Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (IV) *

This paper is the fourth and the last of the series and deals with quotations from the earlier mathematical works occurring in Bhāskara I's commentary on the *Āryabhaṭīya*. Of these quotations, some are taken from such works as were popularly used in the time of Bhāskara I, some are mentioned to point out the approximate character of rules contained in them so as to emphasise the superiority of the corresponding rules given in the *Āryabhaṭīya*, and some are quoted to find fault with them. Some of these quotations are in Prākṛta *gāthās* and seem to have been taken from Jaina sources. These quotations show that in the seventh century when Bhāskara I wrote his commentary on the *Āryabhaṭīya* there existed works on mathematics which were written not only in Sanskrit but also in Prākṛta. These works were probably of the same nature as the *Pāṭīgaṇita* and the *Triśatikā* and were probably written by Maskarī Pūraṇa, Mudgala, and Patana etc. whose names have been mentioned by Bhāskara I.

5 Passages quoted by Bhāskara I from mathematical works

5.1 Quotation 1: Proclaiming twofold nature of mathematics

> आह च–
> संयोगभेदा गुणना गतानि शुद्धेश्च भागो गतमूलयुक्तः ।
> व्याप्तं समीक्ष्योपचयक्षयाभ्यां विद्यादिदं द्व्यात्मकमेव शास्त्रम् ॥ [*Ā*, ii. intro]
>
> Multiplication and involution are the kinds of addition, and division and evolution, of subtraction. Seeing that the science of mathematics is permeated by increase and decrease, this science is indeed of two kinds.

This passage seems to have been taken from the introductory verses of a certain work on *Pāṭīgaṇita*.

* K. S. Shukla, *Ganita*, Vol. 23, No. 2 (December 1972), pp. 41–50.

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5.2 Quotation 2: Giving a rule for squaring a number

In the *Āryabhaṭīya* there is no rule for squaring a number, so Bhāskara I quotes the following rule:

> अन्त्यपदस्य च वर्गं कृत्वा द्विगुणं तदेव चान्त्यपदम् ।
> शेषपदैराहन्याद् उत्सार्योत्सार्य वर्गविधौ ॥[1] [*Ā*, ii. 3 (i)]
>
> Having squared the last digit (on the left) (and then having written that square underneath the last digit), multiply twice of that last digit by the remaining digits of the number, and write the products successively one place ahead (to the right). This is the procedure (to be adopted) in squaring a number.[2]

It is interesting to note that Bhāskara I refers to the above rule as *lakṣaṇa-sūtra*. In all later works on mathematics a rule has been called *karaṇa-sūtra*. Possibly, in the work from which the above passage was taken the rules were called *lakṣaṇa-sūtra*.

5.3 Quotation 3: Giving a formula for the simplification of a fraction of the type $a + \frac{b}{c}$

While solving Ex. 2,[3] Bhāskara I quotes (under *karaṇa*) the following formulatory sentence which probably formed part of some rule given by an earlier author:

> छेदगुणं सांशम् ।
>
> Multiply (the whole number) by the denominator and add the numerator.[4]

---

[1] Similar rules occur in *Triśatikā* (Rule 10, p. 3) and *Gaṇitasārasaṅgraha* (Rule 31, p. 13).

[2] To square 34, for example, the following procedure will be adopted:

		3	4		
Square of the last digit		9			
Product of 2 times 3 multiplied by the next digit 4 written one place ahead		2	4		(One round of the operation is over and the rule is repeated)
Square of the last but one digit i.e., 4, written one place ahead			1	6	
Addition gives	1	1	5	6	which is the required number.

In the actual Hindu process of working, the numbers were not allowed to accumulate. Addition was performed after every step and only one number was allowed to remain on the writing board.

[3] Vide supra 3.1.

[4] That is to say, $a + \frac{b}{c}$ is equal to $\frac{(ac+b)}{c}$.

The similarity of this formula with the corresponding formula "*chedasaṅguṇaṃ sāṃśam*" of Śrīdhara is noteworthy.[5]

5.4 Quotation 4: Giving a rule for cubing a number

On the cubing of a number also there is no specific rule in the *Āryabhaṭīya*. Bhāskara I refers to the following rule:

> "अन्त्यपदस्य घनं स्या"दित्यादि लक्षणसूत्रम् । [*Ā*, ii. 3 (ii)]
>
> "Obtain the cube of the last digit (on the left) etc." is the rule (for the purpose).

