Sources and Studies in the History of Mathematics and Physical Sciences

Aditya Kolachana
K. Mahesh
K. Ramasubramanian *Editors*

Studies in Indian Mathematics and Astronomy

Selected Articles of Kripa Shankar Shukla

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Sources and Studies in the History of Mathematics and Physical Sciences

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॥ पद्यसुमाञ्जलिः ॥

K. S. Shukla
(1918–2007)

दत्तैर्दत्तविभूतिमद्गणितभां प्राप्यावधेशान्मुदा
नैरन्तर्यगवेषणेन मुनिवत् ग्रन्थप्रकाशे रतः ।
ज्योतिर्विज्ञकृतीः व्यभूषयदयं स्वीयानुसन्धानतः
विद्वन्मण्डलकेसरी स्वमहिमा-शुक्लः कृपाशङ्करः ॥

Kripa Shankara Shukla, having received the treasure of the luminescence of mathematics bequeathed by [Bibhutibhusana] Datta through Avadhesha [Narayana Singh], happily dedicating himself to research, relentlessly worked towards the publication of [multiple] treatises like a sage. Through his investigations, he embellished the writings of astronomers, and was truly a lion in the assembly of scholars, shining (*śukla*) through his fame.

॥ ग्रन्थसमर्पणम् ॥

स्थित्वाचार्यपदे हि लक्ष्मणपुरे विश्वाभिरामः कृपा-
शुक्लो ज्योतिषशास्त्रगूढविषयान् लोकाय चोपाहरत् ।
लेखांस्तस्य सुधीप्रियान् सुमनिभान् सङ्गृह्य मालामिमां
सौलभ्याय बुभुत्सुभृङ्गततये दद्मो वयं तुष्टये ॥

While remaining in city of Lucknow (*lakṣmaṇapura*) as an *Ācārya*, Kripa [Shankar] Shukla was indeed a source of delight for the whole world (*viśvābhirāma*), [as] he also presented the subtler aspects of [Indian] astronomy [in a manner comprehensible] to the entire mankind (*loka*). Having collected his articles resembling a bunch of flowers, which are a source of delight to the men of wisdom (*sudhīpriya*), we present them, for the sake of easy accessibility, as a garland for the gratification of the swarm of bees constituted by the [community of] knowledge-seekers (*bubhutsu*).

Contents

Preface

The rich history of the origin and development of science in India and the voluminous literature produced in this pursuit stand in stark contrast to the prevalent ignorance and meagre attention paid to this heritage in current times. Despite the neglect of the discipline of history of science, our understanding of India's scientific heritage—especially that of mathematics and astronomy—has progressed due to the tireless efforts of several stalwarts such as Bapudeva Sastri (1821–1900), Shankar Balakrishna Dikshit (1853–1898), Sudhakara Dvivedi (1855–1910), M. Rangacarya (1861–1916), P. C. Sengupta (1876–1962), B. B. Datta (1888–1958), A. A. Krishnaswamy Ayyangar (1892–1953), A. N. Singh (1901–1954), C. T. Rajagopal (1903–1978), T. A. Saraswati Amma (1918–2000), S. N. Sen (1918–1992), K. S. Shukla (1918–2007) and K. V. Sarma (1919–2005).

Prof. K. S. Shukla was one of India's leading historians of science. Combining a flair for mathematics with a strong grasp of Sanskrit, Prof. Shukla made immense contributions to advancing our understanding of the history and development of mathematics and astronomy in India. On the occasion of his birth centenary, we have taken the opportunity to collate and compile some of Prof. Shukla's most important papers in the form of the volume in front of you. Such a volume naturally demands a brief introduction of the individual who has authored its contents, and in the following sections we set out to do the same. Subsequently, in the final section of the preface we briefly discuss the structure of the volume as well as the editorial practices adopted in compiling it.

Early life

The biographical details pertaining to the early phase of Prof. Shukla's life has been succinctly brought out by Prof. R. C. Gupta, an eminent historian of mathematics himself, and a student of Prof. Shukla on the occasion of the latter's 80th birth anniversary:[1]

[1] See the paper entitled "Dr. Kripa Shankar Shukla, veteran historian of Hindu astronomy and mathematics" by Prof. Gupta in Part I of this volume. Subsequent quotes and many

> Kripa Shankar Shukla's birth took place at Lucknow on July 10, 1918. From the very early years, he was a brilliant student of Mathematics and Sanskrit. He passed the High School Examination of U.P. Board in 1934 in First Division with Distinction in Mathematics and Sanskrit and the Intermediate Examination of that Board again in First Division with Distinction in Mathematics. He had his higher education at Allahabad, passing the B.A. examination in the second division from Allahabad University in 1938. From the same University, he obtained his Master of Arts degree in Mathematics in the First Division in 1941.

Career and contributions

Just as Prof. Shukla was completing his undergraduate studies, Prof. Bibhutibhusan Datta and Prof. Avadhesh Narayan Singh published the second part of their monumental work "*History of Hindu Mathematics*" in 1938. Subsequently, in 1939, Prof. Singh launched a Scheme of Research in Hindu Mathematics in the Department of Mathematics and Astronomy at Lucknow University, where he was a Lecturer. After his post-graduation, Prof. Shukla joined this scheme. Prof. Gupta writes:

> Dr. K. S. Shukla joined the Department and the Scheme in 1941 ... Dr. Shukla investigated thoroughly the works of Bhāskara I and studied other relevant primary and secondary material. Under the supervision of Dr. A. N. Singh, Shukla prepared a thesis on "Astronomy in the Seventh Century India: Bhāskara I and His Works" ... The significance of the thesis lies not only in providing a genuine additional source for the history of early Indian exact sciences but also in bringing to light many new historical and methodological facts.

Over the course of his career at Lucknow University, Prof. Shukla carried forward his investigations into the history of Indian mathematics and astronomy, publishing a number of critically edited texts (often with English translation and commentary), as well as research papers, in addition to supervising the research of five doctoral scholars. Prof. Gupta as well as Prof. Ohashi[2] (another student of Prof. Shukla) have given detailed bibliographies of Prof. Shukla's publications.

The major source works brought out by Prof. Shukla are listed in Table 1. He also wrote over 40 important articles and reviews which not only brought

details in this preface regarding Prof. Shukla's life and work are also borrowed from here.

[2]See Prof. Ohashi's "Obituary" in Part I of this volume. Subsequent quotes from Prof. Ohashi in this preface are also borrowed from this paper.

Table 1: Source works brought out by Prof. Shukla.

No.	Title of the works and their authors	Year
1	*Sūrya-siddhānta* with the commentary of Parameśvara	1957
2	*Pāṭīgaṇita* of Śrīdharācārya	1959
3	*Mahābhāskarīya* of Bhāskara I	1960
4	*Laghubhāskarīya* of Bhāskara I	1963
5	*Dhīkoṭida-karaṇa* of Śrīpati	1969
6	*Bījagaṇitāvataṃsa* of Nārāyaṇa Paṇḍita	1970
7	*Āryabhaṭīya* of Āryabhaṭa	1976
8	*Āryabhaṭīya* of Āryabhaṭa with the commentary of Bhāskara I and Someśvara	1976
9	*Karaṇaratna* of Devācārya	1979
10	*Vaṭeśvarasiddhānta* and *Gola* of Vaṭeśvara (2 Vols)	1985–86
11	*Laghumānasa* of Mañjula	1990
12	*Gaṇitapañcaviṃśī* (published posthumously)	2017

forth the numerous contributions of Indian mathematicians and astronomers, but also served to demolish certain wrong conceptions regarding the origin, technical soundness, and depth of the Indian works. Some of Prof. Shukla's most important contributions include (i) his study of Varāhamihira's *Pañca-siddhāntikā*, (ii) bringing to light Jayadeva's verses on the brilliant *cakravāla* method of solving second order indeterminate equations, (iii) clearing misconceptions among modern scholars regarding the use of the iterated hypotenuse by Indian astronomers in determining the equation of centre, and (iv) revising and publishing the third and final part of the "*History of Hindu mathematics*" by B. B. Datta and A. N. Singh.[3]

Scholarship and commitment

In praise of Prof. Shukla's scholarship, Prof. Gupta notes:

> Working wholeheartedly with single minded devotion for more than half a century, Dr. Shukla's contribution in the field of history of ancient and medieval Indian mathematics forms a pioneer work which will continue to motivate future research and investigations. He gave new interpretations of many obscure Sanskrit passages and corrected misinterpretations and other errors committed by others.

[3]For a detailed discussion on some of Prof. Shukla's seminal contributions, see the articles of Yukio Ohashi (1995) and M. D. Srinivas (2018) in Part I of this volume.

Extolling Prof. Shukla's meticulous way of maintaining notes, and his commitment towards his doctoral students, Prof. Ohashi writes:

> When we read his notes, we feel as if we are being taught by him directly. It should also be mentioned that he noted several parallel statements in other Sanskrit texts in the footnotes of his English translations. So, his English translations can also be used as a kind of annotated index of Sanskrit astronomical and mathematical texts. Only Prof. Shukla could do this ...
>
> I studied the history of Indian astronomy and mathematics under the guidance of Prof. Shukla from 1983 to 1987 as a research scholar (Ph. D. student) of Lucknow University ... Prof. Shukla already had retired but kindly taught me how to read Sanskrit astronomical texts, both printed texts and manuscripts. I saw several people were visiting Lucknow to meet Prof. Shukla.

Tireless efforts to attain perfection

Unlike today, in the India of the 1980s, printing was done using the letterpress, wherein a worker used to compose the text in a metallic frame of a given dimension, employing a variety of metallic fonts stored in a huge type case. Also printing of a volume could not happen all in one go as we do it today. At most 16 or 32 sheets could be printed at one time, and if there were to be any slip in proof reading, it could not be corrected again since the frame would have been dismantled once the pages were printed. Hence the author had to be all the more careful in proof-reading the text. Reminiscing how punctiliously and tirelessly Professor Shukla worked to ensure that the books he edited were error-free, Prof. Ohashi observes:

> When I was in Lucknow, the *Vaṭeśvara Siddhānta and Gola of Vaṭeśvara* was being printed at a press in Lucknow. Prof. Shukla visited the press almost every day, supervised its printing work by himself, and read its proofs very carefully. From this fact, we can understand why his edition is so reliable. These original sources are the most important foundation for future research.

Awards and accolades

In recognition of his scholarship and lifelong contributions, Prof. Shukla received many awards and was associated with several prestigious institutions. Some of these have been detailed by Prof. Gupta:

Dr. Shukla was awarded the Banerji Research Prize of the Lucknow University. He was associated with the editorial work of the Journal *Gaṇita* of the Bhārata Gaṇita Pariṣad (formerly the Benaras Mathematical Society) for many years. He was elected Fellow of the National Academy of Sciences, India in 1984, and the Corresponding Member of the International Academy of History of Science, Paris, in 1988. He served as a member of several national and international committees.

The current volume

Prof. Shukla passed away on September 22, 2007. In his memory, on the occasion of the centenary of his birth, the current volume presents a collection of his papers highlighting the wide range of his scholarship.

Structure of the volume

This volume consists of six parts. Part I consists of five introductory articles which give an overview of the life and work of Prof. Shukla. They include detailed bibliographies of his publications, and reminiscences from his former students Prof. Yukio Ohashi and Prof. R. C. Gupta. The last two articles by Prof. Yukio Ohashi and Prof. M. D. Srinivas highlight the important contributions made by Prof. Shukla to improve our understanding of Indian mathematics and astronomy. Part II consists of a collection of articles penned by Prof. Shukla related to various aspects of Indian mathematics. Part III consists of revised version of articles on Indian mathematics by Bibhutibhusan Datta and Avadhesh Narayan Singh, which together constitute the third unpublished part of their "*History of Hindu Mathematics*". As noted earlier, these articles were revised and updated by Prof. Shukla and published in the Indian Journal for History of Science between 1980 and 1993. Parts IV and V consist of a number of articles penned by Prof. Shukla on different aspects of Indian astronomy. Part VI includes some of Prof. Shukla's reviews of works related to Indian mathematics and astronomy authored by various scholars. This part also includes a few reviews of Prof. Shukla's publications by other scholars, and in one instance, his response to a review.

A note on the editorial practices adopted

While preparing this volume, we have emended the original text in a number of places. These emendations are generally accompanied with an editorial note prefaced with the abbreviation "**ed.**". Occasionally, we have also emended the text silently for a better reading experience. For instance, typographical errors in the English text as well as the Sanskrit verses have been silently emended. In a few instances, tables and figures which originally occur in between running text have been given numbers and placed elsewhere in the text, with the appropriate reference, for better typesetting. Footnotes to mathematical equations have been moved to the adjacent text. In the interests of standardising the style of the volume, we have (i) redrawn all the figures, (ii) presented Sanskrit verses in the Devanāgarī script, (iii) largely made uniform the different styles of transliterating Sanskrit words and the names of Sanskrit texts into the roman script, (iv) modified the section numbers in a few instances, (v) standardised table styles across papers, and (vi) converted all end-notes to footnotes.

We hope that this volume serves to familiarize the reader with the wide range of research carried out by Prof. Shukla, and also inspires young scholars to seriously pursue research in Indian mathematics and astronomy.

विलम्बि-मार्गशिरशुक्लषष्ठी
गतकल्यब्दाः ५११९
कल्यहर्गणः १८७०००० (अज्ञानिनः सदीपाः)
December 13, 2018

Aditya Kolachana
K Mahesh
K Ramasubramanian
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Acknowledgements

At the outset, we would like to express our sincere gratitude to the various publishers of Prof. Shukla's research articles for giving their kind permission to compile this volume. For the sheer volume of articles sourced from these journals, we would like to especially thank the publishers of *Gaṇita*, *Gaṇita-Bhāratī*, and the *Indian Journal of History of Science.*

Given the wide range of scholarship and prolific output of Prof. Shukla, the process of sorting through and selecting the appropriate assortment of his papers for this volume, which are representative of the depth and range of his scholarship, was not an easy task. We are quite grateful to Prof. M. D. Srinivas for his invaluable guidance and helpful insights in this process.

Some of Prof. Shukla's important papers that were published in Lucknow University's Journal *Gaṇita* were not accessible to us as they were not digitally archived. In this connection, we approached Prof. Poonam Sharma, Department of Mathematics and Astronomy, Lucknow University. She readily agreed and immediately arranged to procure copies of these articles. We deeply appreciate her enthusiastic help and express our sincere gratitude to her for this kind gesture. We would also like to thank Prof. Yukio Ohashi for contributing a note on his Reminiscences, heeding to our request in a very short notice.

Currently, the personal library of Prof. Shukla is safeguarded by his son Sri Ratan Shukla. As we wanted to consult this library in connection with the preparation of this volume, we contacted Sri Shukla. He warmly welcomed us into his home, and provided us open access to the entire library as well as Prof. Shukla's personal communications. This gave us greater insights into the working style as well as the personality of Prof. Shukla, and helped us make informed choices while editing this volume. We express our heartfelt thanks to Sri Ratan Shukla for his generous hospitality.

Preparing this volume involved the diligent effort of several people who assisted with typing, drawing figures, typesetting, proof-reading, technical support, and overall coordination. For helping with all these variety of tasks—without which this volume would not have been possible—we are greatly thankful to Dr. Dinesh Mohan Joshi, Smt. Sushma Sonak, Smt. Sreelekshmy Ranjit, Sri G. Periasamy, Sri Vikas Uttekar, and Smt. T. Mehtaj.

We would also like to express our sincere gratitude to the Ministry of Human Resource Development, Government of India, for the generous support extended to carry out research activities on Indian science and technology by way of initiating the Science and Heritage Initiative (SandHI) at IIT Bombay. Finally, we convey our gratitude to the Hindustan Book Agency, New Delhi, for enthusiastically coming forward to publish this volume as a part of their series on Culture and History of Mathematics.

About the Editors

Aditya Kolachana is a post-doctoral researcher at the Indian Institute of Technology Bombay, India, studying various aspects of the history of mathematics and astronomy in India. He obtained his PhD in the history of science from Indian Institute of Technology Bombay, India. He studied at the Indian Institute of Technology Kharagpur and Indian Institute of Management Ahmedabad, India. Earlier, he worked for over four years with Bain & Company, Clinton Foundation, and Procter & Gamble and also briefly taught finance at the Birla Institute of Technology and Science, Pilani, Hyderabad Campus, India.

K. Mahesh is a research scientist at the Cell for the Indian Science and Technology in Sanskrit, Department of Humanities and Social Sciences, Indian Institute of Technology Bombay, India. He acquired his PhD degree from Indian Institute of Technology Bombay, India, in 2010, by working on the Indian astronomy (Siddhānta Jyotiṣa). He went to the Centre Nationale de la Recherche Scientifique (CNRS), Paris, for doing his post-doctoral research in the history of numerical tables. On returning from Paris, he joined the Samskrit Promotion Foundation, New Delhi, and served there for three years. In 2015, the Indian National Science Academy, New Delhi, bestowed upon him the Young Historian of Science Award.

K. Ramasubramanian is a professor at the Cell for the Indian Science and Technology in Sanskrit, Department of Humanities and Social Sciences, Indian Institute of Technology Bombay, India. He holds a doctorate degree in theoretical physics, a master's degree in Sanskrit, and a bachelor's degree in engineering—a weird but formidable combination of subjects to do multidisciplinary research. He was honoured with the coveted title "Vidvat Pravara" by the Shankaracharya of Sri Sringeri Sharada Peetham, Karnataka, India, for completing a rigorous course in Advaita Vedānta (a 14-semester program), in 2003. He is one of the authors who prepared detailed explanatory notes of the celebrated works Gaṇita-yuktibhāṣā (rationales in mathematical astronomy), Tantrasaṅgraha and Karaṇa-paddhati, which bring out the seminal contributions of the Kerala School of Astronomers and Mathematicians. He was conferred with the prestigious award of the Maharṣi

Bādarāyaṇa Vyas Samman by the then President of India, in 2008, and the R.C. Gupta Endowment Lecture Award by the National Academy of Sciences India, in 2010. He is a recipient of several other awards and coveted titles as well. From 2013, he has been serving as an elected council member of the International Union of History and Philosophy of Science and Technology. He is also a member of various other national and international bodies.

Part I

The Oeuvre of Kripa Shankar Shukla

Reminiscences of Prof. K. S. Shukla *

I arrived at Lucknow in 1983. It was my first experience to go abroad. At that time, Prof. Kripa Shankar Shukla already had retired from the Department of Mathematics and Astronomy of Lucknow University, but sometimes came to the department. In the Department of Mathematics and Astronomy, Prof. A. N. Singh started a research centre of Hindu mathematics in 1939, and Prof. Shukla joined this project in 1941. Prof. Shukla took forward the monumental work *History of Hindu Mathematics* (Vol. 1, 1935; Vol. 2, 1938) of B. B. Datta and A. N. Singh, revised the draft of its subsequent parts left by Datta and Singh, and published them in the *Indian Journal of History of Science* (1980–1993).

I studied the history of Indian mathematics and astronomy from 1983 to 1987 as a research scholar under the guidance of Prof. K. S. Shukla, and was awarded Ph.D. degree (History of Mathematics) in 1992. Prof. Shukla taught me how to read Sanskrit and Hindi texts on mathematics and astronomy usually in his house. At that time, I was young, and travelled several places in India to collect source materials. Photocopy was not so popular in India at that time, and I took several photographs of manuscripts by my camera. The most important manuscripts for me were in Lucknow University itself. There was a collection of Sanskrit manuscripts in the Department of Mathematics and Astronomy. And also Tagore Library (central library of Lucknow University) had several Sanskrit and Persian manuscripts.

I started to read the *Yantra-rāja-adhikāra* (Chap. 1 of the *Yantra-kiraṇāvalī*) of Padmanābha, of which two manuscripts are preserved in the Department and Tagore Library. I published it in my paper "Early History of the Astrolabe in India", *Indian Journal of History of Science*, 32(3), 1997, 199–295. Though I could take pictures of the both, but it was much better to copy them from the original manuscripts directly. There are some corrections in manuscripts which cannot be seen clearly by pictures or photocopies. So, I visited Tagore Library several times to copy the manuscripts, and then visited Prof. Shukla's house. At that time, the primary mode of transport to go to Prof. Shukla's house was cycle rickshaw. (By the way, the Indian word "rickshaw" is originated in a Japanese word "jin-riki-sha", which means human-

* Yukio Ohashi, 2018 (Invited contribution, specially for this volume).

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powered transport. "Jin" in Japanese means human, "riki" means force or power, "sha" means car. If it is driven by bicycle, it is called "cycle rickshaw", and if it is driven by motor cycle, it is called "auto rickshaw".)

Some of the manuscripts in the Department are modern manuscripts written on foolscap. How did they come into being? There are several Sanskrit manuscripts written in regional scripts in several places in India. When photocopy could not be used, it was necessary to copy by hand, and it was convenient to read if it was copied in Nāgarī script. So, there are several modern manuscripts in Nāgarī script on foolscap. Still now, there must be several unstudied manuscripts in regional scripts or regional languages in local libraries etc. They are worth studying, but I, an old foreigner, cannot engage myself anymore to study them. I hope some young Indian researchers do research on the regional development of mathematics and astronomy in India!

