Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (II) *

This paper is the second of the series and deals with the mathematical examples set by Bhāskara I in illustration of the rules given in the *Gaṇita* Section of the *Āryabhaṭīya*.

3 Bhāskara I's examples

Below are given the examples set by Bhāskara I in illustration of the various rules of the *Āryabhaṭīya*. The rule under which the particular example occurs is given within square brackets after the statement of the example.

3.1 Examples on arithmetic and mensuration

On the squaring of integral numbers

Ex. 1. "Separately tell (me) the squares of (integral numbers) beginning with 1 and ending in 9, and also the square of 25 and of 100 plus 25."[1]

[*Ā*,ii. 3 (i)]

On the squaring of fractional numbers

Ex. 2. "Tell me the squares of 6 plus $\frac{1}{4}$, 1 plus $\frac{1}{5}$, and 2 minus $\frac{1}{9}$."

[*Ā*, ii. 3 (i)]

On the cubing of integral numbers[2]

Ex. 3. "Tell me separately the cubes of integral numbers beginning with 1 and ending 9, and also the cubes of $(8 \times 8)^2$ and $(25^2)^2$."

[*Ā*, ii. 3 (i)]

* K. S. Shukla, *Gaṇita*, Vol. 22, No. 2 (December 1971), pp. 61–78.

[1] Ex. 1 reappears it Yallaya's commentary on *Ā*, ii. 3.

[2] Exs. 3 and 4 reappear in Yallaya's comm. on *Ā*, ii. 3.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
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On the cubing of fractional numbers

Ex. 4. "If you have clear understanding of cubing a number, say correctly the cubes of 6, 15, and 8 as respectively diminished by $\frac{1}{6}$, $\frac{1}{15}$, and $\frac{1}{8}$ (i.e., the cubes of 6 minus $\frac{1}{6}$, 15 minus $\frac{1}{15}$, and 8 minus $\frac{1}{8}$)." [$\bar{A}$, ii. 3 (ii)]

On extracting the square root of integral numbers

Ex. 5. "I want to know, O friend, the square root of the (square) numbers 1, etc., previously determined, and also of the square number 625."

[$\bar{A}$, ii. 4]

On extracting the square root of fractional numbers

Ex. 6. "Calculate, in accordance with the arithmetic of (Ārya)bhaṭa, the square root of 6 plus $\frac{1}{4}$ and of 13 plus $\frac{4}{9}$ and state the two results."

[$\bar{A}$, ii. 4]

On extracting the cube root of integral numbers

Ex. 7. "Tell me separately the cube roots of the cube numbers 1, etc. Also quickly calculate the cube root of 1728."[3] [$\bar{A}$, ii. 5]

Ex. 8. "Correctly state, in accordance with the rules prescribed in the *Bhaṭa-śāstra* (i.e., *Āryabhaṭīya*), the cube root of 8291469824."[4] [$\bar{A}$, ii. 5]

On extracting the cube root of fractional numbers

Ex. 9. "Correctly calculate in accordance with the arithmetic of Āryabhaṭa, the fractional (cube) root of 13 plus $\frac{103}{125}$." [$\bar{A}$, ii. 5]

On the determination of the area of triangles

Ex. 10. "Tell (me), O friend, the areas of the (three) equilateral triangles whose sides are 7, 8, and 9 (units) respectively, and also the area of the isosceles triangle whose base is 6 (units) and the lateral sides each 5 (units)."[5] [$\bar{A}$, ii. 6 (i)]

Ex. 11. "Carefully say the area of the isosceles triangle in which the two lateral sides are each stated to be 10 (units) and the base is given to be 16 (units)." [$\bar{A}$, ii. 6 (i)]

[3]Exs. 7 and 8 reappear in Yallaya's comm. on $\bar{A}$, ii. 5.
[4]See footnote 3.
[5]Ex. 10 reappears in Yallaya's comm. on $\bar{A}$, ii. 6.

Ex. 12. "O friend, what is the area of the scalene triangle in which one lateral side is 13 (units), the other (lateral side) 15 (units), and the base 14 (units)?"[6] [$\bar{A}$, ii. 6 (i)]

Ex. 13. "Say what is the area of the scalene triangle in which the base is 51 (units), one lateral side is 37 (units), and the other lateral side is stated to be 20 (units)." [$\bar{A}$, ii. 6 (i)]

For finding the area of a triangle, Āryabhaṭa I states the general formula: Area $= \frac{1}{2}$ base $\times$ altitude. This formula is not directly applicable to finding the areas of triangles in which the three sides are given. In order to make use of that formula it is necessary to find the altitude. In the case of equilateral and isosceles triangles, in which the altitude bisects the base, the altitude is easily obtained by the formula:

$$(\text{altitude})^2 = (\text{lateral side})^2 - \left(\frac{\text{base}}{2}\right)^2.$$

In case of scalene triangles, Bhāskara I makes use of the following result:

If a be base and b and c the lateral sides of a triangle, then

$$(\text{altitude})^2 = b^2 - x^2 \text{ or } c^2 - (a-x)^2,$$
$$\text{where} \quad x = \frac{1}{2}\left[a + \frac{(b^2 - c^2)}{a}\right],$$
$$\text{and} \quad a - x = \frac{1}{2}\left[a - \frac{(b^2 - c^2)}{a}\right].$$

This rule occurs in the *Brāhmasphuṭasiddhānta* (xii. 22) also. Brahmagupta has also given the formula:[7]

$$\text{area} = \sqrt{(s-a)(s-b)(s-c)},$$
$$\text{where } 2s = a + b + c,$$

but Bhāskara I has not used this, perhaps because it was irrelevant to him. It must be borne in mind that Bhāskara I aims at illustrating the rules given by Āryabhaṭa I only.

