Hindu geometry *

1 General survey

1.1 Origin of Hindu geometry

The Hindu Geometry originated in a very remote age in connection with the construction of the altars for the Vedic sacrifices. The sacrifices, as described in the Vedic literature of the Hindus, were of various kinds. The performance of some of them was obligatory upon every Vedic Hindu, and hence they were known as *nitya* (or "obligatory", "indispensable"). Other sacrifices were to be performed each with the purpose of achieving some special object. Those who did not aim at the attainment of any such object had no need to perform any of them. These sacrifices were classed as *kāmya* (or "optional", "intentional"). According to the strict injunctions of the Hindu scriptures, each sacrifice must be made in an altar of prescribed shape and size. It was emphasised that even a slight irregularity and variation in the form and size of the altar would nullify the object of the whole ritual and might even lead to an adverse effect. So the greatest care had to be taken to secure the right shape and size of the altar. In this way there arose in ancient India problems of geometry and also of arithmetic and algebra. There were multitudes of altars. Let us take for instance the three primary ones, viz. the *Gārhapatya*, *Āhavanīya* and *Dakṣiṇa*, in which every Vedic Hindu had to offer sacrifices daily. The *Gārhapatya* altar was prescribed to be of the form of a square, according to one school, and of a circle according to another. The *Āhavanīya* altar had always to be square and the *Dakṣiṇa* altar semi-circular. But the area of each had to be the same and equal to one square *vyāma*.[1] So the construction of these three altars involved three geometrical operations: (i) to construct a square on a given straight line; (ii) to circle a square and vice versa; and (iii) to double a circle. The last problem is the same as the evaluation of the surd $\sqrt{2}$. Or it may be considered as a case of doubling a square and then circling it. There were altars of the shape of a falcon with straight or bent wings, of a square, an equilateral triangle, an isosceles trapezium, a circle, a wheel (with or without spokes), a

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 15, No. 2 (1980), pp. 121–188.

[1] 1 *vyāma* = 96 *aṅgulis* (or "finger breadths") = 2 yards.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_15

tortoise, a trough and of other complex forms all having the same area. Again at the second and each subsequent construction of an altar, it was necessary to increase its size by a specified amount, usually one square *puruṣa*,[2] but the shape was always kept similar to that of the first construction. Thus there arose problems of equivalent areas and transformation of areas. The Vedic geometers also treated problems of 'application of areas'.

1.2 Different early schools of geometry

In the course of time, Hindu geometry grew beyond its original sacrificial purpose or the bounds of practical utility and began to be cultivated as a science for its own sake. This happened in the Vedic age when different schools of geometry were founded. More notable ones amongst them were the schools of Baudhāyana, Āpastamba and Kātyāyana. Though the geometrical propositions treated in all of them were almost the same, and there were many things common in the methods of their solution, still there were other things to distinguish one school from another. Even in the solution of elementary propositions such as the construction of a square, rectangle or an isosceles trapezium, different schools had preferential liking for differential methods. The difference appears most marked in the solution of the problems of the division of figures. The large altars, of which the fundamental one was of the shape of a falcon, had to be built with 200 bricks. Geometrically, it was a case of division of a figure into 200 parts. We have described before how the different Vedic Schools of Geometry did this in different ways.

1.3 Intuitive and demonstrative geometry

Early Hindu geometers did not describe proofs of the propositions discovered by them. Only the bare results were recorded and those too in a language as concise as possible, sometimes even to the fault of ellipticity. This was, of course, in keeping with the characteristic of the Hindu race and was manifested in all their early works. Indeed the character of all the sciences of all the early nations is found to be more or less intuitive. Still the Vedic Geometry, as found in the manuals of the *Śulba*, was not wholly intuitional without any semblance of demonstration. In fact we find a kind of proof in case of certain propositions of the *Śulba*. For instance, how to find the area of a trapezium, has been demonstrated by Āpastamba in the course of the mensuration of the *Mahāvedi* which is of the shape of an isosceles trapezium whose altitude, face and base are respectively 36, 24 and 30 *padas* (or *prakramas*). He says:

> The *Mahāvedi* measures (in area) one thousand less twenty-eight (square) *padas*. Draw a straight line from the south-eastern corner

[2] 1 *puruṣa* = 120 *aṅgulis* = $2\frac{1}{2}$ yards.

of the *vedi* to a point 12 *padas* towards the south-western corner. Place the portion thus cut off on the other (i.e. the northern) side of the *vedi* after inverting it. It (the *Mahāvedi*) will then become a rectangle. After that construction the area will be apparent.[3]

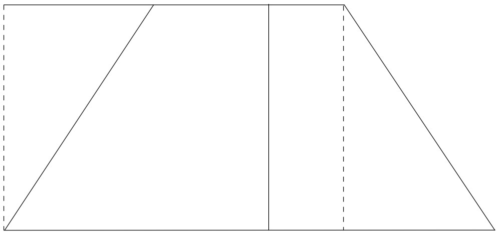

Figure 1

After the general enunciation of the theorem of the square of the diagonal, the so-called Pythagorean theorem, *Baudhāyana* observes that the truth of it will be "realised" in case of certain rational rectangles enumerated. This is an attempt for a kind of demonstration. After describing the constructions necessary in a proposition, the early Hindu geometers are found to have remarked *sa samādhiḥ* (or "This is the construction"). The significance of such an observation is obvious. It emphasises that the construction which was required to be made, has thus been actually made, and indeed corresponds to the expression *Quod Erat Faciendum* (or "what was required to do") occurring at the end of a proposition of Euclid's *Elements*. Further it discloses a rational and demonstrative attitude of the mind of the early Hindu geometers.[4]

1.4 Post-vedic geometry

The Hindu geometry which started in a brilliant way not only by going much in advance of the ancient Egyptian or Chinese geometry but also by anticipating some of the notable discoveries of the posterior Greek geometry, did not make much progress in the post-Vedic period as it ought to have done. In this period there was renaissance of Hindu Mathematics.[5] But compared with arithmetic and algebra, geometry seems to have received little impetus for further development. It will not be true to think that the study of geometry was completely neglected by the Hindus of the early renaissance period. On the other hand, it is found to have become widespread and came to be regarded as a part of general education of the people. In an early Jaina canonical

[3] *Āpastamba Śulba*, v. 7.

[4] See Datta, B., *The Science of the Śulba*, pp. 50f.

[5] See Datta, Bibhutibhusan, "The Scope and Development of the Hindu *Gaṇita*", *Ind. His. Quart.*, V, (1929), pp. 479 ff. We have drawn here heavily on this article.

work, composed circa 300 BC we find the remark, "Geometry is the lotus in Mathematics, ... and the rest is inferior."[6] But it appears strange that we do not find evidence of much progress and improvement in geometry. The notable contributions of this period to geometry are, however, the discovery of the ellipse, elliptic cylinder, the value $\pi = \sqrt{10}$ and certain formulae for the mensuration of the segment of a circle. The value $\pi = \sqrt{10}$, though not a fairly accurate one, is an improvement upon the *Śulba* value. It occurs as early as in the *Sūryaprajñapti* (c. 500 BC).[7] The ellipse is called *viṣama-cakravāla*, in contradistinction to *cakravāla*, meaning "circle" in the *Sūryaprajñapti*,[8] and *parimaṇḍala* in the *Dhammasaṅgani* (before 350 BC)[9] and *Bhagavatī-sūtra* (c. 300 BC).[10] In the last mentioned work its form has been described as the *yavamadhya-vṛttasaṃsthāna* or "the circular figure resembling the middle (longitudinal section) of a barley corn."[11] It seems to have been known that the ellipse is symmetrical about its either axis.[12] The mention of the elliptic cylinder, called *ghana-parimaṇḍala* (or "solid ellipse") in contradistinction to *pratara-parimaṇḍala* ("plane ellipse") occurs in the *Bhagavatī-sūtra*.[13]

1.5 Later Hindu geometry

Later Hindu geometry consists mainly of some mensuration formulae and solution of certain rectilinear figures such as triangles and quadrilaterals of different varieties. In some of these the Hindus undoubtedly showed considerable proficiency and indeed they obtained some remarkable results, e.g. a new proof of the Pythagorean theorem, formulae for the area and diagonals of an inscribed convex quadrilateral and rational solution of triangles and cyclic quadrilaterals. But on the whole their geometry remained empirical. There were no definitions, no postulates, no axioms, no proofs of theorems, in short, no scientific treatment of the subject. It is perhaps noteworthy that the later Hindus included geometry in their treatises of arithmetic (*pāṭīgaṇita*) more particularly in the sections on *kṣetra* ("plane figures"), *khāta* ("excavations"), *citi* ("piles of bricks"), *rāśi* ("maunds of grain") and *krākacika* ("saw"). The last four topics are pertaining to solid figures.

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[6] *Sūtrakṛtāṅga-sūtra*, 2nd *Śrutaskanda*, ch. 1, verse 154.
[7] *Sūtra* 20.
[8] *Sūtra* 19, 25, 100. See Weber, *Indische Studien*, X, p. 274.
[9] Sec. 617.
[10] *Sūtra* 726–7.
[11] *Bhagavatī-sūtra*, *Sūtra* 725. Bhuddhaghosa (350) describes it as *kukkuṭāṇda-saṃsthāna* (or "a figure of the shape of an egg of a hen") and the *Petavattu* commentary as the *āyatavṛtta* (or "the elongated circle").
[12] Compare *Bhagavatī-sūtra, Sūtra* 726.
[13] *Sūtra* 726.

1.6 Euclid's *Elements* in India

Though Hindu geometry is not connected with Euclid's *Elements* in any way, whether directly or indirectly, it will be interesting to know when and how it came to India. The earliest attempt, as far as known, to introduce Euclid's *Elements* into India, in the garb of Sanskrit verses, was made by the eminent Persian mathematician and traveller, Al-Bīrūnī (b. 973). But that attempt did not succeed. With the establishment of Muhammadan supremacy in India towards the close of the twelfth century of the Christian era, Arabic and Persian works on mathematics began to be brought into this country. There were very likely amongst them Arabic versions of the *Elements.* King Firuz Shah Bahmani (1397–1422), we are informed by Ferishta, was used to hear on three days in a week, lectures on botany, geometry and logic.[14] A son of Daud Shah was very fond of *Taḥrīr-u-Uqlidas* (Euclid's *Elements*) and used to teach it regularly to his students.[15] Akbar (1575) included it into the course of study for the school boys.[16] In his *Ain-i-Akbari*, Abul Fazl (1590) has referred to a few propositions of the *Elements* in a way which shows his thorough acquaintance with the work. The work, however, remained confined to the circle of Moslem schools in India. We do not find any trace of its influence in any work of a Hindu writer before the middle of the seventeenth century. In 1658 AD Kamalākara, the court-astronomer of the Emperor Jahangir of Delhi, wrote a voluminous treatise on astronomy entitled *Siddhānta-tattva-viveka.* Certain passages in this work can be easily recognised to have been adapted from Euclid's *Elements.*[17] The first complete translation of the work in Sanskrit was made in 1718 AD under the title *Rekhāgaṇita* ("Mathematics of lines") by Samrāṭa Jagannātha, by the order of his patron King Jaya Siṃha of Jaipur. Another Sanskrit version is known as the *Siddhānta-Cūḍāmaṇi.* The author of this version is still unknown.

2 Hindu names for geometry

The Hindu name for the science of geometry has varied from time to time.[18] The earliest name was *śulba.* It is at least as old as the *Śrautasūtra* of Āpastamba (c. 1000 BC). Geometry was then sometimes also called *rajju*, as is evident from the opening aphorism of the *Śulba* of Kātyāyana, "I shall speak of the collection of (rules regarding) the *rajju*". In the *Mānava Śulba* and

[14] Law, N. N., *Promotion of Learning in India during Muhammadan Rule* (by Muhammadans), 1916, p. 84.
[15] *Ibid*, p. 81, footnote 1.
[16] Abul Fazl's *Ain-i-Akbari*, English translation by Blockmann, p. 279.
[17] See *Siddhānta-tattva-viveka*, iii. 22 ff.
[18] Datta, Bibhutibhusan, "Origin and history of the Hindu names for Geometry", *Quellen und. Studien z. Gesh. d. Math.*, Ab. B; Bd, I, pp. 113–9.

Maitrāyaṇīya Śulba we get the name *Śulba-vijñāna* ("The Science of the *Śulba*") for the science of geometry. In the early canonical works of the Jainas (500–300 BC) the more common name for geometry is found to be *rajju*.

The Sanskrit words *śulba* and *rajju* have the identical significance, which is ordinarily "a rope", "a cord". The word *śulba* (or *śulva*) is derived from the root *śulb* (or *śulv*) meaning "to measure" and hence its etymological significance is "measuring" or "act of measurement". From that it came to denote "a thing measured" and consequently " a line (or surface)" as well as "an instrument of measurement" or "the unit of measurement". Thus the terms *śulba* and *rajju* have four meanings: (i) mensuration—the act and process of measuring; (ii) line (or surface)—the result obtained by measuring; (iii) a measure—the instrument of measuring; and (iv) geometry—the science of measurement. Mention of a linear measure, called *rajju* is found in the *Āpastamba-śulba*, *Mānava-śulba*, *Arthaśāstra* of Kauṭilya and later on in the *Śilpa-śāstra*. In fact in ancient India, there were three kinds of measures—linear, superficial and voluminal—having the same epithet *rajju*. In the Jaina canonical works they are sometimes distinguished as *sūcī-rajju* ("needle-like or linear *rajju*"), *pratara-rajju* ("superficial *rajju*") and *ghana-rajju* ("cubic *rajju*"). In the *Arthaśāstra* of Kauṭilya the superficial unit of *rajju* is called *parideśa* and the cubical unit *nivartana*. In the works on the *Śulba*, we find the use of the word *rajju* in the sense of a measuring tape as also of a line.

In later times, geometry was called by the Hindus *kṣetra-gaṇita* ("Mathematics of the *kṣetra*"). This term appears in the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In this work the term *kṣetra* denotes a plane figure. In the mathematical treatises of Brahmagupta (628), Śrīdhara (900) and Bhāskara II (1150), the section devoted to the treatment of plane figures is called *kṣetra-vyavahāra* ("Treatment of plane figures"). The epithet *kṣetra-gaṇita* occurs as early as the works of Siddhasena Gaṇi (550). There the term *kṣetra* has a wider connotation so as to include both areas and volumes. In the same significance it appears in the title of the Jaina cosmographical works called *kṣetra-samāsa*. We think that the term *kṣetra-gaṇita* had a wider connotation in the beginning so as to include the geometry of plane as well as solid figures. But in later times, when the two branches of geometry began to be treated separately, the old name was reserved only for the geometry of plane figures.

Jagannātha (1718) called his version of Euclid's *Elements* the *Rekhāgaṇita* ("Mathematics of lines"). Bāpūdeva Śāstri preferred the name *kṣetra-miti* ("Measurement of areas and volumes"). He seems to have intended an accurate translation of the Greek name, but it is less scientific. For the Greek science is indeed the geometry of lines, but not the geometry of areas and volumes. Jagannātha's epithet is more in keeping with the spirit of the Greek geometry. He had probably discarded the Greek epithet intentionally as it is a misnomer.

In some of the modern vernacular tongues of India, geometry is now more commonly known as *kṣetra-tattva* ("Principles of areas and volumes") or *jyāmiti.* This latter term is highly interesting because it is very alike the Greek term "geometry", not only phonetically but also in significance, and at the same time it is not a hinduised Greek word. The word *jyāmiti* is a compound of pure Sanskrit origin derived from *jyā*, meaning 'earth' and *miti*, meaning "measure". Hence its literal significance is "earth-measurement". It is thus clearly a translation of the Greek name.

One who was well versed in the science of geometry was called in ancient India as *saṃkhyājña* ('the expert in numbers'), *parimāṇajña* ('the expert in measuring'), *sama-sūtra-nirañchaka* ('uniform-rope-stretcher'), *śulba-vid* ('the expert in the *śulba'*) and *śulba-paripṛcchaka* ('the inquirer into the *śulba*'). In the *Śilpa-śāstra*, he is spoken of as the *sūtra-grāhī* or *sūtra-dhāra* ('rope-holder') and he is further described as an expert in alignment (*rekhājña*, lit. 'one who knows the line'). In the early *Pāli* literature we find the terms *rajjuka* and *rajju-grāhaka* ('rope-holder') for the king's land-surveyor. The first of these terms appears copiously in its various case-endings, in the inscriptions of the Emperor Aśoka (250 BC).

3 Technical terms

3.1 Line

The history of a few technical terms of Hindu geometry will be considered here. There is no attempt to define those terms in any early work. Only in a work of the seventeenth century, *Siddhānta-tattva-viveka* of Kamalākara (1658), we come across some definitions but, as already stated, it was influenced by Euclid's *Elements.* The line is called in the *śulba*, *rekhā* or *lekhā*, both the terms being identical as, according to the rules of Sanskrit grammar, the letters *r* and *l* can replace each other. In posterior geometry we, however, commonly meet with the term *rekhā* only. A straight line is distinguished with the help of the qualifying adjective *ṛju* or *sarala*, meaning "straight".

3.2 Rectilinear figures

In Hindu geometry, we find two different systems of nomenclature for the rectilinear geometrical figures.[19] In one system the naming is according to the number of sides of the figures and the names are formed by juxtaposition of the number names with *bhuja*, meaning "arm", "side"; e.g. *tribhuja* ('trilateral'), *catur-bhuja* ('quadrilateral'), *pañca-bhuja* ('pentalateral'), *ṣaḍ-bhuja* ('hexa-lateral'). In the other, the naming is based on the number of angles

[19]The subject has been treated fully in an article of Datta, B. *JASB* (new series), Vol. XXVI (1930), pp. 283–299; see also his *Śulba*, pp. 221–6.

and corners in the figures, and the names are compounds of number names with *karṇa* or *koṇa*. The Sanskrit word *karṇa* means the ear. Applied to geometrical figures, it implies, the angle.[20] In the *Katyāyana Śulba*[21] (c. 500 BC), we find the terms *trikarṇa* ('triangle'), *pañca-karṇa* ('pentangle'). The word *karṇa* degenerated into *koṇa* in the *Prākṛta* languages.[22] So in the Ardha-Māgadhī work, *Sūryaprajñapti*[23] (c. 500 BC), we get *tri-koṇa* ('trigonon'), *catuṣkoṇa* ('tetragonon'), *pañca-koṇa* ('pentagon'), etc. These terms are, however, accepted in posterior Sanskrit literature.[24] The oldest Hindu compound name for rectilinear figures ending with *srakti* meaning the angle or corner, is *catuḥsrakti* ('quadrangle') which occurs in the *Saṃhitās* and the *Brāhmaṇas* (c. 2000 BC). In the time of the *Śrauta-sūtra* (c. 2000–1500 BC), was introduced another kind of name consisting of compounds of number names with *aśra* or *asra*, e.g. *tryasra*, *caturasra*, etc. Though these words *aśra* and *asra*, ordinarily mean "corner" or "angle", in compound names for rectilinear figures, they are sometimes found to denote "side". It is perhaps noteworthy that like the early Hindus, the early Greeks also followed the usage of naming the rectilinear figures according to the number of sides as well as of angles.[25] But while with the Hindus the angle-nomenclature is older than the side-nomenclature, with the Greeks quite the contrary is the case.[26]

Triangles are classified according to the sides: *sama-tribhuja* ('equilateral triangle'), *dvisama-tribhuja* ('isosceles triangle') and *viṣama-tribhuja* ('scalene triangle'). The classification according to the angles is not found here. Only the right-angled triangle is called by the name *jātya-tribhuja* by Brahmagupta and others.[27] The oblique triangles are grouped according as the perpendicular (*lamba*) from the vertex on the base falls inside or outside the figure, viz. *antarlamba* ('in-perpendicular') and *bahir-lamba* ('out-perpendicular'). In the *Taittirīya Saṃhitā* (c. 3000 BC), the *Brāhmaṇa* (c. 2000 BC) and the *Śulba*, an isosceles triangle is called *prauga*, derived probably from *pra* + *yuga*, meaning "the fore part of the shafts of a chariot". A rhombus is similarly called

[20]The term *karṇa* is used to denote the hypotenuse of a right-angled triangle (*vide infra*).

