Magic squares in Indian mathematics *

The following is a magic square:

4	9	2
3	5	7
8	1	6

In this the sum of numbers in each row is 15. The sum in each column is 15; and the numbers in each diagonal also sum up to 15. The above magic square was known to all ancient people. It is the simplest magic square consisting of 3×3 $(= 9)$ cells. The problem of arranging numbers in squares containing more cells is not easy. Magic squares were used in India about first century AD as charms and reference to them is found in the earliest tantric literature.

The construction of magic squares requires ingenuity because no definite method can be given, at least in some cases. For this reason, the subject has aroused the interest of mathematicians. The Hindu mathematician Nārāyaṇa, who flourished in the fourteenth century AD has included in his work on arithmetic, called *Gaṇitakaumudī*, a chapter on the construction of magic squares and other allied figures. In this respect the *Gaṇitakaumudī* is unique as no other treatise on arithmetic known to us deals with the construction of magic squares. From Nārāyaṇa's work we find that methods of construction of all types of magic squares were known in India.

Methods of constructing magic squares have been given by European and American mathematicians from the 17th century onwards but the Hindu methods appear to be the simplest, although some of the methods recently developed in the west are more general. We propose to give here some simple methods of constructing magic squares and allied figures mostly from Hindu sources.

Classification of magic squares

Magic squares, in accordance with their methods of construction, are divided into three classes:

* K. S. Shukla, in *Interaction between Indian and Central Asian Science and Technology in Medieval Times*, Vol. 1, Indo-Soviet Joint Monograph Series, INSA (1990), pp. 249–270.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_14

(1) those having an odd number of cells in each row

(2) those having $4n$ cells in each row, and

(3) those having $4n + 2$ cells in each row.

1 Odd magic squares

The construction of odd magic squares is the easiest. We give below two methods of constructing such magic squares, These methods have been given in the *Gaṇitakaumudī* of Nārāyaṇa.

1.1 First method

This method is illustrated by the following four squares (**ed.** see Figure 1) in which we will use the natural numbers 1, 2, 3, etc. in succession.

Rule for filling

Begin with the middle cell of the top row and write 1 in the cell. Then proceed along the outward drawn diagonal of the cell. (The direction of the diagonal has been indicated in each figure by arrow-heads). This leads you beyond the square. Now, if the square were wrapped around a cylinder, you would get into the lowest cell of the next column. Therefore, write 2 in it. Then proceed again in the direction of the above outward drawn diagonal and write the next successive numbers 3, 4, 5, etc. in the cells thus encountered. Thus you again reach a cell which leads you beyond the square. Again imagine that the square is wrapped around a cylinder. Thus you reach the first cell of the next row, write the next number in the cell. Then proceed again in the direction of the above outward drawn diagonal and continue the above process till all the cells are filled. Whenever the above process leads you to a cell which is already occupied by a number, write the next number in the cell below and proceed in the direction of the diagonal.

It will be easily seen that the above squares may also be filled by proceeding in the direction of the other outward drawn diagonal of the middle cell of the top row. Similarly, the above process may also be started with the middle cell of the bottom row or with the middle cell of the first or last column. Thus an odd square can be filled up in 8 ways to form a magic square.

The magic squares constructed by the above method are such that the sum of any two numbers which are equidistant from the centre is equal to twice the number at the centre. Such squares have been called perfect by W. S. Andrews (Magic square and cubes, Chicago, 1908).

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

30	39	49	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Figure 1: Odd magic squares: First method.