On the determination of the volume of a triangular pyramid

Ex. 14. "Quickly tell me the more accurate volume and also the measure of the altitude of the solid of the shape of a trapa in which each edge is 12 (units)." [$\bar{A}$, ii. 6 (ii)]

[6]Ex. 12 appears twice in *Gaṇitasārasaṅgraha.* See *GSS*, vii. 10 and 53. It occurs also in the *Triśatikā* of Śrīdharācārya and the *Līlāvatī* (p. 154) of Bhāskara II.

[7]See *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xii. 21.

Ex. 15. "The length of each edge of a trapa is given to be 18 (units). I want to know, O friend, the altitude and the volume thereof." [*Ā*, ii. 6 (ii)]

Āryabhaṭa I's formula for the volume of a pyramid is

$$\text{volume} = \frac{1}{2}\,(\text{area of base}) \times (\text{altitude}).$$

Bhāskara I has made little improvement in this result. His contemporary Brahmagupta has, however, given the correct formula for the volume of a cone.[8]

On the determination of the circumference and area of a circle

Ex. 16. "The diameter (of three circles) are accurately determined by me to be 8, 12, and 6 (units) respectively. Tell me separately the circumference and area of each of these circles." [*Ā*, ii. 7 (i)]

On the volume of a sphere

Ex. 17. "The diameters of (three) spheres are to be known as 2, 5, and 10 (units) respectively. I want to know their volumes briefly." [*Ā*, ii. 7 (ii)]

Āryabhaṭa I's formula for the volume of a sphere is

$$\text{volume} = (\text{area of central circle})^{\frac{3}{2}}.$$

Āryabhaṭa I writes that this is the accurate value for the volume of a sphere. Bhāskara I too holds the same view. In fact, that value is not only inaccurate but also wrong. The correct formula was given by Bhāskara II.[9]

On the determination of the junction-lines[10] and the area of a trapezium

Ex. 18. "(In a trapezium) the base is 14 (units), the face (i.e., the upper side) is 4 units and the lateral sides each 13 (units). Give out the junction-lines and the area."[11] [*Ā*, ii. 8]

Ex. 19. "(In a trapezium) the base, the lateral sides and the face are stated to be 21 (units), 10 (units) each, and 9 (units) respectively. Give out the area and the junction–lines." [*Ā*, ii. 8]

[8] See *BrSpSi*, xii. 44.

[9] See *Līlāvatī* (*Ānandāśrama* Sanskrit Series), p. 201, stanza 201.

[10] By the junction-lines are meant the segments of the altitude through the intersection of the diagonals.

[11] Ex. 18 reappears in the commentaries of Sūryadeva, Yallaya and Raghunātha Rāja on *Ā*, ii. 8.

Ex. 20. "(In a trapezium) the base is 33 (units), and the other sides are each stated to be 17 (units). What is the area thereof and what are the junction-lines?" [$\bar{A}$, ii. 8]

Ex. 21. "(In a trapezium) having 25 (units) for the face, the base is stated to be 60 (units); the lateral sides are 13 (units) multiplied by 4 and 3 respectively. (Find the area and the junction-lines)."[12]

Ex. 22. "(In a trapezium) the altitude is stated to be 12 (units), the base 19 (units) and the face 5 (units). The lateral sides of that are given to be 10 (units) as severally increased by 5 and 3 (units). I want to know the area and the junction-lines correctly."[13] [$\bar{A}$, ii. 8]

On the determination of the area of a rectangle, etc.

Ex. 23. "(Of three rectangles) the breadths are 8, 5, and 10 (units); and the lengths of these are 16, 12, and 14 (units) (respectively). What are the areas of rectangles?" [$\bar{A}$, ii. 9]

Ex. 24. "How will the verification be made in the case of all the areas of triangles, quadrilaterals, and circles which have been determined by theoretical calculation?"[14] [$\bar{A}$, ii. 9]

Ex. 25. "(In a trapezium) one face (i.e., side) is seen to be 11 (units), the opposite (parallel) face is stated to be 9 (units), and the length (=distance) (between them) is 20 (units). What, O mathematician, is the area of that figure?"[15] [$\bar{A}$, ii. 9]

On the determination of the area of a figure resembling the drum-shaped musical instrument *Paṇava*

Ex. 26. "The two (parallel) faces of (a figure resembling) a *Paṇava* are each 8 (units), the central width is 2 (units), and the length (between the faces) is 16 (units). Say what is the area of this figure resembling the (musical instrument) *Paṇava*."[16] [$\bar{A}$, ii. 9]

[12] Ex. 21 reappears in Yallaya's comm. on $\bar{A}$, ii. 8.

[13] This is an example of a trapezium in which the lateral sides are unequal. In such a trapezium, the area and the junction-lines are determined if, besides the sides, the altitude is also known.

[14] According to Bhāskara I, the first half of $\bar{A}$, ii. 9 relates to the verification of areas of rectilinear figures. What is meant is that the given figure should be deformed into a rectangle and then the area should be obtained by multiplying the length of the rectangle by its breadth. A rectangle is chosen because its area is well known. In this connection Bhāskara I has quoted a passage from some unknown mathematical work.

[15] Exs. 25 and 26 reappear in Raghunātha Rāja's comm. on $\bar{A}$, ii. 9.

[16] See footnote 15.

The figure contemplated is a double trapezium obtained by placing two equal trapeziums in juxtaposition in such a way that the smaller of the two parallel sides of the trapeziums forms the central width of the double trapezium. The formula used by Bhāskara I for the area of this figure is

$$\text{area} = \frac{1}{2}\left(\frac{a+b}{2}+c\right)\times l$$

where a, b are the lengths of the parallel faces, l the distance between them, and c the central width.

