The evection and the deficit of the equation of the centre of the Moon in Hindu astronomy *

Section I

1. Dhirendranath Mukhopadhyaya (1930) published a paper entitled "The Evection and the Variation of the Moon in Hindu Astronomy" wherein he showed that the Hindu astronomer Mañjula knew of a lunar correction which is equivalent to the deficit of the equation of the centre and the evection. P. C. Sengupta (1932) published another paper entitled "Hindu Luni-solar Astronomy" in which, among other things, he considered various formulae regarding this dual correction as given by Mañjula (932), Śrīpati (1039), and Candra Śekhara Siṃha (latter half of the 19th century). None of these papers, however, contains a complete or systematic study of this correction and in consequence some errors have crept in. The object of the present paper is to exhibit the central idea underlying the corrections prescribed by various Hindu authors and to explain them more thoroughly in the light of further investigations in the field of Hindu astronomy.

2. The discovery of this correction is one of the greatest achievements of the Hindus in the field of practical astronomy. Early Hindu astronomers made observations and recorded the differences between the observed and computed positions of the heavenly bodies. As early as Vedic times, the Hindus performed sacrifices when the planets occupied specified positions in the heavens. This practice continued for thousands of years. The record of such observations served as the basis for the foundation of the Hindu theoretical astronomy and later on supplied material on the basis of which corrections were made and refinements introduced from time to time. These observations continued over long periods led to the discovery of the above lunar inequality as well as of all the other inequalities.

3. Due to the fact that a good deal of the early Hindu astronomical literature has not been preserved, it is impossible to locate the exact date when the

* K. S. Shukla, *Proceedings of the Benares Mathematical Society*, New Series, Vol. 7, No. 2 (1945), pp. 9–28.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_33

lunar inequality was first detected in India and to trace a regular theoretical history of this subject. The available formulae giving this correction exhibit an advanced state of the subject and it is our belief that they must have taken centuries to develop.

4. At present this correction can be traced back to the time of Vaṭeśvara,[1] the well-known critic of Brahmagupta[2] (628). The works of Vaṭeśvara are not available to us, but from Yallaya's commentary (1482) on the *Laghumānasa* (932) we learn that the *Vaṭeśvarasiddhānta* contained this correction. In his commentary on the *Laghumānasa*, i(c), 1–2, Yallaya has actually quoted Vaṭeśvara's version of this correction. This is as follows:

> एकादशभिर्भागैर्विवर्जितैः शुद्धचन्द्रगतिभागैः ।
> स्फुटसूर्यात् चन्द्रोच्चं त्यक्त्वा तत्कोटिजीवा या ॥
> गुणिता स्याद्गुणकारैर्धनर्णसंज्ञां प्रयात्येषा ।
> शुद्धेन्दौ स्फुटसूर्यं विशोध्य कोटिज्यकां भुजज्यां वा ॥
> ज्ञात्वा तयोर्धनाख्यामृणसंज्ञां वा यथोचितां कृत्वा ।
> भुजकोटिज्ये गुणिते तेन गुणेनैव ते भुजे क्रमशः ॥
> रूपेण पञ्चभिर्ये लिप्त्याद्ये शीतगोश्च तद्भुक्तौ ।
> भवति फलं शशिलिप्त्यां गुणकभुजातुल्यभिन्ननामयुतौ ॥
> कुर्याद्रूपाप्तं यत् धनमृणमिन्दोः क्रमाद्भुक्त्याम् ।
> भिन्नाशाख्यौ स्यातां कोटिगुणौ तद्धनं क्षयं कुर्यात् ॥[3] By the multiplier obtained by subtracting eleven degrees from the Moon's true daily motion, in degrees (*bhāga*), multiply the Rcosine of the Sun's true longitude *minus* the longitude of the Moon's apogee (*ucca*). This comes positive or negative. (Next) having subtracted the Sun's true longitude from the Moon's true longitude and having obtained the Rsine and Rcosine thereof, and (then) having properly ascertained their signs—positive or negative—, multiply the Rsine and Rcosine (thus obtained) by the (previous) product (or epicyclic multiplier). The results should be respectively divided by 1 and 5 and applied as correction, in minutes (*liptās*), to the Moon's true longitude and true daily motion (in the following manner): The

[1] According to Śaṅkara Bālakṛṣṇa Dīkṣita, Vaṭeśvara's time is 899 AD.

[2] In his *Gaṇakataraṅgiṇī*, Sudhākara Dvivedī writes:

> यथा ब्रह्मगुप्तेनार्यभटादीनां खण्डनं कृतं तथैव वटेश्वरेण स्वसिद्धान्ते बहुत्र ब्रह्मगुप्तखण्डनं कृतमस्ति।

[3] Dattarāja gives the last $1\frac{1}{2}$ verses thus:

> भवति फलैः शशिलिप्त्यां गुणकभुजातुल्यभिन्ननामयुतौ ॥
> कुर्याद्रूपाप्तं यत् धनमृणमिन्दोः क्रमाल्लिप्ताम् ।
> भिन्नाशाख्यौ स्यातां कोटिगुणा तद्धनं क्षयं कुर्यात् ॥

Cf. Ketakī-grahagaṇitam, p. 128.

> result which is obtained on dividing by one should be applied as a positive or negative correction to the minutes of the Moon according as the multiplier and the Rsine are of like or unlike signs; (and) the product of the Rcosines is to be applied as a positive or negative correction to the Moon's true daily motion provided their signs satisfy the contrary condition.

5. This correction is also found in the *Laghumānasa* of Mañjula in exactly the same form as stated above by Vaṭeśvara. According to the commentator Yallaya, Mañjula has borrowed this correction from the *Vaṭeśvarasiddhānta* itself. Yallaya gives the following introductory line to Mañjula's stanzas regarding this correction:

> अथ चन्द्रस्य ग्रहसमागमच्छाया शृङ्गोन्नतिदृक्साधने वटेश्वरसिद्धान्तोक्तदृक्कर्मविशेषं श्लोकद्वयेनाह–
>
> Now, in the (next) two verses, (the author) gives a special visibility correction, the same which has been stated in the *Vaṭeśvarasiddhānta* in connection with the calculation of the Moon's conjunction with the planets, the Moon's shadow, the Moon's *śṛṅgonnati* and the Moon's longitude agreeing with observation.