Lucknow is a beautiful city with several historical sites. It was formerly "Awadh" ruled by Nawabs. It has a tradition of delicious cuisine. And also, Bhatkhande Music Institute was situated nearby Lucknow University, and I could enjoy Indian classical music several times. Mr. Ratan Shukla, a son of Prof. Shukla, was kind enough to take me to several historical places, and sometimes we enjoyed taking "biryani".

When I was in Lucknow, the *Vaṭeśvara-siddhānta and Gola of Vaṭeśvara* (2 vols., published by INSA in 1985–1986) was being printed at a press in Lucknow, and Prof. Shukla visited the press several times. I also helped in its proof reading. It was printed by traditional letterpress printing, which has become rare now. So, proof reading was quite hard at that time, but Prof. Shukla read proof sheets very carefully at the press itself.

In 1985, General Assembly of International Astronomical Union was held in New Delhi, and I went to New Delhi with Prof. Shukla by train in order to attend its colloquium on the history of oriental astronomy. It was my only one experience of travel with Prof. Shukla. The colloquium was held in INSA (Indian National Science Academy), and we met several specialists of oriental astronomy from many parts of the world.

In 1987, I returned Japan, and continued to write my Ph.D. thesis. I submitted it to Lucknow University in 1991, and was awarded Ph.D. degree in 1992. My life in India was my most exciting period. Now I am nearing the age of retirement. I was so lucky that I could study in India as a student of Prof. Shukla.

Though I am writing this paper as his student, I am one of you. We shall study the history of Indian astronomy and mathematics, and exchange information. I am old, but I would very much wish to correspond with young researchers!

Dr. Kripa Shankar Shukla, veteran historian of Hindu astronomy and mathematics *

Kripa Shankar Shukla's birth took place at Lucknow on July 10, 1918. From the very early years, he was a brilliant student of Mathematics and Sanskrit. He passed the High School Examination of U.P. Board in 1934 in First Division with Distinction in Mathematics and Sanskrit and the Intermediate Examination of that Board again in First Division with Distinction in Mathematics.

He had his higher education at Allahabad, passing the B.A. examination in the second division from Allahabad University in 1938. From the same University, he obtained his Master of Arts degree in Mathematics in the First Division in 1941. During his M.A. studies in Allahabad, Paṇḍit Devi Datta Shukla (editor of the Hindi monthly *Sarasvatī*) greatly helped K. S. Shukla by arranging the latter's regular meals in his own house. D. D. Shukla regarded K. S. Shukla like his own son and taught him the full *pūjā-paddhati* (ritual worship) of Śrī Bālā Devī.

Dr. Avadhesh Narain (or Narayan) Singh (1905–1954), a student of Prof. Ganesh Prasad, was quite enthusiastic about the study of history of mathematics and was associated with Dr. B. B. Datta (1888–1958) in that field. The *History of Hindu Mathematics*, Part II, by Datta and Singh, was published in 1938 from Lahore (then in India). Dr. Singh, although still a Lecturer in the Department of Mathematics and Astronomy, Lucknow University, was very sincerely interested in promoting the study of history of Indian mathematics. In 1939 he started a Scheme of Research in Hindu Mathematics in the Department. Dr. Oudh (i.e., Avadha) Upadhyaya (1894–1941) who had just returned from France with a D.Sc. (Math.), was appointed in the Scheme (see P. D. Shukla's note on Upadhyaya in *Proc. Benaras Math. Soc. N.S.*, *III*, 95–98).

Dr. K. S. Shukla joined the Department and the Scheme in 1941 and his whole-hearted devotion in the field of study and research in ancient Indian astronomy and mathematics proved very fruitful. His very first research paper on "The Eviction and Deficit of Moon's Equation of Centre" (1945) showed his talent. He concentrated more in studying the works of Bhāskara I, a follower

* Radha Charan Gupta, *Gaṇita Bhāratī*, Vol. 20, Nos. 1–4 (1998), pp. 1–7.

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(but not a direct pupil) of Āryabhaṭa I (born AD 476). As early as in 1950, Dr. Shukla studied Bhāskara I's commentary (AD 629) on the *Āryabhaṭīya* and prepared a full Hindi translation of it (see Introduction, p. cxiii, in Shukla's 1976 edition of the commentary).

Dr. Shukla investigated thoroughly the works of Bhāskara I and studied other relevant primary and secondary material. Under the supervision of Dr. A. N. Singh, Shukla prepared a thesis on "Astronomy in the Seventh Century India: Bhāskara I and His Works". But Dr. Singh died before the Lucknow University awarded the D.Litt. degree on the thesis to Dr. Shukla in 1955. Perhaps by divine plan Singh's death occurred on July 10 which is the date of Shukla's birth in Gregorian Calendar.

Shukla's doctoral thesis was in four parts: (i) Introduction; (ii) Edition and Translation of the *Mahābhāskarīya*; (iii) Edition and Translation of the *Laghubhāskarīya*; and (iv) Bhāskara I's commentary on the *Āryabhaṭīya* with English Translation of *Āryabhaṭīya*. The significance of the thesis lies not only in providing a genuine additional source for the history of early Indian exact sciences but also in bringing to light many new historical and methodological facts. By now most of the material from the thesis has been published in various forms.

In fact, Dr. Shukla proved to be a worthy successor in carrying on the study and research in the field of Hindu astronomy and mathematics. With the help of research assistants like Markandeya Mishra, Dr. Shukla brought out the editions of several Sanskrit texts which were published under the "Hindu Astronomical and Mathematical Texts Series" (= *HAMTS*) of the Department of Mathematics and Astronomy of Lucknow University. Dr. Shukla supervised the research work of a number of theses. Under his guidance the following scholars got their doctoral degree.

(i) Usha Asthana, *Ācārya Śrīdhara and His Triśatikā* (Lucknow University, 1960) (She started her research under A. N. Singh's guidance).

(ii) Mukut Bihari Lal Agrawal, *Contribution of Jaina Ācāryas in the development of mathematics and astronomy* (in Hindi) (Agra Univ. 1973).

(iii) Paramanand Singh, *A Critical Study of the Contributions of Nārāyaṇa Paṇḍita to Hindu Mathematics* (Bihar Univ. 1978).

(iv) Loknath Sharma, *A study of Vedāṅga Jyotiṣa* (L. N. Mithila Univ. 1984).

(v) Yukio Ohashi, *A History of Astronomical Instruments in India* (Lucknow Univ. 1992).

After serving the Lucknow University department with distinction for 38 years, Professor Shukla retired formally under rules on June 30, 1979. But he continued his outstanding and creative works actively in his cherished field for many more years and scholars still continue to get ideas, suggestions, and encouragement from him. One of the tasks he completed after retirement was to bring out a revised edition of the manuscript of Part III of Datta and Singh's *History of Hindu Mathematics*. The manuscript was lying with Dr. Shukla since long (see *Gaṇita Bhāratī*, Vol. 10, 1988, pp. 8–9) but now he found time to publish it in the form of a series of eight articles on Geometry, Trigonometry, Calculus, Magic Squares, Permutations and Combinations, Series, Surds, and Approximate Values of Surds in the *IJHS* Vols. 15 (1980), 121–188; 18 (1983), 39–108; 19 (1984), 95–104; 27 (1992), 51–120; 231–249; and 28 (1993), 103–129; 253–264; 265–275 respectively. It is unfortunate that parts I and II of *HHM* were reprinted (Bombay, 1962) without any revision. Anyway, there is an urgent national need to bring out a consolidated edition of all the three parts possibly after making them up-to-date, and also to take up the writing of a national history of mathematics in India as team work.

Working wholeheartedly with single minded devotion for more than half a century, Dr. Shukla's contribution in the field of history of ancient and medieval Indian mathematics forms a pioneer work which will continue to motivate future research and investigations. He gave new interpretations of many obscure Sanskrit passages and corrected misinterpretations and other errors committed by others. He has worked diligently and is proud of India's scientific heritage. He has been working silently without caring for publicity. Yet he is greatly reputed for his in depth research among the scholars, and the merit of his work is widely recognised as shown by various citations.

Dr. Shukla was awarded the Banerji Research Prize of the Lucknow University. He was associated with the editorial work of the Journal *Gaṇita* of the Bhārata Gaṇita Pariṣad (formerly the Benaras Mathematical Society) for many years. He was elected Fellow of the National Academy of Sciences, India in 1984, and the Corresponding Member of the International Academy of History of Science, Paris, in 1988. He served as a member of several national and international committees.

As a student of the Lucknow University, the writer of the present article (RCG) attended B.Sc. and M.Sc. courses in the Department of Mathematics and Astronomy during 1953–1957; and Dr. Shukla taught him the subject of a paper in M.Sc. Part I. But there was no course available in History of Mathematics or Hindu Mathematics then (and even now). It is a tragedy that our educational set up is deficient in this respect. A course in the history (in wide sense) of any subject should form a part of postgraduate curriculum to justify the award of "Master's" title in that subject. It is also hoped that the glorious tradition of study and research in the field of ancient Indian Mathematics and Astronomy will be maintained in the concerned Lucknow University Department.

A preliminary note on Dr. Shukla's work appeared in "Two Great Scholars", *Gaṇita Bhāratī*, 12 (1990), 39–44 and Dr. Yukio Ohashi discussed "Prof. Shukla's contribution to the study of history of Hindu astronomy", in the same journal, Vol. 17 (1995, 29–44). The present article is a humble tribute and felicitation on the occasion of the 80th birth-anniversary of respected Shuklaji. May God grant him best health, happiness and long life.

Dr. K. S. Shukla's publications

(I) Edited, translated and other books

1. *Hindu Gaṇita-Śāstra kā Itihāsa* being a Hindi translation of B. B. Datta and A. N. Singh's *History of Hindu Mathematics Part I* (Lahore 1935), Hindi Samiti, Lucknow, 1956. Reprinted many times.

2. *The Sūrya-siddhānta with the commentary of Parameśvara* (1431). Edited with an introduction in English. *HAMTS* No. 1, Lucknow, 1957.

3. *Pāṭīgaṇita of Śrīdharācārya* edited with an ancient commentary, introduction, and English translation. *HAMTS* No. 2, Lucknow, 1959.

4. *Mahābhāskarīya* (of Bhāskara I) edited with introduction and translation. *HAMTS* No. 3, Lucknow, 1960.

5. *Laghubhāskarīya* (of Bhāskara I) edited with introduction and translation. *HAMTS* No. 4, Lucknow, 1963.

6. *Dhīkoṭida-karaṇa* (of Śrīpati) edited with introduction and translation. Akhila Bharatiya Sanskrit Parishad, Lucknow, 1969.

7. *Bījagaṇitāvataṃsa* (of Nārāyaṇa Paṇḍita) edited with introduction. Akhila Bharatiya Sanskrit Parishad, Lucknow, 1970.

8. *Āryabhaṭa, Indian Astronomer and Mathematician* (5th century). INSA, New Delhi, 1976.

9. *The Āryabhaṭīya of Āryabhaṭa I* edited (in collaboration with K. V. Sarma) with introduction and translation. INSA, New Delhi, 1976.

10. *The Āryabhaṭīya* with the commentary of Bhāskara I (629 AD) and Someśvara, edited with introduction and appendices. INSA, New Delhi, 1976.

11. *Karaṇa-ratna of Devācārya* (689 AD) edited with introduction and translation, *HAMTS* No. 5, Lucknow, 1979.

12. Late Bina Chatterjee's edition and translation of *Lalla's Śiṣyadhīvṛddhida Tantra* completed and edited. Two volumes, INSA, New Delhi, 1981 (Chatterjee's edition contains the commentary of Mallikārjuna Sūri in Vol. 1 and 17 appendices after the translation in Vol. 2).

13. *Vaṭeśvara Siddhānta and Gola* edited with introduction and translation. Part I (text) and Part II (translation), INSA, New Delhi, 1985–1986.

14. *History of Astronomy in India* edited by S. N. Sen and K. S. Shukla, INSA, New Delhi, 1985 (also issued as *IJHS* Vol. 20).

15. *History of Oriental Astronomy* edited by G. Swarup, A. K. Bag and K. S. Shukla, Cambridge Univ. Press, Cambridge, 1987 (The book constitutes Proceedings of IAU Colloquium No. 91, New Delhi, 1985).

16. *A Critical Study of Laghumānasa of Mañjula* (with edition and translation of the text). INSA, New Delhi, 1990. (It was issued as supplement to IJHS, Vol. 25).

17. *A Text book on Algebra* (for B.A. and B.Sc.) by K. S. Shukla and R. P. Agarwal, Kanpur, 1959.†

18. *A Text book on Trigonometry* (for B.A. and B.Sc.) by K. S. Shukla and R. S. Verma, Allahabad, 1951.

19. *Avakalan Gaṇita* (in Hindi) by M. D. Upadhyay, revised by K. S. Shukla, Hindi Sansthan, Lucknow, 1980.

(II) Research papers and other articles

1. "The evection and the deficit of the equation of the centre of the Moon in Hindu Astronomy". *Proc. Benaras Math. Soc. (N. S.)*, 7(2) (1945), 9–28.

2. "On Śrīdhara's rational solution of $Nx^2 + 1 = y^2$ ". *Gaṇita*, I(2) (1950), 1–12.

3. "Chronology of Hindu Achievements in Astronomy". *Proc. National Inst. Sci. India*, 18(4) (July-August 1952), 336–337 (Summary of a 1950 symposium paper).

4. "The *Pāṭīgaṇita* of Śrīdharācārya" (in Hindi). *Jñānaśikhā* (Lucknow), 2(1) (October 1951), 21–38.

5. "Ācārya Jayadeva, the mathematician". *Gaṇita*, 5(1) (1954), 1–20.

† Information about text-books (serial No. 17, 18, 19) has been provided by Shri Ratan Shukla (son of KSS).

6. "On the three stanzas from the *Pañca-siddhāntikā* of Varāhamihira". *Gaṇita*, 5(2) (1954), 129–136.

7. "A note on the *Rājamṛgāṅka* of Bhoja published by the Adyar Library". *Ibid.*, 149–151.

8. "Indian Geometry" (in Hindi). *Svatantra-Bhārata* (Lucknow), dated 24 November 1957, pp. 1 and 11.

9. "Hindu methods of finding factors or divisors of number". *Gaṇita*, 17(2) (1966), 109–117.

10. "Ācārya Āryabhaṭa's *Ārdharātrika-Tantra*" (in Hindi). *C. B. Gupta Abhinandana Grantha*, New Delhi, 1966, 483–494.

11. "Āryabhaṭa I's astronomy with midnight day reckoning". *Gaṇita*, 18(1) (1967), 83–105.

12. "Early Hindu methods in spherical astronomy". *Ganita*, 19(2) (1968), 49–72.

13. "Astronomy in ancient and medieval India". *IJHS*, 4 (1969), 99–106. (*cf.* no. 15 below).

14. "Hindu mathematics in the seventh century AD as found in Bhāskara I's commentary on the *Āryabhaṭīya*". *Gaṇita*, 22(1) (1971), 115–130; 22(2) (1971), 61–78; 23(1) (1972), 57–79; and 23(2), 41–50.

15. "Ancient and medieval Hindu astronomy" (in Hindi). *Jyotish-Kalp* (Lucknow), 3(6) (March 1972), 32–37. (*cf.* no. 13).

16. "Characteristic features of the six Indian seasons as described by astronomer Vaṭeśvara". *Jyotish-Kalp*, 3(11) (August 1972) 65–74.

17. "Hindu astronomer Vaṭeśvara and his works". *Ganita*, 23(2) (1972), 65–74.

18. "Use of hypotenuse in the computation of the equation of the centre under the epicyclic theory in the school of Āryabhaṭa". *IJHS*, 8 (1973), 43–57.

19. "The *Pañca-siddhāntikā* of Varāhamihira (I)". *Gaṇita*, 24(1) (1973), 59–73; also same in *IJHS*, 8 (1974), 62–76. (*cf.* no. 22 below)

20. "Āryabhaṭa". In *Cultural Leaders of India: Scientists* (edited by V. Raghavan), Ministry of Information and Broadcasting, Delhi, 1976, reprinted 1981, pp. 83–99.

21. "Astronomy in India before Āryabhaṭa". Paper read at the Symposium on Hindu Astronomy, Lucknow, 1976, 11 pages (cyclostyled).

22. "The *Pañca-siddhāntikā* of Varāhamihira (II)". *Gaṇita*, 28 (1977), 99–116. (*cf.* no. 19).

23. "Glimpses from the *Āryabhaṭa-siddhānta*". *IJHS*, 12 (1977), 181–186.

24. "Series with fractional number of terms". *Bhāratī Bhānam* (K. V. Sarma Felicitation Volume) = *Vishveshvaranand Indolog. Jour.*, 18 (1980), 475–481.

25. "Astronomy in ancient India". In *Bhāratīya Saṃskriti*, Bharatīya Saṃskriti Saṃsad, Calcutta, 1982, pp. 440–453.

26. "A note on R. P. Mercier's review of *Karaṇaratna of Devācārya*". *Gaṇita Bhāratī*, 6 (1984), 25–28.

27. "Phases of the Moon, rising and setting of planets and stars and their conjunctions". *IJHS*, 20 (1985), 212–251.

28. "Main characteristics and achievements of ancient Indian astronomy in historical perspective". In *History of Oriental Astronomy* (edited by G. Swarup et al.), Cambridge 1987, 7–22.

29. "The Yuga of the *Yavana-jātaka*: David Pingree's text and translation reviewed". *IJHS*. 24 (1989), 211–223.

30. "*Vedic Mathematics*: The illusive title of Swamiji's book". *Mathematical Education* 5(3) (1989), 129–133. (*cf.* next item)

31. "*Vedic Mathematics*: The deceptive title of Swamiji's book". Pages 31–39 in *Issues in Vedic Mathematics* (edited by H. C. Khare), Delhi, 1991.

32. "Graphic methods and astronomical instruments" being translation (with notes) of Chapter XIV of the *Pañcasiddhāntikā* of Varāhamihira. Pages 261–281 in K. V. Sarma's edition of *Pañcasiddhāntikā with Translation of T. S. Kuppanna Sastry*, Madras 1993.

(III) Book reviews

1. Review of the *Pañcasiddhāntikā* of Varāhamihira (ed. by O. Neugebauer and D. Pingree, Two Parts, Copenhagen), 1970–1971. Journal of the American Oriental Society 93(3) (1973), p. 386.

2. Review of *Census of Exact Sciences in Sanskrit Series A*, Vol. 3, (by D. Pingree, Philadelphia, 1976) *IJHS*, 13, (1978), 72–73.

3. Review of *Candracchāyāgaṇitam* of Nīlakaṇṭha Somayājī, (edited by K. V. Sarma, Hoshiarpur, 1976) *IJHS*, 13, (1978), p. 73.

4. Review of *Siddhānta-darpaṇam* of Nīlakaṇṭha Somayājī, (edited by K. V. Sarma, Hoshiarpur, 1976), *IJHS*, 13 (1978), p. 73–74.

5. Review of *Rāśigolasphuṭanītiḥ* of Acyuta Piṣāraṭi, (in Hindi) ed. by K. V. Sarma, Hoshiarpur, 1977, *IJHS*, p. 74.

6. Review of A. K. Bag, *Mathematics in Ancient and Medieval India* (Varanasi, 1979), *Gaṇita Bhāratī*, 3 (1981), 107–108.

7. Review of R. C. Pandeya (editor), *Grahalāghavaṃ Karaṇam* (Parts 1 and 2, Jammu, 1976 and 1977), *Ibid.*, 108–109.

8. Review of *Census of Exact Sciences in Sanskrit Series A*, Vol. 4, (by D. Pingree, Philadelphia, 1981). *Jour. Hist. Astron.* Vol. 13 (1982), 225–226. Also *IJHS*, 18 (1983), 221–222.

9. Review of *'Prāchīn Bhārat Mein Vijñān'* (in Hindi) (by S. L. Dhani, Panchkula, 1982). *IJHS*, 19 (1984), 86–87.

10. Review of Rahman, A. et al., *Science and Technology in Medieval India - A Bibliography of Source Materials in Sanskrit, Arabic and Persian* (INSA, New Delhi, 1982). *IJHS*, 19 (1984), 412–413.

Obituary: Kripa Shankar Shukla (1918–2007) *

1 Reminiscence

Professor Kripa Shankar Shukla was born on July 10, 1918 at Lucknow. He obtained M.A. degree from Allahabad University in 1941. After obtaining M.A. degree, he joined Lucknow University as a lecturer of mathematics. He was awarded D.Litt. degree for his work "Astronomy in the seventh century in India: Bhāskara I and his works" by Lucknow University in 1955. He retired from Lucknow University as a Professor of Mathematics in 1979.

Systematic study of the history of Indian astronomy and mathematics was carried on by Sankar Balakrishna Dikshit (1853–1898), Sudhakara Dvivedin (1860–1922) etc. around the end of the 19th century, and also by Prabodh Chandra Sengupta (1876–1962), Bibhutibhusan Datta (1888–1958), Avadhesh Narayan Singh (1901–1954) etc. around the first half of the 20th Century. The monumental work *History of Hindu Mathematics* (Vol. 1, 1935; Vol. 2, 1938) of Datta and Singh is well known. Prof. Shukla succeeded this work, revised the draft of its subsequent parts left by Datta and Singh, and published them in the *Indian Journal of History of Science* (1980–1993).

