Surds in Hindu mathematics*

Elementary treatment of surds, particularly their addition, multiplication and rationalisation, is found in the *Śulbasūtras*. Fuller treatment of this subject occurs in the works on Hindu algebra where rules for addition and subtraction, multiplication and involution, separation and extraction of square-root of surds and compound surds are given. The present article gives an account of the treatment of surds in Hindu mathematics.

The Sanskrit term for the surd is *karaṇī*. Śrīpati (1039) defines it as follows:

> The number whose square-root cannot be obtained (exactly) is said to form an irrational quantity *karaṇī*.[1]

Similar definitions are given by Nārāyaṇa (1356) and others.[2] Of course, the number is to be considered a surd when the business is with its square root. A surd number is indicated by putting down the tachygraphic abbreviation *ka* before the number affected. Thus, *ka* 8 means $\sqrt{8}$ and *ka* 450 means $\sqrt{450}$.

1 Origin of surds

The origin of the term *karaṇī* is interesting. Literally it means "making one" or "producing one". It seems to have been originally employed to denote the cord used for measuring (the side of) a square. It then meant the side of any square and was so called because it made a square (*caturaśra-karaṇī*). Hence, it came to denote the square-root of any number. As late as the second century of the Christian era, Umāsvātī (c. 150) treated the terms *mūla* ("root") and *karaṇī* as synonymous. In later times, however, the application of the term has been particularly restricted to its present significance as a surd. Nemicandra (c. 975)[3] has occasionally used the generic term *mūla* to signify a surd, e.g., *daśa-mūla* $= \sqrt{10}$.

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[1] *SiŚe* (=*Siddhānta-śekhara*), xiv. 7.

[2] *NBi* (= Nārāyaṇa's *Bījagaṇita*) I, R. 25. See also the commentaries of Gaṇeśa and Kṛṣṇa on the *Bījagaṇita* of Bhāskara II.

[3] *Gommaṭa-sāra* of Nemicandra, *Jīvakāṇḍa*, *Gāthā* 170.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
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2 Addition and subtraction

For addition of surds, we have the following ancient rule:

> Reducing them by some (suitable) number, add the square-roots of the quotients; the square of the result multiplied by the reducer should be known as the sum of the surds.

We do not know the name of the author of this rule. It is found to have been quoted by Bhāskara I (629) in his commentary on the *Āryabhaṭīya* (ii. 10). A similar rule is given by Brahmagupta (628):

> The surds being divided by a (suitable) optional number, the square of the sum of the square-roots of the quotients should be multiplied by that optional number (in case of addition); and the square of the difference (of the square-roots of the quotients being so treated will give the difference of the surds).[4]

Mahāvīra says:

> After reducing (the surd quantities) by an optional divisor, the square of the sum or difference of the square-roots of the quotients is multiplied by the optional divisor, the square-root (of the product) is the sum or difference of the square-root quantities. Know this to be the calculation of surds.[5]

Śrīpati writes:

> For addition or subtraction, the surds should be multiplied (by an optional number) intelligently (selected), so that they become squares. The square of the sum, or difference of their roots, should then be divided by that optional multiplier. Those surds which do not become squares on multiplication (by an optional number), should be put together (side by side).[6]

Bhāskara II says:

> Suppose the sum of the two numbers of the surds $(a+b)$ as the *mahatī* ('greater') and twice the square-root of their product $(2\sqrt{ab})$ as the *laghu* ('lesser'). The addition or subtraction of these like integers is so (of the original quantities).
>
> Multiply and divide as if a square number by a square number. In addition and subtraction, the square-root of the quotient of the

[4] *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xviii. 38.
[5] *GSS* (=*Gaṇitasārasaṅgraha*), vii. $88\frac{1}{2}$.
[6] *ŚiŚe*, xiv. 8f.

