Ācārya Jayadeva, the mathematician *

1 Introduction

The object of the present paper is to invite attention of historians of science to an important Hindu algebraist, Ācārya Jayadeva, who lived and wrote in the early 11th century of the Christian era (or earlier). His name and quotations from his work on algebra are found to occur in the *Sundarī*, which is the name of Śrīmad Udayadivākara's commentary on the *Laghubhāskarīya* of Bhāskara I (629 AD). The *Sundarī* has not yet seen the light of day but manuscript copies of that work are preserved in H. H. the Maharajah's Palace Library, Trivandrum, and in the Curator's Office Library, Trivandrum. A transcript copy of that work has been very recently procured for our use from the former by the Tagore Library of the Lucknow University. The extracts from Ācārya Jayadeva's work, which have been quoted and explained with illustrations by the commentator, relate to the solution of the indeterminate equation of the second degree of the type $Nx^2+1 = y^2$. These extracts, it may be pointed out, are of immense historical interest as they include rules giving the well known cyclic method of finding the integral solution of the above-mentioned equation. The credit of the first inception of that ingenious method was hitherto given to the twelfth century mathematician Bhāskara II (1150 AD) who himself not only did not claim originality for that method but also ascribed it to earlier writers. The discovery of that method in an anterior work definitely proves that the cyclic method was invented in India much earlier. Jayadeva may or may not have been its inventor but quotations from his work in the *Sundarī* are the earliest sources of our information regarding that method. Another noteworthy feature of the references from Ācārya Jayadeva's composition is the solution of the equation $Nx^2 + C = y^2$, C being positive or negative. This method, though not superior to that suggested by Brahmagupta (628 AD), Bhāskara II, and Nārāyaṇa (1356 AD), is nevertheless different from the known methods. Incidentally we have also given Udayadivākara's method for the solution of the multiple equations, $x + y =$ a square, $x - y =$ a square, $xy + 1 =$ a square. This method, though inferior to those given by Brahmagupta and Nārāyaṇa, deserves attention because of the ingenuity displayed by the author. It also shows that Udayadivākara knew full well how

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to tackle and solve the general indeterminate equation of the second degree of the type $ax^2 + bx + c = y^2$. Equations of that type were hitherto found treated in the *Bījagaṇita* of Bhāskara II, though some of his examples relating to such equations prove his indebtedness to ancient authors.[1]

2 The *Sundarī*

The transcript of the *Sundarī*, which is available to us, is written in *Devanāgarī* characters on paper in foolscap size. It is scribed in good hand but there are the usual imperfections and omissions. The manuscript is practically complete and extends to 252 leaves written on one side only. There are 21 lines to a page and about 24 letters to a line. The beginning and end of the *Sundarī* are as follows:

Beginning

॥श्रीः॥
लघुभास्करीयम्।

उदयदिवाकरप्रणीतया सुन्दर्याख्यया व्याख्यया समेतम्।
नत्वा समस्तजगतामधिपं मुरारि-
माचार्यमार्यभटमप्यभिवन्द्य भक्त्या ।
यद्भास्करेण गुरुणा ग्रहतन्त्रमुक्तं
लघ्वस्य विस्तृततरां विवृतिं विधास्ये ॥

तत्र तावदाचार्यः प्रथममेव भास्करीयं नाम ग्रहकर्मनिबन्धनं प्रतिपाद्य तदेव पुनः संक्षिप्तं चिकीर्षुस्तद्विघ्नोपशान्तये भगवते भास्कराय प्रणाममाद्यश्लोकेनाचष्टे—

भास्कराय ...

Colophon

इति ज्योतिषिकभट्टश्रीमदुदयदिवाकरविरचितायां लघुभास्करीयविवृतौ सुन्दर्याभिधा-नायां मध्यगतिः प्रथमोऽध्यायः।

End

एवं पुनः पुनर्भावनयानीतज्येष्ठमूलेनैवान्यौ राशी स्यातामिति।

Colophon

इति लघुभास्करीयविवृतौ सुन्दर्याभिधानायां नक्षत्रध्रुवग्रहयोगाध्यायोऽष्टमः।

[1] See Datta, B., and Singh, A. N., *History of Hindu Mathematics*, Part II, p. 181.

From the colophons at the ends of the chapters it is clear that the *Sundarī* is a commentary on the *Laghubhāskarīya* and that the name of the commentator is Bhaṭṭa Udayadivākara. The former conclusion is confirmed by the contents of the work.

In the commentary there is no reference to the time of birth of the commentator or of writing the commentary. But at one place in the commentary[2] the commentator cites an example where he states the *ahargaṇa* (i.e., the number of days elapsed since the beginning of Kaliyuga) for Friday, the 10th lunar date, *Vaiśākha*, bright fortnight, Śaka year 995. This epoch corresponds to Friday, April 19, AD 1073. It is usual to give the *ahargaṇa* for the current day. So we infer that the *Sundarī* was written in the year 1073 of the Christian era.

