Free Vibration Analysis of Cracked Orthotropic Rectangular Plates Under Thermal Effect

S. K. Lai and L. H. Zhang

Abstract This paper presents the free vibration analysis of orthotropic rectangular plates with line surface cracks under thermal conditions. The classical plate theory is employed to derive the governing equation of cracked plates, in which the surface cracks located at the plate center are formulated based on a line-spring model. It is assumed that a thermal load is uniformly distributed on the plates throughout its volume, and thus the moments of the plates resulted by the thermal effect are neglected. The deduced governing equation is solved by the discrete singular convolution (DSC) method with the Shannon's delta kernel. The DSC technique is a relatively new method for vibration analysis of plates. It not only possesses flexibility in handling complex geometries and boundary conditions, and also holds a high-level of accuracy. In this study, the treatment for orthotropic cracked plates with various combinations of boundary conditions, namely, simply supported, clamped and free edges is studied. The results are compared with the existing solutions to verify the correctness and reliability. Some first-known results are also presented.

Keywords Orthotropic plates · Surface cracks · Thermal environment · DSC method

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1 Introduction

Composite materials have long been widely used in aerospace, mechanical and civil engineering due to its features of high strength-to-weight and stiffness-to-weight ratios [2]. Such materials can be generally modeled as orthotropic plates by utilizing anisotropic materials and altering the isotropic properties along perpendicular directions in manufacturing processes [21]. In thermal environment, the presence of cracks can accelerate the change of material properties, inducing the loss of stability and reliability. Hence, the understanding of dynamic response of orthotropic plates subject to both cracking and thermal effects is crucial for engineers and researchers.

In the past, various plate theories have been well established for structural analysis [9, 10, 13]. To study the influence of cracks on the dynamic responses of plates, Rice and Levy [14] proposed a ling-spring model (LSM) based on the classical Kirchhoff plate theory, where the part-through crack located at the center of rectangular plates can be represented by a line-spring. Recently, Israr et al. [3] and Joshi et al. [5] extended their works and developed analytical models for vibration analysis of cracked isotropic and orthotropic plates, respectively. However, only a few number of articles have been published on the dynamic analysis of cracked plates subject to thermal conditions. Natarajan et al. [11] analyzed the cracked functionally graded plates under various parameters, such as crack length and temperature variation. In addition, Joshi et al. [6] proposed an analytical model to study the vibration characteristics of heated and cracked thin orthotropic plates.

The prime objective of this work is to present accurate solutions for the prediction of structural responses of cracked plates in thermal environment by using the discrete singular convolution (DSC) method. The DSC method emerges as an efficient numerical method that was firstly proposed by Wei [17]. It is regarded as a local method with good flexibility for dealing complex geometries and boundary conditions, but also it holds a high level of accuracy [12, 19, 20]. To go beyond the restriction of the original DSC technique, the incorporation of the Taylor series expansion method was proposed for the treatment of structural elements with free edges [15, 16]. Although the DSC method has been further explored for solving a variety of plate problems [1, 7], it is still a lack of applications on the analysis of cracked plates. This study firstly attempts to apply this method to fill this knowledge gap. The obtained solutions herein are compared with those from the open literature to validate the accuracy and reliability. Some accurate benchmark solutions are also presented. In addition, this paper aims to share and introduce this work to other participants of the 25th Australasian Conference on Mechanics of Structures and Materials with common research interests. A comprehensive investigation for this research, including thermal buckling analysis, vibration mode shapes and special restrained manner of simply-supported conditions, can be referred to the authors' recent work [8].

2 Theoretical Formulation

2.1 Governing Equation

The governing equation of rectangular orthotropic plates subject to both crack and thermal effects has been rigorously treated by Joshi et al. [6]. Figure 1 shows the geometry and coordinate system of an orthotropic plate, wherein two linear surface cracks with lengths $2a$ and $2b$ are parallel to the x-axis and y-axis, respectively. The plate has a uniform thickness h that is sufficiently thin when comparing to its in-plane dimensions (i.e., length L_1 and width L_2).