Note

The four magic squares, which have been shown above (**ed.** see Figure 1), have been filled by the natural numbers 1, 2, 3, etc. But that is not necessary. A $n \times n$ magic square with total $\underline{S}$ may, in general, be filled either by $\underline{n} \times \underline{n}$ numbers in arithmetical progression and having $n \times S$ for their sum, or else, by $\underline{n}$ sets each of $\underline{n}$ numbers belonging to the same arithmetical progression and having $\underline{n} \times \underline{S}$ for their sum. Thus, for example, a 3×3 magic square with total 33 may be constructed by the numbers

(a) 7, 8, 9, 10, 11, 12, 13, 14, and 15, or
(b) 2, 4, 6, 9, 11, 13, 16, 18, and 20

taken in succession. The series (a) give the magic square (i); and the series (b) give the magic square (ii):

(i)

14	7	12
9	11	13
10	15	8

(ii)

18	2	13
6	11	16
9	20	4

1.2 Second method (superposition)

According to this method, a 3×3 magic square with total 21, say, is constructed as follows:

Take two sets of numbers each containing 3 numbers in arithmetical progression, say, 1, 5, 9 and 0, 1, 2. Multiply the numbers in the second set by

$$\frac{\text{given total} - \text{sum of the first set}}{\text{sum of the second set}} = \frac{21 - 15}{3} = 2.$$

Thus we get 0, 2, 4 as the second set.

Now construct two squares having 3×3 cells. In the cells of the middle row or column of the first square fill the numbers 1, 5, 9 of the first set, in the remaining cells fill the same numbers in a cyclic order as below. Similarly, in the second square fill the numbers 0, 2, 4 of the second set. Thus are obtained the skeleton squares (i) and (ii) below:

(i)

9	1	5
1	5	9
5	9	1

(ii)

4	0	2
0	2	4
2	4	0

Now fold the paper in such a way that the second square may fall on the first and then add the numbers which fall on each other. The gives

11	1	9
5	7	9
5	13	3

which is the required magic square with total 21.

This method of superposition was called by Nārāyaṇa *chādya-chādaka-vidhi*. The method was rediscovered in Europe by M. de la Hire in the beginning of the 18th century.

2 $4\underline{n} \times 4\underline{n}$ magic squares

The simplest squares having $4\underline{n}$ cells in a row are 4×4 magic squares. These are also constructed by means of numbers forming an arithmetical progression, or by 4 sets of 4 contiguous numbers belonging to the same arithmetical progression. We give below three methods for constructing magic squares.

2.1 First method

This method is given by Nārāyaṇa in his *Gaṇitakaumudī*. We illustrate it by constructing 4×4 magic squares with the help of the natural numbers 1, 2, 3 etc. The total in the resulting magic squares will be 34.

Rule

Arrange the above 16 numbers in their order as follows:

$$1, 2, 3, 4, 5, 6, 7, 8;\ \ 9, 10, 11, 12, 13, 14, 15, 16.$$

Now write the numbers in a square directly and inversely as in (a) below. In the upper half of the square (a) interchange the numbers in the last two columns and in the lower half interchange the numbers in the first two columns. This would give the square (b).

(a)

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

(b)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Now fill the numbers in the first and second row of the square (b) in a 4×4 square by knight's move as in chess, starting with 1 in the first cell and with 8 either in a contiguous cell or in the last cell. The numbers in the second half of the square (b) are filled similarly, so that the numbers 25 and 10 occupy cells opposite to 1 and 8 respectively. Thus we can obtain the following four magic squares.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

To obtain another set of four squares, we interchange the numbers in the second and fourth columns in the upper half of the square (b) and the first and third columns in the lower half. This gives the square (c).

The numbers in the square (c) are filled, as before, to give another set of 4 magic squares.

(c)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

The above process may also be started with any one of the following arrangements of the 16 numbers 1, 2, 3 etc.

$$1, 3, 5, 7, 2, 4, 6, 8, 9, 11, 13, 15, 10, 12, 14, 16,$$

or

$$1, 5, 2, 6, 3, 7, 4, 8, 9, 13, 10, 14, 11, 15, 12, 16.$$

Each of these would give 8 magic squares as above.

The total number of magic squares obtained by Nārāyaṇa's method is thus twenty-four.