A. N. Singh was appointed to be a full time lecturer in mathematics at Lucknow University in 1928, and, after the publication of the *History of Hindu Mathematics*, started a project to study original sources of Indian astronomy and mathematics in the Department of Mathematics and Astronomy, Lucknow University in 1939. A. N. Singh collected several manuscripts etc. of the original sources, which are preserved in the Department now. Prof. Shukla joined this project in 1941, and succeeded it. His D.Litt. dissertation is one of its results. Prof. Shukla published several critical editions of original sources from the Department (and also from Indian National Science Academy etc. later). Some of them are accompanied by lucid English translation with detailed mathematical notes. Prof. Shukla sometimes collaborated with K. V. Sarma (1919–2005), who made a great contribution to the study of the history of Kerala astronomy.

I remember that Prof. Shukla did his best to make his mathematical notes easy to understand. When we read his notes, we feel as if we are being taught by him directly. It should also be mentioned that he noted several parallel

* Yukio Ohashi, *Indian Journal of History of Science*, Vol. 43, No. 3 (2008), pp. 475–485.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_3

statements in other Sanskrit texts in the footnotes of his English translations. So, his English translations can also be used as a kind of annotated index of Sanskrit astronomical and mathematical texts. Only Prof. Shukla could do this.

When I was in Lucknow, the *Vaṭeśvara Siddhānta and Gola of Vaṭeśvara* was being printed at a press in Lucknow. Prof. Shukla visited the press almost every day, supervised its printing work by himself, and read its proofs very carefully. From this fact, we can understand why his edition is so reliable. These original sources are the most important foundation for future research.

I studied the history of Indian astronomy and mathematics under the guidance of Prof. Shukla from 1983 to 1987 as a research scholar (Ph. D. student) of Lucknow University. It was my first experience to live abroad, and was the most exciting period during my life. Prof. Shukla already had retired but kindly taught me how to read Sanskrit astronomical texts, both printed texts and manuscripts. I saw several people were visiting Lucknow to meet Prof. Shukla. Prof. Shukla did not create so called "school". It means that his works are open to everybody. Even if you have not met him, you can become his student by studying his works.

Prof. Shukla passed away on September 22, 2007, but we have to continue to study Indian astronomy and mathematics further. We should read his works again and again, and try to develop the study.

In the course of compiling the following list, I received valuable information and/or warmhearted encouragement from Dr. A. K. Bag, Dr. S. M. R. Ansari, Dr. Takao Hayashi, Dr. Sunil Datta, Mr. Ratan Shukla (son of Prof. K. S. Shukla), and several other people. I am grateful to all of them.

2 List of K. S. Shukla's works

Publication of original sources

The *Sūrya-siddhānta with the commentary of Parameśvara*, (Hindu Astronomical and Mathematical Texts Series No. 1), Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1957. [Note: The *Sūrya-siddhānta* is very popular in India, but most of its printed editions are based on Raṅganātha's version (1603 AD). Prof. Shukla's edition is the first publication of its earlier version commented by Parameśvara (1432 AD). The readings of some other early versions are also shown in its footnotes. In this publication, only Sanskrit text is given without English translation, but a detailed introduction is given.]

The *Pāṭī-gaṇita of Śrīdharācārya*, (Hindu Astronomical and Mathematical Texts Series No. 2), Department of Mathematics and Astronomy, Lucknow

University, Lucknow, 1959. [Note: In this publication, Sanskrit text and English translation are given.]

Mahā-Bhāskarīya, Bhāskara I and his works, Part II, (Hindu Astronomical and Mathematical Texts Series No. 3), Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1960. [Note: In this publication, Sanskrit text and English translation are given. In its introduction, it was announced that the "Bhāskara I and his works" would have been divided into 4 parts. Only parts II and III are published in Lucknow. The proposed Part IV (Bhāskara I's commentary on the *Āryabhaṭīya* of Āryabhaṭa I) was later published as the "*Āryabhaṭīya* Critical Edition Series, Part 2" in New Delhi in 1976 (see below). This (1976) edition's introduction may be considered to be the proposed Part I (General introduction).]

Laghu-Bhāskarīya, Bhāskara I and his works, Part III, (Hindu Astronomical and Mathematical Texts Series No. 4), Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1963. [Note: In this publication, Sanskrit text and English translation are given.]

The Dhīkoṭida-Karaṇa of Śrīpati, (originally published in *Ṛtam* 1, 1969); Separately issued: Akhila Bharatiya Sanskrit Parishad, Lucknow, 1969. [Note: In this publication, Sanskrit text and English translation are given.]

Nārāyaṇa Paṇḍita's Bījagaṇitāvataṃsa, Part I, (originally published in *Ṛtam* 1, 1969/70); Separately issued: Akhila Bharatiya Sanskrit Parishad, Lucknow, 1970. [Note: In this publication, only Sanskrit text is given without English translation. Part II of this work was not available in its complete form in the manuscript used, and only its fragment has been appended in this publication.]

Āryabhaṭīya of Āryabhaṭa, critically edited with translation and notes, in collaboration with K. V. Sarma, (*Āryabhaṭīya* Critical Edition Series, Part 1), Indian National Science Academy, New Delhi, 1976. [Note: In this publication, Sanskrit text and English translation are given. This series was published on the occasion of the celebration of the 1500th birth anniversary of Āryabhaṭa on 2nd November, 1976.]

Āryabhaṭīya of Āryabhaṭa, with the commentary of Bhāskara I and Someśvara, (*Āryabhaṭīya* Critical Edition Series, Part 2), Indian National Science Academy, New Delhi, 1976. [Note: In this publication, only Sanskrit text is given without English translation, but a detailed introduction is given. The "*Āryabhaṭīya* Critical Edition Series" consists of 3 parts. Its Part 3 "*Āryabhaṭīya of Aryabhaṭa*, with the commentary of Sūryadeva Yajvan" was edited by K. V. Sarma and published in the same year.]

The Karaṇa-ratna of Devācārya, (Hindu Astronomical and Mathematical Texts Series No. 5), Department of Mathematics and Astronomy, Lucknow University, Lucknow, 1979. [Note: In this publication, Sanskrit text and English translation are given.]

Vaṭeśvara-siddhānta and Gola of Vaṭeśvara, 2 parts, Indian National Science Academy, New Delhi, 1985–1986. [Note: Its Part 1 is Sanskrit text, and Part 2 is English translation.]

A Critical Study of the Laghumānasa of Mañjula, Indian Journal of History of Science, Vol. 25, 1990, Supplement; also separately issued, Indian National Science Academy, New Delhi, 1990. [Note: In this publication, Sanskrit text and English translation are given].

Handbook

Āryabhaṭa: Indian Mathematician and Astronomer (5th Century AD). Indian National Science Academy, New Delhi, 1976. [Note: This handbook is a kind of general introduction to the "*Āryabhaṭīya* Critical Edition Series" published in the same year. The "*Āryabhaṭīya* Critical Edition Series, Part 1" itself also has a detailed introduction].

Research papers

"The evection and the deficit of the equation of the centre of the Moon in Hindu Astronomy", *Proceedings of the Benares Mathematical Society*, New Series 7.2 (1945) 9–28. [Note: The *Proceedings of the Benares Mathematical Society* was succeeded by the *Gaṇita*. See below.]

"On Śrīdhara's rational solution of $Nx^2+1=y^2$", *Gaṇita*, 1.2 (1950), 53–64. [Note: Regarding this paper, also see his edition of the *Pāṭī-gaṇita* (1959), Introduction, p. xxxii, footnote 1. The *Gaṇita* is a journal published by Bhārata Gaṇita Pariṣad, Department of Mathematics, Lucknow University.]

"Chronology of Hindu achievements in astronomy", *Proceedings of the National Institute of Sciences of India*, 18.4 (1952), 336–337. [Note: This is a paper read at the "Symposium on the History of Sciences in South Asia" held in Delhi. It can be said that the modern study of the history of Indian science made great progress since this symposium. The National Institute of Sciences of India started to publish the *Indian Journal of History of Science* in 1966. The National Institute of Sciences of India is the present Indian National Science Academy.]

"Ācārya Jayadeva, the Mathematician", *Gaṇita*, 5.1 (1954), 1–20.

"On three stanzas from the *Pañca-siddhāntikā*", *Gaṇita*, 5.2 (1954), 129–136.

"A note on the *Rāja-mṛgāṅka* of Bhoja published by the Adyar Library", *Gaṇita*, 5.2 (1954), 149–151.

"Series with fractional number of terms", *Mathematical and Statistical Association Magazine*, (Lucknow University), 1 (1958), 30–38.

"Hindu methods for finding factors or divisors of a number", *Gaṇita*, 17.2 (1966), 109–117.

"Āryabhaṭa I's astronomy with midnight day-reckoning", *Gaṇita*, 18.1 (1967), 83–105.

"Early Hindu methods in spherical astronomy", *Gaṇita*, 19.2 (1968), 49–72.

"Astronomy in ancient and medieval India", *Indian Journal of History of Science*, 4.1–2 (1969), 99–106. [Note: This volume is the collection of papers presented at the "Symposium on the History of Sciences of Ancient and Medieval India", held in Delhi, 1968.]

"Hindu Mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya*", *Gaṇita*, 22.1 (1971), 115–130; 22.2 (1971), 61–78; 23.1 (1972), 57–59; 23.2 (1972), 41–50.

"Hindu astronomer Vaṭeśvara and his works", *Gaṇita*, 23.2 (1972), 65–74.

"Characteristic features of the six Indian seasons as described by astronomer Vaṭeśvara", *Jyotiṣa-kalpa*, 3.4 (1972), 43–47. [I have an off-print of this paper, and the volume number is mentioned as "*Varṣa* 3, *aṅka* 4", but the year of publication is not mentioned there. R. C. Gupta's list (1998) (see the section of references below) has this paper, and mentions its volumes and year as "3.11 (Aug 1972)". They are inconsistent, but I tentatively use this year of publication.]

"Use of hypotenuse in the computation of the equation of the centre under the epicyclic theory in the school of Āryabhaṭa I ???", *Indian Journal of History of Science*, 8 (1973), 43–57. [Note: This is a paper to refute the assertion of T. S. Kupanna Shastry.]

"The *Pañca-siddhāntikā* of Varāhamihira (1)", *Gaṇita*, 24.1 (1973), 59–73; and also in the *Indian Journal of History of Science*, 9.1 (1974), 62–76. [Note: This is a paper read at the Seminar organised by the Indian National Science Academy, New Delhi, on the occasion of the 500th Birth Anniversary of Nicolaus Copernicus on February 19–20, 1973. The paper published in the *Indian Journal of History of Science* has a short "Note" at the end which is not included in the paper in the *Gaṇita*.]

"Āryabhaṭa", in *Cultural Leaders of India, Scientists*, Publications Division (Government of India), New Delhi, 1976, pp. 83–99.

"The *Pañca-siddhāntikā* of Varāhamihira (2)", *Gaṇita*, 28 (1977), 99–116.

"Glimpses from the *Āryabhaṭa-siddhānta*", *Indian Journal of History of Science*, 12.2 (1977), 181–186. (This issue is the Proceedings of the Symposium on the 1500th Birth Anniversary of Āryabhaṭa I held in New Delhi, November 2–4, 1976).

"Series with fractional number of terms", in S. Bhaskaran Nair (ed.): *Bhāratībhānam* (Light of Indology), Dr. K. V. Sarma Felicitation Volume, Panjab University Indological Series 26, Vishveshvaranand Vishva Bandhu Institute of Sanskrit and Indological Studies, Hoshiarpur, 1980, pp. 475–481.

"Astronomy in ancient India", in *Bhāratīya Samskriti*, Bhāratīya Samskriti Samsad, Calcutta, 1982, pp. 440–453. [Note: I have not seen this paper, but R. C. Gupta's list (1998) (see the section of references below) mentions this paper.]

"A note on Raymond P. Mercier's review of *Karaṇaratna* of Devācārya", *Gaṇita Bhāratī*, 6 (1984), 25–28.

"Phases of the Moon, rising and setting of planets and stars and their conjunctions", in S. N. Sen and K. S. Shukla (eds.): *History of Astronomy in India*, (originally published in *Indian Journal of History of Science*, Vol. 20, 1985), Indian National Science Academy, New Delhi, 1985, pp. 212–251; Second Revised Edition, 2000, pp. 236–275.

"Main characteristics and achievements of ancient Indian astronomy in historical perspective", in G. Swarup, A. K. Bag and K. S. Shukla (eds.): *History of Oriental Astronomy*, (Proceedings of an International Astronomical Union Colloquium No. 91, New Delhi, India, 13–16 November 1985), Cambridge University Press, Cambridge, 1987, pp. 9–22.

"The Yuga of the *Yavanajātaka*, David Pingree's text and translation reviewed", *Indian Journal of History of Science*, 24.4 (1989), 211–223.

"Vedic Mathematics—The illusive title of Swamiji's book", *Mathematical Education*, 5.3 (1989), 129–133. [Note: This is a paper read at the National Workshop on Vedic Mathematics held at University of Rajasthan, Jaipur, 1988. The same paper was also published as follows.]

"Vedic Mathematics—The deceptive title of Swamiji's book", in *Issues in Vedic Mathematics*, Proceedings of the National Workshop on Vedic Mathematics, 25–28 March, 1988, at the University of Rajasthan, Jaipur, Maharshi

Sandipani Rashtriya Veda Vidya Pratishthan, Ujjain, (in association with Motilal Banarsidass, Delhi), 1991, (reprinted: 1994), pp. 31–39.

"Magic squares in Indian mathematics", in *Interaction between Indian and Central Asian Science and Technology in Medieval Times*, Vol. 1, (Indo-Soviet Joint Monograph Series), Indian National Science Academy, New Delhi, 1990, pp. 249–270.

Hindi papers

[Note: The following Hindi papers are listed in R. C. Gupta's paper (1998) (see the section of references below). They may be incorporated in the above list of research papers, but I quote them separately from Gupta's paper, because I have not seen them, and it is not possible at present to ascertain their original Hindi title.]

"The *Pāṭīgaṇita* of Śrīdharācārya", *Jñanaśikhā* (Lucknow), 2.1 (1951), 21-38.

"Indian Geometry", *Svatantra-Bhārata* (Lucknow), 1957, pp. 1 and 11.

"Ācārya Āryabhaṭa's *Ardharātrika-Tantra*", *C. B. Gupta Abhinandana Grantha*, New Delhi, 1966, pp. 483–494. [Note: According to the bibliography of Prof. Shukla's edition and English translation (1976) of the *Āryabhaṭīya*, this is a Hindi version of his English paper "Āryabhaṭa I's astronomy with midnight day-reckoning" (1967).]

"Ancient and medieval Hindu astronomy", *Jyotiṣa-kalpa* (Lucknow), 3.6, (1972), 32–37.

Book reviews

O. Neugebauer and D. Pingree, *The Pañcasiddhāntikā of Varahamihira*, Kopenhagen, 1970–1971, in *Journal of the American Oriental Society*, 93.3 (1973), 386.

David Pingree, *Census of the Exact Sciences in Sanskrit*, Series A, Vol. 3, Philadelphia, 1976, in *Indian Journal of History of Science*, 13.1 (1978), 72–73.

K. V. Sarma (ed.), *Cañdracchāyāgaṇitam* of Nīlakaṇṭha Somayājī, Hoshiarpur, 1976; *Siddhāntadarpaṇam* of Nīlakaṇṭha Somayājī, Hoshiarpur, 1976; *Rāśigolasphuṭanīti* of Acyuta Piṣāraṭi, Hoshiarpur, 1977, in *Indian Journal of History of Science*, 13.1 (1978), 73–74.

A. K. Bag, *Mathematics in Ancient and Medieval India*, Delhi, 1979, in *Gaṇita Bhāratī*, 3.3–4 (1981), 107–108.

R. C. Pandeya, *Grahalāghavam Karaṇam*, 2 parts, Jammu, 1976–77, in *Gaṇita Bhāratī*, 3.3–4 (1981), 108–109.

David Pingree, *Census of the Exact Sciences in Sanskrit*, Series A, Vol. 4, Philadelphia, 1981, in *Journal of the History of Astronomy*, 13.3 (1982), 225–226; and also *Indian Journal of History of Science*, 18.2 (1983), 221-222.

S. L. Dhani, *Prācīn Bhārat men Vijñan*, Panchkula, 1982, in *Indian Journal of History of Science*, 19.1 (1984), 86–87.

A. Rahman et al., *Science and Technology in Medieval India—A Bibliography of Source Materials in Sanskrit, Arabic and Persian*, New Delhi, 1982, in *Indian Journal of History of Science*, 19.4 (1984), 412–413.

B. V. Subbarayappa and K. V. Sarma, *Indian Astronomy—A Source-Book*, Bombay, 1985, in *Indian Journal of History of Science*, 22.3 (1987), 273–275.

Hindi translation of the book of Datta and Singh

[Avadhesa Narayana Simha and Bibhutibhusana Datta] [translated into Hindi by Kripasamkara Sukla]: *Hindu Gaṇita-śāstra kā Itihāsa*, Part 1, Hindi Samiti, Lucknow, 1956; 2nd ed., 1974. [Note: As far as I know, its Part 2 has not been translated into Hindi. The Hindi Samiti is an organisation to publish academic books in Hindi, and the Hindi translation (1957) of the *Bhāratīya Jyotiṣa* of Śaṅkara Bālakṛṣṇa Dīkṣita and the *Bhāratīya Jyotiṣa kā Itihāsa* (2nd ed.: 1974) of Gorakha Prasāda were also published by this organisation.]

Revision of the papers of Datta and Singh

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Hindu geometry", *Indian Journal of History of Science*, 15.2 (1980), 121–188.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Hindu trigonometry", *Indian Journal of History of Science*, 18.1 (1983), 39–108.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Use of calculus in Hindu mathematics", *Indian Journal of History of Science*, 19.2 (1984), 95–104.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Magic squares in India", *Indian Journal of History of Science*, 27.1 (1992), 51–120.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Use of permutations and combinations in India", *Indian Journal of History of Science*, 27.3 (1992), 231–249.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Use of series in India", *Indian Journal of History of Science*, 28.2 (1993), 103–129.

[Bibhutibhusan Datta and Avadhesh Narayan Singh, Revised by Kripa Shankar Shukla]: "Approximate value of surds in Hindu mathematics", *Indian Journal of History of Science*, 28.3 (1993), 265–275.

Text books

[Note: Several pioneer mathematicians and astronomers were also excellent educators. Most of the Indian students of mathematics must have used texts books written by Gorakh Prasad, who was also a historian of Indian astronomy. Prof. Shukla also wrote two text books.]

[R. S. Verma and K. S. Shukla]: *Text-Book on Trigonometry*, Pothishala Private Limited, Allahabad, 1951; Ninth edition: 1980.

[R. P. Agarwal and K. S. Shukla]: *Text-Book on Algebra*, The City Book House, Kanpur, 1959; Eighth Revised and Enlarged Edition: 1983.

Editorial works

[S. N. Sen and K. S. Shukla (eds.)]: *History of Astronomy in India*, (originally published as *Indian Journal of History of Science*, Vol. 20, 1985), Indian National Science Academy, New Delhi, 1985; Second Revised Edition, 2000. [Note: This book was released to commemorate the IAU Colloquium in 1985 (see the following book.) This book was first released without "Notes and references, Bibliography, Index and Errata" (pp. 437–526), and this portion was later sent to subscribers. It seems that a complete issue with "Notes" etc. was also published. The Second Revised Edition (2000) is published by INSA. K. S. Shukla's paper is also included in this book (see the above section of research papers).]

[G. Swarup, A. K. Bag and K. S. Shukla (eds.)]: *History of Oriental Astronomy*, (Proceedings of an International Astronomical Union Colloquium No. 91, New Delhi, India, 13–16 November, 1985), Cambridge University

Press, Cambridge, 1987. [K. S. Shukla's paper is also included in this book (see the above section of research papers).]

Contribution to other books

Bina Chatterjee: *Śiṣyadhīvṛddhida Tantra of Lalla*, Part 2, Indian National Science Academy, New Delhi, 1981. [Note: The English translation of its chapter 21 (Astronomical Instruments), which was left untranslated by Chatterjee, was supplied by K. S. Shukla.]

T. S. Kuppanna Sastry (posthumously edited by K. V. Sarma): *Pañcasiddhāntikā of Varāhamihira*, P.P.S.T. Foundation, Madras, 1993. [Note: The English translation of its chapter 14 (Graphical Methods and Astronomical Instruments), which was left untranslated by Sastry, was supplied by K. S. Shukla.]

3 References

Sinvhal, S. D.: "Dr. Avadhesh Narain Singh (a life sketch)", *Gaṇita*, 1954, Vol. 5, No. 2, pp. i–vii.

Anonymous, "Professor Kripa Shankar Shukla", *A Date with Mathematicians*, The Mathematical Association of India, Delhi Chapter, 1989, pp. 27–31.

Ohashi, Yukio: "Prof. K. S. Shukla's Contribution to the Study of the History of Hindu Astronomy", *Gaṇita Bhāratī*, 17 (1995), 29–44.