> greater surd number divided by the smaller surd number should be increased or diminished by unity; the result multiplied by itself should be multiplied by the smaller surd number. The product (is the sum or difference of the two surds). If there be no (rational) root, the surds should be stated separately side by side.[7]

Nārāyaṇa says:

> Divide the two surds separately by the smaller or greater among them; add or subtract the square-roots of the quotients; then multiply the square of the result by that divisor. The product is the sum or difference.
>
> Or multiply the two surds by the smaller or greater one among them; add or subtract the square-roots of the products; then dividing the square of the result by that selected multiplier, the quotient is the sum or difference.
>
> Or divide the greater surd by the smaller one; add unity to or subtract unity from the square-root of the quotient; then multiply the result by itself and also by the smaller quantity. The result is the sum or difference (required). Or proceed in the same way with the greater surd.
>
> Or add twice the square-root of the product of the two surds, supposed as if rational, to or subtract that from their sum. The result is the sum or difference. If there be no rational root of the product, then the two surds should be stated severally.
>
> To add up several surds, divide them by an optional number and then take the sum of the square-root of the quotients. This sum multiplied by itself and also by that divisor will give the sum of them.[8]

Thus, we have the following methods for addition or subtraction of surds:

(i) $\sqrt{a} \pm \sqrt{b} = \sqrt{b\left(\sqrt{\frac{a}{b}} \pm 1\right)^2}$,

(ii) $\sqrt{a} \pm \sqrt{b} = \sqrt{\frac{1}{a}(a \pm \sqrt{ab})^2}$,

(iii) $\sqrt{a} \pm \sqrt{b} = \sqrt{c\left(\sqrt{\frac{a}{c}} \pm \sqrt{\frac{b}{c}}\right)^2}$,

(iv) $\sqrt{a} \pm \sqrt{b} = \sqrt{\frac{1}{c}(\sqrt{ac} \pm \sqrt{bc})^2}$,

[7] *BBi* (=*Bhāskara's Bījagaṇita*), pp. 12f.
[8] *NBi* (*Junction of a door*), I, R. 25–30.

(v) $\sqrt{a} \pm \sqrt{b} = \sqrt{(a+b) \pm 2\sqrt{ab}}$.

The optional number c is so chosen that (ac, bc) or $\left(\frac{a}{c}, \frac{b}{c}\right)$ become perfect squares.

Brahmagupta and Mahāvīra teach the method (iii), Śrīpati gives (iv). Bhāskara states (i) and (v). Nārāyaṇa gives (i), (ii), (iii), and (v).

3 Multiplication and involution

For the multiplication of surd expressions, the Hindu works give an algebraic method. Thus, Brahmagupta says:

> Put down the multiplicand horizontally below itself as many times as there are terms in the multiplier; then multiplying by the *khaṇḍa-guṇana* method (i.e., by the method of multiplication by component parts), add the (partial) products.[9]

Thus, to multiply $\sqrt{a} + \sqrt{b}$ by $\sqrt{c} + \sqrt{d}$, one should proceed as follows:

$$\begin{aligned}(\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d}) &= (\sqrt{a} + \sqrt{b}) \times \sqrt{c} + (\sqrt{a} + \sqrt{b}) \times \sqrt{d} \\ &= \sqrt{ac} + \sqrt{bc} + \sqrt{ad} + \sqrt{bd}.\end{aligned}$$

Brahmagupta further notes:

> The squaring of a surd is (finding) the product of two equal (surds).[10]

Śrīpati writes:

> Putting down the multiplicand and multiplier in the manner of the *kapāṭasandhi* multiply according to the method taught before. But those surds should be added, as before, in which the product yields a perfect square.[11]
>
> On multiplying two equal surd quantities, the square of that surd is obtained.[12]

Bhāskara II (1150) observes:

> For abridgement, multiplication or division of surd expressions should be proceeded with after addition (or subtraction) of two or more terms of the multiplier and multiplicand or of the divisor and dividend.[13]

A similar remark has been made by Nārāyaṇa.[14]

[9] *BrSpSi*, xviii. 38.
[10] *BrSpSi*, xviii. 39.
[11] *SiŚe*, xiv. 9.
[12] *SiŚe*, xiv. 11.
[13] *BBi*, p. 13.
[14] *NBi*, I, R. 31.