On the determination of the area of a figure resembling the tusk of an elephant

Ex. 27. "The width (at the base) is stated to be 5 (units), the belly (i.e., inner curved side) is 9 (units), and the back (i.e., outer curved side) is 15 (units). Say, what is the area of this (figure resembling the) tusk of an elephant."[17] [*Ā*, ii. 9]

The figure envisaged is a curvilinear triangle, bounded by a straight base and two curved sides curved in the same direction. The formula used by Bhāskara I for the area of such a figure is

$$\text{area} = \frac{a}{2}\times\frac{b+c}{2},$$

where a is the base and b, c the curved sides.

On the area of a circle

Ex. 28. "Calculate, O friend, according to the *Gaṇita* (of Āryabhaṭa), the nearest approximations to the areas of the circles whose diameters are 2, 4, 7, and 8 respectively." [*Ā*, ii. 10]

On the determination of the diameter of a circle from the given circumference

Ex. 29. "Calculate and tell me the diameters of the circles whose peripheries are 3299 minus $\frac{8}{25}$ and 216000 respectively." [*Ā*, ii. 10]

On the determination of the local latitude from the midday shadow of the gnomon

Ex. 30. "When at an equinox the Sun is on the meridian, the shadow of a gnomom, divided into 12 units, on level ground is seen to be 5, 9, and $3\frac{1}{2}$ (units at three different places). (Find the latitudes of those places)."

[17] Ex. 27 reappears in Raghunātha Rāja's comm. on *Ā*, ii. 9.

[$\bar{A}$, ii. 14]

Ex. 31. "The shadow of the gnomon of 15 *aṅgulas* at midday on an equinox is (seen to be) 6 plus $\frac{1}{4}$ *aṅgulas*. Give out the Rsines of the latitude and the co-latitude." [$\bar{A}$, ii. 14]

Ex. 32. "Say what is the distance of the Sun, whose rays are (profusely) spread all round, from the zenith, when the shadow of a gnomon of 30 *aṅgulas* is observed to be 16 *aṅgulas*."[18] [$\bar{A}$, ii. 14]

On the shadow of a gnomon due to a lamp-post

Ex. 33. "Tell (me the length of) the shadow situated at a distance of 80 (*aṅgulas*) from the foot of the lamp-post of height 72 (*aṅgulas*); and also that of another gnomon situated at a distance of 20 (*aṅgulas*) from a lamp-post of height 30 (*aṅgulas*)."[19] [$\bar{A}$, ii. 15]

Ex. 34. "Say what is the distance of the foot of the lamp-post of height 72 (*aṅgulas*) from the gnomon of 12 (*aṅgulas*) if the shadow (cast by the gnomon) is 16 (*aṅgulas*)."[20] [$\bar{A}$, ii. 15]

Ex. 35. "The shadow of a gnomon, situated at a distance of 50 (*aṅgulas*) from the foot of a lamp-post, is 10 (*aṅgulas*). Say what is the height of the lamp."[21] [$\bar{A}$, ii. 15]

Ex. 36. "(The lengths of) the shadows of two equal gnomons (of 12 *aṅgulas*) are seen to be 10 and 16 (*aṅgulas*) respectively; the distance between the shadow-ends is seen to be 30 (*aṅgulas*). Give out the upright and the base for each (gnomon)."[22] [$\bar{A}$, ii. 16]

The "base" means "the height of the lamp-post" and the "upright" means "the distance of the shadow-end from the foot of the lamp-post". The two gnomons are assumed to be in the same line as seen from the lamp-post.

Ex. 37. "(The lengths of) the shadows of two equal gnomons (of 12 *aṅgulas*) are stated to be 5 and 7 (*aṅgulas*) respectively. The distance between the shadow-ends is observed to be 8 (*aṅgulas*). Give out the base and the upright." [$\bar{A}$, ii. 16]

[18] By saying that the rays of the Sun are profusely spread it is stated that it is midday.
[19] Ex. 33 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 15.
[20] Ex. 34 reappears in the commentaries of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 15.
[21] Ex. 35 reappears in the commentary of Raghunātha Rāja on $\bar{A}$, ii. 16.
[22] Ex. 36 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 16.

On the so called Pythagoras' theorem

Ex. 38. "Give out the hypotenuses (for three right-angled triangles) where the bases and the uprights are 3 and 4, 6 and 8, and 12 and 9 (units) respectively." [$\bar{A}$, ii. 17 (i)]

On the following property of the circle: "If the diameter ABC and the chord LBM of a circle intersect at right angles, then $LB^2 = AB \times BC$," AB and BC being called the arrows and LB the Rsine

Ex. 39. "In a circle of diameter 10 (units), the arrows (i.e., segments of a diameter) are seen by me to be 2 and 8 (units); in the same circle, another set of arrows are 9 and 1 (units). Tell (me) the corresponding Rsines."[23] [$\bar{A}$, ii. 17 (ii)]

The Hawk-and-Rat problems

Ex. 40. "A hawk is siting at the top of a rampart whose height is 12 cubits. The hawk sees a rat at a distance of 24 cubits away from the foot of the rampart; the rat, too, sees the hawk. Thereupon the rat, for fear of him, hastens to his own dwelling situated at (the foot of) the rampart but is killed in between by the hawk who came along a hypotenuse (i.e., along an oblique path). I want to know the distance traversed by the rat and also the (horizontal) motion of the hawk (the speeds of the two being the same)."[24] [$\bar{A}$, ii. 17 (ii)]

Ex. 41. "A hawk is sitting on a pole whose height is 18 (cubits). A rat, who has gone out of his dwelling (at the foot of the pole) to a distance of 81 (cubits), while returning towards his dwelling, afraid of the hawk, is killed by the cruel (bird) on the way. Say how far has he gone towards his hole, and also the (horizontal) motion of the hawk (the speeds of the rat and the hawk being the same)."[25] [$\bar{A}$, ii. 17 (ii)]

The above two examples (Exs. 40 and 41) have been called the "hawk-and-rat problems". Bhāskara I ascribes such problems to previous writers. He writes: "At this very place they narrate the hawk-and-rat problems."