[21]iv. 7–8.

[22]Some writers are of opinion that the word *koṇa* is derived from Greek sources, but we do not think so.

[23]*Sūtra* 19, 25.

[24]See for instance, *Pariśiṣṭas of the Atharva-Veda*, xxiii. 1; 5; xxv, 1, 3, 6, 7, etc.; *Artha-śāstra* of Kauṭilya, ii. 11, 29.

[25]Tropfke, J. *Geschichte der Elementar-Mathematik*, (1923), Bd. IV. pp. 60–1.

[26]The conjecture of S. Gandz that "the observation of the corners and angles and the classification according to their number seem to be distinctly Greek, a specific invention of the Greek science, based upon the introduction of angle-geometry" is erroneous. *Vide* his article on "The origin of angle-geometry" in *Isis*, XII, pp. 452–481; more particularly p. 473.

[27]The Sanskrit word *jātya* means "noble", "well-born", "genuine". The name *jātya-tribhuja* for the right-angled triangle seems to imply that all other triangles are derived from it.

ubhayataḥ prauga (‘*prauga* on both sides’).[28]

In the *Śulba*[29] the diagonal of a rectilinear figure is called the *akṣṇa* or *akṣṇayā* (‘that which goes across or transversely’, i.e. ‘the cross line’); also *karṇa*, meaning ‘the line going across the *karṇa* or angle’, or ‘the line going across from corner to corner’. Referring to the instrument of measurement, it is sometimes termed the *akṣṇayā-veṇu* (‘diagonal bamboo-rod’) or *akṣṇayā-rajju* (‘diagonal cord’). Out of all these only the term *karṇa* has survived, others have become obsolete.

The classification of quadrilaterals according to the sides as well as the angles is found as early as the *Sūryaprajñapti*. There are generally distinguished five kinds of quadrilaterals; *sama-caturbhuja* (‘square’), *āyata-caturbuja* (‘rectangle’), *dvisama-caturbhuja* (‘isosceles trapzium’), *trisama-caturbhuja* (‘equi-trilateral trapezium’), and *viṣama-caturbhuja* (‘quadrilateral of unequal sides’). Similarly we have the *sama-caturasra*, *āyata-caturasra*, *dvisama-caturasra* and *viṣama-caturasra* for those figures (*caturbhuja* = *caturasra* = quadrilateral.) In the *Śulba*, the square is generally called *sama-caturasra* and the rectangle *dīrgha-caturasra* (‘longish quadrilateral’).

3.3 Circle

In early geometry, the circle was termed *maṇḍala* (‘round’) or *pari-maṇḍala* (‘round on all sides’); the circumference, *pariṇāha* (‘surrounding boundary line’); the diameter, *viṣkambha* or *vyāsa* (‘breadth’); and the centre, *madhya* (‘middle’). The last term had, however, wider use so as to denote the middle most point of a square, rectangle or line. So also the terms *viṣkambha* and *vyāsa*. In Prākṛta works of the fourth century before the Christian era, the term *pari-maṇḍala* is used to denote the ellipse.[30] In later geometry, the term for the circle is *vṛtta*[31] and for the centre *kendra*.[32] The significance of the terms *vyāsa* and *viṣkambha* has now become fixed for the diameter of a circle. The radius is called *vyāsārdha* or *viṣkambhārdha* (‘semi-diameter’). These terms occur as early as the works of Umāsvāti (c. 150).[33] Still earlier in the *Āpastamba Śulba*, we find the term *ardha-vyāyāma*, having the identical significance.

[28] Datta, *Śulba*, pp. 223f.
[29] *Ibid*, pp. 224f.
[30] *Dhammasaṅgani* 617; *Bhagavatī-sūtra, Sūtra* 724–6. See Datta, *Hindu Contribution to Mathematics*, p. 8.
[31] See *Bhagavatī-sūtra, Sūtra* 724–6.
[32] In Hindu astronomy the term *kendra* is used to signify the anomaly.
[33] See his *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14; *Jambūdvīpa-samāsa*, ch. iv.

3.4 Surface and area

In the early Hindu geometry, a plane surface bounded by a figure was called by the term *kṣetra* and its area by *bhūmi*. Occasionally, however, the term *kṣetra* was employed also to signify area. In the canonical works of the Jainas (500–300 BC), a plane surface is termed *pratara* ('expanse'), and it is defined as that which is obtained by multiplying line by line. In posterior geometry, the *bhūmi*, together with its synonyms *bhū*, *mahī*, etc., signifying earth, denotes the ground or base of a plane figure; the area is called *kṣetraphala*, *kṣetra-gaṇita* or simply *phala*, or *gaṇita*. These terms carry the concept of specific operations of mensuration by breaking up the figure into smaller portions and calculating them so that the area is what is obtained as the result (*phala*) of such calculation (*gaṇanā*). Another term is more explicit. It is *sama-koṣṭha-miti* ('the measure of like compartments' or 'the measure of the number of equal squares'). A curved surface or surface of a solid is called its *pṛṣṭha* ('back'), from *dharā-pṛṣṭha* (or 'the back of the earth') which is rounded. The term for the superficial area of a solid is *pṛṣṭha-phala*.

4 Typical propositions of early geometry[34]

The *Śulba-sūtras*, which form a part of the Vedic literature of the Hindus, deal with the construction of fire altars for sacrificial purposes. At present we know of seven *Śulba-sūtras*, although it is quite likely that many more such works existed in ancient times. According to European scholars, these *Sūtras* were composed in the period 800 to 500 BC, but they are probably much older. The *vedīs* ('altars') dealt with in these *sūtras* are of various forms. Their construction requires a knowledge of the properties of the square, the rectangle, the rhombus, the trapezium, the triangle and the circle. The geometrical propositions involved in the constructions are the following.

4.1 Constructions

1. To divide a line into any number of equal parts.[35]

2. To divide a circle into any number of equal areas by drawing diameters.[36]

3. To divide a triangle into a number of equal and similar areas.[37]

[34] For details consult Datta, B., *The Science of the Śulba*, Calcutta, (1932).
[35] The knowledge of this construction is throughout assumed. It was probably done by drawing parallels, as in Euclid. The following construction shows this surmise to be correct.
[36] *BŚl*, ii. 73–4; *ĀpŚl*, vii. 13–14.
[37] *BŚl*, iii. 256; See Datta, *Śulba*, p. 46.

4. To draw a straight line at right angles to a given line.[38]
5. To draw a straight line at right angles to a given straight line from a given point on it.[39]
6. To construct a square on a given side.[40]
7. To construct a rectangle of given sides.[41]
8. To construct an isosceles trapezium of given altitude, face and base.[42]
9. To construct a parallelogram having given sides at a given inclination.[43]
10. To construct a square equal to the sum of two different squares.[44]
11. To construct a square equivalent to two given triangles.[45]
12. To construct square equivalent to two given pentagons.[46]
13. To construct a square equal to a given rectangle.[47]
14. To construct a rectangle having a given side and equivalent to a given square.[48]
15. To construct an isosceles trapezium having a given face and equivalent to a given square or rectangle.[49]
16. To construct a triangle equivalent to a given square.[50]
17. To construct a square equivalent to a given isosceles triangle.[51]
18. To construct a rhombus equivalent to a given square or rectangle.[52]
19. To construct a square equivalent to a given rhombus.[53]

[38] *KŚl*, i. 3.
[39] *Ibid.*
[40] *ĀpŚl*, viii. 8–10; xi. 1; i. 7; i. 2; *BŚl*, i. 22–28, 29–35, 42–44; iii. 13. *TS*, v, 2.5.1.; ff. *MaiS*, iii. 2.4; *KṭS*, xx. 3.4; *KapS*, xxxii. 5.6; *ŚBr* x. 2.3.8 (2000 BC), etc.
[41] *BŚl*, i. 36–40.
[42] *BŚl*, i. 41; *ĀpŚl*, v. 2–5.
[43] *ĀpŚl*, xix. 5.
[44] *BŚl*, i. 51–52; *ĀpŚl*, ii. 4–6; *KŚl*, ii. 22, iii. 1.
[45] This follows from the above.
[46] *BŚl*, iii. 68, 288; *KŚl*, iv. 8.
[47] *BŚl*, i. 58, *ĀpŚl*, ii. 7; *KŚl*, iii. 2, 3.
[48] *ĀpŚl*, iii. 1, *BŚl*, i. 53.
[49] *BŚl*, i. 55; *ŚBr*, x. 2.1.4.
[50] *BŚl*, i. 56.
[51] *KŚl*, iv. 5.
[52] *BŚl*, i. 57; *ĀpŚl*, xii. 9; *KŚl*, iv. 4.
[53] *KŚl*, iv. 6.

4.2 Theorems

The following theorems are either expressly stated or the results are implied in the methods of construction of the altars of different shapes and sizes:

1. The diagonals of a rectangle bisect each other. They divide the rectangle into four parts, two and two (vertically opposite) of which are equal in all respects.[54]

2. The diagonals of a rhombus bisect each other at right angles.

3. An isosceles triangle is divided into two equal halves by the line joining the vertex to the middle point of the base.[55]

4. The area of a square formed by joining the middle points of the sides of a square is half that of the original one.

5. A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

6. A parallelogram and rectangle on the same base and within the same parallels have the same area.

7. The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

8. If the sum of the squares on two sides of a triangle be equal to the square on the third side, then the triangle is right-angled.

4.3 The Baudhāyana theorem

Theorem 7 given above has been stated by Baudhāyana (c. 800 BC) in the following words:

> The diagonal of a rectangle produces both areas which its length and breadth produce separately.[56]

Āpastamba[57] and Kātyāyana[58] give the above theorem in almost identical terms. The theorem is now universally associated with the name of the Greek Pythagoras (c. 540 BC) though "no really trustworthy proof exists that it was actually discovered by him".[59] The Chinese knew the numerical relation for

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[54] Implied in *BŚl*, iii. 168–9, 178.
[55] *BŚl*, iii. 256.
[56] *BŚl*, i. 48: दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्गानी च यत्पृथग्भूते कुरुतस्तदुभयं करोति।
[57] *ĀpŚl*, i. 4.
[58] *KŚl*, ii. 11.
[59] Heath, *Greek Math.*, Vol. I, p. 144f.

the particular case $3^2 + 4^2 = 5^2$ probably in the time of Chou-Kong (d. 1105 BC).[60] The *Kahun* Papyrus (c. 2000 BC) contains four similar numerical relations, all of which can be derived from the above one.[61] As for the Hindus, one instance of that kind, $39^2 = 36^2 + 15^2$, occurs in the *Taittirīya Saṃhitā*[62] (before 2000 BC). It should be noted that this instance is different from that known to other early nations.

Although particular instances of the theorem are found amongst several ancient nations, the first enunciation of the theorem in its general form is found in India. It cannot be said what made Baudhāyana give the theorem in the general form. It is not improbable that he possessed a proof of the theorem. But what this proof was will never be known with certainty. Bürk, Hankel, Thibaut and Datta are of opinion that Baudhāyana knew a proof of the theorem.[63] It is conjectured that this proof may have been one of the following.

4.4 Hindu proofs

(i) Let $ABCD$ be a given square. Draw the diagonal AC; produce AB and cut off AE equal to AC (Figure 2). Construct the square $AEFG$ on AE. Join DE and on it construct the square $DHME$. Complete the construction as indicated in Figure 2. Now the square $DHME$ is seen to be comprised of four right-angled triangles each equal to DAE and the small square $ANPQ$. This square will be easily recognised to be equal to the square $CRFS$ and triangles equal to the rectangles $AERD$ and $ABSG$. Therefore, the square $DHME$ is equal to the sum of the squares $ABCD$ and $AEFG$. Hence the theorem.

It might be mentioned that constructions like the above are necessary in the usual course in the *Śulba*.

(ii) Let ABC be a right-angled triangle (**ed.** see Figure 3) of which the angle C is a right-angle. From C draw the perpendicular CD on AB. Then the triangle ABC, ACD and CBD are similar. Therefore,

$$AB : AC :: AC : AD,$$

or $AC^2 = AB \times AD$. Similarly, $CB^2 = AB \times DB$. Adding we get

$$AC^2 + CB^2 = AB^2.$$

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[60] Mikami. Y., *The Development of Mathematics in China and Japan*, Leipzig (1913), p. 7.
[61] These are $1^2 + \left(\frac{3}{4}\right)^2 = \left(1\frac{1}{4}\right)^2$, $2^2 + \left(1\frac{1}{2}\right)^2 = \left(2\frac{1}{2}\right)^2$, $8^2 + 6^2 = 10^2$, $16^2 + 12^2 = 20^2$.
[62] vi. 2.4.6.; It also occurs in the *Śatapatha Brāhmaṇa*, x. 2.3.4.
[63] Datta, *The Science of the Śulba*, ch. ix.

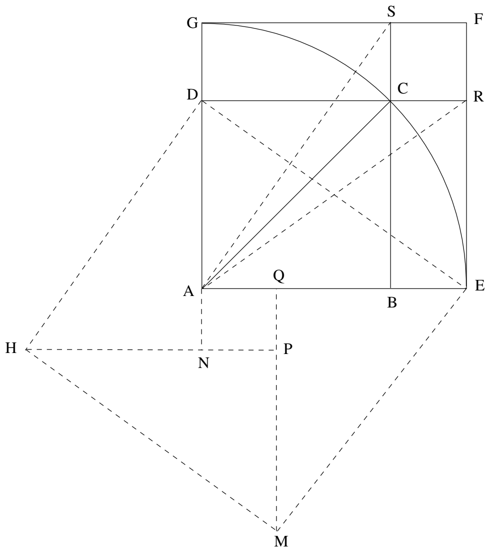


Figure 2

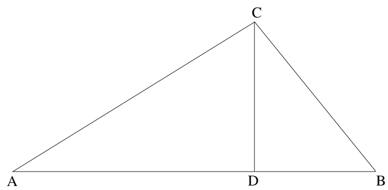


Figure 3

This proof is given by Bhāskarācārya,[64] and does not occur in the west until 1693 when it was rediscovered in Europe by Wallis.

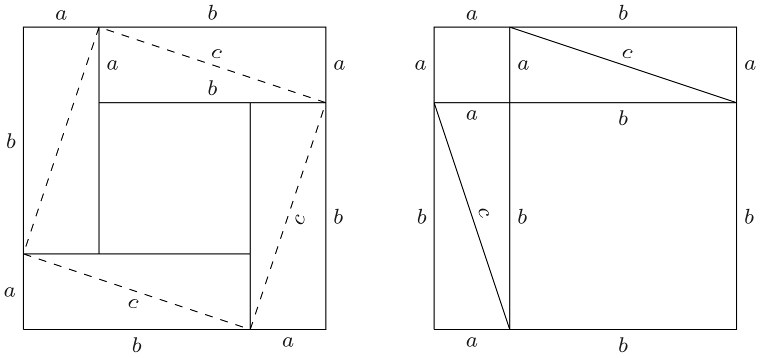
(iii) Let a, b, c be the sides of a right-angled triangle. Taking four such triangles they are arranged as in Figure 4a, inside a square whose side is equal to the hypotenuse of the given triangle. Obviously then,

$$c^2 = 4\left(\frac{ab}{2}\right) + (b-a)^2 = a^2 + b^2.$$

This proof was anticipated by the Chinese by several centuries.[65]

The technique employed in this proof was used by Āpastamba for the enlargement of a square. Thus to construct a square whose side will

[64] *Cf.* Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, London, 1817, pp. 221–2.
[65] Mikami, *l. c.*, p. 5.



(a) **(b)**

Figure 4

exceed a side b of a given square by a, add, says Āpastamba, on the two sides of the given square two rectangles whose lengths are equal to b and breadths to a; then add on the corner a square whose sides are equal to the increment a. Thus will be obtained a square with a side equal to $a + b$ (Figure 4b). A similar method was taught by Baudhāyana.[66]

4.5 Particular case

The particular case of the above theorem relating to the diagonal of a square has been stated thus:

> The diagonal of a square produces an area twice as much.

The statement is given in all the *Śulbasūtras*[67] and the theorem has been used for "doubling the square" at several places. Instances of its use are found in the *Taittirīya* (before 2000 BC) and other *Saṃhitās*, and can be traced back to the *Ṛgveda* (before 3000 BC).

Thibaut says:

> The authors of the *sūtras* do not give us any hint as to the way in which they found their proposition regarding the diagonal of a square; but we suppose that they, too, were observant of the fact that the square of the diagonal is divided by its diagonals into four triangles, one of which is equal to half the first square (Figure 5). This is at the same time an immediately convincing proof of the

[66] See Datta, *The Science of the Śulba*, p. 117.
[67] *BŚl*, i. 45; *ĀpŚl*, i. 5; *KŚl*, ii. 12; etc.

> Pythagorean proposition as far as squares or equilateral rectangular triangles are concerned.[68]

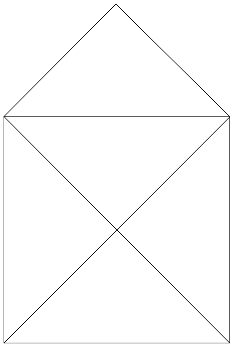


Figure 5

5 Measurement of triangles

5.1 Area of a triangle

The method for finding the area of a triangle that was known in the *Śulba*[69] was

$$\text{Area} = \frac{1}{2}(\text{base} \times \text{altitude}),$$

and that was one of the methods followed in later times. Āryabhaṭa I says:

> The area of a triangle is the product of the perpendicular and half the base.[70]

According to Brahmagupta:

> The product of half the sums of the sides and counter-sides of a triangle or a quadrilateral is the rough value of its area. Half the sum of the sides is severally lessened by the three or four sides, the square-root of the product of the remainders is the exact area.[71]

That is to say, if a, b, c, d, be the four sides of a quadrilateral taken in

[68] Thibaut, *Śulbasūtras*, p. 8.
[69] See Datta, *The Science of the Śulba*, p. 96.
[70] *Ā*, i. 6.
[71] *BrSpSi*, xii. 21.

order, we have

$$\text{Area} = \frac{c+d}{2} \times \frac{a+b}{2}, \text{ roughly;}$$
$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ exactly,}$$

where

$$s = \frac{1}{2}(a+b+c+d).$$

In case of a triangle $d = 0$; so that we get

$$\triangle = \frac{c}{2} \times \frac{a+b}{2}, \text{ roughly;}$$
$$\triangle = \sqrt{s(s-a)(s-b)(s-c)} \text{ exactly.}$$

The second formula was given before by the Greek Heron of Alexandria (c. 200).[72] Pṛthūdakasvāmi calculates by these methods the area of the triangle (14, 15, 13) to be 98 roughly, 84 exactly.

Śridhara says that the exact value of a triangle will be given by the formulae[73]

$$\triangle = \frac{1}{2} \text{ (base} \times \text{altitude),}$$
$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Mahāvīra,[74] Āryabhaṭa II,[75] and Śrīpati[76] teach both these accurate methods as well as the rough one of Brahmagupta. Bhāskara II[77] adopts the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

5.2 Segments and altitudes

Bhāskara I (629) writes:

> In a triangle the difference of the squares of the two sides or the product of their sum and difference is equal to the product of the sum and difference of the segments of the base. So divide it by the base or the sum of the segments; add and subtract the quotient to and from the base and then halve, according to the rule of concurrence. Thus will be obtained the values of the two

[72] Heath, *History of Greek Mathematics*, II, p. 321.
[73] *Triś*, R. 43.
[74] *GSS*, vii. 7, 50.
[75] *MSi*, xv. 66, 69, 78.
[76] *SiŚe*, xiii. 30.
[77] *L*, p. 41.

> segments. From the segments of the base of a scalene triangle, can be found its altitude.[78]

That is to say

$$a^2 - b^2 = (a+b)(a-b) = c_1^2 - c_2^2 = (c_1+c_2)(c_1-c_2),$$

also

$$c_1 + c_2 = c.$$

Therefore

$$c_1 - c_2 = \frac{a^2 - b^2}{c}.$$

Hence

$$c_1 = \frac{1}{2}\left(c + \frac{a^2-b^2}{c}\right),$$
$$c_2 = \frac{1}{2}\left(c - \frac{a^2-b^2}{c}\right),$$
$$h = \sqrt{a^2 - c_1^2} = \sqrt{b^2 - c_2^2}.$$

By means of these formulae Bhāskara I finds the segments (9, 5; 35, 16) of the bases (14, 51), altitudes (12, 12) and areas (84, 306) of the scalene triangles (13, 15, 14) and (20, 37, 51).

