On Śrīdhara's rational solution of $Nx^2 + 1 = y^2$ *

1 Introduction

Śrīdhara is remembered as one of the greatest Hindu Mathematicians.[1] Unfortunately there is no definite evidence to show when and where he lived. Even his works are not all available. His very extensive (*ativistṛta*) treatise on algebra, which has been mentioned and quoted by the celebrated astronomer-mathematician Bhāskara II[2] (1150), is known only by name. Probably it is lost. Of his two works on arithmetic, known to us, the smaller one, known as *Pāṭīgaṇitasāra*, was edited by Sudhakara Dvivedi (1899) and published under the title of *Triśatikā*.[3] An English translation, with notes and introduction, of the rules occurring in this work has also appeared in the *Bibliotheca Mathematica*[4] under the joint authorship of N Ramanujacharia and G. R. Kaye. The bigger work on arithmetic, known as Śrīdhara's *Pāṭīgaṇita*, has not yet appeared in print. This work has been called *Navaśatī* and has been quoted by Makkibhaṭṭa in his commentary on the *Siddhāntaśekhara* of Śrīpati[5] (1039). An incomplete MS of this work is preserved in the Raghunatha Temple Library of His Highness the Maharaja of Jammu and Kashmir. It is a copy of an older MS, is written in modern Kashmīrī script and extends to 157 leaves with 9 lines to a page and about 44 letters to a line. Starting from the very beginning, it runs up to about the middle of the *kṣetra-vyavahāra* and is furnished with a commentary. The name of the commentator does not occur anywhere

* K. S. Shukla, *Ganita*, Vol. 1, No. 2 (1950), pp. 53–64.

[1] The following stanza, which occurs with the colophon at the end of Śrīdhara's *Pāṭīgaṇitasāra* in certain manuscripts, gives an idea of the highest position which Śrīdhara occupied in his time as a mathematician:

> उत्तरतः सुरनिलयं दक्षिणतो मलयपर्वतं यावत् ।
> प्रागपरोदधिमध्ये नो गणकः श्रीधरादन्यः ॥
>
> Up to the abode of the gods (i.e., the Himālayas) towards the north and up to the Malaya mountains towards the south and between the eastern and the western oceans, there is no mathematician (worth the name) except Śrīdhara.

[2] *Cf.* Bhāskara II's *Bījagaṇita*, conclusion. Also see *madhyamāharaṇa*, 1–3 (comm.).

[3] *Triśatikā* or *Triśatī* is another name of Śrīdhara's *Pāṭīgaṇitasāra*.

[4] Vol. XIII (1912–13), p. 203–217.

[5] *Cf.* *Siddhāntaśekhara*, i. 26 (comm.).

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_10

in the MS. A transcript copy of this, in Devanāgarī characters, exists in the Lucknow University Library.

This MS contains a rule for the rational solution of the equation

$$Nx^2 + 1 = y^2,$$

based on the rational solution of the equation $x^2 + y^2 = z^2$ and is fundamentally different from those given by Brahmagupta (628), Śrīpati (1039), and other Hindu mathematicians. The object of the present paper is to explain and illustrate Śrīdhara's method of obtaining this rule and to give analogous rules for the rational solution of certain other equations of a similar nature.

2 Śrīdhara's lemma

Śrīdhara starts with a lemma stating how to construct a rational rectangle (or right triangle). It runs:

> भुजस्य कृतिरिष्टस्य भक्तोनेष्टेन तद्दलम् ।
> कोटिरिष्टाधिका कर्णश्चतुरश्रायतस्य ते ॥[6]
>
> The square of the base (*bhuja*), chosen at pleasure, when divided and diminished by an arbitrary number and then halved gives the perpendicular (*koṭi*), and that increased by the (same) arbitrary number gives the hypotenuse (*karṇa*)—all of them of a rectangle.[7]

That is, if a number b be chosen for the base and ε for the arbitrary number, then there corresponds a rectangle of base b having the rational numbers

$$\frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} \quad \text{and} \quad \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} + \varepsilon$$

for its perpendicular and hypotenuse (or diagonal) respectively.

