{\Gamma(k\nu+1-n)},\ \ n\geq 0. \tag{4.15}$$

They also showed that

$$N_\nu(t,\lambda)\stackrel{d}{=}N(D_\nu(t),\lambda),\ t\geq 0, \tag{4.16}$$

where $\{D_\nu(t)\}_{t\geq 0}$, $0<\nu<1$, is a ν-stable subordinator independent of the Poisson process.

Next, we give brief details of the proof of the above result using the ADM, given in Kataria and Vellaisamy [8].

The difference-differential equations (4.14) can be equivalently written as

$$p_\nu(n,t)=p_\nu(n,0)-\lambda^\nu\int_0^t\sum_{r=0}^{n}(-1)^r\frac{(\nu)_r}{r!}p_\nu(n-r,s)\,\mathrm{d}s,\ \ n\geq 0. \tag{4.17}$$

As done in the case of TFPP, one can show that the result (4.15) holds true for $n=0,1$. Now assume for $m>1$ the following:

$$p_{\nu,k}(m,t)=\frac{(-1)^m}{m!}\frac{(k\nu)_m(-\lambda^\nu t)^k}{k!},\ \ k\geq 0,$$

i.e., (4.15) holds for $n=m$, where $p_\nu(m,t)=\sum_{k=0}^{\infty}p_{\nu,k}(m,t)$.

For $n = m+1$, substituting $p_\nu(m+1,t) = \sum_{k=0}^{\infty} p_{\nu,k}(m+1,t)$ in (4.17) and applying ADM, we get

$$\begin{aligned}p_\nu(m+1,t) = \sum_{k=0}^{\infty} p_{\nu,k}(m+1,t) &= p_\nu(m+1,0)\\ &\quad - \lambda^\nu \sum_{k=0}^{\infty} \int_0^t \sum_{r=0}^{m+1} (-1)^r \frac{(\nu)_r}{r!} p_{\nu,k}(m+1-r,s)\,\mathrm{d}s.\end{aligned}$$

Thus, $p_{\nu,0}(m+1,t) = p_\nu(m+1,0) = 0$ and

$$p_{\nu,k}(m+1,t) = -\lambda^\nu \int_0^t \sum_{r=0}^{m+1} (-1)^r \frac{(\nu)_r}{r!} p_{\nu,k-1}(m+1-r,s)\,\mathrm{d}s, \quad k \geq 1.$$

Hence,

$$\begin{aligned}p_{\nu,1}(m+1,t) &= -\lambda^\nu \int_0^t \sum_{r=0}^{m+1} (-1)^r \frac{(\nu)_r}{r!} p_{\nu,0}(m+1-r,s)\,\mathrm{d}s\\ &= -\lambda^\nu \frac{(-1)^{m+1}}{(m+1)!}(\nu)_{m+1} \int_0^t \mathrm{d}s = \frac{(-1)^{m+1}}{(m+1)!}(\nu)_{m+1}(-\lambda^\nu t),\\ p_{\nu,2}(m+1,t) &= -\lambda^\nu \int_0^t \sum_{r=0}^{m+1} (-1)^r \frac{(\nu)_r}{r!} p_{\nu,1}(m+1-r,s)\,\mathrm{d}s\\ &= \frac{\lambda^{2\nu}(-1)^{m+1}}{(m+1)!} \int_0^t s\,\mathrm{d}s \sum_{r=0}^{m+1} \frac{(m+1)!}{r!(m+1-r)!}(\nu)_r(\nu)_{m+1-r}\\ &= \frac{(-1)^{m+1}}{(m+1)!} \frac{(2\nu)_{m+1}(-\lambda^\nu t)^2}{2!},\end{aligned}$$

where the last step follows from the binomial theorem for falling factorials. Now let

$$p_{\nu,k-1}(m+1,t) = \frac{(-1)^{m+1}}{(m+1)!} \frac{((k-1)\nu)_{m+1}(-\lambda^\nu t)^{k-1}}{(k-1)!}.$$

Then

$$\begin{aligned}p_{\nu,k}(m+1,t) &= -\lambda^\nu \int_0^t \sum_{r=0}^{m+1} (-1)^r \frac{(\nu)_r}{r!} p_{\nu,k-1}(m+1-r,s)\,\mathrm{d}s\\ &= \frac{(-\lambda^\nu)^k(-1)^{m+1}}{(m+1)!(k-1)!} \int_0^t s^{k-1}\,\mathrm{d}s \sum_{r=0}^{m+1} \frac{(m+1)!}{r!(m+1-r)!}(\nu)_r((k-1)\nu)_{m+1-r}\\ &= \frac{(-1)^{m+1}}{(m+1)!} \frac{(k\nu)_{m+1}(-\lambda^\nu t)^k}{k!}, \quad k \geq 0.\end{aligned}$$

Therefore

$$p^{\alpha}(m+1,t)=\frac{(-1)^{m+1}}{(m+1)!}\sum_{k=0}^{\infty}\frac{(-\lambda^{\nu}t)^{k}}{k!}\frac{\Gamma(k\nu+1)}{\Gamma(k\nu-m)},$$

and thus the result holds for $n=m+1$.

5 Application of ADM to Generalized Fractional Poisson Processes

We discuss here some recent results obtained by Kataria and Vellaisamy [8] for space–time fractional Poisson process and Saigo space–time fractional Poisson process, using the ADM.

5.1 Space–Time Fractional Poisson Process (STFPP)

A further generalization of the SFPP, namely, the STFPP (see [19]) $\{N_{\nu}^{\alpha}(t,\lambda)\}_{t\geq 0}$, $0<\alpha\leq 1$, $0<\nu\leq 1$, is defined as the stochastic process whose pmf $p_{\nu}^{\alpha}(n,t)=\Pr\{N_{\nu}^{\alpha}(t,\lambda)=n\}$ satisfies

$$\partial_t^{\alpha}p_{\nu}^{\alpha}(n,t)=-\lambda^{\nu}(1-B)^{\nu}p_{\nu}^{\alpha}(n,t),\quad n\geq 0, \tag{5.1}$$

with initial conditions $p_{\nu}^{\alpha}(0,0)=1$ and $p_{\nu}^{\alpha}(n,0)=0$, $n\geq 1$. Also, $p_{\nu}^{\alpha}(-n,t)=0$, $t\geq 0$, $n\geq 1$. Equivalently, (5.1) can be rewritten as

$$\partial_t^{\alpha}p_{\nu}^{\alpha}(n,t)=-\lambda^{\nu}\sum_{r=0}^{n}\frac{(\nu)_r}{r!}(-1)^r p_{\nu}^{\alpha}(n-r,t),\quad n\geq 0. \tag{5.2}$$

The state probabilities of TFPP (SFPP) can be obtained as special cases of the STFPP, *i*.e., by substituting $\nu=1$ ($\alpha=1$) in (5.1), respectively.

The functional equation obtained on applying RL integral I_t^{α} on both sides of (5.2) is

$$p_{\nu}^{\alpha}(n,t)=p_{\nu}^{\alpha}(n,0)-\lambda^{\nu}I_t^{\alpha}\sum_{r=0}^{n}(-1)^r\frac{(\nu)_r}{r!}p_{\nu}^{\alpha}(n-r,t),\quad n\geq 0.$$

Using ADM, the state probabilities for $n=0,1,2$ can be obtained as

$$p_{\nu}^{\alpha}(0,t)=\sum_{k=0}^{\infty}\frac{(-\lambda^{\nu}t^{\alpha})^{k}}{\Gamma(k\alpha+1)},$$

$$p_{\nu}^{\alpha}(1,t) = -\sum_{k=0}^{\infty}\frac{k\nu(-\lambda^{\nu}t^{\alpha})^{k}}{\Gamma(k\alpha+1)},$$
$$p_{\nu}^{\alpha}(2,t) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k\nu(k\nu-1)(-\lambda^{\nu}t^{\alpha})^{k}}{\Gamma(k\alpha+1)}.$$

Finally, the method of induction is used to obtain the solution of (5.2) as

$$p_{\nu}^{\alpha}(n,t) = \frac{(-1)^{n}}{n!}\sum_{k=0}^{\infty}\frac{(-\lambda^{\nu}t^{\alpha})^{k}}{\Gamma(k\alpha+1)}\frac{\Gamma(k\nu+1)}{\Gamma(k\nu+1-n)},\quad n\geq 0. \tag{5.3}$$

The above result is obtained by Orsingher and Polito [19] using the method of generating function.

Remark 5.1 Let the random variable X_{ν}^{α} be the waiting time of the first space–time fractional Poisson event. Then the following determine the distribution of X_{ν}^{α}:

$$\Pr\{X_{\nu}^{\alpha} > t\} = \Pr\{N_{\nu}^{\alpha}(t,\lambda) = 0\} = E_{\alpha}(-\lambda^{\nu}t^{\alpha}),\quad t\geq 0, \tag{5.4}$$

where $E_{\alpha}(.)$ is the Mittag-Leffler function defined by (2.3).

For $\alpha = 1$ and $\nu = 1$, we get the corresponding waiting times of SFPP and TFPP as

$$\Pr\{X_{\nu} > t\} = e^{-\lambda^{\nu}t},\quad t\geq 0,\quad \text{and}$$
$$\Pr\{X^{\alpha} > t\} = E_{\alpha}(-\lambda t^{\alpha}),\quad t\geq 0,$$

respectively.

Polito and Scalas [22] introduced and studied a further generalization of STFPP which involves the Prabhakar derivative. Kataria and Vellaisamy [8] introduced and studied a more generalized version of STFPP, namely, the Saigo space–time fractional Poisson process whose state probabilities are difficult to obtain using the existing methods of generating function and the Laplace transforms.

5.2 *Saigo Space–Time Fractional Poisson Process (SSTFPP)*

The SSTFPP is a stochastic process $\{N_{\nu}^{\alpha,\beta,\gamma}(t,\lambda)\}_{t\geq 0}$, $0<\alpha,\nu\leq 1$, $\beta<0$, $\gamma\in\mathbb{R}$, whose pmf $p_{\nu}^{\alpha,\beta,\gamma}(n,t) = \Pr\{N_{\nu}^{\alpha,\beta,\gamma}(t,\lambda) = n\}$, satisfies

$$\partial_{t}^{\alpha,\beta,\gamma}p_{\nu}^{\alpha,\beta,\gamma}(n,t) = -\lambda^{\nu}(1-B)^{\nu}p_{\nu}^{\alpha,\beta,\gamma}(n,t),\quad n\geq 0, \tag{5.5}$$

with $p_{\nu}^{\alpha,\beta,\gamma}(-1,t) = 0$ and subject to the initial conditions $p_{\nu}^{\alpha,\beta,\gamma}(0,0) = 1$ and $p_{\nu}^{\alpha,\beta,\gamma}(n,0) = 0, n\geq 1$. Here, $\partial_{t}^{\alpha,\beta,\gamma}$ denotes the regularized Saigo fractional derivative defined by Kataria and Vellaisamy [8] thereby improving a result of Rao et al. [24] as follows:

$$\partial_t^{\alpha,\beta,\gamma} f(t) = I_t^{1-\alpha,-1-\beta,\alpha+\gamma} f'(t). \tag{5.6}$$

Here, $I_t^{\alpha,\beta,\gamma}$ denotes the Saigo integral (see [26]) defined as

$$I_t^{\alpha,\beta,\gamma} f(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{s}{t}\right) f(s)\,\mathrm{d}s, \tag{5.7}$$

where $f(t)$ is a continuous real valued function on $(0,\infty)$ of order $O(t^{\epsilon})$, $\epsilon > \max\{0, \beta-\gamma\} - 1$. Let $\mathbb{Z}_0^-$ denotes the set of nonpositive integers. The Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, \ z \in \mathbb{C},$$

where $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Note that (5.5) can be rewritten as

$$\partial_t^{\alpha,\beta,\gamma} p_\nu^{\alpha,\beta,\gamma}(n,t) = -\lambda^\nu \sum_{r=0}^{n} (-1)^r \frac{(\nu)_r}{r!} p_\nu^{\alpha,\beta,\gamma}(n-r,t), \quad n \geq 0. \tag{5.8}$$

For $\beta = -\alpha$, the SSTFPP reduces to STFPP.

The Laplace transform of the regularized Saigo derivatives is not known and hence the method of generating function is not useful in obtaining the state probabilities of SSTFPP. Kataria and Vellaisamy [8] used the ADM to obtain the state probabilities of SSTFPP for each $n \geq 0$.

The functional equation obtained on applying Saigo integral $I_t^{\alpha,\beta,\gamma}$ on both sides of (5.8) is

$$p_\nu^{\alpha,\beta,\gamma}(n,t) = p_\nu^{\alpha,\beta,\gamma}(n,0) - \lambda^\nu I_t^{\alpha,\beta,\gamma} \sum_{r=0}^{n} (-1)^r \frac{(\nu)_r}{r!} p_\nu^{\alpha,\beta,\gamma}(n-r,t), \quad n \geq 0.$$

Using ADM, we get the following expressions of $p_\nu^{\alpha,\beta,\gamma}(n,t)$ (for $n = 0, 1, 2$):

$$p_\nu^{\alpha,\beta,\gamma}(0,t) = \sum_{k=0}^{\infty} \frac{C_k(-\lambda^\nu t^{-\beta})^k}{\Gamma(1-k\beta)},$$

$$p_\nu^{\alpha,\beta,\gamma}(1,t) = -\sum_{k=0}^{\infty} \frac{k\nu C_k(-\lambda^\nu t^{-\beta})^k}{\Gamma(1-k\beta)},$$

$$p_\nu^{\alpha,\beta,\gamma}(2,t) = \frac{1}{2!} \sum_{k=0}^{\infty} \frac{(k\nu)_2 C_k(-\lambda^\nu t^{-\beta})^k}{\Gamma(1-k\beta)}.$$

Using the method of induction, Kataria and Vellaisamy [8] obtained the pmf $p_\nu^{\alpha,\beta,\gamma}(n,t)$ of SSTFPP as

$$p_\nu^{\alpha,\beta,\gamma}(n,t) = \frac{(-1)^n}{n!}\sum_{k=0}^{\infty}\frac{C_k(-\lambda^\nu t^{-\beta})^k}{\Gamma(1-k\beta)}\frac{\Gamma(k\nu+1)}{\Gamma(k\nu+1-n)},\ n\geq 0, \tag{5.9}$$

where

$$C_k = \prod_{j=1}^{k}\frac{\Gamma(1+\gamma-j\beta)}{\Gamma(1+\gamma+\alpha-(j-1)\beta)}. \tag{5.10}$$

The waiting time $X_\nu^{\alpha,\beta,\gamma}$ of the first Saigo space–time fractional Poisson event satisfies

$$\Pr\{X_\nu^{\alpha,\beta,\gamma} > t\} = \Pr\{N_\nu^{\alpha,\beta,\gamma}(t,\lambda) = 0\} = \sum_{k=0}^{\infty}\frac{C_k(-\lambda^\nu t^{-\beta})^k}{\Gamma(1-k\beta)},\quad t\geq 0.$$

The special case $\beta = -\alpha$ corresponds to Mittag-Leffler distribution, i.e., the first waiting time of STFPP. The probability generating function (pgf) $G_\nu^{\alpha,\beta,\gamma}(u,t) = \mathbb{E}(u^{N_\nu^{\alpha,\beta,\gamma}(t,\lambda)})$ of SSTFPP is given by

$$G_\nu^{\alpha,\beta,\gamma}(u,t) = \sum_{k=0}^{\infty}\frac{C_k(-\lambda^\nu(1-u)^\nu t^{-\beta})^k}{\Gamma(1-k\beta)},\quad |u|<1. \tag{5.11}$$

It is shown that $G_\nu^{\alpha,\beta,\gamma}(u,t)$ satisfies the following Cauchy Problem:

$$\begin{aligned}\partial_t^{\alpha,\beta,\gamma}G_\nu^{\alpha,\beta,\gamma}(u,t) &= -\lambda^\nu G_\nu^{\alpha,\beta,\gamma}(u,t)(1-u)^\nu,\quad |u|<1,\\ G_\nu^{\alpha,\beta,\gamma}(u,0) &= 1.\end{aligned}$$

The pgf of TFPP, SFPP and STFPP can be obtained from (5.11) by substituting $\beta = -\alpha$, $\nu = 1$ and $\beta = -\alpha = -1$, and $\beta = -\alpha$, respectively.

6 Application of ADM to State-Dependent Fractional Poisson Processes

Observe that the TFPP is obtained as a solution of the following fractional difference-differential equation

$$\partial_t^\alpha p^\alpha(n,t) = -\lambda(1-B)p^\alpha(n,t),\quad n\geq 0.$$

Recently, Garra et al. [6] studied three state-dependent fractional point processes, where the orders of the fractional derivative involved in the difference-differential equations depend on the number of events that occur in $(0, t]$.

6.1 State-Dependent Time Fractional Poisson Process-I (SDTFPP-I)

The point process, namely, SDTFPP-I $\{N_1^d(t, \lambda)\}_{t\geq 0}$, $\lambda > 0$, is defined as the stochastic process whose pmf $p^{\alpha_n}(n, t) = \Pr\{N_1^d(t, \lambda) = n\}$ satisfies (see Eq. (1.1), [6])

$$\partial_t^{\alpha_n} p^{\alpha_n}(n, t) = -\lambda(p^{\alpha_n}(n, t) - p^{\alpha_{n-1}}(n-1, t)), \quad 0 < \alpha_n \leq 1, \ n \geq 0, \tag{6.1}$$

with $p^{\alpha_{-1}}(-1, t) = 0, t \geq 0$, and the initial conditions $p^{\alpha_n}(0, 0) = 1$ and $p^{\alpha_n}(n, 0) = 0$, $n \geq 1$. For each $n \geq 0$, $\partial_t^{\alpha_n}$ denotes the fractional derivative in Caputo sense which is defined in (2.4). The order of the Caputo derivative in difference-differential equations (6.1) depends on the number of events till time t. The Laplace transform of the state probabilities of SDTFPP-I is given by

$$\tilde{p}^{\alpha_n}(n, s) = \int_0^\infty p^{\alpha_n}(n, t)e^{-st}\,\mathrm{d}t = \frac{\lambda^n s^{\alpha_0 - 1}}{\prod_{k=0}^n (s^{\alpha_k} + \lambda)}, \quad s > 0. \tag{6.2}$$

It is difficult to invert (6.2) to obtain $p^{\alpha_n}(n, t)$. Only the explicit expressions for $n = 0$ and $n = 1$ are obtained by Garra et al. [6] using the method of Laplace inversion as

$$p^{\alpha_0}(0, t) = E_{\alpha_0}(-\lambda t^{\alpha_0}),$$
$$p^{\alpha_1}(1, t) = \sum_{k=0}^{\infty}(-1)^k \lambda^{k+1} \sum_{r=0}^{k} \binom{k}{r} t^{\alpha_0(k-r)+\alpha_1 r+\alpha_1} E^{k+1}_{\alpha_0+\alpha_1, \alpha_0(k-r)+\alpha_1 r+\alpha_1+1}(-\lambda^2 t^{\alpha_0+\alpha_1}),$$

where $E^{\gamma}_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined by

$$E^{\gamma}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k \Gamma(k+\gamma)}{k!\Gamma(\alpha k + \beta)\Gamma(\gamma)}, \quad \alpha, \beta, \gamma \in \mathbb{R}^+, \ x \in \mathbb{R}.$$

Note that $E^1_{\alpha,1}(x) = E_\alpha(x)$ is the Mittag-Leffler function given by (2.3).

The derivation of $p^{\alpha_1}(1, t)$ is tedious. The explicit expressions for $p^{\alpha_n}(n, t)$, for all $n \geq 0$, can be obtained as follows.

Note that the functional equation obtained on applying RL integral $I_t^{\alpha_n}$ on both sides of (6.1) is

$$p^{\alpha_n}(n, t) = p^{\alpha_n}(n, 0) - \lambda I_t^{\alpha_n}(p^{\alpha_n}(n, t) - p^{\alpha_{n-1}}(n-1, t)), \quad n \geq 0.$$

Using ADM, we get the following expressions of $p^{\alpha_n}(n,t)$ for the case $n = 0, 1, 2$

$$p^{\alpha_0}(0,t) = \sum_{k=0}^{\infty} \frac{(-\lambda t^{\alpha_0})^k}{\Gamma(k\alpha_0 + 1)},$$

$$p^{\alpha_1}(1,t) = -\sum_{k=1}^{\infty}(-\lambda)^k \sum_{\Theta_1^k} \frac{t^{k_0\alpha_0+k_1\alpha_1}}{\Gamma\left(k_0\alpha_0 + k_1\alpha_1 + 1\right)},$$

$$p^{\alpha_2}(2,t) = \sum_{k=2}^{\infty}(-\lambda)^k \sum_{\Theta_2^k} \frac{t^{k_0\alpha_0+k_1\alpha_1+k_2\alpha_2}}{\Gamma\left(k_0\alpha_0 + k_1\alpha_1 + k_2\alpha_2 + 1\right)}.$$

Applying the method of induction, the solution of (6.1) can be derived and is given (see [10] for more details) below.

Let $k_0 \in \mathbb{N}_0, k_j \in \mathbb{N}_0\backslash\{0\}, 1 \le j \le n$. The solution of the governing equations of SDTFPP-I, given in (6.1), with $p^{\alpha_0}(0,0) = 1$ and $p^{\alpha_n}(n,0) = 0, n \ge 1$, is given by

$$p^{\alpha_n}(n,t) = (-1)^n \sum_{k=n}^{\infty}(-\lambda)^k \sum_{\Theta_n^k} \frac{t^{\sum_{j=0}^n k_j\alpha_j}}{\Gamma\left(\sum_{j=0}^n k_j\alpha_j + 1\right)}, \tag{6.3}$$

where $\Theta_n^k = \{(k_0, k_1, \ldots, k_n) | \sum_{j=0}^n k_j = k\}$.

Let $X_0, X_1, \ldots, X_n$ be $n+1$ independent random variables such that X_0 follows exponential distribution with mean 1 and X_j, $1 \le j \le n$, follow the Mittag-Leffler distribution (see [21]) with $F_{X_j}(t) = 1 - E_{\alpha_j}(-t^{\alpha_j}), 0 < \alpha_j \le 1$, as the distribution function. Then, the density function of the convolution $S = X_0 + X_1 + \cdots + X_n$ follows from (6.3) as

$$f_S(t) = \sum_{k=n}^{\infty}(-1)^{n+k} \sum_{\Theta_n^k} \frac{t^{k_0+\sum_{j=1}^n k_j\alpha_j}}{\Gamma(1 + k_0 + \sum_{j=1}^n k_j\alpha_j)}, \quad t \ge 0, \tag{6.4}$$

where $\Theta_n^k = \{(k_0, k_1, \ldots, k_n) : \sum_{j=0}^n k_j = k, \; k_0 \in \mathbb{N}_0, \; k_j \in \mathbb{N}_0\backslash\{0\}, \; 1 \le j \le n\}$.

The distribution of $X_1^{(1)}$, the first waiting time of SDTFPP-I, is given by

$$\Pr\{X_1^{(1)} > t\} = \Pr\{N_1^d(t,\lambda) = 0\} = E_{\alpha_0}(-\lambda t^{\alpha_0}). \tag{6.5}$$

The state probabilities of TFPP can be obtained as a special case of SDTFPP-I, when $\alpha_n = \alpha$ for all $n \ge 0$, as

$$p^{\alpha}(n,t) = \frac{(\lambda t^{\alpha})^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\lambda t^{\alpha})^k}{\Gamma\left((k+n)\alpha + 1\right)}, \quad 0 < \alpha \le 1, \; \lambda > 0, \; n \ge 0. \tag{6.6}$$

6.2 State-Dependent Time Fractional Poisson Process-II (SDTFPP-II)

The point process SDTFPP-II $\{N_2^d(t,\lambda)\}_{t\geq 0}$, $\lambda > 0$, is defined as the stochastic process with independent but nonidentically distributed waiting times W_n (see Sect. 3, [6]) with $\Pr\{W_n > t\} = E_{\beta_n}(-\lambda t^{\beta_n})$, $0 < \beta_n \leq 1$, where $E_{\beta_n}(.)$ is the Mittag-Leffler function defined in (2.3).

Garra et al. [6] showed that the pmf $p^{\beta_n}(n,t) = \Pr\{N_2^d(t,\lambda) = n\}$ of SDTFPP-II satisfies

$$p^{\beta_n}(n,t) = p^{\beta_n}(n,0) - \lambda(I_t^{\beta_n} p^{\beta_n}(n,t) - I_t^{\beta_{n-1}} p^{\beta_{n-1}}(n-1,t)), \quad n \geq 0, \tag{6.7}$$

with $p^{\beta_{-1}}(-1,t) = 0, t \geq 0$, and the initial conditions $p^{\beta_n}(0,0) = 1$ and $p^{\beta_n}(n,0) = 0$, $n \geq 1$. Here, $I_t^{\beta_n}$ denotes the RL fractional integral of order β_n, $n \geq 0$, defined in (4.3).

The Laplace transform of the state probabilities of SDTFPP-II is given by

$$\tilde{p}^{\beta_n}(n,s) = \int_0^\infty p^{\beta_n}(n,t)e^{-st}\,\mathrm{d}t = \frac{\lambda^n s^{\beta_n - 1}}{\prod_{k=0}^n (s^{\beta_k} + \lambda)}, \quad s > 0. \tag{6.8}$$

It is difficult to invert (6.8) to obtain $p^{\beta_n}(n,t)$. However, on solving the functional equation (6.7) using ADM, we get the following expressions of $p^{\beta_n}(n,t)$ for the case $n = 0, 1, 2$

$$p_k^{\beta_0}(0,t) = \frac{(-\lambda t^{\beta_0})^k}{\Gamma(k\beta_0 + 1)},$$

$$p^{\beta_1}(1,t) = -\sum_{k=1}^\infty (-\lambda)^k \sum_{\Omega_1^k} \frac{t^{k_0\beta_0 + k_1\beta_1}}{\Gamma\,(k_0\beta_0 + k_1\beta_1 + 1)},$$

$$p^{\beta_2}(2,t) = \sum_{k=2}^\infty (-\lambda)^k \sum_{\Omega_2^k} \frac{t^{k_0\beta_0 + k_1\beta_1 + k_2\beta_2}}{\Gamma\,(k_0\beta_0 + k_1\beta_1 + k_2\beta_2 + 1)}.$$

The method of induction is used to obtain the following result (see [10]). Let $k_n \in \mathbb{N}_0,\ k_j \in \mathbb{N}_0\backslash\{0\},\ 1 \leq j \leq n-1$ and $0 < \beta_n \leq 1$. The solution of the governing equations of SDTFPP-II, given in (6.7), with $p^{\beta_0}(0,0) = 1$ and $p^{\beta_n}(n,0) = 0$, $n \geq 1$, is

$$p^{\beta_n}(n,t) = (-1)^n \sum_{k=n}^\infty (-\lambda)^k \sum_{\Omega_n^k} \frac{t^{\sum_{j=0}^n k_j\beta_j}}{\Gamma\left(\sum_{j=0}^n k_j\beta_j + 1\right)}, \; n \geq 0, \tag{6.9}$$

where $\Omega_n^k = \{(k_0, k_1, \ldots, k_n) |\ \sum_{j=0}^n k_j = k\}$.

Note that (6.4) also follows from (6.9). The distribution of $X_1^{(2)}$, the first waiting time of SDTFPP-II, is given by

$$\Pr\{X_1^{(2)} > t\} = \Pr\{N_2^d(t, \lambda) = 0\} = E_{\beta_0}(-\lambda t^{\beta_0}). \tag{6.10}$$

When $\alpha_n = \alpha$ and $\beta_n = \beta$ for all $n \geq 0$, the SDTFPP-I and SDTFPP-II reduce to TFPP studied by Beghin and Orsingher [3]. Further, the case $\alpha_n = 1$ and $\beta_n = 1$ for all $n \geq 0$, gives the classical homogeneous Poisson process.

The SDTFPP-I is related to SDTFPP-II by the following relationship (see Eq. (3.4), [6]):

$$p^{\alpha_n}(n, t) = \begin{cases} I_t^{\alpha_n - \alpha_0} \Pr\{N_2^d(t, \lambda) = n\}, & \alpha_n - \alpha_0 > 0, \\ \\ D_t^{\alpha_0 - \alpha_n} \Pr\{N_2^d(t, \lambda) = n\}, & \alpha_n - \alpha_0 < 0, \end{cases}$$

where D_t^{α} denotes the Riemann–Liouville (RL) fractional derivative which is defined by

$$D_t^{\alpha} f(t) := \frac{\mathrm{d}}{\mathrm{dt}} I_t^{1-\alpha} f(t), \quad 0 < \alpha < 1.$$

6.3 State-Dependent Fractional Pure Birth Process (SDFPBP)

A fractional version of the classical nonlinear birth process, namely, fractional pure birth process (FPBP), $\{\mathcal{N}_p(t, \lambda_n)\}_{t \geq 0}$, is introduced by Orsingher and Polito [18], whose state probabilities satisfy

$$\partial_t^{\nu} p^{\nu}(n, t) = -\lambda_n p^{\nu}(n, t) + \lambda_{n-1} p^{\nu}(n-1, t), \quad 0 < \nu \leq 1, \ n \geq 1, \tag{6.11}$$

where $p^{\nu}(n, t) = \Pr\{\mathcal{N}_p(t, \lambda_n) = n\}$ with $p^{\nu}(0, t) = 0, t \geq 0$, and the initial conditions $p^{\nu}(1, 0) = 1$ and $p^{\nu}(n, 0) = 0, n \geq 2$.

Garra et al. [6] studied a third fractional point process by introducing the state dependency in (6.11). The SDFPBP $\{\mathcal{N}_p^d(t, \lambda_n)\}_{t \geq 0}, \lambda_n > 0$, is defined as the stochastic process whose pmf $p^{\nu_n}(n, t) = \Pr\{\mathcal{N}_p^d(t, \lambda_n) = n\}$ satisfies (see Eq. (4.1), [6])

$$\partial_t^{\nu_n} p^{\nu_n}(n, t) = -\lambda_n p^{\nu_n}(n, t) + \lambda_{n-1} p^{\nu_{n-1}}(n-1, t), \quad 0 < \nu_n \leq 1, \ n \geq 1, \tag{6.12}$$

with $p^{\nu_0}(0, t) = 0, t \geq 0$, and the initial conditions $p^{\nu_1}(1, 0) = 1$ and $p^{\nu_n}(n, 0) = 0$, $n \geq 2$. Further, on substituting $\lambda_n = \lambda n$ for all $n \geq 1$ in (6.12) the SDFPBP reduces to a special process known as the state-dependent linear birth process (SDLBP) (see [6]).

The Laplace transform of the state probabilities of SDFPBP is given by

$$\tilde{p}^{\nu_n}(n, s) = \int_0^\infty p^{\nu_n}(n, t)e^{-st}\,\mathrm{d}t = \frac{s^{\nu_1-1}\prod_{k=1}^{n-1}\lambda_k}{\prod_{k=1}^{n}(s^{\nu_k}+\lambda_k)}, \quad s > 0. \tag{6.13}$$

Again, it is difficult to invert (6.13) to obtain $p^{\nu_n}(n, t)$, even for $n = 2$. Using ADM, the state probabilities of SDFPBP are obtained by Kataria and Vellaisamy [10]. Here, we give brief details.

Note that the functional equation obtained on applying RL integral $I_t^{\nu_n}$ on both sides of (6.12) is

$$p^{\nu_n}(n, t) = p^{\nu_n}(n, 0) + I_t^{\nu_n}(-\lambda_n p^{\nu_n}(n, t) + \lambda_{n-1}p^{\nu_{n-1}}(n-1, t)), \quad n \geq 1.$$

The following expressions of $p^{\nu_n}(n, t)$, for the case $n = 1, 2, 3$, is obtained using ADM:

$$p^{\nu_1}(1, t) = \sum_{k=0}^{\infty}\frac{(-\lambda_1 t^{\nu_1})^k}{\Gamma(k\nu_1+1)},$$

$$p^{\nu_2}(2, t) = -\frac{\lambda_1}{\lambda_2}\sum_{k=1}^{\infty}(-1)^k\sum_{\Lambda_2^k}\frac{\lambda_1^{k_1}\lambda_2^{k_2}t^{k_1\nu_1+k_2\nu_2}}{\Gamma(k_1\nu_1+k_2\nu_2+1)},$$

$$p^{\nu_3}(3, t) = \frac{\lambda_1}{\lambda_3}\sum_{k=2}^{\infty}(-1)^k\sum_{\Lambda_3^k}\frac{\lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3}t^{k_1\nu_1+k_2\nu_2+k_3\nu_3}}{\Gamma(k_1\nu_1+k_2\nu_1+k_3\nu_3+1)}.$$

Using the method of induction, the solution to (6.11) can be obtained and is stated next.

Let $k_1 \in \mathbb{N}_0,\ k_j \in \mathbb{N}_0\backslash\{0\},\ 2 \leq j \leq n$. The solution of the governing equation of the SDFPBP, given by (6.12), with $p^{\nu_1}(1, 0) = 1$ and $p^{\nu_n}(n, 0) = 0, n \geq 2$, is

$$p^{\nu_n}(n, t) = (-1)^{n-1}\frac{\lambda_1}{\lambda_n}\sum_{k=n-1}^{\infty}(-1)^k\sum_{\Lambda_n^k}\frac{\lambda_1^{k_1}\lambda_2^{k_2}\dots\lambda_n^{k_n}t^{\sum_{j=1}^{n}k_j\nu_j}}{\Gamma\left(\sum_{j=1}^{n}k_j\nu_j+1\right)}, \tag{6.14}$$

where $\Lambda_n^k = \{(k_1, k_2, \dots, k_n)|\sum_{j=1}^{n}k_j = k\}$.

Let $Y_1, Y_2, \dots, Y_n$ be n independent random variables such that Y_1 follows exponential distribution with mean λ_1 and Y_j, $2 \leq j \leq n$, follow the Mittag-Leffler distribution (see [5]) with $F_{Y_j}(t) = 1 - E_{\nu_j}(-\lambda_j t^{\nu_j})$, $0 < \nu_j \leq 1$, as the distribution function such that $\lambda_n = 1$. Then, the density function of the convolution $T = Y_1 + Y_2 + \cdots + Y_n$ follows from (6.14) as

$$f_T(t) = (-1)^{n-1}\lambda_1\sum_{k=n-1}^{\infty}(-1)^k\sum_{\Lambda_n^k}\frac{t^{k_1+\sum_{j=2}^{n}k_j\nu_j}\prod_{j=1}^{n-1}\lambda_j^{k_j}}{\Gamma(1+k_1+\sum_{j=2}^{n}k_j\nu_j)}, \quad t \geq 0,$$

where $\Lambda_n^k = \{(k_1, k_2, \dots, k_n) : \sum_{j=1}^{n} k_j = k,\ k_1 \in \mathbb{N}_0,\ k_j \in \mathbb{N}_0\backslash\{0\},\ 2 \le j \le n\}$.

Let $\mathcal{X}_p$ denote the time of second event for SDFPBP. Then the distribution of $\mathcal{X}_p$ is given by

$$\Pr\{\mathcal{X}_p > t\} = \Pr\{\mathcal{N}_p^d(t, \lambda) = 1\} = E_{\nu_1}(-\lambda_1 t^{\nu_1}), \tag{6.15}$$

which has the Laplace transform $\tilde{p}^{\nu_1}(1, s) = s^{\nu_1 - 1}/(s^{\nu_1} + \lambda_1)$.

The state probabilities of SDLBP can be obtained by putting $\lambda_n = n\lambda$, $n \ge 1$, in (6.14) as

$$p_l^{\nu_n}(n, t) = \frac{(-1)^{n-1}}{n} \sum_{k=n-1}^{\infty} (-\lambda)^k \sum_{\Lambda_n^k} \frac{1^{k_1} 2^{k_2} \dots n^{k_n} t^{k_1\nu_1 + k_2\nu_2 + \dots + k_n\nu_n}}{\Gamma\left(k_1\nu_1 + k_2\nu_2 + \dots + k_n\nu_n + 1\right)}, \quad n \ge 1.$$

Also, the state probabilities of FPBP (see [18]) can be obtained by substituting $\nu_n = \nu$ for all $n \ge 1$ in (6.14). The pmf of FPBP is given by

$$p^{\nu}(n, t) = (-1)^{n-1} \frac{\lambda_1}{\lambda_n} \sum_{k=n-1}^{\infty} \frac{(-t)^{k\nu}}{\Gamma\left(k\nu + 1\right)} \sum_{\Lambda_n^k} \lambda_1^{k_1} \lambda_2^{k_2} \dots \lambda_n^{k_n}, \quad n \ge 1.$$

Some semi-Markov processes and their connection with state-dependent models are recently studied by Orsingher et al. [20], Ricciuti and Toaldo [25]. They established the semi-Markovian nature of such processes.

7 Conclusions

We have briefly described the ADM and its applications to various fractional versions of the classical homogeneous Poisson process. The state probabilities of the time fractional Poisson process, space fractional Poisson process and the Saigo space–time fractional Poisson process are derived using ADM. The Laplace transform technique is usually applied to evaluate these state probabilities. For certain fractional generalizations of the Poisson process, the Laplace transforms of the state probabilities can be obtained, but their inversion is too difficult. This is especially the case when we deal with state-dependent fractional Poisson processes. But the ADM could still be used to obtain the state probabilities of such processes. This fact is explained for some recently introduced state-dependent processes by Garra et al. [6]. We do hope that this paper would motivate other researchers to explore the utility of the ADM to other generalizations of the fractional Poisson process.

References

1. Adomian, G.: Nonlinear Stochastic Operator Equations. Academic Press, Orlando (1986)
2. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic, Dordrecht (1994)
3. Beghin, L., Orsingher, E.: Fractional Poisson processes and related planar random motions. Electron. J. Probab. **14**, 1790–1826 (2009)
4. Biard, R., Saussereau, B.: Fractional Poisson process: long-range dependence and applications in ruin theory. J. Appl. Probab. **51**, 727–740 (2014)
5. Cahoy, D.O., Uchaikin, V.V., Woyczynski, W.A.: Parameter estimation for fractional Poisson processes. J. Stat. Plann. Inference **140**(11), 3106–3120 (2010)
6. Garra, R., Orsingher, E., Polito, F.: State-dependent fractional point processes. J. Appl. Probab. **52**, 18–36 (2015)
7. Kataria, K.K., Vellaisamy, P.: Simple parametrization methods for generating Adomian polynomials. Appl. Anal. Discret. Math. **10**, 168–185 (2016)
8. Kataria, K.K., Vellaisamy, P.: Saigo space-time fractional Poisson process via Adomian decomposition method. Stat. Probab. Lett. **129**, 69–80 (2017)
9. Kataria, K.K., Vellaisamy, P.: Correlation between Adomian and partial exponential Bell polynomials. C. R. Math. Acad. Sci. Paris **355**(9), 929–936 (2017)
10. Kataria, K.K., Vellaisamy, P.: On distributions of certain state dependent fractional point processes. J. Theor. Probab. (2019). https://doi.org/10.1007/s10959-018-0835-z
11. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theroy and Applications fo Fractional Differential Equations. Elsevier, Amsterdam (2006)
12. Laskin, N.: Fractional Poisson Process. Commun. Nonlinear Sci. Numer. Simul. **8**, 201–213 (2003)
13. Laskin, N.: Some applications of the fractional Poisson probability distribution. J. Math. Phys. **50**(11), 113513 (2009)
14. Maheshwari, A., Vellaisamy, P.: On the long-range dependence of fractional Poisson and negative binomial processes. J. Appl. Probab. **53**(4), 989–1000 (2016)
15. Meerschaert, M.M., Nane, E., Vellaisamy, P.: The fractional Poisson process and the inverse stable subordinator. Electron. J. Probab. **16**, 1600–1620 (2011)
16. Meerschaert, M.M., Scheffler, H.-P.: Limit theorems for continuous-time random walks with infinite mean waiting times. J. Appl. Probab. **41**, 623–638 (2004)
17. Meerschaert, M.M., Straka, P.: Inverse stable subordinators. Math. Model. Nat. Phenom. **8**, 1–16 (2013)
18. Orsingher, E., Polito, F.: Fractional pure birth processes. Bernoulli **16**, 858–881 (2010)
19. Orsingher, E., Polito, F.: The space-fractional Poisson process. Stat. Probab. Lett. **82**, 852–858 (2012)
20. Orsingher, E., Ricciuti, C., Toaldo, B.: On semi-Markov processes and their Kolmogorov's integro-differential equations. J. Funct. Anal. **275**(4), 830–868 (2018)
21. Pillai, R.N.: On Mittag-Leffler functions and related distributions. Ann. Inst. Stat. Math. **42**(1), 157–161 (1990)
22. Polito, F., Scalas, E.: A generalization of the space-fractional Poisson process and its connection to some Lévy processes. Electron. Commun. Probab. **21**, 1–14 (2016)
23. Rach, R.: A convenient computational form for the Adomian polynomials. J. Math. Anal. Appl. **102**, 415–419 (1984)
24. Rao, A., Garg, M., Kalla, S.L.: Caputo-type fractional derivative of a hypergeometric integral operator. Kuwait J. Sci. Eng. **37**, 15–29 (2010)
25. Ricciuti, C., Toaldo, B.: Semi-Markov models and motion in heterogeneous media. J. Stat. Phys. **169**, 340–361 (2017)
26. Saigo, M.: A remark on integral operators involving the Gauss hypergeometric functions. Math. Rep. Kyushu Univ. **11**, 135–143 (1978)

27. Watanabe, S.: On discontinuous additive functionals and Lévy measures of a Markov process. Japan. J. Math. **34**, 53–70 (1964)
28. Zhu, Y., Chang, Q., Wu, S.: A new algorithm for calculating Adomian polynomials. Appl. Math. Comput. **169**, 402–416 (2005)

On Mittag-Leffler Kernel-Dependent Fractional Operators with Variable Order

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Abstract In this work, integration by parts formulas for variable-order fractional operators with Mittag-Leffler kernels are presented and applied to study constrained fractional variational principles involving variable-order Caputo-type Atangana–Baleanu's derivatives, where the variable-order fractional Euler–Lagrange equations are investigated. A general formulation of fractional Optimal Control Problems (FOCPs) and a solution scheme for such class of systems are proposed. The performance index of a FOCP is taken into consideration as function of state as well as control variables.

1 Introduction

Fractional calculus represents a generalization of the classical differentiation and integration of nonnegative integer order to arbitrary order. This type of calculus has recently gained its importance and popularity because of the significant and interesting results which were obtained when fractional operators were utilized to model real-world problems in diversity of fields, e.g., physics, engineering, biology, etc. [1, 6, 12–14, 16, 23–31, 35, 37–39, 42, 43, 46, 47].

The Lagrangian and Hamiltonian mechanics are an alternate of the standard Newtonian mechanics. They are important because they can be used for the sake of solving

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any problem in the traditional mechanics. It is worth mentioning that in the Newtonian mechanics, the conception of force is needed. While, in the Lagrangian and Hamiltonian systems, the terms of energy are required.

Riewe [44, 45] was the first to consider the Lagrangian and Hamiltonian for a given dissipative system in the frame of fractional operators. Agarwal and Baleanu made significant contributions to the concept of fractional variational principles in [2–5, 19, 21].

In [2, 4, 20, 21, 40, 41], variety of optimization problems embodying constant order fractional control problems were considered. In [2, 4], the author proposed a general formulation for fractional optimal control problems and presented a solution scheme for such problems. This formulation was based on variation principles and Lagrange multipliers technique. In [40, 41], the authors extended the classical control theory to diffusion equations involving fractional operators in a bounded domain with and without a state constraints. These works were advanced in to a larger family of fractional optimal control systems containing constant orders in [11, 17]. Other contributions to this field were discussed in [18, 32, 33] and the references therein. Nevertheless, to the last extent of our knowledge, the area of calculus of variations and optimal control of fractional differential equations with variable order has been paid less attention than the case where fractional derivatives with constant orders. (see [15, 22, 36, 37]). This work is an attempt to fill this gap.

Motivated by what was mentioned above, we discuss the Lagrangian and Hamiltonian formalism for constrained systems in the presence of fractional variable order in this work.

Recently, in order to overcome the disadvantage of the existence of the singular kernels involved in the traditional fractional operators, Atangana and Baleanu [8] proposed a derivative with fractional order. The kernel involved in this derivative is nonlocal and nonsingular. Many researches have considered several applications on this fractional derivative (see e.g [7, 9, 48] and the references therein).

In this study, we use the aforementioned fractional derivative with variable order and propose to generalize the concept of equivalent Lagrangian for the fractional case. For a certain class of classical Lagrangian, there are several techniques to find the Euler–Lagrange equations and the associated Hamiltonians. However, the fractional dynamics depending on the fractional derivatives are used to construct the Lagrangian to begin with. Therefore, the existence of several options can be utilized to deal with a certain physical system. From this point of you, applications of the fractional derivative proposed by Atangana and Baleanu to the fractional dynamics may adduce new advantages in studying the constrained systems primarily because of the fact that there exist left and right derivatives of this kind. Addition, the fractional derivative of a function is given by a definite integral and it depends on the values of the function over the entire interval. Therefore, the fractional derivatives proposed are suitable to model systems with long-range interactions in space and/or time (memory) and process with many scales of space and/or time involved.

This work is organized as follows: In Sect. 2, we go over some concepts and definitions, and then we present the integration by parts formula in the framework of variable-order Atangana–Baleanu fractional time derivative. Section 3 includes a tabloid review of the fractional Lagrangian and Hamiltonian approaches in the frame of the proposed variable-order fractional derivatives and some detailed examples. In Sect. 4, we discuss constrained systems in the frame of the proposed derivative and investigate some example in details. In Sect. 5, the Fractional Optimal Control Problem (FOCP) is presented. Section 6 is dedicated to our conclusions.

2 Preliminaries

In this section, we present some definitions and notions related to Atangana–Baleanu fractional derivatives.

Let $L^2(\Omega)$ be the usual Hilbert space fitted to the scalar product $(.,.)$ and let $H^m(\Omega)$, $H_0^m(\Omega)$ denote the usual Sobolev spaces.

First, lets recall the Mittag-Leffler function $E_{\alpha(x),\beta}(u)$ for variable $\alpha(x) \in (0, 1)$ that is used in a great scale in this work and given below

$$E_{\alpha(x),\beta}(u) = \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(k\alpha(x) + \beta)},\ 0 < \alpha(x) < 1, 0 < \beta < 1,$$

$$E_{\alpha(x),1}(u) = E_{\alpha(t)}(u),\ u \in \mathbb{C},$$

where $\Gamma(.)$ denotes the Gamma function defined as

$$\Gamma(\alpha(x)) = \int_0^{\infty} s^{\alpha(x)-1} e^{-s} ds,\ \Re(\alpha(x) > 0.$$

It can be easily noticed that the exponential function is a particular case of the Mittag-Leffler function. In fact,

$$E_{1,1}(u) = e^u,\ E_{2,1}(u) = \cosh\sqrt{u},\ E_{1,2}(u) = \frac{e^u - 1}{u},\ \ E_{2,2}(u) = \frac{\sinh\sqrt{u}}{\sqrt{u}}.$$

A more generalized form of the Mittag-Leffler function is the Mittag-Leffler function with three parameters defined as

$$E^{\lambda}_{\alpha,\beta}(u) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(k\alpha + \beta)} \frac{u^k}{k!},\ \ u, \beta, \lambda \in \mathbb{C},\ \Re(\alpha) > 0,$$

where, $(\lambda)_k$ denotes the familiar Pochhammer symbol.

First of all, we present a new approach in defining variable-order Riemann–Liouville fractional integrals different from those in [10].

Definition 1 Let $\phi(x)$ be an integrable defined on an interval $[a, b]$ and a let $\alpha(x)$ be function such that $0 < \alpha(x) \leq 1$. We define the left Riemann–Liouville fractional integral of order $\alpha(x]$ as

$$ {}_aI^{\alpha(.)}\phi(x) = \frac{1}{\Gamma(\alpha(x))}\int_a^x (x-s)^{\alpha(x)-1}\phi(s)ds \tag{1}$$

and

$$ {}_a\mathscr{I}^{\alpha(.)}\phi(x) = \int_a^x (x-s)^{\alpha(s)-1}\phi(s)\frac{1}{\Gamma(\alpha(s))}ds. \tag{2}$$

In the right case, we have

$$ I_b^{\alpha(.)}\phi(x) = \frac{1}{\Gamma(\alpha(x))}\int_x^b (s-x)^{\alpha(x)-1}\phi(s)ds \tag{3}$$

and

$$ \mathscr{I}_b^{\alpha(.)}\phi(x) = \int_t^b (s-x)^{\alpha(s)-1}\phi(s)\frac{1}{\Gamma(\alpha(s))}ds. \tag{4}$$

To define fractional integral type operators with variable order, we follow [34].

Definition 2

$$ \mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi(x) = \frac{B(\alpha(x))}{1-\alpha(x)}\int_a^x E_{\alpha(x)}\left[\frac{-\alpha(x)}{1-\alpha(x)}(x-s)^{\alpha(x)}\right]\varphi(s)ds, \quad x > a. \tag{5}$$

Similarly, we define the right generalized fractional integral as

$$ \mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi(x) = \frac{B(\alpha(x))}{1-\alpha(x)}\int_t^b E_{\alpha(x)}\left[\frac{-\alpha(x)}{1-\alpha(x)}(s-x)^{\alpha(x)}\right]\varphi(s)ds, \quad x < b. \tag{6}$$

We define the following operators as well:

$$ \mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi(x) = \int_a^x \frac{B(\alpha(s))}{1-\alpha(s)}E_{\alpha(s)}\left[\frac{-\alpha(s)}{1-\alpha(s)}(x-s)^{\alpha(s)}\right]\varphi(s)ds, \quad x > a \tag{7}$$

and

$$ \mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi(x) = \int_x^b \frac{B(\alpha(s))}{1-\alpha(s)}E_{\alpha(s)}\left[\frac{-\alpha(s)}{1-\alpha(s)}(s-x)^{\alpha(s)}\right]\varphi(s)ds, \quad x < b. \tag{8}$$

Now, we present the definitions of the fractional integrals and derivatives of variable order in the point of view of Atangana–Baleanu [7].