And actually we find that Mañjula has only summarised the above verses of Vaṭeśvara[4]. He says:

> इन्दूच्चोनार्ककोटिघ्ना गत्यंशा विभवा विधोः ।
> गुणो व्यर्केन्दुदोःकोट्यो रूपपञ्चाप्तयोः क्रमात् ॥
> फले शशाङ्कतद्गत्योर्लिप्ताद्ये स्वर्णयोर्वधे ।
> ऋणं चन्द्रे धनं भुक्तौ स्वर्णसाम्यवधेऽन्यथा ॥[5]
>
> Multiply the degrees of the Moon's true daily motion[6] as diminished by 11 by the Rcosine of the true longitude of the Sun *minus* the longitude of the Moon's apogee. This is the multiplier of the Rsine and the Rcosine of the true longitude of the Moon diminished by that of the Sun respectively divided by 1 and 5. The

[4] Note the brevity and conciseness of Mañjula's composition. He states the correction in two verses while Vaṭeśvara gives the same in five.

[5] *Cf. Laghumānasa*, i(c), 1–2.

[6] From the word *śuddhacandragatibhāga* used by Vaṭeśvara it is obvious that Mañjula's corresponding word *gatyaṁśa* should mean "the true daily motion, in degrees" and not "the mean daily motion" which has been given as a translation of *gatyaṁśa* by D. Mukhopadhyaya (1930), P. C. Sengupta (1932), and N. K. Majumdar (1944). Sūryadeva Yajvā also observes:

> विधोश्चन्द्रस्य स्फुटभुक्तिं षष्ट्या आरोप्य भागान् कुर्यात् ते गत्यंशा इति उच्च्यन्ते।
>
> *Cf. Laghumānasa*, i(c). 1–2 (comm).

> results (thus obtained) are respectively the corrections, in minutes (*liptās*), of the Moon and its true daily motion. If in the above product one (factor) is positive and the other negative, the correction for the Moon is subtractive and that for its true daily motion additive. If both are of like sign, both positive or both negative, the corrections are contrary.

If S, M, and U respectively denote the true longitudes of the Sun, the Moon, and the Moon's apogee (*mandocca*), then the correction for the Moon's longitude stated above by Vaṭeśvara and Mañjula is

$$\mp\left(8\frac{2}{15}\right)\cos(S-U)\,[\text{Moon's true daily motion, in degrees} - 11]$$
$$\times\left(8\frac{2}{15}\right)\sin(M-S)\text{ minutes,} \qquad (1)$$

according as

$$\left(8\frac{2}{15}\right)\cos(S-U) \quad \text{and} \quad \left(8\frac{2}{15}\right)\sin(M-S)$$

are of unlike or like signs; and the correction for the Moon's true daily motion is

$$\pm\left(8\frac{2}{15}\right)\cos(S-U)\,[\text{Moon's true daily motion, in degrees} - 11]$$
$$\times\frac{\left(8\frac{2}{15}\right)\cos(M-S)}{5}\text{ minutes,} \qquad (2)$$

according as

$$\left(8\frac{2}{15}\right)\cos(S-U) \quad \text{and} \quad \left(8\frac{2}{15}\right)\cos(M-S)$$

are of unlike or like signs.

Expression (2) is clearly an approximate value of the differential of (1). For, R being the radius,

$$d\left\{\left(8\frac{2}{15}\right)\sin(M-S)\right\} = \left(8\frac{2}{15}\right)\cos(M-S)\; d\left\{\frac{M-S}{R}\right\}$$
$$= \frac{\left(8\frac{2}{15}\right)\cos\,(M-S)}{5},\,[7]$$

the term involving the differential of $\left(8\frac{2}{15}\right)\cos(S-U)$ being neglected.[8]

[7] For let $dM = 790'35''$, $dS = 59'8''$, and $R = 3438'$; then

$$\frac{d(M-S)}{R} = \frac{731'27''}{3438'} = \frac{1}{5}\text{ approx.}$$

[8] The error committed is generally negligible.

6. The general form of this correction appears in the *Siddhāntaśekhara* of Śrīpati (1039), who gives it thus:

त्रिभविरहितचन्द्रोच्चोनभास्वद्भुजज्या
गगननृपविनिघ्नी भत्रयज्याविभक्ता ।
भवति परफलाख्यं तत् पृथक्स्थं शरघ्नं
हृतमुडुपतिकर्णत्रिज्ययोरन्तरेण ॥
यदिह फलमवाप्तं तद्धनर्णं पृथक्स्थे
तुहिनकिरणकर्णे त्रिज्यकोनाधिकेऽथ ।
स्फुटदिनकरहीनादिन्दुतो या भुजज्या
स्फुटपरमफलघ्नी भाजिता त्रिज्ययाऽऽप्तम् ॥
शशिनि चरफलाख्यं सूर्यहीनेन्दुगोलात्
तदृणमुतधनं स्यादुच्चहीनार्कगोलः ।
यदि भवति हि याम्यो व्यस्तमेतद्विधेयं
स्फुटगणितदृगैक्यं कर्तुमिच्छद्भिरत्र ॥[9]

Deduct 90° from the longitude of the Moon's apogee and by that diminish the true longitude of the Sun and obtain the Rsine of that. Multiply that by 160′ and divide by the radius. This is known as the *paraphala* (i.e., the maximum correction). Set it down in two places. Multiply one by 5 and divide by the Moon's true distance as divided by its difference with the radius.[10] Add whatever is obtained here to or subtract that from the other placed elsewhere according as the Moon's true distance is less or greater than the radius. (Thus is obtained the *sphuṭaparamaphala*). Now diminish the true longitude of the Moon by that of the Sun and take its Rsine. Multiply it by the *sphuṭaparamaphala* and divide by the radius. Then is obtained the so-called *cara* correction of the Moon. (When

$$\{\text{Sun's true longitude} - (\text{longitude of Moon's apogee} - 90^\circ)\}$$

is less than 6 signs) this is subtractive or additive according as

$$(\text{Moon's true longitude} - \text{Sun's true longitude})$$

is less or greater than 6 signs. When

$$\{\text{Sun's true longitude} - (\text{longitude of Moon's apogee} - 90^\circ)\}$$

[9] *Cf. Siddhāntaśekhara*, xi. 2–4. The text of the above as printed by Babua Misra in his edition of the *Siddhāntaśekhara* (Calcutta University Press) is defective. Emendations have been made by us by comparison with the text of the above as found in the *MS* of Suryadeva Yajvā's commentary on the *Laghumānasa* in the Lucknow University.