For example, choose 3 for the base and 1 for the arbitrary number. Then there corresponds a rectangle whose base is 3, perpendicular $\frac{1}{2}\left\{\frac{3^2}{1} - 1\right\}$ i.e., 4, and hypotenuse 5. Similarly, choosing 10 for the base and 2 for the arbitrary number, we obtain a rectangle whose base is 10, perpendicular $\frac{1}{2}\left\{\frac{10^2}{2} - 2\right\}$ i.e., 24, and hypotenuse 26; and so on.

[6] *Pāṭīgaṇita*, *kṣetra-vyavahāra*.

[7] This rule occurs elsewhere also. For example, see *Brāhmasphuṭasiddhānta* xii. 35; *Siddhāntaśekhara*, xiii. 41; *Gaṇitasārasaṅgraha*, vii. $97\frac{1}{2}$; *Līlāvatī*, *kṣetra-vyavahāra*, rule 5. The text of the above passage closely resembles that of the *Brāhmasphuṭasiddhānta*, xii. 35.

The rationale of this is as follows: Let b, k, and h be the base, perpendicular, and hypotenuse of a rectangle. Then

$$b^2 = h^2 - k^2 = (h - k)(h + k).$$

Let $h - k = \varepsilon$. Then $k = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\}$ and $h = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} + \varepsilon$.

3 The equation $Nx^2 + 1 = y^2$

This equation has been called *vargaprakṛti* (square-nature) by Hindu mathematicians. The number N is called *prakṛti* or *guṇaka* (i.e., the co-efficient or multiplier of the square); x is called *ādyapada* (i.e., the first root) or *kaniṣṭha-pada* (i.e., the lesser root); y is called *anyapada* (i.e., the other root) or *jyeṣṭha-pada* (i.e., the greater root); and the absolute term 1 is called *kṣepa* (additive) or *śodhya* (subtractive) according as it is positive or negative.

4 Rational solution of $Nx^2 + 1 = y^2$

For the rational solution of $Nx^2 + 1 = y^2$, Śrīdhara gives the following rule, making use of the above lemma:[8]

> गुणके वर्गयोर्मध्ये तत्पदाधो भुजश्रुती ।
> केचित् प्राक्कथिते तत्र वज्रकेणाहती तयोः ॥
> अन्तरस्य कृतिः क्षेपः तत्कोटिः प्रथमं पदम् ।
> ऋजुहत्यन्तरं ज्येष्ठं रूपक्षेपेऽन्तरोद्धृते ॥ [9]
>
> The multiplier (*guṇaka*) having been expressed as a difference of two squares (set down their square-roots in the ascending order of magnitude and) below those square-roots set down any (set of the) base and hypotenuse stated there (in the lemma) before. Then obtain their cross-products. The square of the difference between those (cross-products) gives the additive (*kṣepa*); the perpendicular of that (above set) denotes the first square-root (corresponding to that additive) and the difference between their direct (vertical) products denotes the greater square-root. When these (first and greater square-roots) are divided by the difference (between the above cross-products), they correspond to the additive unity.

Let the multiplier N be expressed as the difference $A^2 - B^2$ $(A > B)$ of two squares[10] and let the base, perpendicular, and hypotenuse determined from the above lemma be b, k, and h respectively. Then setting down B and A in the ascending order of magnitude, we have

$$B \qquad\qquad A$$

[8] In fact, he gives this rule as an application of the above lemma.

[9] *Pāṭīgaṇita*, *l. c.*

[10] Making use of Śrīdhara's lemma, N can always be expressed as

$$\left\{\frac{1}{2}\left(\frac{N}{c}+c\right)\right\}^2 - \left\{\frac{1}{2}\left(\frac{N}{c}-c\right)\right\}^2.$$

and below them writing down the base b and the hypotenuse h respectively, we have

$$\begin{array}{cc} B & A \\ b & h \end{array}$$

Multiplying them across, we obtain

$$Bh \quad \text{and} \quad Ab;$$

and multiplying them directly, we have

$$Bb \quad \text{and} \quad Ah.$$

The square of the difference between the cross-products i.e., $(Bh \sim Ab)^2$, then, denotes the so called additive (*kṣepa*); and the perpendicular k and the difference $Ah - Bb$ of the direct products respectively denote the first and greater roots corresponding to that additive. These first and greater roots when divided by the difference $(Bh \sim Ab)$ of the cross-products give the corresponding quantities for additive unity.