According to the classical plate theory, the governing equation of orthotropic rectangular plates with a surface crack (along the x-axis) in thermal environment is expressed as [6]

$$\begin{aligned} & D_x \frac{\partial^4 w}{\partial x^4} + 2B_o \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \\ & \quad = -\rho h \frac{\partial^2 w}{\partial t^2} - N_x^T \frac{\partial^2 w}{\partial x^2} - N_y^T \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 M_x^T}{\partial x^2} - \frac{\partial^2 M_y^T}{\partial y^2} + \frac{\partial^2 \bar{M}_y}{\partial y^2} \\ & \qquad + \bar{N}_y \frac{\partial^2 w}{\partial y^2} + P_z \end{aligned} \tag{1}$$

where $B_o = D_x v_y + G_{xy} h^3/6$, $D_x = E_x h^3/[12(1 - v_x v_y)]$ and $D_y = E_y h^3/[12(1 - v_x v_y)]$ are the flexural rigidities with Young's modulus (E_x, E_y), Poisson's ratio (v_x, v_y) and shear modulus (G_{xy}); N_x^T and N_y^T are the in-plane forces per unit length due to the thermal effect; M_x^T and M_y^T are the moments induced by heating loads; $\bar{N}_y$ and $\bar{M}_y$ represent the in-plane force and moment resulted by the presence of cracks, respectively; and P_z denotes the transverse load per unit area acting on the plate surface.

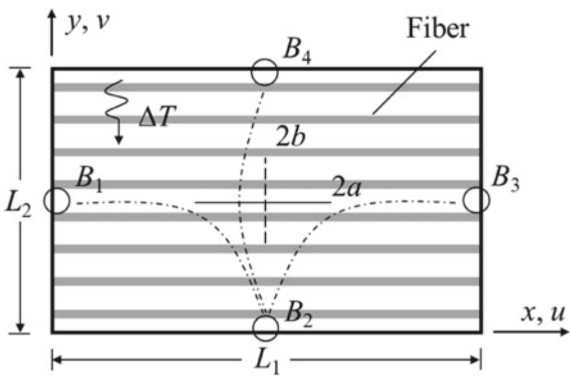

Fig. 1 Geometry and coordinate system of an orthotropic plate with surface part-through cracks subject to thermal loads

2.2 Formulation of Crack Terms

Using the LSM, we transform the cracked plate into a two-dimensional problem. The uniformly distributed tensile stress and the bending stress at the far edges of the plate are written as [14]

$$\sigma_{rs} = \frac{N_{rs}}{h} = \frac{1}{h}\int_{-h/2}^{h/2} \tau_{rs}(x, y, z)dz \text{ and}$$
$$m_{rs} = \frac{6}{h}M_{rs} = \frac{6}{h^2}\int_{-h/2}^{h/2} z\tau_{rs}(x, y, z)dz \tag{2}$$

where $\tau_{rs}(x, y, z)$ is the stress state, r and s are intermediate variables, N_{rs} and M_{rs} are, respectively, the force and moment per unit length in the direction perpendicular to the crack length at the edges of the plate. The crack is represented as a continuous line-spring with compliance, and these compliance coefficients are used to raise the relationship between the tensile stress and the bending stress at the far sides of the plate and the crack location as follows

$$\bar{\sigma}_{rs} = \left[\frac{2a}{(6\alpha_{tb}^o + \alpha_{tt}^o)(1 - v^2)h + 2a}\right]\sigma_{rs} \text{ and}$$
$$\bar{m}_{rs} = \left[\frac{2a}{3(\alpha_{bt}^o/6 + \alpha_{bb}^o)(3 + v)(1 - v)h + 2a}\right]m_{rs} \tag{3}$$

where $\alpha_{bb}^o, \alpha_{tt}^o, \alpha_{bt}^o (= \alpha_{bt}^o)$ denote the non-dimensional bending compliance, stretching compliance and stretching-bending compliance, respectively. Their values depend on the ratio $\zeta = d/h$ (where d is the crack depth and h is the plate thickness) in the range of 0.1–0.7 [14]. The tensile force and the moment caused by a crack along the x-axis can be expressed as