Like the rule of squaring, this rule has also been called by the name *lakṣaṇa-sūtra*, which seems to suggest that both the rules have been extracted from the same source. The striking similarity in the phraseology is noteworthy. The reference of the above rule by indicating its beginning alone proves the popularity of the work from which it has been taken.

5.5 Quotation 5: On the position of the altitude of an equilateral triangles as a line of symmetry

> करणम् – समत्र्यश्रिक्षेत्रे समैवावलम्बकस्थितिरिति । [*Ā*, ii. 6, Ex. 10]
>
> *Process.* (Applying the rule) "In an equilateral triangle, the position of the altitude is that of (the line of) symmetry," (we have that, etc.).

5.6 Quotation 6: Giving an approximate rule for the area of a circle

In order to emphasise the accuracy of Āryabhaṭa I's rule (*Ā*, ii. 7), Bhāskara I quotes the following popular rule and points out that it is only approximate, not accurate. In fact, he says, there is no other accurate rule.

> व्यासार्धकृतिस्त्रिसंगुणा गणितम् । [*Ā*, ii. 7 (i)]
>
> The square of the radius multiplied by 3 is the area (of a circle).

5.7 Quotation 7: Giving an approximate rule for the volume of a sphere

According to Bhāskara I, the rule (*Ā*, ii. 7) of Āryabhaṭa I gives the accurate value of the volume of a sphere.[6] All other rules, he tells us, are only approximate. Of these approximate rules he quotes the following:

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[5] *Cf. Triśatikā*, edited by S. Dvivedi, Banaras (1899), Rule 24, p. 10.

[6] In fact, however, Āryabhaṭa I's rule stated in *Ā*, ii. 7 is wrong.

व्यासार्धघनं भित्वा नवगुणितमयोगुडस्य घनगणितम् । [$\bar{A}$, ii. 7 (ii)]

The cube of the radius divided by two and multiplied by nine is the volume of an iron ball.[7]

It is strange that the accurate formula for the volume of a sphere was not known in India. This seems to suggest that Greek Geometry was not known at all in India and that all geometrical results and mensuration formulae were independently discovered by Hindu mathematicians.

5.8 Quotation 8: On the verification of areas

Bhāskara I is of the opinion that the rule stated in $\bar{A}$, ii. 9 (i) relates to the verification of the area of any rectilinear figure by deforming it into a rectangle. In support of his statement, he cites the following passages:

उक्तञ्च –
करणैरुक्तैर्नित्यं फलमनुगम्यायते तु विज्ञेयम् ।
प्रत्ययकरणं क्षेत्रे व्यक्तं फलमायते यस्मात् ॥ [$\bar{A}$, ii. 9 (i)]

So has it been stated too. Having determined the area in accordance with the prescribed rules, demonstration (verification) should always be made by (deforming the figure into) a rectangle because it is the rectangle only of which the area is obvious.

5.9 Quotation 9: Giving an approximate formula for the area of a plane figure resembling the tusk of an elephant

पृष्ठोदरसमासार्धं विस्तारार्धगुणं फलम् । [$\bar{A}$, ii. 9 (ii)]

Half the sum of the back and the belly multiplied by half the width (at the base) is the area (of the figure that resembles the tusk of an elephant).[8]

5.10 Quotation 10: One-half of a *gāthā* giving a rule for finding the circumference of a circle

विक्खम्भवग्गदहगुणकरणी वट्टस्य परिरओ होदि । [$\bar{A}$, ii. 10]
(विष्कम्भवर्गदशगुणकरणी वृत्तस्य परिणाहः भवति ॥)

The square root of ten times the square of the diameter is the circumference of a circle.

[7] *Cf. Gaṇitasārasaṅgraha* viii. $28\frac{1}{2}$ (i). Also *Cf. Trilokasāra*, *gāthā* 19 (i).
[8] *Vide supra*, 3.1, Ex. 27.