In other words, Śrīdhara's rule amounts to saying that if

$$A^2 - B^2 = N \tag{1}$$

and

$$h^2 - b^2 = k^2 \tag{2}$$

then

$$Nk^2 + (Bh \sim Ab)^2 = (Ah - Bb)^2,$$ [11]

whence

$$N\left\{\frac{k}{Bh \sim Ab}\right\}^2 + 1 = \left\{\frac{Ah - Bb}{Bh \sim Ab}\right\}^2.$$

That is

$$x = \frac{k}{Bh \sim Ab}, \qquad y = \frac{Ah - Bb}{Bh \sim Ab}$$

is the rational solution of

$$Nx^2 + 1 = y^2.$$

[11] This result is easily obtained by multiplying (1) and (2) side by side and bringing the resulting product to the requisite form.

4.1 Illustration

The following example will illustrate the above rule.

Example What is that number whose square-root when multiplied by 55 and then increased by 1 becomes capable of yielding a square-root?

The resulting equation is

$$55x^2 + 1 = y^2.$$

One solution

We have

$$8^2 - 3^2 = 55,$$

$$\text{and} \qquad 5^2 - 3^2 = 4^2.$$

Therefore, setting down 3 and 8 and below them 3 and 5 respectively, we have[12]

$$\begin{array}{cc} 3 & 8 \\ 3 & 5 \end{array}$$

The difference between the cross-products is 9, the perpendicular is 4, and the difference between the direct products is 31. Therefore,

$$55(4)^2 + (9)^2 = (31)^2,$$

$$\text{whence} \qquad 55\left(\frac{4}{9}\right)^2 + 1 = \left(\frac{31}{9}\right)^2.$$

Hence $x = \frac{4}{9}$, $y = \frac{31}{9}$ is a rational solution of the above equation. Thus the required number is $\frac{4}{9}$.

Another solution

Again we have

$$8^2 - 3^2 = 55,$$

$$\text{and} \qquad 10^2 - 8^2 = 6^2.$$

Therefore, setting down 3 and 8 and below them 8 and 10 respectively, we have

$$\begin{array}{cc} 3 & 8 \\ 8 & 10 \end{array}$$

[12] The Hindu way of writing is

3	8
3	5

The difference between the cross-products is 34, the perpendicular is 6, and difference between the direct products is 56. Therefore,

$$55(6)^2 + (34)^2 = (56)^2,$$

$$\text{or} \qquad 55\left(\frac{6}{34}\right)^2 + 1 = \left(\frac{56}{34}\right)^2.$$

Hence $x = \frac{3}{17}$, $y = \frac{28}{17}$ is another rational solution of the above equation. This gives the required number to be $\frac{3}{17}$.

Other solutions

Other rational solutions of the same equation will be obtained by altering the sides of the rational right triangle for every new solution.

5 Another form of rational solution of $Nx^2 + 1 = y^2$

It will be easily seen that when

$$A^2 - B^2 = N$$

$$\text{and} \qquad h^2 - b^2 = k^2,$$

$$\text{then} \qquad Nk^2 + (Bh \overset{+}{\sim} Ab)^2 = (Ah \pm Bb)^2$$

$$\text{or} \qquad N\left\{\frac{k}{Bh \overset{+}{\sim} Ab}\right\}^2 + 1 = \left\{\frac{Ah \pm Bb}{Bh \overset{+}{\sim} Ab}\right\}^2.$$

That is, both

$$\left.\begin{aligned} x &= \frac{k}{Bh + Ab} \\ y &= \frac{Ah + Bb}{Bh + Ab} \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= \frac{k}{Bh \sim Ab} \\ y &= \frac{Ah - Bb}{Bh \sim Ab} \end{aligned}\right\}$$

are the rational solutions of $Nx^2 + 1 = y^2$.