Gupta, Radha Charan: "Dr. Kripa Shankar Shukla, Veteran Historian of Hindu Astronomy and Mathematics ", *Gaṇita Bhāratī*, 20 (1998), 1–7.

Nigam, Aruna: Brief History of the Section of Hindu Mathematics, Department of Mathematics and Astronomy, Lucknow University", *Gaṇita Bhāratī*, 20 (1998), 101–103.

Prof. K. S. Shukla's contribution to the study of the history of Hindu astronomy *

I first visited Lucknow in November 1983 and studied the history of Indian astronomy under the guidance of Prof. Kripa Shankar Shukla until September 1987. Prof. Shukla's contribution to the study of the history of Hindu astronomy is so large and wide that it is beyond my ability to review his work in extenso, and the following are only some aspects of his work.

Those who want to know brief history and main characteristics of Hindu astronomy may first be referred to the following paper of Prof. Shukla.

(I) "Astronomy in Ancient and Medieval India", *Indian Journal of History of Science* (IJHS), Vol. 4, 1969, pp. 99–106.

1 Vedic and post-vedic astronomy

Prof. Shukla's view on the most ancient period of Hindu astronomy is seen in the following paper.

(II) "Main Characteristics and Achievements of Ancient Indian Astronomy in Historic Perspective", in G. Swarup, A. K. Bag and K. S. Shukla (eds.): *History of Oriental Astronomy*, Cambridge University Press, 1987, pp. 9–22.

This is a paper presented at the International Astronomical Union Colloquium held at New Delhi in November 1985. I also participated in this colloquium.

In the first part entitled "Vedic Astronomy" of the paper (II), Prof. Shukla summarises astronomical knowledge found in Vedic *Saṃhitās* and *Brāhmaṇas* and *Vedāṅga-jyotiṣa*. There are some controversial topics of ancient Hindu astronomy, and one topic, the origin of the name of the week days, may be mentioned here. Referring to P. V. Kane's work (1974),[1] Prof. Shukla says that the names of the week days are of Indian origin. The possibility of

* Yukio Ohashi, *Gaṇita-Bhāratī*, Vol. 17, Nos. 1–4 (1995), pp. 29–44. This paper was written as a dedication on the occasion of Platinum Jubilee Year of Dr. Shukla's birth (he was born on July 10, 1918).

[1] Kane, P. V.: *History of Dharmaśāstra*, Vol. V, part I, second ed., Bhandarkar Oriental Research Institute, Poona, 1974, pp. 677–685.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_4

the Indian origin of the names of the week days was as P. V. Kane pointed out, already suggested by A. Cunningham (1885).[2] Usually, however, it is said that the names of the week days are of Hellenistic origin. If the seven planets are arranged according to their distance from the earth in Hellenistic geocentric model as "Saturn, Jupiter, Mars, Sun, Venus, Mercury and Moon", and distributed to each hour, which is of Egyptian origin as the lord of the hour, the planet of the first hour of a day determines the name of the day of the week. However, Cunningham suggested that if the seven planets are arranged in reverse order and distributed to each *ghaṭī* (one sixtieth of a day), which is of Indian origin, the planet of the first *ghaṭī* of a day determines the name of the day of the week. In my opinion, it is difficult to accept Cunningham's suggestion because later Hindu astronomical works mention lords of hours (*horā-īśas*)[3] and not lords of *ghaṭīs*.

In the second part entitled "Post-vedic Astronomy" of the paper (II), Prof. Shukla starts from the discussion of the *Vasiṣṭhasiddhānta* summarised in the *Pañcasiddhāntikā* of Varāhamihira, and proceeds to the *Pauliśasiddhānta* and the *Romakasiddhānta*, both summarised in the *Pañcasiddhāntikā* and Āryabhaṭa's works. In this period, motion of planets was studied besides the sun and moon. As Prof. Shukla has written some specialised papers on these topics, we shall discuss one by one.

2 The *Vasiṣṭhasiddhānta* summarised in the *Pañcasiddhāntikā*

The name of the sage Vasiṣṭha is mentioned in the *Yavana-jātaka* (chap. 79, vs. 3) (AD 269/270) of Sphujidhvaja, and it may be that the *Vasiṣṭhasiddhānta* existed at the time of Sphujidhvaja. The *Vasiṣṭhasiddhānta* was summarised in the *Pañcasiddhāntikā* (the 6th century AD) of Varāhamihira. Among five *siddhāntas* summarised in the *Pañcasiddhāntikā*, the *Paitāmahasiddhānta*, which is the earliest and was written in AD 80, is based on the five-year *yuga* system just like the *Vedāṅga-jyotiṣa*. The *Vasiṣṭhasiddhānta* is the next oldest *siddhānta* to the *Paitāmahasiddhānta*. Varāhamihira only states that the theory of the shadow at the latter part of chapter II of his *Pañcasiddhāntikā* is based on the *Vāsiṣṭha-samāsa-siddhānta*, and it is not clear whether the luni-solar theory at the former part of chapter II and the planetary theory at the former part of chapter XVII are based on the *Vasiṣṭhasiddhānta* or not.[4] In his pa-

[2] Cunningham, A.: "The Probable Indian Origin of the Names of the Week-days", *The Indian Antiquary*, Vol. XIV, 1885, pp. 1–4. This view was criticised by J. Burgess (*The Indian Antiquary*, Vol. XIV, 1885, pp. 322–323.

[3] See, for example, *Āryabhaṭīya* (III. 16), *Sūryasiddhānta* (XII. 79) etc.

[4] In chapter XVII (chap. XVIII of Thibaut and Dvivedin's ed.) of the *Pañcasiddhāntikā*, a colophon after a verse (XVII. 5) reads, "*vāsiṣṭha-siddhānte śukraḥ*", but Varāhamihira

per (II), Prof. Shukla considers that the luni-solar theory and the planetary theory are based on the *Vasiṣṭhasiddhānta*, just like Kuppanna Shastri as well as Neugebauer and Pingree considered so.

Prof. Shukla explained Vasiṣṭha's theory for the moon's motion in the second part of the following paper.[5]

(III) "The *Pañcasiddhāntikā* of Varāhamihira (2)", *Gaṇita*, Vol. 28, 1977, pp. 99–116.

As regards the *Vasiṣṭhasiddhānta*, one topic may be mentioned here. The name of Viṣṇucandra is mentioned in the *Brāhmasphuṭasiddhānta* (XI. 50) (AD 628) of Brahmagupta as the editor of the *Vasiṣṭhasiddhānta.* S. B. Dīkshit (1896) wrote that Viṣṇucandra's version of the *Vasiṣṭhasiddhānta* did not exist at the time of Varahamihira, because he considered that the name Viṣṇucandra is not mentioned in the *Pañcasiddhāntikā*[6]. On the contrary, Prof. Shukla considers that the name of Viṣṇucandra appears in the *Pañcasiddhāntikā.* He discusses Viṣṇucandra and Romaka criticised by Pauliśa in the first part of the following paper.

(IV) "The *Pañcasiddhāntikā* of Varāhamihira (1)", *Gaṇita*, Vol. 24, No. 1, 1973, pp. 59–73: reprinted in *IJHS*, Vol. 9, 1974, pp. 62–76.

In this paper, Prof. Shukla identifies "Vishnu" in the *Pañcasiddhāntikā* (III. 32) with Viṣṇucandra, the editor of the *Vasiṣṭhasiddhānta.* Prof. Shukla remarks that occurrence of criticism of Viṣṇucandra, Romaka etc. in the *Pañcasiddhāntikā* shows that Brahmagupta's critical remarks against them were not totally baseless. This point will have to be investigated further.

3 The *Yuga* of the *Yavana-jātaka*

The *Yavana-jātaka* (AD 269/270) of Sphujidhvaja, edited and translated by David Pingree,[7] is an important text to investigate Greek influence of astronomy and astrology into India. The last chapter (chap. 79) of this work deals

himself does not state the source.

[5]For Vasiṣṭha's theory for the moon's motion, the following papers may also be consulted:
Kharegat, M. P. : "On the Interpretation of certain passages in the Pancha Siddhāntikā of Varāhamihira, an old Hindu Astronomical Work", *The Journal of the Bombay Branch of the Royal Asiatic Society*, Vol. XIX, 1895–97, pp. 109–141; and
Kuppanna Sastri, T. S.: "The Vāsiṣṭha Sun and Moon in Varāhamihira's Pañcasiddhāntikā", *Journal of Oriental Research*, Madras, Vol. XXV, 1955–56, pp. 19–41.

[6]Dikshit, Sankar Balakrishna, tr. by R. V. Vaidya: *Bharatiya Jyotish Sastra*, part II, Calcutta, 1981. David Pingree also thinks that Viṣṇucandra is later than Varāhamihira, because Viṣṇucandra used *mahāyuga* and epicycles, which are absent in Varāhamihira's version of the *Vasiṣṭhasiddhānta* (Neugebauer, O. and D. Pingree: *The Pañcasiddhāntikā of Varāhamihira*, part I, Copenhagen 1970, p. 10.)

[7]Pingree, David: The *Yavana-jātaka of Sphujidhvaja*, 2 vols., Harvard University, Cambridge, Mass., 1978.

with mathematical astronomy on the basis of 165-year *yuga*. In the following paper, Prof. Shukla corrects some errors of Pingree, and explains the *yuga* of the *Yavana-jātaka* in lucid manner.

(V) "The *Yuga* of the *Yavana-jātaka*, David Pingree's text and translation reviewed", *IJHS*, Vol. 24, 1989, pp. 211–223.

Among several points pointed out by Prof. Shukla, I would like to mention the number of *tithis* and civil days in a *yuga* (165 years). Pingree interpreted that the *Yavana-jātaka* (chap. 79, vss. 6–7) states that there are 60265 civil days in a *yuga*, and that there are 61230 *tithis* in a *yuga*. Prof. Shukla has shown that these verses actually state that there are 61230 *tithis* and 60272 civil days in a *yuga*. Prof. Shukla has given mainly textual evidences to prove his interpretation, which are quite sound and understandable. We can also notice that the verses (chap. 79, vss. 8–9) state that the risings of the moon in a *yuga* are 58231, and the number of conjunctions of the sun and moon is 2041. The sum of 58231 and 2041, that is 60272, should be the number of civil days in a *yuga*. This fact shows that Prof. Shukla's reading is correct.

4 The *Pauliśa* and the *Romakasiddhānta* summarised in the *Pañcasiddhāntikā*

Among five *siddhāntas* summarised in the *Pañcasiddhāntikā*, the *Pauliśa* and the *Romakasiddhānta* are considered to be more accurate than the *Paitāmaha* and the *Vasiṣṭhasiddhānta*. Main characteristics of the *Pauliśa*- and the *Romakasiddhānta* are described in the paper (II) of Prof. Shukla. Some particular topics are discussed in his papers (III) and (IV).

In the fourth part of his paper (IV), Prof. Shukla discusses a correction of the Pauliśa school to the longitude of the moon's ascending node. He further points out that the followers of the *Pauliśasiddhānta* fell in with the followers of the *Āryabhaṭasiddhānta* (midnight system), and revised the *Pauliśasiddhānta*, and also adopted the *Pūrva-Khaṇḍakhādyaka* of Brahmagupta as a work of their school. In the first part of his paper (IV), Prof. Shukla discusses Pauliśa's criticism of Viṣṇucandra and Romaka. In the first part of his paper (III), Prof. Shukla discusses the epoch of the *Romakasiddhānta*.

5 The *Āryabhaṭīya* of Āryabhaṭa I

The *Āryabhaṭīya* (AD 499) of Āryabhaṭa (b. AD 476) is the earliest Sanskrit astronomical work whose author and date are definitely known. Prof. Shukla published a critical edition of the *Āryabhaṭīya* with English translation and notes.

(VI) *Āryabhaṭīya of Āryabhaṭa*, critically edited with translation and notes, in collaboration with K. V. Sarma, *Indian National Science Academy* (*INSA*), New Delhi, 1976.

Prof. Shukla also published the text of the *Āryabhaṭīya* with the commentary of Bhāskara I (AD 629) (extant up to IV. 6) and Someśvara (sometime between 968 and 1200 AD) (being a summary of Bhāskara I's commentary, and published after IV. 6).[8]

(VII) *Āryabhaṭīya of Āryabhaṭa*, with the commentary of Bhāskara I and Someśvara, *INSA*, New Delhi, 1976.

Before Prof. Shukla's translation of the *Āryabhaṭīya*, there existed two published complete English translations of the *Āryabhaṭīya*, one by P. C. Sengupta (1927),[9] and the other by W. E. Clark (1930).[10] At their time, only available printed text of the *Āryabhaṭīya* was H. Kern's edition (1874) with the commentary of Parameśvara (the 15th century AD). After that, Nīlakaṇṭha Somayajin's commentary (the early 16th century AD) was also published in the Trivandrum Sanskrit Series (1930–1957).

The significance of Prof. Shukla's work is that he consulted several commentaries, both published and unpublished, and made critical edition in collaboration with K. V. Sarma and translated into English with detailed notes. Especially, Bhāskara I's commentary, which was published by Prof. Shukla for the first time, is important, because it is the earliest extant commentary on the *Āryabhaṭīya*, and Bhāskara I was a follower of Āryabhaṭa school and must have been accessible to several informations handed down to Āryabhaṭa's successors. Sarma edited another commentary.[11]

6 Āryabhaṭa I's midnight system

There were controversies about Āryabhaṭa since the beginning of the study of Indian astronomy and mathematics. H. T. Colebrooke[12] considered that the

[8]Bhāu Dājī (1865) once announced to publish the *Āryabhaṭīya* with the commentary of Someśvara (Bhāu Dājī: "Brief Notes on the Age and Authenticity of the Works of Āryabhaṭa, Varāhamihira, Brahmagupta, Bhaṭṭotpala, and Bhāskarācārya", *Journal of The Royal Asiatic Society*, 1865, 392–418; p. 405.) It could not see the light of day.

[9]Sengupta, P. C.: "The *Āryabhaṭīyam*", *Journal of the Department of Letters, University of Calcutta*, Vol. 16, 1927, art. 6, pp. 1–56.

[10]Clark, Walter Eugene: *The Āryabhaṭīya of Āryabhaṭa*, University of Chicago, 1930. In the preface, he writes that this work was partly based on the work done with him by Baidyanath Sastri for the degree of M.A.

[11]K. V. Sarma (ed.): *Āryabhaṭīya* of Āryabhaṭa with the commentary of Sūryadeva Yajvā, INSA, New Delhi, 1976.

[12]Colebrooke, H. T.: *Algebra with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhāscara*, London, 1817, notes G and I.

Daśagītikā and the *Āryāṣṭaśata* (both of which form what we call *Āryabhaṭīya* of Āryabhaṭa I) are Āryabhaṭa's genuine work, while J. Bentley[13] considered that the *Āryasiddhānta* (which we call *Mahāsiddhānta* of Āryabhaṭa II) is Āryabhaṭa's genuine work. Fitz-Edward Hall (1860)[14] thought that both are genuine, and suspected that there were two Āryabhaṭas. Commenting to Hall's paper, W. D. Whitney[15] wrote that these two Āryabhaṭas were considered to be one person by Brahmagupta, who criticised Āryabhaṭa's inconsistency. Whitney's view is actually wrong, and Āryabhaṭa II is a later person whose date is controversial.[16] Bhāu Dājī (1865)[17] clearly pointed out that there were two Āryabhaṭas, but made a mistake that the only work of Āryabhaṭa known to Brahmagupta etc. was the *Āryabhaṭīya*. He was not aware of Āryabhaṭa I's work of midnight system.[18] After that, S. B. Dikshit[19] and Sudhākara Dvivedin[20] rightly suggested that Āryabhaṭa I might have written two works, that is the *Āryabhaṭīya* and another work of midnight system. P. C. Sengupta (1930)[21] wrote a paper on Āryabhaṭa's lost work of midnight system, and investigated its astronomical constants etc.

Āryabhaṭa's work of midnight system is not extant, but there remain some information in the works of later authors, such as the *Khaṇḍakhādyaka* of Brahmagupta. The *Mahābhāskarīya* of Bhāskara I gave further informations about Āryabhaṭa I's midnight system.[22]

Prof. Shukla made further progress of the study of Āryabhaṭa's midnight system. In the following paper, Prof. Shukla described several aspects of Āryabhaṭa I's midnight system, and published a fragment of the *Yantrādhyāya* (chapter on astronomical instruments) of the *Āryabhaṭasiddhānta* (Āryabhaṭa I's lost work of midnight system), found in Rāmakṛṣṇa Ārādhya's commentary (AD 1472) on the *Sūryasiddhānta*.

[13]Bentley, John: *A Historical View of the Hindu Astronomy*, Calcutta, 1823, part II, section III.

[14]Hall, Fitz-Edward: "On the Āryasiddhānta", *Journal of the American Oriental Society*, Vol. 6, 1866, pp. 556–559.

[15]Committee of Publication (= W. D. Whitney): "Additional Note on Āryabhaṭṭa and his Writings", *Journal of the American Society*, Vol. 6, 1866, pp. 560–564.

[16]J. Bentley and Bhāu Dājī thought it is the 14th century AD, S. B. Dikshit thought the 10th century, D. Pingree thinks between ca. 950 and 1100, and R. Billard thinks the 16th century.

[17]Bhāu Dājī, *op. cit.*

[18]The *Āryabhaṭīya* is based on sunrise system (*audayika*), where a civil day is reckoned from sunrise. In the midnight system (*ārdharātrika*), a civil day is reckoned from midnight.

[19]Dikshit, tr. by Vaidya, *op. cit.*, part II, pp. 58–59.

[20]Dvivedin, Sudhākara (ed.): *Brāhma-sphuṭa-siddhānta*, ed. with the commentary written by Dvivedin, Benares, 1902; commentary on (XI. 13).

[21]Sengupta, P. C.: "Āryabhaṭa's Lost Work", *Bulletin of the Calcutta Mathematical Society*, Vol. 22, 1930, pp. 115–120.

[22]Sengupta, P. C. (tr. into English): *Khaṇḍakhādyaka*, Calcutta, 1934. Introduction, pp. x–xx.

(VIII) "Āryabhaṭa I's astronomy with midnight day-reckoning", *Gaṇita*, Vol. 18, No. 1, 1967, pp. 83–105.

This fragment, published for the first time, is a very important source material of the development of astronomical instruments in India. Prof. Shukla's edition of the fragment is based on a manuscript (deposited in Lucknow University, Acc. no. 45749) of Rāmakṛṣṇa Ārādhya's commentary on the *Sūryasiddhānta*, which is a transcription from a manuscript (no. 2803) of the Government Oriental Library, Mysore.

In the following paper, Prof. Shukla described some informations about the *Āryabhaṭasiddhānta* mentioned in Mallikārjuna Sūri's commentary (AD 1178) on the *Sūryasiddhānta* and Tamma Yajvā's commentary (AD 1599) on the *Sūryasiddhānta.*

(IX) "Glimpses from the *Āryabhaṭasiddhānta*", *IJHS*, Vol. 12, 1977, pp. 181–186.

It is very important to study these early commentaries on the *Sūryasiddhānta*, none of which has been published.

As regards the chronological order of the two works of Āryabhaṭa I, Prof. Shukla says in his paper (VIII) that they were written in the following order: (i) *Āryabhaṭasiddhānta*, and (ii) *Āryabhaṭīya.*

7 The *Sūryasiddhānta* summarised in the *Pañcasiddhāntikā*

According to Varāhamihira, the *Sūryasiddhānta* is the most accurate among the five *siddhāntas* summarised in his *Pañcasiddhāntikā.* This old *Sūryasiddhānta* is different from the modern *Sūryasiddhānta* which is extant now. Differences between these two *Sūryasiddhāntas* are discussed by Prof. Shukla in the Introduction of the following book.

(X) *The Sūryasiddhānta with the commentary of Parameśvara*, (Hindu Astronomical and Mathematical Text Series No. 1), Lucknow, 1957.

In this book (p. 27), Prof. Shukla wrote that the works of Āryabhaṭa I and Lāṭadeva were based on the *Sūryasiddhānta*, and rejected P. C. Sengupta's view that the old *Sūryasiddhānta* was made up-to-date by Varāhamihira by replacing the old constants in it by new ones from Āryabhaṭa I's midnight system. In his papers (VIII) and (IV) also, Prof. Shukla wrote that Āryabhaṭa I's midnight astronomy was based on the old *Sūryasiddhānta.* It seems that Prof. Shukla modified his view later, and wrote in the Introduction of his book (VI) (p. lxiii) that the *Āryabhaṭasiddhānta* is based on the earlier *Sūryasiddhānta*, which is now lost, and that the *Sūryasiddhānta* summarised in the *Pañcasiddhāntikā* is a new version revised by Lāṭadeva in the light of the *Āryabhaṭasiddhānta*. In his paper (II) also, Prof. Shukla wrote that the *Sūryasiddhānta* summarised by Varāhamihira was simply a redaction of the larger work of Āryabhaṭa.

Prof. Shukla corrected some errors in Thibaut and Dvivedin's edition of the *Pañcasiddhāntikā* in the following paper.

(XI) "On three stanzas from the *Pañcasiddhāntikā*", *Gaṇita*, Vol. 5, No. 2, 1954, pp. 129–136.