[10] त्रिज्याकर्णयोरन्तरेण गुणिते स्फुटकलाकर्णेन हृते। (Sūryadeva Yajvā).

> is greater than 6 signs, the correction is reversed. This is the process performed by those who wish to tally computation with observation.[11]

Expressed mathematically, Śrīpati's correction is

$$\mp R\sin\{S-(U-90°)\}\times\frac{160}{R}\times \left[1\pm\frac{5(\text{Moon's true distance, in minutes}\sim R)}{\text{Moon's true distance, in minutes}}\right]\frac{R\sin(M-S)}{R}\ \text{minutes},\quad (3)$$

where, within the square brackets, + or − sign is to be taken according as

$$\text{Moon's true distance in minutes} \lesseqgtr R$$

and the correction is to be applied positively or negatively according as

$$R\sin\{S-(U-90°)\}\quad\text{and}\quad R\sin(M-S)$$

are of unlike or like signs.[12]

It would be easily seen that the correction of Śrīpati may also be stated as

$$\pm\frac{R\cos(S-U)}{R}\,[\text{Moon's true daily motion, in minutes} - 630'35''] \times\frac{R\sin(M-S)}{R}\ \text{minutes approx.},$$

[11]P. C. Sengupta gives a different translation of the above passage which seems to us to be incorrect. For the sake of comparison, however, we quote it here.

> From the moon's apogee subtract 90°, diminish the sun by the remainder left; take the "sine" of the result; multiply it by 160′ and divide by the radius; the result is the *caraphala*. Put it down in another place, multiply it by *śara* (i.e., $R\text{vers}(M-U)$ or versed sine of the Moon's distance from the apogee) and divide by the difference between the moon's distance (hypotenuse) and the radius; the result is called *parama(cara)phala*, which is to be considered to be positive or negative according as the hypotenuse put down in another place is less or greater than the radius. Multiply the "sine" of the moon which has been diminished by the apparent sun, by the apparent *paramaphala* and divide by the radius; the final result is to be called *caraphala* to be applied to the moon negatively or positively, if the moon *minus* the sun and the sun *minus* the moon's apogee (diminished by 90°) be of opposite signs; if these latter quantities be of the same sign the new equation should be applied in the inverse order by those who want to make the calculation of the apparent moon agree with observation.

In consequence, P. C. Sengupta gives the correction in the following form:

$$\mp\frac{160\times R\cos(S-U)\times R\sin(M-S)}{R\times R}\times\frac{\text{Rvers}\,(M-U)}{H-R},$$

H being the Moon's true distance (*mandakarṇa*).

[12]The corresponding correction for the Moon's true daily motion does not occur in the *Siddhāntaśekhara*.

according as

$$R\cos(S-U) \quad \text{and} \quad R\sin(M-S)$$

are of unlike or like signs, which form is analogous to the forms of Vaṭeśvara and Mañjula.

It will be noted that all the formulae stated above are but approximate. The approximation has been preferred because it gives the formulae a particular form. The correct form of the formula of Śrīpati, say, would be

$$\pm R\sin\{S-(U-90^\circ)\} \times \frac{160}{R} \times \left[1 \pm \frac{(\text{Moon's true distance, in minutes} \sim R)}{\text{Moon's true distance, in minutes}}\right] \times \frac{R\sin(M-S)}{R}$$

according as

$$R\sin\{S-(U-90^\circ)\} \quad \text{and} \quad R\sin(M-S)$$

are of unlike or like signs; or

$$\pm\frac{R}{H} \times \frac{160}{R} \times \frac{R\sin(M-S) \times R\cos(S-U)}{R}$$ [13]

according as

$$R\sin(M-S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs, which is equivalent to the form of Nīlakaṇṭha; or,

$$\pm 160 \times \frac{R\sin(M \sim S)}{R} \times \frac{R\cos(S-U)}{R} \times \frac{\text{Moon's true daily motion}}{\text{Moon's mean daily motion}}$$

according as

$$\mathrm{R}\sin(M \sim S) \quad \text{and} \quad \mathrm{R}\cos(S-U)$$

are of unlike or like signs, which exactly conforms to the result of Candra Śekhara Siṃha.

Śrīpati has introduced the number 5 in his formula to make it agree with the forms of Vaṭeśvara and Mañjula.

7. The error committed in the above formulae of Vaṭeśvara, Mañjula, and Śrīpati was recognised by Nīlakaṇṭha (1500), who states his rule in the following manner:

> व्यर्केन्दुबाहुकोटिज्ये हते वीन्दूच्चभास्वतः ।
> कोट्यर्धेन त्रिजीवाप्ते दशघ्नेन्दुकलाश्रुतौ ॥

[13] H denotes the Moon's true distance, in minutes.

> अयनैक्ये च भेदे च स्वर्णं कोटिजमेतयोः ।
> तद्बाहुफलवर्गैक्यमूलमिन्दुधरान्तरम् ॥
> त्रिज्याघ्नं बाहुजं तेन भक्तं स्वर्णं विधोः स्फुटे ।
> कर्केणादौ विधूच्चोनरवौ शुक्लेऽन्यथाऽसिते ॥[14]
>
> Divide by the radius the Rsine and the Rcosine of the Moon's true longitude *minus* the Sun's true longitude severally multiplied by half the Rcosine of the sun's true longitude *minus* the longitude of the Moon's apogee: (the results are, in *yojanas*, the *bāhuphala* and the *koṭiphala*). Add the *koṭiphala* to or subtract that from ten times the true distance of the Moon (viz. the Moon's *mandakarṇa*), in minutes (*kalās*), according as the Rcosines are of like or unlike signs. The square root of the sum of the squares of that and the *bāhuphala* is the distance (in *yojanas*) between (the true positions of) the Moon and the Earth.[15] By that divide the *bāhuphala* as multiplied by the radius and apply it as a positive or negative correction to the Moon according as the Sun *minus* the Moon's apogee is in the six signs beginning with Cancer or in those beginning with Capricorn provided that it is the light half of the lunar month; in the dark half the correction is to be reversed.