In the foregoing rule, Śrīdhara gives the solution in the latter form.

5.1 Other forms

Putting

$$k=\frac{1}{2}\left\{\frac{b^2}{\varepsilon}-\varepsilon\right\},$$

$$h=\frac{1}{2}\left\{\frac{b^2}{\varepsilon}+\varepsilon\right\},$$

$$B=\frac{1}{2}\left\{\frac{N}{c}-c\right\},$$

and $$A=\frac{1}{2}\left\{\frac{N}{c}+c\right\},$$

Śrīdhara's solution may be stated as

$$x=\frac{2c(b^2-\varepsilon^2)}{N(b-\varepsilon)^2\sim c^2(b+\varepsilon)^2}$$

$$y=\frac{N(b-\epsilon)^2+c^2(b+\varepsilon)^2}{N(b-\varepsilon)^2\sim c^2(b+\varepsilon)^2}$$

where b, ε, and c are any numbers.

This may be set in the form

$$x=\frac{4pq}{4p^2N\sim q^2}$$

$$y=\frac{4p^2N+q^2}{4p^2N\sim q^2}$$

in which it was discovered in Europe by John Wallis[13] (1657).

Further, putting $\frac{q}{2p}=r$, this may be written as

$$x=\frac{2r}{N\sim r^2}$$

$$y=\frac{N+r^2}{N\sim r^2}$$

in which form it was given by Śrīpati[14] (1039), Bhāskara II[15] (1150), Nārāyaṇa[16] (1356), Jñānarāja, and Kamalākara[17] (1658), and in Europe by W. Brouncker[18] (1657).

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[13] *Cf. Oeuvres de Fermat*, III, (1896), *Lettre ix*; John Wallis A Kenelm Digby, p. 417 ff. Also *Cf.* Dickson, L. E., *History of the Theory of Numbers*, II, p. 351.

[14] *Cf. Siddhāntaśekhara*, xiv. 33.

[15] *Cf.* His *Bījagaṇita*, *vargaprakṛti*, rule 6.

[16] *Cf.* His *Bījagaṇita*, *I*, rule 77 f.

[17] *Cf.* His *Siddhāntatattvaviveka*, xiii. 216.

[18] *Cf. Oeuvres de Fermat*, III, (1896), *Lettre ix*; John Wallis A Kenelm Digby, p. 417 ff. Also *Cf.* Dickson, L. E.,*History of the Theory of Numbers*, II, p. 351.

6 Rational solution of other equations

Śrīdhara has given simply the rational solution of $Nx^2 + 1 = y^2$. But the method used by him can be easily applied to the determination of the rational solutions of $Nx^2 - 1 = y^2$, $1 - Nx^2 = y^2$, or the general forms $Nx^2 \pm C = y^2$ and $C - Nx^2 = y^2$. In what follows, we propose to give the rational solutions of these equations in accordance with his method.

6.1 Rational solution of $Nx^2 - 1 = y^2$

In this case the rule may be stated as follows:

Rule—Express the multiplier (N) as a sum of two squares.[19] Set down the square-roots of those squares and underneath them the base and the perpendicular of any rectangle (determined form the lemma). Multiplying them across and directly, obtain the sum and the difference of the cross-products and of the direct products. Then corresponding to the square of the sum (or difference) of the cross-products as the subtractive, the hypotenuse of the rectangle (chosen above) is the first square-root and the difference (or sum) of the direct products is the other square-root; and corresponding to the square of the sum (or difference) of the direct products as the subtractive, the hypotenuse of the rectangle is the first square-root and the difference (or sum) of the cross-products is the other square-root. These first and other square-roots when divided by the square-roots of the corresponding subtractives give the first and other square-roots for the subtractive unity.