Brahmagupta (628) gives the same set of formulae. He says:

> The difference of the squares of the two sides being divided by the base, the quotient is added to and subtracted from the base; the results, divided by two, are the segments of the base. The square-root of the square of a side as diminished by the square of the corresponding segment is the altitude.[79]

Pṛthūdakasvāmi proves these formulae in the same way as Bhāskara I and also applies them to the latter's first example (13, 15, 14).

Śrīdhara first finds the area of the triangle by means of the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

and then deduces the segments and perpendicular. His rules are:

> Twice the area of the triangle divided by the base is the altitude. (Then there will be two right-angled triangles of which) the uprights are equal to that altitude, bases are the segments and hypotenuses, the two sides (of the given triangle).[80]

[78] *Vide* his commentary on *Ā*, ii. 6.
[79] *BrSpSi*, xii. 22.
[80] *Triś*, R. 50.

Mahāvīra says:

> Divide the difference between the squares of the two sides by the base. From this quotient and the base, by the rule of concurrence, will be obtained the values of the two segments (of the base) of the triangle; the square-root of the difference of the squares of a segment and its corresponding side is the altitude: so say the learned teachers.[81]

Āryabhaṭa II writes:

> In a triangle, divide the product of the sum and difference of the two sides by the base. Add and subtract the quotient to and from the base and then halve. The results will be the segments corresponding to the greater and smaller sides respectively. The segment corresponding to the smaller side should be considered negative, if it lies outside the figure. The square-root of the difference of the squares of a segment and its corresponding side is the perpendicular.[82]

Similar rules are given by Śrīpati[83] and Bhāskara II.[84] The latter gives in illustration a case of a scalene triangle whose have is 9, and sides 10, and 17. There the segments are 6 and 15, and perpendicular 8.

5.3 Circumscribed circle

Brahmagupta says:

> The product of the two sides of a triangle divided by twice the altitude is the heart-line (*hṛdaya-rajju*). Twice it is the diameter of the circle passing through the corners of the triangle and quadrilateral.[85]

Pṛthūdakasvāmi proves it substantially as follows:

Let ABC be a scalene triangle (**ed.** see Figure 6). Draw AD perpendicular to BC. Produce it to A' making $A'D = AD$. Let O be the centre of the circle circumscribing the triangle ABC. Join OA, OC. Triangles BAA' and OAC are similar. Therefore,

$$AB : OA :: AA' : AC.$$

[81] *GSS*, vii. 49.
[82] *MSi*, xv. 76–7.
[83] *SiŚe*, xii. 29.
[84] *L*, p. 40.
[85] *BrSpSi*, xii. 27.

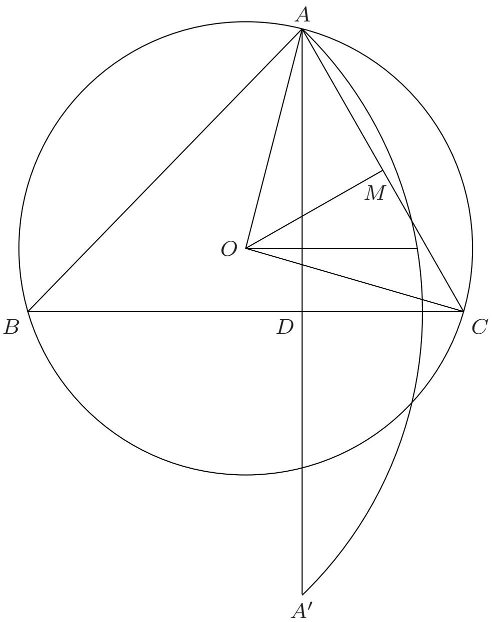


Figure 6

Hence,

$$OA = \frac{AB \times AC}{AA'},$$

or,

$$R = \frac{cb}{2h},$$

where R denotes the radius of the circumscribed circle.

Mahāvīra writes:

> In a triangle, the product of the two sides divided by the altitude is the diameter of the circumscribed circle.[86]

Example:[87] The circum-diameter of the triangle (14, 13, 15) is $16\frac{1}{4}$.

Śrīpati states:

> Half the product of the two sides divided by the altitude is the heart-line.[88]

[86] *GSS*, vii. $213\frac{1}{2}$.
[87] *GSS*, vii. $219\frac{1}{2}$.
[88] *SiŚe*, xiii. 31.

5.4 Inscribed circle

To find the radius of a circle inscribed in a triangle (or quadrilateral, when possible) whose area as well as perimeter are known, Mahāvīra gives the following rule:

> Divide the precise area of a figure other than a rectangle by one-fourth of its perimeter; the quotient is stated to be the diameter of the inscribed circle.[89]

That is to say, if r denote the radius of the circle inscribed within the triangle $(a,\ b,\ c)$, we shall have

$$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)},$$

where

$$2s = a + b + c.$$

5.5 Similar triangles

The properties of similar triangles and parallel lines were known to the ancient Hindus.[90] For example, take the case of the Mount Meru or Mandara. It has been described in the early canonical works of the Jainas as follows:

> At the centre of Jambūdvīpa, there is known to be a mountain, Mandara by name, whose height above (the earth) is 99000 *yojanas*, whose depth below is 1000 *yojanas*, its diameter at the base is $10090\frac{10}{11}$ *yojanas*, at the ground 10000 *yojanas*. Then (its diameter) diminishes by degrees until at the top it is 1000 *yojanas*. Its circumference at the base is $31910\frac{3}{11}$ *yojanas*, at the ground 31623 *yojanas*, and at the top a little over 3162 *yojanas*. It is broader at the base, contracted at the middle and (still) shorter at the top and is of the form of a cow's tail (i.e. a truncated right cone).[91]

To find the diameter of any other section parallel to the base, Jinabhadra Gaṇi (c. 560) gives the following rule:

> Wherever is wanted the diameter (of the Mandara): the descent from the top of the Mandara divided by eleven and then added to a thousand will give the diameter. The ascent from the bottom should be similarly (divided by eleven) and the quotient subtracted

[89] *GSS*, vii. $223\frac{1}{2}$.
[90] See Datta, Bibhutibhusan "Geometry in the Jaina Cosmography", *Quellen und Studien z. Gesch. d. Math.*, Ab. B, Bd. 1., 1930, pp. 249ff.
[91] *Jambūdvīpa-prajñapti*, *Sūtra* 103.

> from the diameter of the base: what remains will be the diameter there (i.e. at that height) of that (Mandara).[92]

It is stated further:

> Half the difference of the diameters at the top and the base should be divided by the height; that (will give) the rate of increase or decrease on one side; that multiplied by two will be the rate of increase or decrease on both sides; in going from either end of the mountain.
>
> Subtract from the diameter of the base of the mountain the diameter at any desired place: what remains when multiplied by the denominator (meaning eleven) will be the height (of that place).[93]

All these rules will follow at once from the following general formulae (**ed.** see Figure 7):

$$a = \frac{D-d}{2h}x,$$
$$\delta = a + \frac{D-d}{h}x,$$
$$y = (D-\delta')\frac{h}{D-d},$$
$$b = \frac{D-d}{2h}y,$$
$$\delta' = D - \frac{D-d}{h}y.$$

Rules similar to those stated above and hence the general properties leading to them, were known to the people long before Jinabhadra Gaṇi. For as early as the second century before the Christian era (or after) Umāsvāti correctly observed that in case of the Mount Meru, "for every ascent of 11000 *yojanas*, the diameter diminishes by 1000 *yojanas*."[94]

Again, "Half the difference between the breadths at the source and the mouth being divided by 45000 *yojanas*, and the quotient multiplied by two will give the rate of increase (of the breadth) on both sides, in case of rivers."[95] (45000 *yojanas* is the length of a river).

They are found even in the early canonical works (500–300 BC). According to the Jaina cosmography, the Salt Ocean is annular in shape, having a breadth of 200000 *yojanas*. In the undisturbed state its height as well as

[92] *Vṛhat Kṣetra-samāsa*, i. 307–8.
[93] *Ibid*, i. 309–11.
[94] *Tattvārthādhigama-sūtra-bhāṣya*, iii. 9.
[95] *Jambūdvīpa-samāsa*, ch, iv.

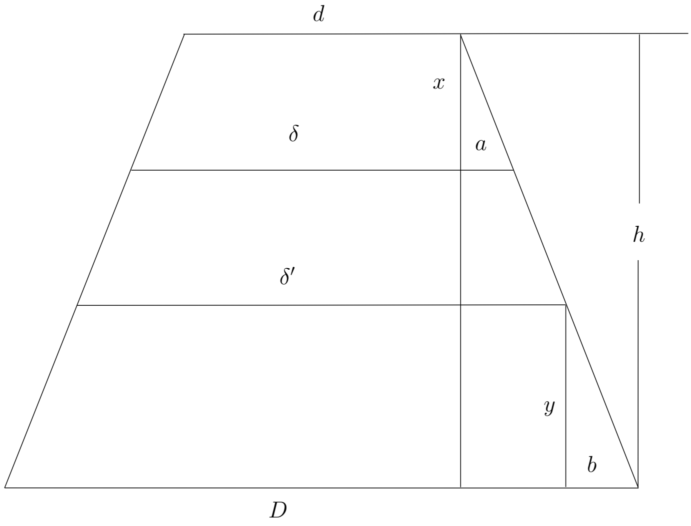


Figure 7

depth are said to be varying continuously from its either banks till at distances of 95000 *yojanas* from the banks where the height is 16000 *yojanas* and the depth 1000 *yojanas*. The radial section of the Salt Ocean in the calm state will be represented by Figure 8, where

$$\begin{aligned} AE &= A'E' = 95000 \text{ } \textit{yojanas}, \\ CE &= C'E' = 16000 \text{ } \textit{yojanas}, \\ ED &= E'D' = 1000 \text{ } \textit{yojanas}, \\ \text{and} \qquad EE' &= 10000 \text{ } \textit{yojanas}. \end{aligned}$$

It is described in the *Jivābhigama-sūtra* that "from either bank of the Salt Ocean, for proceeding every 95 *padas*, the height is known to be increased by 16 *padas* and so on, until on proceeding to 95000 *yojanas*, the height is known to be increased to 16000 *yojanas*".[96]

These can be easily verified thus:

From the properties of similar triangles

$$QR = \frac{ED \times AR}{AE} = \frac{1}{95} AR,$$
$$PR = \frac{EC \times AR}{AE} = \frac{16}{95} AR.$$

If $AR = 95x$, where x is any unit of measurement, then $QR = x$, $PR = 16x$.

[96] *Jivābhigama-sūtra*, *Sūtra* 172.

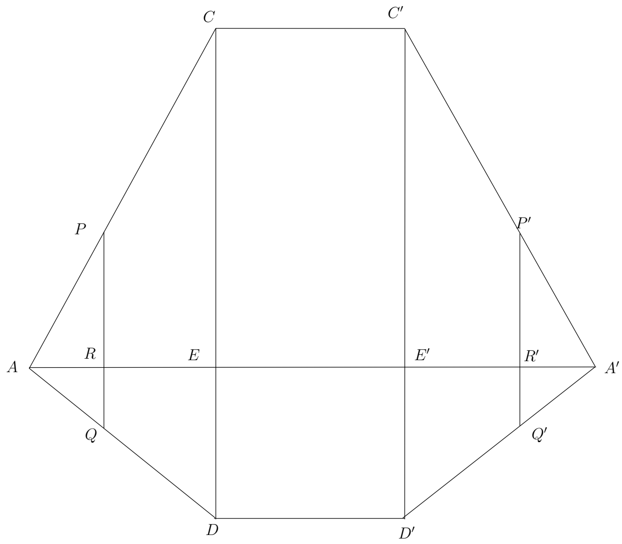


Figure 8

Again it is stated in the *Jambūdvīpa-prajñapti*[97] that at a height of 500 *yojanas* above the ground the breadth of the Mount Mandara is $9954\frac{6}{11}$ *yojanas*, while at 63000 *yojanas* above it is $4272\frac{8}{11}$ *yojanas*. These values, as can be easily verified, tally with the general formulae.

6 Measurement of quadrilaterals

6.1 Area

It should be noted at the outset that four sides alone are not sufficient to determine the true shape of a quadrilateral and consequently its size. For, there can be formed various quadrilaterals with the same four sides. Hence in order to make a quadrilateral determinate we must know, besides the sides, another element such as a diagonal, the altitude of a corner, or an angle. Thus Āryabhaṭa II remarks:

> The mathematician who wishes to tell of the area or the altitudes of a quadrilateral without knowing a diagonal, must be a fool or a blunderer.[98]

Bhāskara II writes:

[97] *Sūtra* 104–5.
[98] *MSi*, xii. 70.

> The diagonals of a quadrilateral (whose four sides are given) are uncertain. How can, then, the area be determinate? The diagonals as calculated by previous teachers will be true only in case of quadrilaterals (of a particular kind) contemplated by them, but not in case of others. For with the same (four) sides, there can be various other pairs of diagonals and consequently the area also is manifold. In a quadrilateral, when two opposite corners are so drawn as to bring the sides contiguous to them inwards, the diagonal joining them is shortened, while the other two corners bulge outwards and consequently their diagonal is lengthened. So it has been stated (just before) that with the same sides there can be other pairs of diagonals. Without specifying one of the altitudes or diagonals, how can one ask to find the other of them and also the area, as these are truly indeterminate? The questioner who does not know the indeterminate nature of a quadrilateral must be a blunderer; still more so is he, who answers such a problem.[99]

6.2 Brahmagupta's formula

To find the area (A) of an inscribed convex quadrilateral whose sides are a, b, c, d, Brahmagupta (628) gives the following formula:[100]

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where

$$2s = a + b + c + d.$$

This formula has been reproduced by Śrīdhara[101] (900), Mahāvīra[102] (850) and Śrīpati[103] (1039). None of these writers has expressly mentioned the limitation that it holds only for an inscribed figure. Still it seems to have been implied by them. So this appears from the particular remark of Bhāskara II that the formula holds only in case of a special kind of quadrilateral contemplated by them. Further we find that the examples of quadrilaterals, viz. (4, 13, 14, 13), (25, 25, 39, 25) and (25, 39, 60, 52) given by Śrīdhara[104] and Pṛthūdakasvāmi[105] and those, namely (14, 36, 61, 36), (169, 169, 407, 169)

[99] *L.* p. 44.
[100] *BrSpSi*, xii. 21.
[101] *Triś*, R. 43.
[102] *GSS*, vii. 50.
[103] *SiŚe*, xiii. 28.
[104] *Triś*, Ex. 78, 79, 80.
[105] *Vide* his commentary on *BrSpSi*, xii. 21. Elsewhere (xii. 26) he finds the circum-radii of these quadrilaterals.

and (125, 195, 300, 260) given by Mahāvīra,[106] in illustration of the above formula, are all of the cyclic variety. Bhāskara II has shown that in the other cases, the above formula gives only an approximate value of the area of a quadrilateral.[107]

6.3 Diagonal, altitude and segment

Āryabhaṭa I says (**ed.** see Figure 9):

> The two sides (severally) multiplied by the altitude and divided by their sum will give the perpendiculars let fall on them from the point of intersection of the diagonals. Half the sum of the two sides multiplied by the altitude should be known as the area.[108]

$$h_1 = \frac{ah}{a+c},$$
$$h_2 = \frac{ch}{a+c},$$
$$\text{Area} = \frac{1}{2}h(a+c).$$

Brahmagupta writes:

> In an isosceles trapezium[109] the square-root of the sum of the products of the sides and counter-sides is the diagonal. The square-root of the square of the diagonal as diminished by the square of half the sum of the face and base, is the altitude.[110]

$$d = \sqrt{ac+b^2}, \quad h = \sqrt{d^2 - \left(\frac{a+c}{2}\right)^2}.$$

> The upper and lower portions of the diagonal or the altitude at the junction of the two diagonals or of a diagonal and an altitude, will be given by the corresponding segments of the base divided by their sum and multiplied again by the diagonal or altitude, as the case may be.[111]

---

[106] *GSS*, vii. 57, 58, 59. Compare also vii. $215\frac{1}{2}$, $216\frac{1}{2}$, $217\frac{1}{2}$ where it is required to find the diameters of the circles circumscribing these very quadrilaterals.

[107] *L*, p. 41.

[108] *Ā*, ii. 8.

[109] The Sanskrit term is *aviṣama-caturasra*, meaning literally "the quadrilateral not of unequal sides". Brahmagupta classifies quadrilaterals (*caturasra*, *caturbhuja*) into five varieties: *sama-caturasra* (square), *āyata-caturasra* (rectangle), *dvisama caturasra* (isosceles trapezium), *trisama caturasra* (trapezium with three equal sides) and *viṣama caturasra* (quadrilateral of unequal sides). Hence *aviṣama caturasra* must mean all except those of the last class. But here more particularly the isosceles trapezium is meant.

[110] *BrSpSi*, xii. 23.

[111] *BrSpSi*, xii. 25.

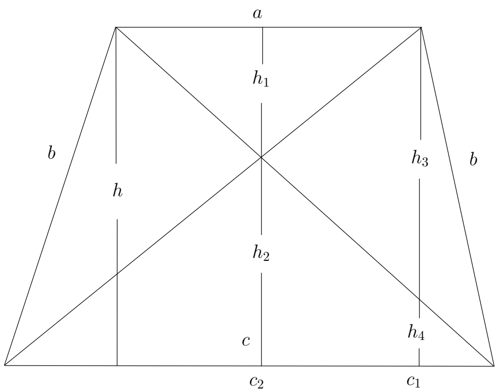


Figure 9

$$h_3 = \frac{c_2 h}{c_1 + c_2}, \quad d_1 = \frac{c_2 d}{c_1 + c_2},$$
$$h_4 = \frac{c_1 h}{c_1 + c_2}, \quad d_2 = \frac{c_1 d}{c_1 + c_2}.$$

For quadrilaterals other than isosceles trapeziums, Brahmagupta gives the following rules:

> Considering two scalene triangles within the quadrilaterals[112] by means of the two diagonals, find separately the segments of the base in them by the method taught before; and thence the two altitudes.[113]
>
> Supposing two scalene triangles within the quadrilateral, with the diagonals as bases, find in each of them separately the segments of the base. They will be the portions of the diagonals above and below their point of intersection. The lower portions of the diagonals are taken to be the sides of another triangle whose base is the same as that of the given quadrilateral. Its altitude is the lower portion of the perpendicular (to the base through the junction of the diagonals). The upper portions of it will be obtained by subtracting this portion from half the sum of the two altitudes.[114]
>
> At the intersection of the diagonals and perpendiculars, the lower segment of a diagonal and of a perpendicular can be found by proportion. On subtracting these segments from the whole, the

[112] The Sanskrit term is *viṣama caturasra*. As pointed out just before, it denotes "a quadrilateral of unequal sides" including a trapezium.
[113] *BrSpSi*, xii. 29.
[114] *BrSpSi*, xii. 30–31.

> upper portions will be found. Such is (the method) also in the needle (i.e. the intersection of two opposite sides produced) and the intersection (of a prolonged side and perpendicular).[115]

Śrīdhara states:

> To find the altitude of a trapezium,[116] suppose a triangle whose base is the difference of the base and face of the trapezium and whose sides are the same as those at the flanks of the given figure; (and then proceed as in the case of finding the altitude of a triangle).[117]

Mahāvīra's rule will be clear from the following problem with reference to which it has been defined (**ed.** see Figure 10):

AB, CD are two vertical pillars. AE, CF are two strings joining the tops A and C of the these pillars to points E and F on the ground. PQ is the perpendicular from the point of intersection of the strings. It has been named "the inner perpendicular."