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[23] Ex. 39 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 17.

[24] Ex. 40 reappears in Raghunātha Rāja's comm. on $\bar{A}$, ii. 17. A similar example occurs in Pṛthūdaka's comm. on *BrSpSi*, xii. 41. See H.T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmegupta and Bhāscara*, London (1817), p. 309, footnote.

[25] Ex. 41 reappears in Yallaya's comm. on $\bar{A}$, ii. 17.

The Hindu method for solving such problems has been explained by Bhāskara I in detail. Following that method, Ex. 41 may be solved as follows:

Draw a circle with centre at O. Let $ABOC$ be the horizontal diameter and LBM a vertical chord intersecting the diameter at B. Imagine that BL is the pole and BC the track of the rat. The hawk is sitting at L and the rat is at C. They see each other. The rat then runs to his hole at B but is killed by the hawk at O, the distance traversed by the hawk (i.e., LO) and by the rat (i.e., CO) being the same.

It is given that $LB = 18$ cubits, and $BC = 81$ cubits. Since $LB^2 = AB \times BC$, therefore $AB = 4$ cubits. Therefore,

$$BO = \frac{1}{2}(BC - AB) = 38\frac{1}{2} \text{ cubits,}$$

$$\text{and} \quad CO = \frac{1}{2}\,(BC + AB) = 42\frac{1}{2} \text{ cubits.}$$

Hence, the distance traversed by the rat is $42\frac{1}{2}$ cubits and the horizontal motion of the hawk is $38\frac{1}{2}$ cubits.

It is interesting to note that Yallaya and Raghunātha Rāja have prescribed the same method for solving the hawk-and-rat problems as described above. The peacock-and-serpent problems given by Bhāskara II, Yallaya, and Raghunātha Rāja are similar to the hawk-and-rat problems.

The Bamboo problems

Ex. 42. "A bamboo of height 18 (cubits) is felled by the wind. It falls at a distance of 6 (cubits) from the root, thus forming a (right-angled) triangle. Where is the break?"[26] [$\bar{A}$, ii. 17 (ii)]

Ex. 43. "A bamboo of 16 cubits is felled by the wind; it falls at a distance of 8 cubits from its root. Say where has it been broken by the wind."

[$\bar{A}$, ii. 17 (ii)]

In the case of the bamboo problems like Exs. 42 and 43, BC (in the figure of Ex. 41) is taken to represent the bamboo which breaks at O and reaches the ground (BL) at L. To find the height of the break, we have to obtain the length BO. As before $BO = \frac{1}{2}(BC - AB)$, where $AB = \frac{LB^2}{BC}$.

Ex. 42 is found to occur in Pṛthūdaka's commentary on the *Brāhmasphuṭa-siddhānta* of Brahmagupta.[27] His method of solution is the same as used by

[26] Ex. 42 reappears in Pṛthūdaka's comm. on *BrSpSi*, xii. 41 and in Raghunātha Rāja's comm. on $\bar{A}$, ii. 17.

[27] See H.T. Colebrooke, *l.c.*, p. 309, footnote.

Bhāskara I.[28] Similar problems are also found to occur in the *Gaṇitasāra-saṅgraha*[29] of Mahāvīra, the *Līlāvatī*[30] and the *Bījagaṇita*[31] of Bhāskara II, and the *Gaṇitakaumudī*[32] of Nārāyaṇa.

The Lotus problems

Ex. 44. "A full blown lotus of 8 *aṅgulas* is seen (just) above the water. Being carried away by the wind it just submerges at a distance of one cubit. Quickly say the height of the lotus plant and the depth of the water."[33]

[*Ā*, ii. 17 (ii)]

Ex. 45. "A lotus flower of 6 *aṅgulas* just dips (into the water) when it advances through a distance of 2 cubits. I want to know the height of the lotus plant and the depth of the water."[34] [*Ā*, ii. 17 (ii)]

Consider a circle with centre at O. Let $ABOC$ be its vertical diameter and LBM a horizontal chord intersecting the vertical diameter at B.

In the case of the lotus problems, the horizontal diameter of the circle is supposed to denote the mud-level; the chord LBM the water-level; O is supposed to be the root of the lotus plant, OB the lotus stalk, AB the lotus flower, and L and M the points where the lotus flower just dips into the water. Then,

$$OA \text{ (i.e., height of lotus plant)} = \frac{1}{2}(BC + AB),$$

where

$$BC = \frac{LB^2}{AB}; \quad \text{and} \quad OB \text{ (i.e., depth of water)} = \frac{1}{2}(BC - AB).$$

The Crane-and-Fish Problems

Ex. 46. "There is a reservoir of water of dimensions 6 × 12. At the east-north corner thereof there is a fish; and at the west-north corner there

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[28] See B. Datta, "On the supposed indebtedness of Brahmagupta to *Chiu-chang Suan-shu*," *Bull. Cal. Math. Soc.*, vol. xxii, p. 41.

[29] vii. $191\frac{1}{2}$–$192\frac{1}{2}$.

[30] See *L* (*Ānandāśrama* Sanskrit Series) p. 141.

[31] See *Bījagaṇita*, ed. by Sudhakara Dvivedi and Muralidhara Jha, Banaras (1927), p. 57.

[32] *Kṣetra-vyavahāra*, Ex. 26.

[33] Ex. 44 reappears in Pṛthūdaka's comm. on *BrSpSi*, xii. 41. (Colebrooke, *l.c.*, p. 309, footnote), and in the comm. of Yallaya and Raghunātha Rāja on *Ā*, ii. 17.