4 Division

Brahmagupta (628) teaches the following method of division of surds:

> Multiply the dividend and divisor separately by the divisor after making an optional term of it negative; then add up the terms. (Do this repeatedly until the divisor is reduced to a single term). Then divide the (modified) dividend by the divisor reduced to a single term.[15]

Śrīpati (1039) writes:

> Reversing the sign, negative or positive, of one of the surds occurring in the denominator, multiply by it both the numerator and the denominator separately and then add together the terms of the (respective) products. Repeat (the operations) until there is left only a single surd in the denominator. By it divide the dividend above. Such is the method of division of surds.[16]

This rule has been almost reproduced by Bhāskara II[17] (1150) and Nārāyaṇa[18] (1356). The latter delivers also another method similar to the division of one algebraic expression by another. He says:

> Multiply the divisor surd so as to make all or some of its terms square such that the sum of their square-roots will be equal to the rational term (in the dividend). Thus will be determined the multiplier surd. Subtract from the dividend the divisor multiplied by that. If there be left a remainder, the sum of the terms of the divisor multiplied by that multiplier should be subtracted from the terms of the dividend. In case of absence of a rational term (in the dividend), that by which the divisor is multiplied and then subtracted for the dividend so as to leave no remainder, will be the quotient.[19]

Example from Bhāskara II[20]

Divide $\sqrt{9} + \sqrt{450} + \sqrt{95} + \sqrt{45}$ by $\sqrt{25} + \sqrt{3}$.

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[15] *BrSpSi*, xviii, 39.
[16] *SiŚe*, xiv. 11.
[17] *BBi*, p. 14.
[18] *NBi*, I, R. 37–8.
[19] *NBi*, I, R. 33–5.
[20] *BBi*, pp. 15–16.

$$\begin{aligned}\frac{\sqrt{9}+\sqrt{450}+\sqrt{95}+\sqrt{45}}{\sqrt{25}+\sqrt{3}} &= \frac{(\sqrt{9}+\sqrt{450}+\sqrt{95}+\sqrt{45})(\sqrt{25}-\sqrt{3})}{(\sqrt{25}+\sqrt{3})(\sqrt{25}-\sqrt{3})}\\ &= \frac{\sqrt{8712}+\sqrt{1452}}{\sqrt{484}}\\ &= \sqrt{18}+\sqrt{3}.\end{aligned}$$

Example from Nārāyaṇa[21]

First method:
Divide $5+\sqrt{90}+\sqrt{180}+\sqrt{648}$ by $\sqrt{5}+\sqrt{36}$.

$$\begin{array}{ccccccccccccccc} \sqrt{5} & + & \sqrt{36} & \Big) & 5 & + & \sqrt{90} & + & \sqrt{180} & + & \sqrt{648} & \Big(& \sqrt{5} & + & \sqrt{18} \\ & & & & 5 & & & + & \sqrt{180} & & & & & & \\ \hline & & & & & & \sqrt{90} & + & \sqrt{648} & & & & & & \\ & & & & & & \sqrt{90} & + & \sqrt{648} & & & & & & \end{array}$$

Second method:
Divide $\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60}$ by $\sqrt{5}+\sqrt{3}+\sqrt{2}$.

$$\begin{aligned}&\frac{\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}\\ &= \frac{(\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60})(\sqrt{5}+\sqrt{3}-\sqrt{2})}{(\sqrt{5}+\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{3}-\sqrt{2})}\\ &= \frac{\sqrt{2100}+\sqrt{1800}+\sqrt{1260}+\sqrt{1080}}{\sqrt{60}+\sqrt{36}}\\ &= \frac{(\sqrt{2100}+\sqrt{1800}+\sqrt{1260}+\sqrt{1080})(\sqrt{60}-\sqrt{36})}{(\sqrt{60}+\sqrt{36})(\sqrt{60}-\sqrt{36})}\\ &= \frac{\sqrt{20160}+\sqrt{17280}}{\sqrt{576}}\\ &= \sqrt{35}+\sqrt{30}.\end{aligned}$$

5 Rule of separation

Bhāskara II gives a rule for an operation converse to that of addition. He says:

> (Find) a square number by which the compound-surd will be exactly divisible. Breaking up the square-root of that (square-number) into parts at pleasure, multiply the square of the parts

[21] *NBi*, I, example on R. 33–5; also Ex. 18.