Stated mathematically, Nīlakaṇṭha's correction takes the following form:

$$\pm \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{R} \text{ minutes} \tag{4}$$

according as

$$R\sin(M-S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs, H_1 being the Moon's second true distance, in minutes.[16]

---

[14] *Cf. Tantrasaṅgraha*, viii. 1–3.

[15] This is also known as Moon's second true distance (*dvitīya-sphuṭa-karṇa*).

[16] As regards the corresponding correction for the Moon's true daily motion, Nīlakaṇṭha does not give any formula analogous to that given by Vaṭeśvara and Mañjula. He has, however, prescribed the following rule (*cf. Tantrasaṅgraha*, viii. 4) for obtaining the second true daily motion of the Moon:

> मध्यभुक्तिर्दशघ्नेन्दोस्त्रिज्याघ्ना योजनैर्हृता ।
> भूचन्द्रान्तरगैर्भुक्तिर्विधोरस्य स्फुटा मता ।।
>
> Ten times the Moon's mean daily motion (in minutes) multiplied by the radius and divided by the distance between (the true positions of) the Earth and the Moon (*bhūcandrāntara*), in *yojanas*, has been stated to be its (second) true daily motion (in minutes).

This gives the following formula:

Moon's second true daily motion =

$$\frac{\text{Moon's mean daily motion, in minutes} \times 10 \times R}{\text{Moon's second true distance, in } \textit{yojanas}} \text{ minutes.}$$

8. This correction also occurs in the *Siddhāntadarpaṇa* of Candra Śekhara Siṃha where it has been called *tuṅgāntara* and stated as follows:

> अभीष्टकालोत्थितचन्द्रमन्दात् पक्षे सिते सत्रिभसूर्यहीनात् ।
> कृष्णे त्रिभोनार्यमवर्जिताद्यत् केन्द्रं तदीया भुजमौर्विका या ॥
> साभ्राङ्गभूघ्नी त्रिगुणेन भक्ता स्फुटार्कचन्द्रान्तरदोर्गुणघ्नी ।
> त्रिज्योद्धृता लब्धमतः कलाद्यं गत्या विनिघ्नं प्रथमस्फुटेन्दोः ॥
> तन्मध्यगत्या विहृतं फलं स्यात् तुङ्गान्तरं तेन विहीनयुक्तः ।
> पर्यायतः सत्रिभवित्रिभार्कहीनेन्दुमन्दोच्चभवोक्तकेन्द्रे ॥
> तुलाधराजादिभषट्कनिष्ठे प्राक्सिद्धचन्द्रो भवति द्वितीयः ।[17]
>
> From the longitude of the Moon's apogee for the desired instant subtract the Sun's true longitude as increased by 3 signs if it is the light half of the lunar month and subtract the Sun's true longitude as diminished by 3 signs if it is the dark half of the lunar month. Treat the remainder as *kendra* and determine the Rsine thereof. Multiply that by 160 and divide by the radius; (again) multiply by the Rsine of the difference between the true longitudes of the Sun and the Moon and divide by the radius. Multiply the quotient, in minutes, thus obtained, by the daily motion of the first true Moon[18] and divide by the Moon's mean daily motion: the result (thus obtained) is known as *tuṅgāntara*. The true longitude of the Moon obtained before, when diminished or increased, *or* increased or diminished by that according as the *kendra* obtained by subtracting the Sun's true longitude as increased by 3 signs *or* decreased by 3 signs from the longitude of the Moon's apogee is in the six signs commencing with Libra or Aries respectively, becomes the second true longitude of the Moon.

This is equivalent to the following correction:

$$\pm\, 160 \times \frac{R\cos(M \sim S)}{R} \times \frac{R\sin(S-U)}{R} \times \frac{\text{daily motion of the first true Moon}}{\text{Moon's mean daily motion}} \text{ minutes,} \tag{5}$$

where $+$ and $-$ signs are chosen according as

$$R\sin(M \sim S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs.

---

[17] *Cf. Siddhāntadarpaṇa, grahagaṇita*, vi. 7–10 (i).

[18] The first true Moon is the same as the true Moon. Similarly the first true longitude and the first true daily motion are the same as the true longitude and the true daily motion.

The corresponding correction for the Moon's true daily motion given by Candra Śekhara Siṃha is contained in the following lines:

> तुङ्गान्तरं यत् फलमत्र सिद्धं त्रिज्याहतं तत् प्रथमेन्दुभान्वोः ।
> विश्लेषदोर्ज्याप्तफलं तदीयान्तरोत्थकोटीगुणसङ्गुणञ्च ॥
> त्रिज्योद्धृतं तत् पुनरर्कचन्द्रगत्यन्तरघ्नं त्रिगुणाप्तलब्धम् ।
> योज्यं तदेव प्रथमेन्दुभुक्तिफले भवेद्भुक्तिफलं द्वितीयम् ॥
> तत् संस्कृतं मध्यगतौ पुरोवद्भवेद्द्वितीया रजनीशभुक्तिः ।[19]
>
> The result known as *tuṅgāntara*, which has been just obtained, should be multiplied by the radius and divided by the Rsine of the difference of the first true longitudes of the Moon and the Sun. The result should be multiplied by the Rcosine of the same difference and divided by the radius. That should be again multiplied by the motion-difference of the Sun and the Moon, and divided by the radius and that should be added to (or subtracted from) the Moon's first *bhuktiphala* (correction for motion): result is the Moon's second *bhuktiphala*. That applied (as a correction—positive or negative—) to the Moon's mean daily motion as before gives the Moon's second (true) daily motion.

Accordingly, the corresponding correction for the Moon's true daily motion is

$$\pm 160 \times \frac{R\cos(M \sim S)}{R} \times \frac{R\cos(S-U)}{R} \times \frac{\text{daily motion of the first true Moon}}{\text{Moon's mean daily motion}} \times \frac{\text{motion-difference of the Moon and the Sun}}{R} \text{ minutes,} \tag{6}$$

$+$ or $-$ sign is to be taken according as

$$R\cos(M \sim S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs.