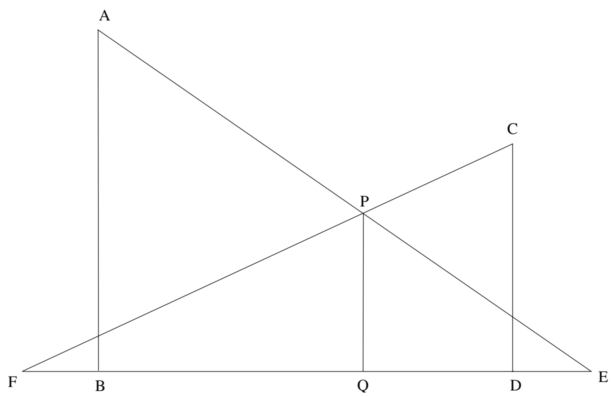


Figure 10

Mahāvīra says:

> Divide each pillar by its distance from (the farthest point of contact of) the string (with the ground), divide again the quotients by their sum and then multiply by the (whole) base. The results are the segments (of the base by the inner perpendicular). These being multiplied by the (first) quotients in the inverse order give the inner perpendicular.[118]

[115] *BrSpSi*, xii. 32.
[116] The Sanskrit term is *ṛjuvadana-caturbhuja* of "the quadrilateral with parallel face."
[117] *Triś*, R. 49.
[118] *GSS*, vii. $180\frac{1}{2}$.

That is to say, we have

$$QF = \frac{\frac{AB}{BE} \times FE}{\frac{AB}{BE} + \frac{CD}{DF}} = \frac{AB \times DF \times FE}{AB \times DF + CD \times BE},$$
$$QE = \frac{\frac{CD}{DF} \times FE}{\frac{AB}{BE} - \frac{CD}{DF}} = \frac{CD \times BE \times FE}{AB \times DF - CD \times BE},$$
$$PQ = \frac{AB}{BE} \times QE = \frac{CD}{DF} \times QF.$$

Example from Mahāvīra:[119] Find the inner perpendicular and the segments of the base caused by it in the quadrilateral (7, 15, 21, 3).

Śrīpati says:

> In an isosceles trapezium, the square-root of the sum of the products of opposite sides is the diagonal. Next I shall speak of quadrilaterals of unequal sides.[120]

Bhāskara II gives several rules. Of them we note the following:

> In a quadrilateral, assume the value of one diagonal. Then in the two triangles lying on either sides of this diagonal, it will be the base and others (i.e. the given sides of the quadrilateral) sides. Now find the perpendiculars and segments (in these triangles). Then the square of the difference of the two segments lying on the same side (i.e. taken from the same corner) being added to the square of the sum of the perpendiculars, the square-root of the resulting sum will be the second diagonal in all quadrilaterals.[121]

Ganeśa has demonstrated the rule substantially as follows (**ed.** see Figure 11):

Let $ABCD$ be a quadrilateral whose diagonal AC as well as the sides are known. Draw BN, DM perpendiculars to AC. Produce BN and draw DP perpendicular to it. Join DB. Then

$$\begin{aligned} DB^2 &= BP^2 + DP^2, \\ &= (BN + DM)^2 + (AN - AM)^2. \end{aligned}$$

> Suppose a triangle whose base is equal to the difference of the face and base of a trapezium, and whose sides are the flank sides of the latter; then as in case of a triangle, find its altitude and segments of the base. Subtract from the base of the given trapezium one

[119] *GSS*, vii. $187\frac{1}{2}$.
[120] *SiŚe*, xiii. 33.
[121] *L*, p. 47f.

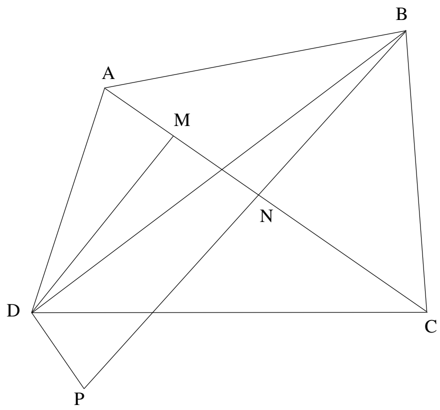


Figure 11

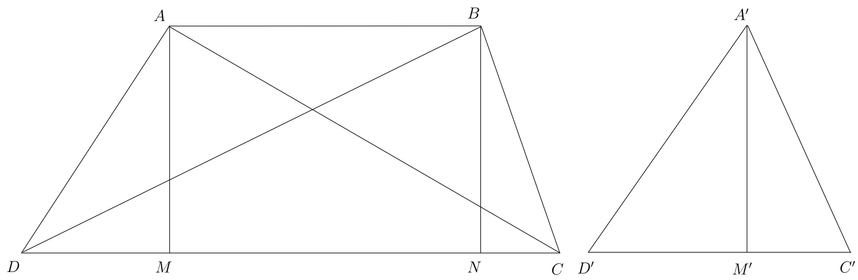


Figure 12

> of the segments. The square of the remainder being added to the square of the perpendicular, the square-root of the sum is the diagonal. In a trapezium, the sum of the base and smaller flank side is greater than the sum of the face and the other flank.[122]

Gaṇeśa's Proof (**ed.** see Figure 12): Let $ABCD$ be a trapezium. Draw the perpendiculars AM, BN. Combine the two triangles ADM and BCN into one triangle $A'C'D'$. Then the altitude $A'M'$ of the new triangle is equal to the altitude of the trapezium.

Join AC and BD. Then

$$AC^2 = AM^2 + MC^2 = A'M'^2 + (DC - D'M')^2,$$
$$BD^2 = BN^2 + DN^2 = A'M'^2 + (DC - C'M')^2.$$

Again

$$A'D' - A'C' < D'C' = DC - AB.$$

[122] *L.* p. 48f.

Therefore

$$DC + A'C' > AD + AB.$$

6.4 Circumscribed circle

To find the radius of the circle described round a quadrilateral, Brahmagupta gives the following rule:

> The diagonal of an isosceles trapezium being multiplied by its flank side and divided by twice its altitude gives its heart line: in case of a quadrilateral of unequal sides it is half the square-root of the sum of the squares of the opposite sides.[123]

Now it has been given by Brahmagupta that

$$h^2 = d^2 - \left(\frac{a+c}{2}\right)^2.$$

Substituting the value of $d^2 = ac + b^2$, we get

$$h = \sqrt{(s-a)(s-c)}.$$

Hence according to the above, the radius of the circle described round the isosceles trapezium (a, b, c, b) is

$$\frac{1}{2}b\sqrt{\frac{ac+b^2}{(s-a)(s-c)}}.$$

In case of a quadrilateral of unequal sides the circum-radius is

$$= \frac{1}{2}\sqrt{a^2+c^2} = \frac{1}{2}\sqrt{b^2+d^2}.$$

This formula holds only in that kind of inscribed convex quadrilaterals in which the diagonals are at right angles.

Mahāvīra says:

> In a quadrilateral, the diagonal divided by the perpendicular and multiplied by the flank side, gives the diameter of the circumscribed circle.[124]

Śrīpati states all the above formulae. He says:

> In a quadrilateral, half the product of a diagonal and flank side divided by the altitude, gives the radius of the circumscribed circle. In a quadrilateral of unequal sides, half the square-root of the sum of the squares of the opposite sides is stated to be the radius and twice it the diameter of the circumscribed circle.[125]

[123] *BrSpSi*, xii. 26.
[124] *GSS*, vii. $213\frac{1}{2}$
[125] *SiŚe*, xiii. 31–2.

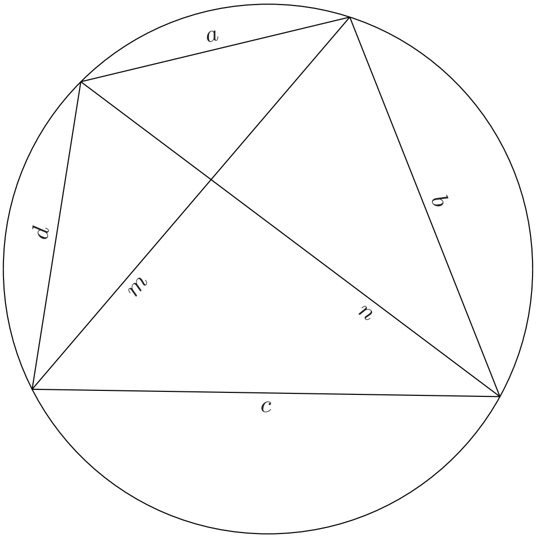


Figure 13

6.5 Inscribed circle

We have already cited Mahāvīra's formula for the diameter of the inscribed circle.

$$\text{Diameter} = \text{Area} \div \frac{\text{Perimeter}}{4}.$$

6.6 Theorems for diagonals

Brahmagupta (628) gives two remarkable theorems for the lengths of the diagonals of an inscribed convex quadrilateral. He says (**ed.** see Figure 13):

> Divide mutually the sums of the products of the sides attached to both the diagonals and then multiply the quotients by the sum of the products of the opposite sides: the square-roots of the results are the diagonals of the quadrilateral.[126]

$$m = \sqrt{\frac{(ab+cd)(ac+bd)}{(ad+bc)}},$$

$$n = \sqrt{\frac{(ad+bc)(ac+bd)}{(ab+cd)}}.$$

Mahāvīra (850) writes:

[126] *BrSpSi*, xii. 28.

> The two flank sides multiplied by the base are added (respectively) to those sides (taken reversely) multiplied by the face. Make the sums (thus obtained respectively) the multiplier and divisor, again the divisor and multiplier of the sum of the products of the opposite sides. The square-roots of the results are the diagonals.[127]

Śrīpati's (1039) enunciation[128] of the theorems is nearly the same as that of Brahmagupta.

It will be noticed that neither Brahmagupta, nor any of the posterior writers mentioned above, has expressly stated the limitation that the theorems hold only in case of inscribed convex quadrilaterals. Did they at all know it will be the question that will be naturally asked. Looking at the context, we think, it will have to be answered in the affirmative. For in the two rules just preceding the one in question, Brahmagupta teaches how to find the radii of the circles circumscribed about a quadrilateral and a triangle respectively. So in the present rule too he has in view a quadrilateral of the type which can be circumscribed by a circle. Illustrative examples given by the commentator Pṛthūdakasvāmi, as also by Mahāvīra, are all of quadrilaterals of that kind. Further Bhāskara II observed in connection with these theorems that they hold in case of quadrilaterals contemplated to be of a particular kind by their author.

7 Squaring the circle

7.1 Origin of the problem

The problem of 'squaring the circle', or what was more fundamental in India, the problem of 'circling the square', originated and acquired special importance in connexion with the Vedic sacrifices, before the earliest hymns of the *Ṛgveda* were composed (before 3000 BC). The three primarily essential sacrificial altars of the Vedic Hindus, namely the *Gārhapatya*, *Āhavanīya* and *Dakṣiṇa*, were constructed so as to be of the same area, but of different shapes, square, circular and semi-circular. Again in constructing the fire-altars called the *Rathacakra-citi*, *Samuhya-citi* and *Paricāyya-citi*, which are mentioned in the *Taittirīya Saṃhitā* (c. 3000 BC) and other works, one had to draw in each case at first a square equal in area to that of the *Śyena-citi*, viz. $7\frac{1}{2}$ square *puruṣas*, and then to transform it into a circle. We find also other instances in the early Hindu works requiring the solution of the problem of circling the square and its converse.[129]

[127] *GSS*, vii. 54.
[128] *SiŚe*, xiii. 34.
[129] See Datta, Bibhutibhusan, *Śulba*, ch. xi, for further informations on the problem.

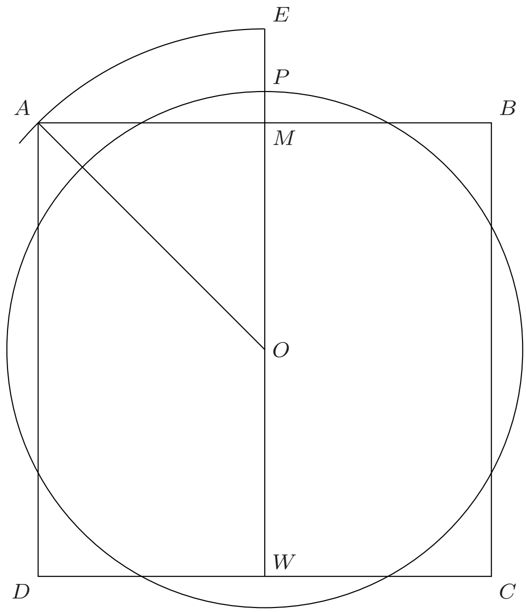


Figure 14

7.2 Circling the square

Baudhāyana writes:

> If you wish to circle a square, draw half its diagonal about the centre towards the east-west line; then describe a circle together with the third part of that which lies outside (the square).[130]

The same method is taught in different words also by Āpastamba[131] and Kātyāyana.[132]

Let $ABCD$ be the square which is to be transformed into a circle (**ed.** see Figure 14). Let O be the central point of the square. Join OA. With centre O and radius OA, describe a circle intersecting the east-west line EW at E. Divide EM at P, such that $EP = 2\,PM$. Then with centre O and radius OP describe a circle. This circle is roughly equal in area to the square $ABCD$.

Let $2a$ denote a side of the given square and r the radius of the circle equivalent to it. Then

$$OA = a\sqrt{2}, \quad ME = (\sqrt{2} - 1)a.$$

[130] *BŚl*, i. 58.
[131] *ĀpŚl*, iii. 2.
[132] *KŚl*. iii. 13.

Hence

$$r = a + \frac{a}{3}\left(\sqrt{2}-1\right) = \frac{a}{3}\left(2+\sqrt{2}\right).$$

Āpastamba observes that the circle thus constructed will be inexact (*anitya*). Now, according to the *Śulba*,

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\times 4} - \frac{1}{3\times 4\times 34}.$$

Therefore

$$r = a \times 1.1380718\ldots$$

7.3 Squaring the circle

Baudhāyana says:

> If you wish to square a circle, divide its diameter into eight parts; then divide one part into twenty-nine parts and leave out twenty-eight of these, and also the sixth part (of the preceding sub-division) less the eighth part (of the last).[133]

That is to say, if $2a$ be the side of a square equivalent to a circle of diameter d, then

$$2a = \frac{7d}{a} + \left\{\frac{d}{a} - \left(\frac{28d}{8\times 29} + \frac{d}{8\times 29\times 6} - \frac{d}{8\times 29\times 6\times 8}\right)\right\},$$

or putting $d = 2r$,

$$a = r - \frac{r}{8} + \frac{r}{8\times 29} - \frac{r}{8\times 29\times 6} + \frac{r}{8\times 29\times 6\times 8}.$$

Baudhāyana further teaches a still rough method of squaring the circle:

> Or else divide (the diameter) into fifteen parts and remove two (of them). This is the gross (value of the) side of the (equivalent) square.[134]

This method is described also by Āpastambā[135] and Kātyāyāna.[136] According to it

$$a = r - \frac{2r}{15}.$$

According to Manu a square of two by two cubits is equivalent to a circle of radius 1 cubit and 3 *aṅgulis*.[137]

[133] *BŚl*, i. 59.
[134] *BŚl*, i. 60.
[135] *ĀpŚl*, iii. 3.
[136] *KŚl*, iii. 14.
[137] *MāŚl*, i. 27.

Dvārakānātha's corrections

Dvārakānātha Yajvā, a commentator of the *Baudhāyana Śulba*, proposed a correction to the above formula for the transformation of a square into a circle. According to him

$$r = \left\{a + \frac{a}{3}\left(\sqrt{2} - 1\right)\right\} \times \left\{1 - \frac{1}{118}\right\},$$

or

$$r = a \times 1.1284272\ldots$$

Similarly he improves the formula for the reverse operation:

$$a = r\left(1 - \frac{1}{8} + \frac{1}{8 \times 29} + \frac{1}{8 \times 29 \times 6} - \frac{1}{8 \times 29 \times 6 \times 8}\right)\left(1 + \frac{1}{2} \times \frac{3}{133}\right).$$

7.4 Later formulae

In the Jaina cosmography, the earth is supposed to be a flat plane divided into successive regions of land and water by a system of concentric circles. The innermost region is one of land and is called Jambūdvīpa. It is a circle of diameter 100000 *yojanas*. Its circumference is given as a little over 316227 *yojanas* 3 *gavyūtis* 128 *dhanus* $13\frac{1}{2}$ *aṅgulas* and its area as 7905694150 *yojanas* 1 *gavyūti* 1515 *dhanus* 60 *aṅgulas*.[138] It will be seen that in calculating these values of the circumference and area from the assumed value of the diameter, the following two formulae have been employed:

$$C = \sqrt{10d^2}, \qquad A = \frac{1}{4}Cd,$$

where d = the diameter of a circle, C = its circumference and A = its area.

Umāsvāti (c. 150 BC or AD) writes:

> The square-root of ten times the square of the diameter of a circle is its circumference. That (circumference) multiplied by a quarter of the diameter (gives) the area.[139]

So does also Jinabhadra Gaṇi (529–589).[140]

Āryabhaṭa I says:

[138] See *Jambūdvīpa-prajñapti*, *Sūtra* 3; *Jīvābhigama-sūtra*, *Sūtra* 82, 124; *Anuyogadvāra-sūtra*, *Sūtra* 146. Compare also *Sūryaprajñapti*, *Sūtra* 20.

[139] *Tattvārthādhigama-sūtra* with the *Bhāṣya* of Umāsvāti, edited by K. P. Mody, Calcutta, 1903, iii. 11 (gloss); *Jambūdvīpa-samāsa*, ch. iv. The latter work of Umāsvāti has been published in the Appendix C of Mody's edition of the former.

[140] *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, Bhavanagara, 1919, i. 7.

> Half the circumference multiplied by the semi-diameter certainly gives the area of a circle.[141]

Brahmagupta:

> Three times the diameter and the square of the semi-diameter give the practical values of the circumference and area (respectively). The square roots of ten times the squares of them are the neat values.[142]

Śrīdhara:

> The square-root of the square of the diameter of a circle as multiplied by ten is its circumference. The square-root of ten times the square of the square of the semi-diameter is the area.[143]

Mahāvīra:

> Thrice the diameter is the circumference. Thrice the square of the semi-diameter is the area ... So said the teachers.[144]
>
> The diameter of a circle multiplied by the square-root of ten, becomes the circumference. The circumference multiplied by the fourth part of the diameter gives the area.[145]

Āryabhaṭa II:

> The square-root of the square of the diameter of a circle as multiplied by ten is the circumference. The fourth part of the square of the diameter being squared and multiplied by ten, the square-root of the product is the area.[146]
>
> The diameter multiplied by 22 and divided by 7 will become nearly equal to the circumference. If the square of the semi-diameter be so treated, the result will be the value of the area as precise as that of the circumference.[147]
>
> Twice the sine of three signs of the zodiac (i.e. 3438) is the diameter and the circumference is then 21600. Multiply the circumference by 191 and divide by 600; the quotient is the diameter.[148]

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[141] *A*, ii. 7.
[142] *BrSpSi*, xii. 40.
[143] *Triś*, R. 45.
[144] *GSS*, vii. 19.
[145] *GSS*, vii. 60.
[146] *MSi*, xv. 88.
[147] *MSi*, xv. 92f.
[148] *MSi*, xvi. 37.

