Use of calculus in Hindu mathematics *

1 Differential calculus

1.1 A controversy

Attention was first drawn to the occurrence of the differential formula

$$\partial(\sin\theta) = \cos\theta\,\partial\theta$$

in Bhāskara II's (1150) *Siddhāntaśiromaṇi* by Pandit Bapu Deva Sastri[1] in 1858. The Pandit published a summarised translation of the passages which involve the use of the above formula. His summary was defective in so far as it did not bring into prominence the idea of the infinitesimal increment which underlies Bhāskara's analysis. Without making clear to his readers, the full significance of Bhāskara's result, the Pandit made the mistake of asserting—what was plain to him—that Bhāskara was fully acquainted with the principles of the differential calculus.

The Pandit was adversely criticised by Spotiswoode,[2] who without consulting the original on which the Pandit based his conclusions, remarked (1) that Bapu Deva Sastri had overstated his case in saying that Bhāskarācārya was fully acquainted with the principles of the differential calculus, (2) that there was no allusion to the most essential feature of the differential calculus, viz. the infinitesimal magnitudes of the intervals of time and space therein employed, and (3) that the approximative character of the result was not realised.

Since the above controversy took place no serious investigation of the subject seems to have been made by any scholar.[3] In order that the reader may be better able to judge the merit of the Hindu claim to the invention of the differential calculus, it is desirable that the problems which required the use of the above differential formula be stated first.

* Bhibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 19, No. 2 (1984), pp. 95–104.

[1] *JASB* (= *Journal of the Asiatic of Bengal*), Vol. 27, 1858, pp. 213–6.

[2] *JARS*, Vol. 17, 1860, pp. 21–2.

[3] Except for a paper by P. C. Sen Gupta in the *Journal of the Department of Letters, Calcutta University*, Vol. XXII (1931). Recently A. K. Bag has included this topic in his book "Mathematics in Ancient and Medieval India", Chaukhambha Orientalia, Varanasi, 1979.

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Sources and Studies in the History of Mathematics and Physical Sciences,
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1.2 Problems in astronomy

The calculation of eclipses is one of the most important problems of astronomy. In ancient days this problem was probably more important than it is now, because the exact time and duration of the eclipses could not be foretold on account of lack of the necessary mathematical equipment on the part of the astronomer. In India, the Hindus observed fast and performed various other religious rites on the occasion of eclipses. Thus their calculation was a matter of national importance. It afforded the Hindu astronomer a means of demonstrating the accuracy of his science and his own ability to the public who patronised him. The problem of the calculation of conjunction of planets and occultation of stars was equally important both from scientific as well as religious view points.

In problems of the above nature it is essential to determine the true instantaneous motion of a planet or star at any particular instant. This instantaneous motion was called by the Hindu astronomers *tātkālika-gati*. The formula giving the *tātkālika-gati* (instantaneous motion) is given by Āryabhaṭa and Brahmagupta in the following form:

$$u' - u = v' - v \pm e(\sin w' - \sin w) \tag{1}$$

where u, v, w are the true longitude, mean longitude, mean anomaly respectively at any particular time and u', v', w' the values of the respective quantities at a subsequent instant; and e is the eccentricity or the sine of the greatest equation of the orbit. The *tātkālika-gati* is the difference $u' - u$ between the true longitudes at the two positions under consideration. Āryabhaṭa and Brahmagupta used the sine table to find the value of $(\sin w' - \sin w)$. The sine table used by them was tabulated at intervals of $3°45'$ and thus was entirely unsuited for the purpose. To get the values of sines of angles not occurring in the table, recourse was taken to interpolation formulae, which were incorrect because the law of variation of the difference was not known.

1.3 A differential formula

Mañjula (932) was the first Hindu astronomer to state that the difference of the sines,

$$\sin w' - \sin w = (w' - w)\cos w,$$

where $(w' - w)$ is small.

He says:

> True motion in minutes[4] is equal to the cosine (of the mean anomaly) multiplied by the difference (of the mean anomalies) and

[4]This clearly shows that the formula is intended for use when difference is small, the result being expressible in minutes.