As regards the authenticity of the *Sundarī* there is little doubt. Reference to that work has been made by Nīlakaṇṭha (1500 AD) who in his commentary on the *Āryabhaṭīya*[3] of Āryabhaṭa I (499 AD) mentions the name *Laghubhāskarīya-vyākhyā Sundarī* and quotes two stanzas from that work. Both of those stanzas are found to occur in the transcript of the *Sundarī* available to us. Moreover, five manuscripts of that work which seems to be derived from different sources are preserved in H. H. the Maharajah's Palace Library, Trivandrum[4] and two in the Curator's Office Library, Trivandrum.[5]

3 Reference to Ācārya Jayadeva

Reference to Ācārya Jayadeva is made in the *Sundarī* in connection with the solution of the indeterminate equation of the second degree, viz.

$$Nx^2 + 1 = y^2,$$

which in Hindu mathematics is called by the name *varga-prakṛti* (square-nature).

In verse 18 of the eighth chapter of the *Laghubhāskarīya* there is an astronomical problem whose solution depends upon the solution of the simultaneous equations

$$8x + 1 = y^2 \tag{1}$$

$$7y^2 + 1 = z^2. \tag{2}$$

As regards the solution of these equations, Udayadivākara tells us that the value of y should be determined from equation (2) by solving it by the method

[2] Comm. on ii. 29.
[3] ii. 17 (ii).
[4] See Descriptive Catalogue, Vol. IV, MSS. Nos. 942, 943, 944, 945, and 977.
[5] See Descriptive Catalogue, Vol. V, MSS. Nos. 761 and 762.

applicable to the *varga-prakṛti*; and then the value of the unknown quantity x should be determined from equation (1) by the method of inversion. In order to give a detailed working of the process, Udayadivākara mentions Ācārya Jayadeva and his rules. He writes:

> In order to demonstrate this (working), we here set forth with exposition and illustration the rules for the *varga-prakṛti*, which were composed by Ācārya Śrī Jayadeva.[6]

4 Quotations from Ācārya Jayadeva's work

Quotations from Ācārya Jayadeva's work comprise 20 stanzas. Below we translate and explain those stanzas.

4.1 Stanza 1. Origin of the name *varga-prakṛti*

> इष्टकृतिरिष्टगुणिताऽभीष्टेन युता विशोधितेष्टा वा ।
> वर्गस्य यतः प्रकृतिर्वर्गप्रकृतिस्ततोऽभिहिता ॥१॥
>
> As (in an equation of the type $Nx^2 \pm C = y^2$) the square of an optional number multiplied by a given number and then the product increased or decreased by another given number is of the nature of a square, so (such an equation) is called *varga-prakṛti* (square-nature).

This proves the significance of the name *varga-prakṛti*.

4.2 Stanza 2. Technical terms explained

> यस्याभीष्टेन कृतिर्विहन्यते तत्कनिष्ठमूलं स्यात् ।
> क्षेपयुताद्रहिताद्वा मूलज्येष्ठं भवति तन्मूलम् ॥२॥
>
> The number whose square is multiplied by the given number is called the lesser root; that product having been increased or decreased by the interpolator (*kṣepa*), the square root thereof is called the greater root.

That is to say, in the equation $Nx^2 \pm C = y^2$, x is the lesser root and y the greater root. N is called *prakṛti* and C the interpolator.

We will see presently that Ācārya Jayadeva calls the lesser and greater roots by the names first root and last root also.

[6] तत्प्रदर्शनायाचार्यजयदेवविरचितवर्गप्रकृतिकरणसूत्राणि सविवरणान्यालिख्यन्ते।

4.3 Stanza 3. Writing down an auxiliary equation

> ईप्सितराशेर्वर्गे चोदितगुणकारताडिते चिन्त्यम् ।
> युक्तेन कृतिः (कियता) कियद्वियुक्तेन वेति धिया ॥३॥
>
> The square of an optionally chosen number having been multiplied by the given multiplier, think out how much be added to or subtracted from that product that it may become a perfect square.

That is, first choose an arbitrary number α for x. Then find out a number k, positive or negative, such that $N\alpha^2 + k$ may become a perfect square, say β^2. Then

$$N\alpha^2 + k = \beta^2$$

is an auxiliary equation. We will see how this equation is helpful in finding a solution of

$$Nx^2 + 1 = y^2.$$

4.4 Stanza 4. *Bhāvanā*

> अशेषकरणव्यापि भावनाकरणं द्विधा ।
> तत्समासविशेषाभ्यां तुल्यातुल्यतयापि च ॥४॥
>
> The process of *bhāvanā*, which pervades all mathematical operations (dealing with the *varga-prakṛti*), is twofold—*samāsa-bhāvanā* and *viśeṣa-bhāvanā*, or *tulya-bhāvanā* and *atulya-bhāvanā*.

