Correlation and Inequality in Weighted Majority Voting Games

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Abstract   In a weighted majority voting game, the weights of the players are deter-
mined based on some socioeconomic parameter. A number of measures have been
proposed to measure the voting powers of the different players. A basic question in
this area is to what extent does the variation in the voting powers reflect the variation
in the weights? The voting powers depend on the winning threshold. So, a second
question is what is the appropriate value of the winning threshold? In this work, we
propose two simple ideas to address these and related questions in a quantifiable
manner. The first idea is to use Pearson's Correlation Coefficient between the weight
vector and the power profile to measure the similarity between weight and power. The
second idea is to use standard inequality measures to quantify the inequality in the
weight vector as well as in the power profile. These two ideas answer the first ques-
tion. Both the weight–power similarity and inequality scores of voting power profiles
depend on the value of the winning threshold. For situations of practical interest, it
turns out that it is possible to choose a value of the winning threshold which maxi-
mizes the similarity score and also minimizes the difference in the inequality scores
of the weight vector and the power profile. This provides an answer to the second
question. Using the above formalization, we are able to quantitatively argue that it is
sufficient to consider only the vector of swings for the players as the power measure.
We apply our methodology to the voting games arising in the decision-making pro-
cesses of the International Monetary Fund (IMF) and the European Union (EU). In
the case of IMF, we provide quantitative evidence that the actual winning threshold
that is currently used is suboptimal and instead proposes a winning threshold which
has a firm analytical backing. On the other hand, in the case of EU, we provide
quantitative evidence that the presently used threshold is very close to the optimal.

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1 Introduction

Voting is arguably the most important aspect of decision-making in a democratic
setup. A committee settles an issue by accepting or rejecting some resolution related
to the issue. While unanimity or consensus is desirable, this may not always be
possible due to the conflicting interests of the different committee members. In such
a situation, a voting procedure among the members is used to either accept or reject
a resolution. A resolution is accepted or passed, if a certain number of persons vote
in its favor, else it fails and is rejected.

In its basic form, each committee member has a single vote. Many scenarios of
practical interest, on the other hand, assign weights to the committee members. These
weights need not be the same for all the members. In the context of weighted voting,
a resolution is accepted, if the sum total of the weights of the members who vote in
its favor cross a previously decided upon threshold. A common example of weighted
voting is a company boardroom, where the members have weights in proportion to
the shares that they hold in the company. Important examples of weighted voting
in the context of public policy are the European Union (EU) and the International
Monetary Fund (IMF).

Voting procedures have been formally studied in the game theory literature under
the name of voting games. Due to its real-life importance, weighted majority voting
games have received a lot of attention. In the literature on voting games, the members
are called players. One of the basic questions is how much influence does a player
have in determining the outcome of a voting procedure? In other words, what is the
power of a player in a voting game? In quantitative terms, it is desirable to measure
the power of a player in a voting game by assigning a nonnegative real number to
the player. A power measure assigns such a number to each player in the game. This
leads to the basic question of what constitutes a good measure of power of a player
in a voting game. The literature contains a number of power measures. Each one
of these measures aims to capture certain aspects of the informal notion of power
in a voting game. We refer to Felsenthal and Machover (1998) for a comprehensive
discussion to voting games and the various power measures. An introduction to the
area can be found in Chakravarty et al. (2015).

Consider the setting of weighted majority voting games. For any such game, the
players are assigned weights based on socioeconomic parameters. As a result, there
is a variation in the weights of the players. Further, given any power measure, we
obtain a variation in the powers of the different players. It is well known that the
variation in the voting powers does not necessarily reflect the variation in the weights.
In this context, the following three questions can be formulated:

1. To what extent does the variation in the voting powers reflect the variation in the weights?
2. Is the inequality present in the weights preserved in the voting powers?
3. How does the value of the winning threshold (i.e., the threshold which is required to be crossed for a motion to be passed) affect the above two questions?

This work addresses the above questions. The questions posed are not merely theoretical. Similar questions have been posed in Leech (2002b) in the context of measurement of voting power in IMF. For example, the following text fragments appear in Leech (2002b):

> ... weighted voting raises the important question of whether the resulting inequality of power over actual decisions is precisely what was intended for the relationship between power and contribution.

> How does the voting power of individual countries compare with their nominal votes? To what extent is the degree of inequality in the distribution of votes reflected in the distribution of voting power?

> Different types of decisions use different decision rules, some requiring a special supermajority. What effect do different decision rules have on the distribution of power and also on the power of the voting body itself to act?

The work Leech (2002b) makes a qualitative analysis of the above issues. Our work allows a quantitative analysis of these issues. In more detail, our work makes the following contributions.

Measurement of Similarity Between Weights and Voting Powers. We propose the use of Pearson's correlation coefficient as a measure of similarity between the weight vector of the players and the vector of voting powers of the players.

Measurement of Inequality in Weights and Voting Powers. There is a large literature on the measurement of inequality in a vector of values obtained from measurement of various social parameters. A survey on measurement of inequality appears in Cowell (2016). The variation of the values in such a vector is captured by an inequality index. A number of inequality indices have been proposed in the literature. We propose the use of such inequality indices to measure the inequality in the weights and also in the voting powers. This allows the comparison of the inequality present in the weights to that present in the voting powers.

Winning Threshold as a Controllable Parameter. Our formalizations of both the similarity between the weights and the voting powers as well as the measurement of inequality in the voting powers have the winning threshold as a parameter. By varying this parameter, both the weight–power similarity and the voting power inequality can be controlled. So, given a vector of weights, the winning threshold can be set to a certain value to maximize the weight–power similarity or to minimize the difference between the inequality in the weights and the inequality in the voting powers.

In this context, we would like to discuss the broader issue of designing games to achieve certain desirable power profiles. This is often called the inverse problem for voting games. Usually, the goal is to determine a set of weights which result in the target powers. For example, in the context of the IMF voting game, an iterative

algorithm to determine weights has been proposed in Leech (2002b). There is one major drawback of this approach. As mentioned earlier, in a weighted majority voting game, the weights often represent a socioeconomic parameter. When the weights are artificially obtained (say, using an iterative algorithm), their interpretation in the socioeconomic context is lost. It then becomes hard to provide a natural justification of the weights.

Our approach of having the winning threshold as a controllable parameter provides an alternative method of designing games. For the complete specification of a game, both the weights and the winning threshold need to be specified. In our approach, the weights do not change, and hence they retain their original interpretation arising from the background socioeconomic application. We only suggest tuning the winning threshold so that the resulting power profile is "imbued" with the intuitive natural justification of the weights. Games designed using such an approach can be much better explained to the general public than games where the weights are artificially obtained.

Detailed Study. We consider seven different voting power measures and two different inequality indices. We show that the scaling invariance property of an inequality measure as well as that of the Pearson's correlation coefficient divides the voting power measures into three groups. The non-normalized Banzhaf measure, the normalized Banzhaf index, and the two Coleman measures fall into one group; the public good measure and public good index defined by Holler fall into a second group, and the Deegan–Packel measure is in the third group. We show that any two power measures in the same group have the same behavior with respect to both the similarity index and the inequality index. This brings down the complexity of the analysis.

There has been a lot of discussion in the literature on the comparative suitabilities of the Banzhaf and the Coleman indices Banzhaf (1965), Brink and Laan (1998), Coleman (1971), Dubey and Shapley (1979), Laruelle and Valenciano (2001), Lehrer (1998), Laruelle and Valenciano (2011), Barua et al. (2009). This discussion has both been qualitative and also formal in the sense of axiomatically deriving the indices Brink and Laan (1998), Lehrer (1998). Our work provides a new perspective to this discussion. The stand-alone values of the powers of the players as measured by any power measure are perhaps not of much interest by themselves. It is only in a relative sense that they acquire relevance. There are two ways to consider this relative sense, in comparison to the weights and in comparison among themselves. We propose to quantify the relative notion in comparison to the weights by the correlation between the weight vector and the power profile and to quantify the relative values of the powers among themselves by an appropriate inequality score. Under both of these quantifications, we prove that the Banzhaf and the Coleman power measures turn out to be the same. Based on this result, we put forward the suggestion that there is perhaps no *essential* difference between these power measures. It is sufficient to consider only the swings for the different players as was originally proposed by Banzhaf (and sometimes called the raw Banzhaf measure). While this may sound a bit radical, our analysis based on correlation and inequality does not leave scope for any other considerations. It is of course possible that there is some other quantifiable

ways of distinguishing between the relative spreads of the Banzhaf and the Coleman measures. This can be a possible future research question.

The literature contains a number of voting power measures which have been proposed as fundamentally different from the swing-based Banzhaf measure. Intuitive arguments have been forwarded as to why these measures are appropriate for certain applications. In our opinion, a basic requirement for any power measure is to reflect the "content" of weights. In addition to the Banzhaf measure, we have also considered the Holler measures and the Deegan–Packel measure. Our simulation experiments as well as computations with real-life data show that the "content" of the weights is best captured by the Banzhaf measure and neither the Holler measure nor the Deegan–Packel measure is good indicators of this "content". Based on this evidence, we put forward the suggestion that it is sufficient to consider the swings as the only measure of power in voting games.

Applications. IMF decision-making procedures have been modeled as voting games Leech (2002b). Decision-making in the EU has also been discussed in the context of voting games Leech (2002a).

The notions of similarity between the variations in the weights and the voting powers as well as the relation between the inequality in the weights and that in the voting powers have been informally discussed. Our proposals for measuring weight–power similarity and the voting power inequality formalizes this intuition. We compute the various measures for the IMF game and (a simplified version of) the EU voting game and suggest that the winning threshold can be used as a parameter in achieving target values of similarity or inequality.

In both the IMF and the EU voting games, there is a "natural" justification for assigning weights to the different players. In the context of IMF, the weights reflect the proportion of financial contribution made by the different countries, while in the case of EU, the weights reflect the population of the different countries. This is reasonable, since the IMF is a financial organization, while the EU is essentially a political organization. In both cases, however, the choice of the winning threshold is not backed by any quantifiable parameter.

Our work provides methods for choosing a winning threshold which has a *quantifiable* justification. There are two options. In the first option, one should choose a value of the winning threshold which maximizes the correlation between the weight vector and an appropriate power profile. In the second option, one should choose a value of the winning threshold which yields an inequality score for an appropriate power profile which is closest to the inequality score of the weight vector. In both the cases of IMF and EU, both the options lead to similar values of the winning threshold. Based on this analysis, we put forward the suggestion that the voting rule for IMF should be modified to reflect the optimal value of the winning threshold. In the case of EU, our results provide evidence that the presently used winning threshold is close to the optimal value.