Śrīpati's rule is the same as the first one of Āryabhaṭa the Younger. Bhāskara II writes:

> When the diameter is multiplied by 3927 and divided by 1250, the result is the nearly precise value of the circumference; but when multiplied by 22 and divided by 7, it is the gross circumference which can be adopted for practical purposes.[149]
>
> In a circle, the one-fourth of the diameter multiplied by the circumference gives the area.[150]
>
> The square of the diameter being multiplied by 3927 and divided by 5000 gives the nearly precise value of the area; or being multiplied by 11 and divided by 14 gives the gross area which can be applied in rough works.[151]

7.5 Values of π

The formulae of Baudhāyana, noted above, yield the following values of π:

$$\pi = \frac{4}{\left\{1 + \frac{1}{3}\left(\sqrt{2} - 1\right)\right\}^2} = 3.0883\ldots$$

$$\pi = 4\left(1 - \frac{1}{8} + \frac{1}{8 \times 29} - \frac{1}{8 \times 29 \times 6} + \frac{1}{8 \times 29 \times 6 \times 8}\right) = 3.0885\ldots$$

$$\pi = 4\left(1 - \frac{2}{15}\right)^2 = 3.004.$$

Baudhāyana has once employed the very rough value, 3. From the rule of Manu, we get

$$\pi = 4\left(\frac{8}{9}\right)^2 = 3.16049\ldots$$

With the corrections of Dvārakānātha, we have

$$\pi = 3.141109\ldots, \quad 3.157991\ldots$$

In the early canonical works of the Jainas (500–300 BC) is employed the value $\pi = \sqrt{10}$.[152] This value has been adopted by Umāsvāti, Varāhamihira (505), Brahmagupta (628), Śrīdhara (c. 900) and others. It is stated in the *Jīvābhigama-sūtra*,[153] that for an increment of 100 *yojanas* in the diameter,

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[149] *L*, p. 54.
[150] *L*, p. 55.
[151] *L*, p. 56f.
[152] See Datta, Bibhutibhusan, "The Jaina School of Mathematics", *BCMS* xxi (1929), p. 13; "Hindu Values of π", *JASB*, xxii (1926), pp. 25–43. The latter article given fuller information on the subject.
[153] *Sūtra* 112.

the circumference increases by 316 *yojanas*. Here has been used the value $\pi = 3.16$.

Āryabhaṭa the Elder (499) gives a remarkably accurate value. His rule is:

> 100 plus 4, multiplied by 8, and added to 62000: this will be the nearly approximate (*āsanna*) value of the circumference of a circle of diameter 20000.[154]

That is to say, we have

$$\pi = \frac{62832}{20000} = \frac{3927}{1250} = 3.1416.$$

This value appears in the works of Lalla[155] (c. 749), Bhaṭṭotpala[156] (966), Bhāskara II and others. We have it on the authority of a writer of the sixteenth century who was in possession of the larger treatise of arithmetic by Śrīdhara that this value of π was adopted there.

The value

$$\pi = \frac{21600}{6876} = \frac{600}{191} = 3.14136\ldots$$

introduced first by Āryabhaṭa the Younger (950) is undoubtedly derived from the value of the Elder Āryabhaṭa. For if the circumference of a circle measures 21600, its diameter will be

$$21600 \times \frac{1250}{3927} = 6875\frac{625}{1309}.$$

Āryabhaṭa takes the value of the diameter to be 6876 in round numbers.[157] This relation (21600 : 6876) between the circumference and diameter of a circle was, however, worked out before by Bhāskara I (629).[158] The value $\pi = \frac{600}{191}$ appears also in the treatises of arithmetic by Gaṇeśa II (c. 1550) and Munīśvara (1656).

It should be particularly noted that the Greek value, $\pi = \frac{22}{7}$, is found in India first in the work of Āryabhaṭa the Younger.[159] Bhāskara II (1150) employs it as a rough approximation suitable for practical purposes.

7.6 Later approximations of π

Later Hindu writers found much closer approximations to the value of π. Nārāyaṇa, a priest of Travancore, gave in 1426, the following rule to construct a temple of circular shape having a given perimeter:

[154] *Ā*, ii. 10.
[155] *ŚiDVṛ*, i. 1, 2; ii. 3; etc.
[156] See his commentary on *Bṛhat Saṃhitā*, p. 53.
[157] *MSi*, xv. 88.
[158] *Vide* his commentary on *Ā*, ii. 10.
[159] *MSi*, xv. 92f

> Divide the given perimeter into 710 parts; with 113 of them as the radius describe a circle and thus construct the circular temple.[160]

Hence he has employed $\pi = \frac{355}{113}$, the Chinese value.

Śaṅkara Vāriyar (c. 1500–60) says:

> The value of the given diameter being multiplied by 104348 and divided by 33215, becomes the accurate value of the circumference. Again from the circumference can be obtained the correct value of the diameter by proceeding reversely; that is, by multiplying the value of the circumference by 33215 and then dividing by 104348, or by multiplying by 113 and dividing by 355.[161]

$$\pi = \frac{104348}{33215} = 3.14159265391\ldots$$
$$\pi = \frac{355}{113} = 3.1415929\ldots$$

The first value is correct up to the ninth place of decimals, the tenth being too large, and the second up to the sixth place of decimals, the seventh being too large.

Mādhava (of Saṅgamagrāma) writes:

> It has been stated by learned men that the value of the circumference of diameter 900000000000 in length is 2827433388233.[162]

Therefore we have

$$\pi = \frac{2827433388233}{900000000000} = 3.141592653592\ldots$$

correct up to the tenth place of decimals, the eleventh being too large.

Putumana Somayājī (c. 1660–1740), the author of the *Karaṇa-paddhati*, observes:

> When the value of the circumference of a circle is multiplied by 10000000000 and divided by 31415926536, the quotient is the value of the diameter. Half that is the radius.[163]

Śaṅkaravarman (1800–38) says:

[160] Nārāyaṇa, *Tantra-samuccaya*, edited by T. Ganapati Sastri, Trivandrum Sanskrit Series, 1919, ii. 65.

[161] *Tantra-saṃgraha*, (commentary in verse, edited by K. V. Sarma), p. 103, vss. 298–9.

[162] Quoted by Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* (ii. 10) edited by K. Sambasiva Sastri, Trivandrum Sanskrit Series, 1930.

[163] *Karaṇa-paddhati*, vi. 7.

> In this way, if the diameter of a great circle measure one *parārdha* (i.e. 10^{17}), its circumference will be 314159265358979324.[164]

Here we have a value of π, 3.14159265358979324, which is correct up to 17 places of decimals.

7.7 Values in series

Śaṅkara Vāriyar (c. 1500–60) gave certain interesting approximations in series for the value of the circumference of a circle in terms of its diameter. He says:

> Multiply the diameter by four and divide by one; subtract from and add to the result alternately the successive quotients of four times the diameter divided severally by the odd numbers 3, 5, etc. Take the even number next to that odd number on division by which this operation is stopped; then as before multiply four times the diameter by the half of that and divide by its square plus unity. Add the quotient thus obtained to the series in case its last term is negative; or subtract if the last term be positive. The result will be very accurate if the division be continued to many terms.[165]

That is to say, if C denotes the circumference and d the diameter, then we shall have

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d(n+1)}{(2n+2)^2+1},$$

where $n = 1, 2, 3 \ldots$

He then continues:

> Now I shall write of certain other correction more accurate than this: In the last term the multiplier should be the square of half the even number together with one, and the divisor four times that, added by unity, and then multiplied by half the even number. After division by the odd numbers 3, 5, etc., the final operation must be made as just indicated.[166]

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d(n^2+2n+2)}{(n+1)(4n^2+8n+9)}.$$

The author seems to have realised the slow convergence of the above infinite series; so in order to get a closer approximation to its value after retaining a

[164] *Sadratna-mālā*, iv. 2.

[165] *Tantra-saṃgraha*, (commentary in verse), p. 101, vss. 271–4. This rule is really that of Mādhava. See *Kriyākramakarī* (Śaṅkara Vāriyar's commentary on *Līlāvatī*), p. 379.

[166] *Tantra-saṃgraha*, (commentary in verse), p. 103, vss. 295–296.

sufficient number of terms, modified the next one in the way described above and then neglected the rest. This series, without the correction in any form, is found also in the *Karaṇa-paddhati*, as follows:

> Divide four times the diameter many times severally by the odd numbers 3, 5, 7, etc. Subtract and add successive quotients alternately from and to four times the diameter. The result is an accurate value of the circumference.[167]

It was rediscovered in Europe two centuries later by Liebnitz (1673) and De Lagney (1682).

Śaṅkara Vāriyar (c. 1500–60) says:

> The square-root of twelve times the square of the diameter is the first result. Divide this by three; again the quotient by three; and so on continuously up to as many times as desired. Then divide the results successively by the odd numbers 1, 3, etc. Of the quotients thus obtained the sum of the odd ones (i.e. 1st, 3rd, etc.) diminished by the sum of the even ones (i.e. 2nd, 4th, etc.) will be the value of the circumference.[168]

That is to say, we shall have

$$C = \sqrt{12d^2}\left(1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \dots\right).$$

The same series is described in slightly difference words in the *Sadratnamālā*.[169] It is also given by Abraham Sharp (c. 1717), who used it for calculating the value of π up to 72 places of decimals.

Śaṅkara Vāriyar writes:

> The fifth powers of the odd numbers 1, 3, etc. are increased by four times their respective roots. Divide sixteen times a given diameter severally by the sums thus obtained and subtract the sum of the even quotients from that of the old ones. The remainder will be the circumference.[170]

That is

$$C = 16d\left(\frac{1}{1^5 + 4 \times 1} - \frac{1}{3^5 + 4 \times 3} + \frac{1}{5^5 + 4 \times 5} - \frac{1}{7^5 + 4 \times 7} + \dots\right).$$

[167] *Karaṇa-paddhati*, vi. 1.

[168] *Tantra-saṃgraha* (commentary in verse), p. 96, vss. 212(c–d)–214(a–b).

[169] *Sadratnamālā*, iv. 2.

[170] *Tantra-saṃgraha* (commentary in verse), p. 102, vss. 287–8.

> Or divide four times the diameter severally by the cubes of the odd numbers beginning with 3, after diminishing each by its respective root; add and subtract the successive quotients alternately to and from thrice the diameter. Hence deduce the value of the circumference also in this way.[171]

$$C = 3d + 4d\left(\frac{1}{3^3 - 3} - \frac{1}{5^3 - 5} + \frac{1}{7^3 - 7} - \dots\right).$$

This infinite series is stated also in the *Karaṇa-paddhati*.[172]

> Or the squares of the even numbers 2, etc. each diminished by unity, are the several denominators. Add and subtract the quotients alternately to and from twice the diameter. Take the odd number next to last even number (at which the series is stopped). The square of it added by two and then multiplied by two should be taken as the divisor at the end.[173]

$$C = 2d + 4d\left\{\frac{1}{2^2 - 1} - \frac{1}{4^2 - 1} + \dots + (-1)^{n-1}\,\frac{1}{(2n)^2 - 1} - (-1)^{n-1}\,\frac{1}{2(2n+1)^2 + 2}\right\}.$$

> Squares of the numbers beginning with two or four and increasing by four, diminished each by unity, are the several denominators; and the numerator in each case is eight times the given diameter. The value of the circumference of the circle is equal in the first case to the sum of the quotients and in the second to half the numerator minus the quotients.[174]

$$C = \left(\frac{8d}{2^2 - 1} + \frac{8d}{6^2 - 1} + \frac{8d}{10^2 - 1} + \dots\right),$$

$$C = 4d - \left(\frac{8d}{4^2 - 1} + \frac{8d}{8^2 - 1} + \frac{8d}{12^2 - 1} + \dots\right).$$

The *Karaṇa-paddhati* adds a new series. It says:

> Or divide six times the diameter by squares of twice the squares of even numbers minus unity as diminished by the squares of the respective even numbers. Thrice the diameter added by these quotients is the value of the circumference.[175]

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[171] *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 290.
[172] *Karaṇa-paddhati*, vi. 2.
[173] *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 292.
[174] *Tantra-saṃgraha* (commentary in verse), p. 103, vss. 293–4.
[175] *Karaṇa-paddhati*, vi. 4.

$$C = 3d + \frac{6d}{(2 \times 2^2 - 1)^2 - 2^2} + \frac{6d}{(2 \times 4^2 - 1)^2 - 4^2} + \frac{6d}{(2 \times 6^2 - 1)^2 - 6^2} + \dots$$

Or,

$$C = 3d + 6d\left(\frac{1}{1 \times 3 \times 3 \times 5} + \frac{1}{3 \times 5 \times 7 \times 9} + \frac{1}{5 \times 7 \times 11 \times 13} + \dots\right).$$

Śaṅkaravarman gives another:

> Take the square-root of twelve times the square of the diameter and also its third part. Divide these continuously by nine. Again divide the quotients (thus obtained) respectively by twice the odd numbers 1, etc. (in the former case) and by twice the even numbers 2, etc. (in the latter case), each as diminished by unity. The difference of the two sums of the final quotients is the value of the circumference of the circle.[176]

$$C = \sqrt{12d^2}\left\{\frac{1}{9(2 \times 1 - 1)} + \frac{1}{9^2(2 \times 3 - 1)} + \frac{1}{9^3(2 \times 5 - 1)} + \dots\right\} - \frac{\sqrt{12d^2}}{3}\left\{\frac{1}{9(2 \times 2 - 1)} + \frac{1}{9^2(2 \times 4 - 1)} + \frac{1}{9^3(2 \times 6 - 1)} + \dots\right\}.$$

8 Measurement of segment of circle

8.1 Data in Jaina canonical works

In the early cosmographical works of the Jainas, we find certain interesting and valuable data relating to the mensuration of a segment of a circle.[177] Jainas suppose that Jambūdvīpa, which has been described before to be a circle of diameter 100000 *yojanas*; is divided into seven *varṣas* ("countries") by a system of six parallel mountain ranges running due East-to-West. The southern region of it is called Bhāratavarṣa. Dimensions of this segment, in

[176] *Sadratnamālā*, iv. 1.

[177] See the article of Datta, Bibhutibhusan, on "Geometry in the Jaina Cosmography" in *Quellen und Studien zur Gesch. d. Math.* Ab. B, Bd. 1, 1930 pp. 245–254, from which extracts are here made.

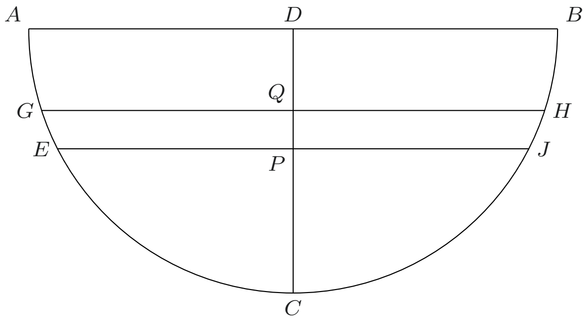

Figure 15

terms of *yojanas*, are as follows (**ed.** see Figure 15):

$$AB = 1447\frac{6}{19} \text{ (a little less)}, \qquad ACB = 14528\frac{11}{19},$$

$$PQ = 50, \qquad GCH = 10743\frac{15}{19},$$

$$CD = 526\frac{6}{19}, \qquad ECJ = 9766\frac{1}{19} \text{ (a little over)},$$

$$CP = QD = 238\frac{3}{19}, \qquad AG = BH = 1892\frac{7}{19} + \frac{1}{33},$$

$$EJ = 9748\frac{12}{19}, \qquad EG = JH = 488\frac{16}{19} + \frac{1}{33},$$

$$GH = 10720\frac{12}{19}.$$

These numerical data will be found to conform to the following formulae for the mensuration of a segment of a circle:

$$c = \sqrt{4h(d-h)},$$

$$d = \frac{c^2}{4h} + h,$$

$$a = \sqrt{6h^2 + c^2},$$

$$a' = \frac{1}{2}\{(\text{bigger arc}) - (\text{smaller arc})\},$$

$$h = \frac{1}{2}(d - \sqrt{d^2 - c^2}),$$

$$\text{or} \qquad h = \sqrt{\frac{(a^2 - c^2)}{6}},$$

where d = the diameter of the circle, c = a chord of it, a = an arc cut off by that chord, h = height of the segment or its arrow and a' = an arc of the circle lying between two parallel chords.

These formulae are not found clearly defined in abstract in any of the early canonical works, though they state in minute details some of the above numerical data.[178]

8.2 Umāsvāti's rules

In his gloss on his own treatise *Tattvārthādhigama-sūtra*, Umāsvāti (c. 150 BC or AD) says:

> The square-root of four times the product of an arbitrary depth and the diameter diminished by that depth is the chord. The square-root of the difference of the squares of the diameter and chord should be subtracted from the diameter: half of the remainder is the arrow. The square-root of six times the square of the arrow added to the square of the chord (gives) the arc. The square of the arrow plus the one-fourth of the square of the chord is divided by the arrow: the quotient is the diameter. From the northern (meaning the bigger) arc should be subtracted the southern (meaning the smaller) arc: half of the remainder is the side (arc).[179]

All these rules have been restated by Umāsvāti in another work, *Jambūdvīpa-samāsa* by name.[180] But there the formula for the arrow is different:

> The square-root of one-sixth of the difference between the squares of the arc and the chord is the arrow.

It is clearly approximate.

8.3 Āryabhaṭa I and Brahmagupta

Āryabhaṭa I writes:

> In a circle, the product of the two arrows is the square of the semi-chord of the two arcs.[181]

Brahmagupta says:

> In a circle, the diameter should be diminished and then multiplied by the arrow; then the result is multiplied by four: the square root of the product is the chord. Divide the square of the chord

[178] For instance see *Jambūdvīpa-prajñapti*, *Sūtra* 3, 10–15; *Jīvābhigama-sūtra*, *Sūtra* 82, 124; *Sūtrakṛtāṅga-sūtra*, *Sūtra* 12.

[179] *Tattvārthādhigama-sūtra*, iii. 11 (gloss).

[180] *Jambūdvīpa-samāsa*, ch. iv.

[181] *Ā*, ii. 17.

> by four times the arrow and then add the arrow to the quotient: the result is the diameter. Half the difference of diameter and the square-root of the difference between the squares of the diameter and chord, is the smaller arrow.[182]

8.4 Jinabhadra Gaṇi's rules

Jinabhadra Gaṇi (529–589) writes:

> Multiply by the depth, the diameter as diminished by the depth: the square-root of four times the product is the chord of the circle.[183]
>
> Divide the square of the chord by the arrow multiplied by four; the quotient together with the arrow should be known certainly as the diameter of the circle. The square of the arrow multiplied by six should be added to the square of the chord; the square-root of the sum should be known to be the arc. Subtract the square of the chord certainly from the square of the arc; the square-root of the sixth part of the remainder is the arrow. Subtract from the diameter the square-root of the difference of the squares of the diameter and chord; half the remainder should be known to be the arrow.[184]
>
> Subtract the smaller arc from the bigger arc; half the remainder should be known to be the side arc. Or add the square of half the difference of the two chords to the square of the perpendicular; the square-root of the sum will be the side arc.[185]

Jinabhadra Gaṇi next cites two formulae for finding the area of a segment of a circle cut off by two parallel chords.

> For the area of the figure, multiply half the sum of its greater and smaller chords by its breadth.[186]

or

> Sum up the squares of its greater and smaller chords; the square-root of the half of that (sum) will be the 'side'. That multiplied by the breadth will be its area.[187]

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[182] *BrSpSi*, xii. 41f.
[183] *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, i. 36.
[184] *Vṛhat Kṣetra-samāsa*, i. 38–41.
[185] *Vṛhat Kṣetra-samāsa*, i. 46–7.
[186] *Ibid*, i. 64.
[187] *Vṛhat Kṣetra-samāsa*, i. 122.

That is to say, if c_1, c_2 be the lengths of the two parallel chords and h, the perpendicular distance between them, then the area of the segment will be given by

$$\text{(i)}\quad \text{Area} = \frac{1}{2}(c_1 + c_2)h,$$

$$\text{(ii)}\quad \text{Area} = \sqrt{\frac{1}{2}(c_1^2 + c_2^2)} \times h.$$

Neither of these formulae, the author thinks, will be available for finding the area of the Southern Bhāratavarṣa which, as has been described before, has only a single chord. So he gives a third formula as follows:

> In case of the Southern Bhāratavarṣa, multiply the arrow by the chord and then divide by four; then square and multiply by ten: the square-root (of the result) will be its area.[188]

$$\text{(iii)}\quad \text{Area} = \sqrt{10\left(\frac{ch}{4}\right)^2}.$$

None of the above formulae will give the desired result to a fair degree of accuracy. Formula (i) indeed gives the area of the isosceles trapezium of which the two parallel chords form the two parallel sides. The result obtained by it will therefore be approximately correct only when the breadth is small. Otherwise as has been observed by the commentator Malayagiri (c. 1200), the formula will give only a wrong result. Jinabhadra Gaṇi seems to have been aware of this limitation of the formula. For he has not followed it in practice. The rationale of formula (ii) which has been followed by our author, cannot be easily determined. Formula (iii) seems to have been derived by analogy with the formula for the finding the area of a semi-circle.