In this paper, Prof. Shukla presented the corrected reading of the *Pañcasiddhāntikā* (XVII. 12)[23] and (IX. 15–16),[24] and made clear that the astronomical constants in the old *Sūryasiddhānta* recorded in them are harmonious with those ascribed to Āryabhaṭa I's midnight system recorded by Bhāskara I.

In the third part of his paper (IV), Prof. Shukla discussed a correction for Mercury and Venus in the old *Sūryasiddhānta*. It may be noted that Prof. Shukla utilised the *Sumati-Mahātantra* of Sumati of Nepal.

8 The *Pañcasiddhāntikā* of Varāhamihira

As we have seen in connection of each *siddhānta* summarised in the *Pañcasiddhāntikā*, Prof. Shukla has written three papers on the *Pañcasiddhāntikā*, viz. papers (XI), (IV), and (III).

In the third part of his paper (III), Prof. Shukla discussed the 30 days of the Parsi calendar mentioned in the *Pañcasiddhāntikā* (I. 23–25). He compared them with the corresponding names given by Vaṭeśvara (AD 904), and verified them. It may be noted that the result is different from readings of Thibaut and Dvivedin, M. P. Kharegat, and Neugebauer and Pingree.

In the second part of his paper (IV), Prof. Shukla discussed the declination table of Varāhamihira.

9 Bhāskara I

Bhāskara I (the 7th century AD), who is a contemporary of Brahmagupta, is a different person from Bhāskara II (the 12th century AD) who wrote the *Siddhānta-śiromaṇi* etc. H. T. Colebrooke was aware of the existence of Bhāskara I cited by Pṛthūdaka Svāmin, but he could not find any work written by him.[25] B. Datta secured the works of Bhāskara I, and wrote a

[23] This is (XVI. 23) in Neugebauer and Pingree's edition.
[24] M. P. Khareghat also proposed similar correction. (See Khareghat, *op. cit.*, pp. 132–134.)
[25] Colebrooke, *op. cit.*, note H.

paper on him (1930).[26] However, Datta misunderstood that Bhāskara I is a direct disciple of Āryabhaṭa I, and that he lived in the first half of the 6th century AD. T. S. Kuppanna Sastri pointed out that Bhāskara I is not a direct disciple of Āryabhaṭa I, but he could not ascertain Bhāskara I's date exactly.[27] Prof. Shukla has shown that Bhāskara I actually lived in the 7th century AD, because Bhāskara I wrote his commentary on the *Āryabhaṭīya* in 629 AD, and accordingly not a direct disciple of Āryabhaṭa I. (See his book (VII), Introduction, pp. xix-xxv). Prof. Shukla also pointed out that Bhāskara I belonged to Aśmaka country lying between the rivers Godāvari and Narmadā, but lived in Valabhī in Saurāṣṭra (in modern Gujarat). (*Ibid.*, pp. xxv-xxx.)

Bhāskara I wrote three works. One is a commentary on the *Āryabhaṭīya*. Other two are the *Mahābhāskarīya* and the *Laghubhāskarīya*, and Prof. Shukla published them with English translation.

(XII) *Mahābhāskarīya*, Lucknow, 1960.

(XIII) *Laghubhāskarīya*, Lucknow, 1963.

There are other editions of the *Mahābhāskarīya*[28] and *Laghūbhāskarīya*,[29] but there is no other English translation.

Prof. Shukla discussed spherical astronomy of Bhāskara I and his contemporary Brahmagupta in the following paper.

(XIV) "Early Hindu Methods in Spherical Astronomy", *Gaṇita*, Vol. 19, No. 2, 1968, pp. 49–72.

He also discussed mathematics of Bhāskara I in the following papers.

(XV) "Hindu Mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya*", (1) *Gaṇita*, Vol. 22, No. 1, 1971, pp. 115–130; (2) *Gaṇita*, Vol. 22, No. 2, 1971, pp. 61–78; (3) *Gaṇita*, Vol. 23, No. 1, 1972, pp. 57–79; (4) *Gaṇita*, Vol. 23, No. 2, 1972, pp. 41–50.

10 Āryabhaṭa School

The *Āryabhaṭīya* of Āryabhaṭa I laid the foundation of the Āryabhaṭa school, of which one of the most eminent astronomer is Bhāskara I, whom we have

[26] Datta, Bibhutibhusan: "The Two Bhāskaras", *The Indian Historical Quarterly*, Vol. VI, 1930, pp. 727–736.

[27] Kuppanna Sastri, T. S.: "*Mahābhāskarīya of Bhāskarācārya*", Madras Government Oriental Series No. cxxx. Madras, 1957, Introduction, pp. xiii–xvii.

[28] Ānandāśrama edition (with Parameśvara's commentary), Pune, 1945; and Kuppanna Sastri's edition (with Govindasvāmin's commentary and Parameśvara's super-commentary). *op. cit.*

[29] Ānandāśrama edition (with Parameśvara's commentary), Pune, 1946; and Trivandrum edition (with Śankaranārāyaṇa's commentary), Trivandrum, 1949.

just discussed. The Āryabhaṭa school flourished in South India, particularly in Kerala, rather than in North India.

T. S. Kuppanna Shastri wrote a paper (1969)[30] on the peculiarities of Āryabhaṭa school, but he misunderstood the computation of the equation of centre in this school. Prof. Shukla criticised Kuppanna Shastri's paper, and explained the computation of the equation of centre of Āryabhaṭa school in the following paper.

(XVI) "Use of Hypotenuse in the Computation of the Equation of the Centre under the Epicyclic Theory in the School of Āryabhaṭa I ???", *IJHS*, Vol. 8, 1973, pp. 43–57.

In this paper, he quotes from the works of astronomers of Āryabhaṭa school, viz. Bhāskara I (AD 629), Govinda Svāmī (c. 800–850), Parameśvara (1430), Nīlakaṇṭha (c. 1500), and Putumana Somayājī (1732).

Prof. Shukla also published the *Karaṇaratna* (AD 689) of Deva, belonging to Āryabhaṭa school, for the first time.

(XVII) *The Karaṇaratna of Devācārya*, Lucknow, 1979.

Deva belonged to South India, probably Kerala. Prof. Shukla points out that the *Karaṇaratna* is the earliest preserved work where three *bīja* corrections, viz. the *Śakābda* correction, the *Kalpa* correction, and the *Manuyuga* correction, are stated, and also it is probably the first work in the Āryabhaṭa school to have given a rule for finding the value of the precession. So, this is a very important work of Hindu astronomy.

11 The *Śiṣyadhīvṛddhidatantra* of Lalla

The *Śiṣyadhīvṛddhidatantra* of Lalla (the 8th or 9th century AD)[31] is also a text following Āryabhaṭa. Bina Chatterjee edited its text with the commentary of Mallikārjuna Sūri (the 12th century AD), and translated into English, but chapter XXI (chapter of astronomical instruments) was left untranslated by Chatterjee who passed away in 1978. So, its translation was supplied by Prof. Shukla, and published as follows:

> Bina Chatterjee: *Śiṣyadhīvṛddhida Tantra of Lalla*, 2 parts, *INSA*, New Delhi, 1981.

[30] Kuppanna Shastri, T. S.: "The School of Āryabhaṭa and the Peculiarities thereof", *IJHS*, Vol. 4, pp. 126–134.

[31] Bina Chatterjee wrote that the date of Lalla is sometime between the 8th and the 11th century, (Introduction of her edition and translation, part II, p. xiv.) Prof. Shukla says that Lalla's date is sometime between AD 665 (*Khaṇḍakhādyaka's* date) and AD 904 (*Vaṭeśvarasiddhānta's* date): see Introduction of his book (VI), p. lx.

Lalla described several instruments, some of which are quite different from those of early authors, and his description is very important.

12 The *Vaṭeśvarasiddhānta* of Vaṭeśvara

The *Vaṭeśvarasiddhānta* (AD 904) of Vaṭeśvara (b. AD 880) is the largest Sanskrit astronomical work. It is well known that Brahmagupta criticised Āryabhaṭa I. Vaṭeśvara reversely criticised Brahmagupta, and defended Āryabhaṭa I.

The first three chapters of the *Vaṭeśvarasiddhānta* were first published by Ram Swarup Sharma and Mukund Misra in 1962,[32] but it was based on a single manuscript. Prof. Shukla discovered another manuscript of the *Vaṭeśvarasiddhānta*, and reported its contents etc. in the following paper.

(XVIII) "Hindu astronomer Vaṭeśvara and his works", *Gaṇita*, Vol. 23, No. 2, 1972, pp. 65–74.

It may be noted that Prof. Shukla identified Vaṭeśvara's place Ānandapura with Vadnagar in northern Gujarat.

Prof. Shukla edited the whole text of the *Vaṭeśvarasiddhānta* based on these two manuscripts, and the fragment of the *Gola* found in the newly discovered manuscript, and translated them into English with detailed commentary.

(XIX) *Vaṭeśvarasiddhānta and Gola of Vaṭeśvara*, 2 parts, *INSA*, New Delhi, 1985–1986.

Prof. Shukla's commentary is so detailed and lucid that it is particularly useful for those who want to understand the theory of Hindu astronomy deeply. Explaining several topics, Prof. Shukla refers to parallel passages in other Sanskrit astronomical works extensively, and this book can be used as a standard reference book of Hindu astronomy. The list of word-numerals, which is appendix II of part I, is perhaps the most exhaustive list of word-numerals.

David Pingree of Brown University, U.S.A, has written a review of this book (XIX). (*IJHS*, Vol. 26, 1991, pp. 115–122.)

It is known that al-Bīrūnī has quoted from the *Karaṇasāra*, a calendrical work of Vateśvara. The *New Catalogus Catalogorum* (Vol. 3, p. 176) of Madras University records a manuscript of the "*Karaṇasāra of Vitteśvara*" in the "State Library", Kota, Rajasthan, but its actual existence has not been ascertained so far. I was suggested this fact by Prof. Shukla, and visited Kota once, but could not find the *Karaṇasāra* during my short stay.

It may be noted that the original idea of the second correction for the moon, which is stated in the *Laghumānasa* of Mañjula as we shall see below,

[32] *Vaṭeśvarasiddhānta*, Vol. I, Indian Institute of Astronomical and Sanskrit Research, New Delhi, 1962.

is attributed to Vaṭeśvara by Yallaya (1482 AD), but it is not found in the extant *Vaṭeśvarasiddhānta*. Prof. Shukla suggests that it must have been mentioned in the *Karaṇasāra* or some other work of Vaṭeśvara. (See p. LIII, Introduction of part II of his book (XIX).)

13 The *Laghumānasa* of Mañjula

The name of Mañjula is sometimes spelt Muñjāla, but, according to Prof. Shukla, Mañjula is the real name.

H. T. Colebrooke (1816)[33] already noticed the notion of the precession of Mañjula quoted in the *Siddhāntaśiromaṇi* (*Gola*, VI. 17–18) of Bhāskara II. According to Bhāskara II, Mañjula stated that the equinox revolves 199669 times in a *kalpa*, that is $59''.9007$ per year. Colebrooke has not seen Mañjula's own work, but we know that Mañjula himself gives the rate of precession as $1'$ per year in his *Laghumānasa*. Reason of this discrepancy is not known.

The *Laghumānasa* (AD 932) of Mañjula was noticed by Sudhākara Dvivedin (1892),[34] and N. K. Majumder (1927)[35] etc. Dvivedin pointed out that the second correction for the moon is mentioned there. The second correction, which is a combination of the deficit of the equation of centre and the evection, was further discussed by D. Mukhopadhyaya (1930)[36] and P. C. Sengupta (1932).[37] Later, N. K. Majumder published an edition and English translation (1940–1951)[38] of the *Laghumānasa*, and Ānandāśrama of Pune published (1944)[39] the text with Parameśvara's commentary.

Prof. Shukla pointed out in the following paper that the interpretations of D. Mukhopadhyaya and P. C. Sengupta contain some errors, and discussed the second correction of Mañjula etc. in detail.

(XX) "The Evection and the Deficit of the Equation of the Centre of the Moon in Hindu Astronomy", *Proceedings of the Benares Mathematical Society*, New Series, Vol. 7, No. 2, 1945, pp. 9–28.

[33] Colebrooke, H. T.: "On the Notion of the Hindu Astronomers concerning the Precession of the Equinoxes and Motion of the Planets", *Asiatic Researches*, Vol. XII, 1816, pp. 209–250; reprinted in his *Miscellaneous Essays*, Vol. II, 1837.

[34] Dvivedin, Sudhākara, *Gaṇaka-taraṅgiṇī*, 1892, section of Muñjāla.

[35] Majumder, N. K.: "*Laghumānasam* of Muñjāla", *Journal of the Department of Letters, University of Calcutta*, Vol. 14, 1927, art. 8, pp. 1–5.

[36] Mukhopadhyaya, Direndranath: "The Evection and the Variation of the Moon in Hindu Astronomy", *Bulletin of the Calcutta Mathematical Society*, Vol. XXII, 1930, pp. 121–132.

[37] Sengupta, P. C.: "Hindu Luni-solar Astronomy", *Bulletin of the Calcutta Mathematical Society*, Vol. 24, 1932, pp. 1–18; reprinted as appendix I of his English translation of the *Khaṇḍakhādyaka*, Calcutta, 1934.

[38] Majumder, N. K.: *Laghumānasam by Muñjalācārya*, Calcutta, 1951. He states in its Introduction that he took up the work in 1940, and published the first instalment in a journal.

[39] *Laghumānasam*, Ānandāśrama Sanskrit Series 123, Pune, 2nd ed., 1952.

According to this paper, Mañjula's second correction for the moon's longitude in terms of minutes can be expressed as follows:

$$\pm\left(8\frac{2}{15}\right)\cos(S-U)[G-11]\times\left(8\frac{2}{15}\right)\sin(M-S)$$

where S, M, U, respectively denote the true longitudes of the sun, the moon, and the moon's apogee, and G the Moon's true daily motion in degrees. Formerly, D. Mukhopadhyaya took S, M, G as the mean longitudes of the sun and the moon, and the mean daily motion of the moon respectively, and P. C. Sengupta and N. K. Majumder (1951) took G as the mean daily motion of the moon, although they took M as the moon's longitude corrected by the first equation. Prof. Shukla says that G should be the *true* daily motion of the moon, because Vaṭeśvara (quoted in Yallaya's commentary on the *Laghumānasa*) states the corresponding term to be the true motion. (As we have discussed, Vaṭeśvara's statement is not found in the extant *Vaṭeśvara-siddhānta*.)

Besides Mañjula, Prof. Shukla explained in his paper (XX) the second correction for the moon in the *Siddhāntaśekhara* (1039 AD) of Śrīpati, the *Tantra-Saṃgraha* of Nīlakaṇṭha (ca. 1500 AD), and the *Siddhāntadarpaṇa* of Candra Śekhara Siṃha (later half of the 19th century). And also, using a figure, Prof. Shukla explained the rationale of this second correction, which is explained in Hindu astronomy as the displacement of the Earth from its natural position.

Recently, Prof. Shukla published a new critical edition and English translation of the *Laghumānasa* of Mañjula with detailed introduction and notes.

(XXI) "*A Critical Study of the Laghumānasa of Mañjula*", *IJHS*, Vol. 25, 1990, Supplement; and also separately issued, *INSA*, New Delhi, 1990.

The *Laghumānasa* is a small but very important work. Prof. Shukla's notes with rationale and examples are quite useful to understand the text.

14 The *Dhīkoṭida-karaṇa* of Śrīpati and the *Rājamṛgāṅka* of Bhoja

Śrīpati wrote three astronomical works, the *Siddhāntaśekhara*, the *Dhīkoṭida-karaṇa* (AD 1039), and the *Dhruvamānasa-karaṇa* (AD 1056).

He also wrote the mathematical work *Gaṇitatilaka*, and several astrological works such as the *Ratnamāla*, the *Jātakapaddhati* etc. The *Siddhāntaśekhara* was published by B. Miśra (1932, 1947),[40] and the *Dhīkoṭida-karaṇa* was

[40] *The Siddhāntaśekhara of Śrīpati*, 2 parts, ed. by Babuāji Miśra, Calcutta University, 1932–1947.

(according to D. Pingree) published by N. K. Majumder (1934),[41] but the *Dhruvamānasa-karaṇa* has not been published.

Prof. Shukla published a critical edition and English translation of the *Dhīkoṭida-karaṇa* with notes and illustrative examples.

(XXII) "*The Dhīkoṭida-karaṇa of Śrīpati*", Akhila Bhāratīya Sanskrit Parishad, Lucknow, 1969.

This is a small work which gives the method of calculation of lunar and solar eclipses. Prof. Shukla has given illustrative examples of the calculation using Śrīpati's method for the eclipses in 1968 AD, and showed that the result is remarkably good.

By the way, it may also be noted that the second correction for the moon in the *Śiddhānta-śekhara* has been discussed in Prof. Shukla's paper (XX).

Another contemporary *karaṇa* work is the *Rājamṛgāṅka* (1042 AD) of Bhoja. Prof. Shukla has written the following comment on the printed text of the *Rājamṛgāṅka.*

(XXIII) "A Note on the Rājamṛgāṅka of Bhoja published by the Adyar Library", *Gaṇita*, Vol. 5, No. 2, 1954, pp. 149–151.

In this paper, Prof. Shukla has shown that K. M. K. Sarma's edition of the *Rājamṛgāṅka* published by the Adyar Library, Madras (1940), may not be the original and full text, but an abridged edition by some later writer.

15 The early versions of the modern *Sūryasiddhānta*

The modern *Sūryasiddhānta* (called "Modern" in contrast with the *Sūryasiddhānta* summarised in the *Pañcasiddhāntikā* of Varāhamihira) is one of the most popular Sanskrit work of astronomy. There are several extant traditional commentaries since the 12th century down to recent time, and also, there are several researches by modern scholars since the end of the 18th century, the earliest of whom is perhaps Samuel Davis (1790).[42] Another early scholar is John Bentley (1799),[43] who analysed the accuracy of the *Sūryasiddhānta*, and

[41] Majumder, N. K.: "Dhīkoṭi-karaṇa of Śrīpati", *Calcutta Oriental Journal*, Vol. I, 1934, pp. 286–299. The calculation in the *Dhīkoṭi-karaṇa* was already explained in Majumder: "Dhīkoṭi-karaṇam of Śrīpati", *Journal of the Asiatic Society of Bengal*, N.S., Vol. XVII, 1921, pp. 273–278. I have not seen his paper of 1934, but have seen his paper of 1921. Differences between his reading and Prof. Shukla's reading exist in the apparent diameters of the sun, the moon, and the shadow of the earth. Perhaps Majumder took the reading "*rasāgni*" (= 36) (in verse 8–d) for the moon's diameter in terms of minutes, while Prof. Shukla takes "*karāgni*" (= 32).

[42] Davis, Samuel: "On the Astronomical Computations of the Hindus", *Asiatic Researches*, Vol. 2, 1790, pp. 175–226.

[43] Bentley, J.: "On the Antiquity of the Sūrya Siddhānta and the Formation of the Astronomical Cycles therein contained", *Asiatic Researches*, Vol. 6, 1799, pp. 540–593.

concluded that it was composed in the eleventh century or so. As regards the date of the modern *Sūryasiddhānta*, Prof. Shukla writes in the Introduction (p. 29) of his book (X) that it is sometime between AD 628 and AD 966, after AD 628 because it is influenced by *Brāhmasphuṭasiddhānta*, and before AD 966 because Bhaṭṭotpala wrote a commentary on it, whose fragment is quoted in a later work.

In the 19th century, the text of the *Sūryasiddhānta* with Raṅganātha's commentary (AD 1603) was published by Fitz Edward Hall and Bāpūdeva Śāstrī (1854–58),[44] and Bāpūdeva Śāstrī translated it into English (1860–62).[45] Ebenezer Burgess also published an English translation of the *Sūryasiddhānta* with the help of W. D. Whitney (1860),[46] and this has become one of the most popular work of Hindu astronomy in English. Burgess' translation is also based on Raṅganātha's commentary. There are some other printed editions of the Sanskrit text of the *Sūryasiddhānta* based on Raṅganātha's version.

There are several earlier extant commentaries of the *Sūryasiddhānta*, such as

(i) Mallikārjuna Sūri (AD 1178)

(ii) Caṇḍeśvara (AD 1185)

(iii) Madanapāla (the 14th century AD)

(iv) Parameśvara (AD 1432)

(v) Yallaya (AD 1472)

(vi) Rāmakṛṣṇa Ārādhya (AD 1472)

(vii) Bhūdhara (AD 1572)

(viii) Tamma Yajvan (AD 1599)

The readings of the text in these early versions are different from Raṅganātha's version at several places. Prof. Shukla published the *Sūryasiddhānta* with Parameśvara's commentary for the first time (1957) as his book (X). In the footnotes of this book, Prof. Shukla gives alternative readings of the text found in the versions of Mallikārjuna Sūri, Yallaya, Rāmakṛṣṇa Ārādhya, and Raṅganātha also. At present this book is only one printed text of an early

[44] Published in the Bibliotheca Indica series of the Asiatic Society, Calcutta.