Previous and Related Works

The Shapley–Shubik power index was introduced in Shapley (1953), Shapley and Shubik (1954), Banzhaf index was introduced in Banzhaf (1965) while Coleman

indices were introduced in Coleman (1971). Later work by Holler (1982) and Holler and Packel (1983) introduced the public good measure/index. Deegan and Packel introduced another power measure in Deegan and Packel (1978). There are other known measures/indices and we refer to Felsenthal and Machover (1998), Chakravarty et al. (2015) for further details.

The first work to address the problem of inequality in voting games is Einy and Peleg (1991). They provided an axiomatic deduction of an inequality index for the Shapley–Shubik power measure. A more general axiomatic treatment of inequality for power measures appears in a paper by Laruelle and Valenciano (2004). This work postulates axioms and deduces an inequality measure for a class of power indices which includes the Banzhaf index. A more recent work by Weber (2016) suggests the use of the Coefficient of Variation as an inequality index for measuring inequality arising from the Banzhaf index. Later, we provide a more detailed discussion of the relationship of these prior works to our contribution.

2 Preliminaries

2.1 *Voting Games*

We provide some standard definitions arising in the context of voting games. For details, the reader may consult Felsenthal and Machover (1998), Chakravarty et al. (2015). In the following, the cardinality of a finite set S will be denoted by $\#S$ and the absolute value of a real number x will be denoted by $|x|$.

Let $N = \{A_1, A_2, \ldots, A_n\}$ be a set of n players. A subset of N is called a *voting coalition*. The set of all voting coalitions is denoted by 2^N. A voting game G is given by its characteristic function $\widehat{G} : 2^N \to \{0, 1\}$, where a winning coalition is assigned the value 1 and a losing coalition is assigned the value 0. Below, we recall some basic notions about voting games:

1. For any $S \subseteq N$ and player $A_i \in N$, A_i is said to be a *swing* in S if $A_i \in S$, $\widehat{G}(S) = 1$ but $\widehat{G}(S \setminus \{A_i\}) = 0$.
2. For a voting game G, the number of swings for A_i will be denoted by $m_G(A_i)$.
3. A player $A_i \in N$ is called a *dummy player* if A_i is not a swing in any coalition, i.e., if $m_G(A_i) = 0$.
4. For a voting game G, the set of all winning coalitions will be denoted by $W(G)$ and the set of all losing coalitions will be denoted by $L(G)$.
5. A coalition $S \subseteq N$ is called a *minimal winning coalition* if $\widehat{G}(S) = 1$ and there is no $T \subset S$ for which $\widehat{G}(T) = 1$.
6. The set of all minimal winning coalitions in G will be denoted by $\mathsf{MW}(G)$ and the set of minimal winning coalitions containing the player A_i will be denoted as $\mathsf{MW}_G(A_i)$.
7. A voting game G is said to be *proper* if for any coalition $S \subseteq N$, $\widehat{G}(S) = 1$ implies that $\widehat{G}(N \setminus S) = 0$. In other words, in a proper game, it is not allowed for both S and its complement to be winning.

Definition 1 Consider a triplet $(N, \mathbf{w}, q)$, where $N = \{A_1, \ldots, A_n\}$ is a set of players; $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ is a vector of nonnegative weights with w_i being the weight of A_i; and q is a real number in $(0, 1)$. Let $\omega = \sum_{i=1}^{n} w_i$. The triplet $(N, \mathbf{w}, q)$ defines a *weighted majority voting game* G given by its characteristic function $\widehat{G} : 2^N \rightarrow \{0, 1\}$ in the following manner. Let $w_S = \sum_{A_i \in S} w_i$ denote the sum of the weights of all the players in the coalition $S \subseteq N$. Then

$$\widehat{G}(S) = \begin{cases} 1 & \text{if } w_S/\omega \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $G = (N, \mathbf{w}, q)$ to denote the weighted majority voting game arising from the triplet $(N, \mathbf{w}, q)$.

For a weighted majority voting game $G = (N, \mathbf{w}, q)$ to be proper, it is necessary that $q > 0.5$. For the technical analysis of weighted majority voting games, we do not restrict to proper games. When considering applications, as is conventional, one should consider only proper games.

2.2 Voting Power

The notion of power is an important concept in a voting system. A *power measure* captures the capability of a player to influence the outcome of a vote.

Given a game G on a set of players N and a player A_i in N, a power measure $\mathcal{P}$ associates a nonnegative real number $v_i = \mathcal{P}_G(A_i)$ to the player A_i. The number v_i captures the power that A_i has in the game G. If $\sum_{A_i \in G} \mathcal{P}_G(A_i) = 1$ for all games G, then $\mathcal{P}$ is called a *power index*. In other words, for a power index, the powers of the individual players sum to 1.

A widely studied index of voting power is the Shapley–Shubik index. This index, however, is defined for a voting game where the order in which the players cast their votes is important. In our application of voting power to the voting games arising in the IMF and EU decision-making processes, the order of casting votes is not important. So, we do not consider the Shapley–Shubik index in this work. Below, we provide the definitions of some of the previously proposed power measures. See Felsenthal and Machover (1998), Chakravarty et al. (2015) for further details.

Banzhaf Power Measures. The *raw Banzhaf power measure* $\mathsf{BR}_G(A_i)$ for a player A_i in the game G is defined as the number of distinct coalitions in which A_i is a swing. Hence,

$$\mathsf{BR}_G(A_i) = m_G(A_i).$$

The *non-normalized Banzhaf power measure* $\mathsf{BZN}_G(A_i)$ is defined as follows:

$$\mathsf{BZN}_G(A_i) = \frac{\mathsf{BR}_G(A_i)}{2^{n-1}} = \frac{m_G(A_i)}{2^{n-1}}.$$

The *Banzhaf normalized power index* $\mathsf{BZ}_G(A_i)$ is defined as follows:

$$\mathsf{BZ}_G(A_i) = \frac{\mathsf{BR}_G(A_i)}{\sum_{j=1}^{n} \mathsf{BR}_G(A_j)} = \frac{m_G(A_i)}{\sum_{j=1}^{n} m_G(A_j)}.$$

Coleman Power Measures. The *Coleman preventive power measure* $\mathsf{CP}_G(A_i)$ for a player A_i in the game G is a measure of its ability to stop a coalition S from achieving $w_S \geq q$. It is defined as follows:

$$\mathsf{CP}_G(A_i) = \frac{m_G(A_i)}{\#W(G)}.$$

The *Coleman initiative power measure* $\mathsf{Cl}_G(A_i)$ for a player A_i in the game G is a measure of its ability to turn an otherwise losing coalition S with $w_S < q$ into a winning coalition with $w_{S \cup \{A_i\}} \geq q$. It is defined as follows:

$$\mathsf{Cl}_G(A_i) = \frac{m_G(A_i)}{\#L(G)} = \frac{m_G(A_i)}{2^n - \#W(G)} = \frac{\mathsf{CP}_G(A_i)}{\frac{2^n}{\#W(G)} - 1}.$$

Holler Public Good Index. Holler proposed the *public good index* $\mathsf{PGI}_{A_i}(G)$ as follows:

$$\mathsf{PGI}_{A_i}(G) = \frac{\#\mathsf{MW}_G(A_i)}{\sum_{A_j \in N} \#\mathsf{MW}_G(A_j)}.$$

The non-normalized version of $\mathsf{PGI}_{A_i}(G)$ is called the *absolute public good measure*. It is defined as

$$\mathsf{PGM}_{A_i}(G) = \frac{\#\mathsf{MW}_G(A_i)}{\#\mathsf{MW}(G)}.$$

Deegan–Packel Power Measure. The *Deegan–Packel power measure* $\mathsf{DP}_G(A_i)$ for a player A_i in the game G is defined to be

$$\mathsf{DP}_G(A_i) = \frac{1}{\#\mathsf{MW}(G)} \sum_{S \in \mathsf{MW}_G(A_i)} \frac{1}{\#S}.$$

Power Profile. Suppose $\mathcal{P}$ is a measure of voting power. Then, $\mathcal{P}$ assigns a nonnegative real number to each of the n players in the game. So, $\mathcal{P}$ is given by a vector of nonnegative real numbers. This vector is called the $\mathcal{P}$-power profile of the game.

Computing Voting Powers. A weighted majority voting game $G = (N, \mathbf{w}, q)$ is completely specified by the set of players N, a weight vector $\mathbf{w}$, and the threshold q. Given this data, it is of interest to be able to compute the $\mathcal{P}$-power profile for any power measure $\mathcal{P}$. There are known dynamic-programming-based algorithms for computing the values of the different voting power indices. We refer to Matsui and Matsui

(2000), Chakravarty et al. (2015) for an introduction to algorithms for computing voting powers. In our work, we have implemented the algorithms for computing the $\mathcal{P}$-power profiles where $\mathcal{P}$ is any of the power measures defined above.

There is a large literature on voting powers. The various indices mentioned above have been introduced to model certain aspects of voting games which are not adequately covered by the other indices. There have been axiomatic characterizations of these indices. A detailed discussion of the relevant literature is not really within the focus of the present work. Instead, we refer to the highly respected monograph Felsenthal and Machover (1998) and the more recent textbook Chakravarty et al. (2015) for such details. *Our concern in this work is how to quantify the efficacy of any particular voting power measure that one may choose for a particular application.*

2.3 Pearson's Correlation Coefficient

Given vectors $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, Pearson's correlation coefficient is the standard measure of linear correlation between these two vectors. It is defined as follows:

$$\mathsf{PCC}(\mathbf{w}, \mathbf{v}) = \begin{cases} 0 & \text{if } w_1 = \cdots = w_n \text{ or } v_1 = \cdots = v_n; \\ \dfrac{\sum_{i=1}^{n}(w_i - \mu_{\mathbf{w}})(v_i - \mu_{\mathbf{v}})}{\sqrt{\sum_{i=1}^{n}(w_i - \mu_{\mathbf{w}})^2}\sqrt{\sum_{i=1}^{n}(v_i - \mu_{\mathbf{v}})^2}} & \text{otherwise.} \end{cases} \quad (1)$$

Here, $\mu_{\mathbf{w}}$ and $\mu_{\mathbf{v}}$ are the means of $\mathbf{w}$ and $\mathbf{v}$, respectively.