[34] Similar examples occur in the works of Bhāskara II (*L*, Ex. 155, p. 145; *BBi*, Ex. 112) and Nārāyaṇa (*GK, Kṣetra-vyavahāra*, Ex. 28).

Problems similar to Exs. 44 and 45 are reported to occur in a Chinese work called *Chiu-chang Suan-shu*, but the Chinese solution to those problems is quite different from that of Bhāskara I. The Hindu solution is based on the property of right-angled triangles which was known in India as early as the Vedic period.

is a crane. For fear of him (i.e., of the crane) the fish, crossing the reservoir, hurriedly went towards the south in an oblique direction but was killed by the crane who came along the sides of the reservoir. Give out the distances travelled by them (assuming that their speeds are the same)."[35] [$\bar{A}$, ii. 17 (ii)]

Ex. 47. "There is a reservoir of water of dimensions 12×10. At the east-south corner there is a crane and at the east-north corner there is a fish. (The crane walks along the sides of the reservoir and the fish swims obliquely). Say, on reaching which point of the western side of the reservoir is the fish killed by the crane."[36] [$\bar{A}$, ii. 17 (ii)]

Following the method of Bhāskara I, the first of the above two examples (i.e., Ex. 46) may be solved as follows:

Let $LBQP$ be the reservoir in which $BQ = LP = 12$, and $LB = PQ = 6$. Also suppose that LB is the east side, PQ the west side, LP the north side, and BQ the south side of reservoir. Initially the fish is at L and the crane at P. After some time the fish swimming along LO reaches O, a point in BQ. In the same time the crane, walking along PQ and then along QB, also reaches O and kills the fish. The speeds of the fish and the crane being the same, $LO = PQ + QO$. Let OC (along OQ produced) be equal to OL. Then the circle drawn with O as centre and OL as radius must pass through C, and we have

$$BC = BQ + PQ = 12 + 6 = 18.$$

If CB produced intersects the circle at A, then

$$AB = \frac{LB^2}{BC} = \frac{36}{18} = 2.$$

Hence, $AC = AB + BC = 20$ giving $OL = 10$. Therefore, the distances traversed by the fish and the crane are each equal to 10.

Proceeding as above, it can be shown that the point required in Ex. 47 divides the western side of the reservoir in the ratio $8\frac{8}{11} : 3\frac{3}{11}$.

An example similar to the above two occurs in the *Gaṇitakaumudī* of Nārāyaṇa. See *kṣetra-vyavahāra*, pp. 38–39, Ex. 29–31.

On the determination of the arrows of the intersecting arcs of the Moon and the shadow when the portion eclipsed is given

Ex. 48. "When 8 out of 32 of (the diameter of) the Moon are eclipsed by the shadow of diameter 80, I want to know then what are the arrows of (the

[35] Ex. 46 reappears in the comm. of Raghunātha Rāja on $\bar{A}$, ii. 17 (ii). A similar example occurs in the comm. of Yallaya also.

[36] Ex. 47 reappears in Raghunātha Rāja's comm. on $\bar{A}$, ii. 17 (ii).

intersecting arcs of) the shadow and the full Moon."[37] [*Ā*, ii. 18]

On the determination of the middle term and the sum of a series in A.P.

Ex. 49. "In a series (in A.P.) the first term is seen to be 2; the successive increase is stated to be 3; and the number of terms is stated to be 5. Tell (me) the middle term and the sum of the series."[38] [*Ā*, ii. 19]

Ex. 50. "In a series (in A.P.) in which the first term is 8, the successive increase is stated to be 5 and the number of terms is seen to be 18. Give out the middle terms and the sum of the series." [*Ā*, ii. 19]

On the determination of the desired term of a series in A.P.

Ex. 51. "(In a series in A.P.) in which the successive increase is 11 and the first term 7, the number of terms is 25. Quickly say the ultimate and penultimate terms of that series and also say what is the twentieth term."[39] [*Ā*, ii. 19]

On the determination of partial sums of a series in A.P.

Ex. 52. "In the month of *Kārtika* a certain king daily gives away some money (in charity) starting with 2 on the first day (of the month) and increasing that by 3 per day. Fifteen days having passed away, there arrived a Brāhmana well-versed in the Vedas. The amount for the next ten days was given to him; that for the (remaining) five days (of the month), to someone else. Say what do the last two persons get." [*Ā*, ii. 19]

Ex. 53. "(In a series in A.P.) in which the first term is 15, the successive increase is stated to be 18 and the number of terms 30. Quickly calculate the sum of the ten middle terms (of that series)." [*Ā*, ii. 19]

On the determination of the sum of a series in A.P. when the first term, the last term, and the number of terms are given

Ex. 54. "(Of 11 conch-shells which are arranged in the increasing order of their prices which are in A.P.) the first conch-shell is acquired for 5

[37] Ex. 48 reappears in Mahāvīra's *Gaṇitasārasaṅgraha*. See *GSS*, vii. $232\frac{1}{2}$. A similar example occurs also in the commentary of Sūryadeva on *Ā*, ii. 18.

[38] Exs. 49 and 51 reappear in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on *Ā*, ii. 19.

[39] See footnote 38.

and the last for 95. Say what is the price of all the 11 conch-shells."[40]
[$\bar{A}$, ii. 19]

Ex. 55. "(In an arithmetic series) the first term is stated to be 1. The last term is declared by the learned to be 100; the same is also stated to be the number of terms. What is the sum of all the terms (of that series)?"
[$\bar{A}$, ii. 19]

On the determination of the number of terms of an arithmetic series when the first term, the common difference, and the sum of the series are given

Ex. 56. "In a series (in A.P.) the first term is stated to be 5; the successive increase is 7 and the sum 95. Say what is the number of terms thereof."
[$\bar{A}$, ii. 20]

Ex. 57. "(In an arithmetic series) in which the successive increase and the first term are 9 and 8 respectively, the sum is stated to be 583. Tell (me) the number of terms found by you." [$\bar{A}$, ii. 20]

On the sum of the series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$

Ex. 58. "There are (three pyramidal) piles (of balls) having respectively 5, 8, and 14 layers which are triangular. Tell me the number of units (balls) (in each of them)."[41] [$\bar{A}$, ii. 21]

In the topmost layer of the pyramidal piles, there is 1 ball; in the second layer from the top, there are $1 + 2 = 3$ balls; in the third layer, there are $1+2+3 = 6$ balls; in the fourth layer, there are $1+2+3+4 = 10$ balls; and so on. Every layer is in the form of a triangle.