The word *bhāvanā* is a technical term. According to Udayadivākara, *bhāvanā* is multiplication.[7] According to B. Datta and A. N. Singh, it means lemma or composition. At any rate the process called *bhāvanākaraṇa* is a special mathematical operation in which multiplication is inherent. The process is described in the next two stanzas.

4.4.1 Stanza 5. *Samāsa-bhāvanā*

> वज्राभ्याससमासात् प्रथमं प्रथमाहतिः प्रकृतिघातात् ।
> अन्त्यपदाभ्यासयुतादितरन्मूलं हतिः क्षिप्तयोः ॥५॥
>
> Summing up the cross products (of the first and last roots) is obtained a (new) first root; multiplying the product of the first roots by the *prakṛti* and then increasing that by the product of the last roots is obtained a (new) last root; and the product of the interpolators (is the corresponding new interpolator).

[7] Compare the term *bhāvita*, which is the name given to an equation of the type $xy = c$ (involving the product of two unknown quantities).

That is to say, if

$$N\alpha^2 + k = \beta^2, \tag{3}$$
$$N\alpha_1^2 + k_1 = \beta_1^2, \tag{4}$$

then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Proof

The auxiliary equations (3) and (4) may be written as

$$N\alpha^2 - \beta^2 = -k,$$
$$N\alpha_1^2 - \beta_1^2 = -k_1.$$

Multiplying these equations side by side, we get

$$N^2\alpha^2\alpha_1^2 + \beta^2\beta_1^2 - N(\alpha^2\beta_1^2 + \alpha_1^2\beta^2) = kk_1,$$

which is the same as

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Actual working explained

Using 4.3 (**ed.** i.e. Section 4.3) we write down two auxiliary equations, say

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2.$$

Now we set down the *prakṛti* and then the lesser roots, the greater roots, and the interpolators corresponding to the two auxiliary equations one under the other as follows:

Prakṛti	Lesser root	Greater root	Interpolator
N	α	β	k
	α_1	β_1	k_1

Now we find out the cross products of the lesser and greater roots and put down their sum underneath the lesser root. Thereafter we obtain the products of the *prakṛti* and the lesser roots and of the greater roots and put down their sum underneath the greater root. And then we write down the product of the interpolators underneath the interpolator. Thus, we get

Prakṛti	Lesser root	Greater root	Interpolator
N	α	β	k
	α_1	β_1	k_1
	$\alpha\beta_1 + \alpha_1\beta$	$N\alpha\alpha_1 + \beta\beta_1$	kk_1

In this way we obtain another auxiliary equation, viz.

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Repeating the above process over and over again, any number of auxiliary equations can be found out.

Note

The above process is called *samāsa-bhāvanā*. Also since the operation has been made on two different auxiliary equations, this may be called *atulya-bhāvanā* (or *atulya-samāsa-bhāvanā*). If everywhere in the above process, α_1 be replaced by α, β_1 by β, and k_1 by k, the above process will be called *tulya-bhāvanā*.

The result of *tulya-samāsa-bhāvanā* may be stated as follows:

$$\begin{aligned} \text{If} \qquad N\alpha^2 + k &= \beta^2, \\ \text{then} \qquad N(2\alpha\beta)^2 + k^2 &= (N\alpha^2 + \beta^2)^2. \end{aligned}$$

Thus we see that the *tulya-bhāvanā* is a particular case of the *atulya-bhāvanā*.

4.4.2 Stanza 6. *Viśeṣa-bhāvanā*

वज्राभ्यासविशेषादादिममाद्याहतिः प्रकृतिघातात् ।
अन्त्यपदाभ्यासेन च विशेषितान्मूलमन्त्यं स्यात् ॥६॥

Taking the difference of the cross products (of the first and the last roots), we get a (fresh) first root; multiplying the product of the first roots by the *prakṛti* and then taking the difference of that and the product of the last roots, we get a (fresh) last root. (The corresponding interpolator is the product of the interpolators).

That is to say, if

$$\begin{aligned} N\alpha^2 + k &= \beta^2, \\ N\alpha_1^2 + k_1 &= \beta_1^2, \end{aligned}$$

then

$$N(\alpha\beta_1 - \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 - \beta\beta_1)^2.$$

The proof and working are as in the previous case.

The rules stated in stanzas 5 and 6 are known as Brahmagupta's lemmas. They occur for the first time in the *Brāhmasphuṭasiddhānta* of Brahmagupta. In Europe they were rediscovered by Euler in 1764 and by Lagrange in 1768.