In other words, if

$$\begin{aligned} & A^2 + B^2 = N \\ \text{and} \quad & b^2 + k^2 = h^2, \\ \text{then} \quad & Nh^2 - (Ak \overset{+}{\sim} Bb)^2 = (Ab \overset{\sim}{+} Bk)^2 \\ \text{and} \quad & Nh^2 - (Ab \overset{+}{\sim} Bk)^2 = (Ak \overset{\sim}{+} Bb)^2. \end{aligned}$$

Or,

$$N\left\{\frac{h}{Ak \overset{+}{\sim} Bb}\right\}^2 - 1 = \left\{\frac{Ab \overset{\sim}{+} Bk}{Ak \overset{+}{\sim} Bb}\right\}^2$$

$$\text{and} \quad N\left\{\frac{h}{Ab \overset{+}{\sim} Bk}\right\}^2 - 1 = \left\{\frac{Ak \overset{\sim}{+} Bb}{Ab \overset{+}{\sim} Bk}\right\}^2.$$

[19] "When unity is the subtractive the solution of the problem is impossible unless the multiplier is the sum of two squares." (Bhāskara II).

"In the case of unity as the subtractive the multiplier must be the sum of two squares. Otherwise, the solution is impossible." (Nārāyaṇa).

That is,

$$\left.\begin{aligned} x &= \frac{h}{Ak+Bb} \\ y &= \frac{Ab\sim Bk}{Ak+Bb} \end{aligned}\right\},\quad \left.\begin{aligned} x &= \frac{h}{Ak\sim Bb} \\ y &= \frac{Ab+Bk}{Ak\sim Bb} \end{aligned}\right\},\quad \left.\begin{aligned} x &= \frac{h}{Ab+Bk} \\ y &= \frac{Ak\sim Bb}{Ab+Bk} \end{aligned}\right\},\ \text{and}\quad \left.\begin{aligned} x &= \frac{h}{Ab\sim Bk} \\ y &= \frac{Ak+Bb}{Ab\sim Bk} \end{aligned}\right\}$$

are the rational solutions of $Nx^2-1=y^2$.

6.1.1 A particular solution

Choosing 1 for the base (b) and also 1 for the arbitrary number (ε) and applying Śrīdhara's lemma, we have

$$b=1,\ k=0,\ \text{and}\ h=1.$$

Substituting these values of b, k, and h in the above rational solutions of $Nx^2-1=y^2$, we obtain

$$\left.\begin{aligned} x &= \frac{1}{B} \\ y &= \frac{A}{B} \end{aligned}\right\}\qquad \text{and} \qquad \left.\begin{aligned} x &= \frac{1}{A} \\ y &= \frac{B}{A} \end{aligned}\right\}$$

as the particular rational solutions of $Nx^2-1=y^2$.

These particular solutions were mentioned by Bhāskara II[20] (1150).

6.2 Rational solution of $1-Nx^2=y^2$

In this case the rule may be stated as follows:

Rule—Express the multiplier (N) as a difference of two squares. Set down the square roots of those squares in the descending order of magnitude and below those square-roots set down the base and hypotenuse of any rectangle (determined from the lemma), in order. Next obtain the cross-products and the direct products. The perpendicular of the rectangle (chosen above) divided by the sum (or difference) of the cross-products, then, denotes the first square-root and the sum (or difference) of the direct products divided by the sum (or difference) of the cross-products denotes the other square-root.

In other words, if

$$A^2-B^2=N$$

$$\text{and}\qquad h^2-b^2=k^2,$$

$$\text{then}\qquad (Ah\pm Bb)^2-Nk^2=(Ab \overset{+}{\sim} Bh)^2$$

$$\text{or}\qquad 1-N\left\{\frac{k}{Ah\pm Bb}\right\}^2=\left\{\frac{Ab\overset{+}{\sim} Bh}{Ah\pm Bb}\right\}^2.$$

[20] *Bījagaṇita*, *cakravāla*, 5(ii)–6.