Definition 3 For a given function $u \in H^1(a,b), b > a, \ \alpha(x) \in (0,1)$, the Atangana–Baleanu fractional integrals (AB integral) of variable order $\alpha(x)$ of a given function $u \in H^1(a,b), b > x > a$ (where A denotes Atangana, B denotes Baleanu) with base point a is defined at a point $x \in (a,b)$ by

$$\begin{aligned} {}^{AB}_{\ a}I^{\alpha(x)}_{x}u(x) &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(t) + \tfrac{\alpha(x)}{B(\alpha(x))\Gamma(\alpha(x))}\int_a^x u(s)(x-s)^{\alpha(x)-1}ds, \quad \text{(left ABI)} \\ &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \tfrac{\alpha(x)}{B(\alpha(x))}\, {}_aI^{\alpha(x)}_x u(x) \end{aligned} \tag{9}$$

and

$$\begin{aligned} {}^{AB}_{\ x}I^{\alpha(x)}_{b}u(x) &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \tfrac{\alpha(x)}{B(\alpha(x))\Gamma(\alpha(x))}\int_x^b u(s)(s-x)^{\alpha(x)-1}ds, \quad \text{(right ABI)} \\ &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \tfrac{\alpha(x)}{B(\alpha(x))}\, {}_xI^{\alpha(x)}_b u(x). \end{aligned} \tag{10}$$

$$\begin{aligned} {}^{AB}_{\ a}\mathscr{I}^{\alpha(x)}_{x}u(x) &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \int_a^x \tfrac{\alpha(s)}{B(\alpha(s))\Gamma(\alpha(s))}u(s)(x-s)^{\alpha(s)-1}ds, \quad \text{(left AB}\mathscr{I}) \\ &= \tfrac{1-\alpha(x)}{B(\alpha(t))}u(x) + {}_a\mathscr{I}^{\alpha(x)}_x \left[\tfrac{\alpha(x)}{B(\alpha(x))}u(x)\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} {}^{AB}_{\ x}\mathscr{I}^{\alpha(x)}_{b}u(x) &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \int_x^b \tfrac{\alpha(s)}{B(\alpha(s))\Gamma(\alpha(s))}u(s)(s-x)^{\alpha(s)-1}ds, \quad \text{(right AB}\mathscr{I}) \\ &= \tfrac{1-\alpha(x)}{B(\alpha(x))}u(x) + \mathscr{I}^{\alpha(x)}_b \left[\tfrac{\alpha(x)}{B(\alpha(x))}u(x)\right]. \end{aligned} \tag{12}$$

Once one takes $\alpha(x) = 0$ in (9), (10) we recover the initial function and when $\alpha(x) = 1$ is considered in (9), (10) we recover the ordinary integral.

The Atangana–Baleanu fractional derivatives in the Riemann–Liouville sense (ABR derivative) of variable order $\alpha(x)$ for a given function $\varphi(x) \in H^1(a,b), b > x > a$ (where R denotes Riemann) with base point a is defined at a point $x \in (a,b)$ by

$${}^{ABR}_{\ \ a}D^{\alpha(x)}_{x}\varphi(x) = \frac{d}{dx}\,\mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi(x) \quad \text{(left ABRD)} \tag{13}$$

$${}^{ABR}_{\ \ x}D^{\alpha(t)}_{b}\varphi(x) = -\frac{d}{dt}\,\mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi(x) \quad \text{(right ABRD)}, \tag{14}$$

$${}^{ABR}_{\ \ a}\mathscr{D}^{\alpha(t)}_{x}\varphi(x) = \frac{d}{dx}\,\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi(x) \quad \text{(left ABR}\mathscr{D}) \tag{15}$$

$${}^{ABR}_{\ \ x}\mathscr{D}^{\alpha(x)}_{b}\varphi(t) = -\frac{d}{dx}\,\mathscr{E}_{\alpha(t),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi(x) \quad \text{(right ABR}\mathscr{D}) \tag{16}$$

and the Caputo Atangana–Baleanu fractional derivatives (ABC derivative) of variable order $\alpha(x)$ for a given function $\varphi(x) \in H^1(a,b)$, $b > x > a$ (where C denotes Caputo) with base point a is defined at a point $x \in (a,b)$ by

$$ {}^{ABC}_{a}D^{\alpha(x)}_{x}\varphi(x) = \mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi'(x) \text{ (left ABRD)} \tag{17} $$

$$ {}^{ABC}_{x}D^{\alpha(x)}_{b}\varphi(t) = -\mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi'(x) \text{ (right ABRD),} \tag{18} $$

$$ {}^{ABC}_{a}\mathscr{D}^{\alpha(t)}_{x}\varphi(x) = \mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}\varphi'(x) \text{ (left ABR}\mathscr{D}\text{)} \tag{19} $$

$$ {}^{ABC}_{x}\mathscr{D}^{\alpha(x)}_{b}\varphi(x) = -\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}\varphi'(x) \text{ (right ABR}\mathscr{D}\text{)} \tag{20} $$

Remark 1 If one replace $\alpha(x)$ in (5) and (6) by $\alpha(x-s)$ and replaces each $\alpha(s)$ in (7) and (8) by $\alpha(x-s)$, then the ABR and ABC fractional derivatives with variable order can be expressed in the convolution form. Analogously, if one replaces $\alpha(x)$ in (9) and (10) by $\alpha(x-s)$ and replaces each $\alpha(s)$ in (11) and (12) by $\alpha(x-s)$, then the second part of the AB fractional integrals with variable order can be expressed in the convolution form.

Lemma 1 *For functions u and v defined on* $[a,b]$ *and* $0 < \alpha(x) \le 1$ *we have*

$$ \int_a^b u(x)\, {}_aI^{\alpha(x)}_x v(x)dx = \int_a^b v(x)\, {}_x\mathscr{I}^{\alpha(x)}_b u(x)dx, \tag{21} $$

$$ \int_a^b u(x)\, {}_a\mathscr{I}^{\alpha(x)}_x v(x)dt = \int_a^b v(x)\, {}_xI^{\alpha(x)}_b u(x)dx. \tag{22} $$

Proof Using Definition 1 and changing the order of integration, we have

$$ \begin{aligned} \int_a^b u(x)\, {}_aI^{\alpha(x)}_x v(x)dx &= \int_a^b u(x)\left(\frac{1}{\Gamma(\alpha(x))}\int_a^x (x-s)^{\alpha(x)-1}v(s)ds\right)dx \\ &= \int_a^b v(s)\left(\int_s^b (x-s)^{\alpha(x)-1}\frac{u(x)}{\Gamma(\alpha(x))}dx\right)ds \\ &= \int_a^b v(s)\, {}_s\mathscr{I}^{\alpha(s)}_b u(s)ds. \end{aligned} $$

On the other side, again using Definition 1 and changing the order of integrations, we get

$$\begin{aligned}\int_a^b u(x)\, {}_a\mathscr{I}_x^{\alpha(x)} v(x)dx &= \int_a^b u(x)\left(\int_a^x (x-s)^{\alpha(s)-1} v(s)\frac{ds}{\Gamma(\alpha(s))}\right)dx \\ &= \int_a^b v(s)\left(\frac{1}{\Gamma(\alpha(s))}\int_s^b u(x)(x-s)^{\alpha(s)-1}dx\right)ds \\ &= \int_a^b v(s)\, {}_xI_b^{\alpha(s)} u(s)ds\end{aligned}$$

Now, benefiting from Lemma 1 we can show that the following integration by parts formulas hold.

Theorem 1 (Integration by parts for AB fractional integrals) *Let* $\alpha(x) > 0$, $p \geq 1$, $q \geq 1$, *and* $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha(x)$ *for all* t. *Then for any* $u(x) \in L^p(a,b)$, $v(x) \in L^q(a,b)$, *we have*

$$\int_a^b u(t)\, {}_a^{AB}I_x^{\alpha(x)} v(x)dt = \int_a^b v(x)\, {}_x^{AB}\mathscr{I}_b^{\alpha(x)} u(x)dx, \tag{23}$$

$$\int_a^b u(x)\, {}_a^{AB}\mathscr{I}_x^{\alpha(x)} v(x)dx = \int_a^b v(x)\, {}_x^{AB}I_b^{\alpha(x)} u(x)dx. \tag{24}$$

Proof From Definition 3 and by applying the first part of Lemma 1, we have

$$\begin{aligned}\int_a^b u(x)\, {}_a^{AB}I_x^{\alpha(x)} v(x)dt &= \int_a^b v(x)\frac{1-\alpha(x)}{B(\alpha(x))}u(x)dx + \int_a^b u(x)\frac{\alpha(x)}{B(\alpha(x))}\, {}_aI_x^{\alpha(x)} v(x)dx \\ &= \int_a^b v(x)\frac{1-\alpha(x)}{B(\alpha(x)}u(x)dx + \int_a^b v(t)\, {}_x\mathscr{I}_b^{\alpha(x)}\left[\frac{u(x)\alpha(x)}{B(\alpha(x))}\right]dx \\ &= \int_a^b v(x)\left(\frac{1-\alpha(x)}{B(\alpha(x))}u(x) + {}_x\mathscr{I}_b^{\alpha(x)}[\frac{\alpha(x)}{B(\alpha(x))}u(x)]\right)dx \\ &= \int_a^b v(x)\, {}_x^{AB}\mathscr{I}_b^{\alpha(x)} u(x)dx.\end{aligned}$$

The proof of the formula in (24) can be done similarly, once we use the second part of Lemma 1.

Lemma 2 *Let* $u(x)$ *and* $v(x)$ *be functions defined on* $[a,b]$ *and let* $0 < \alpha(x) \leq 1$. *Then, we have*

$$\int_a^b u(x)\, \boldsymbol{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+} v(x)dx = \int_a^b v(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-} u(x)dx, \tag{25}$$

$$\int_a^b u(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+} v(x)dx = \int_a^b v(x)\boldsymbol{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-} u(x)dx. \tag{26}$$

Proof The proof can be executed using some definitions and changing the order of integrations. In fact, we have

$$\int_a^b u(x)\ \mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}v(x)dx$$

$$= \int_a^b u(x)\frac{B(\alpha(x))}{1-\alpha(x)}\left(\int_a^x v(s)E_{\alpha(x)}(\frac{-\alpha(x)}{1-\alpha(x)}(x-s)^{\alpha(x)})ds\right)dx$$

$$= \int_a^b v(s)\left(\int_s^b \frac{B(\alpha(x))}{1-\alpha(x)}E_{\alpha(x)}(\frac{-\alpha(x)}{1-\alpha(x)}(x-s)^{\alpha(x)})u(x)dx\right)ds$$

$$= \int_a^b v(s)\mathscr{E}_{\alpha(s),1,\frac{-\alpha(s)}{1-\alpha(s)},b^-}u(s)ds.$$

The formula in (26) can proved similarly.

Theorem 2 *Let* $u(x)$ *and* $v(x)$ *be functions defined on* $[a, b]$ *and let* $0 < \alpha(x) \leq 1$. *We have*

$$\int_a^b u(x)\ {}^{ABC}_{\ a}D_x^{\alpha(x)}v(x)dx = v(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}u(x)\Big|_a^b + \int_a^b v(x)\ {}^{ABR}_{\ x}\mathscr{D}_b^{\alpha(x)}u(x)dx, \tag{27}$$

$$\int_a^b u(x)\ {}^{ABC}_{\ a}\mathscr{D}_x^{\alpha(x)}v(x)dx = v(x)\boldsymbol{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}u(x)\Big|_a^b + \int_a^b v(x)\ {}^{ABR}_{\ t}D_b^{\alpha(x)}u(x)dx, \tag{28}$$

$$\int_a^b u(x)\ {}^{ABC}_{\ x}D_b^{\alpha(x)}v(x)dx = \int_a^b v(x)\ {}^{ABR}_{\ a}\mathscr{D}_x^{\alpha(x)}u(x)dx - v(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}u(x)\Big|_a^b, \tag{29}$$

and

$$\int_a^b u(x)\ {}^{ABC}_{\ x}\mathscr{D}_b^{\alpha(x)}v(x)dx = \int_a^b v(x)\ {}^{ABR}_{\ a}D_x^{\alpha(x)}u(x)dx - v(x)\boldsymbol{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}u(x)\Big|_a^b. \tag{30}$$

Proof The proof follows from Definition 3, Lemma 2 and the classical integration by parts. Bellow, we prove (27) only. The proofs of the rest of the formulas are analogous. Actually, using Definition 3 and applying the first part of Lemma 2 and the traditional integration by parts, we have

$$\begin{aligned}\int_a^b u(x)\ {}^{ABC}_{\ a}D_x^{\alpha(x)}v(x)dx &= \int_a^b u(x)\ \mathbf{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},a^+}v'(x)dx \\ &= \int_a^b v'(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}u(x)dx \\ &= v(x)\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}u(x)\Big|_a^b + \int_a^b v(x)\ {}^{ABR}_{\ x}\mathscr{D}_b^{\alpha(x)}u(x)dx.\end{aligned} \tag{31}$$

3 Fractional Variational Principles in the Frame of Variable-Order Fractional Atangana–Baleanu's Derivatives

In this section, we present Euler–Lagrange and fractional Hamilton equations in the frame of the fractional variable-order Atangana–Baleanu derivatives are.

Theorem 3 *Let $J[z]$ be a functional of the form*

$$J[z] = \int_a^b L(x, z, {}^{ABC}_{a}D_x^{\alpha(x)} z(x))dx \tag{32}$$

defined by the set of functions which have continuous variable-order Atangana–Baleanu fractional derivative in the Caputo sense on the set of order $\alpha(x)$ in $[a, b]$ and which satisfy the boundary conditions $z(a) = z_a$ and $z(b) = z_b$. Then a necessary condition for $J[z]$ to have a maximum for given function $z(x)$ is that $z(x)$ must satisfy the following Euler–Lagrange equation:

$$\frac{\partial L}{\partial z} + {}^{ABR}_{x}\mathscr{D}_b^{\alpha(x)} \left(\frac{\partial L}{\partial ({}^{ABC}_{a}D_x^{\alpha(x)} z(x))} \right) = 0 \tag{33}$$

Proof To obtain the necessary conditions for the extremum, we assume that $z^*(x)$ is the desired function. Let $\varepsilon \in R$ define a family of curves

$$z(x) = z^*(x) + \varepsilon\eta(x) \tag{34}$$

where, $\eta(t)$ is an arbitrary curve except that it satisfies the homogeneous boundary conditions; that is

$$\eta(a) = \eta(b) = 0. \tag{35}$$

To obtain the Euler–Lagrange equation, we substitute equation (34) into Eq. (32) and differentiate the resulting equation with respect to ε and set the result to 0. This leads to the following condition for extremum:

$$\int_a^b \left[\frac{\partial L}{\partial z}\eta(x) + \frac{\partial L}{\partial\, {}^{ABC}_{a}D_z^{\alpha(z)} z(x)}\, {}^{ABC}_{a}D_x^{\alpha(x)} \eta(x) \right] dx = 0. \tag{36}$$

Using Eqs. (30), (36) can be written as

$$\int_a^b \left[\frac{\partial L}{\partial z} + {}^{ABR}_{z}\mathscr{D}_b^{\alpha(x)} \frac{\partial L}{\partial ({}^{ABC}_{a}D_x^{\alpha(x)} z(x))} \right] \eta(x)dx + \\ \eta(x) \cdot \mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-} \left(\frac{\partial L}{\partial ({}^{ABC}_{a}D_x^{\alpha(x)} z(x))} \right)(x) \Big|_a^b = 0. \tag{37}$$

We call $\mathscr{E}_{\alpha(x),1,\frac{-\alpha(x)}{1-\alpha(x)},b^-}(\frac{\partial L}{\partial({}^{ABC}_{\ a}D^{\alpha(x)}_x z(x))})(x)\Big|_a^b = 0$, the natural boundary condition. Now, since $\eta(x)$ is arbitrary, it follows from a well established result in calculus of variations that

$$\frac{\partial L}{\partial z} + {}^{ABR}_{\ \ x}\mathscr{D}^{\alpha(x)}_b \frac{\partial L}{\partial({}^{ABC}_{\ a}D^{\alpha(x)}_x z(x))} = 0 \tag{38}$$

Equation (38) is the Generalized Euler–Lagrange Equation GELE for the Fractional Calculus Variation (FCV) problem defined in terms of the variable-order Atangana–Baleanu Fractional Derivatives ABFD. Note that the Atangana–Baleanu derivatives in the Caputo and Riemann–Liouville sense appears in the resulting differential equations.

Example 1 Consider the following Lagrangian:

$$L = \frac{1}{2}(z + {}^{ABC}_{\ a}D^{\alpha(x)}_x z)^2, \tag{39}$$

then independent fractional Euler–Lagrange equation (38) is given by

$$z + {}^{ABR}_{\ \ x}\mathscr{D}^{\alpha(x)}_b({}^{ABC}_{\ a}D^{\alpha(x)}_x z) = 0 \tag{40}$$

Example 2 We consider now a fractional Lagrangian of the oscillatory system

$$L = \frac{1}{2}m({}^{ABC}_{\ a}D^{\alpha(x)}_x z)^2 - \frac{1}{2}kz^2, \tag{41}$$

where m the mass and k is constant. Then the fractional Euler–Lagrange equation is

$$m\,{}^{ABR}_{\ \ x}\mathscr{D}^{\alpha(x)}_b({}^{ABC}_{\ a}D^{\alpha(x)}_x z) - kz = 0 \tag{42}$$

This equation reduces to the equation of motion of the harmonic oscillator when $\alpha(x) \to 1$.

3.1 Some Generalizations

In this section, we extend the results obtained in give some Theorem 3.1 to the case of n variables $z_1(x), z_2(x), ..., z_n(x)$. We denote by $\mathscr{F}_n$ the set of all functions which have continuous left ABC fractional derivative of order $\alpha(x)$ and right ABC fractional derivative of order β for $x \in [a, b]$ and satisfy the conditions

$$z_i(a) = z_{ia}, z_i(b) = z_{ib}, i = 1, 2, ..., n.$$

The problem can be defined as follows: find the functions $z_1, z_2, ..., z_n$ from $\mathscr{F}_n$, for which the functional

$$J[z_1, z_2, ..., z_n] = \int_a^b L\Big[x, z_1(x), z_2(x), ..., z_n(x),$$

$$ {}^{ABC}_{\ a}D_x^{\alpha(x)} z_1(x), ..., {}^{ABC}_{\ a}D_x^{\alpha(x)} z_n(x), {}^{ABC}_{\ x}D_b^{\alpha(x)} z_1(x), ..., {}^{ABC}_{\ x}D_b^{\alpha(x)} z_n(x)\Big] dx$$

has an extremum, where $L(x, z_1, ..., z_n, y_1, ..., y_n, w_1, ..., w_n)$ is a function with continuous first and second partial derivatives with respect to all its arguments. A necessary condition for $J[z_1, z_2, ..., z_n]$ to admit an extremum is that $z_1(x), z_2(x), ..., z_n(x)$ satisfy Euler–Lagrange equations:

$$\frac{\partial L}{\partial z_i} + {}^{ABR}_{\ x}\mathscr{D}_b^{\alpha(x)} \frac{\partial L}{\partial\, {}^{ABC}_{\ a}D_x^{\alpha(x)} z_i(x)} + {}^{ABR}_{\ a}\mathscr{D}_x^{\alpha(x)} \frac{\partial L}{\partial\, {}^{ABC}_{\ x}D_b^{\alpha(x)} z_i(x)} = 0, \quad i = 1, 2, ..., n. \tag{43}$$

Example 3 Lets consider the system of two planar pendula, both of length l and mass m, suspended from the same distance apart on a horizontal line so that they are moving in the same plane. The fractional counter part of the Lagrangian is $L(t, z_1, z_2, {}^{ABC}_{\ a}D_x^{\alpha(z)} z_1, {}^{ABC}_{\ x}D_b^{\alpha(x)} z_2) =$

$$\frac{1}{2} m \left[({}^{ABC}_{\ a}D_x^{\alpha(x)} z_1)^2 + ({}^{ABC}_{\ x}D_b^{\alpha(x)} z_2)^2 \right] - \frac{1}{2}\frac{mg}{l}(z_1^2 + z_2^2). \tag{44}$$

To obtain the fractional Euler–Lagrange equation, we use

$$\frac{\partial L}{\partial z} + {}^{ABR}_{\ x}\mathscr{D}_b^{\alpha(x)} \frac{\partial L}{\partial\, {}^{ABC}_{\ a}D_x^{\alpha(x)} z(x)} + {}^{ABR}_{\ a}\mathscr{D}_x^{\alpha(t)} \frac{\partial L}{\partial\, {}^{ABC}_{\ x}D_b^{\alpha(x)} z(x)} = 0, \quad . \tag{45}$$

It follows that

$${}^{ABR}_{\ x}\mathscr{D}_b^{\alpha(x)} ({}^{ABC}_{\ a}D_x^{\alpha(x)} z_1) - \frac{g}{l} x_1 = 0, \quad {}^{ABR}_{\ a}\mathscr{D}_x^{\alpha(x)} ({}^{ABC}_{\ x}D_b^{\alpha(x)} z_2) - \frac{g}{l} z_2 = 0 \tag{46}$$

These equation reduces to the equation of motion of the harmonic oscillator when $\alpha(t) \to 1$.

$$z_1'' + \frac{g}{l} z_1 = 0, \quad z_2'' + \frac{g}{l} z_2 = 0 \tag{47}$$

4 Fractional Variational Principles and Constrained Systems in the Frame of Variable-Order Atangana–Baleanu's Derivatives

Lets now consider the following problem: Find the extremum of the functional

$$J[x] = \int_a^b L(x, z, {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x)) dt,$$

subject to the dynamical constraint

$${}^{ABC}_{\ a} D_x^{\alpha(x)} z(x) = \phi(z),$$

with the boundary conditions

$$x(a) = x_a, \quad x(b) = x_b.$$

In this case, we define the functional

$$S[x] = \int_a^b [L + \lambda \Phi] dx,$$

where

$$\Phi(x, z, {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x)) = \phi(z) - {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x) = 0$$

and λ is the Lagrange multiplier. Then Eq. (38) in this case takes the form

$$\frac{\partial S}{\partial z} + {}^{ABR}_{\ x} \mathscr{D}_b^{\alpha(x)} \frac{\partial S}{\partial\, {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x)} = 0 \tag{48}$$

which can be written as

$$\frac{\partial L}{\partial z} + {}^{ABR}_{\ x} \mathscr{D}_b^{\alpha(x)} \frac{\partial L}{\partial\, {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x)} + \lambda \left[\frac{\partial \Phi}{\partial z} + {}^{ABR}_{\ x} \mathscr{D}_b^{\alpha(x)} \frac{\partial \Phi}{\partial\, {}^{ABC}_{\ a} D_x^{\alpha(x)} z(x)} \right] = 0 \tag{49}$$

Example 4 Lets consider

$$J[z] = \int_0^1 ({}^{ABC}_{\ 0} D_x^{\alpha(x)} z(x))^2 dt,$$

with the boundary conditions

$$z(0) = 0, \quad z(1) = 0,$$

$$\int_0^1 z dx = 0, \quad \int_0^1 xz dx = 1.$$

Then we have

$$S[z] = \int_0^1 \left[({}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x))^2 + \lambda_1 z + \lambda_2 xz \right] dx,$$

where λ_1, λ_2 are the Lagrange multipliers. Then Eq. (48) takes the form

$${}^{ABR}_{\ \ x}\mathscr{D}_1^{\alpha(x)} ({}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x)) - \frac{1}{2}(\lambda_1 + \lambda_2 x) = 0. \tag{50}$$

5 Fractional Optimal Control Problem Involving Variable-Order Atangana–Baleanu's Derivatives

Find the optimal control $v(t)$ for a that minimizes the performance index

$$J[v] = \int_0^1 F(z, v, x) dx, \tag{51}$$

subject to the dynamical constraint

$${}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x) = G(z, v, x), \tag{52}$$

with the boundary conditions

$$z(0) = z_0. \tag{53}$$

where $z(x)$ is the state variable, x represents the time, and F and G are two arbitrary functions. Note that Eq. (51) may also include some additional terms containing state variables at the end point. This term in not considered here for simplicity. When $\alpha(x) = 1$, the above problem reduces to the standard optimal control problem. To find the optimal control we follow the traditional approach and define a modified performance index.

Lets define the functional

$$\mathscr{J}[z] = \int_0^1 [F(z, v, x) + \lambda(G(z, v, x) - {}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x))] dx, \tag{54}$$

where λ is the Lagrange multiplier. The variations of Equation (54) give

$$\delta \mathscr{J}[v] = \int_0^1 \Big[\frac{\partial F}{\partial z}\delta z + \frac{\partial F}{\partial v}\delta v + \delta\lambda(G(z, v, x) - {}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x)) \qquad (55)$$
$$+ \lambda\left(\frac{\partial G}{\partial z}\delta z + \frac{\partial F}{\partial v}\delta v - \delta({}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x))\right)\Big]dx,$$

Using Eqs. (27), (55) becomes

$$\int_0^1 \lambda\delta({}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x))dx = \int_0^1 \delta z(x)({}^{ABR}_{\ \ x}\mathscr{D}_1^{\alpha(x)}\lambda)dx, \qquad (56)$$

where $\delta z(0) = 0$ or $\lambda(0) = 0$, and $\lambda z(1) = 0$ or $\lambda(1) = 0$. Because $z(0)$ is specified, we have $\delta z(0) = 0$, and since $z(1)$ is not specified, we require $\lambda(1) = 0$. Using these assumptions, Eqs. (55) and (56) become

$$\delta \mathscr{J}[v] = \int_0^1 \Big[\delta\lambda(G(z, v, x) - {}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x)) + \delta z[\frac{\partial F}{\partial z} + \lambda\frac{\partial G}{\partial z} - {}^{ABR}_{\ \ x}\mathscr{D}_1^{\alpha(x)}\lambda] \qquad (57)$$
$$+ \delta v[\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v}]\Big]dx,$$

Since $\mathscr{J}[v]$ and consequently $J(v)$) is minimized, δz, δv, and $\delta\lambda$ in Eq. (57) are all equal to zero. This gives

$${}^{ABC}_{\ 0}D_x^{\alpha(x)} z(x) = G(z, v, x) \qquad (58)$$

$${}^{ABR}_{\ \ x}\mathscr{D}_1^{\alpha(x)}\lambda = \frac{\partial F}{\partial z} + \lambda\frac{\partial G}{\partial z} \qquad (59)$$

$$\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v} = 0. \qquad (60)$$

and

$$z(0) = z_0 \text{ and } \lambda(1) = 0. \qquad (61)$$

Observe that Eq. (58) contains Left Atangana–Baleanu in Caputo sense FD, whereas Eq. (59) contains Right Atangana–Baleanu in Caputo FD. This clearly indicates that the solution of optimal control problems requires knowledge of not only forward derivatives but also backward derivatives to count on the end conditions. In classical optimal control theories, this issue is either not discussed or they are not clearly stated. This is largely because the backward derivative of order 1 turns out to be the negative of the forward derivative of order 1.

Example 5 Consider

$$J[v] = \frac{1}{2}\int_0^1 [a(x)z^2(x) + b(x)v^2(x)]dx, \qquad (62)$$

where $a(x) \geq 0$, $b(x) > 0$, and,

$$ {}^{ABC}_{\ 0}D^{\alpha(x)}_{x} z(x) = c(x)z(x) + d(x)v. \tag{63} $$

This linear system for $\alpha(t) = 1$ and $0 < \alpha(t) < 1$ was considered before in the literature and formulations and solution schemes for this system within the traditional Riemann–Liouville and Caputo derivatives are addressed in many books and articles (see e.g. [2, 3]. Here, we discuss the same problem in the framework of Atangana–Baleanu fractional derivatives. For $0 < \alpha(t) < 1$, the Euler–Lagrange Equations (58) to (60) gives (63) and

$$ {}^{ABR}_{\ x}\mathscr{D}^{\alpha(x)}_{1} \lambda = a(x)z(x) + c(x)\lambda, \tag{64} $$

and

$$ b(x)v(x) + d(x)\lambda = 0. \tag{65} $$

From (63) and (65), we obtain

$$ {}^{ABC}_{\ 0}D^{\alpha(x)}_{x} z(x) = c(x)z(x) - b^{-1}(x)d^{2}(x)\lambda. \tag{66} $$

Thus, $z(x)$ and $\lambda(x)$ can be computed from (64) and (66).

Example 6 Consider the following time-invariant problem.

Find the control $v(x)$ which minimizes the quadratic performance index

$$ J[v] = \frac{1}{2}\int_{0}^{1} [z^{2}(z) + v^{2}(x)]dx, \tag{67} $$

subject to

$$ {}^{ABC}_{\ 0}D^{\alpha(x)}_{x} z(t) = -z + v, \tag{68} $$

and the initial condition

$$ z(0) = 1. \tag{69} $$

Note that from (5), we have

$$ a(x) = b(x) = -c(x) = d(x) = z_0 = 1, \tag{70} $$

and (64) and (65) read

$$ {}^{ABR}_{\ t}\mathscr{D}^{\alpha(x)}_{1} \lambda = z - \lambda \tag{71} $$

and

$$ v + \lambda = 0. \tag{72} $$

6 Conclusions

In this work, we have tackled some types of optimal control problems in the presence of the newly proposed nonlocal and nonsingular fractional derivatives that involve Mittag-Leffler functions as kernels. In order to obtain Euler–Lagrange equations, we exploited the techniques mentioned in several books and the fractional integration by parts formulas. It turned out, the formulation shewed and the obtained equations are analogous with the ones when the classical variation principles are used; but with slight differences. That is, all the concepts of the classical calculus of variation can be carried to fractional calculus of variation in the frame of either the traditional fractional operators with singular kernels or the newly defined operators involving nonsingular kernel. However, since there is a little advance that has been done in the theory of fractional operators with variable order, there is no big progress in the calculus of variation in the presence of such operators. Therefore, we believe in the need of tackling such operators and that this work may initiate the interest of researches and them as they can also be used in modeling some problems considered in various fields of sciences.

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References

1. Abdeljawad, T., Baleanu, D.: Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. J. Nonlinear Sci. Appl. **10**, 1098–1107 (2017)
2. Agrawal, O.P.: Formulation of Euler-Lagrange equations for fractional variational problems. J. Math. Anal. Appl. **272**, 368–379 (2002)
3. Agrawal, O.P.: A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. **38**, 323–337 (2004)
4. Agrawal, O.P., Baleanu, D.: Hamiltonian formulation and direct numerical scheme for fractional optimal control problems. J. Vib. Control **13**(9–10), 1269–1281 (2007)
5. Agarwal, R.P., Baghli, S., Benchohra, M.: Controllability for semilinear functional and neutral functional evolution equations with infinite delay in Frechet spaces. Appl. Math. Optim. **60**, 253–274 (2009)
6. Ahmad, B., Ntouyas, S.K.: Existence of solutions for fractional differential inclusions with four-point nonlocal Riemann-Liouville type integral boundary conditions. Filomat **27**(6), 1027–1036 (2013)
7. Atangana, A.: Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. Phys. A. **505**, 688–706 (2018)
8. Atangana, A., Baleanu, D.: New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. Therm. Sci. **20**(2), 763–769 (2016)
9. Atangana, A., Gomez-Aguilar, J.F.: Decolonisation of fractional calculus rules: breaking commutativity and associativity to capture more natural phenomena. Eur. Phys. J. Plus **133**, 166 (2018)

10. Atanackovic, T.M., Pilipovic, S.: Hamilton's principle with variable order fractional derivatives. Fract. Calc. Appl. Anal. **14**(1), 94–109 (2011)
11. Bahaa, G.M.: Fractional optimal control problem for variational inequalities with control constraints. IMA J. Math. Control Inform. **33**(3), 1–16 (2016)
12. Bahaa, G.M.: Fractional optimal control problem for differential system with control constraints. Filomat **30**(8), 2177–2189 (2016)
13. Bahaa, G.M.: Fractional optimal control problem for infinite order system with control constraints. Adv. Differ. Equ. (2016). https://doi.org/10.1186/s13662-016-0976-2
14. Bahaa, G.M.: Fractional optimal control problem for differential system with delay argument. Adv. Differ. Equ. (2017). https://doi.org/10.1186/s13662-017-1121-6
15. Bahaa, G.M.: Fractional optimal control problem for variable-order differential systems. Fract. Calc. Appl. Anal. **20**(6), 1447–1470 (2017)
16. Bahaa, G.M., Tang, Q.: Optimal control problem for coupled time-fractional evolution systems with control constraints. J. Differ. Equ. Dyn. Syst. (2017). https://doi.org/10.1007/s12591-017-0403-5
17. Bahaa, G.M., Tang, Q.: Optimality conditions for fractional diffusion equations with weak Caputo derivatives and variational formulation. J. Fract. Calc. Appl. **9**(1), 100–119 (2018)
18. Baleanu, D., Agrawal, O.P.: Fractional Hamilton formalism within Caputo's derivative. Czech. J. Phys. **56**(10–11), 1087–1092 (2000)
19. Baleanu, D., Avkar, T.: Lagrangian with linear velocities within Riemann-Liouville fractional derivatives. Nuovo Cimnto B **119**, 73–79 (2004)
20. Baleanu, D., Jajarmi, A., Hajipour, M.: A new formulation of the fractional optimal control problems involving Mittag-Leffler nonsingular kernel. J Optim. Theory. Appl. **175**, 718–737 (2017)
21. Baleanu, D., Muslih, S.I.: Lagrangian formulation on classical fields within Riemann-Liouville fractional derivatives. Phys. Scr. **72**(2–3), 119–121 (2005)
22. Bota, C., Caruntu, B.: Analytic approximate solutions for a class of variable order fractional differential equations using the polynomial least squares method. Fract. Calc. Appl. Anal. **20**(4), 1043–1050 (2017)
23. Djida, J.D., Atangana, A., Area, I.: Numerical computation of a fractional derivative with non-local and non-singular kernel. Math. Model. Nat. Phenom. **12**(3), 4–13 (2017)
24. Djida, J.D., Mophou, G.M., Area, I., Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular mittag-leffler kernel (2017). arXiv:1711.09070
25. El-Sayed, A.M.A.: On the stochastic fractional calculus operators. J. Fract. Calc. Appl. **6**(1), 101–109 (2015)
26. Frederico Gastao, S.F., Torres Delfim, F.M.: Fractional optimal control in the sense of Caputo and the fractional Noether's theorem. Int. Math. Forum **3**(10), 479–493 (2008)
27. Gomez-Aguilar, J.F.: Space-time fractional diffusion equation using a derivative with nonsingular and regular kernel. Phys. A. **465**, 562–572 (2017)
28. Gomez-Aguilar, J.F., Atangana, A., Morales-Delgado, J.F.: Electrical circuits RC, LC, and RL described by Atangana-Baleanu fractional derivatives. Int. J. Circ. Theor. Appl. (2017). https://doi.org/10.1002/cta.2348
29. Gomez-Aguilar, J.F.: Irving-Mullineux oscillator via fractional derivatives with Mittag-Leffler kernel. Chaos Soliton. Fract. **95**(35), 179–186 (2017)
30. Hafez, F.M., El-Sayed, A.M.A., El-Tawil, M.A.: On a stochastic fractional calculus. Frac. Calc. Appl. Anal. **4**(1), 81–90 (2001)
31. Hilfer, R.: Applications of Fractional Calculus in Physics. Word Scientific, Singapore (2000)
32. Jarad, F., Maraba, T., Baleanu, D.: Fractional variational optimal control problems with delayed arguments. Nonlinear Dyn. **62**, 609–614 (2010)
33. Jarad, F., Maraba, T., Baleanu, D.: Higher order fractional variational optimal control problems with delayed arguments. Appl. Math. Comput. **218**, 9234–9240 (2012)
34. Kilbas, A.A., Saigo, M., Saxena, K.: Generalized Mittag-Leffler function and generalized fractional calculus operators. Int. Tran. Spec. Funct. **15**(1), 31–49 (2004)

35. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Theory and Application of Fractional Differential Equations. North Holland Mathematics Studies, vol. 204. Amsterdam (2006)
36. Liu, Z., Zeng, S., Bai, Y.: Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications. Fract. Calc. Appl. Anal. **19**(1), 188–211 (2016)
37. Lorenzo, C.F., Hartley, T.T.: Variable order and distributed order fractional operators. Nonlinear Dyn. **29**, 57–98 (2002)
38. Machado, J.A.T., Kiryakova, V., Mainardi, F.: Recent history of fractional calculus. Commun. Nonlinear Sci. Numer. Simul. **16**(3), 1140–1153 (2011)
39. Magin, R.L.: Fractional Calculus in Bioengineering. Begell House Publishers, Redding (2006)
40. Mophou, G.M.: Optimal control of fractional diffusion equation. Comput. Math. Appl. **61**, 68–78 (2011)
41. Mophou, G.M.: Optimal control of fractional diffusion equation with state constraints. Comput. Math. Appl. **62**, 1413–1426 (2011)
42. Ozdemir, N., Karadeniz, D., Iskender, B.B.: Fractional optimal control problem of a distributed system in cylindrical coordinates. Phys. Lett. A **373**, 221–226 (2009)
43. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
44. Riewe, F.: Nonconservative Lagrangian and Hamiltonian mechanics. Phys. Rev. E **53**, 1890–1899 (1996)
45. Riewe, F.: Mechanics with fractional derivatives. Phys. Rev. E **55**, 3581–3592 (1997)
46. Ross, B., Samko, S.G.: Integration and differentiation to a variable fractional order. Integr. Transform. Spec. Funct. **1**, 277–300 (1993)
47. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
48. Tateishi, A.A., Ribeiro, H.V., Lenzi, E.K.: The role of fractional time-derivative operators on anomalous diffusion. Front. Phys. **5**, 1–9 (2017)

Analysis of 2-Term Fractional-Order Delay Differential Equations

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Abstract The value of a state variable at past time usually affects its rate of change at the present time. So, it is very natural to consider the *delay* while modeling the real-life systems. Further, the nonlocal fractional derivative operator is also useful in modeling *memory* in the system. Hence, the models involving delay as well as fractional derivative are very important. In this chapter, we review the basic results regarding the dynamical systems, fractional calculus, and delay differential equations. Further, we analyze 2-term nonlinear fractional-order delay differential equation $D^\alpha x + cD^\beta x = f(x, x_\tau)$, with constant delay $\tau > 0$ and fractional orders $0 < \alpha < \beta < 1$. We present a numerical method for solving such equations and present an example exhibiting chaotic oscillations.

1 Introduction

Fractional differential equations involve a derivative of *arbitrary order*. These operators are usually nonlocal. Thus, one has to specify all the values from initial point to evaluate fractional derivative (FD) of a function. This peculiarity of FD is very useful while modeling memory and hereditary properties in the natural systems.

Researchers always have a flexibility to choose a fractional derivative among various (inequivalent) definitions which perfectly suits their needs. Each of these derivatives has its own importance and played an important role in the development of Fractional Calculus. Few examples of FD include Riemann–Liouville derivative [39], Caputo derivative [39], Grunwald–Letnikov derivative, Saigo derivative [33], Agrawal derivative [1], and so on.

Various researchers have worked on the analysis of fractional-order differential equations (FDE). Existence and uniqueness of nonlinear nonautonomous FDEs involving Riemann–Liouville derivative is discussed by Delbosco and Rodino [25].

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Daftardar-Gejji and Babakhani [20] presented the analysis of linear and nonlinear systems of FDEs. Existence of positive solutions of different types of FDEs is discussed by Daftardar-Gejji and coworkers [4, 5, 19, 32]. Existence of solutions for fractional-order boundary value problems is discussed by Ahmad and Nieto [2]. A new iterative method is used by Bhalekar and Daftardar-Gejji [13] to analyze FDEs with Caputo derivative.

Solving nonlinear FDEs is a challenging task due to the nonlocal nature of FDEs. The analytical methods such as Adomian decomposition method and Daftardar-Gejji-Jafari method are proved useful in finding local solutions of FDEs [12, 21]. On the other hand, numerical methods, viz., Fractional Adam's Method (FAM) [27] and New Predictor-Corrector Method (NPCM) [23] provide solutions on larger intervals.

In this chapter, we discuss the dynamics of 2-term fractional-order nonlinear delay differential equations.

2 Preliminaries

In this section, we discuss preliminaries of Dynamical Systems [3, 36], Fractional-Order Dynamical Systems, and Delay Differential Equations.

2.1 Dynamical Systems

Dynamical Systems is a branch of Mathematics, which deals with qualitative and quantitative analysis of various equations.

Definition 2.1 ([36]) An evolution rule that defines a trajectory as a function of a single parameter (time) on a set of states (the phase space) is a dynamical system.

Examples: Differential equations (continuous dynamical systems), Maps (discrete dynamical systems).

2.2 Stability Analysis of Continuous Dynamical Systems

Consider a system of ordinary differential equations

$$\dot{X} = f(X), \tag{1}$$

where $X(t) \in \mathbb{R}^n$, $f \in C^1(E)$ and E is open set in $\mathbb{R}^n$.

2.2.1 Equilibrium Points

Constant (**steady state**) solution of system (1) is called as **equilibrium** point. Thus, the equilibrium points are solutions of $f(X) = 0$.

An equilibrium point X^* of system (1) is said to be

(i) Stable if nearby starting solutions stay nearby.
(ii) Asymptotically stable if it is stable and nearby starting solutions converge to X^*.
(iii) Unstable if it is not stable.

2.2.2 Linearization

Let X be a solution of system (1) starting in the neighborhood of equilibrium X^*. Define $Y = X - X^*$. Using Taylor's approximation, we get

$$\begin{aligned}\dot{Y} &= \dot{X}\\ &= f(X) = f(X^* + Y)\\ &= f(X^*) + Df(X^*)Y.\end{aligned}$$

The system

$$\dot{Y} = AY, \tag{2}$$

where $A = Df(X^*)$ (Jacobian of f evaluated at X^*) is called linearization of (1).

Definition 2.2 An equilibrium X^* of (1) is said to be hyperbolic if no eigenvalue of $Df(X^*)$ has its real part equal to zero.

Definition 2.3 An equilibrium is a sink (respectively, source) if all of the eigenvalues of $Df(X^*)$ have negative (respectively, positive) real parts. Hyperbolic equilibrium is saddle if there is at least one eigenvalue with positive real part and at least one eigenvalue with negative real part. Equilibrium is center if there are purely imaginary eigenvalues.

Theorem 2.1 *Hartman–Grobman Theorem: The behavior of a dynamical system (1) in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization (2).*

Thus, sinks are asymptotically stable and sources are unstable equilibrium points.

Example 2.1 Consider scalar autonomous equation $\dot{y} = y(y - 1)$.

Equilibrium points are $y_1{*} = 0$ and $y_2{*} = 1$. Here $f'(0) = -1 < 0$ and $f'(1) = 1 > 0$. Thus, $y_1{*}$ is sink and $y_2{*}$ is source. The result is illustrated in Fig. 1.

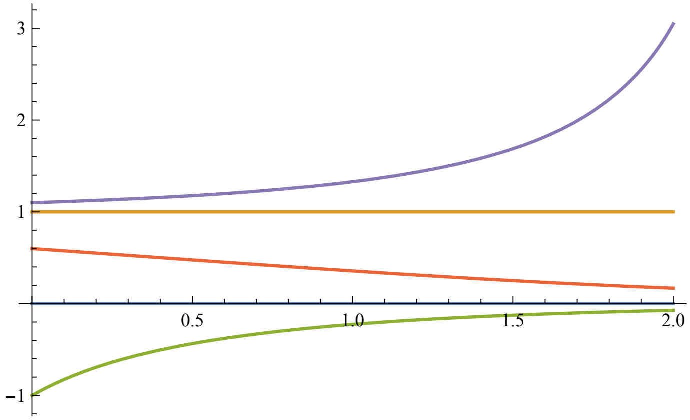

Fig. 1 Stability of $\dot{y} = y(y-1)$

Example 2.2 Consider planar system $\dot{x} = -x$, $\dot{y} = x^2 + y$.

Equilibrium point is $X_* = (0, 0)$. In this case,

$$Df(X) = \begin{pmatrix} -1 & 0 \\ 2x & 1 \end{pmatrix}. \tag{3}$$

Eigenvalues of $Df(0, 0)$ are -1 and 1. Therefore, origin is saddle point (cf. Fig. 2).

2.3 Chaos

Consider an autonomous system of nonlinear differential equations of order three or higher. Solutions of these systems are called chaotic if they have following properties: (i) The solution trajectories are bounded; (ii) The trajectories are aperiodic and (iii) The oscillations in the trajectories never settle. In this case, the solutions are extremely sensitive to initial conditions.

2.4 Delay Differential Equations

The differential equation which contains the delay term $x_\tau = x(t-\tau)$ is called as a delay differential equation (DDE). The delay τ can be a constant, a function of time t or of dependent variable x. The delay can also be used to model memory in the system. Thus, the models involving FD and a delay are crucial.

These equations have found many applications in Control Theory [37, 41], Agriculture [28, 29], Chaos [7, 10, 22], Bioengineering [14, 15], and so on.

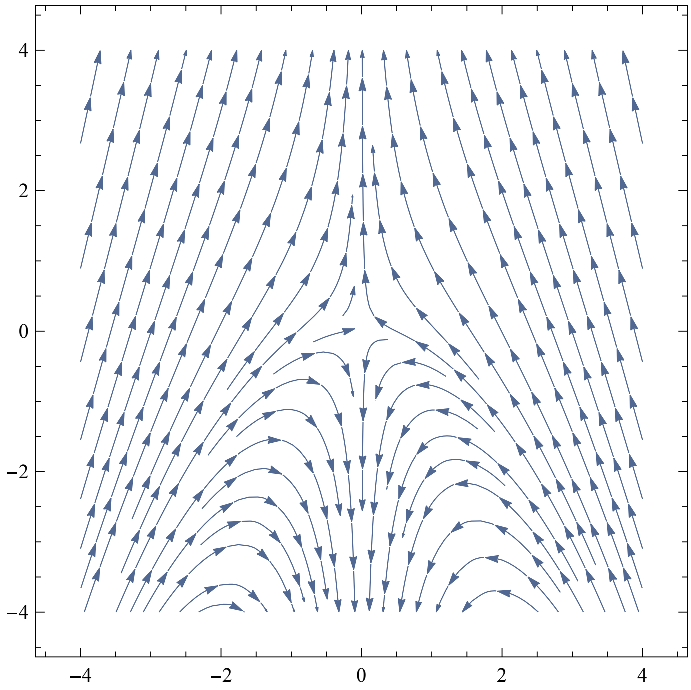

Fig. 2 Vector-field of $\dot{x} = -x$, $\dot{y} = x^2 + y$

Table 1 DDE versus ODE

Property	ODE	DDE
Equation	$\dot{x} = f(x)$	$\dot{x} = f(x, x_\tau)$
Initial condition	$x(0) = x_0$	$x(t) = x_0(t), -\tau \le t \le 0$
Dimension	One-dimensional	Infinite dimensional
Memory	Cannot model memory	Can model memory
Equilibrium x_*	$f(x_*) = 0$	$f(x_*, x_*) = 0$
Characteristic equation	Polynomial $\lambda = f'(x_*)$	Transcendental equation $\lambda = \partial_1 f(x_*, x_*)$ $+\partial_2 f(x_*, x_*)e^{-\lambda\tau}$
Chaos	Doesn't exhibit	Can exhibit

2.4.1 DDE Versus ODE

In Table 1, we discuss the difference between DDE and ODE. Note that $\partial_1 f$ and $\partial_2 f$ are partial derivatives of function f with respect to first and second variables, respectively.

2.5 Fractional Derivative

We consider the Caputo fractional derivative defined by [39, 40]

$$\begin{aligned} D^{\mu} f(t) &= \frac{d^m}{dt^m} f(t), \qquad \mu = m \\ &= I^{m-\mu} \frac{d^m f(t)}{dt^m}, \quad m-1 < \mu < m, \quad m \in I\!N. \end{aligned} \tag{4}$$

The Riemann–Liouville fractional integral of order $\alpha > 0$ is

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0.$$

2.6 Stability of Fractional-Order Systems

Consider the fractional-order system

$$\begin{aligned} D^{\alpha_1} x_1 &= f_1(x_1, x_2, \cdots, x_n), \\ D^{\alpha_2} x_2 &= f_2(x_1, x_2, \cdots, x_n), \\ &\vdots \\ D^{\alpha_n} x_n &= f_n(x_1, x_2, \cdots, x_n), \end{aligned} \tag{5}$$

where $0 < \alpha_i < 1$ are fractional orders. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, then the system (5) is called as a commensurate order system otherwise incommensurate order system. A point $p = \left(x_1^*, x_2^*, \cdots, x_n^*\right)$ is called an equilibrium point of system (5) if $f_i(p) = 0$ for each $i = 1, 2, \cdots, n$.

2.6.1 Asymptotic Stability of the Commensurate Fractional Ordered System

Theorem 2.2 ([35, 43]) *Consider $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_n$ in (5). An equilibrium point p of the system (5) is locally asymptotically stable if all the eigenvalues of the Jacobian matrix $J = \left(\partial_j f_i\right)$ evaluated at p satisfy the following condition:*

$$|arg(Eig(J))| > \alpha\pi/2. \tag{6}$$

2.6.2 Asymptotic Stability of the Incommensurate Fractional Ordered System

Theorem 2.3 ([26, 44]) *Consider the incommensurate fractional ordered system given by (5). Let* $\alpha_i = v_i/u_i,\ (u_i, v_i) = 1,\ u_i,\ v_i$ *be positive integers. Define* M *to be the least common multiple of* u_i*'s. Define* $\Delta(\lambda) = diag\left(\left[\lambda^{M\alpha_1},\ \lambda^{M\alpha_2}, \cdots, \lambda^{M\alpha_n}\right]\right) - J$*. Then* p *is locally asymptotically stable if all the roots of the equation* $det\ (\Delta(\lambda)) = 0$ *satisfy the condition* $|arg(\lambda)| > \pi/(2M)$*.*

This condition is equivalent to the following inequality:

$$\frac{\pi}{2M} - \min_i \{|arg(\lambda_i)|\} < 0. \tag{7}$$

Thus, an equilibrium point p of the system (5) is asymptotically stable if the condition (7) is satisfied. The term $\frac{\pi}{2M} - min_i\ \{|arg(\lambda_i)|\}$ is called as the instability measure for equilibrium points in fractional-order systems (IMFOS). Hence, a necessary condition for fractional-order system (5) to exhibit chaotic attractor is [44]

$$\text{IMFOS} \geq 0. \tag{8}$$

Note that the condition (8) is not sufficient for chaos to exist.

2.7 Fractional DDEs

If the dynamical system involves fractional derivative as well as delay, then it will be very useful in modeling real systems. The stability of linear time-invariant fractional delay systems (LTIFDS) has been studied by many researchers [17, 26, 30, 31, 34, 38].

3 Review of Our Work

In 2010, Prof. Richard Magin (Department of Bioengineering, University of Illinois, Chicago) and Prof. Dumitru Baleanu (Cankaya University, Ankara) discussed with Prof. Varsha Gejji regarding the generalization of Bloch equation

$$\frac{d\ \vec{M}}{dt} = \gamma\ \vec{M} \times \vec{B} - \frac{(M_z - M_0)}{T_1}\widehat{i_z} - \frac{\left(M_x\widehat{i_x} + M_y\widehat{i_y}\right)}{T_2},$$

to include fractional order as well as delay. The equation models the nuclear magnetic resonance (NMR)—the phenomena underlying magnetic resonance imaging (MRI).