2.2 Second method

First fill the numbers

Nil, 1, Nil, 8, Nil, 9, Nil, 2, 6, Nil, 3, Nil, 4, Nil, 7, Nil

in the cells of a 4×4 square as below:

(S)

	1		8
	9		2
6		3	
4		7	

Even Total

If the total is even, say $2\underline{n}$, fill the remaining cells in such a way that in every diagonal the sum of the alternate numbers is equal to $\underline{n}$.

Odd total

If the total is odd, say $(2n + 1)$, fill every blank cell, which is diagonally alternate to the cell containing 1, 2, 3, 4 by $\underline{n}$ minus the diagonally alternate number; and fill the remaining blank cells by $(\underline{n} + 1)$ minus the diagonally alternate number.

$n-3$	1	$n-6$	8
$n-7$	9	$n-4$	2
6	$n-8$	3	$n-1$
4	$n-2$	7	$n-9$

Total $2\underline{n}$

$n-3$	1	$n-5$	8
$n-6$	9	$n-4$	2
6	$n-7$	3	$n-1$
4	$n-2$	7	$n-8$

Total $2\underline{n}+1$

The skeleton square ($\underline{S}$) may, in general, be constructed by filling 8 cells of a 4×4 square by any 8 numbers forming an arithmetical progression or by two set of 4 contiguous numbers belonging to the same arithmetical progression in accordance with the first method above.

2.3 Third method (superposition)

We illustrate this method by constructing a magic square with total 40.

Take two sets of numbers each containing 4 numbers in arithmetical progression, say,

(i) 1, 2, 3, 4; and

(ii) 0, 1, 2, 3.

Multiply the numbers of the second set by

$$\frac{\text{given total} - \text{sum of the first set}}{\text{sum of the second set}} = \frac{40 - 10}{6} = 5.$$

Thus we obtain 0, 5, 10, 15 as second set.

Now fill the numbers of the first set in a square as in (i) and the numbers of the second set in another square as in (ii):

(i)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

(ii)

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Fold the paper in such a way that the square (ii) may cover the square (i) and add the numbers which fall on each other. This gives the following magic square.

17	13	2	8
1	9	16	14
18	12	3	7
4	6	19	11

3 $(4\underline{n}+2) \times (4\underline{n}+2)$ magic squares

The construction of $(4\underline{n}+2) \times (4\underline{n}+2)$ magic squares is comparatively more difficult. Nārāyaṇa has suggested two methods which are helpful in constructing such squares. We illustrate one of these methods by constructing a 6×6 magic square with the help of the natural numbers 1, 2, 3, . . . , 36.

3.1 Nārāyaṇa's method

Fill the numbers 1, 2, 3, etc. in the direct and inverse order as in the square (i) below:

(i)

1	2*	3	4	5*	6
12*	11	10	9	8	7*
13*	14	15	16	17	18*
24*	23	22	21	20	19*
25*	26	27	28	29	30*
36	35*	34	33	32*	31

where the cells marked (*) are known as *śliṣṭa*.

Interchange the numbers 18 and 19 lying in the last two *śliṣṭa* cells of the two middle rows with the corresponding numbers 15 and 22 in the left vertical half of the square; then interchange the numbers in the *śliṣṭa* cells lying in the upper half of the square with those symmetrically lying in the lower half. Then rotate the rectangles containing the numbers 3 and 10, 4 and 9, and 28 and 33 about the centre of the square in the anticlockwise direction and bring each of them to the position of the next rectangle. This gives the following magic square:

1	35	4	33	32	6
25	11	9	28	8	30
24	14	18	16	17	22
13	23	19	21	20	15
12	26	27	10	29	7
36	2	34	3	5	31

where the numbers enclosed within circles do not undergo any change.