8.5 Śrīdhara's rule

In his smaller treatise or arithmetic, Śrīdhara (c. 900) includes a formula for finding the area of a segment of a circle. He says:

> Multiply half the sum of the chord and arrow by the arrow; multiply the square of the product by ten and then divide by nine. The square-root of the result will be the area of the segment.[189]

$$\text{Area} = \sqrt{\frac{10}{9}\left\{h\left(\frac{c+h}{2}\right)\right\}^2}.$$

[188] *Ibid.*, i. 122.
[189] *Triś*, R. 47.

8.6 Mahāvīra's rules

For the mensuration of a segment of a circle, Mahāvīra (850) gives two sets of formulae; the first set gives results serving all practical purposes (*vyāvahārika phala*), while the second set yields nearly precise results (*sūkṣma phala*). He says:

> Multiply the sum of the arrow and chord by the half of the arrow: the product is the area of the segment. The square-root of the square of the arrow as multiplied by five and added by the square of the chord is the arc.[190]
>
> The square-root of the difference between the squares of the arc and chord, as divided by five, is stated to be the arrow. The square-root of the square of the arc minus five times the square of the arrow is the chord.[191]

Thus the rough formulae are:

$$\text{Area} = \frac{1}{2}h(c+h),$$
$$h = \sqrt{\frac{a^2-c^2}{5}},$$
$$c = \sqrt{a^2-5h^2},$$
$$a = \sqrt{5h^2+c^2}.$$

For calculation of nearly precise results his rules are as follow:

> In case of a figure of the shape of (the longitudinal section of) a barley and a segment of a circle, the chord multiplied by one-fourth the arrow and also by the square-root of ten becomes, it should be known, the area.[192]
>
> The square of the arrow is multiplied by six and then added by the square of the chord; the square-root of the result is the arc. For finding the arrow and the chord the process is the reverse of this. The square-root of the difference of the squares of the arc and chord, as divided by six, is stated to be the arrow. The square-root of the square of the arc minus six times the square of the arrow is the chord.[193]

[190] *GSS*, vii. 43.
[191] *GSS*, vii. 45.
[192] *GSS*, vii. $70\frac{1}{2}$.
[193] *GSS*, vii. $74\frac{1}{2}$.

Thus the nearly precise formulae of Mahāvīra are:

$$\text{Area} = \frac{\sqrt{10}}{4} ch,$$
$$h = \sqrt{\frac{a^2 - c^2}{6}},$$
$$a = \sqrt{6h^2 + c^2},$$
$$c = \sqrt{a^2 - 6h^2}.$$

8.7 Āryabhaṭa II's rules

Like Mahāvīra, Āryabhaṭa II (950) too gives two sets of formulae, rough (*sthūla*) as well as nearly precise (*sūkṣma*) for the mensuration of a segment of a circle. But it will be noticed that the rough formulae are the same as the nearly precise ones of his predecessor: one about the area yields distinctly better results. Āryabhaṭa II writes:

> The product of the arrow and half the sum of the chord and arrow is multiplied by itself; the square-root of the result increased by its one-ninth is the rough value of the area of the segment. The square-root of the square of the arrow multiplied by six and added by the square of the chord is the arc. The square-root of the difference of the square of the arc and chord as divided by six, is the arrow. The square-root of the remainder left on subtracting six times the square of the arrow from the square of the arc, is the chord. The half of the arc multiplied by itself is diminished by the square of the arrow; on dividing the remainder by twice the arrow, the quotient will be the value of the diameter.[194]

That is to say, the rough formulae are:

$$\text{Area} = \sqrt{\left(1 + \frac{1}{9}\right) \left\{h \left(\frac{c+h}{2}\right)\right\}^2},$$
$$a = \sqrt{6h^2 + c^2},$$
$$h = \sqrt{\frac{a^2 - c^2}{6}},$$
$$c = \sqrt{a^2 - 6h^2},$$
$$d = \frac{1}{2h} \left(\frac{1}{2}a^2 - h^2\right).$$

Āryabhaṭa II then continues:

[194] *MSi*, xv. 89–92.

> On dividing by 21 the product of half the sum of the chord and arrow, as multiplied by the arrow and again by 22, the quotient will be the nearly precise value of the area of the segment. The square of the arrow being multiplied by 288 and divided by 49, is increased by the square of the chord; the square-root of the result is the near value of the arc. The square-root of the difference of the squares of the arc and chord, as multiplied by 49 and divided by 288, is the arrow. The square-root of what is left on subtracting from the square of the arc, the square of the arrow multiplied by 288 and divided by 49 will be the chord. Multiply the square of the arc by 245 and then divide by 484; divide the quotient as diminished by the square of the arrow, by twice the arrow: the quotient will be the diameter. Similarly the chord will be the square-root of the diameter as diminished by the arrow and then multiplied by four times the arrow. The square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. The square of the semi-chord being added with the square of the arrow, the quotient of the sum divided by the arrow is the diameter.[195]

Hence

$$\text{Area} = \frac{22}{21} h \left(\frac{c+h}{2} \right),$$

$$a = \sqrt{\frac{288}{49} h^2 + c^2},$$

$$h = \sqrt{\frac{49}{288} (a^2 - c^2)},$$

$$c = \sqrt{a^2 - \frac{288}{49} h^2},$$

$$d = \frac{1}{2h} \left(\frac{245}{484} a^2 - h^2 \right),$$

$$e = \sqrt{4h(d-h)},$$

$$h = \frac{1}{2} \left\{ d - \sqrt{d^2 - c^2} \right\},$$

$$d = \frac{1}{h} \left\{ \left(\frac{c}{2} \right)^2 + h^2 \right\}.$$

It should perhaps be noted that the last three formulae are exact, while others are approximate.

[195] *MSi*, xv. 93–99.

8.8 Śrīpati's rules

Śrīpati (c. 1039) states:

The diameter of a circle is diminished by the given arrow and then multiplied by it and also by four: the square-root of the result is the chord. In a circle, the square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. In a circle, the square of the semi-chord being added to the square of the arrow and then divided by the arrow, the result is stated to be the diameter ... Six times the square of the arrow being added to the square of the chord, the square-root of the sum is the arc here. The difference of the squares of the arc and chord being divided by six, the square-root of the quotient is the value of the arrow. From the square of the arc being subtracted the square of the arrow as multiplied by six, the square-root of the remainder is the chord. Twice the square of the arrow being subtracted from the square of the arc, the remainder divided by four times the arrow, is the diameter.[196]

8.9 Bhāskara II's rules

Bhāskara II (1150) does not mention the formulae for the calculation of approximate results, but gives only the exact ones. He writes:

> Find the square-root of the product of the sum and difference of the diameter and chord, and subtract it from the diameter: half the remainder is the arrow. The diameter being diminished and then multiplied by the arrow, twice the square-root of the result is the chord. In a circle, the square of the semi-chord being divided and then increased by the arrow, the result is stated to be the diameter.[197]

These rules have been reproduced by Munīśvara.[198]

8.10 Sūryadāsa's proof

Sūryadāsa (born 1508) proves the formulae for the arrow and diameter as follows (**ed.** see Figure 16):

Let AB be a chord of the circle having its centre at O and CH the arrow of the segment ABC. Join BO and produce it to meet the circumference in

[196] *SiŚe*, xiii. 37–40.
[197] *L*, p. 58.
[198] *Pāṭīsāra*, R. 220–1.

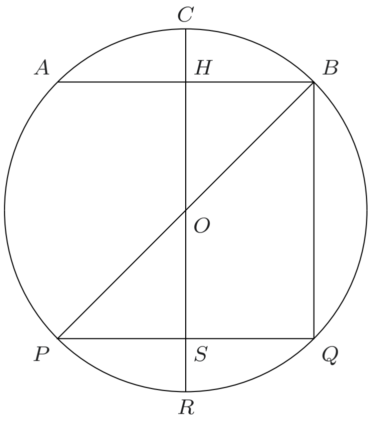

Figure 16

P. Draw PSQ parallel to AB. Join BQ. Then clearly

$$\begin{aligned} CH &= \frac{1}{2}(CR - HS), \\ &= \frac{1}{2}(CR - BQ), \\ &= \frac{1}{2}(CR - \sqrt{BP^2 - PQ^2}). \end{aligned}$$

Hence

$$CH = \frac{1}{2}(CR - \sqrt{CR^2 - AB^2}).$$

Again, since

$$HB^2 = CH \times HR,$$

we get

$$HR = \frac{HB^2}{CH}.$$

Therefore

$$CR = \frac{HB^2}{CH} + CH.$$

8.11 Other formulae for area

For the area of a segment of a circle, Viṣṇu Paṇḍita (c. 1410) and Keśava II (1496) gave the formula:

$$\text{Area} = \left(1 + \frac{1}{20}\right)\left\{h\left(\frac{h+c}{2}\right)\right\}.$$

Gaṇeśa (1545) and Rāmakṛṣṇadeva state:

$$\begin{aligned}\text{Area} &= (\text{area of the sector}) - (\text{area of the triangle})\\ &= \frac{1}{4}ad - \frac{1}{2}c\left(\frac{1}{2}d - h\right).\end{aligned}$$

8.12 Intersection of two circles

When two circles intersect, the common portion cut off is called the *grāsa* ("the erosion"). The origin of the term seems to be connected with the eclipse of the moon (or the sun) which is narrated in the popular mythology of the early Hindus as being caused by the dragon *Rāhu* (earth's shadow) swallowing the moon. The portion swallowed up is the *grāsa.* In fact, the geometrical theorem, just to be described, had its application in the calculation of the eclipse. The common portion is also called *matsya* (fish) as it resembles a fish. (**ed.** see Figure 17.)

Āryabhaṭa I writes:

> (The diameters of) the two circles being severally diminished and then multiplied by (the breadth of) the erosion, the products divided severally by the sum of the diameters (each) as diminished by the erosion, will be the two arrows lying within the erosion.[199]

This rule is nearly reproduced by Mahāvīra.[200]

$$AP \times PA' = PB^2 = DP \times PD',$$

or

$$(AA' - A'P)\, A'P = (DD' - D'P)D'P,$$

or

$$\begin{aligned}AA' \times A'P - DD' \times D'P &= A'P^2 - D'P^2\\ &= (A'P + D'P)(A'P - D'P)\\ &= A'D'(A'P - D'P),\end{aligned}$$

or

$$(AA' - A'D')A'P = (DD' - A'D')D'P.$$

Hence

$$\begin{aligned}\frac{A'P}{DD' - A'D'} &= \frac{D'P}{AA' - A'D'}\\ &= \frac{A'D'}{(DD' - A'D') + (AA' - A'D')}.\end{aligned}$$

[199] *Ā*, ii. 18.
[200] *GSS*, vii. $231\frac{1}{2}$.

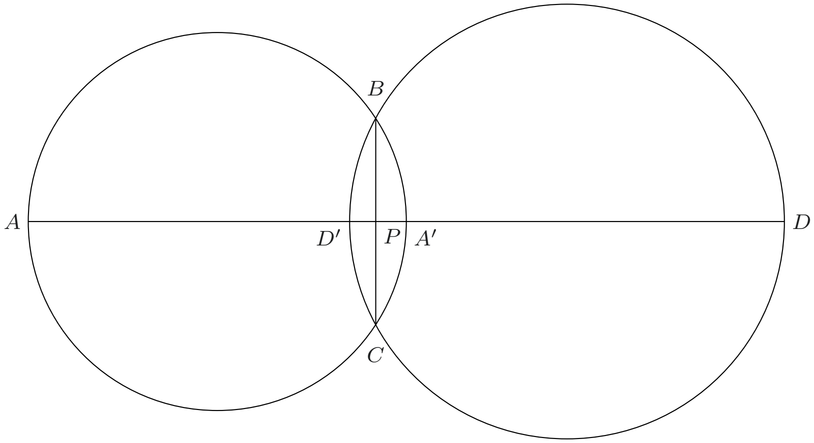

Figure 17

Therefore

$$A'P = \frac{A'D'(DD' - A'D')}{DD' + AA' - 2A'D'},$$
$$D'P = \frac{A'D'(AA' - A'D')}{DD' + AA' - 2A'D'}.$$

Brahmagupta says:

> The erosion being subtracted (severally) from the two diameters, the remainders, multiplied by the erosion and divided by the sum of the remainders, are the arrows.[201]
>
> The square of half the (common) chord being divided severally by the two given arrows, the quotients added with the respective arrows give the two diameters. The sum of the two arrows is the erosion; and that of the quotients is the sum of the diameters minus the erosion.[202]

9 Miscellaneous figures

9.1 Miscellaneous figures

Śrīdhara, Mahāvīra and Āryabhaṭa II have treated the mensuration of certain other plane figures such as of the shape of a barley corn (*yava*), drum (*muraja*, *mṛdaṅga*), elephant's tusk (*gajadanta*), crescent moon (*bālendu*), felloe (*nemi*),

[201] *BrSpSi*, xii. 42.
[202] *BrSpSi*, xii. 43.

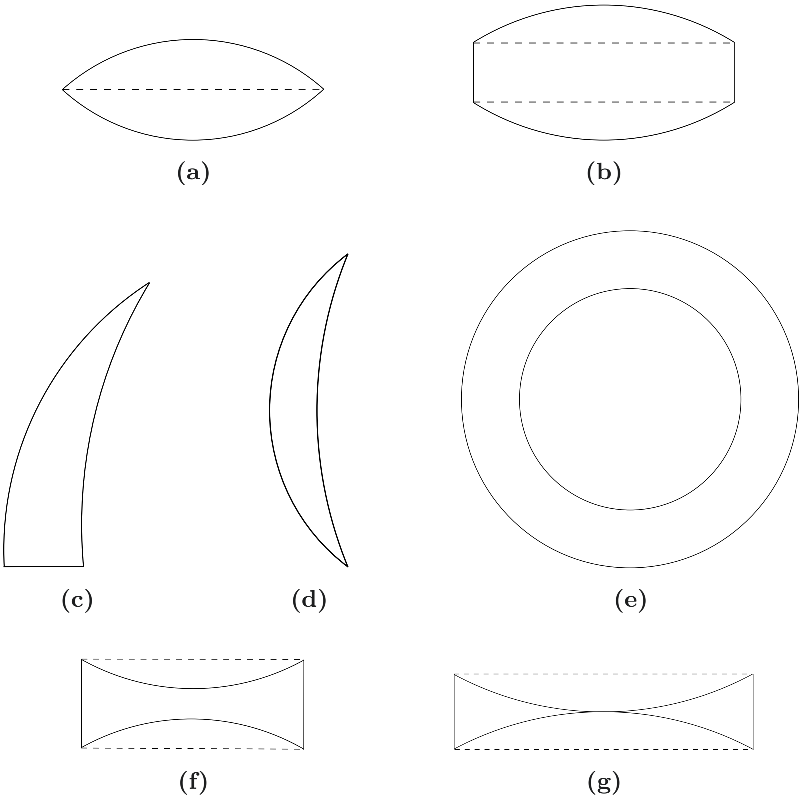

Figure 18: (a) barley corn, (b) drum, (c) elephant's tusk, (d) crescent, (e) felloe, (f) *vajra* (after Śrīdhara and Āryabhaṭa II) or *paṇava* (after Mahāvīra), (g) *vajra* (after Mahāvīra).

thunder-bolt (*vajra*) etc. The formulae described in case of most of them are only roughly approximate and some of them are deduced easily from the results already obtained. It was probably from the point of view of some practical utility that all the results have been stated separately.

9.2 Śrīdhara's rules

Śrīdhara says:

> A figure of the shape of an elephant tusk (may be considered) as a triangle, of a felloe as a quadrilateral, of a crescent moon as two triangles and of a thunderbolt as two quadrilaterals.[203]

[203] *Triś*, R. 44.

A figure of the shape of a drum, should be supposed as consisting of two segments of a circle with a rectangle intervening; and a barley corn only of two segments of a circle.[204]

9.3 Mahāvīra's rules

For finding the gross value of the areas of above figures Mahāvīra gives the following rules:

In a figure of the shape of a felloe, the area is the product of the breadth and half the sum of the two edges. Half that area will be the area of a crescent moon here.[205]

The diameter increased by the breadth of the annulus and then multiplied by three and also by the breadth gives the area of the outlying annulus. The area of an inlying annulus (will be obtained in the same way) after subtracting the breadth from the diameter.[206]

In case of a figure of the shape of a barley corn, drum, *paṇava*, or thunderbolt, the area will be equal to half the sum of the extreme and middle measures multiplied by the length.[207]

For finding the neat values of the areas of them, Mahāvīra has the following rules:

The diameter added with the breadth of the annulus being multiplied by $\sqrt{10}$ and the breadth gives the area of the outlying annulus. The area of the inlying annulus (will be obtained from the same operations) after subtracting the breadth from the diameter.[208]

Find the area by multiplying the face by the length. That added with the areas of the two segments of the circle associated with it will give the area of a drum-shaped figure. That diminished by the areas of the two associated segments of the circle will be the area in case of a figure of the shape of a *paṇava* as well as of a *vajra*.[209]

In case of a felloe-shaped figure, the area is equal to the sum of the outer and inner edges as divided by six and multiplied by the

[204] *Triś*, R. 48.
[205] *GSS*, vii. 7. The formula for the area of the felloe yields, indeed, the accurate value of it.
[206] *GSS*, vii. 28.
[207] *GSS*, vii. 32.
[208] *GSS*, vii. $67\frac{1}{2}$.
[209] *GSS*, vii. $76\frac{1}{2}$.

> breadth and $\sqrt{10}$. The area of a crescent moon or elephant's tusk is half that.[210]

9.4 Āryabhaṭa II's rules

Āryabhaṭa II writes:

> In (a figure of the shape of) the crescent moon, there are two triangles and in an elephant's tusk only one triangle; a barley corn may be looked upon as consisting of two segments of a circle or two triangles.[211]
>
> In a drum, there are two segments of a circle outside and a rectangle inside; in a thunderbolt, are present two segments of two circles and two quadrilaterals.[212]

9.5 Polygons

According to Śrīdhara, regular polygons may be treated as being composed of triangles.[213] Mahāvīra says:

> One-third of the square of half the perimeter being divided by the number of sides and multiplied by that number as diminished by unity will give the (gross) area of all rectilinear figures. One-fourth of that will be the area of a figure enclosed by circles mutually in contact.[214]

That is to say, if $2s$ denote the perimeter of a polygon of n sides, whether regular or otherwise, but without a re-entrant angle, then its area will be roughly given by the formula

$$\text{Area} = \frac{(n-1)s^2}{3n}.$$

Mahāvīra has treated some very particular cases of polygons with re-entrant angles. He says:

> The product of the length and the breadth minus the product of the length and half the breadth is the area of a di-deficient figure; by subtracting half the latter (product form the former) is obtained the area of a uni-deficient figure.[215]

[210] *GSS*, vii. $80\frac{1}{2}$.
[211] *MSi*, xv. 101.
[212] *MSi*, xv. 103.
[213] *Triś*, R. 48.
[214] *GSS*, vii. 39.
[215] *GSS*, vii. 37.

The figures contemplated in this rule are those formed by leaving out two vertically opposite ones or any one of the four portions into which a rectangle is divided by its two diagonals. In the first case, the figure is technically called the *ubhaya-niṣedha-kṣetra* ("di-deficient figure") and in the other the *eka-niṣedha-kṣetra* (uni-deficient figure).

Mahāvīra further says:

> On subtracting the accurate value of the area of one of the circles from the square of a diameter, will be obtained the (neat) value of the area of the space lying between four equal circles (touching each other).[216]

> The accurate value of the area of an equilateral triangle each side of which is equal to a diameter, being diminished by half the area of a circle, will yield the area of the space bounded by three equal circles (touching each other).[217]

> A side of a regular hexagon, its square and its biquadrate being multiplied respectively by 2, 3, and 3 will give in order the value of its diagonal, the square of the altitude, and the square of the area.[218]

Āryabhaṭa II observes:

> A pentagon is composed of a triangle and a trapezium, a hexagon of two trapeziums; in a lotus-shaped figure there is a central circle and the rest are triangles.[219]

9.6 Ellipse

Though the ellipse was known to the Hindus as early as circa 400 BC, we do not find any formula for its mensuration in any of their works on mathematics, except the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In the latter again, we have only roughly approximate results. Mahāvīra says:

> The length of an ellipse being added by half its breadth and multiplied by two, gives the gross value of its circumference. The circumference multiplied by one-fourth the breadth becomes the gross value of the area.[220]

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[216] *GSS*, vii. $82\frac{1}{2}$.
[217] *GSS*, vii. $84\frac{1}{2}$.
[218] *GSS*, vii. $86\frac{1}{2}$.
[219] *MSi*, xv. 102.
[220] *GSS*, vii. 21.