$$\bar{N}_y \equiv -\bar{N}_{rs} = -\left[\frac{2a}{(6\alpha_{tb}^o + \alpha_{tt}^o)(1 - v_x^2)h + 2a}\right]N_{rs} \tag{4}$$

$$\bar{M}_y \equiv -\bar{M}_{rs} = -\left[\frac{2a}{3(\alpha_{bt}^o/6 + \alpha_{bb}^o)(3 + v_x)(1 - v_x)h + 2a}\right]M_{rs} \tag{5}$$

where the negative signs are due to the reduction of the overall stiffness from damages. The bending stress at the far sides of the plate is given by

$$M_{rs} = -D_y\left(\frac{\partial^2 w}{\partial y^2} + v_x\frac{\partial^2 w}{\partial x^2}\right) \tag{6}$$

2.3 Thermal Effect

In this work, the uniformly distributed heating load is considered. The in-plane shear force is then vanished and only the membrane force is considered. The thermal stress parameters are defined as

$$\sigma_x^T = \frac{E_x T(z)}{1 - \nu_x \nu_y}(\alpha_x + \alpha_y \nu_y), \sigma_y^T = \frac{E_y T(z)}{1 - \nu_x \nu_y}(\alpha_y + \alpha_x \nu_x), \tau_{xy}^T = 0 \tag{7}$$

where α_x and α_y are the coefficients of thermal expansion in the x- and y-directions, respectively. The variation of temperature at the plate is assumed as $T(z) = \Delta T$. The in-plane forces and the moments caused by the thermal effect can be written as

$$N_x^T = \frac{E_x h \Delta T(\alpha_x + \alpha_y \nu_y)}{1 - \nu_x \nu_y}, N_y^T = \frac{E_y h \Delta T(\alpha_y + \alpha_x \nu_x)}{1 - \nu_x \nu_y}, M_x^T = M_y^T = 0 \tag{8}$$

Making use of the LSM, only the membrane force due to the change of temperature is considered (i.e., $N^T = N_{rs}$). For generality and simplicity, the dimensionless parameters are defined as

$$X = \frac{x}{L_1}, Y = \frac{y}{L_2}, W = \frac{w}{L_1}, \lambda = \frac{L_1}{L_2}, \Omega = \omega L_1^2 \sqrt{\frac{\rho h}{D_x}} \tag{9}$$

where ω is a circular frequency. Using Eq. (9) and substituting the crack terms and thermal terms stated above into the governing Eq. (1), we have

$$\begin{aligned} &\frac{\partial^4 W}{\partial X^4} + 2\frac{B_o \lambda^2}{D_x}\frac{\partial^4 W}{\partial X^2 Y^2} + \lambda^4 \frac{D_y}{D_x}\frac{\partial^4 W}{\partial Y^4} \\ &= \Omega^2 W - \frac{L_1^2}{D_x}\left[\frac{E_x h \Delta T(\alpha_x + \alpha_y \nu_y)}{1 - \nu_x \nu_y}\frac{\partial^2 W}{\partial X^2} + \lambda^2 \frac{E_y h \Delta T(\alpha_y + \alpha_x \nu_x)}{1 - \nu_x \nu_y}\frac{\partial^2 W}{\partial Y^2}\right] \\ &+ A\frac{D_y}{D_x}\left(\lambda^4 \frac{\partial^4 W}{\partial Y^4} + \nu_x \lambda^2 \frac{\partial^4 W}{\partial X^2 Y^2}\right) - B\frac{L_1^2}{D_x}\frac{E_y h \Delta T(\alpha_y + \alpha_x \nu_x)}{1 - \nu_x \nu_y}\left(\lambda^2 \frac{\partial^2 W}{\partial Y^2}\right) \\ &+ \frac{L_1^4 P_z}{D_x} \end{aligned} \tag{10}$$

where $A = \frac{2a}{3(\alpha_{bt}^o/6 + \alpha_{bb}^o)(3+\nu_x)(1-\nu_x)h + 2a}$ and $B = \frac{2a}{(6\alpha_{tb}^o + \alpha_{tt}^o)(1-\nu_x^2)h + 2a}$. For a free vibration analysis, we assume $P_z = 0$ in Eq. (10).