From (1), it follows that for any two positive real numbers γ and δ,

$$\mathsf{PCC}(\mathbf{w}, \mathbf{v}) = \mathsf{PCC}(\gamma \mathbf{w}, \delta \mathbf{v}). \quad (2)$$

The relation captured in (2) can be considered to be a scale invariance property of the Pearson's correlation coefficient.

2.4 Inequality Indices

The notion of inequality has been considered for social parameters including income, skills, education, health, and wealth Cowell (2016). There are several methods for measuring inequality. At a basic level, the idea of an inequality index $\mathcal{I}$ is the following. Given a vector $\mathbf{a}$ whose components are real numbers, $\mathcal{I}(\mathbf{a})$ produces a nonnegative real number r. In other words, the index $\mathcal{I}$ assigns an inequality score of r to the vector $\mathbf{a}$. There is a large literature on inequality indices including the measurement of multidimensional inequality Chakravarty and Lugo (2016), Chakravarty (2017). In this work, we will consider only basic inequality indices. Some of the most commonly used inequality indices are mentioned below.

Given a vector $\mathbf{a}$ of real numbers, let $\mu_{\mathbf{a}}$ and $\sigma_{\mathbf{a}}$ denote the mean and standard deviation of $\mathbf{a}$. *In the definition of the inequality indices below, we will assume that the entries of $\mathbf{a}$ are nonnegative and $\mu_{\mathbf{a}}$ is positive.*

Gini Index. The value of the Gini index of a vector $\mathbf{a} = (a_1, \ldots, a_n)$ is given by

$$\mathsf{GI}(\mathbf{a}) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_i - a_j|}{2n \sum_{i=1}^{n} a_i}. \tag{3}$$

Coefficient of Variation. For a vector $\mathbf{a} = (a_1, \ldots, a_n)$, the Coefficient of Variation is computed as the ratio of the standard deviation $\sigma_{\mathbf{a}}$ to the mean $\mu_{\mathbf{a}}$ of $\mathbf{a}$.

$$\mathsf{CoV}(\mathbf{a}) = \frac{\sigma_{\mathbf{a}}}{\mu_{\mathbf{a}}} = \frac{\sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} a_i\right)^2}}{\frac{1}{n} \sum_{i=1}^{n} a_i}. \tag{4}$$

Generalized Entropy Index. The generalized entropy index is a measure of inequality based on information theory. For a real number α, the generalized entropy index $\mathsf{GEI}_\alpha(\mathbf{a})$ is defined in the following manner:

$$\mathsf{GEI}_\alpha(\mathbf{a}) = \begin{cases} \frac{1}{\alpha(\alpha-1)} \left(\frac{1}{n} \sum_{A_i \in N} \left(\frac{a_i}{\mu_{\mathbf{a}}}\right)^\alpha - 1\right) & \text{if } \alpha \neq 0, 1; \\ \frac{1}{n} \sum_{A_i \in N, a_i > 0} \left(\frac{a_i}{\mu_{\mathbf{a}}^+}\right) \ln\left(\frac{a_i}{\mu_{\mathbf{a}}^+}\right) & \text{if } \alpha = 1; \\ -\frac{1}{n} \sum_{A_i \in N, a_i > 0} \ln\left(\frac{a_i}{\mu_{\mathbf{a}}^+}\right) & \text{if } \alpha = 0. \end{cases} \tag{5}$$

Here, ln denotes the natural logarithm and $\mu_{\mathbf{a}}^+$ denotes the mean of the positive entries in $\mathbf{a}$. Also, note that for $\alpha = 0, 1$ the sum is over positive values of a_i as otherwise the ln function gets applied to 0. In other words, for $\alpha = 0$ and 1, the computation of inequality considers only the positive entries of $\mathbf{a}$. GEI_1 is called the *Theil Index* and GEI_2 is half the square of CoV.

Remark For application to the context of voting powers, a power of zero implies that the player is a dummy. If GEI_0 or GEI_1 is used to measure inequality, then such dummy players will get ignored. As a result, the inequality in the power profile will not be adequately captured by these two measures. Due to this reason, GEI_0 and GEI_1 are not suitable for measuring inequality in voting powers. GEI_2 is half the square of CoV and will essentially spread out the value of CoV. The relevance of GEI_k for $k > 2$ to the context of voting power is not clear. So, though we have computed, we do not report the values of GEI in this work.

Computing Inequality Indices. It is quite routine to implement an algorithm, which given a vector of nonnegative quantities computes the values of the various inequality indices. In our work, we have implemented algorithms to compute the Gini index and the Coefficient of Variation.

Desirable Properties of an Inequality Index. A few basic and natural properties have been postulated which any reasonable inequality measure should satisfy. Below, we mention these properties. See Cowell (2016) for more details. Let $\mathcal{I}$ be a postulated inequality index and $\mathbf{a} = (a_1, \ldots, a_n)$ be a vector of nonnegative real numbers.

Let π be a bijection from $\{1, \ldots, n\}$ to itself, i.e., π is a permutation of $\{1, \ldots, n\}$. Define $\mathbf{a}_\pi$ to be the vector $(a_{\pi(1)}, \ldots, a_{\pi(n)})$, i.e., $\mathbf{a}_\pi$ is a reordering of the components of $\mathbf{a}$.

Anonymity (ANON): $\mathcal{I}$ is said to satisfy anonymity if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\mathbf{a}_\pi)$ for all permutations π of $\{1, \ldots, n\}$. Anonymity captures the property that inequality depends only on the (multi-)set of values $\{a_1, \ldots, a_n\}$. Information related to ordering or labeling of these values using names are irrelevant for the measurement of inequality.

Egalitarian Principle (EP): $\mathcal{I}$ is said to satisfy the egalitarian principle if $\mathcal{I}(\mathbf{a}) = 0$ for all $\mathbf{a}$ such that $a_1 = \cdots = a_n$. EP captures the property that the inequality is the minimum possible when all components of the vector $\mathbf{a}$ have the same value.

Scale Invariance (ScI). $\mathcal{I}$ is said to satisfy scale invariance if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\gamma\mathbf{a})$ for all real $\gamma > 0$. The idea behind scale invariance is that if all the values are scaled by the same factor then the inequality remains unchanged.

Let $\mathbf{a}^{[k]}$ denote the vector

$$\left(\underbrace{a_1, \ldots, a_1}_{k}, \underbrace{a_2, \ldots, a_2}_{k} \ldots, \underbrace{a_n, \ldots, a_n}_{k} \right).$$

Population Principle (PP). $\mathcal{I}$ is said to satisfy the population principle if $\mathcal{I}(\mathbf{a}) = \mathcal{I}(\mathbf{a}^{[k]})$ for any integer $k \geq 1$. The vector $a^{[k]}$ contains k copies of each of the values $a_1, \ldots, a_n$. PP says that the inequality in such a vector remains the same as in the original vector, i.e., by replicating each of the components of the original vector the same number of times does not change the inequality.

For $1 \leq i < j \leq n$, let $\mathbf{a}_{i,j,\delta}$ be the vector

$$(a_1, \ldots, a_{i-1}, a_i + \delta, \ldots, a_j - \delta, a_{j+1}, \ldots, a_n).$$

Transfer Principle (TP). $\mathcal{I}$ is said to satisfy the transfer principle if $\mathcal{I}(\mathbf{a}) \geq \mathcal{I}(\mathbf{a}_{i,j,\delta})$ for any $1 \leq i < j \leq n$ and $\delta > 0$ such that $a_i < a_j$ and $a_i + \delta \leq a_j - \delta$. The transfer principle says that if δ units are transferred from a richer person to a poorer person without changing their relative ordering, then inequality cannot increase.

Suppose $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are vectors of dimensions $n_1, \ldots, n_k$, respectively, with nonnegative real entries. Let μ_i be the mean of $\mathbf{a}_i$ and define $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$. Let $\mathbf{a}$ be the vector formed by concatenating the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$.

Decomposability (Decom). $\mathcal{I}$ is said to satisfy decomposability if

$$\mathcal{I}(\mathbf{a}) = \sum_{i=1}^{k} n_i \mathcal{I}(\mathbf{a}_i) + \mathcal{I}(\boldsymbol{\mu}).$$

Decomposability captures the following idea. The vector $\mathbf{a}$ is divided into k groups and inequality is measured for each of the groups. Further, the mean of each group is computed and inequality is computed for the vector composing of the means. The inequality for each group is "within-group inequality', whereas the inequality in the vector of means is some kind of "across group inequality". The index $\mathcal{I}$ satisfies decomposability if the overall inequality in the vector can be decomposed into a sum of "within-group inequality" and "across group inequality".

The Gini Index, the Coefficient of Variation, and the Generalized Entropy Indices satisfy ANON, EP, ScI, PP, and TP. It has been shown Shorrocks (1980) that any index which satisfies ANON, ScI, PP, TP, and Decom must necessarily have the form of a generalized entropy index for some value of α.

ANON, EP, ScI, and TP are natural properties that any inequality index should satisfy irrespective of the domain to which it is applied. PP becomes relevant in the context of variable population size. For voting games, the players constitute the population which is fixed. So, the application of PP to voting games is vacuous. On the other hand, it is not clear that Decom is necessarily a desirable property for *all* applications of inequality. In particular, it is not clear that Decom is relevant in the context of voting powers which is the focus of the present work.

3 Weight–Power Similarity

Let $\mathcal{P}$ be a measure of voting power. Suppose this is applied to a weighted majority voting game $G = (N, \mathbf{w}, q)$. Let $\mathbf{v}$ be the resulting power profile. It is of interest to know how similar the power profile vector $\mathbf{v}$ is to the weight vector $\mathbf{w}$. Note that the power profile vector $\mathbf{v}$ depends on the winning threshold q. Based on the Pearson's correlation coefficient, we define the similarity index $\mathcal{P}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$ as follows:

$$\mathcal{P}\text{-}\mathsf{SI}_{\mathbf{w}}(q) = \mathsf{PCC}(\mathbf{w}, \mathbf{v}), \tag{6}$$

where $\mathbf{v}$ is the power profile vector generated by the voting power measure $\mathcal{P}$ applied to the weighted majority voting game $G = (N, \mathbf{w}, q)$.