The model we developed was

$$\begin{aligned}
T^{-\alpha} D^{\alpha} M_x(t) &= \tilde{\omega_0} M_y\,(t-\tau) - \frac{M_x\,(t-\tau)}{T_2},\\
T^{-\alpha} D^{\alpha} M_y(t) &= -\tilde{\omega_0} M_x\,(t-\tau) - \frac{M_y\,(t-\tau)}{T_2},\\
T^{-\alpha} D^{\alpha} M_z(t) &= \frac{M_0 - M_z\,(t-\tau)}{T_1},\\
M_x(t) = 0, \quad & M_y(t) = 100,\ M_z(t) = 0,\ \text{for}\, t \le 0,
\end{aligned}$$

where
$\tilde{\omega_0} = \frac{\omega_0}{T^{\alpha-1}} = (\omega_0 T)T^{-\alpha}$, $\frac{1}{T_1'} = \frac{1}{T^{\alpha-1}T_1} = \frac{T}{T_1}T^{-\alpha}$, $\frac{1}{T_2'} = \frac{1}{T^{\alpha-1}T_2} = \frac{T}{T_2}T^{-\alpha}$.

Though the system was linear, the exact solution was not possible. The numerical method was also not available in the literature. So, we first developed the numerical method [11].

We used this numerical method to analyze linear fractional-order Bloch equation [14] and its generalization involving extended delay [15]. Further, we discussed transient chaos in nonlinear Bloch equation [16] and chaos in the same system involving delay [6].

We also generalized some chaotic fractional-order dynamical systems to include delay [10, 22]. We observed that the chaotic nature of the system depends on the value of delay parameter.

We also observed some interesting things. In [7], we discussed fractional-order Ucar system

$$D^{\alpha} x(t) = \delta x(t-\tau) - \varepsilon\,[x(t-\tau)]^3\,. \tag{9}$$

The **two-scroll** attractor is observed in the system for the range of fractional order $0.5 < \alpha \le 1$. For the range $0.2 \le \alpha \le 0.5$, the same system shows **one-scroll** attractor.

Recently, we used the new iterative method (NIM) to improve this numerical method [23]. The new method called NPCM is proved to be more time efficient than FAM.

The next step was to analyze stability. In the paper [8], the stability of $D^{\alpha} y(t) = af\,(y(t-\tau)) - by(t)$ is discussed.

A more general case is studied in [9].

Theorem 3.1 ([9]) *Suppose x^* is an equilibrium solution of the generalized delay differential equation $D^{\alpha} x(t) = g\,(x(t), x(t-\tau))\,,\ 0 < \alpha \le 1$ and $a = \partial_1 g(x^*, x^*)$, $b = \partial_2 g(x^*, x^*)$.*

1. *If $b \in (-\infty, -|a|)$ then the stability region of x^* in (τ, a, b) parameter space is located between the plane $\tau = 0$ and*

$$\tau_1(0) = \frac{\arccos\left(\frac{\left(a\cos\left(\frac{\alpha\pi}{2}\right)+\sqrt{b^2-a^2\sin^2\left(\frac{\alpha\pi}{2}\right)}\right)\cos\frac{\alpha\pi}{2}-a}{b}\right)}{\left(a\cos\left(\frac{\alpha\pi}{2}\right)+\sqrt{b^2-a^2\sin^2\left(\frac{\alpha\pi}{2}\right)}\right)^{1/\alpha}}.$$

The equation undergoes Hopf bifurcation at this value.

2. *If* $b \in (-a, \infty)$ *then* x^* *is unstable for any* $\tau \geq 0$.
3. *If* $b \in (a, -a)$ *and* $a < 0$ *then* x^* *is stable for any* $\tau \geq 0$.

4 Multi-term Case

Multi-term equations are having physical importance. For example, application of multi-term fractional-order equations to model "Oxygen delivery through capillary to tissues" is given by Srivastava and Rai [42].

A cylindrical capillary of radius R, containing solute, is considered for the physical modeling. The rate of consumption by surrounding tissue is assumed as $k(r, z, t)$. The term $D_t^\alpha C$ indicates the subdiffusion process. The longitudinal diffusion is $D_t^\beta C$. The net diffusion of oxygen to tissues is $D_t^\alpha C - \tau D_t^\beta C$, where τ is the time lag in concentration of oxygen along the z-axis and $0 < \beta < \alpha \leq 1$.

The general equation for conveying oxygen from the capillary to the surrounding tissue is

$$D_t^\alpha C - \tau D_t^\beta C = \nabla\left(d \cdot \nabla C\right) - k,$$

where $C(r, z, t)$ is concentration of oxygen, d is diffusion coefficient of oxygen.

5 2-Term FDDE

We discuss the following 2-term FDDE:

$$D^\alpha x + cD^\beta x = f\left(x, x_\tau\right), \tag{10}$$

$$x(t) = \phi(t), \quad t \leq 0, \tag{11}$$

where $0 < \alpha < \beta \leq 1$, $\tau > 0$, and f is C^1 on $\mathbb{R}^2$.

Existence and uniqueness theorems for multi-term FDDEs are recently discussed by Choudhary and Daftardar-Gejji [18].

If x_* is an equilibrium (steady state) solution of (10) then $D^\alpha x_* = D^\beta x_* = 0$. Thus, the equilibrium points are solutions of $f(x_*, x_*) = 0$.

The characteristic equation for this system in the neighborhood of an equilibrium x_* is given by

$$\lambda^\alpha + c\lambda^\beta = a + bexp(-\lambda\tau), \tag{12}$$

where $a = \partial_1 f(x_*, x_*)$ and $b = \partial_2 f(x_*, x_*)$ are partial derivatives of function f with respect to first and second variables, respectively, evaluated at (x_*, x_*).

6 Stability Analysis

An equilibrium x_* is asymptotically stable (respectively, unstable) if all (respectively, at least one of) the roots of characteristic equation (12) have negative (respectively, positive) real part. Thus, change in stability can occur if the eigenvalue λ crosses the imaginary axis $\lambda = \iota v$.

Substituting $\lambda = \iota v$ in the characteristic equation, we get

$$(\iota v)^\alpha + c\,(\iota v)^\beta = a + bexp(-\iota v\tau). \tag{13}$$

Equivalently,

$$v^\alpha \cos\left(\frac{\alpha\pi}{2}\right) + cv^\beta \cos\left(\frac{\beta\pi}{2}\right) - a = b\cos(v\tau) \tag{14}$$

$$v^\alpha \sin\left(\frac{\alpha\pi}{2}\right) + cv^\beta \sin\left(\frac{\beta\pi}{2}\right) = -b\sin(v\tau). \tag{15}$$

Equation (14) implies

$$\tau = \frac{1}{v}\left[2n\pi \pm \arccos\left(\frac{v^\alpha\cos(\alpha\pi/2) + cv^\beta\cos(\beta\pi/2) - a}{b}\right)\right].$$

We define critical values for τ as

$$\tau_+(n) = \frac{1}{v}\left[2n\pi + \arccos\left(\frac{v^\alpha\cos(\alpha\pi/2) + cv^\beta\cos(\beta\pi/2) - a}{b}\right)\right]$$

and

$$\tau_-(n) = \frac{1}{v}\left[2n\pi - \arccos\left(\frac{v^\alpha\cos(\alpha\pi/2) + cv^\beta\cos(\beta\pi/2) - a}{b}\right)\right].$$

Further, squaring and adding Eqs. (14) and (15), we get

$$\begin{aligned} &v^{2\alpha} + c^2v^{2\beta} + 2cv^{\alpha+\beta}\cos\left((\beta-\alpha)\pi/2\right) - 2av^\alpha\cos\left(\alpha\pi/2\right) \\ &-2acv^\beta\cos\left(\beta\pi/2\right) + a^2 - b^2 = 0. \end{aligned} \tag{16}$$

We have to solve equation (16) and substitute the obtained values in $\tau_{\pm}(n)$ to find critical values of delay τ.

6.1 Stable Region

Let us write $\lambda = u + \iota v$. If $du/d\tau$ is negative on one critical curve and positive on the nearest one then the stability region of equilibrium x_* is located between these two curves.

We differentiate characteristic equation (12) with respect to τ to obtain

$$\begin{aligned}\frac{d\lambda}{d\tau} &= \frac{-\lambda b exp(-\lambda\tau)}{\alpha\lambda^{\alpha-1} + c\beta\lambda^{\beta-1} + b\tau exp(-\lambda\tau)} \\ &= \frac{-\lambda\left(\lambda^{\alpha} + c\lambda^{\beta} - a\right)}{\alpha\lambda^{\alpha-1} + c\beta\lambda^{\beta-1} + \tau\left(\lambda^{\alpha} + c\lambda^{\beta} - a\right)}. \end{aligned} \tag{17}$$

Consider

$$\begin{aligned}\frac{d\lambda}{d\tau}|_{u=0} &= \frac{-\iota v\left((\iota v)^{\alpha} + c(\iota v)^{\beta} - a\right)}{\alpha(\iota v)^{\alpha-1} + c\beta(\iota v)^{\beta-1} + \tau\left((\iota v)^{\alpha} + c(\iota v)^{\beta} - a\right)} \\ &= \frac{z_1 + \iota z_2}{z_3 + \iota z_4} \\ &= \frac{z_1 z_3 + z_2 z_4}{z_3^2 + z_4^2} + \iota\frac{z_2 z_3 - z_1 z_4}{z_3^2 + z_4^2}, \end{aligned} \tag{18}$$

where $z_1 = v\left(v^{\alpha}\sin(\alpha\pi/2) + cv^{\beta}\sin(\beta\pi/2)\right)$, $z_2 = v\left(a - v^{\alpha}\cos(\alpha\pi/2) - cv^{\beta}\cos(\beta\pi/2)\right)$, $z_3 = \alpha v^{\alpha-1}\cos((\alpha-1)\pi/2) + c\beta v^{\beta-1}\cos((\beta-1)\pi/2) + \tau v^{\alpha}\cos(\alpha\pi/2) + \tau c v^{\beta}\cos(\beta\pi/2) - \tau a$ and $z_4 = \alpha v^{\alpha-1}\sin((\alpha-1)\pi/2) + c\beta v^{\beta-1}\sin((\beta-1)\pi/2) + \tau v^{\alpha}\sin(\alpha\pi/2) + \tau c v^{\beta}\sin(\beta\pi/2)$.

Along critical curve,

$$\begin{aligned}\frac{du}{d\tau} &= Re\left(\frac{d\lambda}{d\tau}|_{u=0}\right) \\ &= \frac{z_1 z_3 + z_2 z_4}{z_3^2 + z_4^2}. \end{aligned} \tag{19}$$

The sign of $\frac{du}{d\tau}$ is solely decided by

$$\begin{aligned} z_1 z_3 + z_2 z_4 = {} & \alpha v^{2\alpha} + c^2\beta v^{2\beta} - a\alpha v^{\alpha}\cos(\alpha\pi/2) \\ & -ac\beta v^{\beta}\cos(\beta\pi/2) + cv^{\alpha+\beta}\beta\cos((\beta-\alpha)\pi/2) \\ & +cv^{\alpha+\beta}\alpha\cos((\beta-\alpha)\pi/2) \end{aligned} \tag{20}$$

7 Numerical Solution

In this section, we describe the numerical algorithm based on the new predictor–corrector method (NPCM) [24] developed by Daftardar-Gejji, Sukale and Bhalekar.

Applying I^β to Eq. (10), we get

$$x(t) = x_0\left(1+\frac{t^{\beta-\alpha}}{c\Gamma(\beta-\alpha+1)}\right)-\frac{1}{c\Gamma(\beta-\alpha)}\int_0^t (t-s)^{\beta-\alpha-1}x(s)ds \\ +\frac{1}{c\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f\left(x(s), x(s-\tau)\right) ds, \tag{21}$$

where $x_0 = x(0)$.

We solve (21) in the interval $[-\tau, T]$. The interval $[-\tau, T]$ is divided into $k+N$ subintervals such that the step-size $h = T/N = \tau/k$. The nodes are described by $\{t_n = nh : n = -k, -k+1, \cdots, N\}$. We denote $x_j = x(t_j)$, a numerical approximation. If $j \le 0$ then $x_j = \phi(t_j)$ and $x(t_j - \tau) = x(jh - kh) = x_{j-k}$.

Using product trapezoidal quadrature formula, Eq. (21) can be discretized as

$$x_{n+1} = x_0\left(1+\frac{t_{n+1}^{\beta-\alpha}}{c\Gamma(\beta-\alpha+1)}\right) \\ -\frac{1}{c}\frac{h^{\beta-\alpha}}{\Gamma(\beta-\alpha+2)}\left(x_{n+1}+\sum_{j=0}^{n} a_{j,n+1}x_j\right) \\ +\frac{1}{c}\frac{h^{\beta}}{\Gamma(\beta+2)}\left(f\left(x_{n+1}, x_{n+1-k}\right)+\sum_{j=0}^{n} b_{j,n+1} f\left(x_j, x_{j-k}\right)\right), \tag{22}$$

where

$$a_{j,n+1} = \begin{cases} n^{\beta-\alpha+1}-(n-\beta+\alpha)(n+1)^{\beta-\alpha}, & \text{if } j=0,\\ (n-j+2)^{\beta-\alpha+1}+(n-j)^{\beta-\alpha+1} & \\ -2(n-j+1)^{\beta-\alpha+1}, & \text{if } 1\le j\le n,\\ 1, & \text{if } j=n+1. \end{cases} \tag{23}$$

$$b_{j,n+1} = \begin{cases} n^{\beta+1}-(n-\beta)(n+1)^{\beta}, & \text{if } j=0,\\ (n-j+2)^{\beta+1}+(n-j)^{\beta+1} & \\ -2(n-j+1)^{\beta-\alpha+1}, & \text{if } 1\le j\le n,\\ 1, & \text{if } j=n+1. \end{cases} \tag{24}$$

The three-term approximation of New Iterative Method gives the following predictor–corrector method:

Predictor Terms

$$x_{n+1}^{p} = x_0\left(1+\frac{t_{n+1}^{\beta-\alpha}}{c\Gamma(\beta-\alpha+1)}\right) - \frac{1}{c}\frac{h^{\beta-\alpha}}{\Gamma(\beta-\alpha+2)}\sum_{j=0}^{n} a_{j,n+1}x_j$$
$$+\frac{1}{c}\frac{h^{\beta}}{\Gamma(\beta+2)}\sum_{j=0}^{n} b_{j,n+1} f\left(x_j, x_{j-k}\right), \qquad (25)$$

$$z_{n+1}^{p} = -\frac{1}{c}\frac{h^{\beta-\alpha}}{\Gamma(\beta-\alpha+2)}x_{n+1}^{p} + \frac{1}{c}\frac{h^{\beta}}{\Gamma(\beta+2)} f\left(x_{n+1}^{p}, x_{n+1-k}\right). \qquad (26)$$

Corrector Term

$$x_{n+1}^{c} = x_{n+1}^{p} - \frac{1}{c}\frac{h^{\beta-\alpha}}{\Gamma(\beta-\alpha+2)}\left(x_{n+1}^{p}+z_{n+1}^{p}\right)$$
$$+\frac{1}{c}\frac{h^{\beta}}{\Gamma(\beta+2)} f\left(x_{n+1}^{p}+z_{n+1}^{p}, x_{n+1-k}\right).$$

8 Example

Consider the 2-term FDDE

$$D^{\alpha}x + D^{\beta}x = \frac{4x_{\tau}}{1+x_{\tau}^{9}} - 2x, \qquad (27)$$
$$x(t) = 0.5, \quad t \le 0, \qquad (28)$$

where $0 < \alpha < \beta \le 1$ and $\tau > 0$. If $\alpha = \beta$ then Eq. (27) gets reduced to that studied by Bhalekar and Daftardar-Gejji [11]. We set $\alpha = 0.8 = 4/5$ and $\beta = 0.98 = 49/50$. Let $M = LCM(5, 50) = 50$.

8.1 *Equilibrium Points and Stability*

Equilibrium points of this system are $x_{**} = 0$ and $x_* = 1$.

Stability at $\tau = 0$: The characteristic equation in the neighborhood of $x_* = 1$ is $\lambda^{\alpha} + \lambda^{\beta} = -9$ or $s^{40} + s^{49} = -9$, where $s = \lambda^{1/M}$.

Therefore, $\min |Arg(s)| = 0.069 > \pi/(2M)$. This shows that $x_* = 1$ is asymptotically stable at $\tau = 0$. Similarly, it can be checked that $x_{**} = 0$ is not stable at $\tau = 0$.

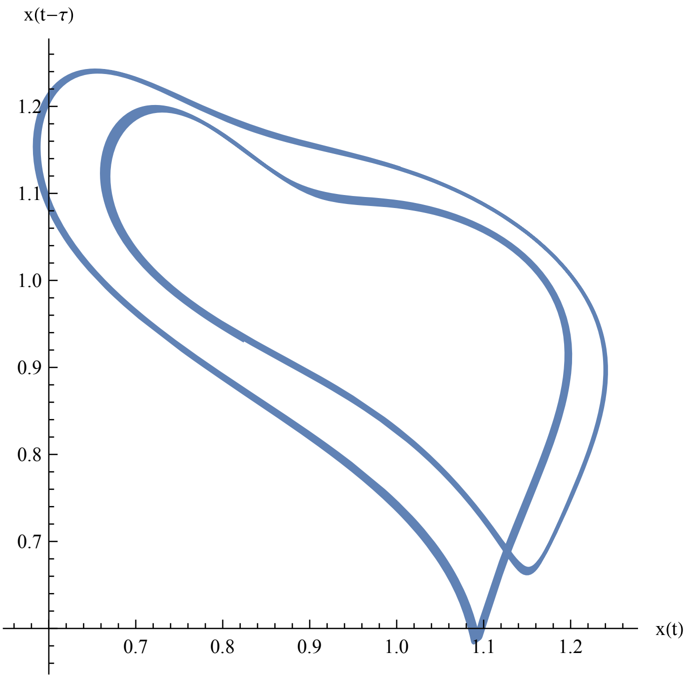

Fig. 3 2-cycle at $\tau = 1.5$

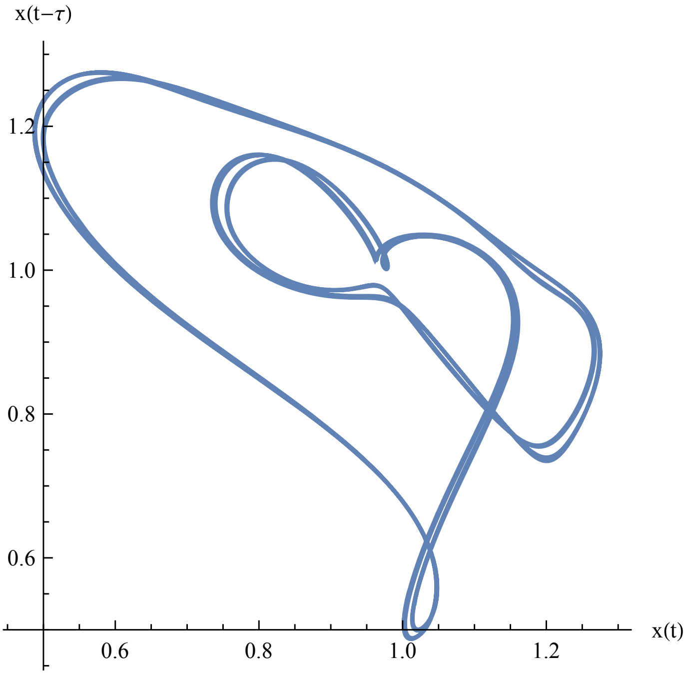

Fig. 4 Period doubling $\tau = 2.1$

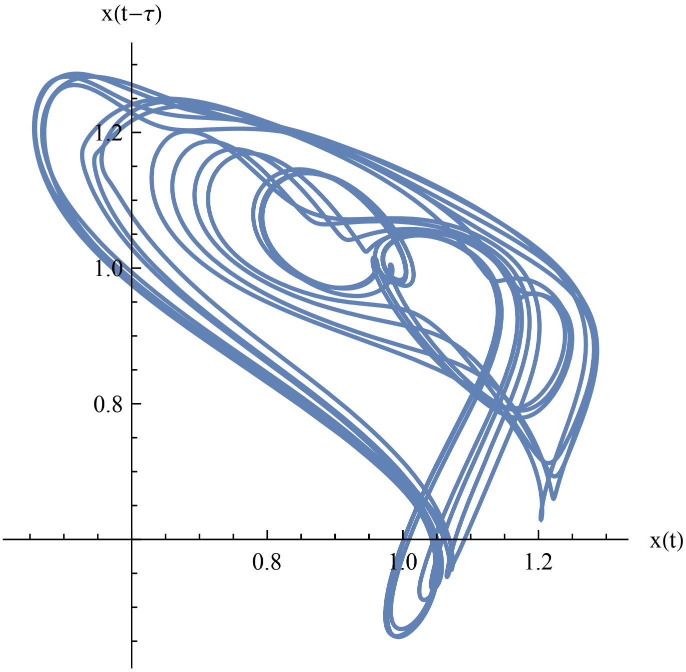

Fig. 5 Chaos $\tau = 2.2$

Stability for $\tau > 0$: If we solve (16) in the neighborhood of $x_* = 1$ then we get $v = 3.70992$. Substituting in (20), we get $z_1 z_3 + z_2 z_4 = 38.5819$. Therefore $\frac{du}{d\tau} > 0$ on all critical curves. The critical value $\tau_+(0) = 0.5426$, in this case. Hence, the equilibrium $x_* = 1$ is stable between $0 \leq \tau < 0.5426$.

The necessary (but not sufficient) condition for existence of chaos in this system is $\tau > 0.5426$. In Fig. 3, the 2-cycle is observed for $\tau = 1.5$. If we increase the value of τ, then we get period doubling bifurcation (cf. Fig. 4) leading to chaos (cf. Fig. 5).

9 Conclusions

In this chapter, we have taken a review of fractional-order dynamical systems and delay differential equations. We have analyzed a nonlinear autonomous delay differential equation involving two fractional-order derivatives. The stability of equilibrium points is discussed using analytical result. An illustrative example is provided to verify the proposed theory. It is observed that the chaos cannot occur in the stability region of the given equation.

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References

1. Agrawal, O.P.: Generalized variational problems and Euler-Lagrange equations. Comput. Math. Appl. **59**, 1852–1864 (2010)
2. Ahmad, B., Nieto, J.J.: Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **58**(9), 1838–1843 (2009)
3. Alligood, K.T., Sauer, T.D., Yorke, J.A.: Chaos: An Introduction to Dynamical Systems. Springer, New York (2008)
4. Babakhani, A., Daftardar-Gejji, V.: Existence of positive solutions of nonlinear fractional differential equations. J. Math. Anal. Appl. **278**(2), 434–442 (2003)
5. Babakhani, A., Daftardar-Gejji, V.: Existence of positive solutions for N-term non-autonomous fractional differential equations. Positivity **9**, 193–206 (2005)
6. Baleanu, D., Magin, R., Bhalekar, S., Daftardar-Gejji, V.: Chaos in the fractional order nonlinear Bloch equation with delay. Commun. Nonlinear Sci. Numer. Simul. **25**(1), 41–49 (2015)
7. Bhalekar, S.: Dynamical analysis of fractional order Ucar prototype delayed system. Signals Image Video Process. **6**(3), 513–519 (2012)
8. Bhalekar, S.: Stability analysis of a class of fractional delay differential equations. Pramana **81**(2), 215–224 (2013)
9. Bhalekar, S.: Stability and bifurcation analysis of a generalized scalar delay differential equation. Chaos **26**(8), 084306 (2016)
10. Bhalekar, S., Daftardar-Gejji, V.: Fractional ordered Liu system with time-delay. Commun. Nonlinear Sci. Numer. Simul. **15**(8), 2178–2191 (2010)
11. Bhalekar, S., Daftardar-Gejji, V.: A predictor-corrector scheme for solving nonlinear delay differential equations of fractional order. J. Fract. Calc. Appl. **1**(5), 1–9 (2011)
12. Bhalekar, S., Daftardar-Gejji, V.: Solving fractional-order logistic equation using a new iterative method. Int. J. Differ. Equ. **2012**, Article number 975829 (2012)
13. Bhalekar, S., Daftardar-Gejji, V.: Existence and uniqueness theorems for fractional differential equations: A new approach. In: Daftardar-Gejji, V., (ed.) Fractional Calculus: Theory and Applications. Narosa Publishing House, New Delhi (2013). ISBN 978-81-8487-333-7
14. Bhalekar, S., Daftardar-Gejji, V., Baleanu, D., Magin, R.: Fractional Bloch equation with delay. Comput. Math. Appl. **61**(5), 1355–1365 (2011)
15. Bhalekar, S., Daftardar-Gejji, V., Baleanu, D., Magin, R.: Generalized fractional order Bloch equation with extended delay. Int. J. Bifurc. Chaos **22**(4), 1250071 (2012)
16. Bhalekar, S., Daftardar-Gejji, V., Baleanu, D., Magin, R.: Transient chaos in fractional Bloch equations. Comput. Math. Appl. **64**(10), 3367–3376 (2012)
17. Chen, Y., Moore, K.L.: Analytical stability bound for a class of delayed fractional-order dynamic systems. Nonlinear Dyn. **29**(1), 191–200 (2002)
18. Choudhari, S., Daftardar-Gejji, V.: Existence uniqueness theorems for multi-term fractional delay differential equations. Fract. Calc. Appl. Anal. **5**(18), 1113–1127 (2015)
19. Daftardar-Gejji, V.: Positive solutions of a system of non-autonomous fractional differential equations. J. Math. Anal. Appl. **302**(1), 56–64 (2005)
20. Daftardar-Gejji, V., Babakhani, A.: Analysis of a system of fractional differential equations. J. Math. Anal. Appl. **293**(2), 511–522 (2004)
21. Daftardar-Gejji, V., Jafari, H.: Adomian decomposition: a tool for solving a system of fractional differential equations. J. Math. Anal. Appl. **301**(2), 508–518 (2005)

22. Daftardar-Gejji, V., Bhalekar, S., Gade, P.: Dynamics of fractional ordered Chen system with delay. Pramana-J. Phys. **79**(1), 61–69 (2012)
23. Daftardar-Gejji, V., Sukale, Y., Bhalekar, S.: A new predictorcorrector method for fractional differential equations. Appl. Math. Comput. **244**, 158–182 (2014)
24. Daftardar-Gejji, V., Sukale, Y., Bhalekar, S.: Solving fractional delay differential equations: A new approach. Fract. Calc. Appl. Anal. **18**(2), 400–418 (2015)
25. Delbosco, D., Rodino, L.: Existence and uniqueness for a nonlinear fractional differential equation. J. Math. Anal. Appl. **204**, 609–625 (1996)
26. Deng, W., Li, C., Lü, J.: Stability analysis of linear fractional differential system with multiple time delays. Nonlinear Dyn. **48**, 409–416 (2007)
27. Diethelm, K., Ford, N.J., Freed, A.D.: A predictorcorrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**, 3–22 (2002)
28. Feliu, V., Rivas, R., Castillo, F.J.: Fractional robust control to delay changes in main irrigation canals. In: Proceedings of the 16th International Federation of Automatic Control World Congress. Czech Republic, Prague (2005)
29. Feliu, V., Rivas, R., Castillo, F.: Fractional order controller robust to time delay variations for water distribution in an irrigation main canal pool. Comput. Electron. Agric. **69**(2), 185–197 (2009)
30. Hotzel, R.: Summary: some stability conditions for fractional delay systems. J. Math. Syst. Estim. Control **8**, 499–502 (1998)
31. Hwang, C., Cheng, Y.C.: A numerical algorithm for stability testing of fractional delay systems. Automatica **42**, 825–831 (2006)
32. Jafari, H., Daftardar-Gejji, V.: Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method. Appl. Math. Comput. **180**(2), 700–706 (2006)
33. Kiryakova, V.: A brief story about the operators of the generalized fractional calculus. Fract. Calc. Appl. Anal. **11**(2), 203–220 (2008)
34. Lazarevic, M.P., Debeljkovic, D.L.: Finite time stability analysis of linear autonomous fractional order systems with delayed state. Asian J. Control **7**(4), 440–447 (2005)
35. Matignon, D.: Stability results for fractional differential equations with applications to control processing. In: Computational Engineering in Systems and Application multiconference, vol. 2, pp. 963–968, IMACS, IEEE-SMC Proceedings. Lille, France (1996)
36. Meiss, J.D.: Differential Dynamical Systems. SIAM, Philadelphia (2007)
37. Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, D.Y., Feliu, V.: Fractional-Order Systems and Controls: Fundamentals and Applications. Springer, London (2010)
38. Moornani, K., Haeri, M.: On robust stability of LTI fractional-order delay systems of retarded and neutral type. Automatica **46**, 362–368 (2010)
39. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
40. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
41. Si-Ammour, A., Djennoune, S., Bettayeb, M.: A sliding mode control for linear fractional systems with input and state delays. Commun. Nonlinear Sci. Numer. Simul. **14**, 2310–2318 (2009)
42. Srivastava, V., Rai, K.N.: A multi-term fractional diffusion equation for oxygen delivery through a capillary to tissues. Math. Comput. Model. **51**, 616–624 (2010)
43. Tavazoei, M.S., Haeri, M.: Regular oscillations or chaos in a fractional order system with any effective dimension. Nonlinear Dyn. **54**(3), 213–222 (2008)
44. Tavazoei, M.S., Haeri, M.: Chaotic attractors in incommensurate fractional order systems. Phys. D **237**, 2628–2637 (2008)

Stability Analysis of Two-Dimensional Incommensurate Systems of Fractional-Order Differential Equations

Oana Brandibur and Eva Kaslik

Abstract Recently obtained necessary and sufficient conditions for the asymptotic stability and instability of the null solution of a two-dimensional autonomous linear incommensurate fractional-order dynamical system with Caputo derivatives are reviewed and extended. These theoretical results are then applied to investigate the stability properties of a two-dimensional fractional-order conductance-based neuronal model. Moreover, the occurrence of Hopf bifurcations is also discussed, choosing the fractional orders as bifurcation parameters. Numerical simulations are also presented to illustrate the theoretical results.

1 Introduction

Due to the fact that fractional-order derivatives reflect both memory and hereditary properties, numerous results reported in the past decades have proven that fractional-order systems provide more realistic results in practical applications [7, 12, 15, 16, 24] than their integer-order counterparts.

Regarding the qualitative theory of fractional-order systems, stability analysis is one of the most important research topics. The main results concerning stability properties of fractional-order systems have been recently surveyed in [21, 31]. It is worth noting that most investigations have been accomplished for linear autonomous commensurate fractional-order systems. In this case, the well-known Matignon's sta-

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bility theorem [25] has been recently generalized in [32]. Analogues of the classical Hartman–Grobman theorem, i.e., linearization theorems for fractional-order systems, have been recently reported in [20, 34].

However, when it comes to incommensurate fractional-order systems, it is worth noticing that their stability analysis has received significantly less attention than their commensurate counterparts. Linear incommensurate fractional-order systems with rational orders have been analyzed in [28]. Oscillations in two-dimensional incommensurate fractional-order systems have been investigated in [8, 30]. BIBO stability of systems with irrational transfer functions has been recently investigated in [33]. Lyapunov functions were employed to derive sufficient stability conditions for fractional-order two-dimensional nonlinear continuous-time systems [17].

Following these recent trends in the theory of fractional-order differential equations, necessary and sufficient conditions for the stability/instability of linear autonomous two-dimensional incommensurate fractional-order systems have been explored in [4, 5]. In the first paper [4], stability properties of two-dimensional systems composed of a fractional-order differential equation and a classical first-order differential equation have been investigated. A generalization of these results has been presented in [5], for the case of general two fractional-order systems with Caputo derivatives. For fractional orders $0 < q_1 < q_2 \leq 1$, necessary and sufficient conditions for the $\mathcal{O}(t^{-q_1})$-asymptotic stability of the trivial solutions have been obtained, in terms of the determinant of the linear system's matrix, as well as the elements a_{11} and a_{22} of its main diagonal. Sufficient conditions have also been explored which guarantee the stability and instability of the system, regardless of the choice of fractional orders $q_1 < q_2$. In this work, our first aim is to further extend the results presented in [5] for any $q_1, q_2 \in (0, 1]$, by exploring certain symmetries in the characteristic equation associated to our stability problem. This leads to improved fractional-order-independent sufficient conditions for stability and instability.

As an application, an investigation of the stability properties of a two-dimensional fractional-order conductance-based neuronal model is presented, considering the particular case of a FitzHugh–Nagumo neuronal model. Experimental results concerning biological neurons [1, 23] justify the formulation of neuronal dynamics using fractional-order derivatives. Fractional-order membrane potential dynamics are known to introduce capacitive memory effects [35], proving to be necessary for reproducing the electrical activity of neurons. Moreover, [11] gives the index of memory as a possible physical interpretation of the order of a fractional derivative, which further justifies its use in mathematical models arising from neuroscience.

2 Preliminaries

The main theoretical results of fractional calculus are comprehensively covered in [18, 19, 29]. In this paper, we are concerned with the Caputo derivative, which is known to be more applicable to real-world problems, as it only requires initial conditions given in terms of integer-order derivatives.

Definition 1 For a continuous function h, with $h' \in L^1_{loc}(\mathbb{R}^+)$, the Caputo fractional-order derivative of order $q \in (0, 1)$ of h is defined by

$$^cD^q h(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} h'(s) ds.$$

Consider the n-dimensional fractional-order system with Caputo derivatives

$$^cD^{\mathbf{q}}\mathbf{x}(t) = f(t, \mathbf{x}) \tag{1}$$

with $\mathbf{q} = (q_1, q_2, ..., q_n) \in (0, 1)^n$ and $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function on the whole domain of definition and Lipschitz-continuous with respect to the second variable, such that

$$f(t, 0) = 0 \quad \text{for any } t \geq 0.$$

Let $\varphi(t, x_0)$ denote the unique solution of (1) satisfying the initial condition $x(0) = x_0 \in \mathbb{R}^n$. The existence and uniqueness of the initial value problem associated to system (1) is guaranteed by the properties of the function f stated above [9].

In general, the asymptotic stability of the trivial solution of system (1) is not of exponential type [6, 14], because of the presence of the memory effect. Thus, a special type of non-exponential asymptotic stability concept has been defined for fractional-order differential equations [22], called Mittag-Leffler stability. In this paper, we are concerned with $\mathcal{O}(t^{-\alpha})$-asymptotic stability, which reflects the algebraic decay of the solutions.

Definition 2 The trivial solution of (1) is called *stable* if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $x_0 \in \mathbb{R}^n$ satisfying $\|x_0\| < \delta$ we have $\|\varphi(t, x_0)\| \leq \varepsilon$ for any $t \geq 0$.

The trivial solution of (1) is called *asymptotically stable* if it is stable and there exists $\rho > 0$ such that $\lim_{t\to\infty} \varphi(t, x_0) = 0$ whenever $\|x_0\| < \rho$.

Let $\alpha > 0$. The trivial solution of (1) is called $\mathcal{O}(t^{-\alpha})$*-asymptotically stable* if it is stable and there exists $\rho > 0$ such that for any $\|x_0\| < \rho$ one has:

$$\|\varphi(t, x_0)\| = \mathcal{O}(t^{-\alpha}) \quad \text{as } t \to \infty.$$

3 Stability and Instability Regions

Let us consider the following two-dimensional linear autonomous incommensurate fractional-order system:

$$\begin{cases} ^cD^{q_1}x(t) = a_{11}x(t) + a_{12}y(t) \\ ^cD^{q_2}y(t) = a_{21}x(t) + a_{22}y(t) \end{cases} \tag{2}$$

where $A = (a_{ij})$ is a real two-dimensional matrix and $q_1, q_2 \in (0, 1)$ are the fractional orders of the Caputo derivatives. Using Laplace transform tools, the following characteristic function is obtained:

$$\Delta_A(s) = \det\left(\text{diag}(s^{q_1}, s^{q_2}) - A\right) = s^{q_1+q_2} - a_{11}s^{q_2} - a_{22}s^{q_1} + \det(A).$$

where s^{q_1} and s^{q_2} represents the principal values (first branches) of the corresponding complex power functions [10].

Based on the Final Value Theorem and asymptotic expansion properties of the Laplace transform [3, 4, 10], the following necessary and sufficient conditions for the global asymptotic stability of system (2) have been recently obtained [5]:

Theorem 1 *1. Denoting $q = \min\{q_1, q_2\}$, system (2) is $\mathcal{O}(t^{-q})$-globally asymptotically stable if and only if all the roots of $\Delta_A(s)$ are in the open left half-plane ($\Re(s) < 0$).*
2. If $\det(A) \neq 0$ and $\Delta_A(s)$ has a root in the open right half-place ($\Re(s) > 0$), system (2) is unstable.

Our next aim is to analyze the distribution of the roots of the characteristic function $\Delta_A(s)$ with respect to the imaginary axis of the complex plane. For simplicity, for $(a, b, c) \in \mathbb{R}^3$, $q_1, q_2 \in (0, 1]$ we denote:

$$\Delta(s; a, b, c, q_1, q_2) = s^{q_1+q_2} + as^{q_2} + bs^{q_1} + c.$$

As in [5], we easily obtain the following result:

Lemma 1 *If $c < 0$, the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ has at least one positive real root.*

In the following, we assume $c > 0$ and we seek to characterize the following sets:

$$S(c) = \{(a, b) \in \mathbb{R}^2 \ : \ \Delta(s; a, b, c, q_1, q_2) \neq 0, \ \forall\, s \in \mathbb{C}^+, \forall\, (q_1, q_2) \in (0, 1]^2\}$$
$$U(c) = \{(a, b) \in \mathbb{R}^2 \ : \ \forall\, (q_1, q_2) \in (0, 1]^2, \ \exists\, s \in \text{Int}(\mathbb{C}^+) \text{ s.t. } \Delta(s; a, b, c, q_1, q_2) = 0\}$$
$$Q(c) = \text{Int}\left(\mathbb{R}^2 \setminus (A_c \cup U_c)\right)$$

where $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re(s) \geq 0\}$ and $(0, 1]^2 = (0, 1] \times (0, 1]$. Based on Theorem 1 and the previous lemma, the link between the stability properties of system (2) and the three sets defined above is given by

Proposition 1 *1. If $\det(A) < 0$, the trivial solution of system is unstable, regardless of the fractional orders $(q_1, q_2) \in (0, 1]^2$.*
2. If $\det(A) > 0$, the trivial solution of system (2) is

a. asymptotically stable, regardless of the fractional orders $(q_1, q_2) \in (0, 1]^2$ if and only if $(-a_{11}, -a_{22}) \in S(\det(A))$.
b. unstable, regardless of the fractional orders $(q_1, q_2) \in (0, 1]^2$ if and only if $(-a_{11}, -a_{22}) \in U(\det(A))$.

c. *asymptotically stable with respect to some (but not all) fractional orders* $(q_1, q_2) \in (0, 1]^2$ *if and only if* $(-a_{11}, -a_{22}) \in Q(\det(A))$.

Lemma 2 *Let* $c > 0$. *The sets* $S(c)$, $U(c)$ *and* $Q(c)$ *are symmetric with respect to the first bisector in the* (a, b)*-plane.*

Proof The statement results from the fact that $\Delta(s; a, b, c, q_1, q_2) = \Delta(s; b, a, c, q_2, q_1)$, for any $(a, b, c) \in \mathbb{R}^3$ and $(q_1, q_2) \in (0, 1]^2$. □

In the following, we give several intermediary lemmas which are obtained by generalizing the results presented in [5]. As the proofs are built up in a similar manner as in [5], they will be omitted.

Lemma 3 *Let* $c > 0$, $q_1, q_2 \in (0, 1]$, $q_1 \neq q_2$, *and consider the smooth parametric curve in the* (a, b)*-plane defined by*

$$\Gamma(c, q_1, q_2) \ : \quad \begin{cases} a = c\rho_1(q_1, q_2)\omega^{-q_2} - \rho_2(q_1, q_2)\omega^{q_1} \\ b = \rho_1(q_1, q_2)\omega^{q_2} - c\rho_2(q_1, q_2)\omega^{-q_1} \end{cases} , \quad \omega > 0,$$

where

$$\rho_1(q_1, q_2) = \frac{\sin \frac{q_1 \pi}{2}}{\sin \frac{(q_2 - q_1)\pi}{2}} \quad , \quad \rho_2(q_1, q_2) = \frac{\sin \frac{q_2 \pi}{2}}{\sin \frac{(q_2 - q_1)\pi}{2}} \ .$$

The curve $\Gamma(c, q_1, q_2)$ *is the graph of a smooth, decreasing, convex bijective function* $\phi_{c,q1,q2} : \mathbb{R} \to \mathbb{R}$ *in the* (a, b)*-plane.*

Lemma 4 *Let* $c > 0$ *and* $q_1, q_2 \in (0, 1]$.

a. *If* $q_1 \neq q_2$, *the function* $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ *has a pair of pure imaginary roots if and only if* $(a, b) \in \Gamma(c, q_1, q_2)$.
   Al the roots of the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ *are in the open left half-plane if and only if* $b > \phi_{c,q1,q2}(a)$.
b. *If* $q_1 = q_2 := q$, *the function* $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ *has a pair of pure imaginary roots if and only if* $(a, b) \in \Lambda(c, q)$, *where* $\Lambda(c, q)$ *is the line defined by*

$$\Lambda(c, q) : \quad a + b + 2\sqrt{c} \cos \frac{q\pi}{2} = 0.$$

   Al the roots of the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ *are in the open left half-plane if and only if* $a + b + 2\sqrt{c} \cos \frac{q\pi}{2} > 0$.

As a consequence of the previous lemma, the following characterization of the set $Q(c)$ is formulated:

Corollary 1 *The set* $Q(c)$ *in the* (a, b)*-plane is the union of all curves* $\Gamma(c, q_1, q_2)$, *for* $(q_1, q_2) \in (0, 1)^2$, $q_1 \neq q_2$ *and all lines* $\Lambda(c, q)$, *for* $q \in (0, 1)$.

Lemma 5 *Let $c > 0$. The region*

$$R_u(c) = \{(a, b) \in \mathbb{R}^2 \ : \ a + b + c + 1 \leq 0\} \cup \{(a, b) \in \mathbb{R}^2 \ : \ a < 0, \ b < 0, \ ab \geq c\}$$

is included in the set $U(c)$.

Proof Let $(a, b) \in R_u(c)$. First, let us notice that $\Delta(1; a, b, c, q_1, q_2) = a + b + c + 1$. Hence, if $a + b + c + 1 \leq 0$, it follows that for any $(q_1, q_2) \in (0, 1]^2$, the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ has at least one positive real root in the interval $[1, \infty)$. Therefore, $(a, b) \in U(c)$.

On the other hand, if $a < 0$, $b < 0$ and $ab \geq c$, as

$$\Delta(s; a, b, c, q_1, q_2) = (s^{q_1} + a)(s^{q_2} + b) + c - ab$$

we see that for $s_0 = |a|^{1/q_1} > 0$, we have $\Delta(s_0; a, b, c, q_1, q_2) = c - ab \leq 0$. Hence, for any $(q_1, q_2) \in (0, 1]^2$, the function $s \mapsto \Delta(s; a, b, c, q_1, q_2)$ has at least one strictly positive real root. It follows that $(a, b) \in U(c)$. □

The following lemma is obtained as in [5]:

Lemma 6 *Let $c > 0$. The region*

$$R_s(c) = \{(a, b) \in \mathbb{R}^2 \ : \ a + b > 0, \ a > -\min(1, c), \ b > -\min(1, c)\}$$

is included in the set $S(c)$.

Based on all previous results, the following conditions for the stability of system (2) with respect to its coefficients and the fractional orders q_1 and q_2 are obtained (Figs. 1 and 2):

Proposition 2 *For the fractional-order linear system (2) with $q_1, q_2 \in (0, 1]$, the following holds:*

1. *If $\det(A) < 0$, system (2) is unstable, regardless of the fractional orders q_1, q_2.*
2. *Assume that $\det(A) > 0$ and $q_1, q_2 \in (0, 1]$ are arbitrarily fixed and $q = \min\{q_1, q_2\}$. If $q_1 \neq q_2$, let $\Gamma = \Gamma(\det(A), q_1, q_2)$, otherwise, if $q_1 = q_2$, let $\Gamma = \Lambda(\det(A), q)$.*
   - *(a) System (2) is $\mathcal{O}(t^{-q})$-asymptotically stable if and only if $(-a_{11}, -a_{22})$ is in the region above Γ.*
   - *(b) If $(-a_{11}, -a_{22})$ is in the region below Γ, system (2) is unstable.*
3. *If $\det(A) > 0$, the following sufficient conditions for the asymptotic stability and instability of system (2), independent of the fractional orders q_1, q_2, are obtained:*
   - *(a) If $a_{11} < \min(1, \det(A))$, $a_{22} < \min(1, \det(A))$ and $Tr(A) < 0$, system (2) is asymptotically stable, regardless of the fractional orders $q_1, q_2 \in (0, 1]$.*
   - *(b) If $Tr(A) \geq \det(A) + 1$ or if $a_{11} > 0$, $a_{22} > 0$ and $a_{12}a_{21} \geq 0$, system (2) is unstable, regardless of the fractional orders $q_1, q_2 \in (0, 1]$.*

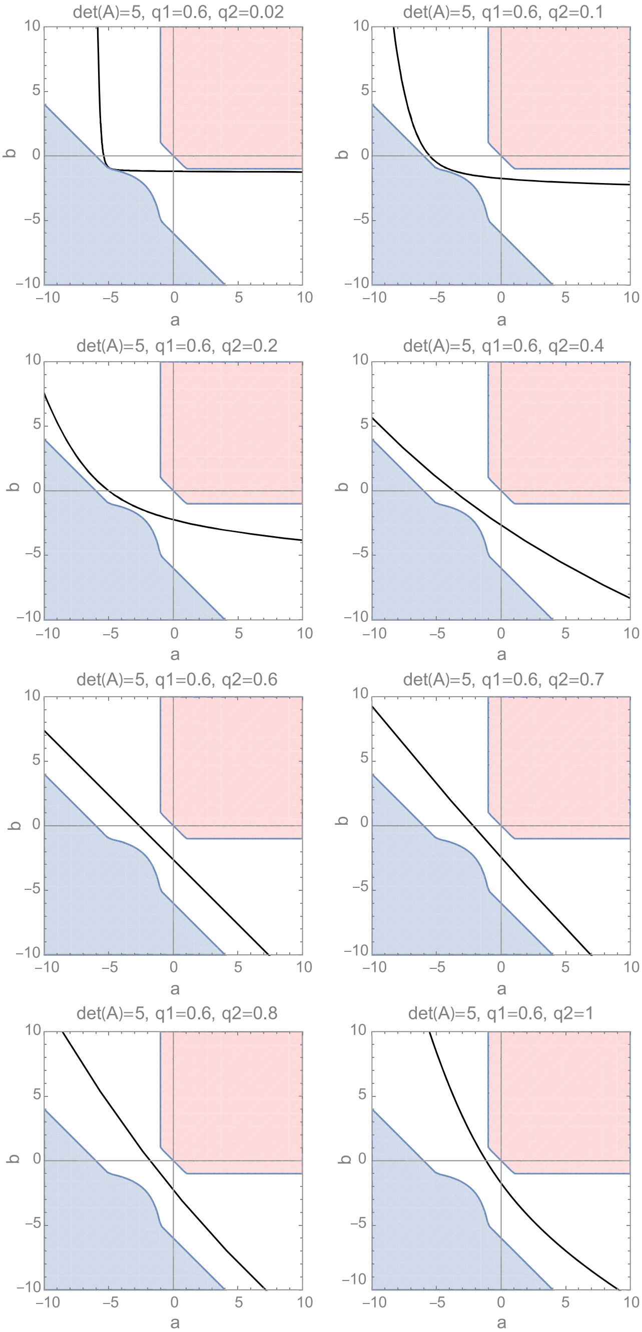


Fig. 1 Individual curves $\Gamma(c, q_1, q_2)$ (black) given by Lemma 3, for fixed values of $c = 5, q_1 = 0.6$, for different values of q_2 in the range 0.02–1. The red/blue shaded regions represent the sets $R_u(c)$ and $R_s(c)$, respectively. These curves represent the boundary of the fractional-order-dependent stability region in each case

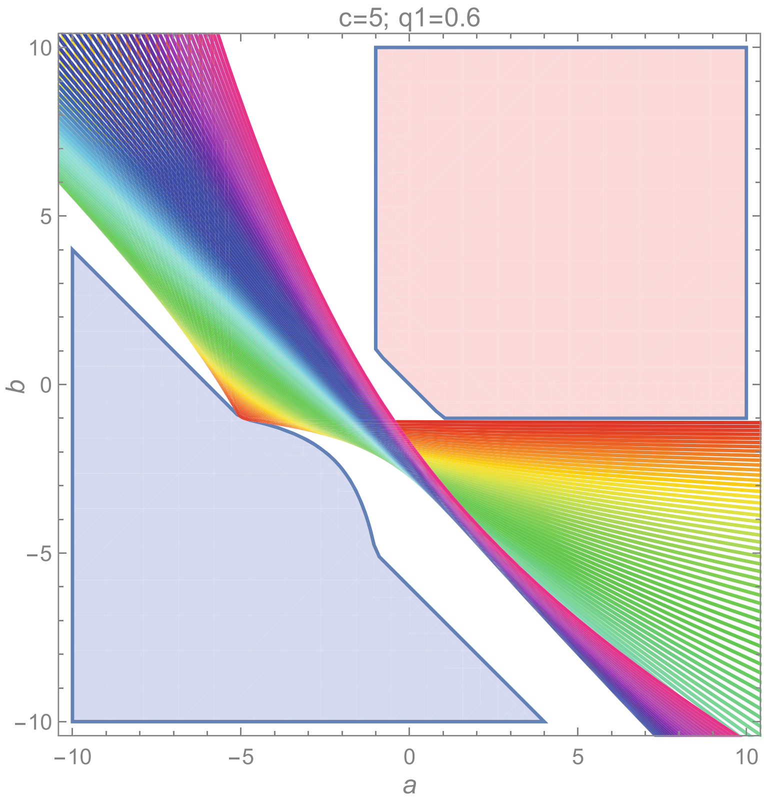


Fig. 2 Curves $\Gamma(c, q_1, q_2)$ given by Lemma 3, for fixed values of $c = 5, q_1 = 0.6$, varying q_2 from 0.01 (red curve) to 1 (violet curve) with step size 0.01. The red/blue shaded regions represent the sets $R_u(c)$ and $R_s(c)$, respectively

The fractional-order-independent sufficient conditions for the asymptotic stability/instability of system (2) obtained in Proposition 2 (point 3.) are particularly useful in the case of the practical applications in which the exact values of the fractional orders used in the mathematical modeling are not known precisely. We conjecture that in fact, these conditions are not only sufficient, but also necessary, i.e., $R_s(c) = S(c)$ and $R_u(c) = U(c)$. The proof of necessity requires further investigation and constitutes a direction for future research.