Expression (6) is clearly an approximate differential of (5), the term involving the differential of $R\cos(S-U)$ having been neglected.

Correspondence between formulae (2) and (6) may be noted.

9. The above discussion clearly shows that there is striking similarity among the rules stated above. The differences are due to different maximum values of the correction taken by different authors.

[19] *Cf. l.c.* vi. 17–19 (i).

10. Although there is a general unity among the rules above, yet it is surprising to note, at first sight, that Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha have deviated from Vaṭeśvara and Mañjula regarding the sign of the correction for the Moon's longitude. Vaṭeśvara and Mañjula apply the correction negatively where Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha apply it positively and vice versa. The reason for this deviation is that all the Hindu astronomers including, of course, Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha agree among themselves in taking $R \sin \theta$ positive or negative according as

$$0 < \theta < 6 \text{ signs}$$

or

$$6 \text{ signs} < \theta < 12 \text{ signs},$$

whereas Vaṭeśvara and Mañjula take $R \sin \theta$ positive or negative according as

$$6 \text{ signs} < \theta < 12 \text{ signs}$$

or

$$0 < \theta < 6 \text{ signs}.$$

Mañjula gives his rule of sign as follows:

> ग्रहः स्वोच्चोनितः केन्द्रं तदूर्ध्वाधोऽर्धजो भुजः ।
> धनर्णं पदशः कोटी धनर्णर्णधनात्मिका ॥[20]
>
> The (mean or true-mean) longitude of the planet diminished by the (mean) longitude of the (*manda* or *śīghra*) *ucca* is known as (*manda* or *śīghra*) *kendra*. There the *bhuja*[21] (and the Rsine thereof) is positive or negative according as the *kendra* is greater or less than half a circle; and the *koṭi* (i.e., the complementary arc of the *bhuja*) (and the Rsine thereof) is *plus*, *minus*, *minus*, and *plus* in the respective quadrants.[22]

---

[20] *Cf. Laghumānasa*, i(b), 1. The text given here agrees with that given by Sūryadeva Yajvā (b. 1191), Parameśvara (1409), and Yallaya (1482). N. K. Majumdar, however, gives *ṣaḍūrdha* instead of *tadūrdhva*. P. C. Sengupta's version is *ṣaḍūrdhvādardhajo*.

[21] "If the (mean or true-mean) planet is in the odd quadrant (the portion of) the *kendra* (which lies in that quadrant) is known as *bhuja* (and the complementary arc as *koṭi*); if the (mean or true-mean) planet is in the even quadrant (the portion of) the *kendra* (which lies in that quadrant) is called *koti* (and the complementary arc as *bhuja*)." (Lalla).

[22] Parameśvara says: केन्द्रे तुलादिषड्राशिगते धनात्मको भुजः मेषादिषड्राशिगते ऋणात्मक इत्यर्थः। Sūryadeva Yajvā says: केन्द्रस्य च उर्ध्वार्धात् अधोऽर्धात् जातश्च भुजः क्रमात् धनसंज्ञं च ऋणसंज्ञं च भवतः। यदा केन्द्रं राशिषट्कादधिकं तदा ऊर्ध्वार्धं वर्तत इति ज्ञेयम्। यदा राशिषट्कादूनं तदा अधोऽर्धं वर्तत इति ज्ञेयम्।

Literally translated the latter part of the above verse would give: "There the *bhuja* arising from the upper half-circle (commencing from the sign Libra) and the lower half-circle (commencing from the sign Aries) is positive and negative and the *koṭi* in the respective quadrants is positive, negative, negative, and positive."

Consequently in the four quadrants the signs of $R \sin\theta$ and $R \cos\theta$ taken by Vaṭeśvara and Mañjula are *minus, minus, plus, plus* and *plus, minus, minus, plus* respectively. As regards the sign of $R \sin\theta$, this convention, as already pointed out, does not agree with the general Hindu convention.

The conception of Vaṭeśvara and Mañjula is based, however, on the following two considerations:

(i) the *bhujaphala* (the equation of the centre) is a function of the Rsine of the *bhuja* while the *koṭiphala* (i.e., the correction for the radius) is a function of the Rsine of the *koṭi*; and

(ii) the *bhujaphala* (i.e., equation of the centre) is subtractive, subtractive, additive, and additive in the four successive quadrants while in the same quadrants the *koṭiphala* (i.e., the correction for the radius) is additive, subtractive, subtractive, and additive.

Following this Vaṭeśvara and Mañjula take the Rsine and the corresponding arc known as *bhuja* negative where the *bhujaphala* (i.e., the equation of the centre) is negative and positive where it is positive. Similarly, where the *koṭiphala* (i.e., the correction for the radius) is negative the Rcosine and likewise the corresponding arc known as *koṭi* is taken as negative and where the *koṭiphala* is positive, the Rcosine and the *koṭi* are taken as positive.

This explains the difference in sign in the corrections for the Moon's longitude given by Vaṭeśvara and Mañjula and those given by Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha.

11. Thus there has been established complete unity among the rules of Vaṭeśvara, Mañjula, Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha.

Section II

12. Vaṭeśvara and Mañjula call the expression

$$\left(8\frac{2}{15}\right)\cos(S-U)[\text{Moon's true daily motion, in degrees} - 11]$$

by the term *guṇa*, which has also been used to denote the epicyclic multiplier. Śrīpati calls

$$\frac{160}{R} \times R\sin\{S-(U-90^\circ)\}$$

paraphala, which corresponds with the *antyaphala* i.e., the radius of the epicycle, and

$$R\sin\{S-(U-90^\circ)\}\times\frac{160}{R}\times \left[1\pm\frac{5(\text{Moon's true distance, in minutes}\sim R)}{\text{Moon's true distance, in minutes}}\right]$$

sphuṭa-parama-phala, which may be translated by the expression "corrected epicyclic radius". Nīlakaṇṭha has actually used the terms *bāhuphala* and *koṭiphala*, and says

$$\textit{bāhuphala}=\frac{R\sin(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ \textit{yojanas},$$

$$\text{and}\qquad \textit{koṭiphala}=\frac{R\cos(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ \textit{yojanas}.$$

These facts clearly indicate that the Hindu astronomers had also an epicyclic representation of the above correction. In what follows we shall explain this point of view.