> of the previous quotient. These will be the several component surds.[22]

A similar rule is stated by Nārāyaṇa:

> Divide the compound-surd by the square of some number so as to leave no remainder. Parts of it multiplied by themselves and also by the quotient will be the (component) terms of the surd.[23]

That is to say, if $N = m^2k$ and $m = a + b + c + d$, then

$$\begin{aligned}\sqrt{N} = \sqrt{m^2k} = m\sqrt{k} &= (a + b + c + d)\sqrt{k}, \\ &= (\sqrt{a^2k} + \sqrt{b^2k} + \sqrt{c^2k} + \sqrt{d^2k}).\end{aligned}$$

6 Extraction of square-root

For the extraction of the square-root of a surd expression, Brahmagupta described the following method:

> The optionally chosen surds being subtracted from the square of the absolute (i.e. rational) term, the square-root of the remainder should be added to and subtracted from the rational term and halved; then the first is considered as a rational term and the second a surd different from the previous. (Such operations should be carried on) repeatedly (if necessary).[24]

Illustrative example from Pṛthūdakasvāmi (860)

To find the square-root of $16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24}$. It has been solved substantially as follows:

Subtract the surd numbers 120, 72, 48 from the square of the rational number, viz., 256; the remainder is $(256 - 120 - 72 - 48) = 16$. Its root is 4; $\frac{1}{2}(16 \pm 4) = 10, 6$. Now subtracting the surd numbers 60 and 24 from 10^2, we get 16; its root is 4; $\frac{1}{2}(10 \pm 4) = 7, 3$. Again subtracting the surd number 40 from 7^2, we have 9; its root is 3; and $\frac{1}{2}(7 \pm 3) = 5, 2$. Hence, the required square-root is $\sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}$.

The same method is taught by Śrīpati (c. 1039)[25] and Bhāskara II (1150). The latter says:

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[22] *BBi*, p. 15.
[23] *NBi*, I, R. 36.
[24] *BrSpSi*, xviii, 40.
[25] *SiŚe*, xiv. 12.

> From the square of the rational number in the (proposed) square-surd, subtract the rational equivalent to one or more of the surd numbers; the square-root of the remainder should be severally added to and subtracted from the rational number; halves of the results will be the two surds in the square-root. But if there be left any more surd term in the (proposed) square surd, the greater surd number amongst those two should again be regarded as a rational number (and the same operations should be repeated).[26]

The above example of Pṛthūdakasvāmi is solved by Bhāskara substantially thus:

$$\sqrt{16^2 - (48+40+24)} = \sqrt{144} = 12; \quad \frac{1}{2}(16 \pm 12) = 14, 2.$$
$$\sqrt{14^2 - (120+72)} = 2, \quad \frac{1}{2}(14 \pm 2) = 8, 6.$$
$$\sqrt{8^2 - 60} = 2, \quad \frac{1}{2}(8 \pm 2) = 5, 3.$$

Therefore,

$$(16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24})^{\frac{1}{2}} = \sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}.$$

For the above rule, all the terms of the surd expression have been contemplated to be positive, as is also clear from the illustrative examples given. For the case in which there is a negative term, Bhāskara II lays down the following procedure:

> If there be a negative surd in the square (expression), the traction of roots should be proceeded with supposing it as if positive: but of the two surds deduced one, chosen at pleasure by the intelligent mathematician, should be taken as negative.[27]

Example from Bhāskara II[28]

To find the square-root of $10 + \sqrt{24} - \sqrt{40} - \sqrt{60}$.

The solution is given substantially as follows:

$$\sqrt{10^2 - (40+60)} = 0, \quad \frac{1}{2}(10 \pm 0) = 5, 5$$
$$\sqrt{5^2 - 24} = 1, \quad \frac{1}{2}(5 \pm 1) = 3, 2.$$

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[26] *BBi*, pp. 17f.
[27] *BBi*, p. 19.
[28] *BBi*, pp. 19f.