The above *gāthā* is almost the same as *gāthā* 185 of the Jaina work, the *Jyotiṣakaraṇḍaka*.[9] The latter runs as follows:

> विक्खम्भवग्गदहगुणकरणी वट्टस्स परिरओ होइ ।

This *gāthā* is also found to occur in the *Trilokasāra*[10] and the *Kṣetrasamāsa*, and has been quoted by Malayagiri in his commentary on the *Jīvābhigama* (*Sūtra* 109).

Bhāskara I has quoted the above *gāthā* as giving the so called accurate value of π, viz. $\sqrt{10}$. Bhāskara I has taken pains to demonstrate that this value was far from being accurate. In this connection he makes use of the rules given in Quotations 11, 12, 13, and 14. In the end he concludes in derision:

> So I bow to $\sqrt{10}$ whose grace is not well conceived.[11]

5.11 Quotation 11: One *gāthā* giving a rule for getting the value of the chord of a circle

> ओगाहूणं विक्कम्भ एगाहेण संगुणं कुर्यात् ।
> चउगुणियस्सतु मूउं सा जीवा सघ्वखत्ताणम् ॥ (MS T)
> ओगाहूणं विक्खम्भ एगाहेण संगुणं कुर्यात् ।
> चउगुणियः स तु (मू) उं सा जीवा सव्वखत्ताणम् ॥ (MS I) [$\bar{A}$, ii. 10]

Probable Sanskrit version:

> अवगाहोनं विष्कम्भमवगाहेण सङ्गुणं कुर्यात् ।
> चतुर्गुणितस्य तु मूलं सा जीवा सर्वक्षेत्राणाम् ॥
>
> Multiply the diameter as diminished by the depth (of the chord) by the depth. The square root of four times the (product) is (the length of) the chord of any (circular) figure.

In the adjoining figure (**ed.** see Figure 1), let AB be a chord of the circle ACB and let CD be the perpendicular diameter intersecting the chord at X. Then CX ($< XD$) denotes the depth of the chord.

If R be taken to denote the radius of the circle, then the rule stated above may be written as

$$\text{chord} = \sqrt{4 \times \text{depth} \times (2R - \text{depth})}.$$

[9] *Cf.* Rājeśvarīdatta Miśra, "Vṛttakṣetra kā gaṇita: Jaina tathā Jainetara ācāryon ke siddhānta," *Jaina-siddhānta-bhāskara* (Jaina Antiquary), Part XV, vol. 2, pp. 105 ff.

[10] *Gāthā* 96.

[11] अतोऽस्यै अविचारितमनोहरायै नभोऽस्तु दशकरण्यै ।

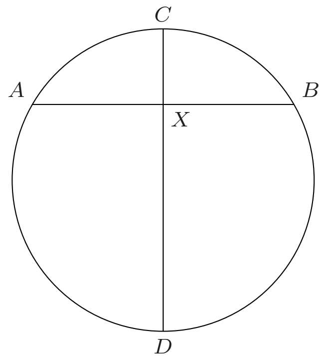

Figure 1

This formula is also mentioned in the *Tatvārthādhigamasūtrabhāṣya* of Umāsvātī.[12] Ācārya Yativṛṣbha puts it in the form[13]

$$\text{chord} = \sqrt{4\{R^2 - (R - \text{depth})^2\}}.$$

5.12 Quotation 12: One *gāthā* giving a rule for the area of a segment of a circle

इसुपायगुणा जीवा दसिकरणि भवेद् विगणिय पदम् ।
धनुपट्ट अम्मिखत्ते एदं करणं तुदा अव्वम् ॥ (MS T)
इषुपायगुणा जीवा दशकरणि भवेद् विगणिय पदम् ।
धनुवट्ट अम्मिखत्ते एदं करणं तुदा अप्पम् ॥[14] (MS I) [$\bar{A}$, ii. 10]

This *gāthā* states that the area of a segment of a circle

$$= \sqrt{10}\left(\frac{1}{4}\right)\text{(arrow)(bounding chord)},$$

[12]See B. Datta, The Jaina School of Mathematics, *Bull. Cal. Math. Soc.*, Vol. XXI, 1929. Umāsvātī, according to the tradition of the Śvetāmbara Jainas, lived about 150 BC. According to the Digambara tradition he is sometimes called Umāsvāmī and is said to have lived between 135 AD and 219 AD. Satischandra Vidyabhushan thinks that he lived in the first century AD. See Datta, *l.c.*

[13]*Cf. Tiloya-paṇṇattī* (*Triloka-prajñapti*), edited by A. N. Upadhye and Hiralal Jain, and published by Jaina Saṃrakṣaka Saṃgha, Sholapur (1943), Part I, iv. 180.