> divided by the *cheda*,[5] added or subtracted contrarily (to the mean motion).[6]

Thus according to Mañjula formula (1) becomes

$$u' - u = v' - v \pm e(w' - w)\cos w, \tag{2}$$

which, in the language of the differential calculus, may be written as

$$\partial u = \partial v \pm e \cos\theta \, \partial\theta.$$

We cannot say exactly what was the method employed by Mañjula to obtain formula (2). The formula occurs also in the works of Āryabhaṭa II (950),[7] Bhāskara II (1150),[8] and later writers. Bhāskara II indicates the method of obtaining the differential of sine θ, His method is probably the same as that employed by his predecessors.

1.4 Proof of the differential formula

Let a point P (See Figure 1) move on a circle. Let its position at two successive intervals be denoted by P and Q. Now, if P and Q are taken very near each other, the direction of motion in the interval PQ is the same as that of the tangent at P. Let PT be measured along the tangent at P equal to the arc PQ. Then PT would be the motion of the point P if its velocity at P had not changed direction.

Discussing the motion of planets, Bhāskarācārya says:

> The difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its (*sphuṭa*)*gati* (true rate of motion) for that interval of time.
>
> This is indeed rough motion (*sthūlagati*). I now describe the fine (*sūkṣma*) instantaneous (*tātkālika*) motion.[9] The *tātkālika-gati* (instantaneous motion) of a planet is the motion which it would have, had its velocity during any given interval of time remained uniform.

During the course of the above statement, Bhāskara II observes that the *tātkālika-gati* is *sūkṣma* ("fine" as opposed to rough), and for that the interval

[5] Here *cheda* (divisor) $= \frac{1}{e}$. According to Hindu astronomers $\frac{1}{e} = \frac{360}{P}$, where P is the periphery of the epicycle.

[6] *Laghumānasa*, ii. 7.

[7] *MSi* (=*Mahā-siddhānta*), iii. 15f.

[8] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36–7.

[9] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36 (c–d).

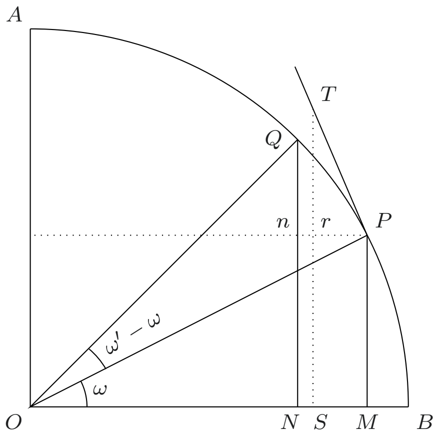

Figure 1

must be taken to be very small, so that the motion would be very small. This small interval of time has been said to be equivalent to a *kṣaṇa*[10] which according to the Hindus is an infinitesimal interval of time (immeasurably small).[11] It will be apparent from the above that Bhāskara did really employ the notion of the infinitesimal in his definition of *tātkālika-gati*.

But in actual practice, the intervals that are considered are not infinitesimal. How are we, then, to apply the notion of *tātkālika-gati* to actual problems? The answer to the above question is given by Bhāskara II as follows:

In equation (1) we have to consider the sine-difference $(\sin w' - \sin w)$. Let an arc of 90° be divided into n parts each equal to A, and let us consider the sine differences $R(\sin A - \sin O)$, $R(\sin 2A - \sin A)$, $R(\sin 3A - \sin 2A)$, etc. These differences are termed *bhogya-khaṇḍa*. Bhāskara II says:

> These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies—on account of deflection of the arc.

In the figure given above, let the arc $PQ = A$. Then

$$R(\sin \angle BOQ - \sin \angle BOP) = QN - PM = Qn$$

which is the *bhogya-khaṇḍa*. Bhāskara introduces the notion of *tātkālika bhogya-khaṇḍa* (instantaneous sine difference) in order to find the variation

[10] The smallest unit of time, according to Bhāskara II is a *truṭi* (*SiŚi*, *Gaṇita*, *Madhyamādhikāra*, *Kālamānādhyāya*, 6), which is equivalent to $\frac{1}{33750}$ of a second. The *kṣaṇa* is smaller, in fact the smallest interval of time that can be imagined.

[11] These remarks are made with reference to the motion of the moon. As the motion of the moon is comparatively quicker, so the *tātkālika-gati* will not give correct result unless the time interval is taken small enough.

of the sine at P. According to him if the arc BP instead of being deflected towards Q, be increased in the direction of the tangent, so that $PT = PQ = A$, then $TS - PM = Tr$ is the *tātkālika bhogya-khaṇḍa* of the sine PT, i.e. the "instantaneous sine difference". By having recourse to this artifice Bhāskara II avoids the use of the infinitesimal in his analysis. It should be borne in mind that the "instantaneous sine difference" for a finite arc PQ, is a purely artificial quantity created with a special end in view, and is different from the actual "sine difference" $R(\sin BOQ - \sin BOP)$.