So, for a fixed q, $\mathcal{P}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$ measures the similarity of the power profile vector to the weight vector by the correlation between these two vectors. Note that $\mathcal{P}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$ is a function of q. As, q changes, the power profile vector $\mathbf{v}$ will also change, though the weight vector $\mathbf{w}$ will not change. So, with change in q, the correlation between $\mathbf{w}$ and $\mathbf{v}$ changes. By varying q, it is possible to study the change in the correlation between $\mathbf{w}$ and $\mathbf{v}$.

Theorem 1 *Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game such that $0 < \#W(G) < 2^n$. Then, for any $q \in (0, 1)$ the following holds:*

1. $\mathsf{BZN}\text{-}\mathsf{SI}_{\mathbf{w}}(q) = \mathsf{BZ}\text{-}\mathsf{SI}_{\mathbf{w}}(q) = \mathsf{CP}\text{-}\mathsf{SI}_{\mathbf{w}}(q) = \mathsf{CI}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$.
2. $\mathsf{PGI}\text{-}\mathsf{SI}_{\mathbf{w}}(q) = \mathsf{PGM}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$.

Proof Let $\mathbf{v} = (m_G(A_1), \ldots, m_G(A_n))$ be the vector of swings for the players $A_1, \ldots, A_n$ in the game G. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ are the power profiles for G corresponding to BZN, BZ, CP, and CI, respectively. Then

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 = \alpha_3 \mathbf{v}_3 = \alpha_4 \mathbf{v}_4,$$

where

$$\alpha_1 = 2^{n-1}, \ \ \alpha_2 = \sum_{j \in N} m_G(A_j), \ \ \alpha_3 = \#W(G) \text{ and } \alpha_4 = \#L(G).$$

In G, the values $2^n, \sum_{j \in N} m_G(A_j), \#W(G)$, and $\#L(G)$ are fixed. So, $\alpha_1, \alpha_2, \alpha_3$, and α_4 are constants. Further, since $0 < \#W(G) < 2^n$, it follows that $0 < \#L(G) < 2^n$ and so $\sum_{j \in N} m_G(A_j) > 0$. This in particular means that $\alpha_2, \alpha_3, \alpha_4 > 0$ and clearly $\alpha_1 > 0$. So, using (2), we have

$$\text{BZN-SI}_\mathbf{w}(q) = \text{PCC}(\mathbf{w}, \mathbf{v}_1) = \text{PCC}(\mathbf{w}, \mathbf{v}/\alpha_1) = \text{PCC}(\mathbf{w}, \mathbf{v}).$$

In a similar manner, it follows that $\text{BZ-SI}_\mathbf{w}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$, $\text{CP-SI}_\mathbf{w}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$, and $\text{CI-SI}_\mathbf{w}(q) = \text{PCC}(\mathbf{w}, \mathbf{v})$.

The argument for $\text{PGI-SI}_\mathbf{w}(q) = \text{PGM-SI}_\mathbf{w}(q)$ is similar. □

From the viewpoint of weight–power similarity, Theorem 1 shows that it is sufficient to consider only $\text{BZ-SI}_\mathbf{w}(q)$, $\text{PGI-SI}_\mathbf{w}(q)$, and $\text{DP-SI}_\mathbf{w}(q)$.

4 Measuring Inequality of Voting Powers

Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game. The weights of all the players are not equal. In fact, in several important practical situations, the voting game is designed in a manner such that the weights are indeed unequal. The inequality in the weights can be captured by applying an appropriate inequality measure. Suppose $\mathcal{I}$ is an inequality measure. Then, $\mathcal{I}(\mathbf{w})$ is the inequality present in the weights.

Let $\mathcal{P}$ be a measure of voting power. Suppose $\mathcal{P}$ is applied to G to obtain the power profile vector $\mathbf{v}$. Then, $\mathbf{v}$ is a vector consisting of nonnegative real numbers. The inequality in the vector $\mathbf{v}$ can be measured by the inequality index $\mathcal{I}$ as $\mathcal{I}(\mathbf{v})$. The value of $\mathcal{I}(\mathbf{v})$ depends on the winning threshold q, whereas the value of $\mathcal{I}(\mathbf{w})$ does not depend on q. So, by varying q, it is possible to vary $\mathcal{I}(\mathbf{v})$ with the goal of making it as close to $\mathcal{I}(\mathbf{w})$ as possible. Then, one can say that the inequality present in the weights is more or less reflected in the inequality that arises in the voting powers.

Given a weighted majority voting game $G = (N, \mathbf{w}, q)$, we define the *weight inequality* of G with respect to an inequality measure $\mathcal{I}$ as

$$\mathcal{I}\text{-WI}(\mathbf{w}) = \mathcal{I}(\mathbf{w}). \tag{7}$$

We consider two different options for $\mathcal{I}$, namely, GI and CoV. This gives rise to two different measures of weight inequality, which are GI-WI and CoV-WI.

Given a weighted majority voting game $G = (N, \mathbf{w}, q)$, a voting power measure $\mathcal{P}$, and an inequality measure $\mathcal{I}$, the *power inequality* of $\mathcal{P}$ as determined by $\mathcal{I}$ is denoted by $(\mathcal{P}, \mathcal{I})$-$\mathsf{PI_w}(q)$ and is defined in the following manner:

$$(\mathcal{P}, \mathcal{I})\text{-}\mathsf{PI_w}(q) = \mathcal{I}(\mathbf{v}), \tag{8}$$

where $\mathbf{v}$ is the power profile vector generated by the power measure $\mathcal{P}$ applied to the weighted majority voting game $G = (N, \mathbf{w}, q)$.

We have considered seven options for $\mathcal{P}$, namely, BZN, BZ, CP, CI, PGI, PGM, and DP. The following results show that under a simple and reasonable condition on an inequality measure $\mathcal{I}$, it is sufficient to consider only three of these.

Theorem 2 *Let $\mathcal{I}$ be an inequality index satisfying scale invariance. Let $G = (N, \mathbf{w}, q)$ be a weighted majority voting game such that $0 < \#W(G) < 2^n$. Then, for any $q \in (0, 1)$, the following holds:*

1. $(\mathsf{BZ}, \mathcal{I})$-$\mathsf{PI_w}(q) = (\mathsf{BZ}, \mathcal{I})$-$\mathsf{PI_w}(q) = (\mathsf{CP}, \mathcal{I})$-$\mathsf{PI_w}(q) = (\mathsf{CI}, \mathcal{I})$-$\mathsf{PI_w}(q)$.
2. $(\mathsf{PGI}, \mathcal{I})$-$\mathsf{PI_w}(q) = (\mathsf{PGM}, \mathcal{I})$-$\mathsf{PI_w}(q)$.

Proof As in the proof of Theorem 1, let $\mathbf{v} = (m_G(A_1), \ldots, m_G(A_n))$ be the vector of swings and $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be the power profiles corresponding to BZN, BZ, CP, and CI, respectively, so that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 = \alpha_2 \mathbf{v}_2 = \alpha_3 \mathbf{v}_3 = \alpha_4 \mathbf{v}_4, \tag{9}$$

where α_1, α_2, α_3, and α_4 are the positive constants defined in the proof of Theorem 1.

Using the scale invariance of $\mathcal{I}$, we have

$$(\mathsf{BZN}, \mathcal{I})\text{-}\mathsf{PI_w}(q) = \mathcal{I}(\mathbf{v}_1) = \mathcal{I}(\mathbf{v}/\alpha_1) = \mathcal{I}(\mathbf{v}).$$

In a similar manner, it follows that $(\mathsf{BZ}, \mathcal{I})$-$\mathsf{PI_w}(q) = \mathcal{I}(\mathbf{v})$ $(\mathsf{CP}, \mathcal{I})$-$\mathsf{PI_w}(q) = \mathcal{I}(\mathbf{v})$ and $(\mathsf{CI}, \mathcal{I})$-$\mathsf{PI_w}(q) = \mathcal{I}(\mathbf{v})$.

The argument for $(\mathsf{PGI}, \mathcal{I})$-$\mathsf{PI_w}(q) = (\mathsf{PGM}, \mathcal{I})$-$\mathsf{PI_w}(q)$ is similar.   □

The Gini Index, the Coefficient of Variation, and the Generalized Entropy Index satisfy the scale invariance property. Based on Theorem 2, if $\mathcal{I}$ is any of these indices, then from the viewpoint of power inequality, it is sufficient to consider $(\mathsf{BZ}, \mathcal{I})$-$\mathsf{PI_w}(q)$, $(\mathsf{PGI}, \mathcal{I})$-$\mathsf{PI_w}(q)$, and $(\mathsf{DP}, \mathcal{I})$-$\mathsf{PI_w}(q)$. For $\mathcal{I}$, we will consider the Gini Index and the Coefficient of Variation. This means that we need to consider six possibilities.

Remark Theorem 2 has been stated for weighted majority voting games. The crux of the argument is based on (9). This relation does require G to be a weighted majority voting game. So, it is possible to rewrite the proof to show that for general voting games (which are not necessarily weighted majority voting games), the inequalities

of the different Banzhaf power profiles and the Coleman power profiles are all equal and also the inequality of the Holler public good index is equal to that of the Holler public good measure.

Comparison to Previous Works. The work Einy and Peleg (1991) on inequality in voting system is concerned with measuring inequality arising in the Shapley–Shubik power index. Since we do not consider this index in our work, we do not comment any further on the work in Einy and Peleg (1991). Instead, we simply remark that our approach can also be applied to the Shapley–Shubik power index.

Laruelle and Valenciano (2004) axiomatically derive an inequality index for a class of power indices which includes the normalized Banzhaf index. We note the following points about the work in Laruelle and Valenciano (2004):

1. The approach works only for an index, i.e., the sum of the powers must sum to one. So, for example, it cannot be applied to measure inequality arising in either of the Coleman power measures.
2. The notion of power considered in the work is based on swings. So, the power measures given by PGI, PGM, and DP are not covered by their work.
3. Among the axioms, ANON and EP are assumed and it is shown that the obtained measure satisfies Scl. On the other hand, PP and TP are not mentioned in the paper and it is not clear whether these two properties hold for the obtained measure.
4. Justification for one of the axioms (namely, Constant Sensitivity to Null Players) is not clear. In the discussion leading up to this axiom, the authors remark: "Thus, at this point any further step is questionable, though necessary to specify an index."