13. The Hindu astronomers believed that the Earth did not always occupy its natural position (which coincides with the centre of the so-called *bhagola*) and that the dual correction above was due to its displacement. Let E (**ed.** See Figure 1) denote the natural position of the Earth's centre (*bhagolaghanamadhya*), the bigger circle round E the Moon's concentric (*kakṣāvṛtta*), the point U the position of the Moon's apogee on the concentric, M the true position of the Moon's centre and ES the direction of the Sun from the Earth's centre. The small circle round E has λ for its radius where λ denotes the maximum value of the dual correction at the Moon's distance given by the following table:

Authority	λ, in minutes (*kalās*)
Vaṭeśvara and Mañjula	144 approx.
Śrīpati and Candra Śekhara Siṃha	160
Nīlakaṇṭha	171.9

K is the point on the small circle opposite to U. KE_1 is perpendicular to SES_1 and E_1P to MEM_1. The point E_1 denotes, according to Hindu astronomers, the displaced position of the Earth's centre and is known as *ghanabhūmadhya*.

The circle with centre E and radius EE_1 (not shown in the figure) is treated as an epicycle and its radius EE_1 is known as the epicyclic radius[23] (*paraphala*). In consequence E_2P is known as *bāhuphala* (or *bhujāphala*) and EP

[23]It will be noted that the size of this epicycle does not remain constant. It depends on the

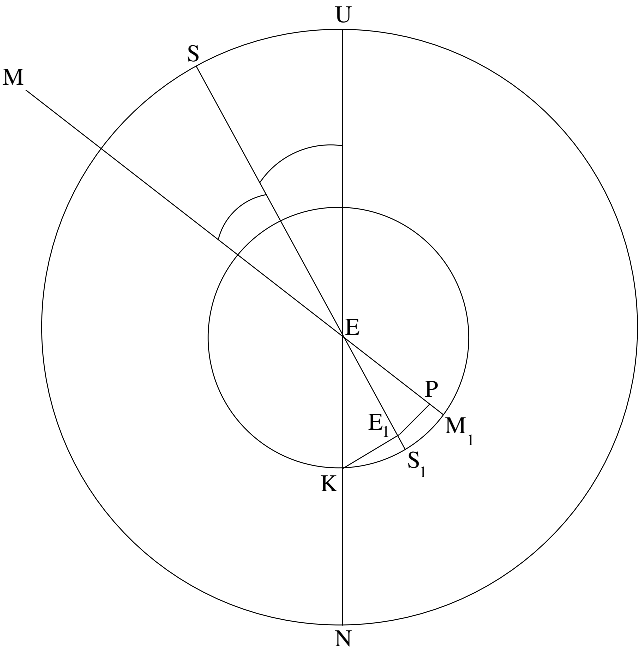

Figure 1

as *koṭiphala.*

$$
\begin{aligned}
& \because \angle KEE_1 = S - U, \\
\text{and} \quad & \angle E_1EP = M - S, \\
& \therefore EE_1 = \lambda\cos(S-U), \\
\text{whence} \quad & E_1P \text{ i.e., } b\bar{a}huphala = \lambda\cos(S-U)\times\sin(M-S), \\
\text{and}^{24} \quad & EP \text{ i.e., } ko\d{t}iphala = \lambda\cos(S-U)\times\cos(M-S).
\end{aligned}
$$

positions of the Sun and the Moon's apogee. When $S-U$ is zero or $180°$, it assumes its greatest size and coincides with the small circle drawn in the figure. When $S \sim U$ equals $90°$, it reduces to the point-circle at E. It will be further noted that EE_1 denotes the epicyclic radius at the Moon's second true distance. When it is reduced to the distance R, it is known as corrected epicyclic radius (*sphuṭa-parama-phala*).

[24] If we put $\lambda = 171.9$ minutes or $\frac{R}{2}$ *yojanas*, we have

$$
b\bar{a}huphala = \frac{R\sin(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ yojanas,
$$

$$
\text{and} \qquad ko\d{t}iphala = \frac{R\cos(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ yojanas,
$$

which agree with the formulae stated by Nīlakaṇṭha.

Due to the displacement of the Earth's centre from E to E_1 the Moon's true distance, measured in minutes, (*mandakarṇa* or *sphuṭa-kalā-karṇa*) changes from EM to E_1M. The distance E_1M is known as the distance between (the true positions of) the Moon and the Earth (*bhūmyantara-karṇa*) or the Moon's second true distance (*dvitīya-sphuṭa-karṇa*) and is obtained by the following formula:

$$\text{Moon's second true distance, in minutes} = \Big[(\text{Moon's true distance, in } \textit{kalās} \pm \textit{koṭiphala}, \text{ in } \textit{kalās})^2 + (\textit{bāhuphala}, \text{ in } \textit{kalās})^2\Big]^{\frac{1}{2}}$$

according as the *koṭiphala* is positive or negative.[25]

Similarly, the dual correction due to the displacement of the Earth's centre from E to E_1 is obviously given by the following:

$$\begin{aligned}\text{the dual correction} &= \angle EME_1 \\ &= R\sin^{-1}\left\{\frac{E_1P \times R}{E_1M}\right\} \\ &= \frac{E_1P \times R}{E_1M} \text{ approx.} \\ &= \frac{R}{H_1} \times \lambda \sin(M-S) \times \cos(S-U)^{26}\end{aligned}$$

where H_1 denotes the Moon's second true distance, in minutes.

Since the Moon's true distance is approximately equal to the Moon's second true distance, the Hindu astronomers have in general used the Moon's true distance, in minutes, in place of the Moon's second true distance, in minutes, in their formulae for the dual correction. The error is negligible. Nīlakaṇṭha, however, used the Moon's second true distance, in minutes.

14. The displacement of the Earth's centre conceived by Hindu astronomers not only changes the Moon's true distance but it also creates a change in the

[25] This corresponds with Nīlakaṇṭha's formula above.

[26] Putting various values of λ in this formula, we obtain the formulae of Vaṭeśvara etc. in their modified form. For example, putting $\lambda = \frac{R}{2}$ *yojanas*, we obtain Nīlakaṇṭha's formula

$$\begin{aligned}\text{the dual correction} &= \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{R} \textit{ yojanas}, \\ &= \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{10 \times R} \text{ minutes.}\end{aligned}$$

distances of all other planets, howsoever small that change may be. Nīlakaṇṭha has considered the two particular cases relating to the lunar and solar distances:

(1) when the longitudes of the Sun and the Moon are the same; and

(2) when the longitudes of the Sun and the Moon differ by 180°.