> The square-root of six times the square of the breadth added with the square of twice the length, will be the neat value of the circumference of an ellipse. That multiplied by one-fourth the breadth will become the neat value of the area.[221]

That is to say if $2a$ be the longer diameter of an ellipse and $2b$ its shorter diameter, then, according to Mahāvīra,

$$\text{Circumference } (\textit{Gross}) = 2(2a+b),$$
$$\text{Circumference } (\textit{Neat}) = \sqrt{16a^2+24b^2},$$
$$\text{Area } (\textit{Gross}) = b(2a+b),$$
$$\text{Area } (\textit{Neat}) = \frac{1}{2}\, b\, \sqrt{16a^2+24b^2}.$$

10 Measurement of volumes

10.1 Solids considered

Things in everyday life of the ancient Vedic Hindus which led them to develop formulae for the measurement of volumes were fire-altars and excavations. Amongst the fire-altars described in the extant works on the *Śulba*, we find that some are right prisms of various cross-sections, and others are right circular cylinders. Only in one case, namely, the fire-altar of the shape of the cemetery, the solid considered resembles a frustum of a pyramid. For the measurement of the latter, the Hindus developed an approximate formula. Though we meet with copious descriptions of pits, caves and mountains etc., of the shape of truncated cones and pyramids, in the early canonical works of the Jainas, there is nothing to indicate that the mensuration of those solids was known to them. In later Hindu treatises of arithmetic, solids generally treated are excavations, mounds of grains and piles of bricks.

10.2 Prism and cylinder

The formula for calculating the volumes of prisms and cylinders is found in the *Śulba*.[222]

$$\text{Volume of a prism or cylinder} = (\text{base}) \times (\text{height}).$$

The same formula is stated in later works.[223]

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[221] *GSS*, vii. 63.
[222] Datta, *Śulba*, p. 101. See also *Jaina Math., Quel, und Stud. z. Gesch. d. Math.* Bd. I. (1930), p. 253.
[223] *BrSpSi*, xii. 44; *Triś*, R. 53; *GSS*, viii. 4; etc.

It may be noted that in later treatises of arithmetic, an excavation (*khāta*) whose depth is uniform is called the *sama-khāta*. The section of the base may be of any form, as it has not been particularly mentioned. The word *sama* (equal) implies that all sections parallel to the face or base are equal.

In the *Veda* and *Saṃhitā*, the prisms whose sections are regular polygons, were named according to the number of edges. Thus in the *Ṛgveda* (c. 3000 BC), the triangular prism is called *trirasri* (three-edged solid; *tri* = three, *asri* = edge), a quadrangular prism *caturasri* (= four edged solid) and so on.[224] But these terms do not seem to have been completely standardised. For in comparatively later times, a cube was called *dvādasāsrika* (= twelve-edged solid).

10.3 Cone and pyramid

The Hindus do not always distinguish between a cone and a pyramid. They include both under a generic name *sūcī*, which means literally "a needle", "a sharp pointed object", and hence, "a solid of the form of the needle", "a sharp pointed solid". Thus the term generally denotes a pyramid with a base of any form; as the base may be a circle it includes a cone as well. A triangular pyramid is, however, distinguished as the *ghana-ṣaḍasri* or simply *ṣaḍasri* (literally, "six-edged solid").

Āryabhaṭa I says:

> Half the product of this area (of the triangular base) and the height is the volume of the six-edged solid.[225]

This formula for the volume of the triangular pyramid is wrong. The correct formula is found in the works of Brahmagupta. He states:

> The volume of the uniform excavation divided by three is the volume of the needle-shaped solid.[226]

That is to say, we shall have

$$\text{Volume of a cone or pyramid} = \frac{1}{3}(\text{base}) \times (\text{height}).$$

This formula reappears in the works of Āryabhaṭa II,[227] Nemicandra,[228] Śrīpati[229] and Bhāskara II.[230]

[224] Datta, "On the Hindu names for the rectilinear geometrical figures", *loc. cit*, pp. 284f.
[225] *Ā*, ii. 6.
[226] *BrSpSi*, xii. 44.
[227] *MSi*, xv. 105.
[228] *Trilokasāra*, *Gāthā* 19.
[229] *SiŚe*, xiii. 44.
[230] *L*, p. 62

For measuring the mounds of grains which approximate to the form of a right circular cone, the Hindus ordinarily employed a rough formula. In such cases, they further assume the height of the mound to be equal to the circumference of the base divided by 9, 10 or 11 according to the kind of grain of which the mound is composed. Thus Brahmagupta says:

> In case of *śuki* grains one-ninth, in case of course grains one-tenth and in case of fine grains one-eleventh of the circumference (of the base) is the height; that multiplied by the square of the sixth part of the circumference will be the volume.[231]

Śrīpati writes:

> Of a heap of grains standing on the plane surface of the earth, the square of one-sixth the circumference multiplied by the height is the volume in terms of *Māgadha Khārikā*. In case of grains known as *syāmāka*, *śāli*, *tila*, *sarṣapa*, etc., the circumference is nine times the height; in case of *godhūma*, *mudga*, *yava*, *dhānyaka*, etc., it is ten times; and in case of *vadara*, *kaṅgu*, *kulattha*, etc., eleven times.[232]

The rough formula was obtained probably thus:

$$\text{Volume of a cone} = \frac{1}{3}(\text{base}) \times (\text{height}).$$

If r denote the radius of the base, we have

$$\text{Base} = \pi r^2 = \frac{2\pi r \times 2\pi r}{4\pi} = \frac{(\text{circumference})^2}{4\pi}.$$

Hence

$$\text{Volume of a cone} = \frac{1}{12\pi}(\text{circumference})^2 \times (\text{height}).$$

Now putting $\pi = 3$ roughly we get,

$$\text{Volume} = \left(\frac{\text{circumference}}{6}\right)^2 \times (\text{height}).$$

This approximate formula is stated also by Srīdhara,[233] Āryabhaṭa II,[234] Nemicandra[235] and Bhāskara II.[236] The ancient commentators have observed that it was intended only for "rough calculation".

[231] *BrSpSi*, xii. 50.
[232] *SiŚe*, xiii. 50–1.
[233] *Triś*, R. 61.
[234] *MSi*, xv. 115.
[235] *Trilokasāra*, *Gāthā*, 22, 23.
[236] *L*, pp. 69f.

10.4 Frustum of a cone

To find the volume of a frustum of a right circular cone, Śrīdhara gives the following formula:

> The square-root of ten times the square of the sum of the squares of the diameters of the face, base and of their sum, being multiplied by the height and divided by twenty-four, will be the volume of a well.[237]

That is to say, if d, d' denote the diameters of the upper and lower faces of the frustum of a right circular cone and h its height, then its volume V will be given by

$$V = \frac{h}{24}\sqrt{10\left\{d^2 + d'^2 + (d+d')^2\right\}^2},$$

or

$$V = \frac{\pi}{3}\left(r^2 + r'^2 + rr'\right)h,$$

where r, r' denote the radii of the upper and lower faces and $\pi = \sqrt{10}$, the value adopted by Śrīdhara. Other writers have included the treatment of the frustum of a cone in that of a more general kind of obelisk.

Example from Śrīdhara:

> The diameter of the top of a well is 16 cubits, and of the bottom 4 cubits; its depth is 12 cubits. Find, O learned man, its volume.[238]

10.5 Obelisk

An approximate formula for calculating the volume of a frustum of a pyramid on a rectangular base is found as early as the works on the *Śulba* by Baudhāyana (800 BC) and others.[239] If (a, b) be the length and breadth of the base of the solid, (a', b') the corresponding sides of the face parallel to it and h the height, then

$$\text{Volume of the frustum} = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) \times h.$$

In later treatises of arithmetic we find the accurate formula for the same. Thus Brahmagupta says:

> The area from half the sum of (the edges of) the face and base, being multiplied by the depth gives *vyāvahārika* volume; half the sum of the areas of the face and base being multiplied by the depth

[237] *Triś*, R. 54.
[238] *Triś*, Ex. 91.
[239] Datta, *Śulba*, p. 103.

> will be the *autra* volume. Subtract the *vyāvahārika* volume from the *autra* volume and divide the remainder by three; the quotient added with the *vyāvahārika* volume will become the truly accurate volume.[240]

It is noteworthy that Brahmagupta does not specify the shape of the face and base of the excavation contemplated by him. His text is *mukhatalayuti-dalagaṇitam* etc., or "the area from the half the sum of the face and base," etc. If we, however, suppose them to be rectangular, then according to the rule, we shall have,

$$V' = \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right)h,$$
$$A = \frac{1}{2}\left(ab + a'b'\right)h,$$
$$V = \frac{1}{3}\left(A - V'\right) + V',$$

where V', A and V denote respectively the *vyāvahārika*, *autra* and accurate volumes of the obelisk. Substituting the values in the last formula, we get

$$V = \frac{h}{6}\left\{(a+a')(b+b') + ab + a'b'\right\}.$$

If the face and base be circular, and if their radii be r' and r respectively, then by the rule

$$V' = \pi\left(\frac{r+r'}{2}\right)^2 h = \frac{\pi}{4}(r+r')^2 h,$$
$$A = \left(\frac{\pi r^2 + \pi r'^2}{2}\right)h.$$

Hence

$$V = \frac{h}{3}\left\{\frac{\pi}{2}\left(r^2 + r'^2\right) - \frac{\pi}{4}(r+r')^2\right\} + \frac{\pi}{4}(r+r')^2 h,$$
$$= \frac{1}{3}\pi h\left(r^2 + r'^2 + rr'\right).$$

10.6 Particular cases

(i) Put $a' = 0 = b'$; then we get

$$\text{Volume of a cone or pyramid} = \frac{1}{3}(\text{base})(\text{height}).$$

[240] *BrSpSi*, xii. 45–6.

(ii) Let $b' = 0$;

$$\text{Volume of a wedge} = \frac{h}{6}(2ab + a'b).$$

(iii) Suppose $a = b$, $a' = b'$; then

$$\text{Volume of a truncated square pyramid} = \frac{h}{3}(a^2 + a'^2 + aa').$$

Pṛthūdakasvāmi has worked out the following example in illustration of the above rule of Brahmagupta:

> There is a square tank whose each side is 10 cubits long at the face and 6 cubits long at the base; it is excavated so as to have a depth of 30 cubits. Tell me its *vyāvahārika*, *autra* and truly accurate volumes.

This example has misled some of the modern historians of mathematics to presume that Brahmagupta's rule was meant for the measurement of the volume of a truncated pyramid on a square base only.[241] But, as already pointed out, there is nothing in the definition of the rule to warrant such a limited application of it.[242]

Mahāvīra writes:

> Of the outer (i.e. at the ground) and various inner sections (of the excavation) the sides of the ground section are added by all the corresponding sides of the other sections and divided (by the number of sections). Multiply these sides (of the average section) mutually in accordance with the method of finding the area of a figure of that shape; the result (thus obtained) multiplied by the depth will be the *karmāntika* volume. Find the areas of those sections (severally), add them together and then divide by the number of sectional areas; the quotient multiplied by the depth will be the *aundra* volume. One third the difference of those two volumes added with the *karmāntika* volume will be the truly accurate volume.[243]

It will be noticed that in finding the average volumes, Mahāvīra takes into consideration several parallel sections of the solid, instead of only two, the

[241] Such is the opinion of Cantor, followed by J. Tropfke and D. E. Smith.

[242] See also the article of Datta, Bibhutibhusan, "On the supposed indebtedness of Brahmagupta to Chiu-chang Suan-Shu" in the *BCMS*, xiii (1930), pp. 39–51; more particularly pp. 45 ff.

[243] *GSS*, viii. 9–11$\frac{1}{2}$.

face and base.[244] In three of the illustrative examples,[245] he actually states three sections of the solid. If however, we take into consideration only the top and base, the formula obtained will be the same as that of Brahmagupta.

In illustration of his rule, Mahāvīra gives examples of excavations of various kinds, which are indeed inverted cases of truncated pyramids on square, rectangular, or equilateral triangular bases, and truncated cones. There is an instance of a truncated wedge:

> (In a well with rectangular sections), the lengths at the top, middle and base are 90, 80 and 70 respectively; and the breadths are 22, 16 and 10. Its depth is 7. (Calculate its volume).[246]

Āryabhaṭa II says:

> Divide the sum of the areas of the face, base and that arising from the sum of (the dimensions of) them by six; the quotient multiplied by the height will be the volume of an excavation such as a well and tank.[247]

That is to say,

$$V = \frac{h}{6}\left\{(a + a')\,(b + b') + ab + a'b'\right\}.$$

This formula reappears in the works of Śrīpati and Bhāskara II. The former says:

> The sum of the areas of the face, base and that arising from the sum of their sides, being divided by six and multiplied by the depth, will be the truly accurate value of the volume.[248]

Bhāskara II writes:

> The sum of the areas from (the linear dimensions of) the face, base and their sums, divided by six gives the area of the equivalent prism (*samaṃ kṣetraphalam*) (of the same height). That multiplied by the depth is the true volume.[249]

---

[244] Hence Raṅgācārya is wrong in supposing that the rule contemplates only the face and base.

[245] *GSS*, xiii. $16\frac{1}{2}$–$18\frac{1}{2}$.

[246] *GSS*, vii. $16\frac{1}{2}$. In the printed text 22 is wrongly stated as 32.

[247] *MSi*, xv. 106.

[248] *SiŚe*, xiii. 49.

[249] *L*, p. 65.

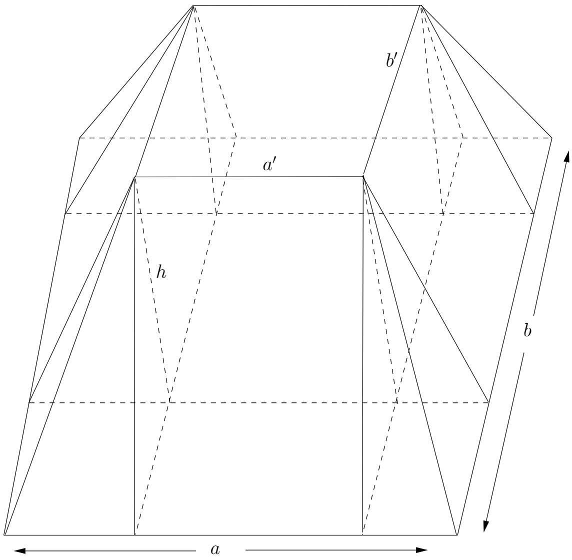

Figure 19

Gaṇeśa's proof

Gaṇeśa demonstrates this formula substantially as follows:

Suppose $(a,\ b)$ and $(a',\ b')$ denote the length and breadth of the base and face of the solid respectively. Let its height be h. Then it is clear from the figure (**ed.** see Figure 19) that

Volume of the obelisk = volume of the prism at the centre
+ volumes of four pyramids at the corners
+ volumes of four prisms on four sides.

Now the four pyramids at the corners can be combined into one of base $(a - a')$ by $(b - b')$ and height h. Hence its volume is

$$\frac{h}{3}(a - a')(b - b').$$

The four side prisms can be combined into two others: (1) one on a triangle of base $(b - b')$ and altitude h, its height being a'; and (2) the other on a triangle of base $(a - a')$ and altitude h; its height will be b'. Therefore their volumes are together equal to

$$\frac{1}{2}(b - b')ha' + \frac{1}{2}(a - a')hb'.$$

Therefore

Volume of the obelisk

$$= a'b'h + \frac{h}{3}(a-a')(b-b') + \frac{1}{2}(b-b')\,ha' + \frac{1}{2}(a-a')\,hb'.$$
$$= \frac{h}{6}(2ab + 2a'b' + a'b + ab')$$
$$= \frac{h}{6}\{(a+a')(b+b') + ab + a'b'\}.$$

Mahāvīra has treated a problem like this : A fort wall of height h and length l, whose extremities are vertical, has its base b in breadth and face a. Its upper portion is blown off by cyclone, obliquely. It is required to calculate the volume of the portion still intact.[250]

Another problem runs as follows:

> The heights (of a certain construction) are 12, 16, and 20 cubits (at one end, middle and other end respectively); the breadths (at those points) are respectively 7, 6 and 5 cubits at the base and 4, 3 and 2 cubits at the top; the length is 24 cubits. (Find the number of bricks employed in the construction.)[251]

10.7 Surface of a sphere

The earliest reference to a formula for the surface of a sphere occurs, so far as known, in the treatise on arithmetic by Lalla (c. 749). That work is now lost. But the relevant passage has survived in a citation by Bhāskara II.[252] It is as follows:

> The area of the circle (of a diametral section) multiplied by its circumference will be equal to the area of the surface of a sphere.

If r be the radius of a sphere, then according to this rule, its surface S will be

$$S = \pi r^2 \times 2\pi r = 2\pi^2 r^3.$$

This formula is clearly inaccurate. So it has been adversely criticised and discarded by Bhāskara II.[253]

Āryabhaṭa II was undoubtedly aware of a formula for the surface of a sphere, though he has not expressly defined it anywhere. For he says, "the diameter

[250] *GSS*, viii. $52\frac{1}{2}$–$54\frac{1}{2}$.
[251] *GSS*, viii. $51\frac{1}{2}$.
[252] *SiŚi*, *Gola*, iii. 57 (*vāsanā*).
[253] *SiŚi*, *Gola*, iii. 53.

of the earth is a little less than 2109; its circumference is 6625; and the area of its surface is 13971849."[254]

Now according to Āryabhaṭa II, $\pi = \frac{21600}{6876}$. Then

$$\text{Diameter of earth} = \frac{6876}{21600} \times (\text{circumference of the earth}) = \frac{6876}{21600} \times 6625.$$

$$\text{Surface} = 6625\left(2109 - \frac{1}{24}\right) = 13971849 - \frac{1}{24}.$$

Thus it seems that Āryabhaṭa II employed the formula

$$\text{Surface of a sphere} = (\text{circumference}) \times (\text{diameter}).$$

This formula is, however, expressly stated by Bhāskara II.[255] He further says:

> That (the area of a diametral section) multiplied by four is the net lying all over a round ball (i.e., the area of the surface of a sphere).[256]

$$S = 4\pi r^2.$$

Bhāskara II has given a demonstration of this formula by means of the method of infinitesimals. We shall describe it later on.

10.8 Volume of a sphere

Āryabhaṭa I writes:

> That (the area of a diametral section) multiplied by its own square-root is the exact volume of a sphere.[257]

That is to say, if r be the radius of a sphere, then according to Āryabhaṭa I,

$$\text{Volume of a sphere} = \pi r^2 \sqrt{\pi r^2}.$$

This formula is inaccurate. Śrīdhara says:

> Half the cube of the diameter of a sphere, then added with its eighteenth part, will give the volume.[258]

$$\text{Volume} = \frac{19 \times (\text{diameter})^3}{18 \times 2}.$$

Mahāvīra writes:

[254] *MSi*, xvi. 35–6.
[255] *SiŚi*, *Gola*, iii. 52, 61.
[256] *L*, p. 55.
[257] *Ā*, ii. 7.
[258] *Triś*, R. 56.