It is known that the fundamental frequency of an intact plate is zero at the critical buckling temperature [6]. This can also be applied to cracked plates. By substituting the general solution $w(x, y) = W_{mn} \sin(m\pi x/L_1) \sin(n\pi y/L_2)$ to Eq. (10), the critical buckling temperature becomes

$$T_{cr} = \frac{h^2\pi^2}{12L_1^2}\left[\frac{D_x m^4 + 2B_o\lambda^2 m^2 n^2 + D_y\lambda^4 n^4 - AD_y(\lambda^4 n^4 + v_x\lambda^2 m^2 n^2)}{D_x(\alpha_x + \alpha_y v_y)m^2 + (1+B)D_y(\alpha_y + \alpha_x v_x)\lambda^2 n^2}\right] \tag{11}$$

where m and n are the number of half sine waves in both directions. Equation (11) can be reduced to the model proposed by Jones [4] for plates without cracks. To satisfy the material properties, the minimum critical buckling temperature can be obtained by setting $m = n = 1$.

3 Solution Procedure

3.1 DSC Algorithm

Following the DSC algorithm, a weighted linear combination of the function values at uniformly distributed points $(2M+1)$ is employed to approximate the nth derivatives of a function $f(x)$. It can be discretized as

$$f^{(n)}(x_i) = \sum_{k=-M}^{M} \delta_{\alpha,\Delta}^{(n)} f(x_k) \tag{12}$$

where Δ is a grid spacing, $\delta_{\alpha,\Delta}$ is a delta kernel of the Dirichlet type. In this work, the regularized Shannon's delta kernel (RSK) [17] is employed as

$$\delta_{\alpha,\Delta}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right] \tag{13}$$

where σ is a controllable parameter to determine the effective computational bandwidth. To formulate the governing equation in terms of the DSC method, a column vector $\mathbf{W}$ is introduced

$$\mathbf{W} = (W_{0,0}, \ldots, W_{0,N_Y}, W_{1,0}, \ldots, W_{N_X,N_Y})^{\mathrm{T}} \tag{14}$$

where each element denotes the transverse displacement of an arbitrary point in the orthotropic plate. A differential matrix $\mathbf{D}_q^n (q = X, Y; n = 1, 2, \ldots)$ with the elements is given by

$$[\mathbf{D}_q^n]_{i,j} = \left[\left(\frac{d}{dq}\right)^n \delta_{\alpha,\Delta}^{(n)}(q - q_j)\right]_{q=q_i} = C_m^n, \quad i,j = 0, 1, \ldots, N_q \tag{15}$$

where $m = (q_i - q_j)/\Delta$. The matrix $\mathbf{D}$ is distributed to $i - j = m = -M, \ldots, 0, \ldots, M$. After that, the governing equation for cracked orthotropic plate can be written as

$$\left(\mathbf{D}_X^4 \otimes \mathbf{I}_Y + 2\frac{B_o\lambda^2}{D_x}\mathbf{D}_X^2 \otimes \mathbf{D}_Y^2 + \lambda^4\frac{D_y}{D_x}\mathbf{I}_X \otimes \mathbf{D}_Y^4\right)\mathbf{W}$$
$$= \Omega^2\mathbf{W} - \frac{L_1^2}{D_x}\left[\frac{E_x h\Delta T\left(\alpha_x + \alpha_y \nu_y\right)}{1 - \nu_x\nu_y}\mathbf{D}_X^2 \otimes \mathbf{I}_Y\right.$$
$$\left. + \lambda^2\frac{E_y h\Delta T\left(\alpha_y + \alpha_x \nu_x\right)}{1 - \nu_x\nu_y}\mathbf{I}_X \otimes \mathbf{D}_Y^2\right]\mathbf{W}$$
$$+ A\frac{D_y}{D_x}\left(\lambda^4\mathbf{I}_X \otimes \mathbf{D}_Y^4 + \nu_x\lambda^2\mathbf{D}_X^2 \otimes \mathbf{D}_Y^2\right)\mathbf{W}$$
$$- B\frac{L_1^2}{D_x}\frac{E_y h\Delta T\left(\alpha_y + \alpha_x \nu_x\right)}{1 - \nu_x\nu_y}\left(\lambda^2\mathbf{I}_X \otimes \mathbf{D}_Y^2\right)\mathbf{W} \tag{16}$$