In view of the above, we feel that it might be preferable to study the behavior of standard inequality indices on both power measures and power indices rather than relying on an axiomatically derived inequality index where at least one of the axioms does not necessarily have a natural justification.

Weber (2016) considered the application[1] of the Coefficient of Variation to the measurement of inequality for essentially the normalized Banzhaf index. In contrast, we consider all the standard inequality indices and a much larger class of power measures/indices. Even for the Coefficient of Variation, the result that the scale invariance property implies that all the swing-based measures have the same inequality is not present in Weber (2016).

5 Variation of Similarity and Inequality with Winning Threshold

We have conducted some experiments to understand the dependence of the similarity and inequality indices of power profiles on the winning threshold.

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[1]The author remarks: "To the best of my knowledge, I am the first to propose a measure of inequality of voting systems that can be used across different constituencies." This is an oversight since the work by Laruelle and Valenciano Laruelle and Valenciano (2004) is much earlier.

In the first experiment, N was taken to be $\{A_1, \ldots, A_{30}\}$ and one hundred weight vectors were generated where the individual weights were chosen to be integers independently and uniformly in the range $[1, \ldots, 100]$. For each of the 100 weight vectors $\mathbf{w}$, the value of the winning threshold q was varied from 0.01 to 0.99 in steps of 0.01. For the game defined by the triplet $(N, \mathbf{w}, q)$, the power profiles for the different power measures were computed. From this, the similarity indices $\mathsf{BZ}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$, $\mathsf{PGI}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$, and $\mathsf{DP}\text{-}\mathsf{SI}_{\mathbf{w}}(q)$ were computed and the inequality indices $\mathcal{I}\text{-}\mathsf{WI}(\mathbf{w})$, $(\mathsf{BZN}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$, $(\mathsf{PGI}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$, and $(\mathsf{DP}, \mathcal{I})\text{-}\mathsf{PI}_{\mathbf{w}}(q)$ were computed where $\mathcal{I}$ was taken to be GI and CoV. All the obtained results show a definite pattern.

A second experiment was conducted with $n = 30$ and nonrandom weights. In particular, two distinct values of weights were used, namely, n_1 of the weights were taken to be 100 and n_2 of the weights were taken to be 1 with $n_1 + n_2 = 30$. The value of q was varied as mentioned above and the corresponding similarity and inequality indices were computed. In this case, no definite pattern was observed and there was a rich variation in the behavior.

To further explain the above experiments, we report three particular cases with $n = 30$.

Case-I: A random weight vector. The actual value of $\mathbf{w}$ (after sorting into descending order) came out to be

$$\{93, 92, 90, 86, 86, 83, 82, 77, 72, 68, 67, 67, 63, 62, 62, 59,$$
$$49, 40, 36, 35, 35, 30, 29, 27, 26, 26, 23, 21, 15, 11\}.$$

The plots of similarity, Gini Index, and the Coefficient of Variation are shown in Figs. 1, 2, and 3, respectively.

Case-II: 15 of the weights were taken to be 100 and the other 15 of the weights were taken to be 1. The plots of similarity, Gini Index and the Coefficient of Variation are shown in Figs. 4, 5 and 6 respectively.
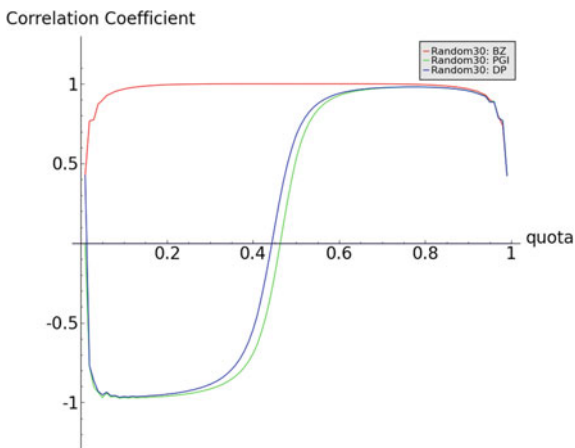

Fig. 1 Similarity indices for Case-I

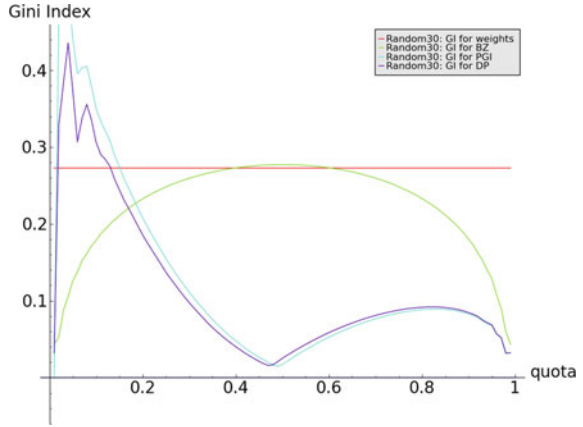**Fig. 2** Gini index for Case-I

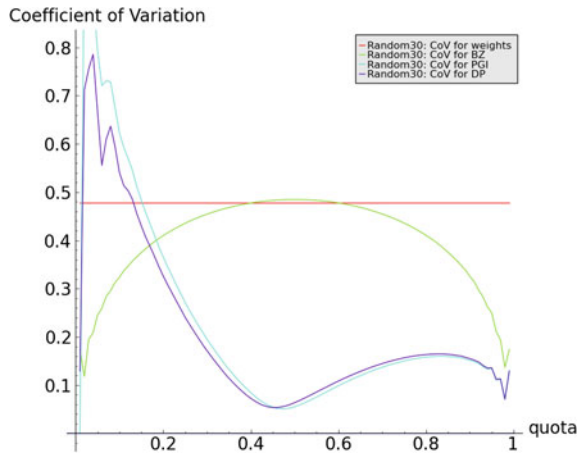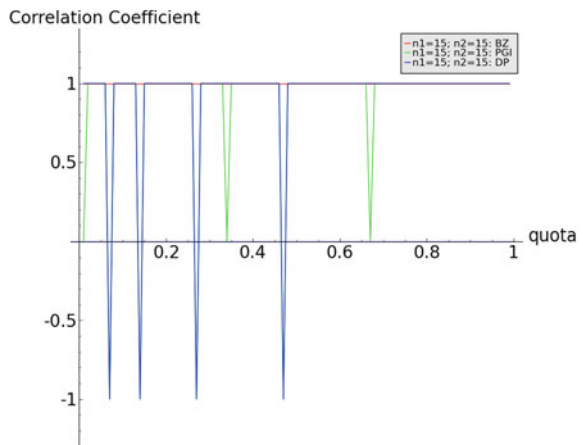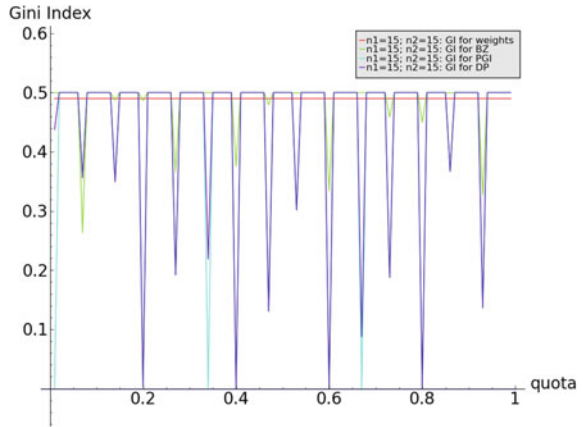**Fig. 3** Coefficient of variation for Case-I

Fig. 4 Similarity indices for Case-II

Fig. 5 Gini index for Case-II

Fig. 6 Coefficient of variation for Case-II
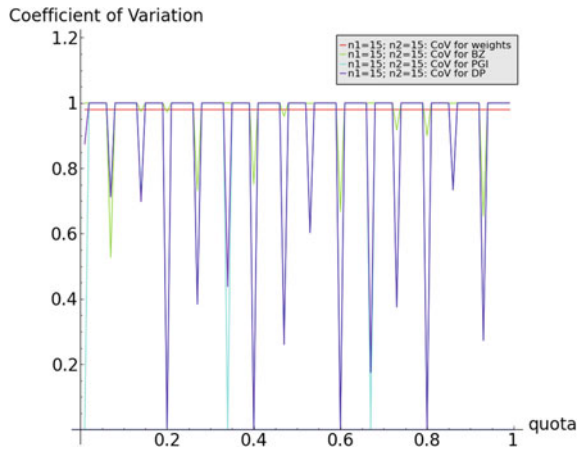
Fig. 7 Similarity indices for Case-III
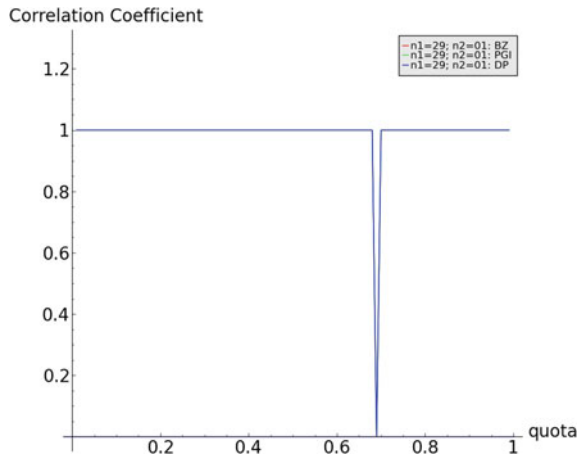
Fig. 8 Gini index for
Case-III
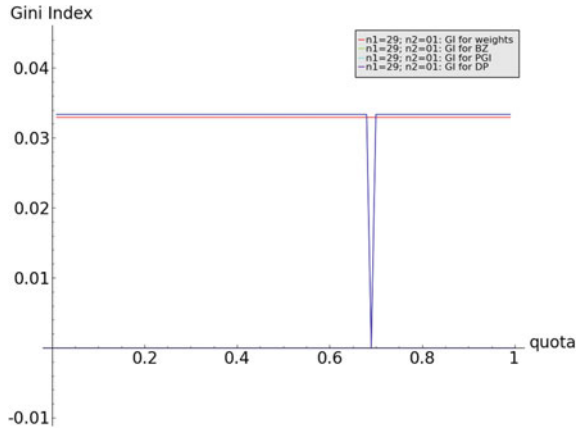
Fig. 9 Coefficient of
variation for Case-III
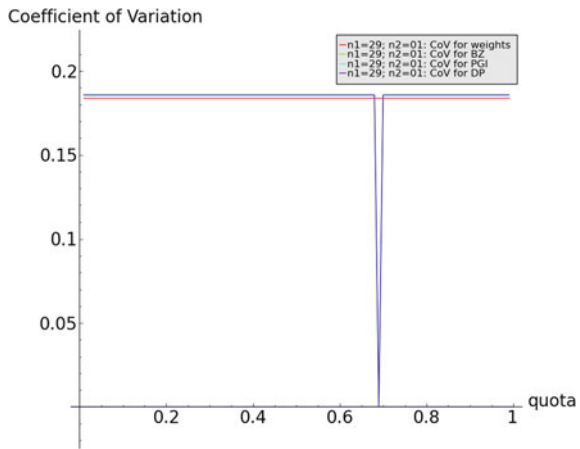
Case-III: 29 of the weights were taken to be 100 and the other weight was taken
to be 1. The plots of similarity, Gini Index, and the Coefficient of Variation are
shown in Figs. 7, 8, and 9, respectively.