4 Other magic figures

4.1 8×4 magic rectangle

This is

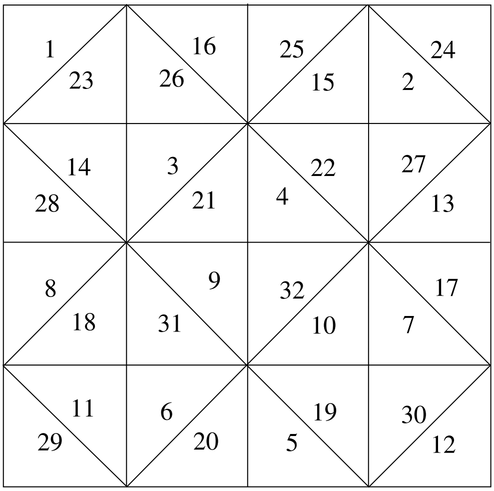
1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Total: rows = 132, columns = 66.

The following magic figures are based on this rectangle:

4.1.1 *Vitana* (canopy)

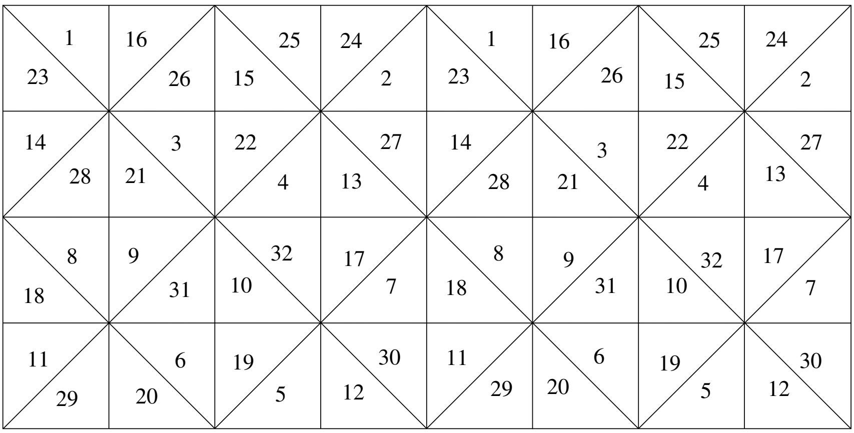
The figure is



Here the sum of 8 numbers taken symmetrically, either horizontally or vertically or diagonally or in a circle or in a square is the same (132). Also the sum of groups of four numbers taken symmetrically is 66.

4.1.2 *Maṇḍapa* (altar)

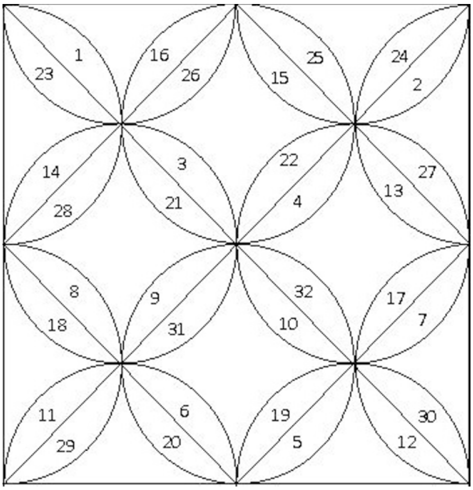
The figure is



Here any set of 8 numbers occurring together, horizontally, vertically or diagonally sum up to 132. The 8 numbers lying in a square also have the same total. There is cylindrical symmetry i.e., if the figure be rolled on a cylinder any continuous 8 numbers or those lying in a square have the total 132. It is easy to find 26 sets of 8 numbers having the same total 132.

4.1.3 *Padma* (Lotus)

The figure is



Here any set of 8 numbers taken vertically, horizontally or in any 4 leaves symmetrically situated gives the same total 132. There is cylindrical symmetry. In this case 32 sets of 8 numbers having the same total can be easily picked up.