Therefore, $(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{3}+\sqrt{2}-\sqrt{5}$,

$$\text{or,}\qquad \sqrt{10^2-(24+60)}=4,\qquad \frac{1}{2}(10\pm 4)=7,3.$$

The greater number, viz., 7 is considered as negative. Then

$$\sqrt{7^2-40}=3,\ \frac{1}{2}(7\pm 3)=5,2.$$

Hence,

$$(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{3}+\sqrt{2}-\sqrt{5}.$$

Also,

$$(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{5}-\sqrt{3}-\sqrt{2}.$$

7 Limitation of the method

Bhāskara II indicates how to test whether a given multinominal surd has a square-root at all or not. "This matter has not been explained at length", observes he, "by previous writers. I do it for the instruction of the dull".[29] He then says:

> In a square surd, the number of irrational terms must be equal to a number same as the sum of the (natural) number 1 etc. In a square surd having three irrational terms, the rational number equal to two of the surd numbers; in a square surd having six irrational number terms, the rational equal to three of them in one of ten irrational terms, integers equal to four of them; and in one of fifteen irrational terms, integers equal to five of them; having been subtracted from the square of the rational term the square-root of the remainder should be extracted. If (done) otherwise (in any case), it will not be proper. The numbers to be subtracted from the square of the rational number (in extracting roots of a square-surd) should be exactly divisible by four times the smaller term in the resulting root-surd. The quotients obtained by this exact division will be the surd terms in the root. If they are not obtained by the last rule, then the (resulting) root is wrong.[30]

He has added the following explanatory notes to the above rule:

> In the square of an expression containing irrational terms, there must be a rational term. In the square of (an expression consisting

[29] *BBi*, p. 20.
[30] *BBi*, pp. 20ff.

of) a singe surd, there will be only a rational term; of two surds, one surd together with a rational term; of three surds, three irrational terms and a rational term; of four surds, six; of five surds, ten; of six surds, fifteen; and so on. Thus, in the square of surd expressions consisting of two or more irrational terms, the number of irrational terms will be equal to the sum of the natural numbers one, etc. respectively, besides the rational term. So if in an example (proposed), the number (of irrational terms present) be not such; then it must be considered as a compound surd. Break it up (into required number of component surds) and then extract the square-root. This is what has been implied. Thus will be clear the significance of the rule, "In a square surd having three irrational terms, the rational number equal to two of the surd numbers, etc."

Illustrative examples with solution from Bhāskara II

Example 1: Find the square-root of $10+\sqrt{32}+\sqrt{24}+\sqrt{8}$.

In this square there being three surd terms, a rational number equivalent to two of the surd numbers is first subtracted from the square of the rational term and the root (of the remainder) extracted. Then proceeding in the same way with (the remaining) one term, no root is found in this case. Hence this (i.e., the proposed expression) does not possess a root expressible in surd terms. If, however, we extract the root by subtracting, contrary to the rule, an integer equivalent to all the surd terms, we get $\sqrt{2}+\sqrt{8}$. But this is wrong as its square is 18.

Or on adding together the surds $\sqrt{32}$ and $\sqrt{8}$, (the expression becomes) $10+\sqrt{72}+\sqrt{24}$. Then (by the rule) we obtain $2+\sqrt{6}$. But that is also erroneous.[31]

Example 2: Find the square-root of $10+\sqrt{60}+\sqrt{52}+\sqrt{12}$.

Here in this square, are present three surd terms; so subtracting a rational number equal to two surd numbers, viz., 52 and 12, the two surd terms for the root are obtained as $\sqrt{8}$ and $\sqrt{2}$; of these the smaller one, namely, 2 multiplied by four, that is 8, does not exactly divide 52 and 12. So they should not be subtracted, for it has been stated, "The numbers to be subtracted from the square of the rational number (in extracting root of a square surd) should be exactly divisible by four times the smaller term in the resulting root-surd." Let it, however, be supposed that the mention of "the

[31] *BBi*, p. 23.

smaller term" here is metaphorical and may sometimes imply also "the greater term" and that it should be considered as "the greater term", if with that root-surd as the rational term other surd terms are deducible. Now on doing so we obtain for the root $\sqrt{2}+\sqrt{3}+\sqrt{5}$. But this is also wrong; for its square is $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$.[32]