Based on the plots, we make the following observations:

1. For the random case, compared to the Holler index and the Deegan–Packel mea-
sure, the Banzhaf index is a much better marker of similarity to the weight vector
and it is also much better at capturing the inequality present in the weights. For
the two nonrandom cases, there is not much difference between the three power
measures.

2. For Case-II, there are sharp spikes in the similarity and inequality plots. These
correspond to choices of q for which the low weight players achieve power similar
to the high weight players.

3. For Case-III, there is one player with low weight. Apart from a small range of q where this player gains significant power, for other values of q this player becomes a dummy. The inequality indices, however, do not reflect this. The inequality scores are generally quite low indicating that there is small inequality in the system. This is a feature of the inequality indices which are not sensitive to low scores of a small number of players.

6 Applications

6.1 IMF Voting Games

The IMF has two decision-making bodies, namely, the Board of Governors (BoG) and the Executive Board (EB).

A total of 189 member countries make up the BoG. Each country has a specified voting share. The voting share or weight of a country is calculated as the sum of a basic weight plus an amount which is proportional to the special drawing rights of the country. The EB consists of 24 Executive Directors (EDs) representing all the 189 member countries. Eight of these directors are nominated by eight member countries while each of the other directors is elected by a group of countries. Each ED has a voting weight which is the sum of the voting weights of the countries that he or she represents.

The BoG is the highest decision-making body of the IMF and is officially responsible for all major decisions. In practice, however, the BoG has delegated most of its powers to the EB.[2] Accordingly, in this work, we will consider only the voting game arising from the EB weights.

Actual weights of the members of the EB are available from the IMF website.[3] These weights range from the minimum of 80157 to the maximum of 831407. Since these values are rather large, for the purposes of computation of voting powers, we have divided these voting weights by 1000 and then rounded to the nearest integer. While this is an approximation, it does not significantly affect the voting powers. In particular, we have checked that no two members with originally unequal weights get the same weight after this rounding off process. The actual weight vector that has been used to compute the voting powers is the following:

$$\mathbf{w}_{\mathsf{imf}} = \{831, 310, 306, 273, 268, 267, 219, 208, 203, 203, 196, 170,$$
$$165, 162, 155, 154, 150, 149, 138, 131, 111, 101, 82, 80\}.$$

The rules specify several winning thresholds.[4] We mention these below:

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[2]https://en.wikipedia.org/wiki/International_Monetary_Fund.

[3]https://www.imf.org/external/np/sec/memdir/eds.aspx.

[4]https://www.imf.org/external/pubs/ft/aa/index.htm.

Except as otherwise specifically provided, all decisions of the Fund shall be made by a majority of the votes cast.

The Fund, by a seventy percent majority of the total voting power, may decide at any time to distribute any part of the general reserve.

The Fund may use a member's currency held in the Investment Account for investment as it may determine, in accordance with rules and regulations adopted by the Fund by a seventy percent majority of the total voting power.

An eighty-five percent majority of the total voting power shall be required for any change in quotas.

So, three possible values of q are used: $q = 0.5$, $q = 0.7$, and $q = 0.85$.

We have computed the similarity and inequality indices for the IMF EB voting game with q varying from 0.01 to 0.99 in steps of 0.01. The plots of $\text{BZ-SI}_{\mathbf{w}_{imf}}(q)$, $\text{PGI-SI}_{\mathbf{w}_{imf}}(q)$, and $\text{DP-SI}_{\mathbf{w}_{imf}}(q)$ are shown in Fig. 10; the plots of $(\text{BZ, GI})\text{-PI}_{\mathbf{w}_{imf}}(q)$, $(\text{PGI, GI})\text{-PI}_{\mathbf{w}_{imf}}(q)$, and $(\text{DP, GI})\text{-PI}_{\mathbf{w}_{imf}}(q)$ are shown in Fig. 11; and the plots of

Fig. 10 Similarity indices for IMF-EB weights
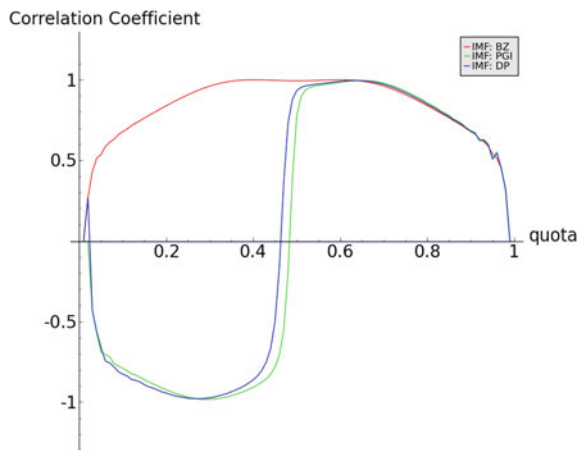
Fig. 11 Gini index for IMF-EB weights
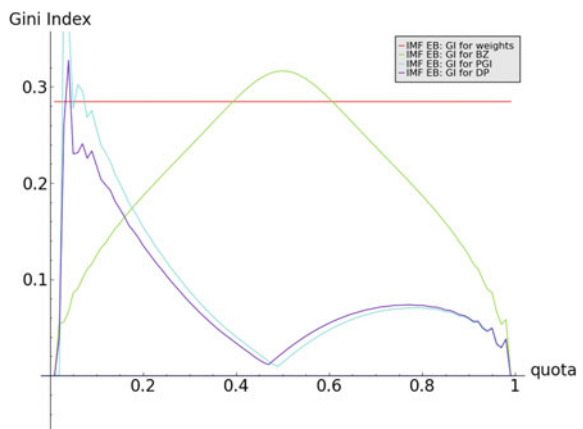
$(\text{BZ, CoV})\text{-PI}_{\mathbf{w}_{imf}}(q)$, $(\text{PGI, CoV})\text{-PI}_{\mathbf{w}_{imf}}(q)$, and $(\text{DP, CoV})\text{-PI}_{\mathbf{w}_{imf}}(q)$ are shown in Fig. 12. The actual values of these indices for the range $[0.5, 0.65]$ along with the values for $q = 0.70$ and $q = 0.85$ are shown in Tables 1 and 2. Based on these data, we have the following observations:

Fig. 12 Coefficient of variation for IMF-EB weights
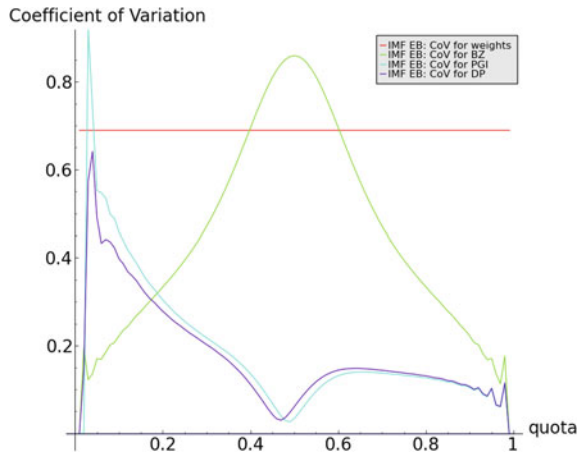
Table 1 IMF-EB similarity indices

q	BZ-SI$_{\mathbf{w}_{imf}}(q)$	PGI-SI$_{\mathbf{w}_{imf}}(q)$	DP-SI$_{\mathbf{w}_{imf}}(q)$
0.50	0.9938690	0.7743330	0.9330398
0.51	0.9939952	0.8965981	0.9531445
0.52	0.9943706	0.9363455	0.9618091
0.53	0.9949599	0.9522328	0.9669979
0.54	0.9957522	0.9600696	0.9708993
0.55	0.9966513	0.9649342	0.9740920
0.56	0.9975953	0.9688469	0.9771681
0.57	0.9985210	0.9730949	0.9806975
0.58	0.9992881	0.9769905	0.9839294
0.59	0.9998168	0.9806887	0.9869004
0.60	**0.9999911**	0.9846956	0.9899614
0.61	0.9996918	0.9882609	0.9924378
0.62	0.9988216	0.9912144	0.9942195
0.63	0.9972257	0.9936152	0.9953372
0.64	0.9948817	0.9951439	0.9954396
0.65	0.9916875	0.9956169	0.9945365
0.70	0.9609858	0.9788570	0.9715726
0.85	0.7634234	0.7787653	0.7721551