[45] Bāpūdeva Śāstrī and Lancelot Wilkinson: *The Sūrya siddhānta, or an Ancient System of Hindu Astronomy followed by the Siddhānta Śiromaṇi*, Asiatic Society, Calcutta, 1860–1862.

[46] Burgess, Ebenezer: "Translation of the Sūryasiddhānta", *Journal of the American Oriental Society*, Vol. 6, 1860, pp. 141–498. Reprinted by Calcutta University in 1935.

version of the *Sūryasiddhānta* before Raṅganātha. So, this is an indispensable work to investigate the early form of the modern *Sūryasiddhānta*.

We also recall that Prof. Shukla published a fragment of the *Āryabhaṭa-siddhānta* of Āryabhaṭa I quoted in Rāmakṛṣṇa Ārādhya's commentary on the *Sūryasiddhānta* in his paper (VIII), and also discussed about the informations about the *Āryabhaṭasiddhānta* found in Mallikārjuna Sūri and Tamma Yajvā's commentaries on the *Sūryasiddhānta* in his paper (IX).

Early commentaries on the *Sūryasiddhānta* are mine of informations of Hindu astronomy, and much more study is necessary.

16 Other works

Papers (I) and (II) may be said to be general papers. Prof. Shukla has written the following paper also.

(XXIV) "Phases of the Moon, Rising and Setting of Planets and Stars and their Conjunctions", in S. N. Sen and K. S. Shukla (eds.): *History of Astronomy in India*, *INSA*, New Delhi, 1985.

This paper is complementary to Arka Somayaji's "The Yuga System and the Computation of Mean and True Longitudes" and S. D. Sharma's "Eclipses, Parallax and Precession of Equinoxes" in the same book.

Prof. Shukla also made several contributions to the study of Hindu Mathematics. He published the *Pāṭīgaṇita* of Śrīdhara (Lucknow, 1959), and the *Bījagaṇitāvataṃsa* of Nārāyaṇa. (Akhila Bharatiya Sanskrit Parishad, Lucknow, 1970), and also revised B. Datta and A. N. Singh's papers on Hindu Geometry, Trigonometry, Calculus, Magic squares, Permutations and combinations, Series, Surds, and Approximate values of surds, and published in *IJHS* (vols. 15, 18, 19, 27, and 28).

17 Conclusion

We have seen that Prof. Shukla's works cover almost all periods of Classical Hindu Astronomy, and are based on several primary sources. Several fundamental Sanskrit texts were critically edited and translated with detailed mathematical and astronomical notes which are lucid and exact. I believe that all students of the history of Indian astronomy should study the works of Prof. Shukla carefully.

The seminal contribution of K. S. Shukla to our understanding of Indian astronomy and mathematics *

Kripa Shankar Shukla[1] was born on June 12, 1918 in Lucknow. He completed his undergraduate and postgraduate studies in mathematics at Allahabad University. In 1941, he joined the research programme on Indian mathematics at the Department of Mathematics, Lucknow University, to work with Prof. Avadhesh Narayan Singh (1905–1954). Prof. Singh, the renowned collaborator of Bibhutibhusan Datta (1888–1958), had joined Lucknow University in 1928. He initiated a research programme on the study of Indian astronomy and mathematics at the University in 1939. He managed to collect a number of manuscripts of important source-works and also attracted many researchers to work with him.

Shukla's first paper, published in 1945, presented a comprehensive survey of the second correction (due to evection) for the Moon in Indian Astronomy. In 1955, Shukla was awarded the D.Litt. degree from Lucknow University for his thesis on "Astronomy in the Seventh Century India: Bhāskara I and His Works". Dr. Shukla became the worthy successor of Prof. Singh to lead the research programme on Indian astronomy and mathematics at Lucknow University. Though he retired as Professor of Mathematics in 1979, he continued to guide researchers and work relentlessly to publish a number of outstanding articles and books—which included an edition and translation of *Vaṭeśvara-siddhānta* (c. 904), the largest known Indian astronomical work with over 1400 verses, brought out by Indian National Science Academy in two volumes during 1985–86. Prof. Shukla passed away on June 22, 2007.

* M. D. Srinivas, 2018 (To appear in *Gaṇita-Bhāratī*).

[1] For a detailed biography of Prof. K. S. Shukla along with a list of his publications, see: R. C. Gupta, "Dr. Kripa Shankar Shukla, Veteran Historian of Hindu Astronomy and Mathematics", *Gaṇita-Bhāratī*, 20 (1998), pp. 1–7. Also, Yukio Ohashi, "Kripa Shankar Shukla (1918–2007)", *Indian Journal of History of Science*, 43 (2008), pp. 475–485.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_5

1 Publications of K. S. Shukla on Indian astronomy and mathematics[2]

Prof. Shukla was famous as a great teacher and expositor of astronomy and mathematics. In the 1950s he wrote popular textbooks on trigonometry and algebra and also published a Hindi translation of Part I of the History of Hindu Mathematics by B. B. Datta and A. N. Singh. It was indeed very unfortunate that there was no course on History of Mathematics or Indian Mathematics taught at the Lucknow University, notwithstanding the presence of a great scholar and teacher such as Prof. Shukla on its faculty.[3] However, Prof. Shukla guided several researchers in their work on Indian astronomy and mathematics. Amongst those who worked with Shukla for their Ph.D. Degree, are the well known scholars, Parmanand Singh (who worked on the *Gaṇitakaumudī* of Nārāyaṇa Paṇḍita), and the Japanese scholar Yukio Ohashi (who worked on the history of astronomical instruments in India).

Shukla brought out landmark editions of twelve important source-works of Indian astronomy and mathematics. Some of them were published from Lucknow University under the Hindu Astronomical and Mathematical Texts Series. Following is a list of the source-works published by Shukla.

1. *Sūryasiddhānta* with commentary of Parameśvara, ed. by K. S. Shukla, Lucknow University, Lucknow 1957.
2. *Pāṭīgaṇita* of Śrīdharācārya, ed. and tr. with notes by K. S. Shukla, Lucknow University, Lucknow 1959.
3. *Mahābhāskarīya* of Bhāskara I, ed. and tr. with notes by K. S. Shukla, Lucknow University, Lucknow 1960.
4. *Laghubhāskarīya* of Bhāskara I, ed. and tr. with notes by K. S. Shukla, Lucknow University, Lucknow 1963.
5. *Dhīkoṭidakaraṇa* of Śrīpati, ed. and tr. with notes by K. S. Shukla, Akhila Bharatiya Sanskrit Parishad, Lucknow 1969.
6. *Bījagaṇitāvataṃsa* of Nārāyaṇa Paṇḍita, ed. by K. S. Shukla, Akhila Bharatiya Sanskrit Parishad, Lucknow 1970.
7. *Āryabhaṭīya* of Āryabhaṭa, ed. and tr. with notes by K. S. Shukla and K. V. Sarma, Indian National Science Academy, New Delhi 1976.
8. *Āryabhaṭīya* of Āryabhaṭa with the commentary of Bhāskara I, ed. by K. S. Shukla, Indian National Science Academy, New Delhi 1976.

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[2] For an insightful overview of the publications of Prof. Shukla on Indian astronomy, see: Yukio Ohashi, "Prof. K. S. Shukla's Contribution to the Study of Hindu Astronomy", *Gaṇita-Bhāratī*, 17 (1995), pp. 29–44.

[3] See R. C. Gupta (1998), p. 3.

9. *Karaṇaratna* of Devācārya, ed. and tr. with Notes by K. S. Shukla, Lucknow University, Lucknow 1979.
10. *Vaṭeśvarasiddhānta* and *Gola* of Vaṭeśvara, ed. and tr. with notes by K. S. Shukla, 2 Volumes, Indian National Science Academy, New Delhi 1985–86.
11. *Laghumānasa* of Mañjula, ed. and tr. with notes by K. S. Shukla, Indian National Science Academy, New Delhi 1990.
12. *Gaṇitapañcaviṃśī*, ed. and tr. by K. S. Shukla, *Indian Journal of History of Science*, 52.4 (2017), pp. S1–S22.

As may be seen from the above list, most of these editions also include lucid English translations and detailed mathematical explanatory notes. This is indeed one of the greatest contributions of Prof. Shukla, since till the 1960s there had been very few editions of the classical source-works of Indian astronomy which also included a translation as well as explanatory notes. As regards the source-works of Indian mathematics, there were the well known translations, along with explanatory notes, authored by Colebrooke[4] and Rangacarya[5] of the mathematics chapters of *Brāhmasphuṭasiddhānta* of Brahmagupta, the *Līlāvatī* and *Bījagaṇita* of Bhāskarācārya and the *Gaṇitasārasaṅgraha* of Mahāvīra. As regards Indian astronomy, while a number of source-works were published by Sudhākara Dvivedi and other scholars, the only texts which were translated into English,[6] along with explanatory notes, were the *Sūryasiddhānta* by Burgess,[7] *Pañcasiddhāntikā* of Varāhamihira by Thibaut,[8] *Āryabhaṭīya* by Sengupta[9] and Clark,[10] and the *Khaṇḍakhādyaka* of Brahmagupta by Sengupta.[11]

The scholarly world is highly indebted to Prof. Shukla for having taken great pains to publish lucid translations, along with detailed mathematical explanatory notes, of some of the most important source-works of Indian astronomy, including works of all the three categories, namely, *Siddhānta*, *Tantra* and

[4] H. T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, John Murray, London 1817.

[5] M. Rangacarya, *The Gaṇitasārasaṅgraha of Mahāvīrācārya with English Translation and Notes*, Government Press, Madras 1912.

[6] There were also translations of *Sūryasiddhānta*, *Āryabhaṭīya*, *Siddhāntaśiromaṇi* of Bhāskara II and *Grahalāghava* of Gaṇeśa into various Indian languages, some of which also included explanatory notes.

[7] E. Burgess, *Translation of the Sūryasiddhānta*, The American Oriental Society, New Haven 1860.

[8] G. Thibaut and Sudhakar Dvivedi, The *Pañcasiddhāntikā*, Medical Hall Press, Benares 1889.

[9] P. C. Sengupta, "The *Āryabhaṭīyam*", Journal of Department of Letters of Calcutta University, 16 (1927), pp. 1–56.

[10] W. E. Clark, *The Āryabhaṭīya of Āryabhaṭa*, University of Chicago Press, Chicago 1930.

[11] P. C. Sengupta, *The Khaṇḍakhādyaka*, University of Calcutta, Calcutta 1934.

Karaṇa, and covering the entire classical *Siddhāntic* period from Āryabhaṭa (c. 499) to Śrīpati (c. 1039). His explanatory notes often include summaries of important discussions found in various commentaries, and also detailed references to similar results or procedures contained in other important texts. Shukla's editions and translations have therefore acquired the status of canonical textbooks which can be profitably used by all those interested in a serious study of Indian astronomical tradition. In collaboration with the renowned scholar Samarendra Nath Sen (1918–1992), Prof. Shukla has also edited a pioneering History of Indian Astronomy in 1985, which continues to be the standard reference work on the subject.[12]

Prof. Shukla has also written over 40 important research articles, which have ushered in an entirely new perspective on the historiography of Indian astronomy and mathematics. We may here make a mention of just a few of his seminal contributions:

(i) Clear exposition of various aspects of the Vasiṣṭha, Romaka and Pauliśa *Siddhāntas* as summarised in *Pañcasiddhāntikā* of Varāhamihira.

(ii) Correction of the faulty readings and translations of some of the crucial verses giving the number of civil days and other parameters of a *yuga*, as presented in the 1978 edition of *Yavanajātaka* by David Pingree.

(iii) Discovery of the verses of *Āryabhaṭasiddhānta* dealing with *yantras* (instruments).

(iv) Correct explanation of the *manda-saṃskāra* (equation of centre) in Indian astronomy, including the computation of the *aviśiṣṭa-mandakarṇa* (iterated *manda*-hypotenuse) and its significance.

(v) Correct explanation of the second lunar correction (incorporating the evection correction) as presented by Mañjulācārya.

(vi) Discovery of the verses of Acārya Jayadeva on the *cakravāla* method for solving quadratic indeterminate equations.

(vii) Detailed exposition of the study of magic squares in Indian mathematics.

(viii) Publication of a revised and updated version of Part III of the 'History of Hindu Mathematics' by B. B. Datta and A. N. Singh.

In what follows, we shall present some highlights of the seminal contribution of Prof. Shukla in relation to items iv, ii and viii of the above list (in that order).

[12] S. N. Sen and K. S. Shukla, (eds.), *History of Indian Astronomy*, Indian National Science Academy, New Delhi 1985 (2nd Revised Edition 2000).

2 Explaining the correct formulation of the *manda-saṃskāra* (equation of centre) in Indian astronomy

In his landmark translations of *Mahābhāskarīya* and *Laghubhāskarīya*, published in 1960 and 1963 respectively, Prof. Shukla explained the correct formulation of the *manda-saṃskāra* or the equation of centre, as expounded by Bhāskara I (c. 629). This corrected a longstanding misconception, as the equation of centre in Indian astronomy was totally misconstrued by modern scholarship for nearly two centuries. In what follows, we shall summarise the formulation of *manda-saṃskāra* or the equation of centre, and the computation of the *manda-karṇa* or the *manda*-hypotenuse, as expounded by Bhāskara in Chapter IV of *Mahābhāskarīya*, following closely the lucid exposition of Prof. Shukla. We shall discuss the *manda-saṃskāra* formulated in terms of an epicycle model.[13]

In Figure 1, O is the centre of the earth, P_0 the mean planet and U the mandocca or the *manda*-apsis. $OP_0 = R$, is the radius of the concentric. P_1 is on the epicycle centred at P_0, with radius equal to the tabulated epicycle radius r_0,[14] such that P_0P_1 is parallel to OU. P_1 is also on the eccentric circle with O' as the centre, where O' is along OU, such that $OO' = r_0$. The true radius of the epicycle r is different from the tabulated radius r_0. Hence, P_1 is not the *manda-sphuṭa* or the true *manda*-corrected planet. Similarly, $OP_1 = K_0$, is only the *sakṛt-karṇa* or the initial hypotenuse and not the true hypotenuse.

The *manda-sphuṭa* or the true *manda*-corrected planet is at P (along P_0P_1) such that $P_0P = r$, which is the true epicycle radius. Correspondingly the true *manda*-hypotenuse is given by $OP = K$. The main feature of this model is that the true epicycle radius r and the true *manda*-hypotenuse K are related by

$$r = \frac{r_0}{R} K. \tag{1}$$

Let θ_0 be the longitude of the mean planet P_0, θ_u the longitude of the *mandocca* U, and θ_{ms} the longitude of the *manda-sphuṭa* or the *manda*-corrected

[13] *Mahābhāskarīya* of Bhāskara I, ed. and tr. with Notes by K. S. Shukla, Lucknow University, Lucknow 1960. The equation of centre for the Sun and the Moon is discussed, following both the epicycle and eccentric-circle models, on pp. 110–119 and pp. 122–126, respectively. The equation of centre for the planets is similar and discussed later on pp. 134–144.

[14] The values of the *manda* and *śīghra* epicycle radii are presented in Chapter VII of *Mahābhāskarīya* (ibid. pp. 206–7). It is important to note that, except in the case of the Sun and the Moon, even these tabulated epicycles are not constant, but vary with the anomaly. Their extreme values are given at the beginning of the odd and even quadrants, and in between they have a periodic variation. Only in the case of the Sun and the Moon, this factor does not come into play, and we can treat the tabulated epicycle as a constant.

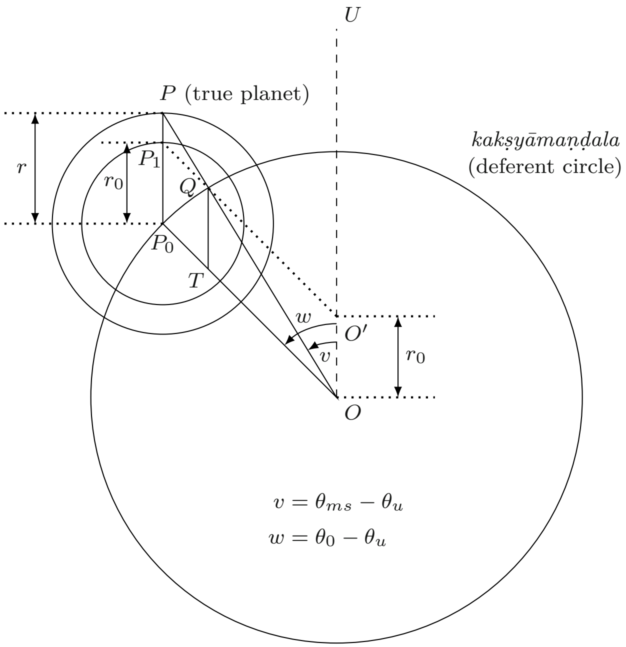

Figure 1: *Manda*-correction and the iterated-*manda*-hypotenuse.

planet P. Hence, the mean anomaly is given by the angle $P_0OU = \theta_0–\theta_u$, and the true anomaly is given by the angle $POU = \theta_{ms}–\theta_u$, and the angle $P_0OP = \theta_0–\theta_{ms}$. From Figure 1, we can easily see that the *manda*-correction for the longitude will be

$$R \sin(\theta_{ms}–\theta_0) = -\left(\frac{r}{K}\right) R \sin(\theta_0–\theta_u). \tag{2}$$

Now, by applying the condition (1), we get the final form of the *manda*-correction

$$R \sin(\theta_{ms}–\theta_0) = -\left(\frac{r_0}{R}\right) R \sin(\theta_0–\theta_u). \tag{3}$$

Thus, the *manda*-correction or the equation of centre (3) involves only the ratio of the mean epicycle radius r_0 and the radius R of the concentric. It does not involve the initial hypotenuse K_0 or even the true hypotenuse K.

The important point to be realised is that the *karṇa* or hypotenuse does not appear in the equation of centre in Indian astronomy—unlike in the case of the *śīghra*-correction (or the so called equation of conjunction) where the correction crucially depends on the *śīghra-karṇa* or the *śīghra*-hypotenuse—because the *manda*-epicycle, in Indian planetary theory, is assumed to be variable and varies in the same way as *karṇa* as shown in relation (1). It is for this reason that the *karṇa* gets replaced by just the radius of the concentric in the equation of centre—not because of any approximation that the radius does not differ too much from the hypotenuse in the *manda-saṃskāra.*

While it may not make its appearance in the equation of centre, the *manda-karṇa* is still very important. For instance, it gives the true distance in the case of the Sun and the Moon. Now, in order to determine both r and K, Bhāskara presents the following iterative process. To start with, the initial hypotenuse (*sakṛt-karṇa*), K_0, is computed in the usual way in terms of the mean anomaly (θ_0–θ_u), using the mean epicycle radius r_0:

$$K_0 = OP_1 = \sqrt{[R\sin(\theta_0–\theta_u)]^2 + [R\cos(\theta_0–\theta_u) + r_0]^2}. \tag{4}$$

Then the next approximation to the epicycle radius, r_1, is computed using

$$r_1 = \frac{r_0}{R}K_0. \tag{5}$$

From r_1 the corresponding hypotenuse K_1 is computed using

$$K_1 = \sqrt{[R\sin(\theta_0–\theta_u)]^2 + [R\cos(\theta_0–\theta_u) + r_1]^2}. \tag{6}$$

And, from K_1, the next approximation r_2 is computed using

$$r_2 = \frac{r_0}{R}K_1. \tag{7}$$

And so on, till there is no appreciable difference between successive results (*aviśeṣa*), which means that, for some m

$$r_{m+1} = \frac{r_0}{R}K_m \approx r_m. \tag{8}$$

Then, it can be seen right away that the iterated radius r_m and the associated hypotenuse K_m, are such that

$$r_m = \frac{r_0}{R}K_m. \tag{9}$$

In other words, they very nearly satisfy the relation (1) that characterises the true epicycle r and the corresponding true hypotenuse K.

In his explanatory notes to the *Mahābhāskarīya*, Shukla explains how the above iteration process actually converges to the true radius r and the true hypotenuse K, satisfying the relation (1). He also identifies the geometrical location P of the *manda*-corrected planet in the following manner. In Figure 1, let O' be the point on OU such that $OO' = r_0$. Let the line $O'P_1$ intersect the concentric at Q. Then, the P the true or *manda*-corrected planet is located at the intersection of the lines OQ and P_0P_1, extended if necessary. Now, draw the line QT parallel to P_0P_1, where T is located on OP_0. From the similar triangles OQT and OPP_0, we get

$$\frac{QT}{OQ} = \frac{P_0P}{OP}. \tag{10}$$

Since, $QT = P_0P_1 = r_0$, $P_0P = r$, $OQ = R$, and $OP = K$, equation (10) reduces to relation (1) as required.

Later, in another landmark article published in 1973,[15] Shukla explains how the above formulation of the *manda-saṃskāra* is the one followed almost universally by all schools of Indian astronomy, except for a few astronomers such as Pṛthūdakasvāmi (c. 860) and some seventeenth century astronomers who had not understood the traditional formulation. Shukla first presents detailed quotations from the astronomers of the school of Āryabhaṭa (such as Lalla (c. 750), Govindasvāmi (c. 800), Sūryadevayajvā (c. 1200) and the later Kerala astronomers Parameśvara (c. 1430), Nīlakaṇṭha Somayājī (c. 1500) and Putumana Somayājī (c. 1600)) to show that all of them clearly hold the view that:

(i) The *manda*-hypotenuse does not appear in the equation of centre because the radius of the epicycle and the hypotenuse vary according to the relation (1) mentioned above,

(ii) The true epicycle radius and the true hypotenuse may be found by an iterative process such as the one discussed in *Mahābhāskarīya*.