4.5 Stanza 7. Rational solution of $Nx^2 + 1 = y^2$

> प्रक्षेपकसंवर्गो वर्गश्चेदस्य वर्गमूलेन ।
> मूले भाज्ये तद्भावने च रूपं भवेत् क्षेपः ॥७॥
>
> When (in the above process) the product of the interpolators becomes a perfect square, by the square root thereof divide the (lesser and greater) roots: then they correspond to the interpolator unity and so they continue to be even when the process of the (*tulya*)*bhāvanā* is applied thereafter.

From what has been said above, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2,$$

then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

If $kk_1 = K^2$, then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + K^2 = (N\alpha\alpha_1 + \beta\beta_1)^2,$$

$$\text{i.e.,} \qquad N\left(\frac{\alpha\beta_1 + \alpha_1\beta}{K}\right)^2 + 1 = \left(\frac{N\alpha\alpha_1 + \beta\beta_1}{K}\right)^2.$$

In other words, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2,$$

and $kk_1 = K^2$, then

$$x = \frac{(\alpha\beta_1 + \alpha_1\beta)}{K},$$
$$y = \frac{(N\alpha\alpha_1 + \beta\beta_1)}{K}$$

is a solution of $Nx^2 + 1 = y^2$. In particular, if

$$N\alpha^2 + k = \beta^2,$$

then

$$x = \frac{2\alpha\beta}{k},$$
$$y = \frac{(N\alpha^2 + \beta^2)}{k},$$

is a solution of $Nx^2 + 1 = y^2$.

Illustration

Solve $7x^2 + 1 = y^2$.

Let the auxiliary equation be

$$7(1)^2 + 2 = 3^2.$$

Then applying the process of *tulya-bhāvanā*, we have

Prakṛti	Lesser root	Greater root	Interpolator
7	1	3	2
	1	3	2
	6	16	4

Thus

$$7(6)^2 + 4 = 16^2,$$
$$\text{or} \quad 7(3)^2 + 1 = 8^2.$$

Hence $x = 3$, $y = 8$ is a solution of the given equation. To get another solution, we treat the equation

$$7(3)^2 + 1 = 8^2$$

as the auxiliary equation. Then applying the process of *tulya-bhāvanā*, we have

7	3	8	1
	3	8	1
	48	127	1

Hence $x = 48$, $y = 127$ is another solution of the same equation. To obtain still another solution, we treat the equations

$$7(3)^2 + 1 = 8^2,$$
$$7(48)^2 + 1 = 127^2,$$

as auxiliary equations. Then applying *samāsa-bhāvanā*, we have

7	3	8	1
	48	127	1
	765	2024	1

Hence $x = 765$, $y = 2024$ is another solution of the same equation. Proceeding like this, we can get any number of solutions.

4.6 Stanzas 8–15. Integral solution of $Nx^2 + 1 = y^2$. The *Cakravāla* or the Cyclic Method

ह्रस्वज्येष्ठक्षेपान् प्रतिराश्य क्षेपभक्तयोः क्षेपात् ।
कुट्टाकारे च कृते कियद्गुणं क्षेपकं क्षिप्त्वा ॥८॥
तावत्कृतेः प्रकृत्या हीने प्रक्षेपकेण संभक्ते।
स्वल्पतरावाप्तिः स्यादित्याकलितोऽपरः क्षेपः ॥९॥
प्रक्षिप्तप्रक्षेपककुट्टाकारे कनिष्ठमूलहते ।
सज्येष्ठपदे प्रक्षेप(के)ण लब्धं कनिष्ठपदम् ॥१०॥
क्षिप्तक्षेपककुट्टागुणितात्तस्मात्कनिष्ठमूलहतम् ।
पाश्चात्यं प्रक्षेपं विशोध्य शेषं महन्मूलम् ॥११॥
कुर्यात् कुट्टाकारं पुनरनयोः क्षेपभक्तयोः पदयोः ।
तत्सेष्टहतक्षेपे सदृशगुणेऽस्मिन् प्रकृतिहीने ॥१२॥
प्रक्षेपः क्षेपाप्ते प्रक्षिप्तक्षेपकाच्च गुणकारात् ।
अल्पघ्नात् सज्येष्ठात् क्षेपावाप्तं कनिष्ठपदम् ॥१३॥
एतत्क्षिप्तक्षेपककुट्टकघातादनन्तरक्षेपम् ।
हित्वाल्पहतं शेषं ज्येष्ठं तेभ्यश्च गुणकादि ॥१४॥
कुर्यात्तावद्यावत् षण्णामेकद्विचतुर्णां पतति ।
इति चक्रवालकरणेऽवसरप्राप्तानि योज्यानि ॥१५॥