4 Investigation of a Fractional-Order Conductance-Based Model

The FitzHugh–Nagumo neuronal model [13] is a simplification of the well-known Hodgkin–Huxley model and it describes a biological neuron's activation and deactivation dynamics in terms of spiking behavior. In this paper, we consider a modified version of the classical FitzHugh–Nagumo neuronal model, by replacing the integer-order derivatives with fractional-order Caputo derivatives of different orders. Mathematically, the fractional-order FitzHugh–Nagumo model is described by the following two-dimensional fractional-order incommensurate system:

$$\begin{cases} {}^cD^{q_1}v(t) = v - \dfrac{v^3}{3} - w + I \\ {}^cD^{q_2}w(t) = r(v + c - dw) \end{cases} \tag{3}$$

where v represents the membrane potential, w is a recovery variable, I is an external excitation current and $0 < q_1 \leq q_2 \leq 1$. For comparison, a similar model has been investigated by means of numerical simulations in [2].

Rewriting the second equation of system (3) it follows that:

$${}^cD^{q_2}w(t) = rd\Big(\frac{1}{d}v + \frac{c}{d} - w\Big) = \phi(\alpha v + \beta - w)$$

where $\phi = rd \in (0, 1)$, $\alpha = \dfrac{1}{d}$ and $\beta = \dfrac{c}{d}$. Thus, system (3) is equivalent to the following two-dimensional conductance-based model:

$$\begin{cases} {}^cD^{q_1}v(t) = I - I(v, w) \\ {}^cD^{q_2}w(t) = \phi(w_\infty(v) - w) \end{cases} \tag{4}$$

where $I(v, w) = w - v + \dfrac{v^3}{3}$ and $w_\infty(v) = \alpha v + \beta$ is a linear function.

4.1 *Branches of Equilibrium States*

For studying the existence of equilibrium states of the fractional-order neuronal model (4), we intend to find the solutions of the algebraic system

$$\begin{cases} I = I_\infty(v) \\ w = w_\infty(v) \end{cases}$$

where

$$I_\infty(v) = I(v, w_\infty(v)) = w_\infty(v) - v + \frac{v^3}{3} = (\alpha - 1)v + \frac{v^3}{3} + \beta.$$

We observe that $I_\infty \in C^1$, $\lim\limits_{v\to-\infty} I_\infty(v) = -\infty$ and $\lim\limits_{v\to\infty} I_\infty(v) = \infty$. Moreover, $I'_\infty(v) = v^2 + \alpha - 1$. Therefore, we can distinguish two cases: $\alpha > 1$ and $\alpha < 1$.

The case $\alpha > 1$ has been studied in [4] and corresponds to the existence of a unique branch of equilibrium states. In this paper, we will focus on the case when $\alpha < 1$.

For $\alpha < 1$, the roots of the equation $I'_\infty(v) = 0$ are $v_{\max} = -\sqrt{1-\alpha}$ and $v_{\min} = \sqrt{1-\alpha}$. The function I_∞ is increasing on the intervals $(-\infty, v_{\max}]$ and $[v_{\min}, \infty)$ and decreasing on the interval $(v_{\max}, v_{\min})$. We denote $I_{max} = I_\infty(v_{\max})$, $I_{min} = I_\infty(v_{\min})$.

The function $I_\infty : (-\infty, v_{\max}] \to (-\infty, I_{max}]$, is increasing and continuous, and hence, it is bijective. We denote $I_1 = I_\infty|_{(-\infty, v_{\max}]}$ the restriction of function I_∞ to the interval $(-\infty, v_{\max}]$ and consider its inverse:

$$v_1 : (-\infty, I_{max}] \to (-\infty, v_{\max}], \quad v_1(I) = I_1^{-1}(I).$$

The first branch of equilibrium states of system (4) is composed of the points of coordinates $(v_1(I), n_\infty(v_1(I)))$, with $I < I_{max}$.

The second and the third branch of equilibrium states are obtained similarly:

$$I_2 = I_\infty|_{(v_{\max}, v_{\min})}, \quad v_2 : (I_{min}, I_{max}) \to (v_{\max}, v_{\min}), \quad v_2(I) = I_2^{-1}(I)$$

$$I_3 = I_\infty|_{[v_{\min}, \infty)}, \quad v_3 : [I_{min}, \infty) \to [v_{\min}, \infty), \quad v_3(I) = I_3^{-1}(I).$$

Remark 1 We have the following situations:

- If $I < I_{min}$ or if $I > I_{max}$, then system (4) has an unique equilibrium state.
- If $I = I_{min}$ or if $I = I_{max}$, then system (4) has two equilibrium states.
- If $I \in (I_{min}, I_{max})$, then system (4) has three equilibrium states.

4.2 Stability of Equilibrium States

For the investigation of the stability of equilibrium states, we consider the Jacobian matrix associated to system (4) at an arbitrary equilibrium state $(v^*, w^*) = (v^*, w_\infty(v^*))$:

$$J(v^*) = \begin{bmatrix} 1-(v^*)^2 & -1 \\ \phi\,\alpha & -\phi \end{bmatrix}$$

The characteristic equation at the equilibrium state (v^*, w^*) is

$$s^{q_1+q_2} - a_{11}s^{q_2} - a_{22}s^{q_1} + \det(J(v^*)) = 0 \tag{5}$$

where

$$\begin{aligned} a_{11} &= 1-(v^*)^2 \\ a_{22} &= -\phi < 0 \\ Tr(J(v^*)) &= 1-(v^*)^2 - \phi \end{aligned}$$

$$\det(J(v^*)) = \phi \cdot I'_\infty(v^*).$$

Considering $\alpha < 1$, the following results are obtained.

Proposition 3 *Any equilibrium state from the second branch of equilibrium states* $(v_2(I), w_\infty(v_2(I)))$ *(with* $I \in (I_{min}, I_{max})$*) of system (4) is unstable, regardless of the fractional order* q_1 *and* q_2*.*

Proof Let $I \in (I_{min}, I_{max})$ and $v^\star = v_2(I) \in (v_\alpha, v_\beta)$. Then $I'_\infty(v^\star) < 0$, so $\det(J(v^*)) < 0$. From Proposition 2 (point 1), the equilibrium state $(v^\star, w^\star) = (v_2(I), w_\infty(v_2(I)))$ is unstable, regardless of the fractional orders q_1 and q_2.

Proposition 4 *Any equilibrium state* $(v^\star, w^\star)$ *of system (4) belonging to the first or the third branch with* $|v^\star| > \sqrt{1-\phi}$ *is asymptotically stable, regardless of the fractional order* q_1 *and* q_2*.*

Proof Let $(v^\star, w^\star)$ be an equilibrium state belonging to the first or the third branch of equilibrium states such that $|v^*| > \sqrt{1-\phi}$. So $Tr(J(v^*)) < 0$ and $a_{11} \leq 1$. Moreover, $\det(J(v^*)) > 0 > a_{22}$. We apply Proposition 2 (point 3a) and we obtain the conclusion. □

Consider the following two subcases:

4.2.1 Case $\alpha \in (0, \phi]$

In this case, the second branch of equilibrium states is completely unstable, regardless of the fractional orders q_1 and q_2 and for the first and third branch of equilibrium states, the following result is obtained (see Fig. 3):

Corollary 2 *Any equilibrium state belonging to the first and the third branch of equilibrium states are asymptotically stable, regardless of the fractional orders* q_1 *and* q_2

Proof Let $(v^\star, w^\star)$ be an equilibrium state belonging to the first or the third branch of equilibrium states. Then $|v^*| > \sqrt{1-\alpha} > \sqrt{1-\phi}$. From Proposition 4, we obtain the conclusion. □

4.2.2 Case $\alpha \in (\phi, 1)$

In this case, we have the following situations (see Figs. 4 and 5):

- any equilibrium point belonging to the first or the third branch with $|v^*| \geq \sqrt{1-\phi}$ is asymptotically stable, regardless of the fractional orders q_1 and q_2;

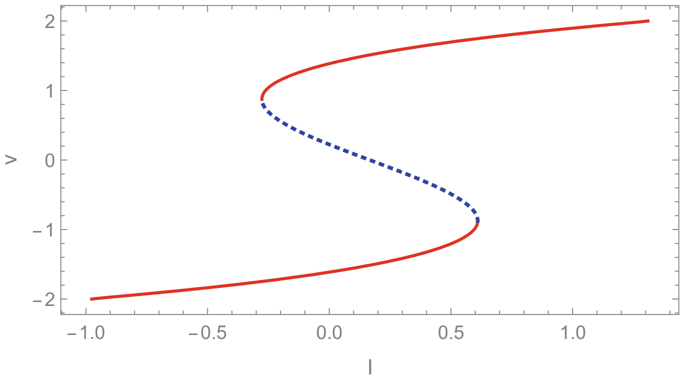


Fig. 3 Membrane potential (v^*) of the equilibrium states (v^*, w^*) of system (3) belonging to the three branches (with parameter values: $r = 0.08$, $c = 0.7$, $d = 4.2$) with respect to the external excitation current I and their stability: red/blue represents asymptotic stability/instability, regardless of the fractional orders q_1 and q_2

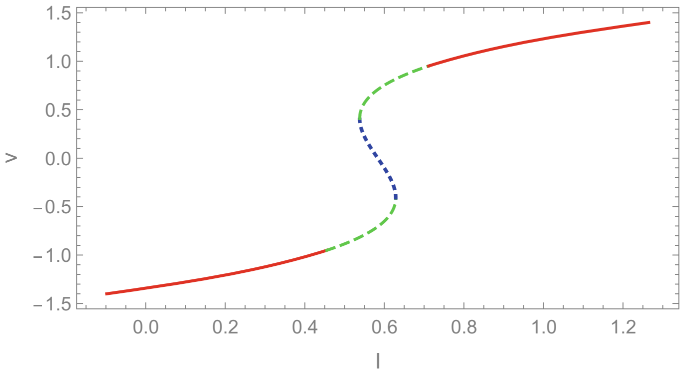


Fig. 4 Membrane potential (v^*) of the equilibrium state (v^*, w^*) of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$) with respect to the external excitation current I and their stability: red represents parts of the first and third branches of equilibrium states which are asymptotically stable, regardless of the fractional orders q_1 and q_2; blue represents the second branch of equilibrium states, which is an unstable region; green represents equilibrium states from the first or the third branch of equilibrium states whose stability depends on the fractional orders q_1 and q_2

- any equilibrium point belonging to the second branch of equilibrium states is unstable, regardless of the fractional orders q_1 and q_2;
- the stability of any equilibrium point belonging to the first branch of equilibrium states with $v^* \in [-\sqrt{1-\phi}, -\sqrt{1-\alpha}]$ or to the third branch of equilibrium states with $v^* \in [\sqrt{1-\alpha}, \sqrt{1-\phi}]$ will depend on the fractional orders q_1 and q_2 (Fig. 6).

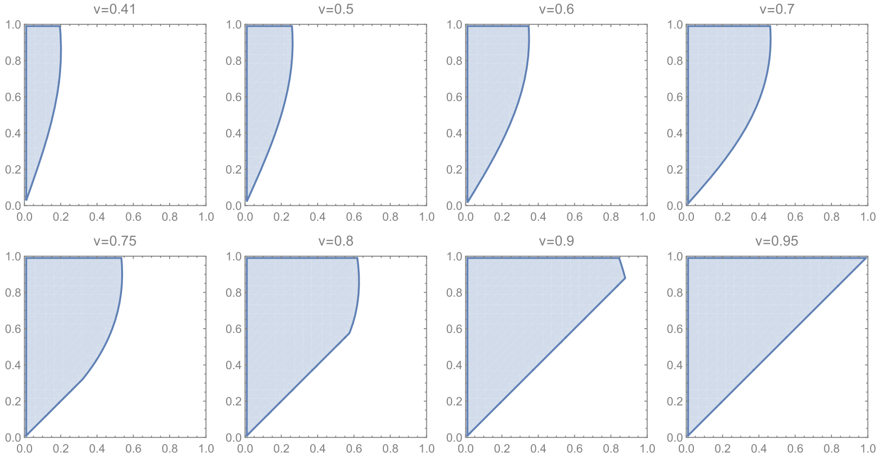


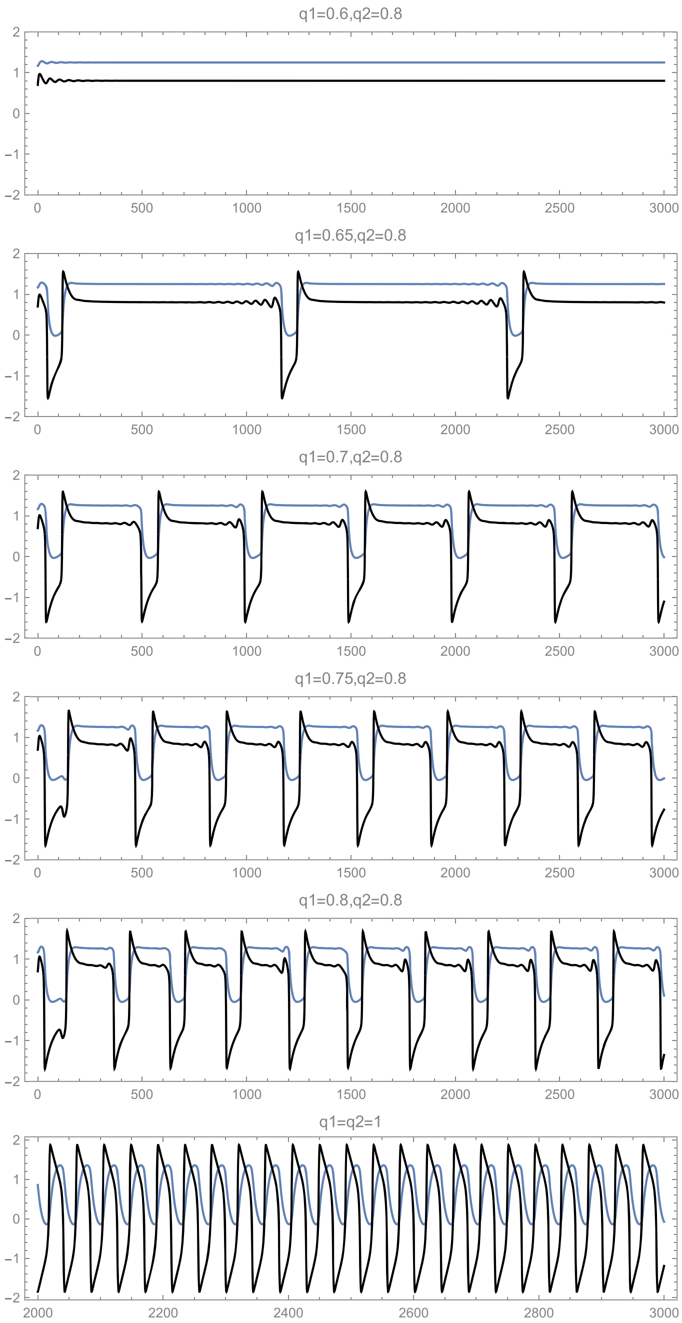
Fig. 5 Stability regions (shaded) in the (q_1, q_2)-plane for equilibrium states (v^*, w^*) of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$), with different values of the membrane potential v^* between $\sqrt{1-\alpha} \approx 0.41$ and $\sqrt{1-\phi} \approx 0.95$. In each case, the part of the blue curve strictly above the first bisector represents the Hopf bifurcation curve in the (q_1, q_2)-plane

5 Conclusions

In this work, recently obtained theoretical results concerning the asymptotic stability and instability of a two-dimensional linear autonomous system with Caputo derivatives of different fractional orders have been reviewed and extended. As a consequence, improved fractional-order-independent sufficient conditions for the stability and instability of such systems have been obtained. Several open problems are identified below, which require further investigation, in accordance to the recent trends in the field of interest of fractional-order differential equations:

- Are the fractional-order-independent sufficient conditions for stability and instability identified in this work, also necessary?
- Complete characterization of the fractional-order-independent stability set and fractional-order-independent instability set, respectively.
- Extension of these results to the case of two-dimensional systems of fractional-order difference equations [26, 27] and to higher dimensional systems.

As an application, the second part of the paper investigated the stability properties of a fractional-order FitzHugh–Nagumo system. Moreover, numerical simulations were provided, exemplifying the theoretical findings and revealing the possible occurrence of Hopf bifurcations when critical values of the fractional orders are encountered.


q1=0.6,q2=0.8
q1=0.65,q2=0.8
q1=0.7,q2=0.8
q1=0.75,q2=0.8
q1=0.8,q2=0.8
q1=q2=1
2
1
0
−1
−2
0
500
1000
1500
2000
2500
3000
2200
2400
2600
2800

◀**Fig. 6** Evolution of the state variables of system (3) (with parameter values: $r = 0.08$, $c = 0.7$, $d = 1.2$ and $I = 1.25$) for different values of the fractional orders. In the first five graphs, the value for fractional order q_2 has been fixed 0.8 and the value of the fractional order q_1 has been increased. Observe that for $q_1 = 0.6$ we have asymptotic stability and for $q_1 = 0.65$ we have oscillations, which means that between those values a Hopf bifurcation occurs. Moreover, we observe that as q_1 is increased, the frequency of the oscillations increases

References

1. Anastasio, T.J.: The fractional-order dynamics of brainstem vestibulo-oculomotor neurons. Biol. Cybern. **72**(1), 69–79 (1994)
2. Armanyos, M., Radwan, A.G.: Fractional-order fitzhugh-nagumo and izhikevich neuron models. In: 2016 13th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology (ECTI-CON), pp. 1–5. IEEE (2016)
3. Bonnet, C., Partington, J.R.: Coprime factorizations and stability of fractional differential systems. Syst. Control. Lett. **41**(3), 167–174 (2000)
4. Brandibur, O., Kaslik, E.: Stability properties of a two-dimensional system involving one caputo derivative and applications to the investigation of a fractional-order Morris-Lecar neuronal model. Nonlinear Dyn. **90**(4), 2371–2386 (2017)
5. Brandibur, O., Kaslik, E.: Stability of Two-Component Incommensurate Fractional-Order Systems and Applications to the Investigation of a FitzHugh-Nagumo Neuronal Model. Math. Methods Appl. Sci. **41**(17), 7182–7194 (2018)
6. Čermák, J., Kisela, T.: Stability properties of two-term fractional differential equations. Nonlinear Dyn. **80**(4), 1673–1684 (2015)
7. Cottone, G., Di Paola, M., Santoro, R.: A novel exact representation of stationary colored gaussian processes (fractional differential approach). J. Phys. A: Math. Theor. **43**(8), 085002 (2010)
8. Datsko, B., Luchko, Y.: Complex oscillations and limit cycles in autonomous two-component incommensurate fractional dynamical systems. Math. Balk. **26**, 65–78 (2012)
9. Diethelm, K.: The Analysis of Fractional Differential Equations. Springer, Berlin (2004)
10. Doetsch, G.: Introduction to the Theory and Application of the Laplace Transformation. Springer, Berlin (1974)
11. Maolin, D., Wang, Z., Haiyan, H.: Measuring memory with the order of fractional derivative. Sci. Rep. **3**, 3431 (2013)
12. Engheia, N.: On the role of fractional calculus in electromagnetic theory. IEEE Antennas Propag. Mag. **39**(4), 35–46 (1997)
13. FitzHugh, R.: Impulses and physiological states in theoretical models of nerve membrane. Biophys. J. **1**, 445–466 (1961)
14. Gorenflo, R., Mainardi, F.: Fractional calculus, integral and differential equations of fractional order. In: Carpinteri, A., Mainardi, F. (eds.) Fractals and Fractional Calculus in Continuum Mechanics. CISM Courses and Lecture Notes, vol. 378, pp. 223–276. Springer, Wien (1997)
15. Henry, B.I., Wearne, S.L.: Existence of turing instabilities in a two-species fractional reaction-diffusion system. SIAM J. Appl. Math. **62**, 870–887 (2002)
16. Heymans, N., Bauwens, J.-C.: Fractal rheological models and fractional differential equations for viscoelastic behavior. Rheol. Acta **33**, 210–219 (1994)
17. Huang, S., Xiang, Z.: Stability of a class of fractional-order two-dimensional non-linear continuous-time systems. IET Control Theory Appl. **10**(18), 2559–2564 (2016)
18. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
19. Lakshmikantham, V., Leela, S., Devi, J.V.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)

20. Li, C., Ma, Y.: Fractional dynamical system and its linearization theorem. Nonlinear Dyn. **71**(4), 621–633 (2013)
21. Li, C.P., Zhang, F.R.: A survey on the stability of fractional differential equations. Eur. Phys. J.-Spec. Top. **193**, 27–47 (2011)
22. Li, Y., Chen, Y.Q., Podlubny, I.: Mittag-Leffler stability of fractional order nonlinear dynamic systems. Automatica **45**(8), 1965–1969 (2009)
23. Lundstrom, B.N., Higgs, M.H., Spain, W.J., Fairhall, A.L.: Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. **11**(11), 1335–1342 (2008)
24. Mainardi, F.: Fractional relaxation-oscillation and fractional phenomena. Chaos Solitons Fractals **7**(9), 1461–1477 (1996)
25. Matignon, D.: Stability results for fractional differential equations with applications to control processing. In Computational Engineering in Systems Applications, pp. 963–968 (1996)
26. Mozyrska, D., Wyrwas, M.: Explicit criteria for stability of fractional h-difference two-dimensional systems. Int. J. Dyn. Control. **5**(1), 4–9 (2017)
27. Mozyrska, D., Wyrwas, M.: Stability by linear approximation and the relation between the stability of difference and differential fractional systems. Math. Methods Appl. Sci. **40**(11), 4080–4091 (2017)
28. Petras, I.: Stability of fractional-order systems with rational orders (2008). arXiv preprint arXiv:0811.4102
29. Podlubny, I.: Fractional Differential Equations. Academic, New York (1999)
30. Radwan, A.G., Elwakil, A.S., Soliman, A.M.: Fractional-order sinusoidal oscillators: design procedure and practical examples. IEEE Trans. Circuits Syst. I: Regul. Pap. **55**(7), 2051–2063 (2008)
31. Rivero, M., Rogosin, S.V., Tenreiro Machado, J.A., Trujillo, J.J.: Stability of fractional order systems. Math. Probl. Eng. **2013** (2013)
32. Sabatier, J., Farges, C.: On stability of commensurate fractional order systems. Int. J. Bifurc. Chaos **22**(04), 1250084 (2012)
33. Trächtler, A.: On BIBO stability of systems with irrational transfer function (2016). arXiv preprint arXiv:1603.01059
34. Wang, Z., Yang, D., Zhang, H.: Stability analysis on a class of nonlinear fractional-order systems. Nonlinear Dyn. **86**(2), 1023–1033 (2016)
35. Weinberg, S.H.: Membrane capacitive memory alters spiking in neurons described by the fractional-order Hodgkin-Huxley model. PloS one **10**(5), e0126629 (2015)

Artificial Neural Network Approximation of Fractional-Order Derivative Operators: Analysis and DSP Implementation

Pratik Kadam, Gaurav Datkhile and Vishwesh A. Vyawahare

Abstract Fractional derivative operators, due to their infinite memory feature, are difficult to simulate and implement on software and hardware platforms. The available limited-memory approximation methods have certain lacunae, viz., numerical instability, ill-conditioned coefficients, etc. This chapter deals with the artificial neural network (ANN) approximation of fractional derivative operators. The input–output data of Gr*ü*nwald–Letnikov and Caputo fractional derivatives for a variety of functions like ramp, power law type, sinusoidal, Mittag-Leffler functions is used for training multilayer ANNs. A range of fractional derivative order is considered. The Levenberg–Marquardt algorithm which is the extension of back-propagation algorithm is used for training the ANNs. The criterion of mean squared error between the outputs of actual derivative and the approximations is considered for validation. The trained ANNs are found to provide a very close approximation to the fractional derivatives. These approximations are also tested for the values of fractional derivative order which are not part of the training data-set. The approximations are also found to be computationally fast as compared to the numerical evaluation of fractional derivatives. A systematic analysis of the speedup achieved using the approximations is also carried out. Also the effect of increase in number of layers (net size) and the type of mathematical function considered on the mean squared error is studied. Furthermore, to prove the numerical stability and hardware suitability, the developed ANN approximations are implemented in real time on a DSP platform.

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1 Introduction

The basic ideas of fractional calculus, the calculus dealing with derivatives and integrals of arbitrary non-integer order, were proposed more than 300 years ago. Riemann, Euler, Fourier, Liouville and Bode were some of the stalwarts who made profound contributions to this area [1]. The differential equations with fractional derivatives are called fractional differential equations (FDEs). The mathematical theory of FDEs is quite matured [2, 3]. Scientists and engineers recognized the importance of FDEs only during last 40 years, especially when it was observed that the mathematical descriptions of some systems are more accurate if modeled using FDEs. Also, the resulting mathematical models are quite compact and provide a more realistic representation for processes with memory and spatial distribution. Further, fractional-order (FO) models provide more degrees of freedom in the model while an unlimited memory is also guaranteed in contrast to integer-order models with limited memory [4, 5]. Some of the areas where FO models have been developed are viscoelasticity, diffusion, Warburg impedance, nuclear reactor, and the voltage–current relation of a semi-infinite lossy transmission line [6]. The theory of linear and nonlinear FO systems is also very well developed [7, 8].

Fractional-order derivative operators, by definition, possess infinite memory. This feature makes them very difficult to simulate/realize using the limited memory software/physical hardware. This problem is usually circumvented by using the various available continuous and discrete limited memory approximations (viz., Oustaloup's Recursive Approximation, Al-Alaoui, Tustin, CFE, etc.). All these techniques result in a rational approximation (continuous- or discrete-time) of the FO derivative or integral operator. These approximations suffer from the following shortcomings/lacunae:

1. The approximations are valid for a particular value of fractional order of the derivative or integral. The same approximation can not be used if there is a change in the value of differentiation/integration order and a new approximate rational transfer function is required.
2. The resulting integer-order transfer functions are generally ill-conditioned (polynomial coefficients of very large value of the order of 1e70). This may lead to computational instability and saturation of the implementation hardware.
3. These approximations may lead to internal instability.

It can be concluded that there is a need for an approach which can overcome the aforementioned limitations. Here the artificial neural network approximations of FO derivative operators are proposed.

There is a vast literature available on realization and implementation of FO operators and FO systems on various electronic hardware platforms. The DSP realization of FO operators is proposed in [9], whereas [10] implements the Grünwald–Letnikov and Caputo fractional derivative definitions on FPGA platform. The FO chaotic system is widely implemented on various embedded platforms like FPGA, microcontroller, DSP, etc. The implementation of chaos and hyperchaos in FO chaotic systems on FPGA is given in [11], whereas the control of chaotic system on FPGA is studied in [12]. The DSP realization of FO Lorenz system based on Adomian Decomposition method is proposed in [13]. All these attempts are based on the finite memory rational integer-order approximations discussed earlier.

Study of the human brain goes back to 200 years ago. With the advent of modern electronics, it was only natural to try to harness this thinking process. The concept of artificial neural networks was a paradigm shift [14]. As their name implies, neural networks take a cue from the human brain by emulating its structure. The neuron is the basic structural unit of a neural network. In the human brain, a biological neuron receives electrical impulses from numerous axons and dendrites which serves as a input to the soma which is processing block of neuron. If there is enough aggregated input to the neuron, it generates electrical pulses of signal to its output synapse activating different regions of human body [15]. An artificial neuron functions similarly. A neuron receives a number of inputs that possess weights based on their importance. Similar to a real neuron, the weighted inputs are summed and output based on a threshold function sent to every neuron downstream [16]. The application of neural networks vast variety of domain from the classification, pattern recognition problems [17] to abstract solution in business analytic [18] to solving problem in finance [19].

There have been some attempts to combine the areas of fractional calculus and artificial neural networks. The dynamics of a FO ANN is discussed in [20]. The solution of ordinary differential equation (ode) and partial differential equations (pde) with ANN has been studied in [21]. Dynamics of FO neural network has been analyzed in [20] and the stability study of these networks using LMI approach is reported in [22]. Use of ANNs for solving FDEs is given in [23]. Design of unsupervised fractional neural network model optimized with interior point algorithm for solving Bagley–Torvik equation is given in [24]. Mittag-Leffler functions are very versatile functions in the field of fractional control theory. A fractional-order neural network system with time-varying delays and reaction-diffusion terms using impulsive and linear controllers is presented in [25]. Introduction of fractional derivatives in fuzzy control theory into cellular neural networks to dynamically enhance the coupling strength and propose a fractional fuzzy neural network model with interactions with Lyapunov stability principle is proposed in [26]. Stability of Riemann–Liouville fractional-order neural networks with time-varying delays with Lyapunov stability approach is studied in [27]. In [28] strength of fractional neural networks (FrNNs) is exploited to find the approximate solutions of nonlinear systems based on Riccati equations of arbitrary order.

This work explores the possibility of using artificial neural networks for the approximation of FO derivative operators. A novel approach is proposed to realize the Gr*ü*nwald–Letnikov and Caputo fractional derivative operators with the help of ANN. The proposed ANN approximations provide a close approximation to FO derivative operators. Moreover, these have simple structures, are numerically stable, and are easy to simulate and implement on any hardware platform. Most importantly, an ANN approximation once designed can be used efficiently for a range of fractional order. This is due to the universal approximation feature of neural networks. The training of neural net is achieved using Levenberg–Marquardt algorithm [29].

The validation is carried out in MATLAB environment, where numerical and ANN approximation results are compared on the basis of relative error and computational time. To prove the "ease of hardware implementation" claim, the proposed approximations are realized on a digital signal processor (DSP TMS320F28335) platform along with validation of results. It is shown that ANN approximations require smaller computation time. Further, a detailed analysis of the effect of parameters like non-integer order of FO derivative and number of hidden layers on the performance and execution time of the proposed approximation is also carried out.

The salient contributions of the proposed work are as follows:

1. Design of ANN approximation of fractional-order derivatives with Grünwald–Letnikov and Caputo definitions for a variety of signals like exponential, power, sinusoidal, Mittag-Leffler. The approximations are valid for a range of fractional derivative order.
2. Verification of numerical simulation of the proposed ANN approximations with analytical solutions.
3. DSP implementation of the approximations.
4. Study of effect of variation in parameters like non-integer derivative order, number of neurons and number of hidden layers on the performance of approximations.
5. Reduction in computational time for hardware implementation of the ANN approximation.

The chapter is organized as follows. Next section discusses the fundamentals of fractional calculus and the definitions of fractional derivative operators. It also gives a list of available continuous and discrete approximations of FO operators. Section 3 introduces the basics of artificial neural network, multilayer NN models and Levenberg–Marquardt algorithm. Simulation results are presented in Sect. 4. The DSP implementation results of the designed ANN approximations are given in Sect. 5. Section 6 provides a detailed analysis of the results and conclusion is given in Sect. 7.

2 Fractional Calculus

In the last few decades, the calculus dealing with derivative and integral operators with non-integer order is gaining popularity among mathematicians, physicists, engineers, and researchers. The fractional calculus, as it is commonly known, has been found to provide a very powerful mathematical tool for modeling a variety of real-world and engineering systems. It has been also found useful in control theory. There are a variety of definitions of a fractional integral or derivative and the well-known laws of classical integer-order calculus cannot be extended straightforwardly to fractional-order integro-differential operators (see [30, 31]).

The fractional-order modeling of processes and systems is now a matured field [32]. This involves the use of fractional differential equations (FDEs) for describing various peculiar phenomena like anomalous diffusion [33], mechanics of hereditary and polymeric materials [34, 35], and many more. An excellent reference [36] presents the chronological development of fractional calculus both as an abstract mathematical field and a mathematical tool for modeling and control. This chapter uses the Caputo definition and Grünwald–Letnikov (GL) definition.

Caputo definition of fractional derivative is defined as [1, 37]:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1} $$

with a and t as the limits of the operation and $\alpha \in \mathbb{R}, n-1 < \alpha < n, n \in \mathbb{Z}^+$, where $f^n(\tau)$ is the nth-order derivative of the function $f(t)$. Caputo fractional derivative is found to be very useful for modeling as the physical initial conditions can be used in the solution of FDEs. This greatly facilitates the analytical and numerical solution of the corresponding fractional-order model.

The other fractional derivative used in this work is Grünwald–Letnikov (GL) definition. Using the concept of short memory given in [37], it is defined as:

$$ {}_aD_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \ {}^\alpha C_j f(t-jh), \tag{2} $$

where $[x]$ means the integer part of x and ${}^\alpha C_j$ is the binomial coefficient. This definition is generally used for the numerical calculations.

The fractional derivative operators are nonlocal in nature and therefore have to be simulated and implemented in rational form. The continuous-time approximations have been obtained using evaluation, interpolation and curve fitting, whereas Lubich's formula, trapezoidal rule, etc., have been used to develop discrete-time approximation. The popular approximation methods of FO operators are as follows:

1. Continuous-time Approximation
    a. Continued fraction expansion (CFE) [38]
    b. Carlson's Method [39]
    c. Matsuda's Method [40]
    d. Oustaloup's Method [41]
    e. Charef's Method [42]
2. Discrete-time Approximation [43]
    a. Tustin (Trapezoidal Rule or Bilinear Transform)
    b. Simpson's Rule
    c. Euler (Rectangular Rule)
    d. Al-Aloui's Operator

In addition to above approximations, the literature survey reveals that there are many more discretization schemes for fractional-order derivative operators. These includes Chen-Vinagre operator, Al-Alaoui Schneider Kaneshige Groutage (Al-AlaouiSKG) operator, Schneider operator, Hsue operator, Barbosa operator, Maione operator and many more. The detailed study of these operators is given in [43, 44].

3 Artificial Neural Networks

The neural network approach to computation has emerged in recent years to tackle problems for which more conventional computational approaches have proven ineffective [45]. To a large extent, such problems arise when a computer is asked to interface with the real world, which is difficult because the real world cannot be modeled with concise mathematical expressions. This section discusses the fundamentals of artificial neural networks, mathematical modeling of single neuron, multilayer networks, and their learning algorithms like Error back propagation, Levenberg–Marquardt algorithm.

3.1 McCulloch–Pitts Model

The early model of an artificial neuron was introduced by Warren McCulloch and Walter Pitts in 1943 [16]. The McCulloch–Pitts neural model is also known as linear threshold gate as depicted in Fig. 1. It allows binary 0 (OFF) or 1 (ON) states only, operates under assumption that system is discrete-time and there exist a full synchronous operation of all neurons in larger networks. The input signal is connected through synaptic connections having fixed weights and there is no interaction among network neurons except for output signal flow [46]. Every single neuron model consists of a set of inputs $x_{1j}, x_{2j}, x_{3j}, ..., x_{nj}$ with their respective weights $w_{1j}, w_{2j}, w_{3j}, ..., w_{nj}$ and one output o_j, where j indicates the neuron number. The signal is unidirectional from input to output. The output of neuron is calculated in

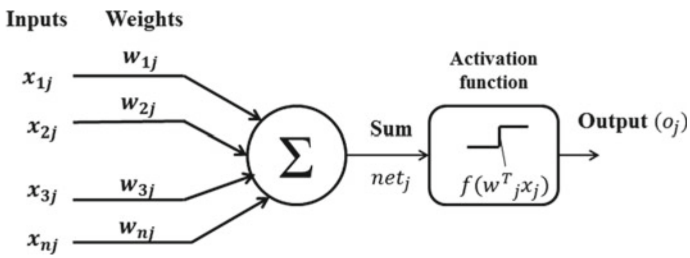

Fig. 1 McCulloch–Pitts (MCP) neuron model [14]

two steps. First step is to calculate the weighted sum of input which is the simple sum of product operation denoted as net_j. The second step is to find the output of simple activation function (f). Thus the output can be defined mathematically as:

$$o_j = f(net_j) \ or \ f(w_j^T x_j), \tag{3}$$

$$o_j = f\left(\sum_{i=1}^{m} x_{ij} w_{ij}\right), \tag{4}$$

The function $f(w_j^T x_j)$ is usually referred as activation function. Its domain is a set of activation values of the neuron model. In early stages, the output o_j was simply in binary format more suitable for classification type problem because of the hard limiting activation functions. But due to advancement in the field of mathematics, researchers developed different types activation function [47] for different application. The activation functions include as unipolar, bipolar, tansigmoidal, gaussian, arctan, exponential, linear, multiquadratics, inverse multiquadratics, and lot more. The applications include classification, curve fitting, pattern recognition, and many more [48] depending upon the type of activation function used for triggering [47, 49].

The most popular activation functions for curve fitting problem are sigmoidal or logistic, hyperbolic tangent and ReLu (Rectified linear units). Each of this has unique feature in curve fitting problem. The sigmoidal is very conventional activation function and have major research literature available. Neural networks with sigmoidal activation function are easy to implement on any conventional low-level embedded platform and does not require device with high computational power. Hence sigmoidal is most suitable for the comparative study for the proposed system. The effect of activation function on learning rate of curve fitting problem is a very vast topic and hence is not discussed in order to simplify the things. The effect of activation on learning is discussed in [50].

3.2 Multilayer Neuron Model

The primary objective of this work is to utilize the curve fitting property of neural nets in order to approximate the fractional derivative of some commonly used functions [51]. A single layer neuron can approximate a single dimension smooth analytic function, but is not able to perform multidimensional function fitting. The remedy is to use multidimensional neural nets. Figure 2 shows a simple multilayer network. The network can be classified into three parts. First is the input normalization (shown in yellow), second is the hidden layer (shown in blue) with bipolar activation function and third is the output layer (shown in orange) with linear activation function which is suitable for curve fitting application [52].

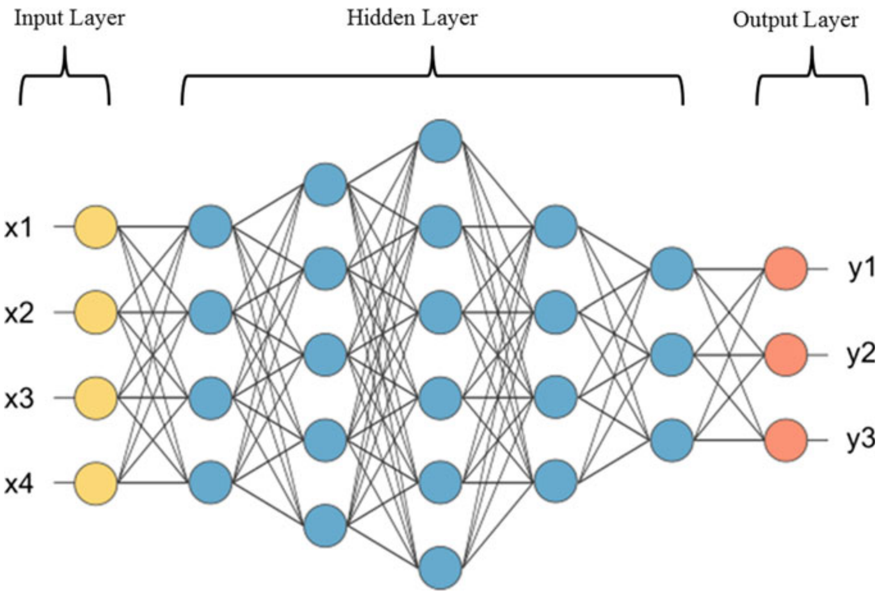

Fig. 2 Multilayer neuron model [14]

To express the ANN mathematically, consider the matrix operator (Γ) for each layer which maps the input space x to output space o_j. The o_j serves as input space for the proceeding layer. The final output matrix o can be expressed as follows:

$$o = \Gamma[Wx], \tag{5}$$

where W is weight matrix (also called as connection matrix):

$$W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{21} & \dots & w_{2n} \\ . & . & \dots & \dots \\ w_{m1} & w_{m1} & \dots & w_{mn} \end{bmatrix}, \tag{6}$$

and

$$\Gamma[.] = \begin{bmatrix} f(.) & 0 & \dots & 0 \\ 0 & f(.) & \dots & 0 \\ . & . & \dots & \dots \\ 0 & 0 & \dots & f(.) \end{bmatrix}. \tag{7}$$

Note that the nonlinear activation function $\Gamma(.)$ is a diagonal matrix operating in component-wise fashion on the activation values *net* of each neuron of the given layer.

3.3 Error Back-Propagation Training (EBBT) Algorithm

The method to train neural nets is similar to the process of the human beings learning any new task. Humans learn from their previous mistake and try to modify the behavior for similar types of action in future. On the same note, a neural net starts the training with arbitrary values. The learning then continues with a set of inputs and the corresponding outputs. The weights of respective nodes are changed according to the error between expected value and outcome. This process is repeated till the overall error is minimized to certain value. This is known as back propogation or delta learning rule [53].

Consider a pth neuron of kth layer of a multilayer neural network. Now the weights of a single neuron are updated. This step is applied to entire network for weight adjustment. The output of neuron is given by

$$\begin{aligned} net_{pk} &= W_{pk} x_p, \\ o_{pk} &= f(net_{pk}), \end{aligned} \tag{8}$$

where W_{pk} is weight matrix of pth neuron of kth layer and x_p are its inputs. The expected output of neuron is d_{pk}. To optimize the results and to design a closely fitting network, the error between d_{pk} and o_{pk} needs to be minimized. The mean squared error is considered as cost function:

$$E_p = \frac{1}{2} \sum_{k=1}^{K} (d_{pk} - o_{pk})^2. \tag{9}$$

A single neuron is connected directly or indirectly to the output by multiple layers or single layer of interlacing functions. To compute change in individual adjustment weight due to error in output is given by

$$\Delta w_{jk} = -\eta \frac{\partial E_p}{\partial w_{kj}}. \tag{10}$$

Taking derivative of (9) with respect to weight we get

$$\frac{\partial E_p}{\partial w_{jk}} = -\sum_{k=1}^{K} (d_{pk} - o_{pk}) . \frac{\partial o_{pk}}{\partial w_{jk}}, \tag{11}$$

where error E_p is defined in (9). The relation between weights and output of neuron is given by (8). Thus,

$$\begin{aligned} \frac{\partial o_{pk}}{\partial w_{jk}} &= \frac{\partial f(net_{pk})}{\partial w_{jk}}, \\ &= \frac{\partial f(net_{pk}),}{\partial net_{pk}} . \frac{\partial net_{pk}}{\partial w_{jk}}. \end{aligned} \tag{12}$$

From (8) and (12), we get

$$\frac{\partial net_{pk}}{\partial w_{jk}} = x_j. \tag{13}$$

Putting (11) and (13) into (10)

$$\Delta w_{jk} = \eta \sum_{k=1}^{K} (d_{pk} - o_{pk}) f_k^{'}(net_k) x_j. \tag{14}$$

Expression (14) gives the change in individual weight after each iteration. The process is repeated till the overall error is reduced below the specified tolerance value.

3.4 Levenberg–Marquardt Algorithm

The Levenberg–Marquardt algorithm (LMA or just LM), also known as the damped least-squares (DLS) method, is used to solve nonlinear least-squares problems [29]. It is a combination of Gradient Descent algorithm and Gauss–Newton method. The basic cost function of optimization is same as mean squared error as given in (9). Next, the algorithm is explained in details.

For given function the output is given by $\mathbf{y}$, its expected output vector as $\hat{\mathbf{y}}$ of vector of n parameters of set of m data points. Then the cost function (χ) of goodness of fit measure is given as

$$\begin{aligned} \chi^2 &= \sum_{i=1}^{m} [y - \hat{y}]^2, \\ &= (y - \hat{y})^T W (y - \hat{y}), \\ &= y^T W y - 2y^T W \hat{y} + \hat{y}^T W \hat{y}, \end{aligned} \tag{15}$$

where W is diagonal matrix called as weight matrix given in (6) and is useful when some particular point has more significance than others. In Gradient Descent or steepest descent method, parameters are updated in opposite direction of gradient of the objective function. This is similar to back propogation and is given by

$$\begin{aligned} \frac{\partial \chi^2}{\partial p} &= 2(y - \hat{y})^T W \frac{\partial}{\partial p}(y - \hat{y}), \\ &= -2(y - \hat{y})^T W \left[\frac{\partial y}{\partial p}\right], \\ &= 2(y - \hat{y})^W J, \end{aligned}$$

where J is the Jacobian matrix of error with respect to change in parameters. The parameter update is given by

$$\Delta w = \alpha J^T W(y - \hat{y}). \tag{16}$$

In Gauss–Newton method, it is assumed that the objective function is quadratic near the optimal region. Hence by applying Taylor expansion, we get

$$\hat{y}(p + h) \approx \hat{y}(p) + \left[\frac{\partial \hat{y}}{\partial p}\right] h = \hat{y} + Jh. \tag{17}$$

Substituting this in (15) gives

$$\chi^2(p + h) \approx y^T W y + \hat{y}^T W \hat{y} - 2y^T W \hat{y} - 2(y - \hat{y})^T W J h + h^T J^T W J h. \tag{18}$$

The condition at optimal solution is $\frac{\partial \chi^2}{\partial h} = 0$, which yields

$$[J^T W J]\Delta w = J^T W(y - \hat{y}). \tag{19}$$

The step size in LMA changes such that initially it follows gradient descent algorithm with small step size. But as it reaches close to the optimal point it changes update procedure as per Gauss–Newton method. The update in weights is given as

$$[J^T W J + \lambda diag(J^T W J)]\Delta w = J^T W(y - \hat{y}). \tag{20}$$

The steps involved in training of ANN are as follows [54]:

1. Decide the minimum MSE error (E_{min}), Maximum number of iteration, performance index.
2. Initialize the weight matrix W.
3. Compute layer response using (8).
4. Compute the MSE for given epoch (E) using (9).
5. Compare calculated MSE (E) with E_{min}. If $E \leq E_{min}$ then stop the training, else continue.
6. Calculate change in weights using (20) and update weights $w_{k+1} = w_k + \delta_w$.
7. Check for halting condition like number of iteration, performance index. If reached these conditions then stop training, else go back to step 3.

At startup the weights are initiated with random values. All the data is normalized in order to remove biasing toward single variable. The output is calculated with these weights and compared with the reference values. The cumulative error of entire dataset is calculated and is compared with the level of fitting (Higher the level of fitting, lower will be cumulative error). If the error is below specified level the training is terminated else changes in individual weights are calculated with the successive weight updation formula (20). The halting conditions are checked: if not satisfied, the algorithm repeats itself until maximum number of epochs have been encountered. The transition from one method to another allow Levenberg–Marquardt algorithm to reach local minimum faster as compared to individual algorithms.

4 Simulation Performance

In order to justify the performance of ANN for approximation of FO derivatives, the proposed theory was simulated in MATLAB2015a environment. To create an ANN which can mimic the fractional derivative of a given function, it is required to train for standard input and output data-sets. The fundamental working principle of neural network is analogous to human brain: it learns, adapts and evolves from its mistake. Hence the neural nets are interpolative networks, they can perform very well for the operation point which lies within its training data-set, but their performance for operating point outside training data-set is non-predictable. Thus the training data-set needs to be accurate and covering wide range of possible input values. The relationship between the time for training of ANN is inversely proportional to the size of database, hence creating a training data-set too big or too small is not a good option.

Considering all the prerequisites for generating training data-sets, numerical solutions of Grünwald–Letnikov derivatives and Caputo derivatives are vectorized in MATLAB. The functions used in this work are given in Table 1.

The fractional derivative order (α) is considered as a variable, therefore a single network can calculate fractional derivative of different derivative orders. The training data-set consists of numerical values of variables which are given as inputs to the neural network (time, derivative order, etc.). The inputs are created in nested fashion in row vector format. The outputs for the same is also calculated and are also arranged in a single row vector. Then the ANNs are trained with algorithm described in Sect. 3.4. The process is repeated for different initial weights and compared with the optimized results. The selection is done on the basis of cost function, i.e., Mean squared error (9).

The network with lowest MSE is considered as the optimized network. The network size of trained neural network is given as $[m \times n]$, where m is the number of neurons in hidden layer and n is the number of hidden layers. The process is repeated

Table 1 Functions under study

FO derivative	Functions
Grünwald–Letnikov derivatives	t, t^{μ}, $sin(at)$, $cos(at)$, e^{at}, 1- and 2-parameter Mittag-Leffler function
Caputo derivatives	t, t^{μ}, $sin(at)$, $cos(at)$, e^{at}

for network with increase in number of neurons or number of layers again till the optimized solution is obtained within the range specified. The trained ANNs are also tested for input parameters which are not part of their training data-set and are checked for under-fitting, perfect fitting, and over-fitting.