According to A. N. Upadhye and Hiralal Jain, Ācārya Yativṛṣbha lived sometime between 473 AD and 609 AD. See *Tiloya-paṇṇattī*, Part II, edited by Hiralal Jain and A. N. Upadhye, and published by Jaina Saṃrakṣaka Saṃgha, Sholapur (1951), p. 7.

[14]The following is the probable Sanskrit version:

इषुपादगुणा जीवा दशकरणीभिर्भवेद् विगुण्य फलम् ।
धनुपट्टेऽस्मिन् क्षेत्रे एतत्करणं तु ज्ञातव्यम् ॥

the arrow being the depth of the chord (explained above).

The occurrence of the above *gāthā* in Bhāskara I's commentary shows that rules for obtaining the area of a segment of a circle were devised in India prior to the seventh century AD. Datta once remarked:

> We do not find amongst the Hindus, as far as is known, any expression for the area of a segment of a circle before the time of Śrīdhara (c. 750) though it was known in Greece and China long before.[15]

The rule described in the above *gāthā* is also found to occur in the *Gaṇitasāra-saṅgraha*[16] of Mahāvīra (850) and in the *Laghu-kṣetra-samāsa* of Ratneśvara Sūri (1440 AD).

The rule, Datta rightly observes,[17] is incorrect and was probably derived by analogy from the rule for the area of a circle.

5.13 Quotation 13: One *gāthā* giving a rule for the the addition of two surds

> औवट्टि अदस्सकेण इमूलसमासस्समोत्थवत् ।
> ओं पट्टणाय गुणियं करणिसमासं तुणा अव्वम् ॥ (MS T)
> औवट्टि असस्सकेण इमूलसमासः समोत्थवत् ।
> ओवट्टणाय गुणियं करणिसमासं तुणा अव्वम् ॥[18] (MS I) [$\bar{A}$, ii. 7]

A. N. Singh has given the following translation:[19]

> Reducing them (i.e. the two surds) by some suitable number, add the square roots of the quotients: the square of the result multiplied by the reducer should be known as the sum of the surds.

In other words,

$$\sqrt{\alpha} + \sqrt{\beta} = \sqrt{c\left(\sqrt{\frac{\alpha}{c}} + \sqrt{\frac{\beta}{c}}\right)^2}$$

where $\frac{\alpha}{c}$ and $\frac{\beta}{c}$ are assumed to be perfect squares. This rule is found to occur in several later works also.[20]

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[15] *Cf.* B. Datta, "The Jaina School of Mathematics."

[16] vii. $70\frac{1}{2}$.

[17] *l.c.*

[18] The following is the probable Sanskrit version:

> अपवर्त्याभीप्सितेन करणीमूलसमासोत्थवर्गो यः ।
> अपवर्तनेन गुणितः करणिसमासस्तु ज्ञातव्यः ॥

[19] *Cf.* A. N. Singh, "On the arithmetic of surds amongst the ancient Hindus," *Mathematica*, Vol. XII, p. 104.

[20] See, for instance, *Brāhmasphuṭasiddhānta*, xviii. 38 and *Gaṇitasārasaṅgraha*, vii. $88\frac{1}{2}$.

5.14 Quotation 14: Containing a rule for the length of an arc of a circle, when the arc is less than a semi-circle

ज्यापादशरार्धयुतिः स्वगुणा दशसङ्गुणा करण्यस्ताः । [$\bar{A}$, ii. 10]

The sum of one-fourth of the chord and one-half of the arrow, multiplied by itself, and then by 10: the square root of so much (is the length of the corresponding arc).

That is to say,

$$\text{arc} = \left(\frac{\text{chord}}{4} + \frac{\text{arrow}}{2}\right) \times \sqrt{10}. \qquad (1)$$

This formula has not been traced in any other Jaina or Hindu work although other formulae for the arc of a circle are found to occur elsewhere.