He says:

> उच्चोनशशिकोटिज्यादलं पर्वान्तजं स्फुटम् ।
> स्फुटयोजनकर्णे स्वं जह्यात् कर्क्यादिजं ततः ॥
> स भूम्यन्तरकर्णः स्यात् तेन बिम्बकलां नयेत् ।
> स्फुटयोजनकर्णे स्वे मासान्ते शशिवद्रवेः ॥
> व्यस्तं पक्षान्तजं कार्यं रविभूम्यन्तराप्तये ।[27]
>
> The half of the true value of the Rcosine of the Moon's longitude *minus* the longitude of the Moon's apogee (*mandocca*) corresponding to the instant of geocentric conjunction or opposition of the Sun and the Moon should be added to the true distance of the Moon, in *yojanas*, (when the Moon is in the six anomalistic signs commencing with Capricorn) and subtracted from that when the Moon is in the six anomalistic signs beginning with Cancer. This gives the distance, in *yojanas*, between (the true positions of) the Moon and the Earth (*bhūmyantarakarṇa*). This is to be used in calculating the Moon's diameter, in minutes (*kalās*), (for the instant of geocentric conjunction or opposition). In order to obtain the distance (in *yojanas*) between (the true positions of) the Sun and the Earth, apply the same as a positive or negative correction to the Sun's true distance, in *yojanas*, as in the case of the Moon, provided that it is the end of the lunar month; if it is the end of the fifteenth lunar date, apply the same reversely.

From figures similar to that drawn above for the general case, it will be seen that, when the Sun and the Moon are in geocentric conjunction,

(i) the Sun and the Moon are in the same direction from the natural position of the Earth's centre (*bhagolaghanamadhya*) while the displaced position of the Earth's centre is in the contrary or the same direction according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer; and

(ii) $M - S = 0$, whence

$$b\bar{a}huphala = 0 \quad \text{and} \quad ko\d{t}iphala = \frac{1}{2} R\cos(M-U) \ \ yojanas.$$

[27] *Cf. Tantrasaṅgraha*, iv. 12–14 (i).

Consequently

$$\text{Sun's second true distance, in } yojanas = \text{Sun's true distance, in } yojanas \pm \frac{1}{2} R\cos(M-U)\ yojanas,$$

and

$$\text{Moon's second true distance, in } yojanas = \text{Moon's true distance, in } yojanas \pm \frac{1}{2} R\cos(M-U)\ yojanas,$$

according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer.

When the Sun and the Moon are in geocentric opposition, it will be, similarly, seen that

(i) the Moon and the Sun are on the opposite sides of the natural position of the Earth's centre (*bhagolaghanamadhya*) and the displaced position of the Earth is directed towards that of the Sun or the Moon according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer; and

(ii) $M - S = 180$, whence

$$b\bar{a}huphala = 0 \quad \text{and} \quad ko\d{t}iphala = \frac{1}{2} R\cos(M-U)\ yojanas.$$

Consequently

$$\text{Sun's second true distance, in } yojanas = \text{Sun's true distance, in } yojanas \mp \frac{1}{2} R\cos(M-U)\ yojanas,$$

and

$$\text{Moon's second true distance, in } yojanas = \text{Moon's true distance, in } yojanas \pm \frac{1}{2} R\cos(M-U)\ yojanas,$$

according as the Moon is in the six anomalistic signs beginning with Capricorn or Cancer.

Hence Nīlakaṇṭha's rules above.

15. In the above graphical method of the Hindus the true position of the Moon remains unaffected whereas the position of the Earth goes on changing from time to time. Also the size of the epicycle ascribed to the Earth does not remain invariable. It depends upon the positions of the Sun and the Moon's apogee. It is maximum when the Sun crosses the Moon's line of apsides (*uccanīcarekhā*) and minimum when the Sun is at right angles to it. This variation in the size of the Earth's epicycle causes a variation in the eccentricity of the Moon's path which, it will be noted, always assumes its maximum value when the Sun crosses the Moon's line of apsides and its minimum value when the Sun is at right angles to it. This variation in the eccentricity of the Moon's orbit is obviously related to the dual correction. In fact the dual correction depends upon it. Young (1889) has actually said that the evection "depends upon the alternate increase and decrease of the eccentricity of the Moon's orbit, which is always a maximum when the sun is passing the moon's line of apsides, and a minimum when the sun is at right angles to it." According to Mañjula and Vaṭeśvara the maximum value of the eccentricity of the Moon's orbit comes out to be roughly about 0.0652 and according to Śrīpati and Candra Śekhara Siṃha about 0.0674; the minimum Hindu value of the Moon's eccentricity is about 0.0442. The corresponding maximum and minimum values given by Horrocks (1640) are 0.06686 and 0.04362 respectively. According to Young the eccentricity of the Moon's orbit varies from $\frac{1}{14}$ to $\frac{1}{22}$.

Section III

16. The Greek astronomer Ptolemy (140 AD) was also aware of this dual correction of the Moon.[28] He is said to have constructed an instrument by means of which he observed the Moon in all parts of its orbit and found

(i) that the computed positions of the Moon were generally different from the observed ones, the maximum amount of this difference noted by him being 159 minutes, and

(ii) that the difference between the observed and computed positions of the Moon attained its maximum value when $|M-S|$ equalled 90° and $S-U$

[28]In modern works we find that instead of this dual correction being attributed to Ptolemy, the evection is generally attributed to him. It should be noted that Ptolemy did not detect the evection alone but a mixture which contains the deficit of the equation of the centre and the evection. And this is what can be naturally expected from the ancient astronomers who took the maximum value of the Moon's equation of the centre smaller than its actual value. Some writers also make the erroneous statement that the ancient astronomers detected the evection because it can affect the time of an eclipse by about 6 hours at its maximum. The fact is that the ancient astronomers did not detect the evection separately, and if we call this dual correction by the name of evection, we find that it does not make any difference in the time of an eclipse. Even Ptolemy says that this correction is zero when $(M - S)$ is zero or 180°.

was either zero or 180°, and that it vanished altogether when $M - S$ equalled zero or 180°.