The number of balls in the first pile having five layers

$$= 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + 4 + 5)$$
$$= \frac{5 \times 6 \times \ 7}{6} \quad \text{or} \quad 35.$$

Similarly, the number of balls in the other two piles are 120 and 560 respectively.

The series $1 + 2 + 3 + \ldots$ has been called by Āryabhaṭa I by the name *citi* or *upaciti* and the series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$ by the name *citighana*. Their sums are also called by the same terms. Bhāskara I calls the

[40]Ex. 54 reappears in Yallaya's comm. on $\bar{A}$, ii. 19.

[41]Ex. 58 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 21. Also see *GSS*, *miśraka-vyavahāra*, Ex. $331\frac{1}{2}$.

sum of the former by the term *saṅkalanā* and that of the latter by the term *saṅkalanā-saṅkalanā*. In the *Brāhmasphuṭasiddhānta* (xii. 19) and other later works, they are called a *saṅkalita* and *saṅkalita-saṅkalita* respectively.

On the determination of the sum of the series $1^2 + 2^2 + 3^2 + \dots$ to any number of terms

Ex. 59. "There are (three pyramidal) piles on square bases having 7, 8, and 17 layers which are also squares. Say the number of units therein (i.e., the number of bricks of unit size used in each of them)."[42] [$\bar{A}$, ii. 22]

In the topmost layer there is one brick, in the next there are four, in the next nine, and so on. The number of bricks used in the three piles are 140, 204, and 1785 respectively.

The sum of the series $1^2 + 2^2 + \cdots + n^2$ has been called *vargacitighana* by Āryabhaṭa I. Bhāskara I calls it *vargasaṅkalanā*. It is generally known as *vargasaṅkalita*.[43]

On the determination of the sum of the series $1^3 + 2^3 + 3^3 + \cdots + n^3$

Ex. 60. "There are (three pyramidal) piles having 5, 4, and 6 cubodial layers. They are constructed of cubodial bricks (of unit dimensions) with one brick in the topmost layer. (Find the number of bricks used in each of them)."[44] [$\bar{A}$, ii. 22]

There is 1^3 brick in the topmost layer, 2^3 bricks in the next layer, 3^3 bricks in the next, and so on. The number of bricks in the three piles are 225, 100, and 441 respectively.

The sum of a series of cubes of natural numbers has been called *ghanacitighana* by Āryabhaṭa I. Bhāskara I calls it *ghanasaṅkalanā*. In later works it is called *ghanasaṅkalita*. The same term has been used by Brahmagupta.[45]

On finding the product of two given numbers by the formula

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$$

Ex. 61. "What are the products of 5 and 4, of 7 and 9, and of 8 and 10? Quickly say separately." [$\bar{A}$, ii. 23]

[42] Ex. 59 reappears in Sūryadeva's comm. on $\bar{A}$, ii. 22.
[43] See e.g. *BrSpSi*, xii. 20.
[44] Ex. 60 reappears in Yallaya's comm. on $\bar{A}$, ii. 22.
[45] See *BrSpSi*, xii. 20.

On the determination of two numbers whose difference and product are known

Ex. 62. "The product (of two numbers) is clearly seen to be 8; their difference is 2. (Of other two numbers) the product being 18, the difference is 7. Tell (me) the numbers multiplied in the two cases." [$\bar{A}$, ii. 24]

On interest

Ex. 63. "I do not know the interest on 100, but I do know that the interest plus interest on interest accruing on 100 in 4 months is 6. Give out the monthly interest on 100."[46] [$\bar{A}$, ii. 25]

Ex. 64. "I do not know the monthly interest on 25 (*rūpas*). But the monthly interest on 25 (*rūpas*) lent out at the same rate (of interest) is seen to amount to 3 (*rūpas*) minus $\frac{1}{5}$ in 5 months. I want to know the monthly interest on 25 (*rūpas*) as also the interest for 5 months on the interest of 25 (*rūpas*)." [$\bar{A}$, ii. 25]

Ex. 65. "The monthly interest on 100 (*rūpas*) is not known, but the interest on 100 (*rūpas*) lent out elsewhere (at the same rate of interest) is seen to amount with interest thereon to 15 (*rūpas*) in 5 months. I want to know — what is the interest on 100 (*rūpas*) as also what is the interest that accrued in 5 months on 100 (*rūpas*)?" [$\bar{A}$, ii. 25]

On the rule of three

Ex. 66. "5 *palas* of sandalwood are purchased by me for 9 *rūpakas*. How much of sandalwood will, then, be purchased for one *rūpaka*?"[47]

[$\bar{A}$, ii. 26–27 (i)]

Ex. 67. "If one *bhāra* of ginger is sold for 10 plus $\frac{1}{5}$ (*rūpakas*), tell me quickly the price of 100 plus $\frac{1}{2}$ *palas* of ginger."[48] [$\bar{A}$, ii. 26–27 (i)]

Ex. 68. "$1\frac{1}{2}$ *palas* of musk are had for 8 plus $\frac{1}{3}$ (*rūpakas*). Let Kṛtavīrya find out how much of musk will be had for 1 plus $\frac{1}{5}$ (*rūpakas*)."[49]

[$\bar{A}$, ii. 26–27 (i)]

Ex. 69. "A serpent of 20 cubits in length enters into a hole, moving forward at the rate of $\frac{1}{2}$ of an *aṅgula* per *muhūrta*[50] and backward at the rate of

[46] Ex. 63 reappears in the commentaries on $\bar{A}$, ii. 25 of Yallaya and Raghunātha Rāja.
[47] Ex. 66 reappears in Yallaya's comm. on $\bar{A}$ ii. 26–27 (i).
[48] 1 *bhāra*=2000 *palas*.
[49] Ex. 68 reappears in Yallaya's comm. on $\bar{A}$, ii. 26–27 (i).
[50] 1 *muhūrta* = 48 minutes.