4.2 12×4 magic rectangle

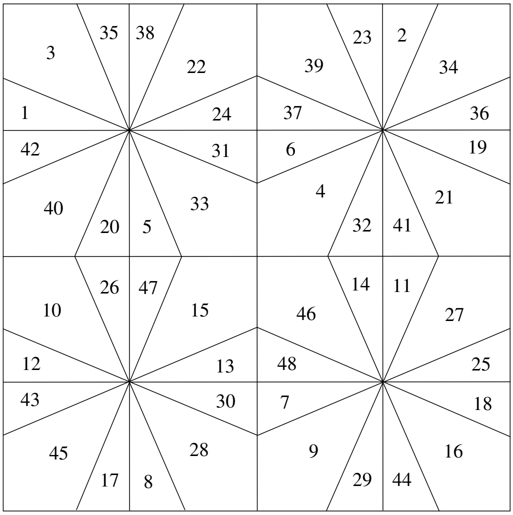
This is

1	24	37	36	2	23	38	35	3	22	39	34
42	31	6	19	41	32	5	20	40	33	4	21
12	13	48	25	11	14	47	26	10	15	46	27
43	30	7	18	44	29	8	17	45	28	9	16

The following magic figures are based on this rectangle:

4.2.1 *Dvādaśakara* (twelve hands)

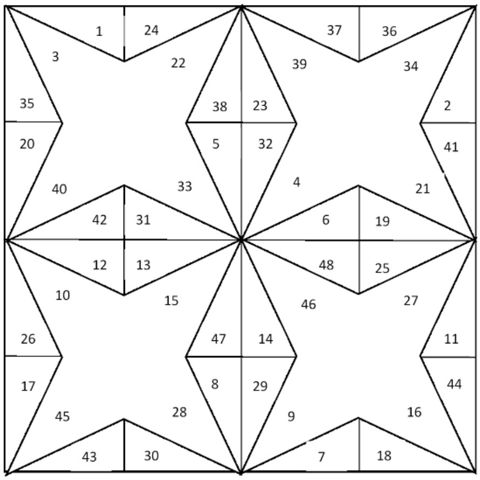
The figure is



Here all groups of 12, of 8 or of 4 numbers have even totals 294, 196 and 98 respectively.

4.2.2 *Vajra Padma*

The figure is



Here every group of 4 numbers occurring in a line or in cell has the total 98;

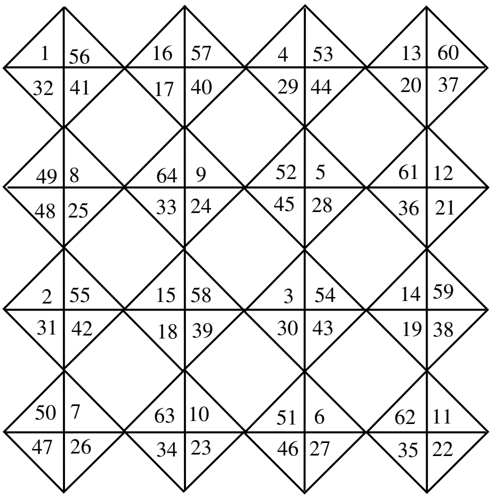
every group of 8 numbers has the total 196 and every group of 12 numbers taken horizontally, vertically or in a circle has the total 294.

4.3 8×8 magic square

This is

1	56	16	57	4	53	13	60
38	41	17	40	29	44	20	37
49	8	64	9	52	5	61	12
48	25	33	24	45	28	36	21
2	55	15	58	3	54	14	59
31	42	18	39	30	43	19	38
50	7	63	10	51	6	62	11
47	26	34	23	46	27	35	22

On this is based the following magic figure, square called the *sarvatobhadra*:



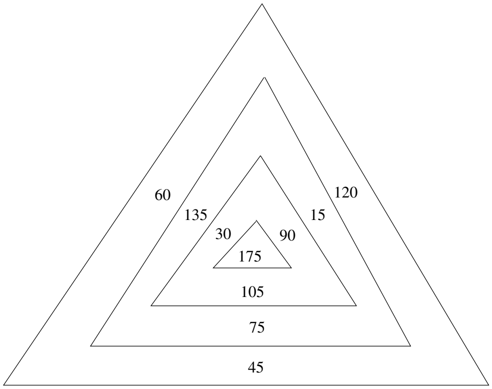
In this figure the totals of all four, eight and sixteen numbers are 130, 260 and 520 respectively.