where $\mathbf{I}_{\mathrm{q}}$ is the $(N_q + 1) \times (N_q + 1)$ unit matrix and $\otimes$ denotes the tensional product.

3.2 Boundary Conditions

For the treatment of rectangular plates with simply supported and clamped boundaries, the anti-symmetric and symmetric extension methods can be applied, respectively [7, 18, 19]. As the primitive version of the DSC method is limited by dealing with the vibration of plates with free edges, a new scheme that incorporates the DSC method with the Taylor series expansion technique was reported to overcome this issue [8, 15, 16]. Hence, the imposition of boundary constraints is different for the following three general supporting conditions:

$$\text{Simply supported edges (S): } w_x = M_x = 0 \text{ and } w_y = M_y = 0 \tag{17}$$

$$\text{Clamped edges (C): } w_x = \frac{\partial w}{\partial x} = 0 \text{ and } w_y = \frac{\partial w}{\partial y} = 0 \tag{18}$$

$$\text{Free edges (F): } Q_x = M_x = 0, \text{ and } R = 0 \tag{19}$$

where w_i $(i = x, y)$ are the transverse displacements, M_i $(i = x, y)$ are the bending moments, Q_i $(i = x, y)$ are the shear forces and R is the corner force.

Consider the above boundary conditions, the governing equation of cracked orthotropic plates can be expressed in a compact form as

$$\begin{bmatrix} \mathbf{K_{II}} & \mathbf{K_{IA}} \\ \mathbf{K_{AI}} & \mathbf{K_{AA}} \end{bmatrix}\begin{Bmatrix} \mathbf{W_I} \\ \mathbf{W_A} \end{Bmatrix} = \Omega^2\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{W_I} \\ \mathbf{W_A} \end{Bmatrix} \tag{20}$$

where $\mathbf{W_I}$ and $\mathbf{W_A}$ denote the transverse displacements of the inner points and the additional degree-of-freedom (DOF) points, respectively. If the boundary is a simply supported or a clamped edge, the corresponding elements are set to zero in $\mathbf{W_A}$. By properly rearranging the displacement vectors, Eq. (20) can be further simplified by vanishing the vector $\mathbf{W_A}$ as

$$[\bar{\mathbf{K}}]W_{\mathbf{I}} = \Omega^2\mathbf{W_I} \tag{21}$$

which can be solved by a standard eigenvalue solver.

4 Analysis Results and Discussion

In this study, the material properties and fundamental frequencies of orthotropic plates are presented in Tables 1, 2 and 3 [8]. The natural frequencies are expressed in terms of a non-dimensional form as $\Omega = \omega L_1^2\sqrt{\rho h/D}$. The uniform rise in temperature is expressed as a non-dimensional variation of temperature $T^* = \Delta T/T_{cr}$, where T_{cr} is critical buckling temperature. In all cases, the value of $\zeta = d/h$ in the LSM is assumed to 0.6. The number of grid points and the half bandwidth used in the DSC algorithm are $N = 32$ and $M = 25$, respectively.