In the first century Umāsvāti in his *Tatvārthādhigamasūtrabhāṣya* gave the formula

$$\text{arc} = \sqrt{6(\text{arrow})^2 + (\text{chord})^2}. \qquad (2)$$

If in (1) and (2), we put arrow $= r$ and chord $= 2r$, ($r =$ radius), both (1) and (2) reduce to

$$\text{arc (of the semi-circle)} = r\sqrt{10}.$$

This shows that both (1) and (2) have been derived from the formula for the arc of a semi-circle, using $\pi = \sqrt{10}$. But the methods used in the derivations are obviously different.

Other formulae for an arc of a segment of a circle are known to occur in the *Gaṇitasārasaṅgraha*,[21] and in the *Mahāsiddhānta*[22] of Āryabhaṭa II (950). The formulae given in those works have been derived in the same way as (2) but with different values of π (viz. $\pi = 3$, $\sqrt{10}$, and $\frac{22}{7}$).

It is noteworthy that the term *jīvā*, which usually means a half-chord, has been used in the above quotations in the sense of a chord.

5.15 Quotation 15: On the arrangement of the three quantities in a problem on the Rule of Three

उक्तञ्च –
आद्यन्तयोस्तु सदृशौ विज्ञेयौ स्थापनासु राशीनाम् ।
असदृशराशिर्मध्ये त्रैराशिकसाधनाय बुधैः ॥ [$\bar{A}$, ii. 26–27 (i)]

So has been stated —

In order to solve a problem involving three quantities, the learned should note that of the three quantities the two of like denomination should be set down in the beginning and the end, and the

[21] vii. 43, $73\frac{1}{2}$.
[22] xv. 90, 94, 95.

third quantity of unlike denomination should be written in the middle (of those two).

For example in Ex. 66,[23] the quantities "9 *rūpakas*" and "one *rūpaka*" are of like denomination (because both of them are *rūpakas*) and the third quantity "5 *palas*" is of unlike denomination. These quantities are therefore to be written as

9 5 1

Similar direction regarding the arrangement (*sthāpana*) of the three quantities is given by Śrīdhara and others.

5.16 Quotation 16: One *gāthā* stating how to perform subtraction when the minuend and the subtrahend are both positive, both negative, or one positive and the other negative

सोज्झं भूणारधनं अणं अणदो न यमदो न यमदो सोज्झं ।
विपरीते साधण एषो झं वाकिवगुहोल ॥ (MS T)
सोज्झं भ्रूणारयणं भ्रणं अणदो न यमदो न यमदो सोज्झम् ।
विपरीते सायण एषो ज्झं वा किं व्व गुहोल ॥[24] (MS I) [*Ā*, ii. 30]

Both the readings are defective. From the context and the nature of the *gāthā* it is clear that it deals with the subtraction of positive and negative quantities. We are sure that the rules given therein are the same as stated in the following stanzas of the *Brāhmasphuṭasiddhānta*:

ऊनमधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेपम् ॥

From the greater should be subtracted the smaller; (the final result is) positive, if positive from positive, and negative, if negative from negative. If however, the greater is subtracted from the less, the difference is reversed (in sign), negative becomes positive and positive becomes negative. When positive is to be subtracted from negative or negative from positive, then they must be added together.[25]

[23] Vide *Supra*, 3.1.

[24] The probable Sanskrit version is as follows:

शोध्यमृणादृणं धनं धनतः न धनतो न ऋणतः शोध्यम् ।
विपरीते शोधनमेव धनं न किमपि गूढमत्र ॥

[25] B. Datta and A. N. Singh, *History of Hindu Mathematics*, Part II, pp. 21–22. Also see, *Brāhmasphuṭasiddhānta* xviii. 31–32.

The *gāthā* mentioned above has been quoted by Bhāskara I in connection with the solution of the equation

$$9x - 24 = 2x + 18.$$

(Ex. 92). Application of Āryabhaṭa I's rule ($\bar{A}$, ii. 30) gives

$$x = \frac{18 - (-24)}{9 - 2}.$$

The *gāthā* is meant to show how -24 is to be subtracted from 18. Using the rule mentioned therein, one gets

$$x = \frac{18 + 24}{7} = 6.$$

The above *gāthā* clearly shows that negative numbers were introduced in analysis in India and methods were also devised for subtracting greater numbers from smaller ones or positive numbers from negative numbers or vice versa much before the time of Bhāskara I.