That is, both

$$\left.\begin{aligned} x &= \frac{k}{Ah+Bb} \\ y &= \frac{Ab+Bh}{Ah+Bb} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{k}{Ah-Bb} \\ y &= \frac{Ab \sim Bh}{Ah-Bb} \end{aligned}\right\}$$

are the rational solutions of $1 - Nx^2 = y^2$.

6.3 Rational solution of $Nx^2 \pm C = y^2$ and $C - Nx^2 = y^2$

These are the general forms of the equations considered above. When N and C are both non-square integers, Śrīdhara's method is not applicable to their solution. When, however, at least N or C is a perfect square, Śrīdhara's method may be used to obtain the rational solutions of the above forms. When C is a perfect square, the above forms easily reduce to the forms discussed above. It is sufficient, therefore, to consider only the two forms *viz.*

$$\text{(i)} \quad a^2x^2 \pm C = y^2$$
$$\text{and} \qquad \text{(ii)} \quad C - a^2x^2 = y^2.$$

6.3.1 Rational Solution of $a^2x^2 \pm C = y^2$

Rule—Express the additive or subtractive as a difference of two squares. Set down the square-roots of these squares in the ascending or descending order of magnitude according as the *kṣepa* is additive or subtractive and below them set down the base and the hypotenuse of any rational rectangle in order. Obtain the cross-products and the direct products. The sum or difference of the cross-products divided by the product of the square-root of the multiplier and the perpendicular of the rectangle, then, gives the first square-root and the sum or difference of the direct products divided by the perpendicular of the rectangle gives the other square-root.

In other words, if

$$A^2 - B^2 = C$$
$$\text{and} \qquad h^2 - b^2 = k^2,$$

$$\text{then} \qquad a^2\left\{\frac{Bh \overset{+}{\sim} Ab}{ak}\right\}^2 + C = \left\{\frac{Bb \overset{+}{\sim} Ah}{k}\right\}^2$$

$$\text{and} \qquad a^2\left\{\frac{Ah \pm Bb}{ak}\right\}^2 - C = \left\{\frac{b \overset{+}{\sim} h}{k}\right\}^2.$$

That is

$$\left.\begin{aligned} x &= \frac{Bh+Ab}{ak} \\ y &= \frac{Bb+Ah}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Bh \sim Ab}{ak} \\ y &= \frac{Bb \sim Ah}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2+C=y^2$; and

$$\left.\begin{aligned} x &= \frac{Ah+Bb}{ak} \\ y &= \frac{Ab+Bh}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ah-Bb}{ak} \\ y &= \frac{Ab \sim Bh}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2-C=y^2$.

Another form

If

$$A^2-B^2=\pm C$$
$$\text{and} \quad h^2-b^2=k^2,$$

then, as shown above

$$\left.\begin{aligned} x &= \frac{Ab+Bh}{ak} \\ y &= \frac{Ah+Bb}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ab \sim Bh}{ak} \\ y &= \frac{Ah-Bb}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2 \pm C=y^2$.

Choosing

$$A=\frac{\pm C+1}{2}$$
$$B=\frac{\pm C-1}{2}$$

and putting

$$k=\frac{1}{2}\left\{\frac{b^2}{\varepsilon}-\varepsilon\right\}$$
$$h=\frac{1}{2}\left\{\frac{b^2}{\varepsilon}+\varepsilon\right\}$$

in one of these solutions, say in the first, we have

$$x=\frac{1}{2a}\left\{\frac{\pm C(b+\varepsilon)}{b-\varepsilon}-\frac{b-\varepsilon}{b+\varepsilon}\right\}$$
$$y=\frac{1}{2}\left\{\frac{\pm C(b+\varepsilon)}{b-\varepsilon}+\frac{b-\varepsilon}{b+\varepsilon}\right\}$$

or, on setting $\frac{b-\varepsilon}{b+\varepsilon} = \lambda$,

$$\left.\begin{aligned} x &= \frac{1}{2a}\left\{\frac{\pm C}{\lambda} - \lambda\right\} \\ y &= \frac{1}{2}\left\{\frac{\pm C}{\lambda} + \lambda\right\} \end{aligned}\right\},$$ [21]

which is another form of the rational solution of $a^2x^2 \pm C = y$. This was given by Brahmagupta[22] (628), Bhāskara II[23] (1150), and Nārāyaṇa[24] (1356).