Table 2 presents the fundamental frequency of SSSS orthotropic plates for various crack length ratios under the thermal condition of $T^* = 0$. The results of the DSC method are very close to the existing results from the Galerkin's method [6]. In Table 3, a higher variation of temperature reduces the natural frequency of the orthotropic plate intensively. It is found that the analysis results obtained by the DSC method show good agreement with those from the publication. In Fig. 2, the variation of natural frequencies of the rectangular orthotropic plates with free edges (i.e., FFFF and CSFF cases) due to the crack effect is first studied. A reduction of the

Table 1 Properties of an orthotropic rectangular plate [8]

v_x	v_y	ρ (kg/m^3)	E_x (GPa)	E_y (GPa)	G_{xy} (GPa)	α_x (/°C)	α_y (/°C)
0.23	0.0208	2000	208	19.8	5.7	7.1×10^{-6}	2.3×10^{-5}

Table 2 Fundamental frequencies for cracked SSSS orthotropic plates ($h = 0.01$ m)

Crack length ratio ($2a/L_1$)	Aspect ratio ($\lambda = 1$)		Aspect ratio ($\lambda = 2$)	
	DSC method [8]	Joshi et al. [6]	DSC method [8]	Joshi et al. [6]
0	10.998	10.99	17.256	17.25
0.20	10.685	10.66	14.296	14.11
0.40	10.611	10.59	13.518	13.37
0.60	10.578	10.56	13.157	13.04
0.80	10.559	10.54	12.948	12.85
1.00	10.547	10.53	12.812	12.73

Table 3 Fundamental frequencies for cracked SSSS orthotropic plates in thermal environment ($L_1 = L_2 = 1$ m, $h = 0.01$ m)

Temperature variation (T^*)	Crack length ratio ($2a/L_1 = 0$)		Crack length ratio ($2b/L_2 = 0.02$)	
	DSC method [8]	Joshi et al. [6]	DSC method [8]	Joshi et al. [6]
0	10.998	10.998	10.430	10.352
0.2	9.8367	9.8366	9.3653	9.3064
0.4	8.5188	8.5188	8.1626	8.1278
0.6	6.9556	6.9556	6.7489	6.7459
0.8	4.9184	4.9184	4.9465	4.9956
0.9	3.4778	3.4778	3.7320	3.8316

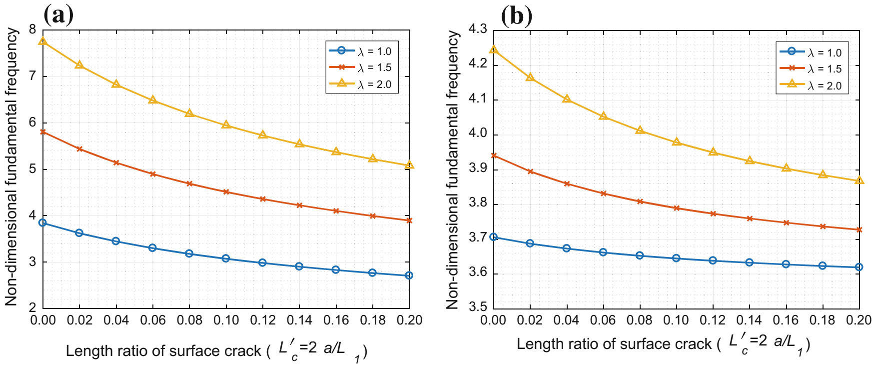

Fig. 2 Effect of crack length on fundamental frequencies: **a** FFFF plate; **b** CSFF plate

fundamental frequency is observed as the crack length increases for different aspect ratios, $\lambda = 1$, 1.5 and 2.

5 Conclusions

This work presents the free vibration analysis of orthotropic plates under both crack and thermal effects. The surface cracks on orthotropic plates are simulated using the line-spring model, and the temperature heating load is considered as a uniformly distributed effect. Based on the mathematical model, the DSC method is first applied to address this problem. By incorporating with the Taylor series expansion approach, the limitation of the DSC method for the treatment of plate problems with free edges has been overcome. The effects of boundary condition, aspect ratio, crack length and thermal load on the dynamic responses of orthotropic plates are considered herein. The analysis results indicate that the presented scheme can achieve a high level of reliability and accuracy. As the temperature rises and the crack length increases, the vibration frequency of the plates would decrease. This is mainly due to the change of material properties under these effects.