4.1 Grünwald–Letnikov ANN Approximation

The simulation performance of a variety of functions mentioned in Table 1 for GL derivative operation is shown in Fig. 3. The system parameter for simulation are given in Table 2. The network size for which the simulation was done is

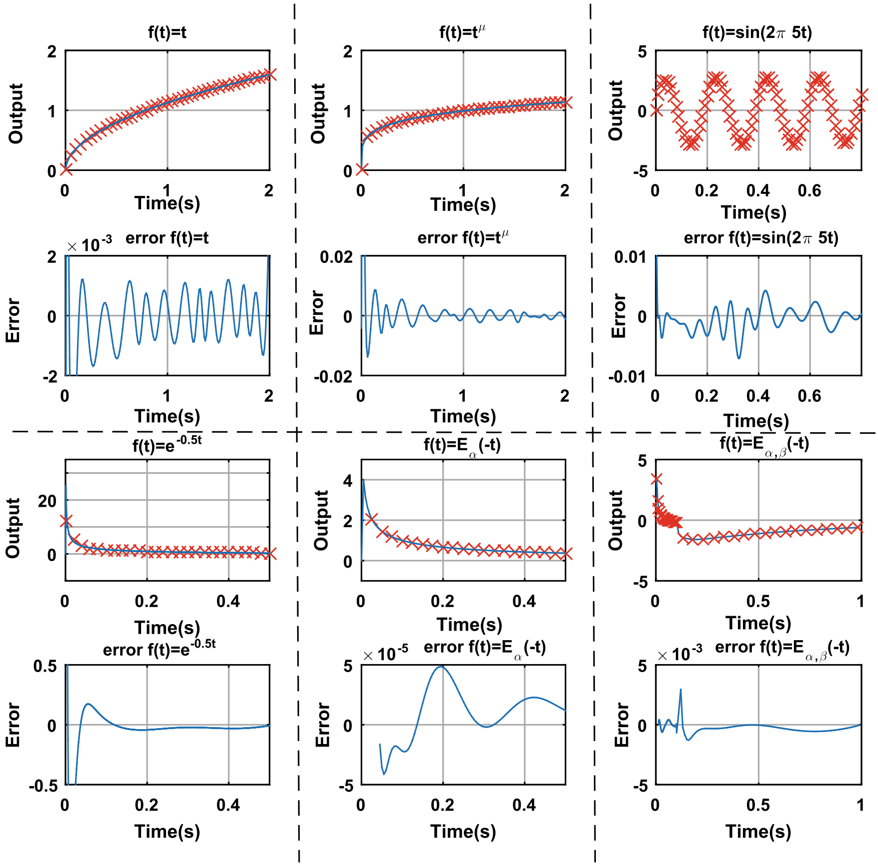

Fig. 3 GL ANN approximation with net size [10 × 3] **a** Fitting plot for $f(t) = t$, **b** Fitting plot for $f(t) = t^{0.7}$, **c** Fitting plot for $f(t) = sin(2\pi 5t)$, **d** Error plot for $f(t) = t$, **e** Error plot for $f(t) = t^{0.7}$, **f** Error plot for $f(t) = sin(2\pi 5t)$, **g** Fitting plot for $f(t) = e^{-0.5t}$, **h** Fitting plot for $f(t) =$ 1-parameter MLF, **i** Fitting plot for $f(t) =$ 2-parameter MLF, **j** Error plot for $f(t) = e^{-0.5t}$, **k** Error plot for $f(t) =$ 1-parameter MLF, **i** Error plot for $f(t) =$ 2-parameter MLF

Table 2 GL training and testing data

Case	Parameter	Values
Training	Time interval	0:0.01:2
	Range for α	0.1:0.01:0.9
Testing	Net size	[10 × 2], [05 × 3], [10 × 3]

[10 × 3], with good accuracy and faster response. The function under study are t, $t^0.7$, $sin(2\pi 5t)$, $e^{-0.5t}$, $1 - parameter\ MLF\ with\ \alpha = 0.4$ and $2 - parameter\ MLF\ with\ \alpha = 0.7, \beta = 0.25$ whose derivative order α were 0.5, 0.5, 0.3, 0.6, 0.4 and 0.7 respectively. From Fig. 3 it can be seen that the ANNs were able to approximate the GL derivative operator with negligible error.

4.2 Caputo ANN Approximation

The simulation of various function under study as mentioned in Table 1 for Caputo derivative operation is shown in Fig. 4. The system parameter for simulation are tabulated in Table 3. The function under study are t, $t^{0.5}$, $sin(2\pi 4t)$, $cos(2\pi 4t)$ and $e^{-0.5t}$. The value of fractional derivative order is taken to be $\alpha = 0.2645$, which is not part of the training data-set. Also for the sinusoidal function, in Fig. 4h and k the derivative order is 0.3816. Even though the derivative order is not part of training data-set, it can be seen from error plot in Fig. 4 that the ANNs were able to approximate Caputo derivative operator with very small error of order less than 10^{-3}.

4.3 Time Performance of ANN over Numerical Solutions

The comparative analysis of the computational time required for the original fractional-order derivative and its ANN approximation is carried out. For this, the ANN approximations with optimized result were used. The simulations of the original fractional derivative definitions and optimized ANN approximation were repeated 100 times. The speedup was calculated as follows:

$$\text{Minimum Speedup} = \frac{\text{Minimum simulation time for numerical definition}}{\text{Maximum simulation time for ANN approximation}},$$

$$\text{Maximum Speedup} = \frac{\text{Maximum simulation time for numerical definition}}{\text{Minimum simulation time for ANN approximation}},$$

$$\text{Average Speedup} = \frac{\text{Average simulation time for numerical definition}}{\text{Average simulation time for ANN approximation}}.$$

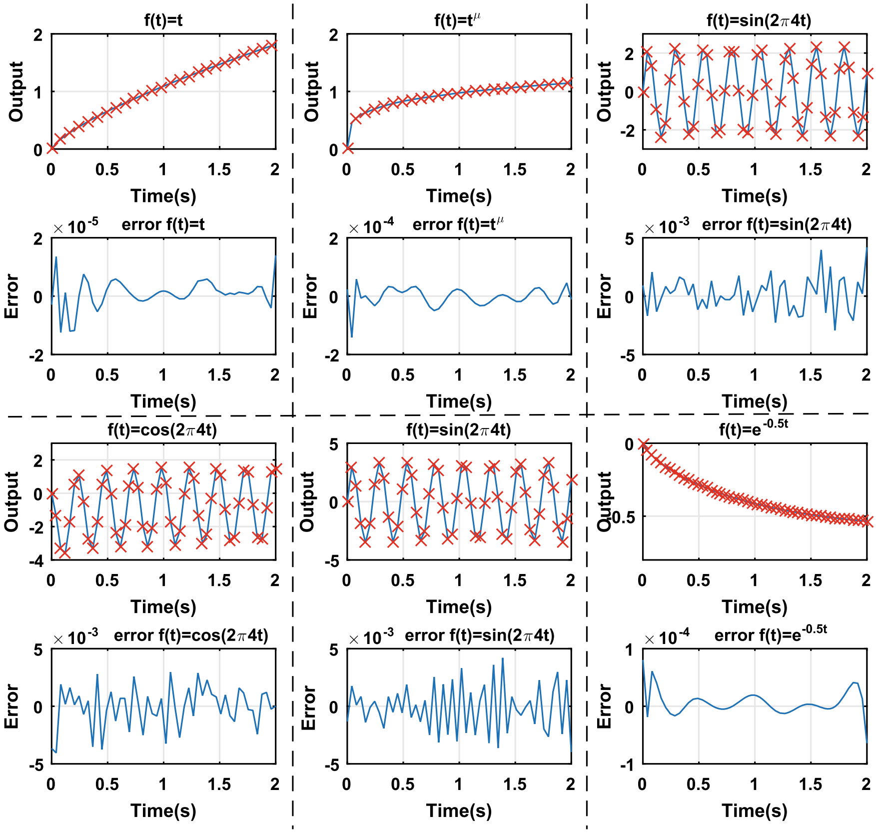

Fig. 4 Caputo ANN approximation with net size [10 × 3] **a** Fitting plot for $f(t) = t$, **b** Fitting plot for $f(t) = t^{0.5}$, **c** Fitting plot for $f(t) = sin(2\pi 4t)$, **d** Error plot for $f(t) = t$, **e** Error plot for $f(t) = t^{0.5}$, **f** Error plot for $f(t) = sin(2\pi 4t)$, **g** Fitting plot for $f(t) = cos(2\pi 4t)$, **h** Fitting plot for $f(t)sin(2\pi 4t) =$ and $\alpha = 0.3816$, **i** Fitting plot for $f(t) = e^{-0.5t}$, **j** Error plot for $f(t) = cos(2\pi 4t)$, **k** Error plot for $f(t) = sin(2\pi 4t)$ and $\alpha = 0.3816$, **i** Error plot for $f(t) = e^{-0.5t}$

Table 3 Caputo training and testing data

Case	Parameter	Values
Training	Time interval	0:0.01:2
	Range for α	0.1:0.01:0.4
Testing	Net size	[10 × 2], [05 × 3], [10 × 3]

The speedup achieved by ANN approximation for GL and Caputo fractional derivative operator are tabulated in Appendix. The machine used for calculation was Intel(R) Core(TM) i3-4005U CPU 1.70 GHz processor, 4.00 GB RAM, 64-bit operating system on Windows OS platform. In order to optimize the simulation performance of machine all the auxiliary process including internet, Wi-fi, system application, etc., were terminated. The simulation was performed in MATLAB2015a environment.

The procedure followed for the time performance calculation is as follows:

1. Initially the testing data-set is generated. If the step size for the training was h_1, then the testing data-set is created with step size $h_2 \approx h_1/10$.
2. All the other processes/software were closed and the system was disconnected from all peripherals.
3. The solution of fractional-order derivative for given function is calculated. The MATLAB tic-toc command is used to calculate time required for the computation. This process is repeated 100 times and the results are stored in one single dimensional array.
4. Similar process is repeated with ANN approximation of FO derivatives and the results are stored in a single dimensional array.
5. In order to calculate the performance parameter, the time-array generated using step 3 and 4 are used.
6. The process is repeated for different net size and the results are summarized and discussed.

From Tables 6 and 7, it is observed that the ANNs are computationally faster than numerical executions of the original definitions. Especially the ANNs are faster for calculating Caputo derivative as compared to GL derivatives. The speed of ANN approximations reduces if we increase number of layers or number of neurons. But still the ANN performs significantly better than the conventional vectorized method (numerical method).

5 Hardware Performance

To realize fractional derivative in digital environment we use different finite element approximations. The higher the order the closer the approximation at cost of higher calculation time. ANN allows us to realize the higher order approximation of fractional derivative (it can be analytic solution or finite approximation) with very small network of neurons thereby reducing the time of calculation. Once trained, neural networks are very easy to implement on any hardware platform with addition and multiplication functions, hence can be implemented on lower level hardware also.

In this section, the implementation of ANN approximations on DSP TMS320F-28335 are discussed. The TMS320F28335 is Digital Signal Processor by Texas Instruments with High Performance Static CMOS Technology. The clock speed

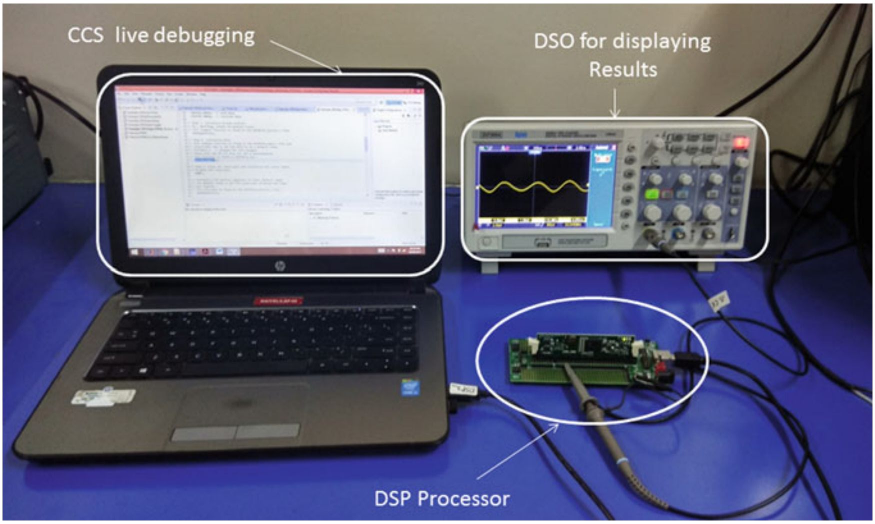

Fig. 5 Hardware assembly DSP board interfaced with CCS

is 150 MHz with 32-Bit CPU embedded with IEEE-754 Single-Precision Floating-Point Up to 6 Event Capture Inputs Unit (FPU), allowing floating-point operations. The trained networks from Sect. 4 are coded in serial sequential manner. Code Composer Studio (CCS) v6.1.1 is used for coding and analysis. The number of samples are kept same as that of used in validation in MATLAB environment. The results are compared with numerical solution calculated in MATLAB.

Figure 5 shows the hardware assembly. The CCS was installed in computer machine with Intel(R) Core(TM) i3-4005U CPU @1.70 GHz with 64-bit Windows8.1 Operating system. All the simulation and hardware results were processed on the same platform.

The procedure followed to implement ANN approximations on DSP platform:

1. The numerical solutions are obtained in MATLAB and ANN networks are trained using the algorithm described in Sect. 3.4.
2. The weights of optimized network are exported to the parameters in DSP processor. The size of net, number of inputs, number of neurons in single layer, number of layers are defined in CCS environment.
3. Value of derivative orders (α), which are not the part of training data-set are chosen.
4. Output of input layer is calculated and successive outputs of neurons are calculated in cascaded manner.
5. The process is repeated in nested fashion to calculate fractional derivative in real-time environment.
6. The data is stacked in array format which is later exported to the PC.
7. Comparative study is carried out using MATLAB.

Table 4 GL training and testing parameters

Case	Parameter	Values
Training	Time interval	0:0.01:2
	Range for α	0.1:0.01:0.9
Testing	α	[0.12345, 0.46375, 0.8537]
	Net size	$[10 \times 3]$

5.1 DSP Implementation Results for Grünwald–Letnikov Fractional Derivative

As discussed in Sect. 2 Grünwald–Letnikov definitions of fractional derivative is one of the fundamental definition of fractional calculus [2]. Now the hardware implementation results for ANN approximation of GL fractional derivative will be presented for different function. The parameters for training and testing of GL derivative approximation which are used for hardware implementation are listed in Table 4. The networks were trained up to 2 decimal digit accuracy, whereas the value of α is with 4 decimal precision. Due to interpolative property of neural network even though the ANN were not trained for given particular value gives very close approximated results.

5.1.1 Example 1: Ramp Function

It is given by

$$f(t) = t, \quad t \in [0, 2].$$

The implementation details are given in Table 4. The DSP implementation results are shown in Fig. 6. It can be seen that the output of ANN approximation exactly matches with the analytical calculation for the three values of α that were not used in training. The MSE is negligible.

5.1.2 Example 2: Power Law Type Function

It is given by

$$f(t) = t^{\mu}, \quad \mu = 0.7 \quad t \in [0, 2].$$

The implementation details are given in Table 4. The DSP implementation results are shown in Fig. 7. It can be observed from the figure that the ANN approximates the GL derivative operation very closely with very small MSE.

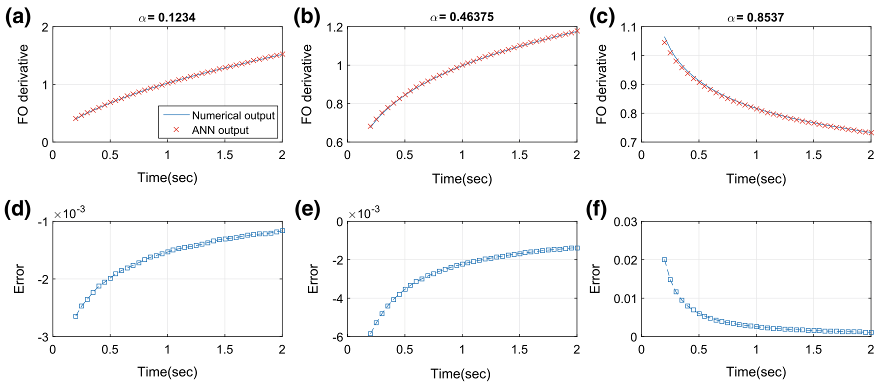

Fig. 6 Network performance for GL derivative for $D^{\alpha}t$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.46375$ **c** Network fitting for $\alpha = 0.8537$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.46375$ **f** Error curve for $\alpha = 0.8537$

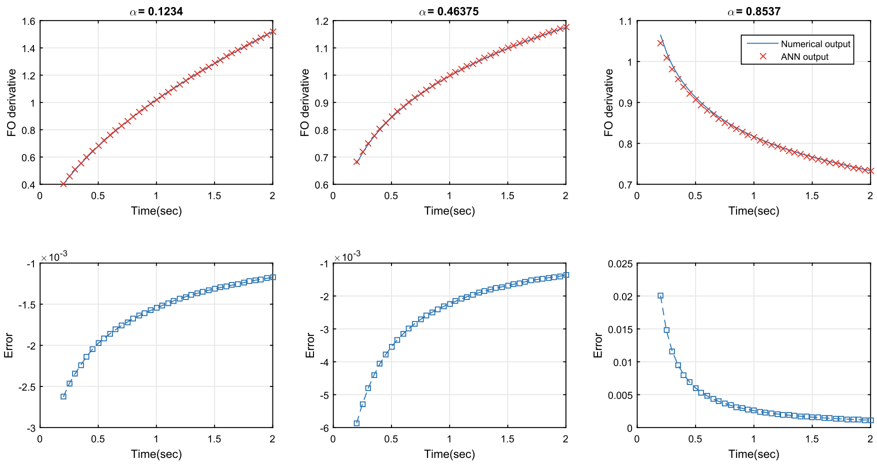

Fig. 7 Network performance for GL derivative for $D^{\alpha}t^{0.7}$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.46375$ **c** Network fitting for $\alpha = 0.8537$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.46375$ **f** Error curve for $\alpha = 0.8537$

5.1.3 Example 3: Sine and Cosine Functions

It is given by

$$f(t) = sin(2\pi 5t) \ and \ f(t) = cos(2\pi 5t).$$

The implementation details are given in Table 4. Both functions were approximated but the results for sinusoidal function only are shown for illustrative purpose. The DSP implementation results are shown in Fig. 8. Even though the values of α

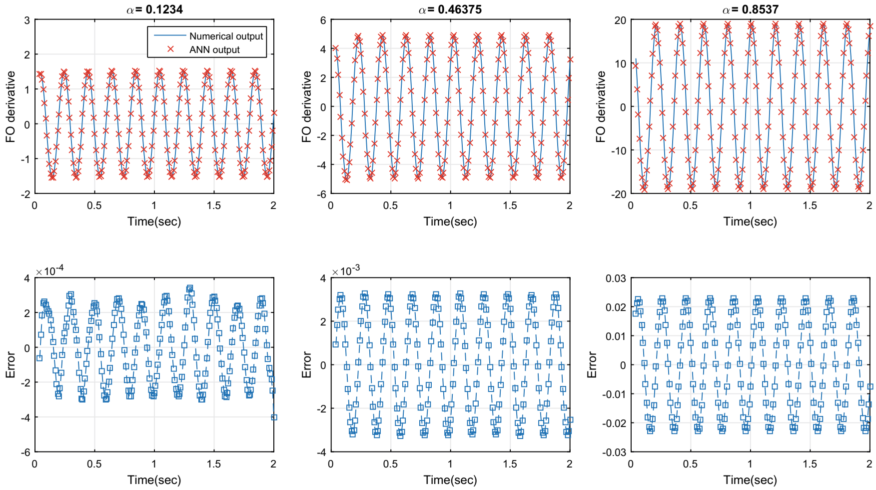

Fig. 8 Network performance for GL derivative for $D^{\alpha} sin(2\pi 5t)$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.46375$ **c** Network fitting for $\alpha = 0.8537$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.46375$ **f** Error curve for $\alpha = 0.8537$

were not a part of training set, due to interpolative property of ANN the approximation is very accurate with minimal MSE.

5.1.4 Example 4: Mittag-Leffler Function

Mittag-Leffler Function is special type of entire function. It naturally occurs as a solution of fractional-order derivatives and integrations. During last two decades, this function has gained popularity among mathematics community due to its vast potential in solving fractional-order differential equations used for modeling in biology, physics, engineering, earth science, etc. [55]. Mathematically, MLF is defined as 1-parameter Mittag-Leffler function:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(1+\alpha k)}, \alpha \varepsilon \mathbb{C}, \Re(\alpha) > 0, z \varepsilon \mathbb{C} \tag{21}$$

and its generalized form is given as 2-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta+\alpha k)}, \alpha, \beta \varepsilon \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, z \varepsilon \mathbb{C} \tag{22}$$

where $\mathbb{C}$ is set of complex number.

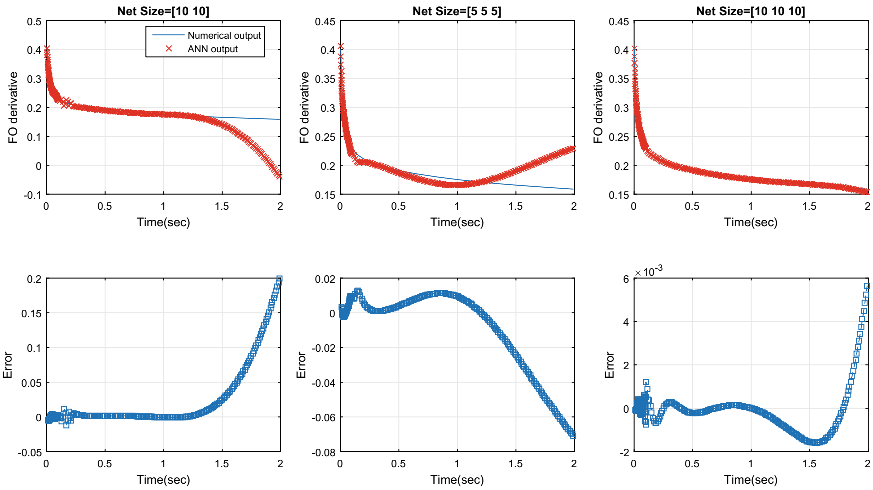

Fig. 9 Network performance for GL derivative for 2-parameter Mittag-Leffler Function (with $\beta = 0.3526$) for α which are not part of the training set **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.46375$ **c** Network fitting for $\alpha = 0.8537$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.46375$ **f** Error curve for $\alpha = 0.8537$

The ANN approximation is designed for GL fractional derivative of 1- and 2-parameter MLFs. The output of 2-parameter MLF is shown in Fig. 9. The training range for ANN was $0.1 \leq \alpha \leq 0.9$ and $0.1 \leq \beta \leq 0.5$. It can be deduced that the ANN approximation of fractional derivative of complex function like Mittag-Leffler with two input variable α and β is within the accuracy level of 10^{-6}. Hence ANN approximations can be used for practical implementation of fractional differentiation of Mittag-Leffler type function.

5.2 DSP Implementation Results for Caputo Derivative

As discussed in Sect. 2, the Caputo definition of fractional derivative is one of the fundamental definitions of fractional calculus [2]. Now the hardware implementation results for ANN approximation of Caputo fractional derivative are presented for different function. The parameters for training, testing of Caputo derivative approximation and hardware implementation are listed in Table 5. The order of derivative is kept within the limit of 0.1–0.4.

Table 5 Caputo training and testing parameters

Case	Parameter	Values
Training	Time interval	0:0.01:2
	Range for α	0.1:0.01:0.4
Testing	α	[0.12345, 0.2645, 0.3816]
	Net size	$[10 \times 3]$

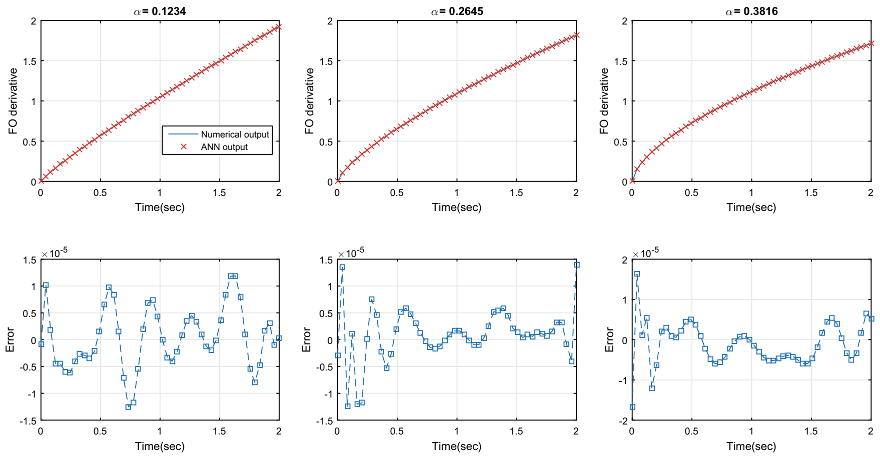

Fig. 10 Network performance for Caputo derivative for $D^{\alpha}t$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.2645$ **c** Network fitting for $\alpha = 0.3816$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.2645$ **f** Error curve for $\alpha = 0.3816$

5.2.1 Example 1: Ramp Function

It is given by

$$f(t) = t, \quad t \in [0, 2].$$

The implementation details are given in Table 5. The DSP implementation results are shown in Fig. 10. It should be noted that the output of ANN approximation exactly matches with the analytical calculation for the three values of α that were not used in training. It is also seen that the MSE is negligible.

5.2.2 Example 2: Power Law Type Function

It is given by

$$f(t) = t^{\mu}, \quad \mu = 0.7 \quad t \in [0, 2].$$

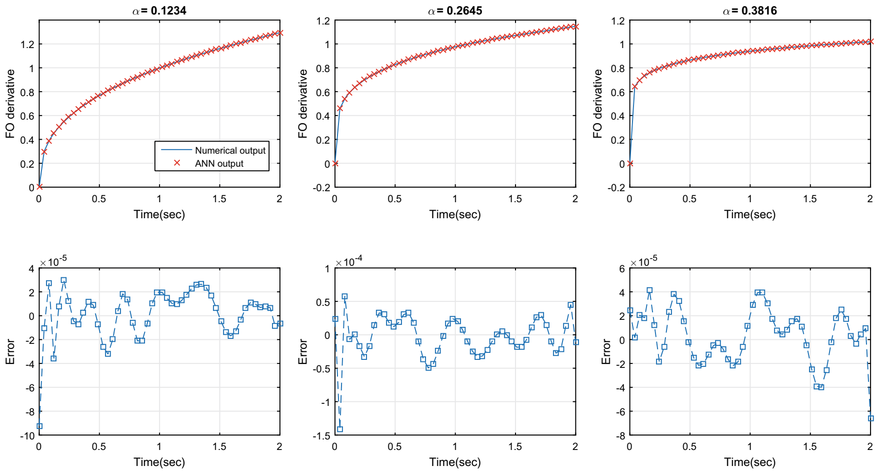

Fig. 11 Network performance for Caputo derivative for $D^{\alpha}t^{0.5}$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.2645$ **c** Network fitting for $\alpha = 0.3816$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.2645$ **f** Error curve for $\alpha = 0.3816$

The implementation details are given in Table 5. The DSP implementation results are shown in Fig. 11. It is observed from the figure that the ANNs are able to approximate Caputo derivation of power law type function within the MSE range of 10^{-4} up to 10^{-6}. Hence it can be concluded that numerically ANNs are able to closely approximate the Caputo derivative operators.

5.2.3 Example 3: Sine and Cosine Functions

It is given by

$$f(t) = sin(2\pi 4t) \; and \; f(t) = cos(2\pi 4t).$$

The approximation was done for both sine and cosine signals. The results for sinusoidal function are presented. The implementation details are given in Table 5. From Fig. 12, it can be deduced that the approximation of Caputo derivative of trigonometric functions is possible with ANN technique. The maximum error is of the order 10^{-4}, whereas MSE is of order 10^{-6}.

5.2.4 Example 4: Exponential Function

It is given by

$$f(t) = e^{-0.5t}, \quad t \in [0, 2].$$

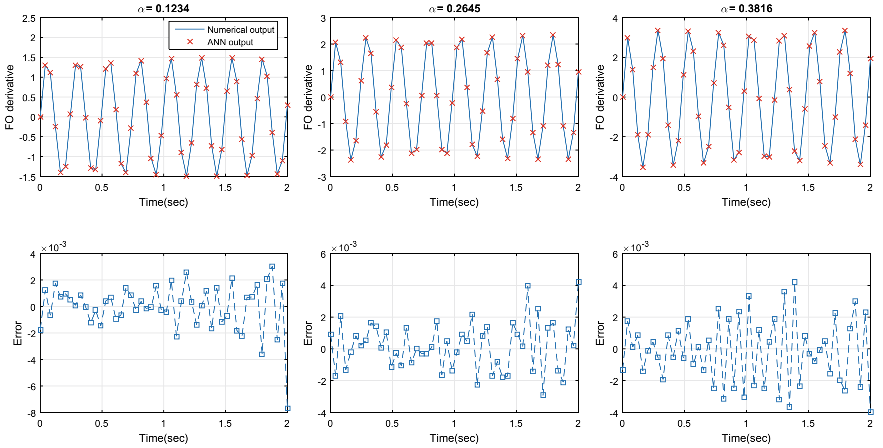

Fig. 12 Network performance for Caputo derivative for $D^{\alpha} sin(2\pi 4t)$ for untrained values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.2645$ **c** Network fitting for $\alpha = 0.3816$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.2645$ **f** Error curve for $\alpha = 0.3816$

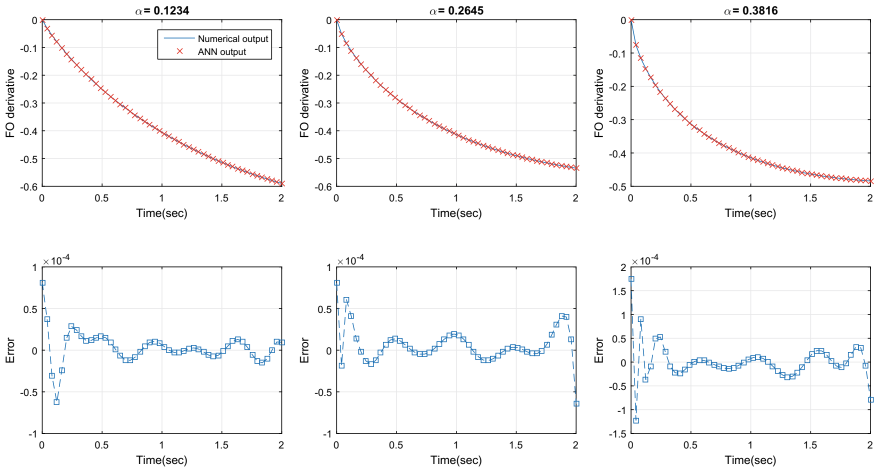

Fig. 13 Network performance for Caputo derivative for $D^{\alpha} e^{-0.5t}$ not used in training values of α **a** Network fitting for $\alpha = 0.1234$ **b** Network fitting for $\alpha = 0.2645$ **c** Network fitting for $\alpha = 0.3816$ **d** Error curve for $\alpha = 0.1234$ **e** Error curve for $\alpha = 0.2645$ **f** Error curve for $\alpha = 0.3816$

The implementation details are given in Table 5. From Fig. 13 it can inferred that the ANN is able to approximate the Caputo derivative of any order between the training set that is not part of training data-set. The error is negligibly small of the order of 10^{-4}, which is good enough for practical implementation in real-time environment.

Table 6 Time analysis of ANN with Grünwald–Letnikov derivatives. (Time is in seconds)

Size of net	Max time for ANN	Min time for ANN	Avg time for ANN	Max time for GL	Min time for GL	Avg time for GL	Max speedup	Min speedup	Average speedup
Example 1: $\alpha = [0.1, 0.9]$	$D^{\alpha} t$								
[10 × 1]	0.221	0.0115	0.0164	0.2692	0.1612	0.1688	23.3756	0.7295	10.2848
[05 × 2]	0.0211	0.0129	0.0141	0.2315	0.1614	0.1719	17.8852	7.6654	12.2223
[10 × 2]	0.0181	0.0156	0.0162	0.2223	0.1615	0.1676	14.2707	8.9342	10.319
[05 × 3]	0.0188	0.0157	0.0163	0.1805	0.1604	0.1649	11.5298	8.5504	10.1115
[10 × 3]	0.025	0.0196	0.0208	0.1921	0.1604	0.1679	9.791	6.4298	8.0624
[10 × 5]	0.0651	0.028	0.0338	0.4485	0.1599	0.1914	16.0021	2.4551	5.6637
Example 2: $\alpha = [0.1, 0.9]$	$D^{\alpha} t^{0.7}$								
[10 × 1]	0.2627	0.0135	0.0225	0.3191	0.1892	0.236	23.69	8.39	10.47
[05 × 2]	0.0351	0.0153	0.0206	0.3199	0.1865	0.2362	20.88	9.04	11.44
[10 × 2]	0.0439	0.0188	0.0253	0.3141	0.1879	0.2523	16.68	7.42	9.97
[05 × 3]	0.0421	0.0182	0.0244	0.3507	0.1831	0.247	19.31	7.5	10.11
[10 × 3]	0.0476	0.0238	0.032	0.3278	0.1962	0.243	13.78	6.13	7.59
[10 × 5]	0.0655	0.034	0.0445	0.3217	0.1908	0.2433	9.47	4.29	5.47
Example 3a: $\alpha = [0.1, 0.9]$	$sin(2\pi 5t)$								
[10 × 1]	0.3853	0.013	0.025	0.3386	0.1858	0.2443	26.062	0.4822	9.77
[05 × 2]	0.0315	0.0131	0.0218	0.3553	0.1708	0.2485	27.0725	5.4245	11.39
[10 × 2]	0.0213	0.0156	0.017	0.1776	0.1585	0.1678	11.4176	7.4496	9.85
[05 × 3]	0.019	0.0156	0.0168	0.1828	0.159	0.1647	11.6887	8.3858	9.79
[10 × 3]	0.0326	0.0196	0.0229	0.2235	0.1584	0.1792	11.393	4.8597	7.82
[10 × 5]	0.0463	0.0278	0.0301	0.1808	0.1593	0.1645	6.4987	3.4406	5.47
Example 3b: $\alpha = [0.1, 0.9]$	$cos(2\pi 5t)$								
[10 × 1]	0.2077	0.0112	0.016	0.2351	0.1596	0.1635	20.9605	0.7686	10.2319
[05 × 2]	0.0179	0.0127	0.0132	0.1793	0.1595	0.164	14.0865	8.9281	12.3843
[10 × 2]	0.0223	0.0154	0.016	0.1734	0.1588	0.1624	11.2905	7.1233	10.1511
[05 × 3]	0.0174	0.0154	0.0159	0.1706	0.1596	0.1622	11.0615	9.1903	10.206
[10 × 3]	0.0269	0.0194	0.0203	0.1741	0.1593	0.1634	8.9745	5.9176	8.0487
[10 × 5]	0.0309	0.0277	0.0285	0.1681	0.1589	0.1619	6.0592	5.1481	5.6837
Example 4: $\alpha = [0.1, 0.9]$	$D^{\alpha} e^{-0.5t}$								
[10 × 1]	0.2246	0.0113	0.0178	0.2352	0.1611	0.1791	20.8736	0.7174	10.0522
[05 × 2]	0.0233	0.0127	0.0144	0.265	0.1601	0.1762	20.8606	6.8575	12.2177
[10 × 2]	0.0274	0.0154	0.0172	0.2147	0.159	0.1687	13.9622	5.8021	9.8228
[05 × 3]	0.0219	0.0155	0.0169	0.1849	0.1602	0.1684	11.9488	7.3323	9.9366
[10 × 3]	0.0258	0.0196	0.0211	0.1806	0.1599	0.1683	9.1921	6.1987	7.9925
[10 × 5]	0.0354	0.028	0.0298	0.1809	0.1598	0.166	6.4577	4.5186	5.578

(continued)

Table 6 (continued)

Size of net	Max time for ANN	Min time for ANN	Avg time for ANN	Max time for GL	Min time for GL	Avg time for GL	Max speedup	Min speedup	Average speedup
Example 5: $\alpha = [0.1, 0.9]$	$D^{\mu} E_{\alpha}(-t)$								
[10 × 1]	0.2047	0.0073	0.0178	10.3472	10.1819	10.2841	1408.6715	49.7476	577.584
[05 × 2]	0.0091	0.0083	0.0086	10.3333	10.1167	10.2373	1251.0447	1115.493	1188.7105
[10 × 2]	0.0106	0.0089	0.0094	11.4368	10.1048	10.3486	1285.8881	956.2055	1100.609
[05 × 3]	0.0114	0.0094	0.0099	12.8626	9.9005	10.8763	1366.9657	865.5357	1100.0548
[10 × 3]	0.0144	0.01	0.0105	11.1043	9.6992	9.9795	1109.4236	675.0052	951.0624
[10 × 5]	0.0243	0.0132	0.0141	12.0715	9.6858	10.0272	912.1519	397.9716	710.2943
Example 6: $\alpha = [0.1, 0.9]$ $\beta = [0.1, 0.5]$	$D^{\mu} E_{\alpha,\beta}(-t)$								
[10 × 1]	0.2385	0.0075	0.0206	2.2375	1.5526	1.8553	296.9263	6.5106	89.954
[05 × 2]	0.0094	0.0084	0.0088	1.5735	1.5443	1.5564	186.81	163.8847	177.5684
[10 × 2]	0.0101	0.0091	0.0094	1.6381	1.5469	1.5685	179.8173	152.7168	167.2176
[05 × 3]	0.0103	0.0098	0.01	1.6161	1.5422	1.5705	164.9284	149.7618	156.5101
[10 × 3]	0.0124	0.011	0.0114	1.7908	1.5483	1.5911	163.5087	124.4195	139.8641
[10 × 5]	0.0255	0.0145	0.0153	1.8109	1.5438	1.5892	124.5665	60.5899	104.0366

6 Result Analysis

The performance of ANN approximated GL derivative for different functions is presented in Table 6 and the error analysis is shown in Figs. 6, 7, 8, and 9. The speedup from the data collected, the performance of ANN network for approximation of GL derivative is analyzed. The effect of number of layers on MSE and on speedup in calculations is analyzed and shown in Figs. 14 and 15, respectively. The performance of Caputo derivative for different function is given in Table 7 and the error analysis is shown in Figs. 10, 11, 12, and 13. The speedup from the data collected the performance of ANN network for approximation of caputo derivative is analyzed. The effect of number of layer on MSE and on speedup in calculations is analyzed and shown in Figs. 16 and 17, respectively.

The error analysis for ANN approximation like MSE, standard deviation, and variance for net size of [10 × 3] is summarized in Table 8. It can be deduced that the ANN networks are approximating the GL and Caputo definition with very negligible error.

1. The differences between ANN outputs and numerical solutions are very small. The errors are of the order of $10^{-4} - 10^{-5}$. The plots for ANN approximation and original definition of GL and Caputo FO derivatives almost coincide.
2. As the error between ANN solutions and numerical solution is close to zero. Hence the MSE values and variance values are equal.

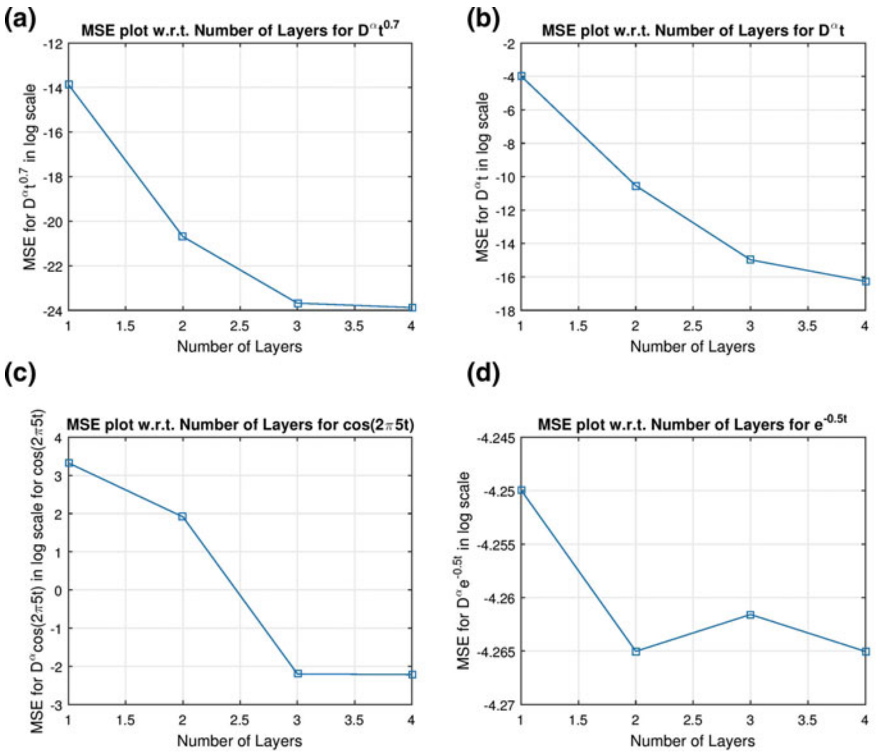

Fig. 14 Effect of ANN layers on error of GL derivative **a** $D^{\alpha}t^{0.7}$ **b** $D^{\alpha}t$ **c** $D^{\alpha}cos(2\pi 4t)$ **d** $D^{\alpha}e^{-0.5t}$

3. Mean squared error reduces as the number of layers between input and output neurons of ANN are increased. There is an initial decrease in MSE as the number of layers increases as seen from Figs. 14 and 16. But after a certain threshold, there is no significant reduction in MSE with the increase in hidden layers.
4. The computational time for ANN network is much smaller than their numerical solution. Even the minimum speedup noted gives sufficiently faster performance for practical applications. See Tables 6 and 7.
5. Improvement in performance does not only depend on size of ANN network but, mostly upon the type of function. For simple function like t, t^{μ}, $sin(t)$, etc., the speedup recorded is small. But for complex function like e^{at}, polynomial function, Mittag-Leffler Function, etc., the speedup is large.
6. The weights of all the trained networks always lie between –10 and 10, hence the system is numerically stable. The neural networks are interpolative systems, hence as long as the input is bounded within the range of training data-set, the output will always be bounded by the extremas of the output data-set used for the training.

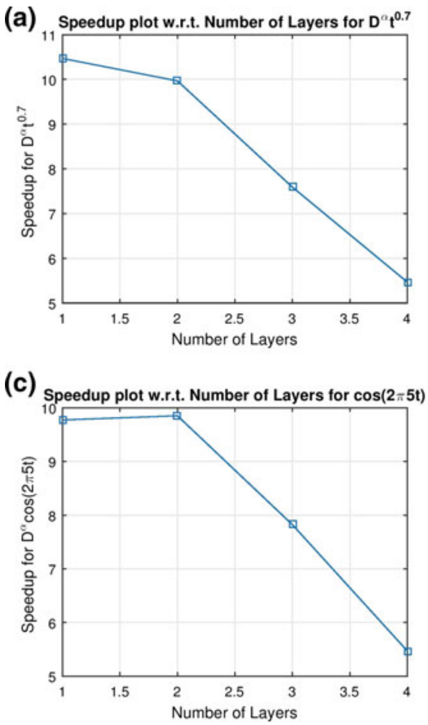

Fig. 15 Effect of ANN Layers on speedup of GL Derivative **a** $D^\alpha t^{0.7}$ **b** $D^\alpha t$ **c** $D^\alpha cos(2\pi 4t)$ **d** $D^\alpha e^{-0.5t}$

7. The interpolative property of ANN makes it robust to any change in sampling of signals. The sampling rate used for training and that of the data-set need not be same.

Thus, the ANN approximations to FO derivatives can be considered as a powerful and efficient alternative to the presently available continuous and discrete-time approximations.

7 Conclusion

A novel approach for approximation of Gr*ü*nwald–Letnikov and Caputo definitions of fractional derivatives is proposed. The input–output data of fractional derivatives of various functions with different derivative orders is used for training the ANNs. The resulting ANN approximations are verified using both simulation and DSP platform implementation. It is shown that the computational time required for ANN approximation is much lesser than the numerical implementation of fractional derivative

Table 7 Time analysis of ANN with Caputo derivatives. (Time is in seconds)

Size of net	Max time for ANN	Min time for ANN	Avg time for ANN	Max time for caputo	Min time for caputo	Avg time for caputo	Max speedup	Min speedup	Average speedup
Example 1: $\alpha = [0.1, 0.4]$	$D^{\alpha} t$								
[10 × 1]	0.2056	0.0065	0.0108	13.5127	11.0283	12.3038	2064.1634	53.6424	1135.1255
[05 × 2]	0.0089	0.0076	0.0079	13.5127	11.0283	12.3038	1789.5392	1237.1672	1560.4588
[10 × 2]	0.0087	0.0081	0.0082	13.5127	11.0283	12.3038	1678.5455	1264.3752	1494.3919
[05 × 3]	0.0098	0.0088	0.009	13.5127	11.0283	12.3038	1536.7203	1121.6722	1372.375
[10 × 3]	0.0106	0.0095	0.0097	13.5127	11.0283	12.3038	1423.8915	1044.3646	1265.3708
[10 × 5]	0.0134	0.0128	0.0129	13.5127	11.0283	12.3038	1059.4452	821.1617	954.0566
Example 2: $\alpha = [0.1, 0.4]$	$D^{\alpha} t^{0.5}$ $\mu = 0.5$								
[10 × 1]	0.3003	0.0086	0.0159	49.0094	37.7902	42.9231	5689.221	125.843	2692.6945
[05 × 2]	0.0152	0.0096	0.0115	49.0094	37.7902	42.9231	5113.4126	2489.7942	3734.1837
[10 × 2]	0.0228	0.0104	0.0132	49.0094	37.7902	42.9231	4698.863	1659.1176	3252.9887
[05 × 3]	0.0175	0.0114	0.0124	49.0094	37.7902	42.9231	4306.3569	2160.569	3458.8496
[10 × 3]	0.0206	0.0125	0.0141	49.0094	37.7902	42.9231	3915.6971	1834.3295	3046.0334
[10 × 5]	0.0304	0.0165	0.0188	49.0094	37.7902	42.9231	2978.0945	1243.3373	2278.6191
Example 3a: $\alpha = [0.1, 0.4]$	$D^{\alpha} sin(2\pi 4t)$								
[10 × 1]	0.2067	0.0065	0.0108	48.0211	34.1206	41.5817	7375.2998	165.096	3857.5078
[05 × 2]	0.0087	0.0076	0.0079	48.0211	34.1206	41.5817	6339.4894	3922.8094	5266.8347
[10 × 2]	0.0088	0.008	0.0082	48.0211	34.1206	41.5817	5988.519	3887.0387	5082.6847
[05 × 3]	0.0093	0.0088	0.0089	48.0211	34.1206	41.5817	5465.3227	3677.0003	4664.029
[10 × 3]	0.0103	0.0096	0.0097	48.0211	34.1206	41.5817	5024.9026	3323.8507	4281.5417
[10 × 5]	0.0136	0.0127	0.0129	48.0211	34.1206	41.5817	3793.3459	2506.643	3221.1382
Example 3b: $\alpha = [0.1, 0.4]$	$D^{\alpha} cos(2\pi 4t)$								
[10 × 1]	0.2019	0.0065	0.0106	38.9906	34.3829	37.3661	5995.1726	170.3297	3518.9036
[05 × 2]	0.0089	0.0075	0.0077	38.9906	34.3829	37.3661	5214.3244	3873.1999	4864.9736
[10 × 2]	0.0089	0.008	0.0082	38.9906	34.3829	37.3661	4901.5898	3871.662	4558.5822
[05 × 3]	0.0096	0.0088	0.0091	38.9906	34.3829	37.3661	4412.7609	3567.7939	4125.9745
[10 × 3]	0.0108	0.0095	0.0098	38.9906	34.3829	37.3661	4085.9864	3189.1351	3828.3921
[10 × 5]	0.0134	0.0127	0.0129	38.9906	34.3829	37.3661	3073.0633	2565.4548	2894.9409
Example 4: $\alpha = [0.1, 0.4]$	$D^{\alpha} e^{-0.5t}$								
[10 × 1]	0.2559	0.0072	0.0131	14.4644	11.9954	13.6968	2007.2498	46.8817	1043.2826
[05 × 2]	0.0143	0.0083	0.0102	14.4644	11.9954	13.6968	1747.6208	836.0134	1347.5892
[10 × 2]	0.0224	0.0087	0.0128	14.4644	11.9954	13.6968	1659.5928	535.9479	1068.533
[05 × 3]	0.0171	0.0095	0.012	14.4644	11.9954	13.6968	1526.6154	701.0453	1143.9065
[10 × 3]	0.0178	0.0103	0.0121	14.4644	11.9954	13.6968	1403.4061	675.0183	1134.5651
[10 × 5]	0.0237	0.0136	0.0154	14.4644	11.9954	13.6968	1066.1232	506.3818	890.5274

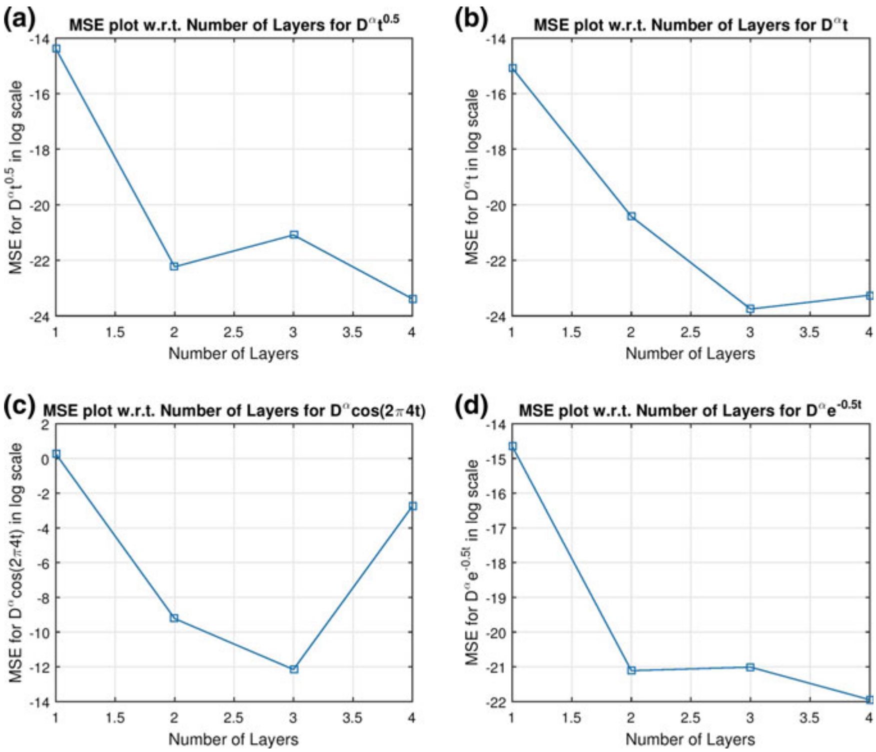

Fig. 16 Effect of ANN layers on error of caputo derivative **a** $D^{\alpha}t^{0.5}$ **b** $D^{\alpha}t$ **c** $D^{\alpha}cos(2\pi 4t)$ **d** $D^{\alpha}e^{-0.5t}$

definitions. Further the effect of increase in the number of hidden layers on the accuracy of approximation is also studied in details. The present work will greatly facilitate the tasks of simulation and real-time implementation of fractional-order systems and controllers.