> Nine times the half of the cube of the semi-diameter is the *vyāvahārika* volume of a sphere. Nine-tenth of that will be the very accurate volume.[259]

Āryabhaṭa II:

> The cube of the diameter of a sphere being halved and then added with its eighteenth part, will give its volume in cubic cubits: such is the formula taught (by the ancient teachers).[260]

This formula was given before by Śridhara. It reappears also in the works of Śrīpati.[261] All the above-mentioned formulae for the volume of a sphere are more or less approximate. The truly accurate formula is, however, given by Bhāskara II. He says:

> That area of the surface multiplied by the diameter and divided by six, will be the accurate value of the volume of a sphere.[262]

That is to say, we shall have

$$\text{Volume} = \frac{1}{6}(\text{surface}) \times (\text{diameter}).$$

Now according to Bhāskara II,

$$\begin{aligned} \text{Surface} &= 4\ (\text{area of a diametral section}), \\ \text{Area of a diametral section} &= \frac{1}{4}(\text{circumference}) \times (\text{diameter}), \\ \text{Circumference} &= \frac{22}{7}(\text{diameter}). \end{aligned}$$

Therefore

$$\begin{aligned} \text{Volume} &= \frac{22}{42}D^3, \\ &= \left(1 + \frac{1}{21}\right) \frac{D^3}{2}. \end{aligned}$$

Hence Bhāskara II writes:

> Half the cube of the diameter being added with its twenty-oneth part becomes the volume of a sphere.[263]

He has further observed that the volume of a sphere obtained by this formula is "rough" (*sthūla*). This is clearly so because that formula is derived with the rough value $\frac{22}{7}$ of π instead of its accurate value $\frac{3927}{1250}$.

[259] *GSS*, viii. $28\frac{1}{2}$.
[260] *MSi*, xvi. 108.
[261] *SiŚe*, xiii. 46.
[262] *L*, p. 55.
[263] *L*, p. 57.

10.9 Average value

In measuring the volume of an excavation whose length, breadth or depth is different at different portions, the other two dimensions remaining the same, the Hindus take for all practical purposes the arithmetic mean of the varying elements. This mean value is technically called *sama-rajju* ("mean measure") by Brahmagupta, *samīkaraṇa* ("equalising value") by Mahāvīra, *sāmya* ("equability", i.e. "equivalent value") by Śrīpati and *samamiti* ("average value") by Bhāskara II.

Brahmagupta says:

> In an excavation having the same breadth at the face and bottom, the aggregates (of the partial products of lengths and depths) divided by the total (length) will be the mean measure (*sama-rajju*) of the depth.[264]

Example from Pṛthūdakasvāmi:

> A tank 30 cubits in length and 8 cubits in breadth contains within it five different excavations which subdivide the length into five portions of lengths four, five etc. (cubits). The depths (of these portions) are successively 9, 7, 7, 3 and 2. Tell at once what is the mean depth of the excavation.

$$\text{Mean depth} = \frac{4\times 9+5\times 7+6\times 7+7\times 3+8\times 2}{4+5+6+7+8} = \frac{150}{30} = 5.$$

Therefore the volume of the tank $= 8\times 30\times 5 = 1200$.

Mahāvīra writes:

> Find the half of the top and bottom dimensions; the sum of all the halves divided by the number of them will be the equivalent value.[265]

> The sum of the depths (measured at different places) divided by the number of places will be the average depth.[266]

According to Bhāskara II,

> Calculate the breadth at several places: the sum of them divided by the number of places is the average value. Do in the same way in case of the length and depth.[267]

[264] *BrSpSi*, xii. 44.
[265] *GSS*, viii. 4.
[266] *GSS*, viii. $23\frac{1}{2}$.
[267] *L*, p. 64.

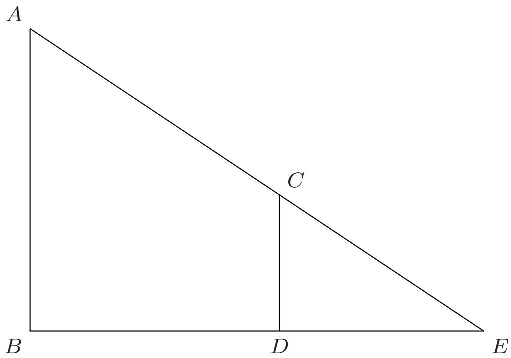

Figure 20

11 Measurement of heights and distances

11.1 Shadow reckoning

The *chāyā*, meaning literally "shadow", but implying truly the measurement by means of shadow of a gnomon, is a common topic for discussion in the Hindu treatises of mathematics. It is applied for measurement of time as well as of heights and distances. We shall, however, notice here only those rules which are related to its application in this latter aspect.[268]

Āryabhaṭa I says:

> Multiply the distance between the gnomon and the lamp-post[269] by the length of the gnomon and divide by the difference between the lengths of the gnomon and the lamp-post. The result will be the length of the shadow of the gnomon measured from its base.[270]

(**ed.** In Figure 20:)

$$AB = \text{the lamp-post},$$
$$CD = \text{the gnomon},$$
$$DE = \text{the shadow of the gnomon},$$
$$DE = \frac{BD \times DC}{AB - CD}.$$

[268]The measurement of time by means of a gnomon is more fully treated in treatises on astronomy.

[269]The Sanskrit original is *bhuja*. Ordinarily the term denotes a side of a triangle (or any rectilinear figure). All the commentators agree in interpreting it as implying here the lamp-post. Latter rules are quite explicit.

[270]$\bar{A}$, ii. 15.

Similar rules are given by Brahmagupta,[271] Mahāvīra,[272] Śrīpati[273] and Bhāskara II.[274] Some later writers[275] have described separately the formulae for calculating the height of the lamp from the length of the shadow and the distance of the gnomon, and the distance from the height of the lamp and the length of the shadow, though the same follow at once from the formula stated above.

11.2 Heights and distances

Another and more useful problem is to find the height and distance of a far off object. By way of illustration of the method employed a high light-post is generally taken into consideration. Then two gnomons of equal heights or the same gnomon successively, being set up in two places at a known distance apart, the two shadows are measured.

Āryabhaṭa I writes:

> The distance between the tips of the two shadows being multiplied by the length of a shadow and divided by the difference between the lengths of the two shadows gives the *koṭi*. That *koṭi* multiplied by the length of the gnomon and divided by the length of the shadow corresponding to it will be the height of the lamp-post.[276]

AB is the lamp-post to be measured (**ed.** see Figure 21); CD, $C'D'$ = the gnomon in its two positions; and DE, $D'E'$ = the shadows respectively. Then the rule says:

$$BE = \frac{EE' \times DE}{D'E' - DE}, \quad BE' = \frac{EE' \times D'E'}{D'E' - DE},$$
$$AB = \frac{BE \times CD}{DE} = \frac{BE' \times CD}{D'E'}.$$

These formulae are stated also by Brahmagupta[277] and Bhāskara II.[278]

11.3 Brahmagupta's rules

The procedure to be adopted in actual practice in measuring the height of a distant object has been indicated by Brahmagupta as follows:

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[271] *BrSpSi*, xii. 53.
[272] *GSS*, ix. $40\frac{1}{2}$.
[273] *SiŚe*, xiii. 54.
[274] *L*, p. 73.
[275] See *GSS*, viii. 43, 45; *SiŚe*, xiii. 55; *L*, p. 74.
[276] *Ā*, ii. 16.
[277] *BrSpSi*, xii. 54.
[278] *L*, p. 75.

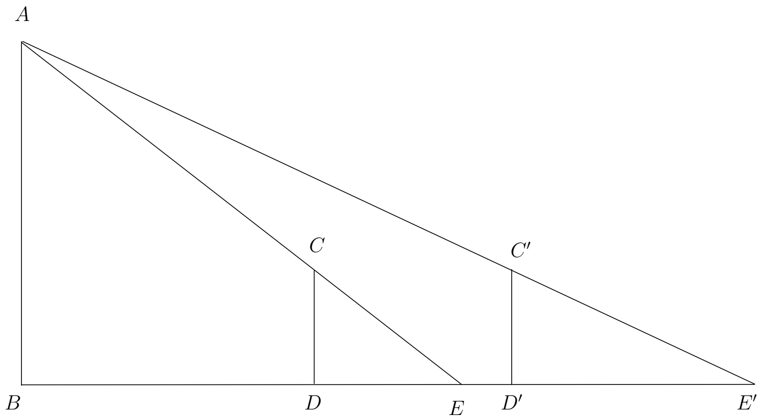


Figure 21

1. Selecting a plane ground, the gnomon is fixed vertically in the position CD (**ed.** see Figure 21). Now the eye is put at the level of the ground at such a place E that E, C and A are in the same straight line. Then the distance DE of the eye from the gnomon is measured. It is called as *dṛṣṭi* (sight) by Brahmagupta. Similar observations are taken with the gnomon in a different position $C'D'$ and the eye E'. The formulae to be applied then are the same as those stated above.

   Brahmagupta re-describes them as follows:

   > The displacement (of the eye) multiplied by a *dṛṣṭi* and divided by the difference of the two *dṛṣṭis* will give the distance of the base. The distance of the base multiplied by the length of the gnomon and divided by its own *dṛṣṭi* will give the height.[279]

2. Observations may also be taken, thinks Brahmagupta, by placing the gnomon horizontally on the level ground (**ed.** see Figure 22). In this case a graduated rod CR is fixed vertically at the extremity C of the gnomon CD nearer to the object to be measured. Then placing the eye at the other end D, the graduation P which is in a straight line with the tip of the object is noted. This gives the altitude CP. Brahmagupta calls it by the term *śalākā* (rod). Observations are taken again with the gnomon in the position $C'D'$.

   Then Brahmagupta says:

   > The displacement (of the gnomon) multiplied by the other *śalākā* and divided by the difference of the two *śalākās* will give the

[279] *BrSpSi*, xxii. 33.

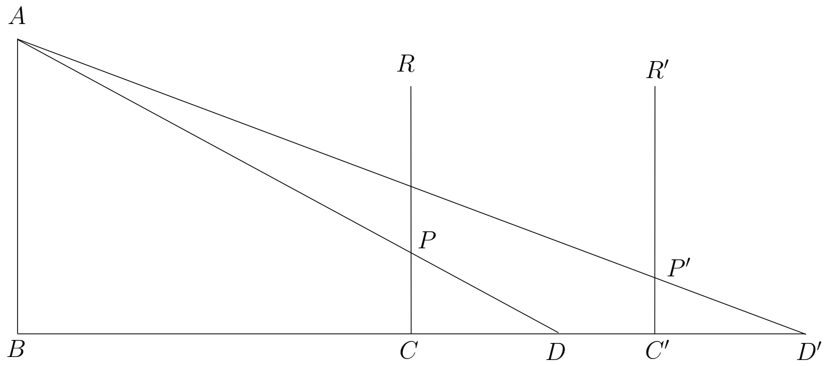


Figure 22

> distance of the base. The distance of the base multiplied by the *śalākā* corresponding to it and divided by the length of the gnomon will give the height of the house etc.[280]

$$BD = \frac{DD' \times C'P'}{CP - C'P'}, \quad BD' = \frac{DD' \times CP}{CP - C'P'},$$
$$AB = \frac{BD \times CP}{CD} = \frac{BD' \times C'P'}{CD}.$$

3. Brahmagupta then gives a different method (**ed.** see Figure 23): Placing the eye at E, the gnomon is first directed towards the base B of the object and then towards its tip A. From the front extremities G, G' of the gnomon in the two positions draw the perpendiculars GN, $G'N'$ to the ground. Also draw the perpendicular EM. Measure the distances MN, MN'.

   Now it can be proved easily that

$$BM = \frac{ME \times MN}{ME - GN},$$

   and

$$AB = ME + \frac{BM(G'N' - ME)}{MN'} \text{ in Figure 23a,}$$

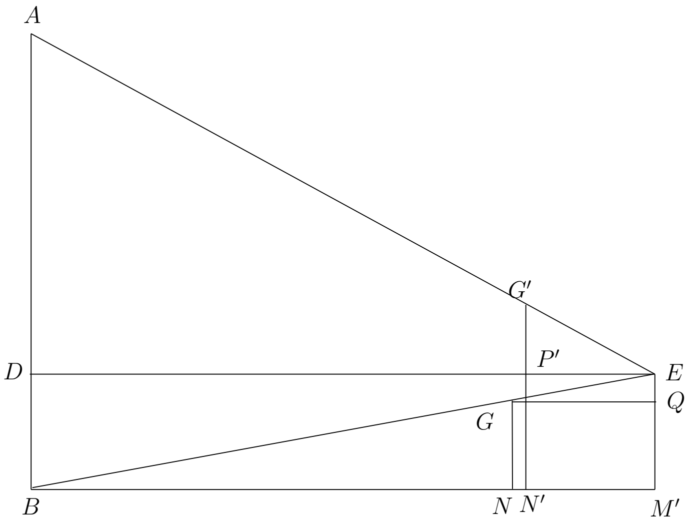
   or

$$AB = ME - \frac{BM(ME - G'N')}{MN'} \text{ in Figure 23b.}$$

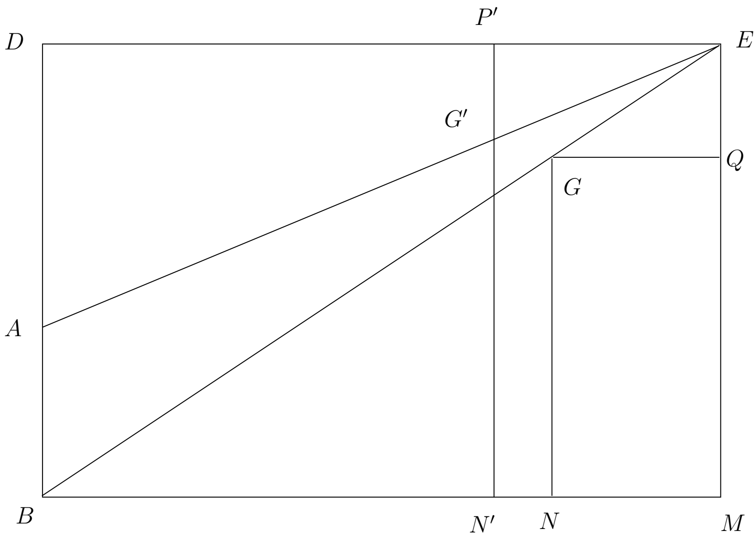
   Hence Brahmagupta says:

   > The distance between the feet of the altitudes (of the eye and the front extremity of the gnomon in the first observation) being

[280] *BrSpSi*, xxii. 32.



(a)



(b)

Figure 23

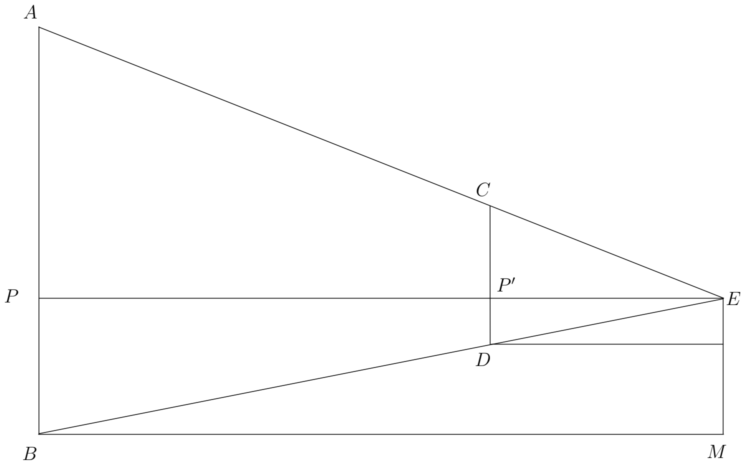


Figure 24

divided by the difference between the altitudes and multiplied by the greater (altitude) gives the distance of the base. Multiply the distance of the base by the difference between the altitudes (of the eye and the front extremity of the gnomon in the second observation) and divide by the distance between the feet of these altitudes. Then subtract the quotient from the altitude of the eye, if the altitude of the front extremity of the gnomon (in the second observation) be less than the altitude of the eye; or add, if greater. The result gives the height of the house. (Thus the height and distance of an object can be determined) by means of observations of its base and tip.[281]

4. Another method of Brahmagupta is as follows: Placing the eye at E, at an altitude ME over the ground, then fix the gnomon CD in front in such a position that its lower end D will be in the line of sight of the bottom of the object AB and its upper end C in the line of sight of the top of the object (Figure 24). Also note the portion DP' of the gnomon below EP, the horizontal line of sight and the distance EP' of the eye from the gnomon. Then, says Brahmagupta:

   The distance of the eye from the gnomon multiplied by the altitude of the eye and divided by the portion of the gnomon below (the horizontal line of sight) will be the distance of the base. The distance of the base multiplied by the whole gnomon and divided

[281] *BrSpSi*, xxii. 34–5.

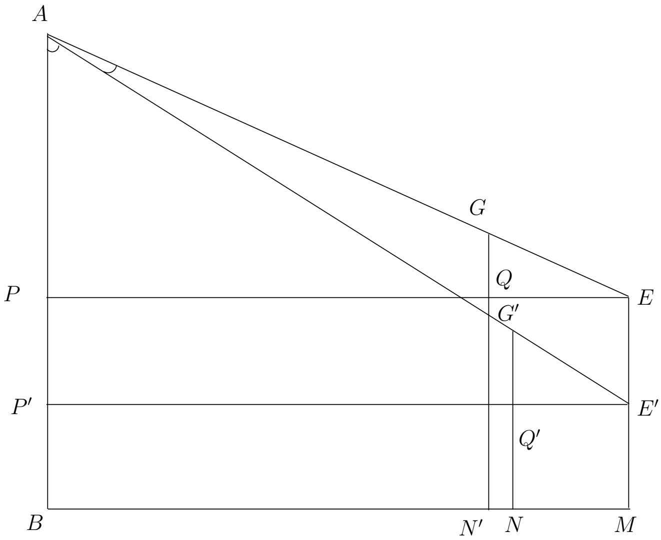


Figure 25

by the distance of the eye from the gnomon will be the height.[282]

$$BM = \frac{EP' \times ME}{DP'}, \quad AB = \frac{BM \times CD}{EP'}.$$

11.4 Bhāskara II

For measuring the heights and distances of far-off objects, Bhāskara II gives two methods, one of which is taken from Brahmagupta. He remarks in general that observations should be made on a plane horizontal ground. Directing the gnomon towards the distant object perpendiculars are drawn from its two extremities on the plane of observation. The horizontal distance between them is the base (*bhuja*), the difference between them is the upright (*koṭi*) and the gnomon itself is the hypotenuse (*karṇa*) of the triangle of observation, says Bhāskara.

(a) Thus observing the bottom of the bamboo, multiply the base (of the triangle of observation) by the altitude of the eye and divide by the upright: the result is the horizontal distance between the self and the bamboo. Then observing the top of the bamboo, multiply the horizontal distance by the upright the divide by the base; the result added with the altitude of the eye is the height of the bamboo.[283] (Figure 25.)

[282] *BrSpSi*, xxii. 36.
[283] *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 43–4.

(b) Observe the top (of the bamboo) first in the standing posture and then again in the sitting posture. Divide each altitude by its base. The difference of the altitudes of the eye divided by the difference of those quotients gives the horizontal distance. The height of the bamboo can then be determined separately as before.[284]

$$PE = \frac{ME - ME'}{\frac{G'Q'}{E'Q'} - \frac{GQ}{EQ}},$$
$$AB = ME + \frac{PE \times GQ}{EQ} = ME' + \frac{PE \times G'Q'}{E'Q'}.$$

Abbreviations

Ā	*Āryabhaṭīya*	*MaiS*	*Maitrāyāṇīya Saṃhitā*
ĀpŚl	*Āpastamba Śulba*	*MāŚl*	*Mānava Śulba*
BrSpSi	*Brāhmasphuṭasiddhānta*	*MSi*	*Mahā-siddhānta*
BŚl	*Baudhāyana Śulba*	*ŚBr*	*Śatapatha Brāhmaṇa*
GSS	*Gaṇita-sāra-saṃgraha*	*ŚiDVṛ*	*Śiṣyadhī-vṛddhida*
KapS	*Kapisthala Saṃhitā*	*SiŚe*	*Siddhāntaśekhara*
KŚl	*Kātyāyana Śulba*	*SiŚi*	*Siddhāntaśiromaṇi*
KṭS	*Kāṭhaka Saṃhitā*	*Triś*	*Triśatikā*
L	*Līlāvatī*	*TS*	*Tantrasaṃgraha*

[284] *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 45–6.