Shukla next presents quotations from Brahmagupta (c. 628), Śrīpati (c. 1039), Bhāskara II (c. 1150) and the *Ādityapratāpa-siddhānta* to show that they also subscribe to the same formulation of the *manda-saṃskāra* as outlined above. He then refers to the view of Caturveda Pṛthūdakasvāmi (c. 860) the commentator of Brahmagupta, that the hypotenuse is not used in the *manda*-correction because the difference between the radius of the concentric and the hypotenuse is small so that the latter is approximated by the radius itself.[16]

> अतः स्वल्पान्तरत्वात् कर्णो मन्दकर्मणि न कार्यः इति।
> So, there being little difference in the result, the hypotenuse-proportion should not be used in the *manda-saṃskāra*.

[15] K. S. Shukla, "Use of Hypotenuse in the Computation of the Equation of Centre Under the Epicyclic Theory in the School of Āryabhaṭa???", *Indian Journal of History of Science*, 8 (1973), pp. 44–57. The provocative title of the paper is due to the fact that it was written in response to an erroneous claim made by the renowned scholar T. S. Kuppanna Shastri (1900–1982) in his article "The School of Āryabhaṭa and the Peculiarities Thereof" (*Indian Journal of History of Science*, 4 (1969), pp. 126–134).

[16] Shukla (1973), p. 52, citing Pṛthūdaka's commentary on *Brāhmasphuṭasiddhānta* XXI.29.

Shukla also discusses the refutation of the above view of Pṛthūdakasvāmi by Bhāskara II in his *Vāsanābhāṣya* on *Siddhāntaśiromaṇi*.[17] Shukla then considers the case of the *Sūryasiddhānta* and remarks:[18]

> The method prescribed in the *Sūryasiddhānta* for finding the equation of the centre is exactly the same as given by the exponents of the schools of Āryabhaṭa I and Brahmagupta and there is no use of the hypotenuse-proportion. The author of the *Sūryasiddhānta* has not even taken the trouble of finding the *manda* hypotenuse. So it may be presumed that the views of the author of the *Sūryasiddhānta* on the omission of the use of the hypotenuse in finding the equation of the centre were similar to those obtaining in the schools of Āryabhaṭa and Brahmagupta.

Shukla perhaps forgot to mention in this context the important fact that his own 1957 edition of the *Sūryasiddhānta* with the commentary of Parameśvara has a verse (verse IV.2 of the Chapter IV dealing with lunar eclipse) which states that the distance of the Sun or the Moon is proportional to the corresponding "*manda-śravaṇa*" or the *manda*-hypotenuse. And the commentator Parameśvara glosses *manda-śravaṇa* as "*mandasphuṭasiddhakarṇaḥ*", the hypotenuse determined by the location of the true *manda*-corrected planet. Parameśvara also notes that these distances are used for computing diameters of the Sun and the Moon. Shukla notes that this verse is not found in other versions of *Sūryasiddhānta*. All the versions however present an alternate rule for computing the diameters as being inversely proportional to the *sphuṭabhukti* or the true velocity.[19]

In this context, we may also mention that some of the astronomers in North India in the seventeenth century seem to have failed to comprehend the traditional formulation of the *manda-saṃskāra* as expounded by Bhāskara I, Brahmagupta and others. We can see this for instance in the commentary *Gūḍharthaprakaśaka* of Raṅganātha on *Sūryasiddhānta* which was composed in the year 1603. While explaining the verse II.39, which merely prescribes that the radius of the concentric should be the denominator in the expression for the equation of centre, Raṅganātha seems to be following Pṛthūdakasvāmi when he argues that:[20]

> **मन्दकर्णस्य त्रिज्यासन्नत्वेन स्वल्पान्तरेण त्रिज्यातुल्यत्वेनाङ्गीकारात्।**
> [The hypotenuse is not used in the *manda-saṃskāra*] because the

[17] *Ibid.* pp. 52–3.
[18] *Ibid.* p. 54.
[19] *Sūryasiddhānta* with commentary of Parameśvara, ed. by K. S. Shukla, Lucknow University, Lucknow 1957, p. 58.
[20] *Sūryasiddhānta* with commentary *Gūḍharthaprakaśaka* of Raṅganātha, ed. By F. E. Hall and Bāpū Deva Śāstrin, Baptist Mission Press, Calcutta 1859, pp. 77–8.

> *manda*-hypotenuse is close to the radius [of the concentric] and it can be accepted to be the equal to the radius with a slight difference.

Shukla mentions in his 1973 paper that Munīśvara (c. 1646), the son of Raṅganātha, and his famous rival Kamalākara (c. 1658) also did not follow the traditional view on *mandasaṃskāra.* Instead of considering a variable epicycle, they seem to have advocated the use of just the tabulated epicycle and also division by the *sakṛt-karṇa* or the first hypotenuse (K_0 of equation (4)).

In conclusion Shukla notes:[21]

> From what has been said above it is clear that the hypotenuse has not been used in Hindu astronomy in the computation of the equation of the centre under the epicyclic theory. It is also obvious that with the single exception of Caturvedācārya Pṛthūdaka all the Hindu astronomers are unanimous in their views regarding the cause of omission of the hypotenuse. According to all of them the *manda* epicycles stated in the works on Hindu astronomy correspond to the radius of the planet's mean orbit and are therefore false.
>
> Since the *manda* epicycle stated in the Hindu works corresponded to the radius of the planet's mean orbit, the true *manda* epicycle corresponding to the planet's true distance (in the case of the Sun and the Moon) or true-mean distance (in the case of the planets Mars, etc.) was obtained by the process of iteration. The planet's true or true-mean distance (*manda-karṇa*) was also like wise obtained by the process of iteration.
>
> Direct methods for obtaining the true *manda-karṇa* or true *manda* epicycle were also known to later astronomers. Mādhava (c. 1340–1425) is said to have given the following formula for the true *manda-karṇa*:[22]
>
> $$\text{true } \textit{manda-karṇa} \text{ (or iterated } \textit{manda-karṇa}) = \frac{R^2}{\sqrt{R^2 - (\textit{bhujāphala})^2} \mathbin{\tilde{+}} \textit{koṭiphala}},$$
>
> $\sim$ or $+$ sign being taken according as the planet is in the half orbit beginning with the anomalistic sign Capricorn or in that sign beginning with the anomalistic sign Cancer.

[21] Shukla (1973), p. 54.

[22] The reference is to Nīlakaṇṭha's commentary on *Āryabhaṭīya*, III.17–21, and *Tantrasaṅgraha* II.44.

The exact analytical expression of Mādhava for the iterated-*manda-karṇa*, mentioned above by Shukla, can be recast in the form

$$K = \frac{R^2}{\sqrt{R^2 - [r_0 \sin(\theta_0 - \theta_u)]^2} - r_0 \cos(\theta_0 - \theta_u)}. \tag{11}$$

Here, it may be noted the above result (11) can be easily derived[23] by using the similarity of the triangles OQT and OPP_0 in Figure 1. We have, $OP = OQ \times \frac{OP_0}{OT}$, which may be recast in the form:

$$K = \frac{R^2}{R_v}, \tag{12}$$

where, $OT = R_v$ is the so called inverse hypotenuse or *viparīta-karṇa*, which can easily be shown to be given by

$$R_v = \sqrt{R^2 - [r_0 \sin(\theta_0 - \theta_u)]^2} - r_0 \cos(\theta_0 - \theta_u). \tag{13}$$

Using Mādhava's exact expression for the iterated-*manda-karṇa*, we can also obtain the exact equation satisfied by the orbit of a planet which is moving on a variable epicycle as specified in the *manda-saṃskāra*. It is seen that the orbit is no longer an eccentric circle but a general oval figure.

3 How modern scholarship has misconstrued the equation of centre in Indian astronomy

Shukla's detailed explanation of the *manda-saṃskāra* was indeed path-breaking since, for nearly two centuries, modern scholarship had totally misinterpreted this and other aspects of Indian planetary theory. One of the earliest accounts of Indian planetary theory was the 1790 article of Samuel Davis (1760–1819), which was largely based on *Sūryasiddhānta* and its commentary *Gūḍhārthaprakāśaka* of Raṅganātha. While discussing the equation of centre for the Sun and the Moon, Davis remarks that while the hypotenuse is used in Indian astronomy for computing the retrogressions of planets (through the equation of conjunction or *śīghra-saṃskāra*), they do not do so while computing the equation of centre. He cites the commentator (Raṅganātha, whom

[23] For a detailed discussion of Mādhava's exact expression for the iterated *manda*-hypotenuse, see: *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*, ed. and tr., with notes by K. Ramasubramanian and M. S. Sriram, Hindustan Book Agency, New Delhi 2011, pp. 96–107, pp. 496–7. Also, *Madhyamānayanaprakāraḥ of Mādhava*, ed. and tr. with notes by U. K. V. Sarma, R. Venketeswara Pai and K. Ramasubramanian, *Indian Journal of History of Science*, 46.1 (2011), pp. T1–T29.

we have cited earlier) as attributing this to the small difference between the hypotenuse and the radius of the concentric:[24]

> It is, however, only in computing the retrogradations and other particulars respecting the planets Mercury, Venus, Mars, Jupiter, and Saturn, where circles greatly excentric are to be considered, that the Hindus find the length of the *carṇa*, or hypotenuse ...; in other cases, as for the anomalistic equations of the sun and the moon, they are satisfied to take ..., their difference, as the commentator on the [*Sūrya*] *Siddhānta* observes, being inconsiderable.

The next major discussion on Indian planetary theories is found in the 1816 article of Henry Thomas Colebrooke (1765–1837), who had access to many more source-works of Indian astronomy. Colebrooke first reiterates what Davis had noted regarding the equation of centre based on his study of the *Sūryasiddhānta*. Colebrooke then notes that Brahmagupta and Bhāskara II held a different view that the reason why the hypotenuse does not appear in the equation of centre is not due to any approximation being made, but because the epicycle itself varies with the hypotenuse. However, at the same time, Colebrooke also mentions that the commentators of Brahmagupta (Pṛthūdakasvāmi) and Bhāskara (Munīśvara) do not agree with this view:[25]

> The Hindus, who have not any of Ptolemy's additions to Hipparchus, have introduced a different modification of the hypothesis, for they give an oval form to the excentric or the equivalent epicycle, as well as to the planet's proper epicycle. That is they assume that the axis of the epicycle is greater at the end of the (*sama*) even quadrants of anomaly ..., and least at the end of the (*viśama*) odd quadrants ...
>
> A further difference of theory, though not of practice, occurs among the Hindu astronomers ... A reference to Mr. Davis essay ... will render intelligible what has been already said and what now remains to be explained. It is there observed ... that for the anomalistic motion of the sun and moon they are satisfied to take ... the sine of the mean anomaly reduced to its dimensions in the epicycle in parts of the radius of the concentric, equal to the sine of the anomalistic equation. The reason is subjoined: 'The differ-

[24] S. Davis "On the Astronomical Computations of the Hindus", *Asiatic researches*, 2 (1790), pp. 225–287. The quotation appears on page 251 and refers to a diagram on page facing 249.

[25] H. T. Colebrooke, "On the notions of the Hindu Astronomers concerning the Precession of the Equinoxes and the Motions of the Planets", *Asiatic researches*, 16 (1816), pp. 209–250. The quotation appears on pp. 235–238.

> ence as the commentator on the *Sūrya Siddhānta* observes being inconsiderable.'
>
> Most of the commentators on the *Sūrya Siddhānta* assign that reason; but some of them adopt Brahmegupta's explanation. This astronomer maintains that, the operation of finding the *carṇa* is rightly omitted ... His hypothesis as briefly intimated by himself, and as explained by Bhāscara, supposes the epicycle, which represents the excentric, to be augmented in proportion which *carṇa* (or the distance of the planet's place from the earth's centre) bears to the radius of the concentric; and it is on this account, and not as a mere approximation that the finding of the *carṇa*, with subsequent operation to which it is applicable is dispensed with.
>
> The scholiast of Brahmegupta objects to his author's doctrine on this point, that upon the same principle, the process of finding the *carṇa* ... should in like manner be omitted in the proper epicycle of the five minor planets; and he concludes therefore, that the omission of that process has no other ground, but the very inconsiderable difference of the result in the instance of a small epicycle. For as remarked by another author ([Munīśvara] in the *Marīci*[26] [commentary on *Siddhāntaśiromaṇi*]), treating on the same subject, the equation itself and its sine are very small near the line of the apsides; and at a distance from that line, the *carṇa* and the radius approach to equality.

The first English translation, along with detailed explanation, of an Indian astronomical text appeared nearly fifty years later. The translation of *Sūrya-siddhānta* due to Ebenezer Burgess (1805–1870) (as revised by William Dwight Whitney (1827–1894)) was published in 1860. This again noted that the equation of centre in Indian astronomy was based on the approximation that the hypotenuse was nearly equal to the radius. It also claimed that the *manda*-corrected planet was not on any epicycle or eccentric, but was always located on the concentric itself:[27]

> The world wide difference between the spirit of the Hindu astronomy and that of the Greek ... the one is purely scientific, devising

[26] The reference is to the following statement in Munīśvara's commentary *Marīci* (on *Siddhāntasiromaṇi*, *Golādhyāya*, *Chedyakādhikara*, verses 36–37): "तथा च यत्र परमफलासन्नं फलं तत्र कर्णस्य त्रिज्यासन्नत्वेनाल्पान्तरं यत्र च कर्णस्य बह्वन्तरेण त्रिज्यातोऽधिकत्वं न्यूनत्वं वा तत्र फलस्यैवाल्पत्वेनाल्पान्तरत्वमिति भावः।" (*Siddhāntaśiromaṇi* of Bhāskarācārya, *Vāsanābhāṣya* and *Marīci* by Munīśvara, ed., Dattātreya Viṣṇu Āpaṭe, Volume I, Ānandāśrama Press, Poona 1943, p. 190).

[27] E. Burgess, *Translation of the Sūryasiddhānta*, The American Oriental Society, New Haven 1860, p. 48, pp. 64–5.

> methods for representing and calculating the observed motions and attempting nothing further; the other is not content without fabricating a fantastic and absurd theory respecting the superhuman powers which occasion the movements with which it is dealing. The Hindu method has this convenient peculiarity, that it absolves from all necessity of adapting the disturbing forces to one another, and making them form one consistent system, capable of geometrical representation and mathematical demonstration; it regards the planets as actually moving in circular orbits, and the whole apparatus of epicycles ... as only a device for estimating the amount of the force and of its resulting motion, exerted at any given point by the disturbing cause ...
>
> Now as the dimensions of the epicycle in all cases are small, ... may be without any considerable error may be assumed to be equal to ...; this assumption is accordingly made and ... gives the equation concerned.

Nearly a hundred years later, in 1956, when translations of the *Pañca-siddhāntikā*, *Āryabhaṭīya* and the *Khaṇḍakhādyaka* had also became available, the renowned historian of astronomy Otto Neugebauer (1899–1990), presented an analysis of the "Hindu Planetary Theory". He noted that:

> Ignoring the theory of latitude the model which forms the basis for the methods followed, e.g., by the *Sūryasiddhānta*, or in the *Khaṇḍakhādyaka*, is an eccentric epicycle. A model of this type (*cf.* Figure 2) is determined by the radius r of the epicycle, the eccentricity e, and the longitude λ_A of the apogee A' of the deferent of radius R.

Neugebauer also discussed the four-step process of combining the *manda* and the *śīghra-saṃskāras* (equations of centre and conjunction) and found that it was an interesting way of combining these equations—especially when they were given in the form of tables—which is different from the Ptolemaic method of interpolation between extreme values. However, as regards the equation of centre, he repeated what was by then the standard view that it was an approximation:[28]

> Hindu astronomy, however, operates in the case of the correction ... for eccentricity with an approximate formula ... There is no reason to treat the effect of the eccentricity with so much less

[28] O. Neugebauer, "The Transmission of Planetary Theories in Ancient and medieval Astronomy", *Scripta Mathematica*, 22 (1956), pp. 165–192. The quotations appear on pages pp. 176–180.

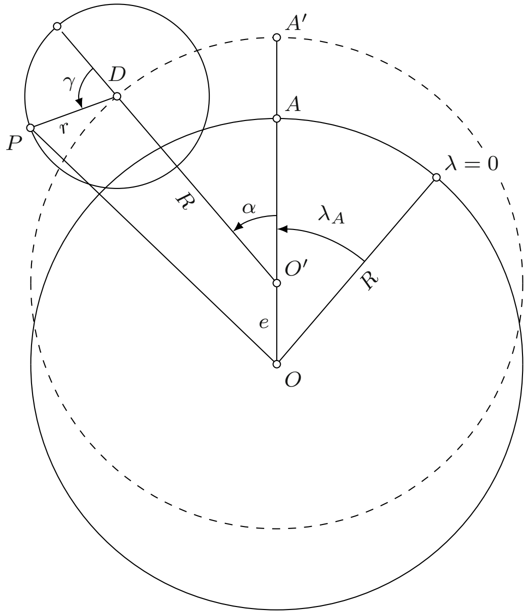

Figure 2: Eccentric epicycle model of Neugebauer.

> accuracy than the effect of the anomaly, except for the fact that usually e [the eccentricity] is smaller than r [the ratio of the radius of the *śīghra* epicycle to that of the concentric]. It may be that in the course of the historical development of planetary theory greater emphasis was attached to the phenomena caused by the anomaly than to those due to the eccentricity, but we know so little about the history of planetary theories that we hardly have any choice except to register the facts.

Starting from the late 1950s, David Pingree (1933–2005), another distinguished scholar of history of exact sciences and a junior collaborator of Neugebauer (whom he succeeded as Professor of History of mathematics at Brown University), brought out a number of studies of Indian astrology and astronomy. Pingree was a reputed scholar of Sanskrit, Akkadian, Arabic and of course Greek and Latin. One of Pingree's main concerns was the transmission of theories and techniques of exact sciences, especially between Mesopotamia, Greece and India in ancient times. Hence, even while a large number of source-works of Indian astronomy—of the classical *Siddhāntic* period (c. 500–1200) and of the medieval Kerala School—had become available (including some of the classic works of Shukla), Pingree, in his studies of Indian planetary theories, seems to have focussed mainly on ancient texts such as the *Paitāmaha-*

siddhānta of *Viṣṇudharmottarapurāṇa*, *Yavanajātaka*, *Pañcasiddhāntikā* and the *Sūryasiddhānta*. It is important to keep in mind that, except for the *Sūryasiddhānta* (which, in any case, is considered to be a later text), all the other texts relied upon by Pingree are in the form of brief summaries that too available only in parts, and the available manuscripts are such that the text had to be substantially emended at several places.

Amidst the large corpus of writings by Pingree on Indian astronomy, we shall here focus only on his analysis of the *manda-saṃskāra* or the equation of centre and some related issues. In 1971, Pingree wrote a paper "On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle". Here, Pingree claims that the "common Indian model for the motion of the star planets" was a "double epicycle model", which "involves two concentric epicycles" and reaffirms the old view of Burgess and Whitney that the planet always moved on a concentric or a deferent circle:[29]

> It is my intention here to investigate the Greek background of the common Indian model for the star planets which involves two concentric epicycles.
>
> In the *Paitāmaha-siddhānta* of the *Viṣṇudharmottarapurāṇa*, which is our earliest extent exponent of the Indian double epicycle model (it was probably composed in the first half of the fifty century AD) the pattern was set for all later texts ...
>
> These two epicycles must be regarded simply as devices for calculating the amounts of the equations by which the mean planet on its concentric orbit is displaced to its true position. This interpretation is confirmed by the explanation offered in early texts of the mechanics of the unequal motions of the planets: demons stationed at the *manda* and *śighra* points on their respective epicycles pull at the planets with chords of wind.[30] The computation of the total effect of these two independent forces upon the mean planet varies somewhat from one school (*pakṣa*) of astronomers to another, or even from astronomer to astronomer within a *pakṣa*; but the fundamental concept remains clear: the planet is always situated on the circumference of a deferent circle concentric with the centre of the earth while two epicycles (one each for the [case of] Sun and Moon) revolve about it.

It is not clear whether any geometric model—not to mention a model where the planets are moving on the concentric—can be inferred at all from the available *Paitāmaha-siddhānta*. What is indisputable is that the vast literature on

[29]D. Pingree, "On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle", *Journal of History of Astronomy*, 2 (1971), pp. 80–85.

[30]The reference here is to *Sūryasiddhānta* II.2.

Siddhāntic astronomy, starting from the *Āryabhaṭīya*, clearly talks of the true *manda*-corrected planet being located on the epicycle or the eccentric. Pingree however thinks that these planetary models were "seldom ... used in computation":[31]

> Āryabhaṭa ... correctly describes an eccentric-epicyclic model and indicates the different directions a planet must travel on an epicycle to produce the differing effects of the equation of the anomaly and the equation of the centre. Though a number of later Indian astronomers acquainted with the *Āryabhaṭīya* or derivative texts of the *Āryapakṣa* refer to the eccentric model, it seems seldom to have been used in computation.