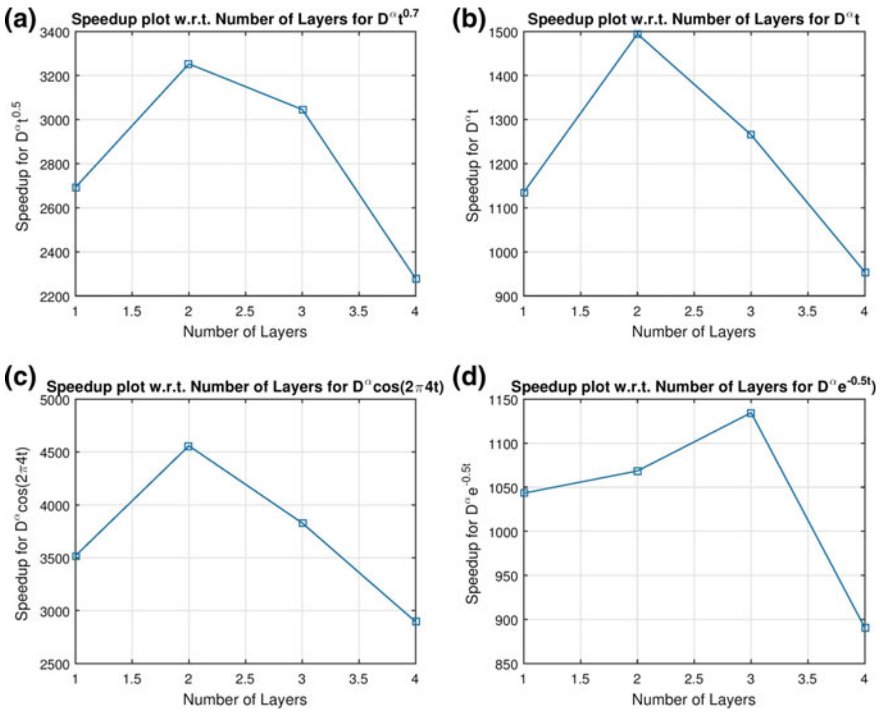

Fig. 17 Effect of ANN layers on speedup of caputo derivative **a** $D^{\alpha}t^{0.5}$ **b** $D^{\alpha}t$ **c** $D^{\alpha}cos(2\pi 4t)$ **d** $D^{\alpha}e^{-0.5t}$

Table 8 Error analysis of ANN approximation

Definition	Error (MSE)						
	$f_1 = t$	$f_2 = t^{\mu}$	$f_{3a} = sin(t)$	$f_{3b} = cos(t)$	$f_4 = e^{-0.5t}$	$f_5 = MLF1$	$f_6 = MLF2$
GL	5.21e-11	4.31e-11	1.10e-01	1.11e-01	1.15e-03	1.41e-02	3.12e-07
Caputo	4.83e-11	6.94e-10	1.63e-10	5.27e-06	7.51e-10	–	–
Definition	Error (standard deviation)						
	$f_1 = t$	$f_2 = t^{\mu}$	$f_{3a} = sin(t)$	$f_{3b} = cos(t)$	$f_4 = e^{-0.5t}$	$f_5 = MLF1$	$f_6 = MLF2$
GL	7.21e-6	7.2184e-6	0.0843	0.0932	0.00378	3.20e-5	4.968e-4
Caputo	6.49e-6	3.45e-4	6.43e-4	7.38e-2	3.26e-4	–	–
Definition	Error (variance)						
	$f_1 = t$	$f_2 = t^{\mu}$	$f_{3a} = sin(t)$	$f_{3b} = cos(t)$	$f_4 = e^{-0.5t}$	$f_5 = MLF1$	$f_6 = MLF2$
GL	5.21e-11	4.31e-11	1.10e-01	1.11e-01	1.15e-03	1.41e-02	3.12e-07
Caputo	4.83e-11	6.94e-10	1.63e-10	5.27e-06	7.51e-10	–	–

References

1. Oldham, K.B., Spanier, J.: The Fractional Calculus. Dover Publications, USA (2006)
2. Podlubny, I.: Fractional Differential Equations. Academic, USA (1999)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Netherlands (2006)
4. Vyawahare, V., Nataraj, P.S.V.: Fractional-order Modeling of Nuclear Reactor: from Subdiffusive Neutron Transport to Control-oriented Models: A Systematic Approach. Springer Singapore (2018)
5. Chen, Y., Petras, I., Xue, D.: Fractional order control - a tutorial. In: 2009 American Control Conference, St. Louis, MO, pp. 1397–1411 (2009)
6. Vyawahare, V.A., Nataraj, P.S.V.: Analysis of fractional-order point reactor kinetics model with adiabatic temperature feedback for nuclear reactor with subdiffusive neutron transport. In: Obaidat, M.S. Ören, T., Kacprzyk, J., Filipe, J. (eds.) Simulation and Modeling Methodologies, Technologies and Applications, pp. 153–172. Springer International Publishing, Cham (2015)
7. Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, D., Feliu, V.: Fractional-order Systems and Control: Fundamentals and Applications. Springer, London Limited, UK (2010)
8. Das, S.: Functional Fractional Calculus for System Identification and Controls. Springer, Germany (2011)
9. Singhaniya, N.G., Patil, M.D., Vyawahare, V.A.: Implementation of special mathematical functions for fractional calculus using DSP processor. In: 2015 International Conference on Information Processing (ICIP), India, pp. 811–816 (2015)
10. Tolba, M.F., AbdelAty, A.M., Said, L.A., Elwakil, A.S., Azar, A.T., Madian, A.H., Ounnas, A., Radwan, A.G.: FPGA realization of Caputo and Grunwald-Letnikov operators. In: 2017 6th International Conference on Modern Circuits and Systems Technologies (MOCAST), Thessaloniki, Greece, pp. 1–4 (2017)
11. Li, Chunguang, Chen, Guanrong: Chaos and hyperchaos in fractional-order Rössler equations. Phys. A: Stat. Mech. Its Appl. **341**, 55–61 (2004)
12. Li, C., Chen, G.: Chaos in the fractional order Chen system and its control. Chaos, Solitons Fractals **22**
13. Wang, Huihai, Sun, Kehui, He, Shaobo: Characteristic analysis and DSP realization of fractional-order simplified Lorenz system based on Adomian decomposition method. Int. J. Bifurc. Chaos **25**(06), 1550085 (2015)
14. Zurada, J.M.: Introduction to Artificial Neural Systems, vol. 8. West St. Paul, India (1992)
15. Schmidhuber, Jrgen: Deep learning in neural networks: an overview. Neural Netw. **61**, 85–117 (2015)
16. Maren, A.J., Harston, C.T., Pap, R.M.: Handbook of Neural Computing Applications. Academic, New York (2014)
17. Bose, N.K., Liang, P.: Neural Network Fundamentals with Graphs, Algorithms and Applications. Series in Electrical and Computer Engineering. McGraw-Hill, New York (1996)
18. Wong, B.K., Bodnovich, T.A., Selvi, Y.: Neural network applications in business: a review and analysis of the literature (1988-95). **19**(04), 301–320 (1997)
19. Wong, B.K., Selvi, Y.: Neural network applications in finance: a review and analysis of literature (1990–1996). Inf. Manag. **34**(3), 129–139 (1998)
20. Kaslik, E., Sivasundaram, S.: Dynamics of fractional-order neural networks. In: The 2011 International Joint Conference on Neural Networks, pp. 611–618 (2011)
21. Lagaris, I.E., Likas, A., Fotiadis, D.I.: Artificial neural networks for solving ordinary and partial differential equations. IEEE Trans. Neural Netw. **9**(5), 987–1000 (1998)
22. Zhang, S., Yu, Y., Yu, J.: Lmi conditions for global stability of fractional-order neural networks. IEEE Trans. Neural Netw. Learn. Syst. **28**(10), 2423–2433 (2017)
23. Pakdaman, M., Ahmadian, A., Effati, S., Salahshour, S., Baleanu, D.: Solving differential equations of fractional order using an optimization technique based on training artificial neural network **293**(01), 81–95 (2017)

24. Raja, M.A.Z., Samar, R., Manzar, M.A., Shah, S.M.: Design of unsupervised fractional neural network model optimized with interior point algorithm for solving Bagley–Torvik equation. Math. Comput. Simul. **132**, 139–158 (2017)
25. Stamova, Ivanka, Stamov, Gani: Mittag-Leffler synchronization of fractional neural networks with time-varying delays and reaction-diffusion terms using impulsive and linear controllers. Neural Netw. **96**, 22–32 (2017)
26. Ma, W., Li, C., Wu, Y., Wu, Y.: Synchronization of fractional fuzzy cellular neural networks with interactions. Chaos: Interdiscip. J. Nonlinear Sci. **27**(10), 103106 (2017)
27. Zhang, H., Ye, R., Cao, J., Ahmed, A., Li, X., Wan, Y.: Lyapunov functional approach to stability analysis of Riemann-Liouville fractional neural networks with time-varying delays. Asian J. Control **12**(35–42)
28. Lodhi, S., Manzar, M.A., Zahoor Raja, M.A.: Fractional neural network models for nonlinear Riccati systems. Neural Comput. Appl. 1–20 (2017)
29. Moré, J.J.: The Levenberg-Marquardt algorithm: implementation and theory. In: Numerical Analysis, pp. 105–116. Springer, India (1978)
30. Podlubny, I.: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. **5**(4), 367–386 (2002)
31. Heymans, N., Podlubny, I.: Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. Rheol. Acta **45**(5), 765–772 (2006)
32. Carpenteri, A., Mainardi, F. (eds.): Fractals and Fractional Calculus in Continuum Mechanics. Springer, USA (1997)
33. Compte, A., Metzler, R.: The generalized Cattaneo equation for the description of anomalous transport processes. J. Phys. A: Math. Gen. **30**, 7277–7289 (1997)
34. Ross, B. (ed.): Fractional Calculus and its Applications: Proceedings of the International Conference Held at the University of New Haven (USA), June 1974. Springer, USA (1975)
35. Kiryakova, V.: Generalized Fractional Calculus and Applications. Longman Science and Technology, UK (1994)
36. Machado, J.T., Kiryakova, V., Mainardi, F.: Recent history of fractional calculus. Commun. Nonlinear Sci. Numer. Simul. **16**(3), 1140–1153 (2011)
37. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Gordon and Breach Science Publishers, Netherlands (1997)
38. Chen, Y.Q., Vinagre, B.M., Podlubny, I.: Continued fraction expansion approaches to discretizing fractional order derivativesan expository review. Nonlinear Dyn. **38**(1–4), 155–170 (2004)
39. Carlson, G., Halijak, C.: Approximation of fractional capacitors (1/s)$\wedge$(1/n) by a regular newton process. IEEE Trans. Circuit Theory **11**(2), 210–213 (1964)
40. Khoichi, M., Hironori, F.: H-∞ optimized waveabsorbing control: analytical and experimental result. J. Guid., Control, Dyn. **16**(6), 1146–1153 (1993)
41. Oustaloup, A., Levron, F., Mathieu, B., Nanot, F.M.: Frequency-band complex noninteger differentiator: characterization and synthesis. IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. **47**(1), 25–39 (2000)
42. Charef, A., Sun, H.H., Tsao, Y.Y., Onaral, B.: Fractal system as represented by singularity function. IEEE Trans. Autom. Control. **37**(9), 1465–1470 (1992)
43. Machado, J.A.: Discrete-time fractional-order controllers. Fract. Calc. Appl. Anal. **4**, 47–66 (2001)
44. Vinagre, B.M., Podlubny, I., Hernandez, A., Feliu, V.: Some approximations of fractional order operators used in control theory and applications. Fract. Calc. Appl. Anal. **3**(3), 231–248 (2000)
45. Kumar, Satish: Neural Networks: A Classroom Approach. Tata McGraw-Hill Education, India (2004)
46. Dayhoff, Judith E.: Neural Network Architectures: An Introduction. Van Nostrand Reinhold Co., New York (1990)
47. Karlik, B., Olgac, A.V.: Performance analysis of various activation functions in generalized MLP architectures of neural networks. Int. J. Artif. Intell. Expert. Syst. **1**(4), 111–122 (2011)

48. Carpenter, G.A.: Neural network models for pattern recognition and associative memory. Neural Netw. **2**(4), 243–257 (1989)
49. DasGupta, B., Schnitger, G.: The power of approximating: a comparison of activation functions. In: Hanson, S.J., Cowan, J.D., Giles, C.L. (eds.) Advances in Neural Information Processing Systems, vol. 5, pp. 615–622. Morgan-Kaufmann (1993)
50. Glorot, X., Bengio, Y.: Understanding the difficulty of training deep feedforward neural networks. In: Teh, Y.W., Titterington, M. (eds.) Proceedings of the Thirteenth International Conference on Artificial Intelligence and Statistics. Proceedings of Machine Learning Research, vol. 9, pp. 249–256, Chia Laguna Resort, Sardinia, Italy, 13–15 (2010). (PMLR)
51. Hornik, Kurt, Stinchcombe, Maxwell, White, Halbert: Multilayer feedforward networks are universal approximators. Neural Netw. **2**(5), 359–366 (1989)
52. Psaltis, D., Sideris, A., Yamamura, A.A.: A multilayered neural network controller. IEEE Control Syst. Mag. **8**(2), 17–21 (1988)
53. Buscema, Massimo: Back propagation neural networks. Substance Use Misuse **33**(2), 233–270 (1998)
54. Karnin, E.D.: A simple procedure for pruning back-propagation trained neural networks. IEEE Trans. Neural Netw. **1**(2), 239–242 (1990)
55. Rogosin, Sergei: The role of the mittag-leffler function in fractional modeling. Mathematics **3**(2), 368–381 (2015)

Theory of Fractional Differential Equations Using Inequalities and Comparison Theorems: A Survey

J. V. Devi, F. A. McRae and Z. Drici

1 Introduction

In this chapter, we present a survey of the qualitative theory pertaining to fractional differential equations (FDEs) developed using differential inequalities and comparison theorems. Differential inequalities help in finding bounds for the solution of the nonlinear fractional differential equation, and once the bounds are known the constructive techniques of Quasilinearization and Monotone Iterative Technique provide the solution.

In Sect. 2, the basic concepts of lower and upper solutions are introduced and the fundamental lemma needed in the comparison theorems is given. Next, the concept of dominating component solution is introduced and existence results pertaining to these solutions are stated.

Section 3 begins with a result relating the solutions of the Caputo and the Riemann–Liouville differential equations. This is followed first by a result relating the solutions of fractional differential equations to those of ordinary differential equations and then by a variation of parameters formula for solutions of perturbed fractional differential equations in terms of solutions of ordinary differential equations. Next, a stability result using Dini derivatives is presented.

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Section 4 covers the concept of fractional trigonometric functions developed using fractional differential equations. It also covers the generalization of these results to fractional trigonometric-like functions.

Section 5 deals with impulsive fractional differential equations of two types, impulsive fractional differential equations with fixed moments of impulse and impulsive fractional differential equations with variable moments of impulse. For each type of equation, an existence and uniqueness result is given. In the case of fixed moments of impulse, the result presented was obtained using the Generalized Quasilinearization (GQL) method. Note that the Quasilinearization (QL) method is a special case of the GQL method. See [8] for an existence and uniqueness result obtained using this method. In the case of variable moments of impulse, the result was obtained using the method of lower and upper solutions and the Monotone Iterative Technique (MIT).

Results pertaining to periodic boundary value problem of Caputo fractional integro-differential equations form the content of Sect. 6.

2 Comparison Theorems, Existence Results, and Component Dominating Solution

2.1 Basic Concepts

The comparison theorems in the fractional differential equations setup require Holder continuity [22–24]. Although this requirement was used to develop iterative techniques such as the monotone iterative technique and the method of quasilinearization, there is no feasible way to check whether the functions involved are Holder continuous. However, the comparison results can be obtained using the weaker condition of continuity. In a subsequent paper [38], it was shown that the same results hold under the less restrictive condition of continuity. Similarly, in [11], differential inequalities, comparison theorems, and existence results were established under a continuity condition for impulsive fractional differential equations. Since Lemma 2.3.1 in [24] is essential in establishing the comparison theorems, we provide a sketch of the proof of this result under this weaker hypothesis. The basic differential inequality theorems and required comparison theorems are also stated.

We begin with the definition of the class $C_p[[t_0, T], \mathbb{R}]$.

Definition 2.1 $m \in C_p[[t_0, T], \mathbb{R}]$ means that $m \in C[(t_0, T], \mathbb{R}]$ and $(t - t_0)^p m(t) \in C[[t_0, T], \mathbb{R}]$ with $p + q = 1$

Definition 2.2 For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann–Liouville derivative of $m(t)$ is defined as

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1} m(s) ds \tag{2.1}$$

Lemma 2.3 *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in [t_0, T]$ we have $m(t_1) = 0$ and, $m(t) < 0$ for $t_0 \le t < t_1$, then it follows that*

$$D^q m(t_1) \ge 0.$$

Proof Consider $m \in C_p[[t_0, T], \mathbb{R}]$, such that $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \le t < t_1$.

Since $m(t)$ is continuous on $(t_0, T]$, given any t_1 such that $t_0 < t_1 < T$, there exists a $k(t_1) > 0$ and $h > 0$ such that

$$-k(t_1)(t_1 - s) \le m(t_1) - m(s) \le k(t_1)(t_1 - s) \tag{2.2}$$

for $t_0 < t_1 - h \le s \le t_1 + h < T$. Set $H(t) = \int_{t_0}^{t} (t - s)^{p-1} m(s) ds$ and consider $H(t_1) - H(t_1 - h) = \int_{t_0}^{t_1-h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds + \int_{t_1-h}^{t_s} (t_1 - s)^{p-1}$ $m(s)ds$.

Let $I_1 = \int_{t_0}^{t_1-h} [(t_1 - s)^{p-1} - (t_1 - h - s)^{p-1}] m(s) ds$ and $I_2 = \int_{t_1-h}^{t_1} (t_1 - s)^{p-1}$ $m(s)ds$. Since $t_1 - s > t_1 - h - s$ and $p - 1 < 0$, we have $(t_1 - s)^{p-1} < (t_1 - h - s)^{p-1}$.

This coupled with the fact that $m(t) \le 0$, $t_0 < t \le t_1$, implies that $I_1 \ge 0$. Now, consider $I_2 = \int_{t_1-h}^{t_0} (t_1 - s)^{p-1} m(s) ds$. Using (2.2) and the fact that $m(t_1) = 0$, we obtain

$$m(s) \ge -k(t_1)(t_1 - s),$$

and $I_2 \ge -k(t_1) \int_{t_1-h}^{t_1} (t_1 - s)^p ds = -k(t_1) \dfrac{h^{p+1}}{p+1}$, for $s \in (t_1 - h, t_1 + h)$. Thus we have

$$H(t_1) - H(t_1 - h) \ge -\frac{k(t_1)(h^{p+1})}{p+1}$$

and

$$\lim_{h \to 0} \left[\frac{H(t_1) - H(t_1 - h)}{h} + \frac{k(t_1) h^{p+1}}{h(p+1)} \right] \ge 0.$$

Since $p \in (0, 1)$, we conclude that $\dfrac{dH(t_1)}{dt} \ge 0$, which implies that $D^q m(t_1)$ ≥ 0. □

We next state the fundamental differential inequality result in the set up of fractional derivative, which is Theorem 2.3.2 in [24] with a weaker hypothesis of continuity.

Theorem 2.4 *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ and*

$$D^q v(t) \le f(t, v(t)),$$

$$D^q w(t) \ge f(t, w(t)),$$

$t_0 < t \le T$. Assume f satisfies the Lipschitz condition

$$f(t, x) - f(t, y) \le L(x - y),\ x \ge y,\ L > 0. \tag{2.3}$$

Then $v^0 \le w^0$, where $v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t - t_0)^{1-q}|_{t=t_0}$, implies $v(t) \le w(t),\ t \in [t_0, T]$.

Now, we define the Caputo fractional derivative, which we need in Sect. 3.

Definition 2.5 The Caputo derivative, denoted by ${}^c D^q u$, is defined as

$$ {}^c D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t - s)^{-q} u'(s) ds. \tag{2.4}$$

If $u(t)$ is Caputo differentiable, then we write $u \in C^q[[t_0, T], \mathbb{R}]$.

We now state the comparison theorem in terms of the Caputo derivative.

Theorem 2.6 *Assume that $m \in C^q[[t_0, T], \mathbb{R}]$ and*

$$ {}^c D^q m(t) \le g(t, m(t)),\ t_0 \le t \le T,$$

where $g \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$. Let $r(t)$ be the maximal solution of the initial value problem (IVP)

$$ {}^c D^q u = g(t, u),\ u(t_0) = u_0, \tag{2.5}$$

existing on $[t_0, T]$ such that $m(t_0) \le u_0$. Then, we have $m(t) \le r(t),\ t_0 \le t \le T$.

The results in Sects. 2.2 and 2.3 are taken from [39].

2.2 Dominating Component Solutions of Fractional Differential Equations

Consider the IVP for the Caputo differential equation given by

$$ {}^c D^q x = f(t, x), \tag{2.6}$$

$$ x(t_0) = x_0, \tag{2.7}$$

for $0 < q < 1,\ f \in C[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$.

If $x \in C^q[[t_0, T], \mathbb{R}^n]$ satisfies (2.6) and (2.7) then it also satisfies the Volterra fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds, \tag{2.8}$$

for $t_0 \leq t \leq T$.

Next, we present a class of functions that are possible solutions of the IVP of FDEs, and which under certain conditions satisfy the relations

$${}^c D^{q^+} |x(t)| \leq |{}^c D^q x(t)|$$

and

$$D^{q^+} |x(t)| \leq |D^q x(t)|,$$

where ${}^c D^{q^+}$ is the Caputo Dini derivative and D^{q^+} is the Riemann–Louiville (RL) fractional Dini derivative, which are defined as follows.

Definition 2.7 The Caputo fractional Dini derivative of a function $x(t)$ is defined as

$${}^c D^{q^+} x(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} D^+ x(s) ds$$

where D^+ is the usual Dini derivative defined in [25]. For more details on fractional Dini derivatives, see [21, 24].

Definition 2.8 The RL fractional Dini derivative is defined as

$$D^{q^+} x(t) = \frac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q} x(s) ds.$$

Definition 2.9 A continuous function $x : I \to \mathbb{R}^n$ is said to be a dominating component function (DCF) if there exists $i \in \{1, 2, \ldots, n\}$ such that $|x_j(s)| \leq x_i(t)$ and $|x'_j(t)| \leq x'_i(t)$ for all $t \in I = [t_0, T],\ j = 1, 2, \ldots, n$.

Definition 2.10 A continuous function $x : I \to \mathbb{R}^n$ is said to be a weakly dominating component function (WDCF) if there exists $i \in \{1, 2, \ldots, n\}$ such that $|x'_j|(t)| \leq x'_i(t)$ for all $t \in I,\ j = 1, 2, \ldots, n$.

Remark 2.11 Every DCF is a WDCF. For example, $x(t) = (\sqrt{t}, t)),\ t \in [1, 2]$ is a DCF and a WDCF whereas $x(t) = (\frac{1}{2}t^2, \frac{-1}{2}t, \frac{-1}{3}t, t),\ t \in [1, 2]$, is a WDCF.

Definition 2.12 By a weakly dominating component solution of the IVP (2.6) and (2.7), we mean a weakly dominating component function which satisfies the IVP (2.6) and (2.7).

We now state a comparison theorem in terms of the Caputo fractional Dini derivative. Note that it is essential to use Dini derivatives when we use an absolute value function or a norm function.

Theorem 2.13 *Assume that $f \in C[[I \times \mathbb{R}^n, \mathbb{R}^n]$ and satisfies the relation*

$$|f(t,x)| \leq g(t,|x|), \tag{2.9}$$

where $g \in C[[I \times \mathbb{R}_+, \mathbb{R}_+]$. Let $r(t)$ be the maximal solution of the scalar Caputo FDE

$$^cD^q u = g(t,u),\ u(t_0) = u_0. \tag{2.10}$$

If $x(t)$ is the weakly dominating solution of the IVP (2.6) and (2.7), then

$$|x(t,t_0,x_0)| \leq r(t,t_0,u_0),$$

$t \in I$ provided $|x_0| \leq u_0$.

Proof Set $m(t) = |x(t)|$ for $t \in I$. Then, using the definition of the Caputo fractional Dini derivative and the fact that $x(t)$ is WDCF of (2.6) and (2.7), we get

$$\begin{aligned}
^cD^{q^+}m(t) &= {}^cD^{q^+}|x(t)| \\
&= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} D^+|x(s)|ds, \\
&\leq \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} |x'(s)|ds, \\
&= \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} \max_j |x'_j(s)|ds, \\
&\leq \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x'_i(s)ds, \\
&\leq \max_j \left|\frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} x'_j(s)ds\right|
\end{aligned}$$

$$= |^cD^q x(t)| = |f(t,x(t))| \leq g((t,|x(t)|) = g((t,m(t)).$$

Now, with $m(t_0) = |x_0|$, the conclusion follows from the hypothesis and the application of Theorem 2.6, which yields

$$|x(t, t_0, x_0)| \leq r(t, t_0, x_0),\ \ t \in I.$$

Thus, the proof is complete. □

Remark 2.14 If $n = 1$, the above theorem states that the result holds if the solution belongs to the set of all increasing functions. In this case, one can observe that the Caputo FDE

$$^cD^q x = Lx, \qquad x(t_0) = x_0$$

has a solution, the Mittag-Leffler function, which is also a weakly dominating component solution.

2.3 Dominating Component Solutions for Riemann–Liouville FDE

Consider the IVP given by

$$D^q x = f(t, x) \tag{2.11}$$

$$x(t_0) = x^0 = x(t)(t - t_0)^{1-q}|_{t=t_0} \tag{2.12}$$

where $f \in C[I \times \mathbb{R}^n, \mathbb{R}^n]$.

For the sake of completeness we give the following definitions from [24].

Definition 2.15 Let $0 < q < 1$ and $p = 1 - q$. The function space $C_p[[t_0, T], \mathbb{R}^n] = \{u \in C[[(t_0, T], \mathbb{R}^n]$ and $(t - t_0)^p u(t) \in C[[t_0, T], \mathbb{R}^n]\}$

Definition 2.16 A function $x(t)$ is said to be a solution of the IVP (2.11) and (2.12) if and only if $x \in C_p[[(t_0, T], \mathbb{R}^n]$, $D^q x(t)$ exists and $x(t)$ is continuous on $[t_0, T]$ and satisfies the relations (2.11) and (2.12).

Definition 2.17 A function $x(t)$ is said to be dominating component solution of the IVP (2.11) and (2.12) if $x(t)$ is a dominating component function and further satisfies the IVP (2.11) and (2.12).

Theorem 2.18 *Assume that f in (2.11) satisfies*

$$|f(t, x(t)| \leq g((t, |x(t)|), \tag{2.13}$$

where $g \in C[I \times \mathbb{R}_+, \mathbb{R}_+]$. Let $r(t)$ be the maximal solution of the scalar Riemann–Liouville FDE

$$D^q u = g((t, u),\ u(t_0) = u^0 = u(t)(t - t_0)^{1-q}|_{t=t_0} \tag{2.14}$$

Further, if $x(t)$ is the dominating component solution of (2.11) and (2.12) then,

$$|x(t, t_0, x^0| \leq r(t, t_0, u^0),$$

$t \in I$, provided $|x^0| \leq u^0$.

Proof Set $m(t) = |x(t)|,\ t \in I$. Using the definition of the RL fractional derivative and the fact that $x(t)$ is a dominating component function, we get

$$\begin{aligned} D^{q^+}m(t) &= \tfrac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q}|x(s)|ds \\ &= \tfrac{1}{\Gamma(1-q)} D^+ \int_{t_0}^{t} (t-s)^{-q} x_i(s)ds \\ &= \tfrac{1}{\Gamma(1-q)} \tfrac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} x_i(s)ds \\ &= f_i(t, x(t)) \\ &= |f(t, x(t))| \\ &\leq g(t, |x(t)|) = g(t, m(t)). \end{aligned}$$

Then,with $m(t_0) = |x^0|$, a result for Riemann–Liouville FDEs, parallel to Theorem 2.6, yields

$$|x(t, t_0, x^0)| \leq r(t, t_0, x^0),$$

$t \in I$. □

Remark 2.19 Note that, in case of Riemann–Liouville FDE, for $n = 1$, we need the solutions to be positive and also increasing. Thus, it is clear that Riemann–Liouville FDEs are more complex than Caputo FDEs.

Next, we give criteria that will guarantee the existence of a dominating component solution for Riemann–Liouville FDE (2.11) and (2.12). Since, as will be shown in Sect. 3, any result that holds for solutions of Riemann–Liouville FDE also holds for solutions of the corresponding Caputo FDE, we obtain a sufficiency condition for the Riemann–Liouville FDEs to have a dominating component solution.

Theorem 2.20 *Suppose that $f \in C^1[I \times \mathbb{R}^n, \mathbb{R}^n]$ in (2.11) is a dominating component-bounded function, that is, there exists an $i \in \{1, 2, 3 \ldots, n\}$ such that*

$$|f_j(t, x)| \leq f_i(t, x) < M \tag{2.15}$$

$$|\frac{d}{dt} f_j(t, x)| \leq \frac{d}{dt} f_i(t, x) \tag{2.16}$$

where $(t, x) \in I \times \mathbb{R}^n,\ j = 1, 2, 3 \ldots, n$. Further, for the above fixed i assume the following criteria hold

(i) $x_i^0 = \max\{x_1^0, x_2^0, x_3^0, \ldots, x_n^0\}$ *and* $|x_j^0| < x_i^0,\ j = 1, 2, 3, \ldots, n.$ (2.17)

(ii) For every neighborhood of t_0, the following relation holds

$$(t - t_0) f_i(t, x^0) > x_i^0(1 - q) \tag{2.18}$$

(iii) For all $j \neq i$, the following relations hold in every neighborhood of t_0,

$$f_i(t_0, x^0) + f_j(t_0, x^0) \geq \frac{(1-q)}{(t-t_0)}(x_i^0 + x_j^0), \tag{2.19}$$

$$f_j(t_0, x^0) - f_i(t_0, x^0) \leq \frac{(1-q)}{(t-t_0)}(x_i^0 - x_j^0). \tag{2.20}$$

Then, there exists a dominating component solution for the IVP of Riemann–Liouville FDE (2.11) and (2.12).

3 The Variational Lyapunov Method and Stability Results

Next, we give a relation between ordinary differential equations (ODEs) and fractional differential equations (FDEs), then present the variation of parameters formula for FDEs in terms of ODEs. This is an important result, as obtaining the variation of parameters formula for FDEs in terms of fractional derivatives is still an open problem. Then, we present a stability result using the variational Lyapunov method. In order to establish the above results, a relation between the solutions of Caputo and Riemann–Liouville fractional differential equations is needed, which we give in the next section.

3.1 Relation Between the Solutions of Caputo and Riemann–Liouville Fractional Differential Equations

In this section, we begin with a relation between the solutions of Caputo FDEs and those of Riemann–Liouville FDEs. This relation leads to the observation that the solutions of Caputo FDEs have the same properties as the solutions of the Riemann–Liouville FDEs [11].

Consider the Caputo fractional differential equation and the corresponding Volterra integral differential equation given by

$${}^cD^q x(t) = F(t, x), \ x(t_0) = x_0 \tag{3.1}$$

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} F(s, x(s)) ds. \tag{3.2}$$

The aforementioned relation is established by observing that

$$ {}^{c}D^{q}x(t) = D^{q}[x(t) - x(t_0)]. \tag{3.3} $$

Setting $y = x - x_0$, we have

$$ {}^{c}D^{q}y = D^{q}x = F(t, x) = F(t, y + x_0) $$

which gives

$$ {}^{c}D^{q}y = \hat{F}(t, y) \tag{3.4} $$

and

$$ y^{0} = [x(t) - x_0](t - t_0)^{1-q}|_{t=t_0} = 0, \tag{3.5} $$

The integral equation corresponding to (3.4) and (3.5) is given by

$$ y(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} \hat{F}(s, y(s))ds. \tag{3.6} $$

Suppose $y(t)$ is a solution of the Volterra fractional integral equation (3.6). Then $y(t)$ also satisfies the corresponding Riemann–Liouville fractional differential equation (3.4). Letting $y(t) = x(t) - x_0$, we obtain

$$ x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} F(s, x(s))ds, $$

which implies that $x(t)$ satisfies the integral equation (3.2) and hence is a solution of both the Caputo fractional differential equation and its corresponding Volterra integral equation.

Thus, a given Caputo FDE can be transformed into a Riemann–Liouville FDE, and hence solutions of Caputo fractional differential equations have properties similar to the properties of solutions of Riemann–Liouville fractional differential equations.

3.2 Relation Between Ordinary Differential Equations and Fractional Differential Equations

The method of variation of parameters provides a link between unknown solutions of a nonlinear system and the known solutions of another nonlinear system, and, as such, is a useful tool for the study of the qualitative behavior of the unknown solutions.

We now present a relation between ODEs and FDEs which was developed in [24]. Then, we use the variation of parameters formula to link the solutions of the two systems. Using this relation and the properties of the solutions of ODEs, which are relatively easy to find, the properties of the solutions of the corresponding FDEs can be investigated.

Consider the IVP

$$D^q x = f(t, x),\ x^0 = x(t)(t - t_0)^q|_{t=t_0}, \tag{3.7}$$

where $f \in C[([t_0, T] \times \mathbb{R}^n, \mathbb{R}^n), x \in C_p([t_0, T], \mathbb{R}^n)$, $D^q x$ is the Riemann–Liouville fractional differential operator of order $q, 0 < q < 1,\ 1 - q = p$, and assume the existence and uniqueness of the solution $x(t, t_0, x^0)$ of (3.7).

To obtain a relation between fractional and ordinary differential equations, we tentatively write

$$x(t) = x(s) + \phi(t - s), t_0 \le s \le T, \tag{3.8}$$

with the function $\phi(t - s)$ to be determined. Substituting this expression in the Riemann–Liouville fractional differential equation, we get

$$\begin{aligned} D^q x(t) &= \frac{1}{\Gamma(p+1)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{p-1}[x(t) - \phi(t-s)]ds \\ &= \frac{1}{\Gamma(p+1)} \frac{d}{dt}[x(t)(t-t_0)^p] - \eta(t, p, \phi). \end{aligned} \tag{3.9}$$

where

$$\eta(t, p, \phi) = \frac{1}{\Gamma(p+1)} \frac{d}{dt}[\int_{t_0}^{t} (t-s)^{p-1}\phi(t-s)ds]. \tag{3.10}$$

Setting $y(t) = \dfrac{x(t)(t-t_0)^p}{\Gamma(1+p)}$, where $x(t)$ is any solution of IVP (3.7), we arrive at the IVP for ordinary differential equation, namely,

$$y'(t) = \frac{dy}{dt} = F(t, y(t)) + \eta(t, p, \phi),\ y(t_0) = x^0 \tag{3.11}$$

where

$$F(t, y) = f(t, \Gamma(1+p)y(t)(t-t_0)^{-p}). \tag{3.12}$$

We consider the unperturbed system

$$y'(t) = F(t, y(t)),\ y(t_0) = x^0, \tag{3.13}$$

and the perturbed system (3.11) and use perturbation theory to obtain the estimates of $|y(t)|$. The nonlinear variation of parameters formula is also a very useful tool to study perturbation theory. It was developed for fractional differential equation in terms of ordinary differential equations in [24] and is presented below.

Suppose $F_y(t, y)$ exists and is continuous on $[t_0, T] \times \mathbb{R}^n$. It is known, (*see Theorem 2.5.3 in* [25]), that the solution $y(t, t_0, x^0)$ of IVP (3.13) satisfies the identity

$$\frac{\partial}{\partial t_0} y(t, t_0, x^0) + \frac{\partial}{\partial x_0} y(t, t_0, x^0) F(t_0, x_0) \equiv 0, \tag{3.14}$$

where $\frac{\partial}{\partial t_0} y(t, t_0, x^0)$ and $\frac{\partial}{\partial x_0} y(t, t_0, x^0) F(t_0, x^0)$ are solutions of the linear system

$$z' = F_y(t, y(t, t_0, x^0))z,$$

with the corresponding initial conditions $z(t_0) = -F(t_0, x^0)$ and $z(t_0) = I$, the identity matrix. Using this information, the nonlinear variation of parameters formula for the solutions of IVP (3.11) was obtained. Setting $p(s) = y(t, s, z(s))$, where $z(t)$ is the solution of the perturbed IVP (3.11), and using (3.13) we have

$$\begin{aligned} \frac{d}{ds} p(s) &= \frac{\partial}{\partial t_0} y(t, s, z(s)) + \frac{\partial}{\partial x^0} y(t, s, z(s))[F(s, z(s)) + \eta(s, t_0, \phi_0)] \\ &= \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_o). \end{aligned}$$

Integrating from t_0 to t yields the desired nonlinear variation of parameters formula, which links the solutions of the fractional differential equation to the solutions of the generated ordinary differential equation:

$$z(t, t_0, x^0) = y(t, t_0, x^0) + \int_{t_0}^{t} \frac{\partial}{\partial x^0} y(t, s, z(s))\eta(s, t_0, \phi_0) ds.$$

3.3 Variational Lyapunov Method and Stability

In order to present the stability results, the Caputo fractional Dini derivative of the Lyapunov function is defined using the Grunwald–Letnikov fractional derivative, taking advantage of the series in its definition.

Definition 3.1 The Grunwald–Letnikov (GL) fractional derivative is defined as

$$D_0^q x(t) = \lim_{\substack{h \to 0_+ \\ nh = t - t_0}} \frac{1}{h^q} \sum_{r=0}^{n} (-1)^r \, {}_qC_r x(t - rh) \tag{3.15}$$

or

$$D_0^q x(t) = \lim_{h\to 0_+} \frac{1}{h^q} x_h^q(t),$$

where

$$\begin{aligned} x_h^q(t) &= \frac{1}{h^q}\sum_{r=0}^{n}(-1)^r\ {}_qC_r x(t-rh) \\ &= \frac{1}{h^q}[x(t) - S(x,h,r,q)] \end{aligned} \tag{3.16}$$

with

$$S(x,h,r,q) = \sum_{r=1}^{n}(-1)^{r+1}\ {}_qC_r x(t-rh). \tag{3.17}$$

Now, using (3.15) we define the GL fractional Dini derivative by

$$D_{0_+}^q x(t) = \limsup_{h\to 0_+} \frac{1}{h^q}\sum_{r=0}^{n}(-1)^r\ {}_qC_r x(t-rh). \tag{3.18}$$

Since the Caputo fractional derivative and GL fractional derivative are related by the equation

$${}^cD^q x(t) = D_0^q[x(t) - x(t_0)],$$

we define the Caputo fractional Dini derivative by

$${}^cD_+^q x(t) = D_{0_+}^q[x(t) - x(t_0)]. \tag{3.19}$$

Consider the Caputo differential equation

$${}^cD^q x = f(t,x),\ \ x(t_0) = x_0. \tag{3.20}$$

Then, using relations (3.19) and (3.20), we get

$$\begin{aligned} f(t,x) &= \limsup_{h\to 0_+} \frac{1}{h^q}\sum_{r=0}^{n}(-1)^r\ {}_qC_r[x(t-rh) - x_0] \\ &= \limsup_{h\to 0_+} \frac{1}{h^q}[x(t) - x_0 - S(x,h,r,q)] \end{aligned}$$

where $S(x,h,r,q) = \sum_{r=1}^{n}(-1)^{r+1}\ {}_qC_r[x(-rh) - x_0]$. This yields

$$S(x,h,r,q) = x(t) - x(t_0) - h^q f(t,x) - \epsilon(h^q)], \tag{3.21}$$

where $\dfrac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$. The definition of the Caputo fractional Dini derivative for the Lyapunov function is given below.

Definition 3.2 Let $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ where $S(\rho) = \{x : |x| < \rho\}$. Let $V(t, x)$ be locally Lipschitzian in x. The Grunwald–Letnikov fractional Dini derivative of $V(t, x)$ is defined by

$$D^q_{0_+} V(t, x) = \limsup_{h \to 0_+} \frac{1}{h^q}[V(t, x) - \sum_{r=1}^{n} (-1)^{r+1} {}_qC_r V(t - rh, S(x, h, r, q))]$$

where $S(x, h, r, q) = x(t) - h^q f(t, x) - \epsilon(h^q)$ with $\dfrac{\epsilon(h^q)}{h^q} \to 0$ as $h \to 0$. Then, the Caputo fractional Dini derivative of $V(t, x)$ is defined as

$${}^cD^q_+ V(t, x) = \limsup_{h \to 0_+} \frac{1}{h^q}[V(t, x) - V(t - h, x - h^q f(t, x)) - V(t_0, x_0)].$$

Definition 3.3 The zero solution of (3.1) is said to be

(i) *stable* if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $|x_0| < \delta$ implies $|x(t; t_0, x_0)| < \epsilon$ for $t \geq t_0$;
(ii) *uniformly stable* if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, for $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ with $|x_0| < \delta$, the inequality $|x(t; t_0, x_0)| < \epsilon$ holds for $t \geq t_0$;
(iii) *uniformly attractive* if for $\beta > 0$ and for every $\epsilon > 0$ there exists $T = T(\epsilon) > 0$ such that for any $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ with $|x_0| < \beta$, the inequality $|x(t; t_0, x_0)| < \epsilon$ holds for $t \geq t_0 + T$;
(iv) *uniformly asymptotically stable* if the zero solution is uniformly stable and uniformly attractive.

Now we present a comparison theorem, which uses the variation of parameters formula and relate the solutions of a perturbed system to the known solutions of an unperturbed system in terms of the solution of a comparison scalar fractional differential equation.

Consider the two fractional differential systems given by

$${}^cD^q y = f(t, y), \ y(t_0) = y_0, \tag{3.22}$$
$${}^cD^q x = F(t, x), \ x(t_0) = x_0 \tag{3.23}$$

where $f, F \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}^n]$, and assume the following assumption relative to system (3.22).

(H) The solutions $y(t, t_0, x_0)$ of (3.22) exist for all $t \geq t_0$, are unique and continuous with respect to the initial data, and $|y(t, t_0, x_0|$ is locally Lipschitzian in x_0.

Let $|x_0| < \rho$ and suppose that $|y(t, t_0, x_0)| < \rho$ for $t_0 \leq t \leq T$. For any $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ and for any fixed $t \in [t_0, T]$, we define the Grunwald–Letnikov fractional Dini derivative of V by

$$D^q_{0_+} V(s, y(t, s, x))$$
$$= \limsup_{h \to 0_+} \frac{1}{h^q} \{V(s, y(t, s, x)) - \sum_{r=1}^{n} (-1)^{r+1} {}_qC_r V(s - rh, s - h^q F(s, x))\}.$$

The Caputo fractional Dini derivative of the Lyapunov function $V(s, y(t, s, x))$, for any fixed $t \in [t_0, T]$, any arbitrary point $s \in [t_0, T]$ and $x \in \mathbb{R}^n$, is given by

$$^cD^q_+ V(s, y(t, s, x))$$
$$= \limsup_{h \to 0_+} \frac{1}{h^q} \{V(s, y(t, s, x)) - V(s - h, y(t, s - h, x - h^q F(s, x)))\},$$

where

$$V(s - h, y(t, s - h, x - h^q F(s, x)))$$
$$= \sum_{r=1}^{n} (-1)^{r+1} {}_qC_r \; V(s - rh, y(t, s - rh, x - h^q F(s, x))).$$

Theorem 3.4 *Assume that assumption (H) holds. Suppose that*

(i) $V \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$, $V(t, x)$ *is locally Lipschitzian in x with Lipschitz constant* $L > 0$, *and for* $t_0 \le s \le t$ *and* $x \in S(\rho)$,

$$^cD^q_+ V(s, y(t, s, x)) \le g(s, V(s, y(t, s, x)); \tag{3.24}$$

(ii) $g \in C[\mathbb{R}^2_+, \mathbb{R}]$ *and the maximal solution* $r(t, t_0, u_0)$ *of*

$$^cD^q u = g(t, u), \;\; u(t_0) = u_0 \ge 0 \tag{3.25}$$

exists for $t_0 \le t \le T$.
Then, if $x(t) = x(t, t_0, x_0)$ *is any solution of (3.23), we have* $V(t, x(t, t_0, x_0)) \le r(t, t_0, u_0), \;\; t_0 \le t \le T$, *provided* $V(t_0, y(t, t_0, x_0)) \le u_0$.

The following stability result is an application of Theorem 3.4.

Theorem 3.5 *Assume that (H) holds and condition (i) of Theorem 3.4 is satisfied. Suppose that* $g \in C[\mathbb{R}^2, \mathbb{R}], \;\; g(t, 0) = 0, \;\; f(t, 0) = 0, \;\; F(t, 0) = 0$ *and for* $(t, x) \in \mathbb{R}_+ \times S(\rho)$,

$$b(|x|) \le V(t, x) \le a(|x|)$$

$a, b \in \mathbb{K} = \{c \in C[[0, \rho), \mathbb{R}_+] : c(0) = 0$ *and c is monotonically increasing}. Further suppose that the trivial solution of (3.22) is uniformly stable and* $u \equiv 0$ *of (3.25) is asymptotically stable. Then, the trivial solution of (3.23) is uniformly asymptotically stable.*

4 Fractional Trigonometric Functions

It is well known that trigonometric functions play a vital role in understanding physical phenomena that exhibit oscillatory behavior. The generalization of trigonometric functions has been made through differential equations. In this section, we give a brief summary of the work done in order to introduce fractional trigonometric functions and their generalizations through fractional differential equations of a specific type [35]. Fractional hyperbolic functions and their generalizations are also described in a similar fashion in [36].

Consider the following αth order homogeneous fractional initial value with Caputo derivative

$$ {}^{c}D^{\alpha}x(t) + x(t) = 0,\ 1 < \alpha < 2,\ t \geq 0, \tag{4.1}$$

$$ x(0) = 1,\ {}^{c}D^{q}x(0) = 0,\ \text{where } \alpha = 2q, 0 < q < 1. \tag{4.2}$$

The general solution of (4.1) and (4.2) is given by $c_1x(t) + c_2y(t)$, where c_1 and c_2 are arbitrary constants, and where $x(t)$ and $y(t)$ are infinite series solutions of the form

$$ x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)},\ y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)},\ t \geq 0,\ 0 < q < 1.$$

We designate these series by $\cos_q t$ and $\cos_q t$, respectively. Then,

$$ \cos_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2kq}}{\Gamma(1+2kq)}, \tag{4.3}$$

$$ \sin_q t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(2k+1)q}}{\Gamma(1+(2k+1)q)}, \tag{4.4}$$

which we denote $M^q_{2,0}(t)$ and $M^q_{2,1}(t)$, respectively, for future convenience. Observe that if $q = 1$, $\cos_q t = \cos t$ and $\sin_q t = \sin t$. Using the FDE (4.1) and the initial condition (4.2), one can prove the following properties of $x(t)$ and $y(t)$:

(1) $x^2(t) + y^2(t) = 1,\ t \geq 0$
(2) $x(t)$ and $y(t)$ have at least on zero in $\mathbb{R}_+$.
(3) The zeros of $x(t)$ and $y(t)$ interlace each other, i.e., between any two consecutive zeros of $y(t)$ there exists one and only one zero of $x(t)$.
(4) For $t \geq 0$ and $\eta \geq 0$

$$ y(t+\eta) = y(t)x(\eta) + y(\eta)x(t)$$

$$ x(t+\eta) = x(t)x(\eta) + y(\eta)y(t)$$

(5) $x(t)$ is an even function, but for $q \neq 1$, $y(t)$ is not an odd function
(6) Euler's Formulae:
The solutions of FDE (4.1) can also be expressed as $E_q(it^q)$ and $E_q(-it^q)$ where $\pm i$ are the roots of $\lambda^2 + 1 = 0$. $E_q(-it^q)$ can be expressed in terms of $M^q_{2,0}(t)$ *and* $M^q_{2,1}(t)$ as

$$(i)\ E_q(it^q) = 1 - \frac{t^{2q}}{\sum(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots + i\left(\frac{t^q}{\sum(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \ldots\right),$$

$$= M^q_{2,0}(t) + iM^q_{2,1}(t)$$

$$(ii)\ E_q(-it^q) = 1 - \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{4q}}{\Gamma(1+4q)} - \cdots - i\left(\frac{t^q}{\Gamma(1+q)} - \frac{t^{3q}}{\Gamma(1+3q)} + \ldots\right).$$
$$= M^q_{2,0}(t) - iM^q_{2,1}(t)$$

$$\text{Thus,}\quad M^q_{2,0}(t) = \frac{1}{2}(E_q(it^q) + E_q(-it^q)), \text{ and}$$

$$M^q_{2,1}(t) = \frac{1}{2i}(E_q(it^q) - E_q(-it^q)), t \in \mathbb{R}^+.$$

The following theorem generalizes the notion of fractional trigonometric functions using an αth order fractional differential equation of the type considered in (4.1).

Theorem 4.1 *Consider the* αth *order fractional IVP of the form*

$$^cD^\alpha x(t) + x(t) = 0,\ x(0) = 1,\ ^cD^q x(0) = 0, \ldots,\ ^cD^{(n-1)_q} x(0) = 0 \qquad (4.7)$$

where $n < \alpha < n + 1$, *with* $\alpha = nq$, $0 < q < 1$, *n fixed.*

The general solution of this equation is given by $c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$ *where* $c_1, c_2, \ldots, c_n$ *are arbitrary constants and* $x_1(t), x_2(t), \ldots, x_n(t)$ *are infinite series of the form*

$$x_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{nkq}}{\Gamma(1+nkq)}$$
$$x_2(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+1)q}}{\Gamma(1+(nk+1)q)}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$x_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)},$$

which are denoted by $M^q_{n,0}(t)$, $M^q_{n,1}(t), \ldots, M^q_{n,n-1}(t)$, *respectively.*

More generally, let

$$M^q_{n,r}(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{(nk+r)q}}{\Gamma(1+(nk+r)q)}, \qquad n \in N, t \geq 0.$$

These are the n linearly independent solutions of the Caputo FDE (4.7).

Let $x_1(t),\ x_2(t), \ldots, x_n(t)$ be n solutions of the nth order Caputo FDE for $t \in \mathbb{R}^+$. Then, the Wronskian $W(t)$ of the n solutions is defined as

$$W(t) = \begin{vmatrix} x_1 & x_2 \cdots & x_n \\ -x_n & x_1 \cdots & x_{n-1} \\ -x_{n-1} & -x_n \quad \cdots & x_{n-2} \\ \vdots & \vdots \quad \vdots & \vdots \\ -x_2 & -x_3 \quad \cdots & x_1 \end{vmatrix}(t).$$

Theorem 4.2 *Let $x_1(t),\ x_2(t), \ldots, x_n(t)$ be n solutions of (4.7). Then, these solutions are linearly independent on $\mathbb{R}_+$ if and only if $W(t) \neq 0$ for every $t \in \mathbb{R}_+$.*

Finally, we give the addition formula of the solutions of (4.7) for $\eta \geq 0$ and $t \geq 0$,

$$M^q_{n,r}(t+\eta) = \sum_{k=0}^{r} M^q_{n,k}(t) M^q_{n,r-k}(\eta) - \sum_{k=r+1}^{n-1} M^q_{n,k}(t) M^q_{n,n+r-k}(\eta).$$

5 Impulsive Differential Equations

It is well established that many evolutionary processes exhibit impulses, which are perturbations whose duration is negligible compared to the duration of the process. Thus, differential equations with impulses are appropriate mathematical models for the study of physical phenomena exhibiting sudden change. As fractional differential equations are considered better models of processes that have memory and hereditary properties, it is natural to use FDEs with impulses to study perturbations or sudden changes in these systems.