Set down the lesser root, the greater root, and the interpolator at two places. (At one place divide the lesser and greater roots by the interpolator. Treating the remainder of the lesser root as the dividend, the remainder of the greater root as the addend, and the interpolator as the divisor of an indeterminate equation of the first degree (*kuṭṭākāra*), solve that equation.) The *kuṭṭākāra*[8] having been (thus) determined from those (lesser and greater roots) divided out by the interpolator and the interpolator, ascertain how many times the interpolator be added to it so that the square of that sum being diminished by the *prakṛti* and then divided by the interpolator may yield the least value. The least value thus obtained is the new interpolator. The *kuṭṭākāra* as increased by (the chosen multiple of) the interpolator when multiplied by the lesser root, then increased by the greater root, and then divided by the interpolator, the quotient is the new lesser root. That (new lesser root) should be multiplied by the *kuṭṭākāra* as increased by (the chosen multiple of) the interpolator and from the product should be subtracted the new interpolator as multiplied by the lesser root; the remainder (thus obtained) is the new greater root.

From these (new lesser and greater) roots divided out by the

[8] In the indeterminate equation of the first degree $\frac{(ax+c)}{b} = y$, a is called the dividend, b the divisor, c the addend, and x the *kuṭṭākāra*.

(new) interpolator again find out the *kuṭṭākāra* (as before). Increase it by the proper multiple of the interpolator: the square of that (sum) being diminished by the *prakṛti* and then divided by the interpolator, the quotient is the (fresh) interpolator. The *kuṭṭākāra* (*guṇakāra*) increased by the chosen multiple of the interpolator being multiplied by the lesser root and increased by the greater root and then divided by the interpolator, the quotient is the fresh lesser root. This (fresh lesser root) being multiplied by the *kuṭṭākāra*, to which the chosen multiple of the interpolator has been added, and the product being diminished by the product of the fresh interpolator and the lesser root, the remainder is the fresh greater root.

From them again calculate the *kuṭṭākāra* etc. and continue the process till the interpolator comes out to be one of the six numbers ± 1, ± 2, and ± 4.

(One of these numbers having been obtained as the interpolator) in the (above) cyclic process (*cakravāla*), necessary operations should be made (to get the integral solution for unit interpolator).

Lemma of the Cyclic Method

The above method is based on the following lemma:

$$\text{If} \qquad N\alpha^2 + k = b^2,$$

where a, b, k are integers, k being positive or negative, then

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2.$$

Proof

Treating

$$\begin{aligned} N\alpha^2 + k &= b^2, \\ \text{and} \qquad N(1)^2 + (t^2 - N) &= t^2, \end{aligned}$$

as auxiliary equations, and applying the process of *samāsa-bhāvanā*, we have

$$\begin{array}{cccc} N & a & b & k \\ & 1 & t & t^2 - N \\ \hline & at + b & Na + bt & k(t^2 - N) \end{array}$$

Therefore,

$$N(at+b)^2 + k(t^2-N) = (Na+bt)^2$$

$$\text{or} \qquad N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left(\frac{Na+bt}{k}\right)^2,$$

which is the same as

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2.$$

The Cyclic Process explained

Suppose that an auxiliary equation is

$$Na^2 + k = b^2,$$

where a, b, and k are integers, k being positive or negative. Then, from the above lemma,

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2. \qquad (5)$$

Now we choose t such that $\frac{at+b}{k}$ is a whole number, and $\left|\frac{(t^2-N)}{k}\right|$ is as small as possible. Let that value be T. Then let

$$a_1 = \frac{aT+b}{k},$$

$$b_1 = T\left(\frac{aT+b}{k}\right) - a\left(\frac{T^2-N}{k}\right),$$

$$k_1 = \frac{T^2-N}{k}.$$

The numbers a_1, b_1, k_1 are all integral.[9] The equation (5) then becomes

$$Na_1^2 + k_1 = b_1^2. \qquad (6)$$

[9] From the form of b_1 it is clear that it will be an integer provided a_1 and k_1 are integers. But a_1 is an integer by assumption. So we have only to show that k_1 is an integer. Now if we eliminate b between

$$a_1 = \frac{aT+b}{k}, \qquad \text{and} \qquad b_1 = \frac{bT+Na}{k},$$

we get

$$\frac{k}{a}(a_1T - b_1) = T^2 - N.$$

Since the right side is integral, therefore the left side is also so. But k and a are prime to each other. Therefore, $a_1T - b_1$ must be perfectly divisible by a. Hence

$$\frac{a_1T - b_1}{a} = \frac{T^2-N}{k} = k_1 = \text{an integer}.$$

Now treating this as the auxiliary equation, and proceeding as above, we derive from (6) a new equation of the same kind

$$Na_2^2 + k_2 = b_2^2,$$

where again a_2, b_2, k_2 are whole numbers. Successive repetition of this process would, according to Ācārya Jayadeva, lead us to an equation in which the interpolator k is ± 1, ± 2, ± 4, and in which a, b are integers. And by the process of *samāsa-bhāvanā* such an equation easily leads us to an equation of the type

$$N\alpha^2 + 1 = \beta^2,$$

where α, β are integers.[10] Thus we get $x = \alpha$, $y = \beta$ as an integral solution of $Nx^2 + 1 = y^2$.