To represent this dual correction Ptolemy imagined an eccentric in the circumference of which the centre of the epicycle moved while the Moon moved on the circumference of the epicycle. Later on it was discovered by Copernicus (1543) that the lunar distances resulting from Ptolemy's hypothesis were totally at variance with the observations of the Moon's apparent diameter. Consequently he gave another method of representing the lunar inequality which is known as Copernicus's hypothesis.

17. Ptolemy had previously discovered that in quadrature when the equation of the centre assumed its maximum value viz. $5°1'$, the dual correction increased it to $7°40'$, which happened when the apse-line (*ucca-nīca-rekhā*) coincided with the direction of the Sun from the Earth's centre, but when the Sun's direction was perpendicular both to the apse–line and Moon's direction, the equation of the centre vanished with the dual correction. Consequently Ptolemy had fixed $6°20\frac{1}{2}'$ as the value of the mean of the two corrections. Copernicus took it as the corrected value of the maximum equation of the centre and treated it as the radius of the Moon's first epicycle. Thus the deficit of the equation of the centre was unconsciously added to it. The radius of the first epicycle conceived by Copernicus was likewise equal to M_1O (**ed.** See Figure 2). In order to account for the remaining correction viz. the evection, Copernicus took Om for the radius of the Moon's second epicycle and supposed the Moon M to move on it in the anti-clockwise direction from the point m in such a way that, at any instant,

$$\angle MOm = 2(M_1 - S_1),$$

where M_1 and S_1 are the mean positions of the Moon and the Sun respectively.

Copernicus's hypothesis also leads to the same form of expression for the Moon's dual correction as given by Hindu astronomers but it does not explain the variation of the eccentricity of the Moon's orbit which really causes the dual correction.[29]

19. According to modern lunar theory the relevant terms of the Moon's longitude are given by

$$\begin{aligned}\text{Moon's longitude} = M_1 &- 377' \operatorname{Sin}(M_1 - U) \\ &- 76' \operatorname{Sin}\{2(M_1 - S_1) - (M_1 - U)\}\,. \qquad (7)\end{aligned}$$

[29]**ed.** As per the sequence, the subsequent point of discussion should be numbered **18**. However, since in the original it jumps by one number, we have just retained it.

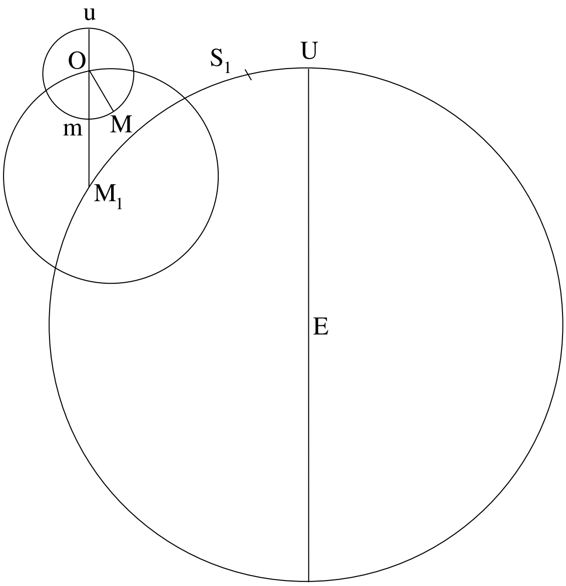

Figure 2

M_1 and S_1 denote the mean longitudes of the Moon and the Sun and U the longitude of the Moon's apogee.

The term $-377'$ Sin $(M_1 - U)$ denotes the equation of the centre and the term $-76'$ Sin $\{2(M_1 - S_1) - (M_1 - U)\}$ is known as evection.

If the equation of the centre viz. $-377'$ Sin $(M_1 - U)$ be broken up into two components $-301'$ Sin $(M_1 - U)$ and $-76'$ Sin $(M_1 - U)$ and the second component be combined with the evection-term, then formula (7) would become

$$\begin{aligned} \text{Moon's longitude} = M_1 &- 301' \text{ Sin}\,(M_1 - U) \\ &- 152' \text{ Cos}\,(S_1 - U) \times \text{Sin}\,(M_1 - S_1). \end{aligned} \tag{8}$$

Here, if the term $-301'$ Sin $(M_1 - U)$ be taken for the equation of the centre, then the term $-152'$ Cos $(S_1 - U) \times$ Sin $(M_1 - S_1)$ would give the deficit of the equation of the centre and the evection. The term $-301'$ Sin $(M_1 - U)$ corresponds to the Hindu equation of the centre and the term $-152'$ Cos $(S_1 - U) \times$ Sin $(M_1 - S_1)$ to the dual correction discussed above. Comparison of this term with the expressions for the dual correction given by Hindu astronomers proves the correspondence between the two as also the perfectness of the Hindu form. It would also be noted that the difference between the Moon's longitudes calculated according to Vaṭeśvara or Mañjula and by formula (7) would never exceed about 5 minutes.

20. The above discussion proves conclusively the soundness of the formulae for the Moon's dual correction given by various Hindu astronomers. We have also seen that this correction in its perfect form was known in India in the time of Vaṭeśvara (c. 899) or earlier. In view of the advanced state of the available Hindu formulae, we have every reason to believe that the correction was known much earlier in India, specially when we see that in Europe it was known in its proper from about 1400 years after it was actually detected. The graphic method of the Hindus which not only explains the dual correction but also the variation of the eccentricity of the Moon's orbit was known to the Hindus long before Copernicus gave his own. Hindus were, thus, the first to give the Moon's dual correction in its perfect form and the first to explain it properly.

21. I take this opportunity to express my thanks to Prof. A. N. Singh for help and guidance.

References

1. Godfray, H., (1871), *An Elementary Treatise on the Lunar Theory.*
2. Majumdar, N. K., (1944), *Laghumānasam.*
3. Mukhopadhyaya, D., (1930), *Bull. Cal. Math. Soc.*, 21, 121–132.
4. Sengupta, P. C., (1932), *Bull. Cal. Math. Soc.*, 24, 1–18.
5. Young, C. A., (1889), *A Text-Book of General Astronomy.*