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References

1. Civalek Ö (2007) Free vibration and buckling analyses of composite plates with straight-sided quadrilateral domain based on DSC approach. Finite Elem Anal Des 43(13):1013–1022
2. Gay D, Hoa SV, Tsai SW (2002) Composite materials: design and applications. CRC Press
3. Israr A, Cartmell MP, Manoach E, Trendafilova I, Krawczuk M, Arkadiusz Ĺ (2009) Analytical modeling and vibration analysis of partially cracked rectangular plates with different boundary conditions and loading. J Appl Mech 76(1):011005
4. Jones RM (2005) Thermal buckling of uniformly heated unidirectional and symmetric cross-ply laminated fiber-reinforced composite uniaxial in-plane restrained simply supported rectangular plates. Compos A Appl Sci Manuf 36(10):1355–1367
5. Joshi PV, Jain NK, Ramtekkar GD (2015) Analytical modelling for vibration analysis of partially cracked orthotropic rectangular plates. Eur J Mech/Solids 50:100–111
6. Joshi PV, Jain NK, Ramtekkar GD, Virdi GS (2016) Vibration and buckling analysis of partially cracked thin orthotropic rectangular plates in thermal environment. Thin-Walled Struct 109:143–158
7. Lai SK, Xiang Y (2009) DSC analysis for buckling and vibration of rectangular plates with elastically restrained edges and linearly varying in-plane loading. Int J Struct Stab Dyn 9(3):511–531
8. Lai SK, Zhang LH (2018) Thermal effect on vibration and buckling analysis of thin isotropic/orthotropic rectangular plates with crack defects. Eng Struct 177:444–458
9. Liew KM, Wang CM, Xiang Y, Kitipornchai S (1998) Vibration of Mindlin plates: programming the p-Version Ritz method. Elsevier
10. Leissa AW (1969) Vibration of plates. Scientific and Technical Information Office, National Aeronautics and Space Administration Washington
11. Natarajan S, Baiz PM, Ganapathi M, Kerfriden P, Bordas S (2011) Linear free flexural vibration of cracked functionally graded plates in thermal environment. Comput Struct 89(15–16):1535–1546
12. Ng CHW, Zhao YB, Wei GW (2004) Comparison of discrete singular convolution and generalized differential quadrature for the vibration analysis of rectangular plates. Comput Methods Appl Mech Eng 193(23–26):2483–2506
13. Reddy JN (2004) Mechanics of laminated composite plates and shells: theory and analysis. CRC Press
14. Rice JR, Levy N (1972) The part-through surface crack in an elastic plate. J Appl Mech 39(1):185–194
15. Wang X, Xu S (2010) Free vibration analysis of beams and rectangular plates with free edges by the discrete singular convolution. J Sound Vib 329:1780–1792
16. Wang X, Yuan Z (2017) Discrete singular convolution and Taylor series expansion method for free vibration analysis of beams and rectangular plates with free boundaries. Int J Mech Sci 122:184–191
17. Wei GW (1999) Discrete singular convolution for the solution of the Fokker-Planck equation. J Chem Phys 110(18):8930–8942

18. Wei GW, Zhao YB, Xiang Y (2001) The determination of natural frequencies of rectangular plates with mixed boundary conditions by discrete singular convolution. Int J Mech Sci 43(8):1731–1746
19. Wei GW, Zhao YB, Xiang Y (2002) Discrete singular convolution and its application to the analysis of plates with internal supports. Part 1: theory and algorithm. Int J Numer Meth Eng 55:913–946
20. Xiang Y, Zhao YB, Wei GW (2002) Discrete singular convolution and its application to the analysis of plates with internal supports. Part 2: Applications. Int J Numer Meth Eng 55:947–971
21. Xing YF, Liu B (2009) New exact solutions for free vibrations of thin orthotropic rectangular plates. Compos Struct 89(4):567–574