4.4 Magic triangle

The following is a magic triangle together with its key square.

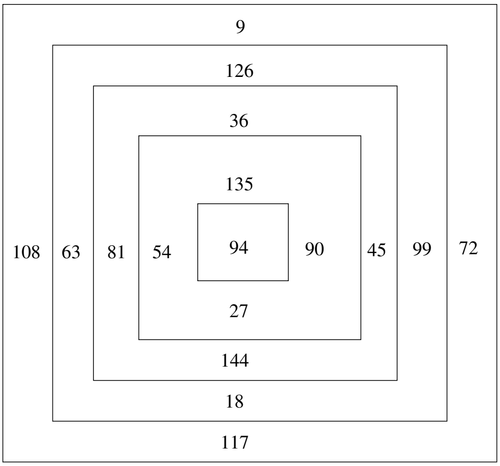
120	45	60
15	75	135
90	105	30

Total 225



4.5 Magic cross

This figure is



(Total 400)

This is based on the following 4×4 magic square:

9	72	117	108
126	99	18	63
36	45	144	81
135	90	27	54

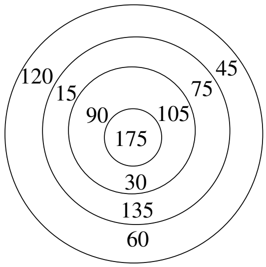
(Total 306)

4.6 Magic circle

Nārāyaṇa has given a number of magic circles. We give below one of these together with the key square:

120	45	60
15	75	135
90	105	30

Total 225



Total 400

5 Magic squares used by Muslims

We shall now give a brief account of the magic squares which were used by the Muslims in India in medieval times as amulets and charms. These magic squares include the following varieties: *Dūpāyā* ("two-legged"), *Sulsī* (3×3), *Rubā'ī* (4×4), *Khamsī* (5×5), *Musaddas* (6×6), *Musabba'* (7×7), *Muṣamman* (8×8), *Mustassa* (9×9), and *Ma'shsher* (10×10). Sometimes they were so large as to have 100 cells in a row or column.

The following is a *Dūpāyā* square with total 12:

3	8	1
2	4	6
7		5

To construct a *Dūpāyā* square, divide the given total by 12. Fill the cells with the quotient increasing by itself in every next cell as you proceed (see the above figure). In case the division by 10 yields any (non-zero) remainder it is to be added to the number in the sixth cell. For example, let the total be 786. Division by 12 yields 65 as the quotient and 6 as the remainder. This gives the following square:

195	526	55
130	260	396
461		325

The following is the *Sulsī* square with total 15:

4	9	2
3	5	7
8	1	6

To construct a *Sulsī* square, subtract 12 from the given total, then divide the remainder by 3 and with the quotient fill up the 9 cells of 3×3 square until the whole square is filled up. The filling may be started from the central cell of a bordering row or column depending on the elements earth, water, air, or fire, thus:

Air

2	7	6
9	5	1
4	3	8

Fire

4	9	2
3	5	7
8	1	6

Earth

6	7	2
1	5	9
8	3	4

Water

6	1	8
7	5	3
2	9	4

To construct a *Rubā'ī* square deduct 30 from the given total, then divide the remainder by 4, and with the quotient fill up the 16 cells of 4×4 square. If 1 remains over, add one to the 13th cell; if 2, add 1 to the 9th cell; if 3, add 1 to the 5th.