In this section, we present known existence and uniqueness results for impulsive fractional differential equation with both fixed and variable moments of impulse.

In both cases, we use the theory of inequalities and comparison theorems, the method of lower and upper solutions and the iterative methods of quasilinearization (QL) and monotone iterative technique (MIT). In order to illustrate this approach, we present an existence and uniqueness result for impulsive FDEs using the generalized QL method for fixed moments of impulse and using the method lower and upper solutions and the MIT for variable moments of impulse.

5.1 FDE with Fixed Moments of Impulse

We begin with the basic notation and a definition of the solution of a FDE with fixed moments of impulse and then proceed to the generalized QL method.

Definition 5.1 Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC_p[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C_p-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t, y) = h(t_k^+, x)$$

exists for $k = 1, 2, \ldots, n-1$.

Definition 5.2 Let $0 \leq t_0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$. Then we say that $h \in PC^q[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ if $h : (t_{k-1}, t_k] \times \mathbb{R}^n \to \mathbb{R}^n$ is C^q-continuous on $(t_{k-1}, t_k] \times \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$

$$\lim_{(t,y)\to(t_k^+,x)} h(t, y) = h(t_k^+, x)$$

exists for $k = 1, 2, \ldots, n-1$.

Consider the impulsive Caputo fractional differential system defined by

$$\begin{cases} {}^cD^q x = f(t, x), \ t \neq t_k, \\ x(t_k^+) = I_k(x(t_k)), \ k = 1, 2, \ldots, n-1, \\ x(t_0) = x_0, \end{cases} \tag{5.1}$$

where $f \in PC[[t_0, T] \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \ldots, n-1$.

Definition 5.3 By a solution $x(t, t_0, x_0)$ of system (5.1), we mean a PC^q continuous function $x \in PC^q[[t_0, T], \mathbb{R}^n]$, such that

$$x(t) = \begin{cases} x_0(t, t_0, x_0), & t_0 \leq t \leq t_1, \\ x_1(t, t_1, x_1^+), & t_1 \leq t \leq t_2, \\ \cdot \\ \cdot \\ \cdot \\ x_k(t, t_k, x_k^+), & t_k < t \leq t_{k+1}, \\ \cdot \\ \cdot \\ x_{n-1}(t, t_{n-1}, x_{n-1}^+), & t_{n-1} < t \leq T, \end{cases} \tag{5.2}$$

where $0 \leq t_0 < t_1 < t_2 < \cdots < t_{n-1} \leq T$ and $x_k(t, t_k, x_k^+)$ is the solution of the following fractional initial value problem

$$\begin{array}{c} {}^cD_x^q = f(t, x), \\ x_k^+ = x(t_k^+) = I_k(x(t_k)) \end{array}.$$

Definition 5.4 $\alpha, \beta \in PC^q[[t_0, T], \mathbb{R}]$ are said to be lower and upper solutions of equation (5.1), if and only if they satisfy the following inequalities:

$$\begin{cases} {}^cD^q\alpha \leq f(t,\alpha) + g(t,\alpha), t \neq t_k, \\ \alpha(t_k^+) \leq I_k(\alpha(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \alpha(t_0) \leq x_0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} {}^cD^q\beta \geq f(t,\beta) + g(t,\beta), t \neq t_k, \\ \beta(t_k^+) \geq I_k(\beta(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \beta(t_0) \geq x_0, \end{cases} \tag{5.4}$$

respectively.

We first state two lemmas [9] that are needed to prove the main theorem.

Lemma 5.5 *The linear, nonhomogeneous impulsive Caputo initial value problem*

$$\begin{cases} {}^cD^q x = M(x-y) + f(t,y) + g(t,y), \ t \neq k, \\ x(t_k^+) = (I_k(x(t_k)), \ k = 1, 2, \ldots, n-1, \\ x(t_0) = x_0, \end{cases}$$

has a unique solution on the interval $[t_0, T]$.

Lemma 5.6 *Suppose that*

(i) $\alpha_0(t)$ *and* $\beta_0(t)$ *are lower and upper solutions of the hybrid Caputo fractional differential equation (5.1).*
(ii) $\alpha_1(t)$ *and* $\beta_1(t)$ *are the unique solutions of the following linear, impulsive Caputo initial value problems,*

$$\begin{cases} {}^cD^q\alpha_1 = f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha-\alpha_0) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0), t \neq t_k, \\ \alpha_1(t_k^+) = I_k(\alpha_0(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \alpha_1(t_0) = x_0, \end{cases} \tag{5.5}$$

and

$$\begin{cases} {}^cD^q\beta_1 = f(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g(t,\beta_0) + g_x(t,\beta_0)(\beta_1-\beta_0), t \neq t_k, \\ \beta_1(t_k^+) = I_k(\beta_0(t_k)), k = 1, 2, 3, \ldots, n-1, \\ \beta_1(t_0) = x_0, \end{cases} \tag{5.6}$$

respectively;
(iii) $I_k(x)$ *is nondecreasing function in* x *for each* $k = 1, 2, 3, \ldots, n-1$*;*
(iv) f_x, g_x *are continuous and Lipschitz in* x *on* $[t_0, T]$.

Then, $\alpha_0(t) \leq \alpha_1(t) \leq \beta_1(t) \leq \beta_0(t)$ *on* $[t_0, T]$.

We now state the main result.

Theorem 5.7 *Assume that*

(i) $f, g \in PC[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $\alpha_0, \beta_0 \in PC^q[[t_0, T], \mathbb{R}]$ *are lower and upper solutions of (5.1) such that* $\alpha_0(t) \leq \beta_0(t), t \in [t_0, T]$*;*

(ii) *$f_x(t,x)$ exists, is increasing in x for each t, $f(t,x) \geq f(t,y) + f_x(t,y)(x-y)$, $x \geq y$ and $|f_x(t,x) - f_x(t,y| \leq L_1|x-y|$, and further suppose that $g_x(t,x)$ exists, is decreasing in x for each t, $g(t,x) \geq g(t,y) + g_x(t,y)(x-y)$, $x \geq y$ and $|g_x(t,x) - g_x(t,y| \leq L_2|x-y|$;*

(iii) *I_k is increasing and Lipschitz in x for each $k = 1, 2, 3, \ldots, n-1$.*

Then, there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \to \rho, \beta_n \to r$, as $n \to \infty$, uniformly and monotonically to the unique solution $\rho = r = x$ of (5.1) on $[t_0, T]$, and the convergence is quadratic.

Remark 5.8 Observe that if we set $I_k \equiv 0$ for all k, then (5.1) reduces to a Caputo fractional differential equation, for which the generalized quasilinearization for this type of equations has been studied in [24], under the assumption of a Holder continuity. However, Theorem 5.7, with $I_k \equiv 0$, shows that those results also hold with the weakened hypothesis of C^q-continuity.

5.2 *Impulsive Differential Equation with Variable Moments of Impulse*

Consider a sequence of surfaces $\{S_k\}$ given by $S_k : t = \tau_k(x),\ k = 1, 2, 3, \ldots;\ \tau_k : \mathbb{R} \to \mathbb{R}$ such that $\tau_k(x) < \tau_{k+1}(x)$ and $\lim\limits_{k\to\infty} \tau_k(x) = \infty$. Then, the impulsive Caputo FDE with variable moments of impulse is given by

$$\left.\begin{array}{l} {}^cD^q x = f(t,x),\ t \neq \tau_k(x) \\ x(t^+) = x(t) + I_k(x(t)),\ t = \tau_k(x). \end{array}\right\} \tag{5.7}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$ is an open set, $\tau_k \in C[\Omega, (0,\infty)]$, $I_k(x(t)) = \triangle(x(t)) = x(t^+) - x(t^-)$, and $I_k \in C[\Omega, \mathbb{R}]$, $k = 1, 2, 3, \ldots$.

In this case, the moments of impulse depend on the solutions satisfying $t_k = \tau_k(x(t_k))$, for each k. Thus, solutions starting at different points will have different points of discontinuity. Also, a solution may hit the same surface $S_k : t = \tau_k(x)$ several times and we shall call such a behavior "*pulse phenomenon*". In addition, different solutions may coincide after some time and behave as a single solution thereafter. This phenomenon is called "*confluence*".

In order to construct the method of lower and upper solutions in the given interval, we have to ensure that the solution does not exhibit a pulse phenomenon. The following theorem gives a simple set of sufficient conditions for any solution to meet each surface exactly once and shows the interplay between the functions f, τ_k, and I_k [14]. In the rest of the section,we shall assume that the solution of (5.7) exists for $t \geq t_0$ and is C_p continuous.

Theorem 5.9 *Assume that*

(i) *$f \in C[[t_0, T] \times \Omega, \mathbb{R}]$, $t_0 \geq 0$, $I_k \in C[\Omega, \mathbb{R}]$, $\tau_k \in C[\Omega, (0,\infty)]$, is linear and bounded, and $\tau_k(x) < \tau_{k+1}(x)$ for each k;*

(ii) *(a)* $\dfrac{\partial \tau_k(x)}{\partial x} f(t,x) < \dfrac{(t-\tilde{t})^p}{\Gamma(p+1)}$, *whenever* $t = \tau_k(x(t,\tilde{t},\tilde{x}))$,

(b) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_k(x))\right) I_k(x) < 0$, *and*

(c) $\left(\dfrac{\partial \tau_k}{\partial x}(x + sI_{k-1}(x))\right) I_{k-1}(x) \geq 0, 0 \leq s \leq 1,\ \ x + I_k(x) \in \Omega$ *whenever* $x \in \Omega$.

Then, every solution $x(t) = x(t, t_0, x_0)$ *of IVP (5.7), such that* $0 \leq t_0 < \tau_1(x_0)$, *meets each surface* S_k *exactly once.*

Next, we consider the following initial value problem:

$$\begin{aligned} &{}^cD^q x = f(t,x), t \neq \tau(x), \\ &x(t^+) = x(t) + I(x(t)),\ \ t = \tau(x) \\ &x(t_0^+) = x_0, \end{aligned} \tag{5.8}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R}, \mathbb{R}]$, and $\tau \in C^q[\mathbb{R}, (0, \infty)]$, with $J = [t_0, T]$, $t_0 \geq 0$, $\tau(x)$ is linear of the form $\lambda_0 x + \lambda_1$, $\lambda_0 \in \mathbb{R}^+$, $\lambda_1 \in \mathbb{R}$, and $\tau(x)$ is increasing.

The lower and upper solutions of (5.8) are defined as follows:

Definition 5.10 A function $v \in C_p[J, \mathbb{R}]$ is said to be a lower solution of (5.8) if it satisfies the following inequalities

$$\begin{aligned} &{}^cD^q v \leq f(t,v), t \neq \tau(v(t)), \\ &v(t^+) \leq v(t) + I(v(t)),\ \ t = \tau(v(t)) \\ &v(t_0^+) \leq x_0, \end{aligned} \tag{5.9}$$

Definition 5.11 A function $w \in C_p[J, \mathbb{R}]$ is said to be an upper solution of (5.8), if it satisfies the following inequalities

$$\begin{aligned} &{}^cD^q w \geq f(t,w), t \neq \tau(w(t)), \\ &w(t^+) \geq w(t) + I(w(t)),\ \ t = \tau(w(t)) \\ &w(t_0^+) > x_0, \end{aligned} \tag{5.10}$$

The following result is the fundamental inequality theorem in the theory of Caputo fractional differential inequalities with variable moments of impulse [13].

Theorem 5.12 *Assume that*

(i) $v, w \in C_p[J, R]$ *are lower and upper solutions of (5.8), respectively;*
(ii) $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R}, \mathbb{R}]$, $\tau \in C^q[\mathbb{R}, (0, \infty)]$, τ *is linear and increasing;*
(iii) $\tau_x(v + sI(v))I(v) < 0,\ t = \tau(v(t)),\ 0 \leq s \leq 1$*;*
(iv) $\tau_x(w + sI(w))I(w) > 0,\ t = \tau(w(t)),\ 0 \leq s \leq 1$*;*

(v) $\tau_x(v)f(t,v) < \dfrac{(t-t_1)^p}{p}$, *whenever* $t = \tau(v(t,t_1,v_1))$, *where* $v(t,t_1,v_1)$ *is the lower solution of (5.8) starting at* (t_1,v_1), $t_1, t \in J$;

(vi) $\tau_x(w)f(t,w) > \left\{\dfrac{(t-t_1)^p}{p}\right\}$, *whenever* $t = \tau(w(t,t_1,w_1))$, *where* $w(t,t_1,w_1)$ *is the upper solution of (5.8) starting at* (t_1,w_1), $t_1, t \in J$.

(vii) $f(t,x) - f(t,y) \le L(x-y), x \ge y, L > 0$.

Then, $v(t_0) \le w(t_0)$ *implies* $v(t) \le w(t),\ t_0 \le t \le T$.

Next, we state an existence result based on the existence of upper and lower solutions.

Theorem 5.13 *Let* $v, w \in C_p[J,\mathbb{R}]$ *be lower and upper solutions of (5.8), respectively, such that* $v(t) \le w(t)$ *on* J. *Suppose that* $w(t)$ *hits the surface* $S : t = \tau(x)$ *only once at* $t = t_* \in (t_0, T]$ *and* $w(t_*) < w(t_*^+)$. *Also, assume*

(i) $f \in C[J \times \mathbb{R}, \mathbb{R}], \tau \in C^q[\mathbb{R}, (0,\infty)],$ *τ is linear and increasing for* $v(t) \le x \le w(t), t \in J$;

(ii) $\tau_x(x + sI(x))I(x) < 0,\ 0 \le s \le 1,\ t = \tau(x),\ v(t) \le x \le w(t),\ t \in J$;

(iii) $\tau_x(x)f(t,x) < \dfrac{(t-t_1)^p}{p}$ *whenever* $t = \tau(x(t,t_1,x_1)),\ v(t) \le x \le w(t),\ t, t_1 \in J$,

(iv) For any (t,x) *such that* $t = \tau(x), v(t) \le x \le w(t)$ *implies* $v(t) \le x^+ \le w(t)$, $t \in J$.

Then, there exists a solution $x(t)$ *of (5.8) such that* $v(t) \le x(t) \le w(t)$ *on* J.

The method of upper and lower solutions, described previously, gives a theoretical result, namely, the existence of a solution of (5.8) in a closed sector, whereas the monotone iterative technique is a constructive method, which gives a sequence that converges to a solution of (5.8). In the case of impulsive Caputo fractional differential equations with variable moments of impulsive, this practical method involves working with sequences of solutions of a simple linear Caputo fractional differential equation of order $q, 0 < q < 1$, with variable moments of impulse. This result is given in the following theorem [12].

Theorem 5.14 *Assume that*

(i) $v_0,\ w \in PC_p[J,\mathbb{R}]$ *are lower and upper solutions of (5.8) respectively, such that* $v_0(t) \le w(t)$ *on* J, *and* $w(t)$ *hits the surface* $S : t = \tau(x)$ *only once at* $t = t_* \in (t_0, T]$ *and* $w(t_*) < w(t_*^+)$, $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $I \in C[\mathbb{R},\mathbb{R}], \tau \in C^q[\mathbb{R}, (0,\infty)]$ *and* $\tau(x)$ *is linear and increasing for* $v_0(t) \le x \le w(t), t \in J$;

(ii) $\tau_x(x + sI(x))I(x) < 0, 0 \le s \le 1, t = \tau(x), v_0(t) \le x \le w(t), t \in J$;

(iii) $\dfrac{\partial \tau}{\partial x} f(t,x) < \dfrac{(t-t_1)^p}{p}$, *whenever* $t = \tau(x(t,t_1,x_1)), v_0(t) \le x \le w(t)$;

(iv) $f(t,x) - f(t,y) \ge -M(x-y),\ v_0(t) \le y \le w(t),\ t \in J, M > 0$;

(v) for any (t,x) *such that* $t = \tau(x),\ v_0(t) \le x \le w(t)$ *implies* $v_0(t) \le x^+ \le w(t), t \in J$,

Then, there exists a monotone sequence $\{v_n\}$ *such that* $v_n \to \rho$ *as* $n \to \infty$ *monotonically on* J. *Also,* ρ *is the minimal solution of (5.8).*

6 Fractional Integro-Differential Equations

It is well known that integro-differential equations are used to mathematically model physical phenomena,where past information is necessary to understand the present. On the other hand, fractional differential equations play an important role in studying processes that have memory and hereditary properties. Fractional integro-differential equations combine these two topics. In this section, we present a summary of results involving periodic-boundary value problems (PBVP) for fractional integro-differential equations using inequalities and comparison theorems [37].

Consider the following Caputo fractional integro-differential equation

$$^{c}D^{q}u = f(t, u, I^{q}u) \tag{6.1}$$

$$u(0) = u_0, \tag{6.2}$$

where $f \in C[J \times \mathbb{R} \times \mathbb{R}^{+}, \mathbb{R}],\ u \in C^{1}[J, \mathbb{R}], J = [0, T]$,

$$\text{and} \qquad I^{q}u(t) = \frac{1}{\Gamma(q)}\int_{0}^{t}(t-s)^{q-1}u(s)ds. \tag{6.3}$$

The following theorem gives the explicit solution of the linear Caputo fractional integro-differential initial value problem.

Theorem 6.1 *Let* $\lambda \in C^{1}([0, T], \mathbb{R})$. *The solution of* $^{c}D^{q}\lambda(t) = L\lambda(t) + MI^{q}\lambda(t)$ *is given by*

$$\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{{}_{n+k}C_{k}M^{n}L^{k}2^{n+k}\lambda(0)}{\Gamma[(2n+1)q+1]}t^{(2n+1)q}$$

where $L, M > 0$.

The following comparison theorem is needed to prove the main result.

Theorem 6.2 *Let* $J = [0, T],\ f \in C[J \times \mathbb{R} \times \mathbb{R}^{+}, \mathbb{R}], v, w \in C^{1}[J, \mathbb{R}]$ *and suppose that the following inequalities hold for all* $t \in J$.

$$^{c}D^{q}v(t) \le f(t, v(t), I^{q}v(t)), v(0) \le u_0 \tag{6.4}$$

$$^{c}D^{q}w(t) \ge f(t, w(t), I^{q}w(t)), w(0) \ge u_0. \tag{6.5}$$

Suppose further that $f(t, u(t), I^{q}u(t))$ *satisfies the following Lipschitz-like condition,*

$$f(t, x, I^{q}x) - f(t, y, I^{q}y) \le L(x-y) + M(I^{q}x - I^{q}y), \tag{6.6}$$

for $x \ge y, L, M > 0$. *Then,* $v(0) \le w(0)$ *implies that* $u(t) \le w(t), 0 \le t \le T$.

Corollary 6.3 *Let $m \in C^1[J, \mathbb{R}]$ be such that*

$$^cD^q m(t) \le L\, m(t) + MI^q m(t),\ m(0) = m_0 \le 1,$$

then

$$m(t) \le \lambda(t)$$

for $0 \le t \le T,\ L, M > 0;\ \lambda(0) = 1$ *and* $\lambda(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{{}_{n+k}C_k M^n L^k 2^{n+k}}{\Gamma[(2n+1)q+1]} t^{(2n+1)q}.$

Proof We have

$$\begin{aligned} ^cD^q m(t) &\le L\, m(t) + MI^q m(t), \\ ^cD^q \lambda(t) + 2L\, \lambda(t) + 2MI^q \lambda(t) & \\ &\ge L\lambda(t) + MI^q\lambda(t), \\ \text{for } m(0) = m_0 \le 1 = \lambda(0). \end{aligned}$$

Hence, from Theorem 6.2 we conclude that $m(t) \le \lambda(t), t \in J$. □

The result in the above corollary is true even if $L = M = 0$, which we state below.

Corollary 6.4 *Let* $^cD^q m(t) \le 0$ *on* $[0, T]$. *If* $m(0) \le 0$ *then* $m(t) \le 0, t \in J$.

Proof By definition of $^cD^q m(t)$ and by hypothesis,

$$^cD^q m(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} m'(s) ds \le 0,$$

which implies that $m'(t) \le 0$, on $[0, T]$. Therefore $m(t) \le m(0) \le 0$ on $[0, T]$. The proof is complete. □

Next, we present a result which uses the generalized monotone iterative technique in order to obtain minimal and maximal solutions of the Caputo fractional integro-differential equation

$$^cD^q u = F(t, u, I^q u) + G(t, u, I^q u), \tag{6.7}$$

with the boundary condition

$$g(u(0), u(T)) = 0, \tag{6.8}$$

where $F, G \in C[J \times \mathbb{R} \times \mathbb{R}^+, \mathbb{R}]$, $u \in C^1[J, \mathbb{R}]$.

Definition 6.5 Let $v_0, w_0 \in C^1[J, \mathbb{R}]$. Then v_0 and w_0 are said to be coupled lower and upper solutions of Type I of (6.7) and (6.8) if

$$\begin{aligned} {}^cD^q v_0(t) &\le F(t, v_0(t), I^q v_0(t)) + G(t, w_0(t), I^q w_0(t)), \\ g(v_0(0), v_0(T)) &\le 0 \end{aligned} \tag{6.9}$$

$$\begin{aligned} {}^cD^q w_0(t) &\le F(t, w_0(t), I^q w_0(t)) + G(t, v_0(t), I^q v_0(t)), \\ g(w_0(0), w_0(T)) &\ge 0 \end{aligned} \tag{6.10}$$

The monotone iterative technique for (6.7) and (6.8) was developed using sequences of iterates which are solutions of linear fractional integro-differential initial value problems. Since the solution of a linear Caputo fractional differential equation is unique, the sequence of iterates is a unique sequence converging to a solution of (6.7) and (6.8). In this approach,it is not necessary to prove the existence of a solution of the Caputo fractional integro-differential equation as it follows from the construction of the monotone sequences.

In the following theorem, coupled lower and upper solutions of Type I are used to obtain monotone sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (6.7) and (6.8).

Theorem 6.6 *Suppose that*

(i) v_0, w_0 *are coupled lower and upper solutions of Type I for (6.7) and (6.8) with* $v_0(t) \le w_0(t)$ *on* J*;*

(ii) the function $g(u, v) \in C[\mathbb{R}^2, \mathbb{R}]$ *is nonincreasing in* v *for each* u*, and there exists a constant* $M > 0$ *such that*

$$g(u_1, v) - g(u_2, v) \le M(u_1 - u_2),$$

for $v_0(0) \le u_2 \le u_1 \le w_0(0),\ v_0(T) \le v \le w_0(T)$*;*

(iii) $F, G \in C[J \times \mathbb{R} \times \mathbb{R}_+, \mathbb{R}]$ *and* $F(t, x_1, x_2)$ *is nondecreasing in* x_1 *for each* $(t, x_2) \in J \times \mathbb{R}_+$ *and is nondecreasing in* x_2 *for each* $(t, x_1) \in J \times \mathbb{R}$*; Further,* $G(t, y_1, y_2)$ *is nonincreasing in* y_1 *for each* $(t, y_2) \in J \times \mathbb{R}_+$ *and is nonincreasing in* y_2 *for each* $(t, y_1) \in J \times \mathbb{R}$*.*

Then, the iterative scheme given by

$$\begin{aligned} {}^cD^q v_{n+1} &= F(t, v_n, I^q v_n) + G(t, w_n, I^q w_n), \\ v_{n+1}(0) &= v_n(0) - \frac{1}{M} g(v_n(0), v_n(T)) \\ {}^cD^q w_{n+1} &= F(t, w_n, I^q w_n) + G(t, v_n, I^q v_n), \\ w_{n+1}(0) &= w_n(0) - \frac{1}{M} g(w_n(0), w_n(T)), \end{aligned}$$

yields two monotone sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *such that*

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0.$$

Further, $v_n \to \rho$ and $w_n \to r$ in $C^1[J, \mathbb{R}]$, uniformly and monotonically, such that ρ and r are,respectively, the coupled minimal and maximal solutions of (6.7) and (6.8), that is, ρ and r satisfy the coupled system

$$\begin{aligned} {}^cD^q\rho &= F(t, \rho, I^q\rho) + G(t, r, I^qr), \\ &g(\rho(0), \rho(T)) = 0, \\ {}^cD^qr &= F(t, r, I^qr) + G(t, \rho, I^q\rho), \\ &g(r(0), r(T)) = 0. \end{aligned}$$

7 Conclusion

Our aim in this chapter was to give a brief survey of the qualitative theory of fractional differential equations developed using the fundamental concepts of differential inequalities and comparison theorems, as well as constructive monotone iterative methods. The results presented here constitute only a representative sample of the work done using these tools. For additional results see, for example, Abbas and Bechohra [1], Agarwal et al. [3–6], Jankowski [16–20], Lin et. al. [26], Nanware [29], Sambadham et al. [31, 32], Vatsala et al. [10, 30, 33, 34], Wang et al. [40–44], Yakar et al. [45, 46], and Zhang [46].

The main results in Sects. 2 and 3 are from Devi et al. [38, 39], Drici et al. [11] and Lakshmikantham et al. [22–24]. The main result in Sect. 4 is from Devi et al. [35]. The main results in Sect. 5 are from Giribabu et al. [12–14] and Devi and Radhika [8, 9]. The result in Sect. 6 is from Devi and Sreedhar [37].

References

1. Abbas, S., Bechohra, M.: Upper and lower solutions method for impulsive partial hyperbolic differential equations with fractional order. Nonlinear Anal.: Hybrid Syst. **4**, 406–413 (2010)
2. Adjabi, Y., et al.: On generalized fractional operators and a gronwall type inequality with applications. Filomat **31**(17), 5457–5473 (2017)
3. Agarwal, R., Hristova, S., O'Regan, D.: Caputo fractional differential equations with non-instantaneous impulses and strict stability by Lyapunov functions. Filomat **31**(16), 5217–5239 (2017)
4. Agarwal, R., O'Regan, D., Hristova, S.: Stability with initial time difference of Caputo fractional differential equations by Lyapunov functions. Z. Anal. Anwend. **36**(1), 49–77 (2017)
5. Agarwal, R., O'Regan, D., Hristova, S.: Stability of Caputo fractional differential equations by Lyapunov functions. Appl. Math. **60**(6), 653–676 (2015)
6. Agarwal, R.P., Benchohra, M., Hamani, S., Pinelas, S.: Upper and lower solutions method for impulsive differential equations involving the Caputo fractional derivative. Mem. Differ. Equ. Math. Phys. **53**, 1–12 (2011)
7. Bushnaq, S., Khan, S.A., Shah, K., Zaman, G.: Mathematical analysis of HIV/AIDS infection model with Caputo-Fabrizio fractional derivative. Cogent Math. Stat. **5**(1), 1432521 (2018)
8. Devi, J.V., Radhika, V.: Quasilinearization for hybrid Caputo fractional differential equations. Dyn. Syst. Appl. **21**(4), 567–581 (2012)

9. Devi, J.V., Radhika, V.: Generalized quasilinearization for hybrid Caputo fractional differential equations. Dyn. Contin. Discret. Impuls. Syst. Ser. A Math. Anal. **19**(6), 745–756 (2012)
10. Denton, Z., Ng, P.W., Vatsala, A.S.: Quasilinearization method via lower and upper solutions for Riemann-Liouville fractional differential equations. Nonlinear Dyn. Syst. Theory **11**(3), 239–251 (2011)
11. Drici, Z., McRae, F., Devi, J.V.: On the existence and stability of solutions of hybrid Caputo differential equations. Dyn. Contin. Discret. Impuls. Syst. Ser. A **19**, 501–512 (2012)
12. Giribabu, N., Devi, J.V., Deekshitulu, G.V.S.R.: Monotone iterative technique for Caputo fractional differential equations with variable moments of impulse. Dyn. Contin. Discret. Impuls. Syst. Ser. B **24**, 25–48 (2017)
13. Giribabu, N., Devi, J.V., Deekshitulu, G.V.S.R.: The method of upper and lower solutions for initial value problem of caputo fractional differential equations with variable moments of impulse. Dyn. Contin. Discret. Impuls. Syst. Seri. A **24**, 41–54 (2017)
14. Giribabu, N.: On pulse phenomena involving hybrid caputo fractional differential equations with variable moments of impulse. GJMS Spec. Issue Adv. Math. Sci. Appl.-13 GJMS **2**(2), 93–101 (2014)
15. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**(2), 1075–1081 (2007)
16. Jankowski, T.: Systems of nonlinear fractional differential equations. Fract. Calc. Appl. Anal. **18**(1), 122–132 (2015)
17. Jankowski, T.: Fractional problems with advanced arguments. Appl. Math. Comput. **230**, 371–382 (2014)
18. Jankowski, T.: Boundary problems for fractional differential equations. Appl. Math. Lett. **28**, 14–19 (2014)
19. Jankowski, T.: Existence results to delay fractional differential equations with nonlinear boundary conditions. Appl. Math. Comput. **219**(17), 9155–9164 (2013)
20. Jankowski, T.: Fractional equations of Volterra type involving a Riemann-Liouville derivative. Appl. Math. Lett. **26**(3), 344–350 (2013)
21. Jalilian, Y.: Fractional integral inequalities and their applications to fractional differential inequalities. Acta Math. Sci. **36B**(5), 1317–1330 (2016)
22. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. **69**, 2677–2682 (2008)
23. Lakshmikantham, V., Vatsala, A.S.: Theory of fractional differential inequalities and applications. Commun. Appl. Anal. **11**(3–2), 395–402 (2007)
24. Lakshmikantham, V., Leela, S., Devi, J.V.: Theory of Fractional Dynamic Systems. Cambridge Scientific Publishers, Cambridge (2009)
25. Lakshmikantham, V., Leela, S.: Differential and Integral Inequalities, vol. I. Academic Press, New York (1969)
26. Lin, L., Liu X., Fang, H.: Method of upper and lower solutions for fractional differential equations. Electron. J. Differ. Equ. 1–13 (2012)
27. Andric, M., Barbir, A., Farid, G., Pecaric, J.: Opial-type inequality due to AgarwalPang and fractional differential inequalities. Integr. Transform. Spec. Funct. **25**(4), 324–335 (2014). https://doi.org/10.1080/10652469.2013.851079
28. Sarikaya, M.Z., Set, E., Yaldiz, H., Baak, N.: HermiteHadamards inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
29. Nanware, J.A., Dhaigude, D.B.: Existence and uniqueness of solutions of differential equations of fractional order with integral boundary conditions. J. Nonlinear Sci. Appl. **7**(4), 246–254 (2014)
30. Pham, T.T., Ramirez, J.D., Vatsala, A.S.: Generalized monotone method for Caputo fractional differential equations with applications to population models. Neural Parallel Sci. Comput. **20**(2), 119–132 (2012)
31. Sambandham, B., Vatsala, A.S.: Basic results for sequential Caputo fractional differential equations. Mathematics **3**, 76–91 (2015)

32. Sambandham, B., Vatsala, A.S.: Numerical results for linear Caputo fractional differential equations with variable coefficients and applications. Neural Parallel Sci. Comput. **23**(2–4), 253–265 (2015)
33. Sowmya, M., Vatsala, A.S.: Generalized iterative methods for Caputo fractional differential equations via coupled lower and upper solutions with superlinear convergence. Nonlinear Dyn. Syst. Theory **15**(2), 198–208 (2015)
34. Stutson, D.S., Vatsala, A.S.: Riemann Liouville and Caputo fractional differential and integral inequalities. Dyn. Syst. Appl. **23**(4), 723–733 (2014)
35. Devi, J.V., Deo, S.G., Nagamani, S.: On fractional trigonometric functions and their generalizations. Dyn. Syst. Appl. **22** (2013)
36. Devi, J.V., Namagani, S.: On fractional hyperbolic functions and their generalizations. Nonlinear Stud. **20**(3), 1–19 (2013)
37. Devi, J.V., Sreedhar, C.V.: Generalized monotone iterative method for Caputo fractional integro-differential equations. Eur. J. Pure Appl. Math. **9**(4), 346–359 (2016)
38. Devi, J.V., McRae, F., Drici, Z.: Variational Lyapunov method for fractional differential equations. Comput. Math. Appl. **64**, 2982–2989 (2012)
39. Devi, J.V, Kishore, M.P.K., Ravi Kumar, R.V.G.: On existence of component dominating solutions for fractional differential equations. Nonlinear Stud., **21**(1), 45–52 (2014)
40. Wang, G.: Monotone iterative technique for boundary value problems of nonlinear fractional differential equations with deviating arguments. J. Comput. Appl. Math. **236**, 2425–2430 (2012)
41. Wang, G., Agarwal, R.P., Cabada, A.: Existence results and the monotone iterative technique for systems of nonlinear fractional differential equations. Appl. Math. Lett. **25**, 1019–1024 (2012)
42. Wang, G., Baleanu, D., Zhang, L.: Monotone iterative method for a class of nonlinear fractional differential equations. Fract. Calc. Appl. Anal. **15**(2), 244–252 (2012)
43. Wang, X.: Wang, L., Zeng, Q.: Fractional differential equations with integral boundary conditions. J. Nonlinear Sci. Appl. **8**, 309–314 (2015)
44. Yakar, C.: Fractional differential equations in terms of comparison results and Lyapunov stability with initial time difference. Abstr. Appl. Anal. (2010)
45. Yakar, C., Yakar, A.: Monotone iterative technique with initial time difference for fractional differential equations. Hacet. J. Math. Stat. **40**(2), 331–340 (2011)
46. Zhang, L., Ahmad, B., Wang, G.: The existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative. Appl. Math. Lett. **31**, 1–6 (2014)

Exact Solutions of Fractional Partial Differential Equations by Sumudu Transform Iterative Method

Manoj Kumar and Varsha Daftardar-Gejji

Abstract Developing analytical methods for solving fractional partial differential equations (FPDEs) is an active area of research. Especially, finding exact solutions of FPDEs is a challenging task. In the present chapter, we extend Sumudu transform iterative method to solve a variety of time and space FPDEs as well as systems of them. We demonstrate the utility of the method by finding exact solutions to a large number of FPDEs.

1 Introduction

Nonlinear fractional partial differential equations (FPDEs) play an important role in science and technology as they describe various nonlinear phenomena especially dealing with memory. To obtain physical information and deeper insights into the physical aspects of the problems, one has to find their exact solutions which usually is a difficult task. For solving linear FPDEs, integral transform methods are extended successfully [1, 2]. Various decomposition methods have been developed for solving the linear and nonlinear FPDEs such as Adomian decomposition method (ADM) [3], Homotopy perturbation method (HPM) [4], Daftardar-Gejji and Jafari method (DJM) [5–8], and so on. Further, combinations of integral transforms and decomposition methods have proven to be useful. A combination of Laplace transform and DJM (Iterative Laplace transform method) has been developed by Jafari et al. [9]. A combination of HPM and Sumudu transform yields homotopy perturbation Sumudu transform method [10]. Similarly, a combination of Sumudu transform and ADM termed as Sumudu decomposition method has been developed [11]. Recently,

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Sumudu transform iterative method (STIM), which is a combination of Sumudu transform and DJM has been introduced and applied for solving time-fractional Cauchy reaction–diffusion equation [12]. Further, a fractional model of nonlinear Zakharov–Kuznetsov equations also has been solved using STIM [13].

In this chapter, we extend STIM to solve time and space FPDEs as well as systems of them. A variety of problems have been solved using the STIM. In some cases, STIM yields an exact solution of the time and space FPDEs as well as systems of them which can be expressed in terms of the well-known Mittag-Leffler functions or fractional trigonometric functions. Further, it has been observed that semi-analytical techniques with Sumudu transform require less CPU time to calculate the solutions of nonlinear fractional models, which are used in applied science and engineering. STIM is a powerful technique to solve different classes of linear and nonlinear FPDEs. STIM can reduce the time of computation in comparison to the established schemes while preserving the accuracy of the approximate results.

The organization of this chapter is as follows: In Sect. 2, we give basic definitions related to fractional calculus and Sumudu transform. In Sect. 3, we extend STIM for time and space FPDEs. In Sect. 4, we apply extended STIM to solve various time and space FPDEs. Further, in Sect. 5 we extend STIM for system of time and space FPDEs. In Sect. 6, we apply extended STIM for system of time and space FPDEs. Conclusions are summarized in Sect. 7.

2 Preliminaries and Notations

In this section, we give some basic definitions, notations, and properties of the fractional calculus ([1, 14]), which are used further in this chapter.

Definition 2.1 Riemann–Liouville fractional integral of order $\alpha > 0$, of a real-valued function $f(t)$ is defined as

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2.2 Caputo derivative of order $\alpha > 0$ $(n-1 < \alpha < n)$, $n \in \mathbb{N}$ of a real-valued function $f(t)$ is defined as

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) = I_t^{n-\alpha}\Big[\frac{d^n f(t)}{dt^n}\Big],$$
$$= \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n f(s)}{ds^n} ds, & n-1 < \alpha < n, \\ \frac{d^n f(t)}{dt^n}, & \alpha = n. \end{cases}$$

Note:

1. $\frac{d^\alpha c}{dt^\alpha} = 0$, where c is a constant.

2. For $\lceil \alpha \rceil = n,\ n \in \mathbb{N}$,

$$\frac{d^\alpha t^p}{dt^\alpha} := \begin{cases} 0, & if\ p \in 0, 1, 2, ..., n-1, \\ \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & if\ p \in \mathbb{N}\ and\ p \geq n,\ or\ p \notin \mathbb{N}\ and\ p > n-1. \end{cases}$$

Definition 2.3 Riemann–Liouville time-fractional integral of order $\alpha > 0$, of a real-valued function $u(x, t)$ is defined as

$$I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x, s) ds.$$

Definition 2.4 The Caputo time-fractional derivative operator of order $\alpha > 0$ $(m-1 < \alpha < m)$, $m \in \mathbb{N}$ of a real-valued function $u(x, t)$ is defined as

$$\begin{aligned} \frac{\partial^\alpha u(x, t)}{\partial t^\alpha} &= I_t^{m-\alpha}\Big[\frac{\partial^m u(x, t)}{\partial t^m}\Big], \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1} \frac{\partial^m u(x,y)}{\partial y^m} dy, & m-1 < \alpha < m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m. \end{cases} \end{aligned}$$

Similarly, the Caputo space fractional derivative operator $\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ of order $\beta > 0$ $(m-1 < \beta < m)$, $m \in \mathbb{N}$ can be defined.

Note that: In the present chapter, fractional derivative $\frac{\partial^{l\beta} u(x,t)}{\partial x^{l\beta}}$, $l \in \mathbb{N}$ is taken as the sequential fractional derivative [15], i.e.,

$$\frac{\partial^{l\beta} u}{\partial x^{l\beta}} = \underbrace{\frac{\partial^\beta}{\partial x^\beta} \frac{\partial^\beta}{\partial x^\beta} \cdots \frac{\partial^\beta u}{\partial x^\beta}}_{l-times}$$

Definition 2.5 Mittag-Leffler function with two parameters α and β is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha) > 0, z, \beta \in \mathbb{C}.$$

Note that:

1. The $\alpha-$th order Caputo derivative of $E_\alpha(at^\alpha)$ is

$$\frac{d^\alpha}{dt^\alpha} E_\alpha(at^\alpha) = a E_\alpha(at^\alpha),\ \alpha > 0, \ \ a \in \mathbb{R}.$$

2. Generalized fractional trigonometric functions for $\lceil \alpha \rceil = n$ are defined as [16]

$$\left.\begin{aligned} \cos_\alpha(\lambda t^\alpha) &= \Re[E_\alpha(i\lambda t^\alpha)] = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} t^{(2k)\alpha}}{\Gamma(2k\alpha+1)}, \\ \sin_\alpha(\lambda t^\alpha) &= \Im[E_\alpha(i\lambda t^\alpha)] = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} t^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \end{aligned}\right\}$$

3. The Caputo derivative of fractional trigonometric functions are defined as

$$\left.\begin{aligned} \frac{d^\alpha}{dt^\alpha}\cos_\alpha(\lambda t^\alpha) &= -\lambda \sin_\alpha(\lambda t^\alpha), \\ \frac{d^\alpha}{dt^\alpha}\sin_\alpha(\lambda t^\alpha) &= \lambda \cos_\alpha(\lambda t^\alpha). \end{aligned}\right\}$$

Definition 2.6 ([17]) The Sumudu transform over the set of functions $A = \{f(t) \mid \exists\, M, \tau_j > 0, j = 1, 2, \text{ such that } |f(t)| < Me^{|t|/\tau_j}\ if\ t \in (-1)^j \times [0, \infty)\}$ is defined as

$$S[f(t)](\omega) = F(\omega) = \int_0^\infty \frac{1}{\omega} e^{-\frac{t}{\omega}} f(t)dt = \int_0^\infty e^{-t} f(\omega t)dt,\ \omega \in (-\tau_1, \tau_2).$$

Definition 2.7 ([17]) The inverse Sumudu transform of $F(\omega)$ is denoted by $f(t)$, and defined by the following integral:

$$f(t) = S^{-1}[F(\omega)] = \frac{1}{2\pi i}\int_{z-i\infty}^{z+i\infty} \frac{1}{\omega} e^{\frac{t}{\omega}} F(\omega)d\omega,$$

where $\Re(1/\omega) > z$ and $z \in \mathbb{C}$.

One of the basic properties of Sumudu transform is

$$S\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big] = \omega^\alpha, \quad \alpha > -1. \tag{1}$$

Sumudu inverse transforms of ω^α is defined as

$$S^{-1}[\omega^\alpha] = \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad \alpha > -1. \tag{2}$$

Note that the Sumudu transform of Caputo time fractional derivative of $f(x,t)$ of order $\gamma > 0$ is [18]

$$\left.\begin{aligned} S\Big[\frac{\partial^\gamma f(x,t)}{\partial t^\gamma}\Big] &= \omega^{-\gamma} S[f(x,t)] - \sum_{k=0}^{m-1}\Big[\omega^{-\gamma+k}\frac{\partial^k f(x,0)}{\partial t^k}\Big], \\ &\quad m-1 < \gamma \le m,\ m \in \mathbb{N}. \end{aligned}\right\} \tag{3}$$

3 STIM for Time and Space FPDEs

In this section, we extend STIM [12] for solving time and space FPDEs.

We consider the following general time and space FPDE:

$$\left.\begin{array}{c}\frac{\partial^{\gamma}u}{\partial t^{\gamma}}=\mathscr{F}\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big),\ m-1<\gamma\leq m,\\ n-1<\beta\leq n,\ l,m,n\in\mathbb{N},\end{array}\right\} \tag{4}$$

along with the initial conditions

$$\frac{\partial^k u(x,0)}{\partial t^k}=h_k(x),\quad k=0,1,2,...,m-1, \tag{5}$$

where $\mathscr{F}\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big)$ is a linear/nonlinear operator and $u=u(x,t)$ is the unknown function.

Taking the Sumudu transform of both sides of Eq. (4) and simplifying, we get

$$S[u(x,t)]=\sum_{k=0}^{m-1}\Big[\omega^k\frac{\partial^k u(x,0)}{\partial t^k}\Big]+\omega^{\gamma}S\Big[\mathscr{F}\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big)\Big]. \tag{6}$$

The inverse Sumudu transform of Eq. (6) leads to

$$u(x,t)=S^{-1}\Big(\sum_{k=0}^{m-1}\Big[\omega^k\frac{\partial^k u(x,0)}{\partial t^k}\Big]\Big)+S^{-1}\Big[\omega^{\gamma}S\Big(\mathscr{F}\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big)\Big)\Big]. \tag{7}$$

Equation (7) can be written as

$$u(x,t)=f(x,t)+N\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big), \tag{8}$$

where

$$\left.\begin{array}{c}f(x,t)=S^{-1}\Big(\sum_{k=0}^{m-1}\Big[\omega^k\frac{\partial^k u(x,0)}{\partial t^k}\Big]\Big),\\ N\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big)=S^{-1}\Big[\omega^{\gamma}S\Big(\mathscr{F}\Big(x,u,\frac{\partial^{\beta}u}{\partial x^{\beta}},...,\frac{\partial^{l\beta}u}{\partial x^{l\beta}}\Big)\Big)\Big],\end{array}\right\} \tag{9}$$

here f is known function and N is a linear/nonlinear operator.

Functional equations of the form (8) can be solved by the DGJ decomposition method introduced by Daftardar-Gejji and Jafari [5].

DJM represents the solution as an infinite series:

$$u = \sum_{i=0}^{\infty} u_i, \tag{10}$$

where the terms u_i are calculated recursively. The operator N can be decomposed as

$$\begin{aligned} N\Big(x, \sum_{i=0}^{\infty} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{\infty} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{\infty} u_i)}{\partial x^{l\beta}}\Big) &= N\Big(x, u_0, \frac{\partial^{\beta} u_0}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta} u_0}{\partial x^{l\beta}}\Big) \\ &+ \sum_{j=1}^{\infty}\Big(N\Big(x, \sum_{i=0}^{j} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{l\beta}}\Big)\Big) \\ &- \sum_{j=1}^{\infty}\Big(N\Big(x, \sum_{i=0}^{j-1} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{l\beta}}\Big)\Big). \end{aligned} \tag{11}$$

$$\left.\begin{aligned} S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{\infty} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{\infty} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{\infty} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big] \\ = S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, u_0, \frac{\partial^{\beta} u_0}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta} u_0}{\partial x^{l\beta}}\Big)\Big)\Big] \\ + \sum_{j=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{j} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big] \\ - \sum_{j=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{j-1} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big]. \end{aligned}\right\} \tag{12}$$

Using Eqs. (10), (12) in Eq. (8), we get

$$\begin{aligned} \sum_{i=0}^{\infty} u_i = S^{-1}\Big(\sum_{k=0}^{m-1}\Big[\omega^{k} \frac{\partial^{k} u(x,0)}{\partial t^{k}}\Big]\Big) + S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, u_0, \frac{\partial^{\beta} u_0}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta} u_0}{\partial x^{l\beta}}\Big)\Big)\Big] \\ + \sum_{j=1}^{\infty}\Big(S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{j} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big] \\ - S^{-1}\Big[\omega^{\gamma} S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{j-1} u_i, \frac{\partial^{\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{\beta}}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{j-1} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big]\Big). \end{aligned}$$

We define the recurrence relation as follows:

$$\left.\begin{aligned}
u_0 &= S^{-1}\Big(\sum_{k=0}^{m-1}\Big[\omega^k \frac{\partial^k u(x,0)}{\partial t^k}\Big]\Big),\\
u_1 &= S^{-1}\Big[\omega^\gamma S\Big(\mathscr{F}\Big(x, u_0, \frac{\partial^\beta u_0}{\partial x^\beta}, \ldots, \frac{\partial^{l\beta} u_0}{\partial x^{l\beta}}\Big)\Big)\Big],\\
u_{r+1} &= S^{-1}\Big[\omega^\gamma S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{r} u_i, \frac{\partial^\beta(\sum_{i=0}^{r} u_i)}{\partial x^\beta}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{r} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big]\\
&\quad - S^{-1}\Big[\omega^\gamma S\Big(\mathscr{F}\Big(x, \sum_{i=0}^{r-1} u_i, \frac{\partial^\beta(\sum_{i=0}^{r-1} u_i)}{\partial x^\beta}, \ldots, \frac{\partial^{l\beta}(\sum_{i=0}^{r-1} u_i)}{\partial x^{l\beta}}\Big)\Big)\Big],\\
&\quad for \quad r \geq 1.
\end{aligned}\right\} \tag{13}$$

The r-term approximate solution of Eqs. (4), (5) is given by $u \approx u_0 + u_1 + \cdots + u_{r-1}$. For the convergence of DJM, we refer the reader to [19].

4 Illustrative Examples

In this section, we solve various nonlinear time and space FPDEs using STIM derived in Sect. 3.

Example 4.1 Consider the following time and space fractional equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 - u\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big), \quad t > 0, \ \alpha, \beta \in (0, 1], \tag{14}$$

along with the initial condition

$$u(x, 0) = 3 + \frac{5}{2} E_\beta(x^\beta). \tag{15}$$

Taking the Sumudu transform of both sides of Eq. (14), we get

$$S\Big[\frac{\partial^\alpha u}{\partial t^\alpha}\Big] = S\Big[\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 - u\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)\Big].$$

Using the property (3) of Sumudu transform, we get

$$S[u(x, t)] = u(x, 0) + \omega^\alpha\Big(S\Big[\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 - u\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)\Big]\Big). \tag{16}$$

Now taking the inverse Sumudu transform of both sides of Eq. (16)

$$u(x,t) = S^{-1}[u(x,0)] + S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\Big(\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)^2 - u\Big(\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big]\Big)\Big).$$

Using the recurrence relation (13)

$$\begin{aligned}
u_0 &= S^{-1}[u(x,0)] = 3 + \frac{5}{2}E_{\beta}(x^{\beta}),\\
u_1 &= S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\Big(\frac{\partial^{\beta} u_0}{\partial x^{\beta}}\Big)^2 - u_0\Big(\frac{\partial^{\beta} u_0}{\partial x^{\beta}}\Big)\Big]\Big)\Big) = -\frac{15t^{\alpha}E_{\beta}(x^{\beta})}{2\Gamma(\alpha+1)},\\
u_2 &= S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\Big(\frac{\partial^{\beta}(u_0+u_1)}{\partial x^{\beta}}\Big)^2 - (u_0+u_1)\Big(\frac{\partial^{\beta}(u_0+u_1)}{\partial x^{\beta}}\Big)\Big]\Big)\Big),\\
&\quad - S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\Big(\frac{\partial^{\beta} u_0}{\partial x^{\beta}}\Big)^2 - u_0\Big(\frac{\partial^{\beta} u_0}{\partial x^{\beta}}\Big)\Big]\Big)\Big),\\
&= \frac{45t^{2\alpha}E_{\beta}(x^{\beta})}{2\Gamma(2\alpha+1)},\\
u_3 &= -\frac{135t^{3\alpha}E_{\beta}(x^{\beta})}{2\Gamma(3\alpha+1)},\\
u_4 &= \frac{405t^{4\alpha}E_{\beta}(x^{\beta})}{2\Gamma(4\alpha+1)},\\
&\vdots
\end{aligned}$$

Hence, the series solution of Eq. (14) along with the initial condition (15) is given by

$$\begin{aligned}
u(x,t) &= 3 + \frac{5}{2}E_{\beta}(x^{\beta}) - \frac{15t^{\alpha}E_{\beta}(x^{\beta})}{2\Gamma(\alpha+1)} + \frac{45t^{2\alpha}E_{\beta}(x^{\beta})}{2\Gamma(2\alpha+1)} - \frac{135t^{3\alpha}E_{\beta}(x^{\beta})}{2\Gamma(3\alpha+1)}\\
&\quad + \frac{405t^{4\alpha}E_{\beta}(x^{\beta})}{2\Gamma(4\alpha+1)} - \cdots .
\end{aligned}$$

This leads to the following closed-form solution:

$$u(x,t) = 3 + \Big[\frac{5}{2}E_{\alpha}(-3t^{\alpha})\Big]E_{\beta}(x^{\beta}),$$

which is the same as obtained in [20].