$\frac{1}{5}$ of an *aṅgula* (per *muhūrta*): in how many days does he get into the hole completely?"[51] [*Ā*, ii. 26–27 (i)]

On proportion and partnership

Ex. 70. "(Out of 11 cattle) 8 are tamed and 3 untamed — so are the cattle described. Out of 1001 cattle, then, how many are tamed and how many untamed?"[52] [*Ā*, ii. 26–27 (i)]

Ex. 71. "15 merchants collaborate (in a business); the capitals invested by them are in A.P. with 1 as the first capital and also 1 as the successive increase. The profit that accrued (on the whole capital) amounts to 1000. Say what should be given to whom." [*Ā*, ii. 26–27 (i)]

Ex. 72. "The combined profit of three merchants, whose investments are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{8}$ respectively, amounts to 70 minus 1. What is whose profit (individually)?"[53] [*Ā*, ii. 26–27 (i)]

On the rule of five

Ex. 73. "Given that 100 increases by 5 in a month, say, if you are versed in (Ārya)bhaṭa's *Gaṇita*, by how much will 20 increase in 6 months."[54].

[*Ā*, ii. 26–27 (i)]

Ex. 74. "100 invested for two months increases by 5; by how much will 25 invested for two months increase?" [*Ā*, ii. 26–27 (i)]

Ex. 75. "If $4\frac{1}{2}$ *rūpakas* be the increase (interest) on 100 (*rūpakas*) for $3\frac{1}{2}$ months, what will be the increase on 50 (*rūpakas*) for 10 months?"[55]

[*Ā*, ii. 26–27 (i)]

Ex. 76. "A sum of 20 plus $\frac{1}{2}$ (*rūpakas*) increases by 1 plus $\frac{1}{3}$ *rūpakas* in 1 plus $\frac{1}{5}$ months. (Say) after carefully understanding "the method of elimination of divisors" from the aphorism of the (*Ārya*)*bhaṭa-tantra*,

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[51] Ex. 69 reappears in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26. Raghunātha Rāja has, however, put the example in a slightly different form. A similar example is found to occur in the *Bakhshālī Manuscript*. *Cf.* G.R. Kaye, *Bakhshālī Manuscript*, Arch. Survey of India, New Imperial Series, Vol. XLIII, Parts I and II, 1927, Ex. 99, p. 51.

[52] Exs. 70 and 71 reappear in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26.

[53] Ex. 72 reappears in Yallaya's comm. on *Ā*, ii. 26.

[54] Ex. 73 reappears in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26.

[55] Ex. 75 reappears in Yallaya's comm. on *Ā*, ii. 26.

what will be the increase of 7 minus $\frac{1}{4}$ (*rūpakas*) in 6 plus $\frac{1}{10}$ months."[56]
[$\bar{A}$, ii. 26–27 (i)]

On the rule of seven

Ex. 77. "If 9 *kuḍavas* of pure parched and flattened rice are obtained daily for an elephant whose height is 7 (cubits), periphery 30 (cubits), and length 9 (cubits), say how much of parched and flattened rice will be obtained for an elephant whose height is 5 (cubits), length 7 (cubits), and periphery 28 (cubits)."[57] [$\bar{A}$, ii. 26–27 (i)]

Ex. 78. "If 2 and a half *kuḍavas* of kidney beans (*māṣa*) are obtained for an excellent elephant whose height is 4 cubits, length 6 (cubits), and breadth 5 (cubits), how much should be obtained for an elephant whose height is 3 (cubits), length 5 (cubits), and breadth $4\frac{1}{2}$ (cubits)?[58]
[$\bar{A}$, ii. 26–27 (i)]

On inverse proportion

Ex. 79. "When one *pala* is equivalent to 5 *suvarṇas*, a certain quantity of gold weighs 16 *palas*, what will the same gold weigh when one *pala* is equivalent to 4 *suvarṇas*?"[59] [$\bar{A}$, ii. 26–27 (i)]

Ex. 80. "8 baskets are seen (to contain the whole grain) when each (basket) contains 14 *prasṛtis*[60] (of grain); say how many baskets would be (required) when each (basket) can contain 8 *prasṛtis* (of grain) (only)."[61]
[$\bar{A}$, ii. 26–27 (i)]

On the simplification of fractions

Ex. 81. "$\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, and $\frac{1}{4}$ being respectively added together (two at a time), say what is the aggregate."[62] [$\bar{A}$, ii. 27(ii)]

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[56] Exs. 76 and 77 reappear in the commentaries of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 26.

[57] See footnote 56.

[58] After solving this example, Bhāskara I adds: "Similarly, (the rules of Āryabhaṭa I) should be applied to problems involving nine quantities or more." This shows that the so called rules of nine and eleven, etc. were well known in the time of Bhāskara I.

[59] Ex. 79 reappears in Yallaya's comm. on $\bar{A}$, ii. 26.

[60] *Prasṛti* is a measure of grain, equivalent to one handful. According to *Anuyogadvārasūtra*, 2 *prasṛtis* are equivalent to 1 *setikā*. See Section 2 (6) of this paper. (**ed.** in Part I of this paper.)