Illustration

Find an integral solution of $7x^2 + 1 = y^2$.

Let the auxiliary equation be

$$7(1)^2 - 3 = 2^2.$$

The process of *tulya-bhāvanā* does not lead to an integral solution. So we apply the Cyclic Process.

From the auxiliary equation

$$\text{lesser root} = 1,$$
$$\text{greater root} = 2,$$
$$\text{interpolator} = -3.$$

Therefore solving the equation

$$\frac{(t+2)}{-3} = \text{a whole number},$$

we get $t = -3\lambda + 1$. Putting $\lambda = 0$, we get $t = 1$ which gives to $\left|\frac{(t^2-7)}{-3}\right|$ the smallest value 2. Therefore,

$$\text{new interpolator} = 2,$$
$$\text{new lesser root} = -1,$$
$$\text{new greater root} = -3.$$

Since the new interpolator is 2, therefore the cyclic process stops here. Applying the *tulya-bhāvanā*, we have

[10] This fact was known to Brahmagupta. For details see Datta, B., Singh, A. N., *History of Hindu Mathematics*, Part II, pp. 157 ff.

7	-1	-3	2
	-1	-3	2
	6	16	4
or	3	8	1

Hence one integral solution of the given equation is

$$x = 3, \quad y = 8.$$

4.7 Stanzas 16–20. Solution of $Nx^2 + C = y^2$, C being positive or negative

प्रकृतौ तावद्दद्याद् यावति वर्गो भवेत् क्षेपात् ।
तावद् वर्गः शोध्यो यथाकृतयथोक्तयोरनयोः ॥१६॥
स्वक्षेपशोधनाभ्यां वज्राहतियोगतोऽपि वर्गः स्यात् ।
एवं मूलं कुर्यात् सक्षेपप्रकृतिमूलेन ॥१७॥
शोधनमूलगुणेनोनाधिकता ततश्च शेषाभ्याम् ।
प्रकृतिक्षेपेणाप्ते मूले लघुनी भवेतां द्वे ॥१८॥
संयुतिगुणमूलहते प्रतिराशि तयोस्तयोः क्षिपेदूने ।
क्षेपकशोधनमूलं विशोधयेदधिकतः क्रमशः ॥१९॥
मूले महती स्यातामतो महीयांसमायोज्यम् ।
प्रकटितमतिगहनमिदं मरुतिमुखे मक्षिकाकरणम् ॥२०॥

Add such a number to the *prakṛti* as makes the sum a perfect square. Then from the interpolator subtract a square number which is chosen in such a way that when the *prakṛti* and the interpolator, as obtained after the said addition and subtraction or as they are stated, are cross multiplied by the additive and the subtractive quantities, the sum of the cross products is again a square number. Then extract the square root of that (square number). Then by the product of the square root of the increased *prakṛti* and the square root of the subtractive (square number) (severally) decrease and increase that square root. The two numbers (thus obtained) being divided by the number added to the *prakṛti* become the two lesser roots. Set them down at two places and multiply both of them by the square-root of the increased *prakṛti*. Then respectively add the square-root of the number subtracted from the interpolator to the lesser one and subtract the same from the greater one. Then they become the two greater roots. A large number of lesser and greater roots may then be determined.

Thus we have revealed a determination which is as difficult as setting a fly against the wind.

Exposition

In order to solve the equation $Nx^2 + C = y^2$, choose a number a such that $N + a$ may become a perfect square. Then choose a number b such that the sum of the cross products of

$$\begin{array}{cc} N+a & c-b^2 \\ a & b^2 \end{array} \quad \text{or} \quad \begin{array}{cc} N & c \\ a & b^2 \end{array}$$

i.e., $(N+a)b^2 + (c-b^2)a$ or $Nb^2 + Ca$ also may become a perfect square. Let

$$\begin{aligned} N + a &= P^2, \\ \text{and} \quad Nb^2 + Ca &= Q^2. \end{aligned}$$

Then according to the rule the two lesser roots are

$$\frac{(Q - Pb)}{a} \quad \text{and} \quad \frac{(Q + Pb)}{a};$$

and the corresponding greater roots are

$$\frac{P(Q - Pb)}{a} + b \quad \text{and} \quad \frac{P(Q + Pb)}{a} - b.$$

That is to say, the two solutions of $Nx^2 + C = y^2$ are

$$\left.\begin{aligned} x &= \frac{(Q - Pb)}{a} \\ y &= \frac{P(Q - Pb)}{a} + b \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= \frac{(Q + Pb)}{a} \\ y &= \frac{P(Q + Pb)}{a} - b \end{aligned}\right\}.$$