Example 4.2 Consider the following time space fractional heat equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big), \quad t > 0, \ \alpha, \beta \in (0, 1], \tag{17}$$

with the initial condition

$$u(x,0) = a + bx^{\beta}, \quad a, b \in \mathbb{R}. \tag{18}$$

Taking the Sumudu transform of both sides of Eq. (17), we get

$$\begin{aligned} S\Big[\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\Big] &= S\Big[\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big], \\ \implies S[u(x,t)] &= u(x,0) + \omega^{\alpha}\Big(S\Big[\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big]\Big), \\ \implies u(x,t) &= S^{-1}[u(x,0)] + S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big]\Big)\Big). \end{aligned}$$

Now using the recurrence relation (13)

$$\begin{aligned} u_0 &= S^{-1}[u(x,0)] = a + bx^{\beta}, \\ u_1 &= S^{-1}\Big(\omega^{\alpha}\Big(S\Big[\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u_0\frac{\partial^{\beta} u_0}{\partial x^{\beta}}\Big)\Big]\Big)\Big), \\ &= b^2(\Gamma(\beta+1))^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ u_i &= 0, \quad \forall\, i \geq 2. \end{aligned}$$

Hence, the solution turns out to be:

$$u(x,t) = a + bx^{\beta} + b^2(\Gamma(\beta+1))^2\frac{t^{\alpha}}{\Gamma(\alpha+1)}.$$

Example 4.3 Consider the following time and space fractional thin film equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = -u\Big(\frac{\partial^{4\beta} u}{\partial x^{4\beta}}\Big) + \eta\Big(\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big(\frac{\partial^{3\beta} u}{\partial x^{3\beta}}\Big) + \zeta\Big(\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\Big)^2, \; t > 0 \; \alpha, \beta \in (0,1], \tag{19}$$

along with the initial condition

$$u(x,0) = a + bx^{\beta} + cx^{2\beta} + dx^{3\beta}, \quad a, b, c, d \in \mathbb{R}. \tag{20}$$

Taking the Sumudu transform of both sides of Eq. (19), we get

$$S\Big[\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\Big] = S\Big[-u\Big(\frac{\partial^{4\beta} u}{\partial x^{4\beta}}\Big) + \eta\Big(\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big(\frac{\partial^{3\beta} u}{\partial x^{3\beta}}\Big) + \zeta\Big(\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\Big)^2\Big].$$

After simplification, we get

$$\begin{aligned} u(x,t) &= S^{-1}[u(x,0)] \\ &\quad + S^{-1}\Big(\omega^{\alpha} S\Big[-u\Big(\frac{\partial^{4\beta} u}{\partial x^{4\beta}}\Big) + \eta\Big(\frac{\partial^{\beta} u}{\partial x^{\beta}}\Big)\Big(\frac{\partial^{3\beta} u}{\partial x^{3\beta}}\Big) + \zeta\Big(\frac{\partial^{2\beta} u}{\partial x^{2\beta}}\Big)^2\Big]\Big). \end{aligned}$$

In view of the recurrence relation (13),

$$
\begin{aligned}
u_0 &= S^{-1}[u(x,0)] = a + bx^{\beta} + cx^{2\beta} + dx^{3\beta}, && (21)\\
u_1 &= S^{-1}\Big(\omega^{\alpha} S\Big[-u_0\Big(\frac{\partial^{4\beta}u_0}{\partial x^{4\beta}}\Big) + \eta\Big(\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\Big)\Big(\frac{\partial^{3\beta}u_0}{\partial x^{3\beta}}\Big) + \zeta\Big(\frac{\partial^{2\beta}u_0}{\partial x^{2\beta}}\Big)^2\Big]\Big),\\
&= \frac{b\eta d\Gamma(\beta+1)\Gamma(3\beta+1)t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\eta cd\Gamma(2\beta+1)\Gamma(3\beta+1)t^{\alpha}x^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}\\
&+ \frac{\eta d^2\Gamma(3\beta+1)^2 t^{\alpha}x^{2\beta}}{\Gamma(\alpha+1)\Gamma(2\beta+1)} + \frac{c^2\zeta\Gamma(2\beta+1)^2 t^{\alpha}}{\Gamma(\alpha+1)}\\
&+ \frac{2c\zeta d\Gamma(2\beta+1)\Gamma(3\beta+1)t^{\alpha}x^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{\zeta d^2\Gamma(3\beta+1)^2 t^{\alpha}x^{2\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)^2}. && (22)
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= \frac{\eta^2 cd^2\Gamma(2\beta+1)\Gamma(3\beta+1)^2 t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2\zeta^2 d^3\Gamma(2\beta+1)\Gamma(3\beta+1)^3 t^{2\alpha}x^{\beta}}{\Gamma(2\alpha+1)\Gamma(\beta+1)^3}\\
&+ \frac{\eta^2 d^3\Gamma(3\beta+1)^3 t^{2\alpha}x^{\beta}}{\Gamma(2\alpha+1)\Gamma(\beta+1)} + \frac{4\eta c\zeta d^2\Gamma(2\beta+1)\Gamma(3\beta+1)^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&+ \frac{\eta\zeta d^3\Gamma(2\beta+1)\Gamma(3\beta+1)^3 t^{2\alpha}x^{\beta}}{\Gamma(2\alpha+1)\Gamma(\beta+1)^3} + \frac{2\eta\zeta d^3\Gamma(3\beta+1)^3 t^{2\alpha}x^{\beta}}{\Gamma(2\alpha+1)\Gamma(\beta+1)}\\
&+ \frac{2c\zeta^2 d^2\Gamma(2\beta+1)^2\Gamma(3\beta+1)^2 t^{2\alpha}}{\Gamma(2\alpha+1)\Gamma(\beta+1)^2}\\
&+ \frac{2\eta\zeta^2 d^4\Gamma(2\alpha+1)\Gamma(2\beta+1)\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(\beta+1)^2}\\
&+ \frac{\eta^2\zeta d^4\Gamma(2\alpha+1)\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}\\
&+ \frac{\zeta^3 d^4\Gamma(2\alpha+1)\Gamma(2\beta+1)^2\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)\Gamma(\beta+1)^4}, && (23)\\
u_3 &= \frac{\eta^3 d^4\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{\eta^2\zeta d^4\Gamma(2\beta+1)\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\beta+1)^2}\\
&+ \frac{2\eta^2\zeta d^4\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{2\eta\zeta^2 d^4\Gamma(2\beta+1)\Gamma(3\beta+1)^4 t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\beta+1)^2}, && (24)\\
u_i &= 0 \;\; \forall \;\; i \geq 4. && (25)
\end{aligned}
$$

Hence, we obtain the exact solution of Eqs. (19), (20) as

$$
u(x,t) = \sum_{i=0}^{\infty} u_i,
$$

where $u_i' s$ are given in Eqs. (21)–(25).

Example 4.4 Consider the following time and space fractional-dispersive Boussinesq equation

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} = \frac{\partial^{2\beta}u}{\partial x^{2\beta}} - \eta\frac{\partial^{2\beta}(u^2)}{\partial x^{2\beta}} - \zeta\frac{\partial^{4\beta}(u^2)}{\partial x^{4\beta}} - \mu\frac{\partial^{6\beta}(u^2)}{\partial x^{6\beta}},\ t > 0,\ \alpha, \beta \in (0, 1], \tag{26}$$

where $\eta = 4[\zeta - 4\mu]$, ζ and μ are constants, along with the initial conditions

$$u(x, 0) = a + b\sin_\beta(x^\beta) + c\cos_\beta(x^\beta), u_t(x, 0) = 0,\ \ a, b, c \in \mathbb{R}. \tag{27}$$

Taking the Sumudu transform of both sides of Eq. (26), we get

$$S\Big[\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}\Big] = S\Big[\frac{\partial^{2\beta}u}{\partial x^{2\beta}} - \eta\frac{\partial^{2\beta}(u^2)}{\partial x^{2\beta}} - \zeta\frac{\partial^{4\beta}(u)^2}{\partial x^{4\beta}} - \mu\frac{\partial^{6\beta}(u)^2}{\partial x^{6\beta}}\Big].$$

Using the property (3) of Sumudu transform, we get

$$\begin{aligned} S[u(x, t)] &= u(x, 0) \\ &\quad + \omega^{2\alpha}\Big(S\Big[\frac{\partial^{2\beta}u}{\partial x^{2\beta}} - \eta\frac{\partial^{2\beta}(u^2)}{\partial x^{2\beta}} - \zeta\frac{\partial^{4\beta}(u)^2}{\partial x^{4\beta}} - \mu\frac{\partial^{6\beta}(u)^2}{\partial x^{6\beta}}\Big]\Big). \end{aligned} \tag{28}$$

Taking the inverse Sumudu transform of both sides of Eq. (28)

$$\begin{aligned} u(x, t) &= S^{-1}[u(x, 0)] \\ &\quad + S^{-1}\Big(\omega^{2\alpha}\Big(S\Big[\frac{\partial^{2\beta}u}{\partial x^{2\beta}} - \eta\frac{\partial^{2\beta}(u^2)}{\partial x^{2\beta}} - \zeta\frac{\partial^{4\beta}(u)^2}{\partial x^{4\beta}} - \mu\frac{\partial^{6\beta}(u)^2}{\partial x^{6\beta}}\Big]\Big)\Big). \end{aligned}$$

Using the recurrence relation (13), we get

$$\begin{aligned} u_0 &= S^{-1}[u(x, 0)] = a + b\sin_\beta(x^\beta) + c\cos_\beta(x^\beta), \\ u_1 &= S^{-1}\Big(\omega^{2\alpha}\Big(S\Big[\frac{\partial^{2\beta}u_0}{\partial x^{2\beta}} - \eta\frac{\partial^{2\beta}(u_0^2)}{\partial x^{2\beta}} - \zeta\frac{\partial^{4\beta}(u_0)^2}{\partial x^{4\beta}} - \mu\frac{\partial^{6\beta}(u_0)^2}{\partial x^{6\beta}}\Big]\Big)\Big) \\ &= \frac{t^{2\alpha}(6a(\zeta - 5\mu) - 1)(b\cos_\beta(x^\beta) + c\sin_\beta(x^\beta))}{\Gamma(2\alpha + 1)}. \\ u_2 &= \frac{t^{4\alpha}(1 - 6a(\zeta - 5\mu))^2(b\cos_\beta(x^\beta) + c\sin_\beta(x^\beta)}{\Gamma(4\alpha + 1)}, \\ u_3 &= \frac{t^{6\alpha}(6a(\zeta - 5\mu) - 1)^3(b\cos_\beta(x^\beta) + c\sin_\beta(x^\beta))}{\Gamma(6\alpha + 1)}, \\ &\ \vdots \end{aligned}$$

Hence, the series solution of Eqs. (26), (27) converges to

$$u(x,t) = a + b\sin_{\beta}(x^{\beta})E_{2\alpha}(\delta t^{2\alpha}) + c\cos_{\beta}(x^{\beta})E_{2\alpha}(\delta t^{2\alpha}),$$

where $\delta = (6a(\zeta - 5\mu) - 1)$.

Example 4.5 Consider the following general time–space fractional diffusion–convection equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \Big(\frac{\partial^{\beta}u}{\partial x^{\beta}}\Big)^2\Big(\frac{\partial f(u)}{\partial u}\Big) + f(u)\frac{\partial^{2\beta}u}{\partial x^{2\beta}} - \frac{\partial^{\beta}u}{\partial x^{\beta}}\Big(\frac{\partial g(u)}{\partial u}\Big), \quad t > 0, \alpha, \beta \in (0,1], \tag{29}$$

where f, g are the functions of u. Here we consider some particular cases:

Case 1: Let $f(u) = u,\ g(u) = k_1 =$ constant, then Eq. (29) reduces to

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} = \Big(\frac{\partial^{\beta}u}{\partial x^{\beta}}\Big)^2 + u\frac{\partial^{2\beta}u}{\partial x^{2\beta}}, \tag{30}$$

along with the initial condition

$$u(x,0) = a + bx^{\beta}, \tag{31}$$

Taking the Sumudu transform of both sides of Eq. (30), we get

$$S\Big[\frac{\partial^{\alpha}u}{\partial t^{\alpha}}\Big] = S\Big[\Big(\frac{\partial^{\beta}u}{\partial x^{\beta}}\Big)^2 + u\frac{\partial^{2\beta}u}{\partial x^{2\beta}}\Big].$$

Using the property (3) of Sumudu transform

$$S[u(x,t)] = u(x,0) + \omega^{\alpha}S\Big[\Big(\frac{\partial^{\beta}u}{\partial x^{\beta}}\Big)^2 + u\frac{\partial^{2\beta}u}{\partial x^{2\beta}}\Big]. \tag{32}$$

Taking inverse Sumudu transform of both sides of Eq. (32)

$$u(x,t) = S^{-1}[u(x,0)] + S^{-1}\Big(\omega^{\alpha}S\Big[\Big(\frac{\partial^{\beta}u}{\partial x^{\beta}}\Big)^2 + u\frac{\partial^{2\beta}u}{\partial x^{2\beta}}\Big]\Big). \tag{33}$$

Using the recurrence relation (13), we get

$$\begin{aligned} u_0 &= S^{-1}[u(x,0)] = a + bx^{\beta},\\ u_1 &= S^{-1}\Big(\omega^{\alpha}S\Big[\Big(\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\Big)^2 + u_0\frac{\partial^{2\beta}u_0}{\partial x^{2\beta}}\Big]\Big) = \frac{b^2\Gamma(\beta+1)^2t^{\alpha}}{\Gamma(\alpha+1)},\\ u_i &= 0\ \forall\ i \geq 2. \end{aligned}$$

Hence, the exact solution of (30), (31) is given by

$$u(x,t) = a + \frac{b^2\Gamma(\beta+1)^2 t^\alpha}{\Gamma(\alpha+1)} + bx^\beta.$$

Case 2: Let $f(u) = \eta u$ and $g(u) = \frac{\zeta}{2}u^2$, where η and ζ are constants and $\eta = \frac{\zeta}{2}$, then Eq. (29) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \eta\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 + \eta u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - \zeta u \frac{\partial^\beta u}{\partial x^\beta}, \tag{34}$$

along with the initial condition

$$u(x,0) = a + bE_\beta(x^\beta),\ \ a, b \in \mathbb{R}. \tag{35}$$

Taking the Sumudu transform of both sides of Eq. (34)

$$S\Big[\frac{\partial^\alpha u}{\partial t^\alpha}\Big] = S\Big[\eta\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 + \eta u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - \zeta u \frac{\partial^\beta u}{\partial x^\beta}\Big].$$

Using the property (3) of Sumudu transform, we get

$$S[u(x,t)] = u(x,0) + \omega^\alpha S\Big[\eta\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 + \eta u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - \zeta u \frac{\partial^\beta u}{\partial x^\beta}\Big]. \tag{36}$$

Taking inverse Sumudu transform of both sides of Eq. (36)

$$u(x,t) = S^{-1}[u(x,0)] + S^{-1}\Big(\omega^\alpha S\Big[\eta\Big(\frac{\partial^\beta u}{\partial x^\beta}\Big)^2 + \eta u \frac{\partial^{2\beta} u}{\partial x^{2\beta}} - \zeta u \frac{\partial^\beta u}{\partial x^\beta}\Big]\Big).$$

Using the recurrence relation (13), we get

$$\begin{aligned}
u_0 &= S^{-1}[u(x,0)] = a + bE_\beta(x^\beta),\\
u_1 &= S^{-1}\Big(\omega^\alpha S\Big[\eta\Big(\frac{\partial^\beta u_0}{\partial x^\beta}\Big)^2 + \eta u_0 \frac{\partial^{2\beta} u_0}{\partial x^{2\beta}} - \zeta u_0 \frac{\partial^\beta u_0}{\partial x^\beta}\Big]\Big),\\
&= -\frac{E_\beta(x^\beta)ab\zeta t^\alpha}{2\Gamma(\alpha+1)},\\
u_2 &= \frac{E_\beta(x^\beta)a^2b\zeta^2 t^{2\alpha}}{4\Gamma(2\alpha+1)},\\
u_3 &= -\frac{E_\beta(x^\beta)a^3b\zeta^3 t^{3\alpha}}{8\Gamma(3\alpha+1)},\\
&\ \vdots
\end{aligned}$$

Hence, the series solution of Eqs. (34), (35) is given by

$$u(x,t) = a + bE_\beta(x^\beta) - \frac{E_\beta(x^\beta)ab\zeta t^\alpha}{2\Gamma(\alpha+1)} + \frac{E_\beta(x^\beta)a^2b\zeta^2t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{E_\beta(x^\beta)a^3b\zeta^3t^{3\alpha}}{8\Gamma(3\alpha+1)} + ...,$$

which is equivalent to the following closed from:

$$u(x,t) = a + bE_\beta(x^\beta)E_\alpha(-a\frac{\zeta}{2}t^\alpha).$$

5 STIM for System of Time and Space FPDEs

In this section we extend STIM to solve system of time and space fractional PDEs.

Consider the following system of time and space FPDEs:

$$\left.\begin{aligned} \frac{\partial^{\gamma_i} u_i}{\partial t^{\gamma_i}} = \mathscr{G}_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta}\bar{u}}{\partial x^{l\beta}}\Big),\ m_i - 1 < \gamma_i \le m_i, \\ i = 1, 2, ..., q,\ n-1 < \beta \le n,\ m_i, l, n, q \in \mathbb{N}, \end{aligned}\right\} \quad (37)$$

along with the initial conditions

$$\frac{\partial^j u_i(x,0)}{\partial t^j} = g_{ij}(x), \quad j = 0, 1, 2, ..., m_i - 1, \quad (38)$$

where $\bar{u} = (u_1, u_2, ..., u_q)$ and $\mathscr{G}_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta}\bar{u}}{\partial x^{l\beta}}\Big)$ is a linear/nonlinear operator.

After taking the Sumudu transform of both sides of Eq. (37) and using Eq. (38), we get

$$S[u_i(x,t)] = \sum_{j=0}^{m_i-1}\Big[\omega^j g_{ij}(x)\Big] + \omega^{\gamma_i} S\Big[\mathscr{G}_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta}\bar{u}}{\partial x^{l\beta}}\Big)\Big]. \quad (39)$$

The inverse Sumudu transform of Eq. (39) yields the following system of equations:

$$u_i(x,t) = S^{-1}\Big(\sum_{j=0}^{m_i-1}\Big[\omega^j g_{ij}(x)\Big]\Big) + S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathscr{G}_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta}\bar{u}}{\partial x^{l\beta}}\Big)\Big)\Big], i = 1, 2, ..., q. \quad (40)$$

Equation (40) is of the following form:

$$u_i(x,t) = f_i(x,t) + M_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta} \bar{u}}{\partial x^{l\beta}}\Big), \tag{41}$$

where

$$\left.\begin{aligned} f_i(x,t) &= S^{-1}\Big(\sum_{j=0}^{m_i-1}\Big[\omega^j g_{ij}(x)\Big]\Big), \\ M_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta} \bar{u}}{\partial x^{l\beta}}\Big) &= S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathscr{G}_i\Big(x, \bar{u}, \frac{\partial^\beta \bar{u}}{\partial x^\beta}, ..., \frac{\partial^{l\beta} \bar{u}}{\partial x^{l\beta}}\Big)\Big)\Big]. \end{aligned}\right\} \tag{42}$$

Here f_i is known function and M_i is a linear/nonlinear operator. Functional equations of the form (41) can be solved by the DJM decomposition method introduced by Daftardar-Gejji and Jafari [5]. The DJM represents the solution as an infinite series:

$$u_i = \sum_{j=0}^{\infty} u_i^{(j)}, \quad 1 \le i \le q, \tag{43}$$

where the terms $u_i^{(j)}$ are calculated recursively.

Note that henceforth we use the following abbreviations:

$$\begin{aligned} \bar{u}^{(j)} &= (u_1^{(j)}, u_2^{(j)}, ..., u_q^{(j)}), \\ \sum_{j=0}^{r} \bar{u}^{(j)} &= \Big(\sum_{j=0}^{r} u_1^{(j)}, \sum_{j=0}^{r} u_2^{(j)}, ..., \sum_{j=0}^{r} u_q^{(j)}\Big), \; r \in \mathbb{N} \cup \{\infty\}, \\ \frac{\partial^{k\beta}(\sum_{j=0}^{r} \bar{u}^{(j)})}{\partial x^{k\beta}} &= \Big(\frac{\partial^{k\beta}(\sum_{j=0}^{r} u_1^{(j)})}{\partial x^{k\beta}}, \frac{\partial^{k\beta}(\sum_{j=0}^{r} u_2^{(j)})}{\partial x^{k\beta}}, ..., \frac{\partial^{k\beta}(\sum_{j=0}^{r} u_q^{(j)})}{\partial x^{k\beta}}\Big), k \in \mathbb{N}. \end{aligned}$$

The operator M_i can be decomposed as

$$\begin{aligned} & M_i\Big(x, \sum_{j=0}^{\infty} \bar{u}^{(j)}, \frac{\partial^\beta(\sum_{j=0}^{\infty} \bar{u}^{(j)})}{\partial x^\beta}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{\infty} \bar{u}^{(j)})}{\partial x^{l\beta}}\Big) = \\ & \qquad M_i\Big(x, \bar{u}^{(0)}, \frac{\partial^\beta \bar{u}^{(0)}}{\partial x^\beta}, ..., \frac{\partial^{l\beta} \bar{u}^{(0)}}{\partial x^{l\beta}}\Big) + \\ & \sum_{p=1}^{\infty}\Big(M_i\Big(x, \sum_{j=0}^{p} \bar{u}^{(j)}, \frac{\partial^\beta(\sum_{j=0}^{p} \bar{u}^{(j)})}{\partial x^\beta}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p} \bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big) - \\ & \sum_{p=1}^{\infty}\Big(M_i\Big(x, \sum_{j=0}^{p-1} \bar{u}^{(j)}, \frac{\partial^\beta(\sum_{j=0}^{p-1} \bar{u}^{(j)})}{\partial x^\beta}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p-1} \bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big). \end{aligned} \tag{44}$$

Therefore,

$$\left.\begin{aligned} S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{\infty}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{\infty}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{\infty}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big] \\ = S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \bar{u}^{(0)}, \frac{\partial^{\beta}\bar{u}^{(0)}}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}\bar{u}^{(0)}}{\partial x^{l\beta}}\Big)\Big)\Big] \\ +\sum_{p=1}^{\infty} S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{p}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{p}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big] \\ -\sum_{p=1}^{\infty} S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{p-1}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{p-1}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p-1}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big]. \end{aligned}\right\} \tag{45}$$

Using Eqs. (43), (45) in Eq. (41), we get

$$\begin{aligned} \sum_{j=0}^{\infty} u_i^{(j)} &= S^{-1}\Big(\sum_{j=0}^{m_i-1}\Big[\omega^{j} g_{ij}(x)\Big]\Big) + S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \bar{u}^{(0)}, \frac{\partial^{\beta}\bar{u}^{(0)}}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}\bar{u}^{(0)}}{\partial x^{l\beta}}\Big)\Big)\Big] \\ &+ \sum_{p=1}^{\infty}\Big(S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{p}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{p}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big] \\ &- S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{p-1}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{p-1}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{p-1}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big]\Big). \end{aligned} \tag{46}$$

We define the recurrence relation as follows:

$$\left.\begin{aligned} u_i^{(0)} &= S^{-1}\Big(\sum_{j=0}^{m_i-1}\Big[\omega^{j} g_{ij}(x)\Big]\Big), \\ u_i^{(1)} &= S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \bar{u}^{(0)}, \frac{\partial^{\beta}\bar{u}^{(0)}}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}\bar{u}^{(0)}}{\partial x^{l\beta}}\Big)\Big)\Big], \\ u_i^{(m+1)} &= S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{m}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{m}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{m}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big] \\ &- S^{-1}\Big[\omega^{\gamma_i} S\Big(\mathcal{G}_i\Big(x, \sum_{j=0}^{m-1}\bar{u}^{(j)}, \frac{\partial^{\beta}(\sum_{j=0}^{m-1}\bar{u}^{(j)})}{\partial x^{\beta}}, ..., \frac{\partial^{l\beta}(\sum_{j=0}^{m-1}\bar{u}^{(j)})}{\partial x^{l\beta}}\Big)\Big)\Big], \\ &\quad for \quad m \geq 1. \end{aligned}\right\} \tag{47}$$

The m-term approximate solution of Eqs. (37), (38) is given by $u_i \approx u_i^{(0)} + u_i^{(1)} + \cdots + u_i^{(m-1)}$ or $u_i \approx u_{i0} + u_{i1} + \cdots + u_{i(m-1)}$.

6 Illustrative Examples

In this section, we solve nonlinear system of time and space FPDEs using STIM derived in Sect. 5.

Example 6.1 Consider the following system of time and space fractional Boussinesq PDEs ($t > 0, 0 < \alpha_1, \alpha_2, \beta \leq 1$):

$$\begin{aligned}
\frac{\partial^{\alpha_1} u_1}{\partial t^{\alpha_1}} &= -\frac{\partial^{\beta} u_2}{\partial x^{\beta}}, \\
\frac{\partial^{\alpha_2} u_2}{\partial t^{\alpha_2}} &= -m_1 \frac{\partial^{\beta} u_1}{\partial x^{\beta}} + 3u_1\Big(\frac{\partial^{\beta} u_1}{\partial x^{\beta}}\Big) + m_2 \frac{\partial^{3\beta} u_1}{\partial x^{3\beta}},
\end{aligned} \tag{48}$$

along with the following initial conditions:

$$u_1(x, 0) = a + bx^{\beta}, u_2(x, 0) = c, \quad a, b, c \in \mathbb{R}. \tag{49}$$

Taking the Sumudu transform on both sides of Eq. (48)

$$\begin{aligned}
S\Big[\frac{\partial^{\alpha_1} u_1}{\partial t^{\alpha_1}}\Big] &= S\Big[-\frac{\partial^{\beta} u_2}{\partial x^{\beta}}\Big], \\
S\Big[\frac{\partial^{\alpha_2} u_2}{\partial t^{\alpha_2}}\Big] &= S\Big[-m_1 \frac{\partial^{\beta} u_1}{\partial x^{\beta}} + 3u_1\Big(\frac{\partial^{\beta} u_1}{\partial x^{\beta}}\Big) + m_2 \frac{\partial^{3\beta} u_1}{\partial x^{3\beta}}\Big].
\end{aligned}$$

In view of (3), we get

$$\begin{aligned}
S[u_1(x, t)] &= u_1(x, 0) + \omega^{\alpha_1} S\Big[-\frac{\partial^{\beta} u_2}{\partial x^{\beta}}\Big], \\
S[u_2(x, t)] &= u_2(x, 0) + \omega^{\alpha_2} S\Big[-m_1 \frac{\partial^{\beta} u_1}{\partial x^{\beta}} + 3u_1\Big(\frac{\partial^{\beta} u_1}{\partial x^{\beta}}\Big) + m_2 \frac{\partial^{3\beta} u_1}{\partial x^{3\beta}}\Big].
\end{aligned} \tag{50}$$

Taking the inverse Sumudu transform on both sides of Eq. (50)

$$\begin{aligned}
u_1(x, t) &= S^{-1}[u_1(x, 0)] + S^{-1}\Big(\omega^{\alpha_1} S\Big[-\frac{\partial^{\beta} u_2}{\partial x^{\beta}}\Big]\Big), \\
u_2(x, t) &= S^{-1}[u_2(x, 0)] \\
&\quad + S^{-1}\Big(\omega^{\alpha_2} S\Big[-m_1 \frac{\partial^{\beta} u_1}{\partial x^{\beta}} + 3u_1\Big(\frac{\partial^{\beta} u_1}{\partial x^{\beta}}\Big) + m_2 \frac{\partial^{3\beta} u_1}{\partial x^{3\beta}}\Big]\Big).
\end{aligned} \tag{51}$$

The recurrence relation (47) yields

$$\begin{aligned}
u_{10} &= S^{-1}[u_1(x, 0)] = a + bx^{\beta}, \\
u_{20} &= S^{-1}[u_2(x, 0)] = c,
\end{aligned}$$

$$
\begin{aligned}
u_{11} &= S^{-1}\Big(\omega^{\alpha_1} S\Big[-\frac{\partial^{\beta} u_{20}}{\partial x^{\beta}}\Big]\Big) = 0,\\
u_{21} &= S^{-1}\Big(\omega^{\alpha_2} S\Big[-m_1\frac{\partial^{\beta} u_{10}}{\partial x^{\beta}} + 3u_{10}\Big(\frac{\partial^{\beta} u_{10}}{\partial x^{\beta}}\Big) + m_2\frac{\partial^{3\beta} u_{10}}{\partial x^{3\beta}}\Big]\Big),\\
&= \frac{3ab\Gamma(\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \frac{3b^2\Gamma(\beta+1)t^{\alpha_2}x^{\beta}}{\Gamma(\alpha_2+1)} - \frac{bm_1\Gamma(\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)},\\
u_{12} &= -\frac{3b^2\Gamma(\beta+1)^2 t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)},\\
u_{22} &= 0,\\
u_{13} &= 0,\\
u_{23} &= -\frac{9b^3\Gamma(\beta+1)^3 t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)},\\
u_{1i} &= 0, i \geq 4,\\
u_{2i} &= 0, i \geq 4.
\end{aligned}
$$

Hence, the exact solution of the system (48) along with the initial conditions (49) is given by

$$
\begin{aligned}
u_1(x,t) &= u_{10} + u_{11} + u_{12} + u_{13},\\
&= a - \frac{3b^2\Gamma(\beta+1)^2 t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + bx^{\beta},\\
u_2(x,t) &= u_{20} + u_{21} + u_{22} + u_{23},\\
&= c + \frac{3ab\Gamma(\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)} - \frac{9b^3\Gamma(\beta+1)^3 t^{\alpha_1+2\alpha_2}}{\Gamma(\alpha_1+2\alpha_2+1)}\\
&\quad + \frac{3b^2\Gamma(\beta+1)t^{\alpha_2}x^{\beta}}{\Gamma(\alpha_2+1)} - \frac{bm_1\Gamma(\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)}.
\end{aligned}
$$

When $a = e$, $b = 2$, and $c = 3/2$, this solution is the same as obtained using invariant subspace method in [21].

Example 6.2 Consider the following two-coupled time and space fractional diffusion equations:

$$
\begin{aligned}
\frac{\partial^{\alpha_1} u_1}{\partial t^{\alpha_1}} &= \frac{\partial^{2\beta} u_1}{\partial x^{2\beta}} + \mu\frac{\partial^{\beta}}{\partial x^{\beta}}\Big(u_2\frac{\partial^{\beta} u_2}{\partial x^{\beta}}\Big) + \xi u_2^2,\\
\frac{\partial^{\alpha_2} u_2}{\partial t^{\alpha_2}} &= \frac{\partial^{2\beta} u_2}{\partial x^{2\beta}} + \eta\frac{\partial^{2\beta} u_1}{\partial x^{2\beta}} + \zeta u_1 + \delta u_2, t > 0, 0 < \alpha_1, \alpha_2, \beta \leq 1,
\end{aligned}
\tag{52}
$$

where $\mu, \xi, \eta, \zeta, \delta$ are arbitrary constants, μ and ξ are not simultaneously zero, we consider $\xi = -2\mu\lambda^2$, $\zeta = \eta\kappa^2$, along with the initial conditions

$$u_1(x,0) = a\cos_\beta(\kappa x^\beta) + b\sin_\beta(\kappa x^\beta),\, u_2(x,0) = cE_\beta(-\lambda x^\beta),\ a, b, c, \lambda, \kappa \in \mathbb{R}. \tag{53}$$

Taking the Sumudu transform on both sides of Eq. (52)

$$\begin{aligned} S\Big[\frac{\partial^{\alpha_1}u_1}{\partial t^{\alpha_1}}\Big] &= S\Big[\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \mu\frac{\partial^\beta}{\partial x^\beta}\Big(u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \xi u_2^2\Big], \\ S\Big[\frac{\partial^{\alpha_2}u_2}{\partial t^{\alpha_2}}\Big] &= S\Big[\frac{\partial^{2\beta}u_2}{\partial x^{2\beta}} + \eta\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \zeta u_1 + \delta u_2\Big]. \end{aligned} \tag{54}$$

After using the property (3) of Sumudu transform, we get

$$\begin{aligned} S[u_1(x,t)] &= u_1(x,0) + \omega^{\alpha_1}S\Big[\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \mu\frac{\partial^\beta}{\partial x^\beta}\Big(u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \xi u_2^2\Big], \\ S[u_2(x,t)] &= u_2(x,0) + \omega^{\alpha_2}S\Big[\frac{\partial^{2\beta}u_2}{\partial x^{2\beta}} + \eta\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \zeta u_1 + \delta u_2\Big]. \end{aligned}$$

Taking the inverse Sumudu transform

$$\begin{aligned} u_1(x,t) &= S^{-1}[u_1(x,0)] + S^{-1}\Big(\omega^{\alpha_1}S\Big[\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \mu\frac{\partial^\beta}{\partial x^\beta}\Big(u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \xi u_2^2\Big]\Big), \\ u_2(x,t) &= S^{-1}[u_2(x,0)] + S^{-1}\Big(\omega^{\alpha_2}S\Big[\frac{\partial^{2\beta}u_2}{\partial x^{2\beta}} + \eta\frac{\partial^{2\beta}u_1}{\partial x^{2\beta}} + \zeta u_1 + \delta u_2\Big]\Big). \end{aligned}$$

Using the recurrence relation (47), we get

$$\begin{aligned} u_{10} &= S^{-1}[u_1(x,0)] = a\cos_\beta(\kappa x^\beta) + b\sin_\beta(\kappa x^\beta), \\ u_{20} &= S^{-1}[u_2(x,0)] = cE_\beta(-\lambda x^\beta), \\ u_{11} &= S^{-1}\Big(\omega^{\alpha_1}S\Big[\frac{\partial^{2\beta}u_{10}}{\partial x^{2\beta}} + \mu\frac{\partial^\beta}{\partial x^\beta}\Big(u_{20}\frac{\partial^\beta u_{20}}{\partial x^\beta}\Big) + \xi u_{20}^2\Big]\Big), \\ &= -\frac{a\kappa^2 t^{\alpha_1}\cos_\beta(\kappa x^\beta)}{\Gamma(\alpha_1+1)} - \frac{b\kappa^2 t^{\alpha_1}\sin_\beta(\kappa x^\beta)}{\Gamma(\alpha_1+1)}, \\ u_{21} &= S^{-1}\Big(\omega^{\alpha_2}S\Big[\frac{\partial^{2\beta}u_{20}}{\partial x^{2\beta}} + \eta\frac{\partial^{2\beta}u_{10}}{\partial x^{2\beta}} + \zeta u_{10} + \delta u_{20}\Big]\Big), \\ &= \frac{c\delta t^{\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma(\alpha_2+1)} + \frac{c\lambda^2 t^{\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma(\alpha_2+1)}, \end{aligned}$$

$$\begin{aligned} u_{12} &= \frac{a\kappa^4 t^{2\alpha_1}\cos_\beta(\kappa x^\beta)}{\Gamma(2\alpha_1+1)} + \frac{b\kappa^4 t^{2\alpha_1}\sin_\beta(\kappa x^\beta)}{\Gamma(2\alpha_1+1)}, \\ u_{22} &= \frac{c\delta^2 t^{2\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma(2\alpha_2+1)} + \frac{2c\delta\lambda^2 t^{2\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma(2\alpha_2+1)} + \frac{c\lambda^4 t^{2\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma(2\alpha_2+1)}, \end{aligned}$$

$$u_{13} = -\frac{a\kappa^6 t^{3\alpha_1}\cos_\beta(\kappa x^\beta)}{\Gamma\,(3\alpha_1+1)} - \frac{b\kappa^6 t^{3\alpha_1}\sin_\beta(\kappa x^\beta)}{\Gamma\,(3\alpha_1+1)},$$

$$u_{23} = \frac{c\delta^3 t^{3\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma\,(3\alpha_2+1)} + \frac{3c\delta^2\lambda^2 t^{3\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma\,(3\alpha_2+1)} + \frac{3c\delta\lambda^4 t^{3\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma\,(3\alpha_2+1)} + \frac{c\lambda^6 t^{3\alpha_2}E_\beta(-\lambda x^\beta)}{\Gamma\,(3\alpha_2+1)},$$

$$\vdots$$

Hence, the series solution of two-coupled time and space fractional diffusion system (52), (53) converges to the following closed form:

$$u_1(x,t) = [a\cos_\beta(\kappa x^\beta) + b\sin_\beta(\kappa x^\beta)]E_{\alpha_1}(-\kappa^2 t^{\alpha_1}),$$
$$u_2(x,t) = cE_{\alpha_2}[(\delta+\lambda^2)t^{\alpha_2}]E_\beta(-\lambda x^\beta).$$

Example 6.3 Consider the following two-coupled time and space fractional PDE:

$$\frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \eta u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \zeta u_2^2,$$
$$\frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \delta u_1 + \tau u_2,\quad t>0, 0<\alpha,\beta\le 1, \tag{55}$$

here η, ζ, δ, τ all are arbitrary constants, η and ζ are not simultaneously zero (taking $\zeta = -2\eta$), along with the initial conditions

$$u_1(x,0) = bE_\beta(-x^\beta),\ u_2(x,0) = dE_\beta(-x^\beta),\quad b,d\in\mathbb{R}. \tag{56}$$

Taking the Sumudu transform on both sides of Eq. (55)

$$S\Big[\frac{\partial^\alpha u_1}{\partial t^\alpha}\Big] = S\Big[\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \eta u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \zeta u_2^2\Big],$$
$$S\Big[\frac{\partial^\alpha u_2}{\partial t^\alpha}\Big] = S\Big[\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \delta u_1 + \tau u_2\Big].$$

After using the property (3) of Sumudu transform, we get

$$S[u_1(x,t)] = u_1(x,0) + \omega^\alpha S\Big[\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \eta u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \zeta u_2^2\Big],$$
$$S[u_2(x,t)] = u_2(x,0) + \omega^\alpha S\Big[\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \delta u_1 + \tau u_2\Big]. \tag{57}$$

Taking the inverse Sumudu transform on both sides of Eq. (57)

$$u_1(x,t) = S^{-1}[u_1(x,0)] + S^{-1}\Big(\omega^\alpha S\Big[\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \eta u_2\frac{\partial^\beta u_2}{\partial x^\beta}\Big) + \zeta u_2^2\Big]\Big),$$
$$u_2(x,t) = S^{-1}[u_2(x,0)] + S^{-1}\Big(\omega^\alpha S\Big[\frac{\partial^{4\beta}u_1}{\partial x^{4\beta}} + \delta u_1 + \tau u_2\Big]\Big). \tag{58}$$

In view of the recurrence relation (47)

$$u_{10} = S^{-1}[u_1(x,0)] = E_\beta(-x^\beta),$$
$$u_{20} = S^{-1}[u_2(x,0)] = E_\beta(-x^\beta),$$
$$u_{11} = S^{-1}\Big(\omega^\alpha S\Big[\frac{\partial^\beta}{\partial x^\beta}\Big(\frac{\partial^{4\beta}u_{10}}{\partial x^{4\beta}} + \eta u_{20}\frac{\partial^\beta u_{20}}{\partial x^\beta}\Big) + \zeta u_{20}^2\Big]\Big),$$
$$= -\frac{bE_\beta(-x^\beta)t^\alpha}{\Gamma(\alpha+1)},$$

$$u_{21} = S^{-1}\Big(\omega^\alpha S\Big[\frac{\partial^{4\beta}u_{10}}{\partial x^{4\beta}} + \delta u_{10} + \tau u_{20}\Big]\Big),$$
$$= \frac{b\delta E_\beta(-x^\beta)t^\alpha}{\Gamma(\alpha+1)} + \frac{bE_\beta(-x^\beta)t^\alpha}{\Gamma(\alpha+1)} + \frac{d\tau E_\beta(-x^\beta)t^\alpha}{\Gamma(\alpha+1)},$$
$$u_{12} = \frac{bE_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_{22} = \frac{b\delta\tau E_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{b\delta E_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b\tau E_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$- \frac{bE_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{d\tau^2 E_\beta(-x^\beta)t^{2\alpha}}{\Gamma(2\alpha+1)},$$
$$u_{13} = -\frac{bE_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$u_{23} = \frac{b\delta\tau^2 E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{b\delta\tau E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{b\delta E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{b\tau^2 E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$- \frac{b\tau E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{bE_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{d\tau^3 E_\beta(-x^\beta)t^{3\alpha}}{\Gamma(3\alpha+1)},$$
$$u_{14} = \frac{bE_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)},$$
$$u_{24} = \frac{b\delta\tau^3 E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{b\delta\tau^2 E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{b\delta\tau E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$- \frac{b\delta E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{b\tau^3 E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{b\tau^2 E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{b\tau E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)}$$
$$- \frac{bE_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{d\tau^4 E_\beta(-x^\beta)t^{4\alpha}}{\Gamma(4\alpha+1)},$$
$$\vdots$$

Hence, the series solution of two-coupled time and space fractional PDE (55) along with the initial conditions (56) converges to

$$u_1(x,t) = bE_\alpha(-t^\alpha)E_\beta(-x^\beta), \quad \tau \neq 1, \tag{59}$$
$$u_2(x,t) = \Big(dE_\alpha(\tau t^\alpha) + b\Big(\frac{1+\delta}{1+\tau}\Big)\Big[E_\alpha(\tau t^\alpha) - E_\alpha(-t^\alpha)\Big]\Big)E_\beta(-x^\beta). \tag{60}$$

Note: For $\alpha = \beta = 1$, the solutions (59), (60) match with as discussed in [22].

Example 6.4 Consider the following time and space fractional system of PDEs:

$$\begin{aligned}\frac{\partial^{\alpha_1} u_1}{\partial t^{\alpha_1}} &= \frac{\partial^{3\beta} u_2^2}{\partial x^{3\beta}} + \eta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_1^2\frac{\partial^\beta u_2}{\partial x^\beta}\Big),\\ \frac{\partial^{\alpha_2} u_2}{\partial t^{\alpha_2}} &= \frac{\partial^{3\beta} u_1^2}{\partial x^{3\beta}} + \zeta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_2^2\frac{\partial^\beta u_1}{\partial x^\beta}\Big),\ t > 0, 0 < \alpha_1, \alpha_2, \beta \le 1,\end{aligned} \tag{61}$$

where $\eta, \zeta \neq 0$ are arbitrary constants, along with the initial conditions

$$u_1(x,0) = a + bx^\beta,\ \ u_2(x,0) = c + dx^\beta,\ a, b, c, d \in \mathbb{R}. \tag{62}$$

Taking the Sumudu transform on both sides of Eq. (61)

$$\begin{aligned} S\Big[\frac{\partial^{\alpha_1} u_1}{\partial t^{\alpha_1}}\Big] &= S\Big[\frac{\partial^{3\beta} u_2^2}{\partial x^{3\beta}} + \eta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_1^2\frac{\partial^\beta u_2}{\partial x^\beta}\Big)\Big],\\ S\Big[\frac{\partial^{\alpha_2} u_2}{\partial t^{\alpha_2}}\Big] &= S\Big[\frac{\partial^{3\beta} u_1^2}{\partial x^{3\beta}} + \zeta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_2^2\frac{\partial^\beta u_1}{\partial x^\beta}\Big)\Big].\end{aligned}$$

Using the property (3) of Sumudu transform, we get

$$\begin{aligned} S[u_1(x,t)] &= u_1(x,0) + \omega^{\alpha_1} S\Big[\frac{\partial^{3\beta} u_2^2}{\partial x^{3\beta}} + \eta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_1^2\frac{\partial^\beta u_2}{\partial x^\beta}\Big)\Big],\\ S[u_2(x,t)] &= u_2(x,0) + \omega^{\alpha_2} S\Big[\frac{\partial^{3\beta} u_1^2}{\partial x^{3\beta}} + \zeta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_2^2\frac{\partial^\beta u_1}{\partial x^\beta}\Big)\Big].\end{aligned} \tag{63}$$

Taking inverse Sumudu transform on both sides of Eq. (63)

$$\begin{aligned} u_1(x,t) &= S^{-1}[u_1(x,0)] + S^{-1}\Big(\omega^{\alpha_1} S\Big[\frac{\partial^{3\beta} u_2^2}{\partial x^{3\beta}} + \eta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_1^2\frac{\partial^\beta u_2}{\partial x^\beta}\Big)\Big]\Big),\\ u_2(x,t) &= S^{-1}[u_2(x,0)] + S^{-1}\Big(\omega^{\alpha_2} S\Big[\frac{\partial^{3\beta} u_1^2}{\partial x^{3\beta}} + \zeta\frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_2^2\frac{\partial^\beta u_1}{\partial x^\beta}\Big)\Big]\Big).\end{aligned}$$

In view of the recurrence relation (47) we get

$$\begin{aligned} u_{10} &= S^{-1}[u_1(x,0)] = a + bx^\beta,\\ u_{20} &= S^{-1}[u_2(x,0)] = c + dx^\beta,\end{aligned}$$

$$
\begin{aligned}
u_{11} &= S^{-1}\Big(\omega^{\alpha_1} S\Big[\frac{\partial^{3\beta} u_{20}^2}{\partial x^{3\beta}} + \eta \frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_{10}^2 \frac{\partial^{\beta} u_{20}}{\partial x^{\beta}}\Big)\Big]\Big),\\
&= \frac{\eta b^2 d \Gamma(\beta+1)\Gamma(2\beta+1)t^{\alpha_1}}{\Gamma(\alpha_1+1)},\\
u_{21} &= S^{-1}\Big(\omega^{\alpha_2} S\Big[\frac{\partial^{3\beta} u_{10}^2}{\partial x^{3\beta}} + \zeta \frac{\partial^{2\beta}}{\partial x^{2\beta}}\Big(u_{20}^2 \frac{\partial^{\beta} u_{10}}{\partial x^{\beta}}\Big)\Big]\Big),\\
&= \frac{\zeta b d^2 \Gamma(\beta+1)\Gamma(2\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)},\\
u_{1n} &= 0, n \geq 2,\\
u_{2n} &= 0, n \geq 2.
\end{aligned}
$$

Thus, the exact solution of the fractional system (61) along with initial conditions (62) is given by

$$
\begin{aligned}
u_1(x,t) &= a + bx^{\beta} + \frac{\eta b^2 d \Gamma(\beta+1)\Gamma(2\beta+1)t^{\alpha_1}}{\Gamma(\alpha_1+1)},\\
u_2(x,t) &= c + dx^{\beta} + \frac{\zeta b d^2 \Gamma(\beta+1)\Gamma(2\beta+1)t^{\alpha_2}}{\Gamma(\alpha_2+1)}.
\end{aligned}
$$

7 Conclusions

Sumudu transform iterative method is developed by combining Sumudu transform and DJM [5]. This approach is suitable for getting exact solutions of time and space FPDEs and as well as systems of them. We demonstrate its applicability by solving a large number of nontrivial examples. Although the combination of Sumudu transform with other decomposition methods such as HPM and ADM has been proposed in the literature [10, 11], the combination of Sumudu transform with DJM gives better and more efficient method as we do not need to construct homotopy or find Adomian polynomials.

References

1. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198. Academic Press, New York (1998)
2. Debnath, L., Bhatta, D.D.: Solutions to few linear fractional inhomogeneous partial differential equations in fluid mechanics. Fract. Calc. Appl. Anal. **7**(1), 21–36 (2004)
3. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer, Boston (1994)
4. He, J.-H.: Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. **178**(3), 257–262 (1999)

5. Daftardar-Gejji, V., Jafari, H.: An iterative method for solving nonlinear functional equations. J. Math. Anal. Appl. **316**(2), 753–763 (2006)
6. Bhalekar, S., Patade, J.: An analytical solution of fishers equation using decomposition method. Am. J. Comput. Appl. Math. **6**(3), 123–127 (2016)
7. AL-Jawary, M.A., Radhi, G.H., Ravnik, J.: Daftardar-Jafari method for solving nonlinear thin film flow problem. Arab. J. Basic Appl. Sci. **25**(1), 20–27 (2018)
8. Jafari, H.: Numerical solution of time-fractional Klein-Gordon equation by using the decomposition methods. J. Comput. Nonlinear Dyn. **11**(4), 041015 (2016)
9. Jafari, H., Nazari, M., Baleanu, D., Khalique, C.: A new approach for solving a system of fractional partial differential equations. Comput. Math. Appl. **66**(5), 838–843 (2013)
10. Singh, J., Devendra, S.: Homotopy perturbation sumudu transform method for nonlinear equations. Adv. Theor. Appl. Mech **4**(4), 165–175 (2011)
11. Kumar, D., Singh, J., Rathore, S.: Sumudu decomposition method for nonlinear equations. Int. Math. Forum **7**, 515–521 (2012)
12. Wang, K., Liu, S.: A new sumudu transform iterative method for time-fractional cauchy reaction-diffusion equation. SpringerPlus **5**(1), 865 (2016)
13. Prakash, A., Kumar, M., Baleanu, D.: A new iterative technique for a fractional model of nonlinear Zakharov-Kuznetsov equations via sumudu transform. Appl. Math. Comput. **334**, 30–40 (2018)
14. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach, Yverdon (1993)
15. Miller, K.S., Ross, B.: An introduction to the fractional calculus and fractional differential equations. Wiley, New York (1993)
16. Bonilla, B., Rivero, M., Rodríguez-Germá, L., Trujillo, J.J.: Fractional differential equations as alternative models to nonlinear differential equations. Appl. Math. Comput. **187**(1), 79–88 (2007)
17. Belgacem, F.B.M., Karaballi, A.A.: Sumudu transform fundamental properties investigations and applications. Int. J. Stoch. Anal. (2006)
18. Amer, Y., Mahdy, A., Youssef, E.: Solving systems of fractional nonlinear equations of Emden Fowler type by using sumudu transform method. Glob. J. Pure Appl. Math. **14**(1), 91–113 (2018)
19. Bhalekar, S., Daftardar-Gejji, V.: Convergence of the new iterative method. Int. J. Differ. Equ. (2011)
20. Choudhary, S., Daftardar-Gejji, V.: Invariant subspace method: a tool for solving fractional partial differential equations. Fract. Calc. Appl. Anal. **20**(2), 477–493 (2017)
21. Choudhary, S., Daftardar-Gejji, V.: Solving systems of multi-term fractional PDEs: Invariant subspace approach. Int. J. Model. Simul. Sci. Comput. **10**(1) (2019)
22. Sahadevan, R., Prakash, P.: Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations. Commun. Nonlinear Sci. Numer. Simul. **42**, 158–177 (2017)