With total 34, there will result 4 *Rubā'ī* squares depending on the elements, viz.

Earth

8	11	14	1
13	2	7	12
3	16	9	6
10	6	4	15

Water

14	4	1	15
7	9	12	6
11	5	8	10
2	16	13	3

Air

15	1	4	14
10	8	5	11
6	12	9	7
3	13	16	2

Fire

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

To construct a *Khamsī* square, subtract 60 from the given total, then divide the remainder by 5 and with the quotient fill up the 25 cells of 5×5 square. If 1 remains over, 1 is to be added to the 21st cell; if 2, to the 16th ; if 3, to the 11th; if 4, to the 6th.

7	13	19	25	1
20	21	2	8	14
3	9	15	16	22
11	17	23	4	10
24	5	6	12	18

Khamsī square with total 65

To construct a *Musaddas* square, deduct 105 from the given total, then divide by 6 and with the quotient fill up the square. If 1 remains over, add 1 to the 31st cell; if 2, to the 35th; if 3, to the 19th; if 4, to the 13th; if 5, to the 7th.

36	18	30	19	7	1
13	26	2	34	24	12
5	9	22	29	15	31
25	6	14	8	35	23
21	32	10	17	3	28
11	20	33	4	27	16

Musaddas square with total 111

To construct a *Musabba*ʻ square, deduct 168 from the given total, then divide by 7 and with the quotient fill up 7×7 squares. If from 1 to 5 remain as the remainder, add 1 to the 43rd cell.

40	23	13	45	35	18	1
32	15	5	37	27	10	49
24	14	46	29	19	2	41
16	6	38	28	11	43	33
8	47	30	20	3	42	25
7	39	22	12	44	34	17
48	31	21	4	36	26	9

Musabba square with total 175

To construct a *Muṣamman* square, subtract 252 from the given total, then divide by 8 and with the quotient fill up 8×8 square. If 1 to 7 are obtained as the remainder, add 1 to the number in the 75th cell.

36	43	35	32	27	60	26	1
41	4	49	59	21	17	45	24
37	15	11	10	58	51	50	28
23	47	57	52	12	9	18	42
3	46	8	13	53	56	19	62
25	63	54	55	7	14	2	40
31	20	16	6	44	48	61	34
64	22	30	33	38	5	39	29

Muṣamman square with total 260

To construct a *Mustassa* square, subtract 360 from the given total, then divide by 9 with the quotient fill up 9×9 square. If 1 to 8 are obtained as the remainder, add 1 to the 73rd cell.

70	59	27	16	76	55	43	22	1
50	39	28	6	66	54	33	12	81
40	18	7	67	56	34	13	73	61
60	29	17	77	46	44	23	2	71
20	19	78	57	45	24	3	72	51
30	8	68	47	25	14	74	62	41
9	79	58	37	35	4	64	52	31
10	69	48	36	15	75	53	42	21
80	49	38	26	5	65	63	32	11

Mustassa square with total 369

To construct a *Ma'shshar* square, subtract 495 from the given total, then divide by 10 and with the quotient fill up 10×10 square. If 1 to 9 remain as the remainder, add 1 to the 91st cell.

28	60	42	61	39	70	98	72	34	1
33	4	26	74	76	95	84	24	21	68
69	83	13	92	10	90	86	12	18	32
2	79	14	50	53	56	43	87	22	99
71	96	85	55	44	49	54	16	5	30
66	19	8	45	58	51	48	93	82	35
36	20	94	52	47	46	57	7	81	65
37	23	89	9	91	11	15	88	78	64
63	80	75	27	25	6	17	77	97	38
100	41	59	40	62	31	3	29	67	73

Ma'shshar square with total 505

For further information regarding Muslim magic squares, see "*Islam in India* or *Qānūn-i-Islām*" by Sa'far Sharīf, translated by G. A. Herklots.