Example 3: Extract the root of

$$13+\sqrt{48}+\sqrt{60}+\sqrt{20}+\sqrt{44}+\sqrt{32}+\sqrt{24}.$$

There being six surd terms in this, an integer equal to three of the surd terms should be first subtracted from the square of the rational term and the root (of the remainder) found; next an integer equal to two of the surd terms and then an integer equal to one surd term (should be subtracted). But on so doing, no root is found in this instance. If we, however, proceed in a different way and subtract from the square of the rational term first an integer equal to the first surd term, then an integer equal to the second and third terms and lastly an integer equal to the remaining surd terms, we get for the root $\sqrt{1}+\sqrt{2}+\sqrt{5}+\sqrt{5}$. But this is incorrect, since its square is $13+\sqrt{8}+\sqrt{80}+\sqrt{160}$.[33]

Bhāskara then observes in general:

This is certainly a defect of those (ancient writers) who have not defined the limitations of this method of extracting the square root of a surd. In case of such square surds, the roots should be found by taking the roots of the surd terms by the method for finding the approximate values of the roots and then combining them with the rational term.[34]

Further he says:

The mention of "the greater surd" is metaphorical, for sometimes it might imply the less.

Example from Bhāskara II[35] (1150)

To find the root of $17+\sqrt{40}+\sqrt{80}+\sqrt{200}$.

[32] *BBi*, p. 23.
[33] *BBi*, p. 24.
[34] *BBi*, p. 24.
[35] *BBi*, p. 24.

Here

$$\sqrt{17^2-(80+200)}=3,\ \frac{1}{2}(17\pm 3)=10,7$$
$$\sqrt{7^2-40}=3,\ \frac{1}{2}(7-3)=5,2.$$

Therefore,

$$(17+\sqrt{40}+\sqrt{80}+\sqrt{200})^{\frac{1}{2}}=\sqrt{10}+\sqrt{5}+\sqrt{2}.$$

8 Nārāyaṇa's rules

For finding the square-root of a surd expression, Nārāyaṇa (1350) gives the following rules:

> The number of irrational terms in the square of a surd expression is equal to the sum of natural numbers: this is the usual rule. In the square of a single surd term, there is only a rational number. In the square of an expression consisting of two surds terms, there is one surd term together with a rational number; of three, three; of four, six; of five, ten; and in the square of an expression consisting of six surd terms, there will be as many as fifteen surd terms; so it should be known. In an expression having the number of surd terms equal to the sum of the natural numbers, subtract from the square of the rational term a rational number equal to the sum of that number of surd numbers and then extract the square-root of the remainder. Add and subtract this to the rational number and halve. The results are the two surd terms. If further terms remain to be operated upon, regard the greater of these two as a rational number and find the other terms (of the root) by proceeding as before. If the number of surd terms in any expression be not equal to the sum of the natural numbers, the (requisite) number should be made up by breaking up some of the terms and then the square-root should be extracted. If that is not possible, the problem is wrong.[36]
>
> Increase twice the number of surd terms (in a given expression) by one fourth and then extract the square-root. Subtract half from that. The residue will give the number of terms (the sum of which is to be subtracted from the square of the rational term).[37]
>
> Or divide all the surd numbers (present in an expression) by four and arrange the quotients in the descending order. Divide the

[36] *NBi*, I, R. 41–5.
[37] *NBi*, I, R. 50.

product of the two surds nearest to the first surd (in the series) by the latter. The square-root of the quotient will be a surd term (in the root). Those two surds divided by this root will give another two surd terms (of the root). By these (three surds) divide next (three) terms of the series and the quotient will be another surd of the root. Again by these should be divided the other terms and the quotient is another surd; and so on. If now the square of the sum of surd numbers (in the root) be subtracted from the rational term (in the given expression) no remainder will be left. If it be not so (i.e., if a remainder is left), then the (given) square expression is a compound surd and it should be broken up into other surds by the rule of separation.[38]

[38] *NBi*, I, R. 46–9.