Rationale

Let $N + a$ be equal to P^2. Then

$$\begin{aligned} Nx^2 + C &\equiv (N + a)x^2 + (C - b^2 - ax^2) + b^2 \\ &\equiv (Px)^2 + (C - b^2 - ax^2) + b^2. \end{aligned} \tag{7}$$

Therefore, let

$$Nx^2 + C = (Px \pm b)^2. \tag{8}$$

Then from (7) and (8), we have

$$\begin{aligned} (Px \pm b)^2 &= (Px)^2 + (C - b^2 - ax^2) + b^2 \\ \text{or} \quad ax^2 \pm 2Pbx &= (C - b^2) \\ \text{or} \quad (ax)^2 \pm 2Pbax &= a(C - b^2). \end{aligned}$$

Adding $(Pb)^2$ to both the sides, we have

$$\begin{aligned}(ax \pm Pb)^2 &= a(C-b^2) + (Pb)^2\\ &= a(C-b^2) + (N+a)b^2\\ &= Nb^2 + Ca\\ &= Q^2, \text{ say.}\\ \therefore \qquad ax \pm Pb &= Q\\ \text{or} \qquad x &= \frac{(Q-Pb)}{a} \quad \text{or} \quad \frac{(Q+Pb)}{a}.\end{aligned}$$

Consequently,

$$\begin{aligned}y &= Px + b \text{ or } Px - b \text{ respectively}\\ &= \frac{P(Q-Pb)}{a} + b \text{ or } \frac{P(Q+Pb)}{a} - b.\end{aligned}$$

Hence the rule.

Alternative rationale

Let

$$\begin{aligned}N(1)^2 + a &= P^2,\\ N(b)^2 + Ca &= Q^2.\end{aligned}$$

Now treating these as auxiliary equations and applying the process of *samāsa-bhāvanā*, we have

$$\begin{array}{cccc} N & 1 & P & a\\ & b & Q & Ca\\ \hline & Q+Pb & Nb+PQ & Ca^2\\ \text{or} & \left(\dfrac{Q+Pb}{a}\right) & \left(\dfrac{Nb+PQ}{a}\right) & C \end{array}$$

Therefore, one solution of $Nx^2 + C = y^2$ is

$$\begin{aligned}x &= \frac{Q+Pb}{a}\\ y &= \frac{(Nb+PQ)}{a} \quad \text{i.e.,} \quad \frac{P(Q+Pb)}{a} - b,\end{aligned}$$

Similarly, applying the process of *viśeṣa-bhāvanā*, we get

$$\begin{aligned}x &= \frac{(Q-Pb)}{a},\\ y &= \frac{P(Q-Pb)}{a} + b,\end{aligned}$$

as another solution of the same equation.

Illustration

> If you know (the method for solving) the *vargaprakṛti*, say what is that number whose square being multiplied by 60 and then increased by 8 times 20 is again a perfect square.

Here we have to solve the equation

$$60x^2 + 160 = y^2.$$

Obviously, a is 4, so that $P = 8$. Now b is to be chosen in such a way that $Nb^2 + Ca$ i.e., $60b^2 + 640$ may be a perfect square. By trial, we get $b = 4$, so that $Q = 40$.

Hence two solutions of the above equation are

$$x = 2,\ y = 20; \qquad x = 18,\ y = 140.$$

To get another solution, we may proceed as follows. Taking

$$60(18)^2 + 160 = 140^2$$

as an auxiliary equation and applying the process of *samāsa-bhāvanā*, we have

$$\begin{array}{cccc} 60 & 18 & 140 & 160 \\ & 18 & 140 & 160 \\ \hline & 5040 & 39040 & 25600 \\ \text{or} & \frac{63}{2} & 244 & 1 \end{array}$$

Now taking

$$60(18)^2 + 160 = 140^2, \qquad \text{and} \qquad 60\left(\frac{63}{2}\right)^2 + 1 = 244^2$$

as auxiliary equations, and applying the process of *samāsa-bhāvanā* we have

$$\begin{array}{cccc} 60 & 18 & 140 & 160 \\ & 63/2 & 244 & 1 \\ \hline & 8802 & 68180 & 160 \end{array}$$

Therefore, $x = 8802$, $y = 68180$ is another solution of the same equation.

Similarly, any number of solutions may be written down.

N. B.—The solution $x = 18$, $y = 140$ may also be derived from $x = 2$, $y = 20$ in the same way as $x = 8802$, $y = 68180$ has been derived from $x = 18$, $y = 140$.