Now from the similar triangles PTr and PMO, we at once derive the proportion[12]

$$R : PT :: R\cos w : Tr. \tag{3}$$

Therefore $Tr = PT \cos w$. But

$$Tr = R(\sin w' - \sin w) \quad \text{and} \quad PT = R(w' - w).$$

Therefore

$$(\sin w' - \sin w) = (w' - w)\cos\, w.$$

Thus the *tātkālika bhogya-khaṇḍa* (the instantaneous sine difference) in modern notation is

$$\partial(\sin\, \theta) = \cos\theta\, \partial\theta.$$

This formula has been used by Bhāskara to calculate the *ayana-valana* ("angle of position").[13]

If the above were the only result occurring in Bhāskara II's work, one would be justified in not accepting the conclusions of Pandit Bapu Deva Sastri. There is however other evidence in Bhāskara II's work to show that he did actually know the principles of the differential calculus. This evidence consists partly in the occurrence of the two most important results of the differential calculus:

(i) He has shown that when a variable attains the maximum value its differential vanishes.

(ii) He shows that when a planet is either in apogee or in perigee the equation of the centre vanishes. Hence he concludes that for some intermediate position the increment of the equation of centre (i.e. the differential) also vanishes.[14]

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[12] It should be noted that for the purpose of the following proof, it is immaterial, whether we take PQ small or not, because it is PT that we are considering and not PQ. Bhāskara actually takes the $\angle POQ = \left(3\frac{3}{4}\right)^\circ = 225'$ for exhibiting equation (3). The notion of the infinitesimal is here involved in the definition of *tātkālika bhogya-khaṇḍa*.

[13] *SiŚi*, *Golādhyāya*, *Grahaṇa*, *Grahaṇa-vāsanā*; see also Sen Gupta, *l.c.*, p. 11 ff.

[14] These results occur in the *Golādhyāya*, *Spaṣṭādhikāra vāsanā* of the *Siddhāntaśiromaṇi*, and were first noted by Sudhakara Dvivedi.

The second of the above results is the celebrated Rolle's Theorem, the mean value theorem of the differential calculus.

1.5 Remarks

The use of a formula involving differentials in the works of ancient Hindu mathematicians has been established beyond the possibility of any doubt. That the notion of instantaneous variation of motion entered into the Hindu idea of differentials as found in works of Mañjula, Āryabhaṭa II, and Bhāskara II is apparent from the epithet *tātkālika* (instantaneous) *gati* (motion) to denote these differentials. The main contribution of Bhāskara II to the theory of these differentials, which were already worked out by his predecessors, seems to be his proof of the formula by the rule of proportion without actually using the infinitesimal or varying quantities. He has, however, made it quite clear that the differentials give true results only when very small variations are concerned.

1.6 Nīlakaṇṭha's result

Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* has given proofs, on the theory of proportion (similar triangles) of the following results:

(i) The sine-difference $\sin(\theta + \partial\theta) - \sin\theta$ varies as the cosine and decreases as θ increases.

(ii) The cosine-difference $\cos(\theta + \partial\theta) - \cos\theta$ varies as the sine negatively and numerically increases as θ increases.

He has obtained the following formulae:

(i) $\sin(\theta + \partial\theta) - \sin\theta = 2\sin\frac{\partial\theta}{2}\ \cos\left(\theta + \frac{\partial\theta}{2}\right)$

(ii) $\cos(\theta + \partial\theta) - \cos\theta = -2\sin\frac{\partial\theta}{2}\sin\left(\theta + \frac{\partial\theta}{2}\right)$.

The above results are true for all values of $\partial\theta$ whether big or small. There is nothing new in the above results. They are simply expressions as products of sine and cosine differences.

But what is important in Nīlakaṇṭha's work is his study of the second differences. These are studied geometrically by the help of the property of the circle and of similar triangles. Denoting by $\Delta_2(\sin\theta)$, and $\Delta_2(cos\theta)$, the second differences of these functions, Nīlakaṇṭha's results may be stated as follows:

(i) The difference of the sine-difference varies as the sine negatively and increases (numerically) with the angle.