[61] Ex. 80 reappears in Yallaya's comm. on $\bar{A}$, ii. 26.

[62] Ex. 81, in different words, is found to occur in Pṛthūdaka's comm. on *BrSpSi*, xii. 8.

Ex. 82. "What are the sums of $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$, and $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{20}$ and $\frac{1}{5}$?" [$\bar{A}$, ii. 27(ii)]

Ex. 83. "Calculate, O mathematician, what the following sums amount to: $\frac{1}{2}$ minus $\frac{1}{6}$; $\frac{1}{5}$ minus $\frac{1}{7}$; and $\frac{1}{3}$ minus $\frac{1}{4}$." [$\bar{A}$, ii. 27(ii)]

On the method of inversion

Ex. 84. "A number is multiplied by 2; then increased by 1; then divided by 5; then multiplied by 3; then diminished by 2; and then divided by 7: the result (thus obtained) is 1. Say what is the initial number."[63] [$\bar{A}$, ii. 28]

Ex. 85. "What is that number which when multiplied by 3, then diminished by 1, then halved, then increased by 2, then divided by 3 and finally diminished by 2, yields 1?" [$\bar{A}$, ii. 28]

3.2 Examples on Algebra

On simultaneous linear equations

Ex. 86. "In a forest there are (four) herds of elephants consisting (severally) of elephants in rut, elephants not in rut, female elephants, and young elephants. The sums of the elephants in the four herds excepting one (herd) at a time are known to be 30, 36, 49, and 50 (respectively). Correctly state the total number of elephants and also the number in each herd separately."[64] [$\bar{A}$, ii. 29]

Ex. 87. "The Sums of the numbers of elephants, horses, goats, asses, camels, mules, and cows neglecting one of those animals at a time, are respectively 28, and the same number (i.e. 28) successively diminished by 1, the last number (thus obtained) being further diminished by 1. If you have read the whole of the (chapter on) *Gaṇita* composed by Āryabhaṭa from a teacher, correctly state the total number of the animals and also the numbers of the different animals separately."[65] [$\bar{A}$, ii. 29]

[63] Ex. 84 reappears in the commentaries of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 28.

[64] Ex. 86 reappears in the commentaries of Sūryadeva and Raghunātha Rāja on $\bar{A}$, ii. 29. It requires the solution of the simultaneous equations: $x_2+x_3+x_4=30$, $x_3+x_4+x_1=36$, $x_4+x_1+x_2=49$, $x_1+x_2+x_3=50$, where x_1, x_2, x_3, and x_4 denote the numbers of animals in the four herds. See B. Datta and A. N. Singh, *History of Hindu Mathematics*, Part II, pp. 47 ff.

[65] If x_1, x_2, x_3, x_4, x_5, x_6, and x_7 are the numbers of the various animals and S their sum, then we have to solve the simultaneous equations: $S-x_1=28$, $S-x_2=27$, $S-x_3=26$, $S-x_4=25$, $S-x_5=24$, $S-x_6=23$, $S-x_7=21$.

On simple equations

Ex. 88. "(There are two merchants.) With the first merchant are seen by me 7 stout horses bearing auspicious marks and money amounting to 100 (*rūpakas*) in hand; with the second (merchant) there are 9 horses and money amounting to 80 (*rūpakas*). If the two merchants be equally rich and the price of each horse be the same, tell (me) the price of one horse and also the equal wealth (with them)."[66] [$\bar{A}$, ii. 30]

Ex. 89. "A certain person has 8 *palas* of saffron and money amounting to 90 *rūpakas*; another person possesses 12 *palas* of saffron and 30 *rūpakas*; (and the two persons are equally rich). If the two persons have bought the saffron at the same rate per *pala*, I want to know the price of one *pala* (of saffron) and also the equal wealth of the two." [$\bar{A}$, ii. 30]

Ex. 90. "7 *yāvattāvat* + 7 *rūpaka* = 2 *yāvattāvat* + 12 *rūpaka*. What is the value of 1 *yāvattāvat*?" [$\bar{A}$, ii. 30]

Ex. 91. "9 *gulikā* + 7 *rūpaka* = 3 *gulikā* + 3 *rūpaka*. What is the price of 1 *gulikā*?" [$\bar{A}$, ii. 30]

Ex. 92. "9 *gulikā* − 24 *rūpaka* = 2 *gulikā* + 18 *rūpaka*. Say what is the price of 1 *gulikā*." [$\bar{A}$, ii. 30]

Ex. 93. "One (man) goes from Valabhī at the speed of $1\frac{1}{2}$ *yojanas* a day; another (man) comes (along the same route) from Harukaccha at the speed of $1\frac{1}{4}$ *yojanas* a day. The distance between the two (places) is known to be 18 *yojanas*. Say, O mathematician, after how much time (since start) they meet each other."[67] [$\bar{A}$, ii. 31]

Ex. 94. "One man goes from Valabhī to the Ganges at the speed of $1\frac{1}{2}$ *yojanas* a day, and another from Śivabhāgapura at the speed of $\frac{2}{3}$ *yojanas* a day. The distance between the two (places) has been stated by the learned to be 24 *yojanas*. If they travel along the same route, after how much time will they meet (each other)?"[68] [$\bar{A}$, ii. 31]

[66] Similar examples occur in Raghunātha Rāja's comm. on $\bar{A}$, ii. 30.

[67] If t denotes the required time in days, then $1\frac{1}{2}t + 1\frac{1}{4}t = 18$, giving $t = 6\frac{6}{11}$ days.

[68] If t denotes the required time in days, then

$$1\frac{1}{2}t - \frac{2}{3}t = 24,$$

giving $t = 28\frac{4}{5}$ days. Exs. 93 and 94 reappear in Raghunāntha Rāja's comm. on $\bar{A}$, ii. 31.