5 Udayadivākara's procedure for solving the multiple equations

$$x + y = \text{a perfect square},$$
$$x - y = \text{a perfect square},$$
$$xy + 1 = \text{a perfect square}.$$

[Udayadivākara works out these equations because their solution is required in a problem set in the *Laghubhāskarīya* (viii. 17). His method does not give the general solution of the problem but it certainly throws light on the technique employed by early Hindu astronomers in solving algebraical equations. Udayadivākara's method under consideration deserves mention here because it is based on a previous rule of Ācārya Jayadeva.]

To begin with, Udayadivākara assumes that

$$xy + 1 = (2y + 1)^2,$$

so that he gets

$$x = 4y + 4.$$

Thus

$$x - y = 3y + 4.$$

Udayadivākara, now assumes that

$$3y + 4 = (3z + 2)^2.$$

Thus he gets

$$y = 3z^2 + 4z,$$
$$\text{making} \qquad x = 12z^2 + 16z + 4.$$

Therefore,

$$x + y = 15z^2 + 20z + 4.$$

To make $x + y$ a perfect square, Udayadivākara puts

$$15z^2 + 20z + 4 = u^2$$

which, after multiplication and transposition, he writes as

$$900z^2 + 1200z + 400 = 60u^2 + 160$$
$$\text{or} \qquad (30z + 20)^2 = 60u^2 + 160.$$

This equation can be written as a pair of equations

$$60u^2 + 160 = \lambda^2, \tag{9}$$

$$30z + 20 = \lambda. \tag{10}$$

Udayadivākara solves equation (9) in the same way as we have solved it under Section 4.7 above. He gets the solutions

$$u = 2, \quad \lambda = 20$$
$$u = 18, \quad \lambda = 140$$
$$u = 8802, \quad \lambda = 68180.$$

Making use of the values $\lambda = 140$ and $\lambda = 68180$, he gets $z = 4$, and $z = 2272$. Likewise he obtains

$$x = 64, \quad y = 260$$
$$x = 15495040, \quad y = 61980164$$

as two solutions of the proposed multiple equations.

6 Conclusion

The most interesting feature of the stanzas discussed above is the cyclic method of finding the integral solution of the equation $Nx^2 + 1 = y^2$. That method has been called *cakravāla* and is the same as given by Bhāskara II[11] and Nārāyaṇa[12] (1356). As regards the cyclic method, H. Hankel has remarked:

> It is above all praise; it is certainly the finest thing which was achieved in the theory of numbers before Lagrange.[13]

As already mentioned the cyclic method was hitherto found to occur for the first time in the *Bījagaṇita* of Bhāskara II, so the credit of that method was attributed to him. But Bhāskara II ascribed the name *cakravāla* to previous writers[14] which shows that the cyclic method was not actually devised by him. The discovery of that method in a work written about a century earlier confirms his admissions and takes away the credit of that method from him. But who is to be given the credit of that method? In this connection we must quote the following stanza which occurs at the end of Bhāskara II's *Bījagaṇita*.

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[11] *Cf. Bījagaṇita, cakravāla*, 1–4.
[12] *Cf. Gaṇitakaumudī, vargaprakṛti*, 8–11; *Bījagaṇita*, I, Rule 79–82.
[13] *Cf.* Zur Geschichte der Math. in Altertum und Mittelalter, Leipzig, 1874, pp. 203–204.
[14] The original text is "चक्रवालमिदं जगुः". The commentator Kṛṣṇa explains: "आचार्या एतद्गणितं चक्रवालमिति जगुः" i.e., "the learned professors call this method of calculation the *cakravāla*".

> As the works on algebra of Brahmagupta, Śrīdhara, and Padmanābha are very extensive, so for the satisfaction of the students I have taken the essence of those works and compiled this small work with demonstrations.

This stanza shows that the *Bījagaṇita* of Bhāskara II was drawn mostly from the works on algebra of Brahmagupta, Śrīdhara, and Padmanābha. The cyclic method does not occur in the works of Brahmagupta: it is likely that the work of Śrīdhara or Padmanābha or both contained that method.[15] There is no doubt that Bhāskara II got that method from some earlier work. If that was Jayadeva's work, Bhāskara II must have mentioned his name along with the other names mentioned by him. But he has not made even a single reference to Jayadeva. At the same time it cannot be said definitely whether the algebraical works of Śrīdhara and Padmanābha contained the cyclic method. In fact, we have absolutely no information about them. It is simply by chance that we have come across the name of and quotations from Ācārya Jayadeva who is otherwise unknown to us. Under these circumstances the question of this invention cannot be decided until we receive some more light in this direction.

[15] P. C. Sengupta (1944) expressed the hope

> that further researches may show that this achievement is to be ascribed to Padmanābha, if his work be ever brought to light.

See the Presidential Address delivered by him at the Technical Sciences Section of the Twelfth All India Oriental Conference held at Banaras, 1944. I fail to understand why Sri Sengupta has shown special favour to Padmanābha against Śrīdhara whose claims are equally good if not greater.