(ii) The difference of the cosine-difference varies as the cosine negatively and decreases (numerically) with the angle.

For $\Delta_2(\sin\theta)$, Nīlakaṇṭha[15] has obtained the following formula:

$$\Delta_2(\sin\theta) = -\sin\theta\left(2\sin\frac{\Delta\theta}{2}\right)^2.$$

Besides the above, Nīlakaṇṭha, has made use of a result involving the differential of an inverse sine function.[16] This result, expressed in modern notation, is

$$\partial(\sin^{-1} e\sin w) = \frac{e\cos w}{\sqrt{1-e^2\sin^2 w}}\partial w.$$

In the writings of Acyuta (1550–1621 AD) we find use of the differential of a quotient[17] also

$$\partial\left[\frac{e\sin w}{1\pm\cos w}\right] = \frac{e\cos w \pm [(e\sin w)^2/(1\pm e\cos w)]}{1\pm e\cos w}\partial w.$$

2 Method of infinitesimal-integration

2.1 Surface of the sphere

For calculating the area of the surface of a sphere Bhāskara II (1150) describes two methods which are almost the same as we usually employ now for the same purpose.

First method

> Make a spherical ball of clay or of wood. On it take a (vertical) circumference circle and divide this into 21600 parts. Mark a point on the top of it. With that point as the centre and with the radius equal to the 96th part of the circumference, i.e. to 225′, describe a circle. Again with same point as the centre with twice that arc as radius describe another circle; with thrice that another circle; and so on up to 24 times. Thus there will be 24 circles in all. The radii of these circles will be the *jyā* 225′ (= $R\sin 225'$), etc. From them the lengths of the circles can be determined by proportion. Now the length of the extreme circle is 21600′ and its radius is 3438′. Multiplying the *R*sines (of 225′, 450′, etc.) by 21600 and dividing by 3438, we shall obtain the lengths of the circles. Between two

[15] This together with the results given above are proved by Nīlakaṇṭha in the commentary on the *Āryabhaṭīya*, ii. 12.
[16] *Cf. Tantrasaṅgraha*, ii. 53–4.
[17] *Cf. Sphuṭa-nirṇaya-tantra*, iii, 19–20; *Karaṇottama*, ii. 7.

> of these circles there lie annular strips and there are altogether 24 such. They will be many more in case of many *R*sines being taken into consideration (*bāhujyā-pakṣe-bahūni syuḥ*). In each annulus considering the larger circle at the lower end as the base and the smaller circle at the top as the face and 225′ as the altitude (of the trapezium), find its area by means of the rule 'half the sum of the base and the face multiplied by the altitude etc.'[18] Similarly the areas of all the annular figures severally can be found. The sum of all these areas is equal to the area of the surface of half of the sphere. So twice that is the area of the surface of the whole sphere. And that is equal to the product of the diameter and the circumference.[19]

In other words, if T_n denotes the nth *jyā* (or *R*sine), C_n the circumference of the corresponding circle, A_n the area of the nth annulus, and S the area of the surface of the sphere, then we shall have

$$\begin{aligned} C_n &= \frac{21600}{3438} \times T_n \\ A_n &= \frac{C_n + C_{n-1}}{2} \times 225 \\ &= \frac{225 \times 21600}{2 \times 3438}(T_n + T_{n+1}). \end{aligned}$$

Therefore,

$$\frac{1}{2}S = \sum A_n = \frac{225 \times 21600}{2 \times 3438} \sum (T_n + T_{n+1})$$

the summation being taken so as to include all the *R*sines in a quadrant of the circle. Since there are ordinarily 24 *R*sines in a Hindu trigonometrical table, we have

$$\begin{aligned} \frac{1}{2}S &= \frac{225 \times 21600}{3438} \sum (T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}) \\ &= \frac{21600 \times 225 \times 52513}{3438} \\ &= 21600 \times 3436.7\ldots \end{aligned}$$

Hence approximately

$$S = 21600 \times 2 \times 3437.$$

Bhāskara II states:

Area of the surface = circumference × diameter.

[18] The rule quoted here for finding the area of a trapezium is that given by Śrīdhara (*Triś*, R. 42). Bhāskara II's rule is defined slightly differently (*vide L*, p. 44).

[19] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 55–7 (gloss).