Table 2 Inequality as measured by GI and CoV in the IMF Game. In the table, $\mathfrak{f}_1(q) = $ (BZ, GI)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$, $\mathfrak{f}_2(q) = $ (PGI, GI)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$, $\mathfrak{f}_3(q) = $ (DP, GI)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$, $\mathfrak{g}_1(q) = $ (BZ, CoV)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$, $\mathfrak{g}_2(q) = $ (PGI, CoV)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$, $\mathfrak{g}_3(q) = $ (DP, CoV)-$\mathrm{PI}_{\mathbf{w}_{imf}}(q)$. Note GI-$\mathrm{WI}(\mathbf{w}_{imf}) = \mathbf{0.285042}$ and CoV-$\mathrm{WI}(\mathbf{w}_{imf}) = \mathbf{0.689527}$

q	$\mathfrak{f}_1(q)$	$\mathfrak{f}_2(q)$	$\mathfrak{f}_3(q)$	$\mathfrak{g}_1(q)$	$\mathfrak{g}_2(q)$	$\mathfrak{g}_3(q)$
0.50	0.316535	0.013684	0.023564	0.859515	0.033982	0.061049
0.51	0.316212	0.017718	0.027366	0.857502	0.046205	0.074206
0.52	0.315214	0.021654	0.031084	0.851387	0.059712	0.086957
0.53	0.313531	0.025291	0.034457	0.841354	0.072727	0.098447
0.54	0.311215	0.028989	0.037852	0.827460	0.085147	0.108920
0.55	0.308401	0.032545	0.041122	0.810689	0.096331	0.118073
0.56	0.305067	0.035747	0.044020	0.791192	0.105924	0.125614
0.57	0.301236	0.038982	0.046930	0.769085	0.114385	0.132059
0.58	0.297133	0.042029	0.049670	0.745825	0.121321	0.137161
0.59	0.292743	0.044794	0.052123	0.721395	0.126885	0.141047
0.60	0.288019	0.047504	0.054517	**0.695812**	0.131366	0.144021
0.61	**0.283235**	0.050075	0.056795	0.670527	0.134712	0.131366
0.62	0.278285	0.052419	0.058857	0.645342	0.137090	0.147386
0.63	0.273201	0.054555	0.060691	0.620092	0.138635	0.147978
0.64	0.268155	0.056714	0.062592	0.596040	0.139694	0.148302
0.65	0.263088	0.058572	0.064200	0.572826	0.140068	0.148049
0.66	0.257927	0.060254	0.065613	0.550134	0.139974	0.147422
0.67	0.252875	0.061895	0.067025	0.528952	0.139751	0.146796
0.68	0.247854	0.063419	0.068331	0.508843	0.139248	0.145961
0.69	0.242726	0.064596	0.069269	0.489429	0.138348	0.144745
0.70	0.237753	0.065816	0.070276	0.471442	0.137616	0.143776
0.85	0.158497	0.066284	0.068153	0.278829	0.116985	0.120198

1. The Holler and the Deegan–Packel indices are not good indicators of either the similarity to or the inequality present in the weights. So, we focus only on the Banzhaf index.
2. The following holds for the Banzhaf index:

   – The plots of the two inequality indices have bell curve shapes. To a lesser extent, the same is also true of the similarity index.
   – The maximum similarity is achieved for $q = 0.60$.
   – For the Gini Index, the inequality in the power profile is closest to the inequality in the weights for $q = 0.61$.
   – For the Coefficient of Variation, the inequality in the power profile is closest to the inequality in the weights for $q = 0.60$.

From Tables 1 and 2, we note that in comparison to $q = 0.60$, the choices $q = 0.50$, $q = 0.70$, and $q = 0.85$ are suboptimal.

Table 3 Voting powers of the players under various power measures for the IMF-EB game with $q = 0.60$. Instead of the players, the voting powers are shown against the weights of the players

Wts	BZN	BZ	CP	CI	PGI	PGM	DP
831	0.4033509	0.1661335	0.8949349	0.2603447	0.0662802	0.8888586	0.0682873
310	0.1508728	0.0621420	0.3347491	0.0973816	0.0435712	0.5843174	0.0442821
306	0.1488262	0.0612990	0.3302082	0.0960606	0.0434846	0.5831563	0.0441693
273	0.1322311	0.0544638	0.2933878	0.0853492	0.0427837	0.5737562	0.0432420
268	0.1297327	0.0534347	0.2878445	0.0837366	0.0426849	0.5724312	0.0431133
267	0.1292485	0.0532353	0.2867701	0.0834240	0.0426597	0.5720929	0.0430820
219	0.1055294	0.0434658	0.2341434	0.0681144	0.0416246	0.5582125	0.0417465
208	0.1001409	0.0412464	0.2221877	0.0646364	0.0414027	0.5552361	0.0414634
203	0.0976979	0.0402401	0.2167671	0.0630595	0.0413069	0.5539514	0.0413394
203	0.0976979	0.0402401	0.2167671	0.0630595	0.0413069	0.5539514	0.0413394
196	0.0942787	0.0388318	0.2091809	0.0608526	0.0411606	0.5519899	0.0411522
170	0.0816418	0.0336269	0.1811427	0.0526961	0.0406361	0.5449551	0.0404690
165	0.0792183	0.0326287	0.1757655	0.0511318	0.0405488	0.5437851	0.0403568
162	0.0777606	0.0320283	0.1725313	0.0501909	0.0404977	0.5430995	0.0402881
155	0.0743695	0.0306316	0.1650074	0.0480022	0.0403450	0.5410517	0.0400980
154	0.0738934	0.0304355	0.1639510	0.0476948	0.0403227	0.5407521	0.0400702
150	0.0719494	0.0296347	0.1596377	0.0464400	0.0402490	0.5397639	0.0399749
149	0.0714759	0.0294397	0.1585871	0.0461344	0.0402247	0.5394390	0.0399449
138	0.0661672	0.0272532	0.1468086	0.0427079	0.0399681	0.5359976	0.0396272
131	0.0627846	0.0258599	0.1393032	0.0405246	0.0397938	0.5336591	0.0394117
111	0.0531555	0.0218939	0.1179389	0.0343095	0.0389966	0.5229683	0.0384651
101	0.0483502	0.0199146	0.1072770	0.0312079	0.0383484	0.5142761	0.0377462
82	0.0392314	0.0161588	0.0870447	0.0253221	0.0361261	0.4844734	0.0353936
80	0.0382680	0.0157619	0.0849071	0.0247002	0.0356770	0.4784506	0.0349375

Based on the above analysis, we put forward the suggestion that the winning threshold of $q = 0.60$ be seriously considered for any future possible change in voting rule. For $q = 0.60$, the actual values of the different power measures are shown in Table 3.

6.2 EU Voting Games

Until Brexit is effective, the European Union Council has 28 members. It votes on different types of matters in three different ways.[5] The first is the unanimity voting where all members have to vote in favor or against for the motion to be passed or

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Table 4 Population percentages of the countries in the European Union

| Country	Pop (%)	w_{eu}
Germany	16.06	1606
France	13.05	1305
United Kingdom	12.79	1279
Italy	12.00	1200
Spain	9.09	909
Poland	7.43	743
Romania	3.87	387
Netherlands	3.37	337
Belgium	2.21	221
Greece	2.11	211
Czech Republic	2.04	204
Portugal	2.02	202
Sweden	1.96	196
Hungary	1.92	192
Austria	1.71	172
Bulgaria	1.40	140
Denmark	1.12	112
Finland	1.07	107
Slovakia	1.06	106
Ireland	0.91	91
Croatia	0.82	82
Lithuania	0.57	57
Slovenia	0.40	40
Latvia	0.39	39
Estonia	0.26	26
Cyprus	0.17	17
Luxembourg	0.11	11
Malta	0.09	9

dismissed. In nonlegislative issues, a simple majority voting is done where at least 15 out of the 28 members have to vote in favor. For most (80%) of the issues that are voted upon in the EU Council, the "qualified majority" method is used. This is stated as follows[6]:

> A qualified majority needs 55% of member states, representing at least 65% of the EU population.

It is the qualified majority voting rule that we consider in the context of weighted majority voting games. The population percentages of the individual countries are

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[6]http://www.consilium.europa.eu/en/council-eu/voting-system/qualified-majority/.

Fig. 13 Similarity indices
for EU weights
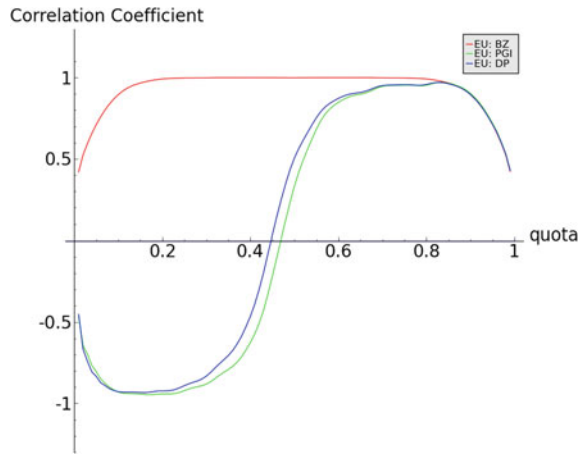
**Fig. 14** Gini index for EU
weights
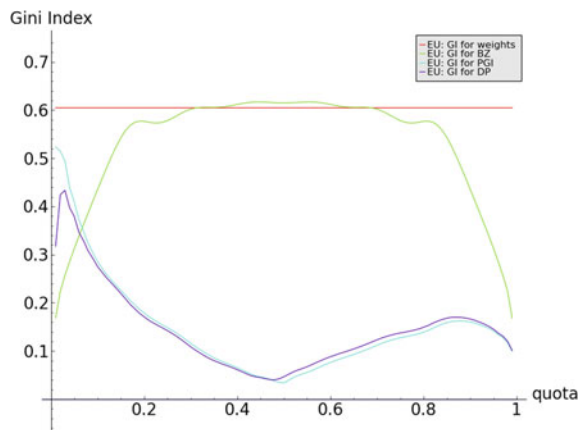
available[7] and are reproduced in Table 4. These percentages are the weights of the
individual countries. For computation of the voting powers, the percentage values are
multiplied by 100 to convert these into integers which are then used as the weights.
This scaling does not affect the decision-making process.

In the qualified majority voting, the passing rule is a joint condition, one on the
number of member states which are involved and the other on the population percent-
age. While analyzing the joint condition would be more accurate, for the purpose of
this work, we have worked with the simpler setting where only the winning condition
on the population percentage is considered. This leads to a weighted majority voting
game where the weight vector $\mathbf{w}_{eu}$ is specified in Table 4 and the winning threshold
is $q = 0.65$.

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[7]http://www.consilium.europa.eu/en/council-eu/voting-system/voting-calculator/.