6.3.2 Rational solution of $C - a^2x^2 = y^2$

A rational solution of this equation is possible by the above method if C is capable of being expressed as a sum of two squares. In that case, we are led to the following rule for the rational solution of the above equation:

Rule—Express C as a sum of two squares, set down their square-roots and below them the base and the perpendicular of any rational rectangle. Obtain their cross-products and direct products. The sum or difference of the cross-products (or of the direct products) divided by the product of the square-root of the multiplier and the hypotenuse of the rectangle, then, gives the first square-root and the sum or difference of the direct products (or of the cross-products) divided by the hypotenuse of the rectangle gives the other square-root.

In other words, if

$$A^2 + B^2 = C$$

$$\text{and} \qquad b^2 + k^2 = h^2,$$

$$\text{then} \qquad C - a^2\left\{\frac{Ak \overset{+}{\sim} Bb}{ah}\right\}^2 = \left\{\frac{b \overset{+}{\sim} Bk}{h}\right\}$$

$$\text{and} \qquad C - a^2\left\{\frac{Ab \overset{+}{\sim} Bk}{ah}\right\}^2 = \left\{\frac{Ak \overset{+}{\sim} Bb}{h}\right\}^2.$$

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[21] This solution can be easily derived from Śrīdhara's lemma. For, $\pm C$, $(ax)^2$, and $(y)^2$ may be algebraically treated as the squares of the base, perpendicular, and hypotenuse of a rectangle; so that, choosing an arbitrary number λ and making use of Śrīdhara's lemma we have

$$x = \frac{1}{2a}\left\{\frac{\pm C}{\lambda} - \lambda\right\}, \qquad y = \frac{1}{2}\left\{\frac{\pm C}{\lambda} - \lambda\right\}.$$

[22] *Cf.* *Brāhmasphuṭasiddhānta*, xviii. 69.

[23] *Cf.* His *Bījagaṇita*, *cakravāla*, rule 8.

[24] *Cf.* His *Bījagaṇita*, I, rule 85.

That is,

$$\left.\begin{aligned} x &= \frac{Ak+Bb}{ah} \\ y &= \frac{Ab+Bk}{h} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{Ak\sim Bb}{ah} \\ y &= \frac{Ah\sim Bk}{h} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{Ab+Bk}{ah} \\ y &= \frac{Ak+Bb}{h} \end{aligned}\right\}, \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ab\sim Bk}{h} \\ y &= \frac{Ak\sim Bb}{h} \end{aligned}\right\}$$

are the rational solutions of $C - a^2x^2 = y^2$.

7 Conclusion

The rectangular method discussed above seems to be Śrīdhara's own contribution to the subject. This method occurs exclusively in Śrīdhara's *Pāṭīgaṇita.* It is not found in any other Hindu mathematical work so far known nor has it been included among the various solutions of the equations of the form $Nx^2 + 1 = y^2$ surveyed by L. E. Dickson (1920) in his *History of the Theory of Numbers, Vol. II.* It is fundamentally different from the tentative method stated by Datta and Singh (1938) in their *History of Hindu Mathematics, Part II*, and from the method given by Śrīpati (1039) and followed by Bhāskara II (1150), Nārāyaṇa (1356), Jñānarāja (1503), and Kamalākara (1658).

Śrīdhara's method is extremely general. It is applicable to the equations of the forms $Nx^2 \overset{+}{\sim} C = y^2$ for obtaining their rational solutions. These solutions, it may be added, are most general. In the above discussion we have shown that the rational solutions given by Brahmagupta (628), Śrīpati (1039), Bhāskara II (1150), Nārāyaṇa (1356), Jñānārāja (1503), and Kamalākara (1658), as also those given by John Wallis and W. Brouncker (1657) are deducible from the rational solutions obtained from Śrīdhara's method.