Second method

> Suppose the (horizontal) circumference-circle on the surface of the sphere to be divided into parts as many as four times the number of *R*sines (in a quadrant). As the surface of an emblic myrobalan is seen divided into *vapras* (i.e. lunes) by lines passing through its face (or top) and bottom, so the surface of the sphere should be divided into lunes by vertical circles as many as the parts of the above mentioned (horizontal) circumference-circle. Then the area of each lune should be determined by (breaking it up into) parts. And this area of a lune is equal to the sum of all the *R*sines diminished by half the radius and divided by the semi-radius. Since that is again equal to the diameter of the sphere, so it has been said that the area of the surface of a sphere is equal to the product of its circumference and diameter.[20]

The method has been further elucidated by him in his gloss thus:

> As many as are *R*sines in the table of any particular work selected, take four times that number, and suppose the (horizontal) circumference-circle on the sphere is to be divided into, as many parts. Like the natural lines seen on the surface of a round emblic myrobalan passing through its face and base and thus dividing it into lunes, draw circles on the surface of the given sphere, passing through its top and bottom and thereby dividing it into lunes as many as the number of parts into which the (horizontal) circumference-circle is divided. Next the area of each lune has to be determined. It can be done thus: For instance in the *Dhīvrddhida*,[21] there are 24 *R*sines. So suppose the (horizontal) circumference-circle measures 96 cubits. On drawing the vertical circles through every cubit, there will be as many lunes. Then the upper half of any one lune on drawing the transverse arcs at distances of every cubit, will be divided into portions equal to the number of *R*sines, that is, 24. The lengths of these transverse lines will be obtained by dividing the *R*sines severally by the radius. Of these the lowest line measures one cubit; but the upper and upper ones are a little smaller and smaller according to the *R*sines. But the altitude is all along one cubit in length. Now by finding the area of each portion in accordance with the rule, "half the sum of the top and the base multiplied by the altitude etc." they should be added together. This sum gives the area of half a lune; twice

[20] *Ibid*, verses 58–61.
[21] That is *Śiṣyadhīvṛddhida* of Lalla.

> that is the area of a lune. For the determination of that the rule is, "the sum of all the Rsines minus half the radius etc." Now the sum of all the Rsines, 225 etc., is 54233.[22] This diminished by the semi-radius becomes 52514. Dividing the result by the semi-radius we get the area of each lune as 30;33. Now 30;33 is equal to the diameter of a circle whose circumference measures 96. And as the number of lunes is equal to the number of portions of the circumference it is consequently proved that the area of the surface of a sphere is equal to the product of its circumference and diameter.

If l_n denotes the length of the nth transverse arc, we have

$$l_n = \frac{T_n \times 1}{R}.$$

Therefore,

$$\begin{aligned}\text{area of a lune} &= 2 \times \sum \frac{1}{2}(l_n + l_{n+1}) \times 1 \\ &= 2 \sum \frac{1}{2R}(T_n + T_{n+1})\end{aligned}$$

the summation being taken so as to include all the Rsines. Hence

$$\begin{aligned}\text{area of a lune} &= 2 \times \frac{1}{R}\left(T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}\right) \\ &= \frac{1}{R/2}\left(T_1 + T_2 + \cdots + T_{24} - \frac{R}{2}\right) \\ &= 30;32,94\ldots\end{aligned}$$

Now

$$96 \times \frac{1250}{3927} = 30;33,46\ldots$$

Hence the area of a lune is numerically equal to the diameter of the sphere. As the number of lunes is equal to the number of parts of the circumference of the sphere, we get

$$\text{Area of the surface} = \text{circumference} \times \text{diameter}.$$

2.2 Volume of the sphere

To find the volume of a sphere Bhāskara II states the following method:

> Consider on the surface of the sphere pyramidal excavations, each of a base of a unit area having unit sides and of a depth equal to

[22]According to Lalla the sum of the Rsines is 54233.

> the radius, as many as the number of units of area in the surface. The apices of these pyramids meet at the centre of the sphere. The sum of the volumes of the pyramids is equal to the volume of the sphere. So it is proved (that the volume of a sphere is equal to the sixth part of the product of the surface area and diameter).[23]

The above results are the nearest approach to the method of the integral calculus in Hindu Mathematics. It will be observed that the modern idea of the "limit of a sum" is not present. This idea, however, is of comparatively recent origin so that credit must be given to Bhāskara II for having used the same method as that of the integral calculus, although in a crude form.

[23] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 58–61, (gloss).