Fig. 15 Coefficient of variation for EU weights
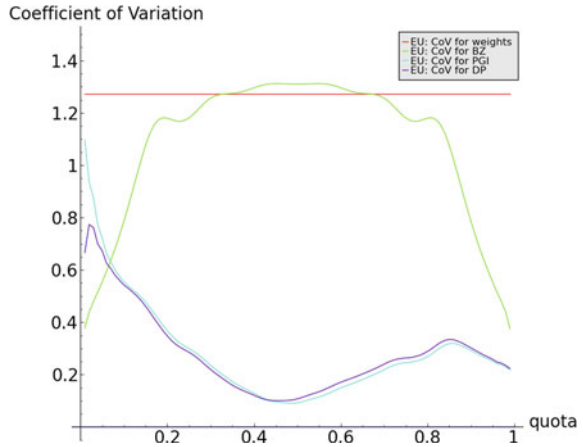
Table 5 EU game similarity indices

q	BZ-SI$_{\mathbf{w}_{eu}}(q)$	PGI-SI$_{\mathbf{w}_{eu}}(q)$	DP-SI$_{\mathbf{w}_{eu}}(q)$
0.50	0.9988826	0.3320853	0.5051382
0.51	0.9989581	0.4181647	0.5640333
0.52	0.9991501	0.4929785	0.6151950
0.53	0.9993725	0.5596608	0.6631804
0.54	0.9995411	0.6230357	0.7099130
0.55	0.9996231	0.6830571	0.7540455
0.56	0.9996430	0.7350513	0.7910700
0.57	0.9996501	0.7765455	0.8199512
0.58	0.9996763	0.8063575	0.8409968
0.59	0.9997123	0.8289370	0.8576475
0.60	0.9997249	0.8477416	0.8722319
0.61	0.9996959	0.8642583	0.8846472
0.62	0.9996486	0.8777664	0.8944232
0.63	0.9996354	0.8870017	0.9006101
0.64	0.9996914	0.8932824	0.9048869
0.65	0.9997886	0.8976327	0.9086210
0.66	**0.9998332**	0.9039397	0.9150038
0.67	0.9997214	0.9135211	0.9242301
0.68	0.9994175	0.9258465	0.9351369
0.69	0.9989968	0.9381400	0.9448867
0.70	0.9986065	0.9465521	0.9507084

Table 6 Inequality as measured by GI and CoV in the EU Game. In the table $\mathfrak{h}_1(q) = $ (BZ, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{h}_2(q) = $ (PGI, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{h}_3(q) = $ (DP, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_1(q) = $ (BZ, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_2(q) = $ (PGI, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, $\mathfrak{k}_3(q) = $ (DP, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$. Note GI-WI$(\mathbf{w}_{eu}) = \mathbf{0.605671}$ and CoV-WI$(\mathbf{w}_{eu}) = \mathbf{1.272825}$

q	$\mathfrak{h}_1(q)$	$\mathfrak{h}_2(q)$	$\mathfrak{h}_3(q)$	$\mathfrak{k}_1(q)$	$\mathfrak{k}_2(q)$	$\mathfrak{k}_3(q)$
0.50	0.614849	0.033960	0.046614	1.309745	0.091907	0.107286
0.51	0.614993	0.039425	0.051749	1.309901	0.095374	0.112346
0.52	0.615391	0.044681	0.056573	1.310338	0.100199	0.118139
0.53	0.615945	0.049083	0.060556	1.310962	0.105176	0.123691
0.54	0.616483	0.052608	0.063782	1.311545	0.109910	0.128951
0.55	0.616834	0.055576	0.067481	1.311731	0.114808	0.134668
0.56	0.616822	0.059365	0.071789	1.311070	0.120243	0.140936
0.57	0.616302	0.063896	0.076268	1.309147	0.126922	0.148409
0.58	0.615243	0.068081	0.080370	1.305796	0.134171	0.156036
0.59	0.613694	0.072437	0.084545	1.301119	0.142004	0.163861
0.60	0.611813	0.076339	0.088256	1.295536	0.149118	0.170797
0.61	0.609851	0.079853	0.091521	1.289678	0.155445	0.176837
0.62	0.608078	0.083431	0.094969	1.284198	0.161705	0.183060
0.63	0.606708	0.086467	0.097851	1.279626	0.167582	0.188834
0.64	0.605893	0.089760	0.101116	1.276296	0.174214	0.195451
0.65	0.605616	0.093045	0.104334	1.274202	0.181046	0.202085
0.66	**0.605689**	0.096528	0.107787	**1.272960**	0.188107	0.209023
0.67	0.605797	0.100171	0.111427	1.271778	0.195381	0.216243
0.68	0.605497	0.103949	0.115322	1.269457	0.202594	0.223577
0.69	0.604333	0.107898	0.119249	1.264638	0.210268	0.231500
0.70	0.601994	0.111935	0.123198	1.256239	0.218490	0.239777

For the weight vector $\mathbf{w}_{eu}$ given in Table 4, we have computed the similarity and inequality indices with q varying from 0.01 to 0.99 in steps of 0.01. The plots of BZ-$\mathsf{SI}_{\mathbf{w}_{eu}}(q)$, PGI-$\mathsf{SI}_{\mathbf{w}_{eu}}(q)$, and DP-$\mathsf{SI}_{\mathbf{w}_{eu}}(q)$ are shown in Fig. 13; the plots of (BZ, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, (PGI, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, and (DP, GI)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$ are shown in Fig. 14; and the plots of (BZ, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, (PGI, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$, and (DP, CoV)-$\mathsf{PI}_{\mathbf{w}_{eu}}(q)$ are shown in Fig. 15. The actual values of these indices for the range $[0.5, 0.7]$ are shown in Tables 5 and 6.

Based on these data, we have the following observations:

1. As in the case of the IMF-EB voting game, the Holler and the Deegan–Packel indices are not good indicators of either the similarity to or the inequality present in the weights. So, again we focus only on the Banzhaf index.
2. The following holds for the Banzhaf index:

   – The plots of the two inequality indices as well as the similarity index have a somewhat bell-shaped nature.

Table 7 Voting powers of the players under various power measures for the EU game with $q = 0.66$. Instead of the players, the voting powers are shown against the weights of the players

Wts	BZN	BZ	CP	CI	PGI	PGM	DP
1606	0.2459032	0.1582166	0.7950917	0.1454426	0.0537148	0.8173260	0.0554496
1305	0.2044021	0.1315144	0.6609039	0.1208962	0.0479912	0.7302363	0.0496147
1279	0.2003657	0.1289173	0.6478528	0.1185089	0.0475521	0.7235548	0.0491300
1200	0.1881286	0.1210439	0.6082861	0.1112711	0.0463214	0.7048273	0.0477734
909	0.1444251	0.0929245	0.4669771	0.0854221	0.0410962	0.6253212	0.0423199
743	0.1115583	0.0717777	0.3607073	0.0659826	0.0368236	0.5603092	0.0372597
387	0.0600651	0.0386465	0.1942116	0.0355263	0.0379704	0.5777581	0.0380942
337	0.0523199	0.0336631	0.1691686	0.0309453	0.0371804	0.5657380	0.0372361
221	0.0343284	0.0220872	0.1109957	0.0203039	0.0355657	0.5411688	0.0354550
211	0.0327751	0.0210878	0.1059733	0.0193852	0.0354097	0.5387953	0.0352811
204	0.0316896	0.0203894	0.1024638	0.0187432	0.0353493	0.5378758	0.0352135
202	0.0313774	0.0201886	0.1014543	0.0185586	0.0353118	0.5373050	0.0351703
196	0.0304473	0.0195901	0.0984469	0.0180085	0.0352300	0.5360610	0.0350762
192	0.0298266	0.0191907	0.0964398	0.0176413	0.0352006	0.5356133	0.0350399
171	0.0265634	0.0170911	0.0858887	0.0157112	0.0349022	0.5310729	0.0347002
140	0.0217525	0.0139958	0.0703336	0.0128658	0.0345472	0.5256718	0.0343204
112	0.0173997	0.0111951	0.0562592	0.0102913	0.0341762	0.5200262	0.0338917
107	0.0166246	0.0106964	0.0537533	0.0098328	0.0341039	0.5189262	0.0338115
106	0.0164687	0.0105961	0.0532489	0.0097406	0.0341017	0.5188932	0.0338069
91	0.0141386	0.0090969	0.0457151	0.0083625	0.0338729	0.5154111	0.0335409
82	0.0127383	0.0081959	0.0411873	0.0075342	0.0336632	0.5122204	0.0332955
57	0.0088505	0.0056945	0.0286168	0.0052347	0.0329459	0.5013056	0.0324475
40	0.0062171	0.0040002	0.0201021	0.0036772	0.0321913	0.4898234	0.0315918
39	0.0060615	0.0039000	0.0195989	0.0035851	0.0321094	0.4885777	0.0314986
26	0.0040399	0.0025993	0.0130624	0.0023894	0.0300802	0.4577017	0.0292896
17	0.0026410	0.0016992	0.0085393	0.0015621	0.0273734	0.4165147	0.0264409
11	0.0017116	0.0011013	0.0055342	0.0010123	0.0238031	0.3621886	0.0228048
9	0.0013991	0.0009002	0.0045238	0.0008275	0.0214120	0.3258058	0.0204462

– The maximum similarity is achieved for $q = 0.66$.
– For the Gini Index, the inequality in the power profile is closest to the inequality in the weights for $q = 0.66$.
– For the Coefficient of Variation, the inequality in the power profile is closest to the inequality in the weights for $q = 0.66$.

So, the value of $q = 0.66$ is the best in terms of similarity and also for inequality measured by either the Gini Index or the Coefficient of Variation.

The value of q actually used in the EU voting games is $q = 0.65$. The corresponding similarity value and the values for Gini Index and Coefficient of Variation are shown in Tables 1 and 2, respectively. These values show that in comparison to $q = 0.66$, the choice $q = 0.65$ is suboptimal but, very close. For $q = 0.66$, the actual values of the different power measures are shown in Table 7.

Unlike the case of the IMF voting game, our analysis shows that for the weighted majority game arising in the context of EU, the winning threshold is very close to the optimal value. So, our analysis provides some quantitative backing to the actual winning threshold used in the EU game.

7   Conclusion

In this paper, we have addressed the problem of quantifying whether a power profile adequately captures the natural variation in the weights of a weighted majority voting game. Ideas based on Pearson's correlation coefficient and standard inequality measures such as the Gini Index and the Coefficient of Variation have been used in the formalization. These ideas have been applied to the voting games arising in the context of the IMF and the EU. We provide concrete quantitative evidence that the actual winning threshold used in the IMF games is suboptimal and instead propose a new value of the winning threshold which has firm theoretical justification. In the case of the EU game, the actual value of the winning threshold used is close to the optimal value. So, in this case, our analysis provides some quantitative backing to the value of the winning threshold that is actually used.

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