Pingree further claims that his version of the Indian double epicycle model "fits most closely into the attempts of Peripatetics [a group of Greek philosophers owing allegiance to Aristotle] in the late first and second century to preserve concentricity while explaining some of the phenomena."[32]

Pingree also notes that:[33]

> The Indians had to still take into account the problem of the varying distances of the Sun and the Moon whose computation is essential for the prediction of eclipse magnitudes. These distances they made to vary with the true instantaneous velocity of the luminaries.[34] Thereby, of course, as was inevitable, strict concentricity is lost. This fact, however, does not militate against the theory of the peripatetic origin of the Indian double-epicycle model.

In some of his later review articles also, Pingree has reiterated his conception of the Indian planetary model being a double epicycle model with two concentric epicycles (see Figure 3).[35] It continues to be cited in the literature as "the planetary model of the Indian tradition".[36]

Following up on his concentric double-epicycle model, Pingree wrote an article on 'concentric with equant' in 1974, where he makes the even more fantastic claim that the verses IV. 9–12 and IV. 19–21 of *Mahābhāskarīya* give the procedure for computing the motion of a body moving along a circle

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[31] *Ibid.* p. 81.

[32] *Ibid.* p. 83.

[33] *Ibid.* p. 84.

[34] The reference here is to *Paitāmaha-siddhānta* V.3–4.

[35] See for instance, D. Pingree, "Mathematical Astronomy in India", in C. G. Gillespie (ed.), *Dictionary of Scientific Biography*, Vol. XV, New York 1978, pp. 533–633. Also, D. Pingree, "Astronomy in India", in C. Walker (ed.), *Astronomy before Telescope*, British Museum Press, London 1996, pp. 123–42.

[36] See for instance, T. Knudsen, *The Siddhāntasundara of Jñanarāja*, Johns Hopkins University Press, Baltimore 2014, pp. 184–5.

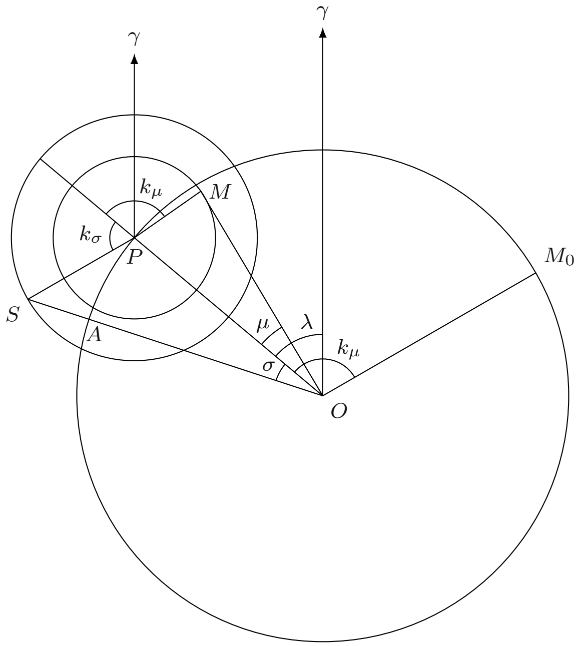

Figure 3: Double epicycle model of Pingree.

but executing uniform motion with respect to a point (equant) displaced from the centre of the circle:[37]

> One purpose of the present article is to point out that a procedure for solving a concentric with equant is described in ... the *Mahābhāskarīya* ...; its second purpose is to suggest a pre-Ptolemaic, Peripataetic origin of the model, and therefore of the equant as well.
>
> In *Mahābhāskarīya* IV. 19–21, is found the method of computing the effect of a concenrtric with equant by means of an eccentric with varying eccentricity. ...
>
> In IV. 9–12, Bhāskara gives an equivalent solution employing an epicycle of varying radius.

It was indeed observed by B. L. van der Waerden and I. V. M. Krishna Rav in the 1950s (whose work has also been cited by Pingree) that the expression for the equation of centre used in Indian astronomy is the same as what would

[37]D. Pingree, "Concentric with Equant", Archives Internationalles d'histoire des Sciences, 24 (1974), pp. 26–28.

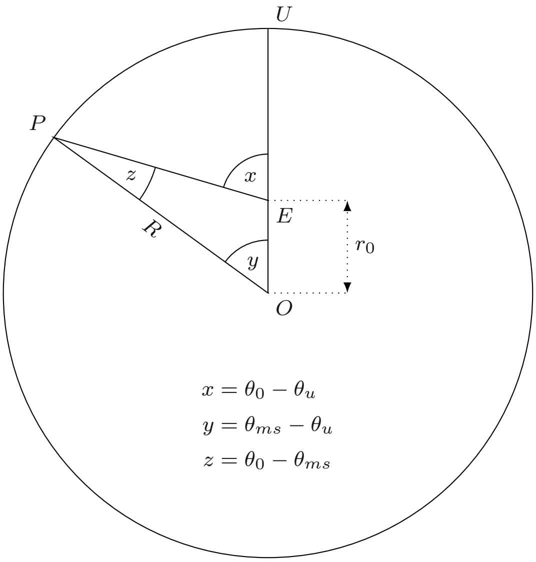

Figure 4: Concentric with equant.

be obtained in the case of a body moving under the hypothesis of 'motion along a concentric with equant'.[38]

In Figure 4, the planet P is moving on the circle with centre O. U is the *ucca* or the apsis and E is the equant point on OU such that $OE = r_0$, the tabulated epicycle radius. If the planet moves uniformly as seen from the equant point, then the angle PEU is the mean anomaly $\theta_0 - \theta_u$, and the angle POU is the true anomaly $\theta_{ms} - \theta_u$, and it can easily be seen that the equation of centre will have the same form as given by equation (3):

$$R\sin(\theta_{ms} - \theta_0) = -\left(\frac{r_0}{R}\right) R\sin(\theta_0 - \theta_u).$$

However, this equivalence is only with respect to the computation of the longitudes of the planets and not their geocentric orbits which also involves the variation of their distance from the centre of the earth. In fact, what Bhāskara is describing in verses IV. 9–12 and IV. 19–21 of *Mahābhāskarīya* is an iterative method to compute the varying *manda-karṇa* or the hypotenuse

[38] See B. L. van der Waerden, "Tamil Astronomy", *Centaurus*, 4 (1956), pp. 221–234, and the references cited there. The title of the paper is due to the fact that the investigations of van der Waerden and Krishna Rav were aimed at understanding the *vākyas* giving the longitudes of the Sun and Moon. It so happens that the *vākya* system of South India had been wrongly characterised as "Tamil Astronomy" by Neugebauer in 1952 (O. Neugebauer, "Tamil Astronomy: A Study in the History of Astronomy in India", *Osiris*, 10 (1952), pp. 252–276).

drawn from the centre of the concentric to the planet on either an epicycle (of variable radius) or an eccentric (of variable eccentricity). That *manda-karṇa*, as the commentator Govindasvāmi notes, is *grahaghanabhūmadhyāntaram*,[39] the distance between the planet and the centre of the earth. There is thus no way that the *manda-saṃskāra* of Indian Astronomy can be conflated with the 'concentric with equant' model of planetary motion—irrespective of whether such a model was known to the Peripatetics (as Pingree suspects) or not.

Pingree, however, has reiterated his claim that, in *Mahābhaskarīya*, "the epicyclic and eccentric models are considered and both are used to solve the concentric with equant model by iteration" in his review article of 1978.[40] Again we find this being echoed in the current literature in statements such as:[41]

> In the early Indian texts the anomalies of the Sun and Moon are both modelled with concentric equant (the Earth is at the centre of the deferent). ...
>
> Bhāskara explains the equivalence of the concentric equant and an oscillating eccentric model by computing one from the other.

It is indeed unfortunate that such distorted views—concerning the formulation of *manda-saṃskāra* and the meaning of *manda-karṇa*—continue to prevail amongst the scholars studying Indian astronomy, notwithstanding the fact that these issues have been dealt with very clearly and conclusively in the books and articles of Prof. Shukla.

4 Correcting the verses giving the *yuga* parameters in Pingree's edition of *Yavanajātaka*

In 1989, Prof. Shukla wrote a seminal article[42] where he examined and corrected the text and translation of about ten verses, which presented the basic

39 *Mahābhaskarīya with Bhāṣya of Govindasvāmi and Siddhāntadīpikā of Parameśvara*, ed. by T. S. Kuppanna Sastri, Government Oriental Manuscripts Library, Madras 1957, p. 190.

40 D. Pingree (1978), p. 593.

41 D. Duke, "Were Planetary Models of India Strongly Influenced by Greek Astronomy?" in J. M. Steele, ed., *The Circulation of Astronomical Knowledge in the Ancient World*, Brill, Leiden 2018, pp. 559–575. The quotations appear on pages pp. 562, 570–1. It may also be noted that in this and some of his earlier articles Duke has shown that the four-step process used in Indian planetary models gives a better approximation to the Ptolemaic model with the equant, than is the case with a simple eccentric-epicycle model.

42 K. S. Shukla,"The Yuga of the *Yavanajātaka*: David Pingree's Text and Translation Reviewed", *Indian Journal of History of Science*, 34 (1989), pp. 211–223.

parameters of the yuga, in chapter 79 of the famous critical edition and translation of *Yavanajātaka* of Sphujidhvaja published by Prof. Pingree in 1978.[43]

The publication of *Yavanajātaka* of Sphujidhvaja, was an important milestone that established Prof. Pingree as a leading scholar of history of astrology and astronomy in the ancient world. In his preface, Pingree recounts the importance of the work, how hard he had to work for getting the manuscript and editing and translating it over nearly two decades, and about his confidence that his "main conclusions are unassailable":[44]

> Sphujidhvaja first attracted my attention over twenty years ago, when I read the brief account of the *Yavanajātaka* given by Mahāmahopādhyāya Haraprasād Śāstri ... I spent the academic year 1957–58 in India ... In December of 1957, I travelled to Nepal to attempt to see the manuscript of the *Yavanajātaka*, but this privilege was not granted to me. Fortunately, in the spring of 1958, Mahāmahopādhyāya Pandurang Vaman Kane, with the utmost kindness and generosity allowed me to copy a transcript that he had acquired of ff. 2–19 and ff. 98–103. On the basis of this fragment I recognised both the Greek origin of the treatise which had been previously surmised from its title, and the Babylonian character of its planetary theory.
>
> It was not however until 1961 that a microfilm of the complete manuscript (lacking, however, f. 102) was obtained through the good offices of my guru Professor Daniel Ingalls of Harvard University, and the then ambassador to India and Nepal from the United States, Professor John Kenneth Galbraith. ... During the years 1961–67, ... I transcribed the Kathmandu Manuscript, established a text, translated it and wrote the commentary; the work then was completed essentially a decade ago. In the interim I have tried to keep the commentary up to date, though I have not been totally successful in this effort. But whatever falsehoods or misrepresentations may persist, I am confident that the main conclusions are unassailable: The greater part of the *Yavanajātaka* was directly transmitted (with some necessary adjustments) from Roman Egypt to Western India, and this text is one of the principle sources for the long tradition of horoscopic astrology in India.

In this context, Pingree also referred to the communication he had received

[43] D. Pingree, *The Yavanajātaka of Sphujidhvaja*, Vols. I, II, Harvard University Press, Cambridge 1978.

[44] *Ibid.* Vol. I, pp. v–vi.

from Prof. Shukla mentioning the citations of *Yavanajātaka* found in the *Āryabhaṭīya-bhāṣya* of Bhāskara I:[45]

> As one further evidence of its influence on India science I quote from a letter written to me by Professor Kripa Shankar Shukla, dated Lucknow 26 January 1977. He informs me that in his *Āryabhaṭīya-bhāṣya* written in 629 (of which important work Professor Shukla is publishing a long-awaited critical edition this year) Bhāskara cites from 'Sphujidhvajayavaneśvara' verses 55–57 of Chapter 79 and from 'Yavaneśvara' *pādas* a–b of Verse 89 of Chapter 1.

Pingree reiterated some of these points in his introduction also:[46]

> For an estimate of how much the Brāhmaṇas borrowed from the Greeks and for an evaluation of how they developed what they borrowed, no text is more pertinent than Sphujidhvaja's *Yavanajātaka* (*The Horoscopy of the Greeks*). Its importance in the history of ancient science has led me, despite difficulties, to edit here all that can be recovered of the work and to accompany the edition with a translation and commentary. ... What we have in *Yavanajātaka*, then is the clearest evidence that has yet come to light of the direct transmission of scientific knowledge from the ancient world of the Mediterranean to the ancient world of India.

Pingree also mentions the difficulty that he had in editing the manuscript and the method he adopted:[47]

> The difficulty of editing and understanding Sphujidhvaja arises from the fact that for most of the text we have only one very incorrectly written manuscript to rely on. The errors of [the main manuscript] N occur, on the average, at least once in every line. Often the expanded version of Mīnarāja [*Vṛddhayavanajātaka*] or some other testimonium comes to our aid; sometimes a knowledge of Sanskrit grammar or idiom suggests the right reading, although Sphujidhvaja was not so exact in his use of Sanskrit as to make this criterion infallible. So we are forced occasionally simply to guess. And I am aware that I must have missed guesses that will occur to others, and that in some cases I will have guessed wrongly. Non omnia possumus omnes [citation from *Aeneid* of Virgil, meaning 'we cannot all do everything'].

[45] *Ibid.* Vol. I, p. vi.
[46] *Ibid.* Vol. I, p. 3.
[47] *Ibid.* Vol I, pp. 22–3.

Pingree's edition of *Yavanajātaka* was highly acclaimed for the detailed critical apparatus and the enormous amount of historical and other data that he had put together. However, the work was not critically reviewed for its contents from a technical point of view. The review by Prof. Shukla was perhaps the first serious review of the book, especially of the 79th chapter which dealt with mathematical astronomy—an unusual feature in what is otherwise a work on *Jātaka* or horoscopy. Shukla notes in the introduction that:[48]

> The *Yavanajātaka* written by Sphujidhvaja Yavaneśvara in the third century AD was edited and translated into English by Prof. David Pingree in 1978. The last chapter (ch. 79) of this work is called *Horāvidhi* and deals with luni-solar astronomy on the basis of a period of 165 years called *yuga* and the synodic motion of the planets. The text is marred by faulty editing, the incorrect readings being adopted and the correct ones given in the apparatus criticus, with the result that the translation is incorrect at places and the meaning really intended by the author is lost.
>
> The object of the present paper is to study this chapter so as to bring out the meaning really intended by the author.

The verses 3–10 of Chapter 79 of *Yavanajātaka* present the basic relations characterising the luni-solar *yuga*[49] of 165 years adopted in the text. Based on his reading and translation of the text, Pingree arrived at the following set of relations, which he presented in his review article of 1978:[50]

$$\begin{aligned} &165 \text{ solar years} = 1{,}980 \textit{ saura} \text{ months} = 2{,}041 \text{ synodic months} \\ &\quad = 58{,}231 \text{ risings of the Moon} = 61{,}230 \textit{ tithis} = 60{,}265 \text{ civil days.} \end{aligned} \tag{14}$$

One can already notice a problem with the above relations (14)—though it was not noticed by Pingree—namely, that the sum of the number of synodic months (2,041) and the risings of the Moon (58,231) is not equal to the number of civil days (60,265). One of the consequence of (14), that was noted by Pingree, is that the length of the solar year turns out to be $6,5;14,32 =$ 365.2424 civil days, which is very close to the tropical year of Hipparchus and Ptolemy (6,5;14,42 civil days). Pingree also noted that his edition and translation of the verses 5, 11–13 and 34 led to values of synodic month,

[48] K. S. Shukla (1989), p. 211.

[49] A luni-solar *yuga*, unlike the *yuga* in *Siddhāntic* texts, is a number of years for which an integral number of sidereal revolutions of the Sun and the Moon are specified along with the number of civil days. The *Vedāṅga-jyotiṣa* uses a luni-solar *yuga* of 5 years. The planetary theory of *Yavanajātaka* on the other hand is based on relations characterizing the synodic motions of planets like in the case of *Vasiṣṭha-siddhānta* of *Pañcasiddhāntikā*.

[50] Pingree (1978), p. 538, equation (III.1).

sidereal month, solar month, etc., which were not consistent with the above relations (14) characterising the luni-solar yuga.[51]

While analysing the verses of Chapter 79, Shukla realised that Pingree had failed to understand the internal logic of the luni-solar *yuga* of Sphujidhvaja, as a result of which he had gone about adopting incorrect readings in place of correct readings found in the manuscript and given as a part of apparatus criticus. Shukla also noticed that Pingree had often misunderstood or misinterpreted various numerical expressions that occurred in the text.

The crucial errors were in the edition and translation of verses 6, 7. The first half of the verse 6 dealt with the notion of *tithi* and its importance. The second half mentioned the number of 'them' ('*teṣām*') in a *yuga*. Pingree chose to interpret this as a reference to the number of civil days, and after emending the readings came up with the interpretation that a *yuga* consisted of 60,265 civil days. Shukla noticed that the verse should be interpreted as giving the number of *tithis* and, using the correct readings that were given in the apparatus, he edited the verse and the translation leading to the interpretation that the *yuga* consisted of 60,230 *tithis*.

The first half of the verse 7 deals with the fact that a *dinarātra* (nychthemeron, civil day) consists of 30 *muhūrtas* and it begins with sunrise. The second half gives their (*teṣām*) number in a *yuga*. Now, Pingree chose to interpret this as referring to the number of *tithis* in a yuga. He emended the manuscript readings again to arrive at the interpretation that a *yuga* had 60,230 *tithis*. Here, Shukla noticed that the verse should be interpreted as giving the number of civil days and, using the correct readings that were given in the apparatus, he edited the verse and the translation leading to the interpretation that the *yuga* consisted of 60,272 civil days.

After his analysis of verses 6 and 7, Shukla remarks:[52]

> Pingree is aware of the fact that the second half of vs. 6 should contain the number of *tithis* in a *yuga* and the second half of vs. 7 the number of civil days in a *yuga*, but his text has landed him in trouble and he remarks: 'A more logical order might be achieved by interchanging 6c–d with 7c–d.' He also complains about Sphujidhvaja Yavaneśvara's way of expressing numbers in verse: 'The extreme clumsiness with which Sphujidhvaja expresses numbers is a reflection of the fact that a satisfactory and consistent method of versifying them had not yet been devised in the late third century.' But these remarks are uncalled for, as it is all due to the faulty edited text.

[51] *Ibid.* p. 538, Tables III.2, III.3.
[52] Shukla (1989), p. 216.

The basic relations that characterise the luni-solar *yuga* of Sphujidhvaja, according to Shukla, are

$$\begin{aligned}&165 \text{ solar years} = 1{,}980 \textit{ saura} \text{ months} = 2{,}041 \text{ synodic months}\\ &\quad = 58{,}231 \text{ risings of the moon} = 61{,}230 \textit{ tithis} = 60{,}272 \text{ civil days.} \qquad (15)\end{aligned}$$

Here, we see that these basic parameters are consistent and the sum of the number of synodic months and the risings of the Moon is indeed equal to the number of civil days. Equally important is the fact that the solar year now turns out to be $(365 + \frac{47}{165}) = 365.28485$ days, fairly close to the standard sidereal year used in *Siddhāntic* astronomy.

In Table 1, we have summarised the corrections made by Shukla to the reading and/or the translation of verses 5, 6, 7, 11, 12, 13, 19, 28, 29 and 34. Shukla noted that they were all consistent with the basic relations (15) characterising the luni-solar *yuga* of Sphujidhvaja. He also restored most of the faulty emendations done by Pingree by readings based on the apparatus, and carefully corrected the translation of each of these verses. One of the important corrections made by Shukla was pertaining to the verses 28–29, which dealt with time measures. Here, Pingree's emendation had resulted in the relation 1 *Nāḍikā* = 30 *Kalās*, which is not attested anywhere in the ancient texts. Shukla restored the reading given in the apparatus to arrive at the correct relation 1 *Nāḍikā* = 10 *Kalās*. Shukla noted that this is the relation given by Parāśara and Suśruta and close to the relation (1 *Nāḍikā* = $10\frac{1}{20}$ *Kalās*) found in the *Vedāṅga-jyotiṣa*. It also makes the values given in verses 11–13 (where fractions of a day are specified in terms of *Kalās* etc.) consistent with the basic *yuga* relations (15).

Finally, at the end of his paper, Shukla noted that the basic *yuga* parameters given by (15) and the values of solar year, synodic month and sidereal year as corrected by him are indeed close to the values specified in the *Sūryasiddhānta*.

Following the corrections to the text and translation of the verses giving the *yuga* parameters worked out by Shukla in his pioneering article of 1989, Harry Falk in 2001 pointed out another major flaw in the edition and translation of the verse 14 of Chapter 79.[53] Falk showed that Pingree had wrongly emended this verse and its meaning to conclude that the epoch of the work was 23 March 144 CE. From the available manuscript reading in the apparatus, Falk showed that the epoch should be 21 March 22 CE.

A spectacular breakthrough in the study of *Yavanajātaka* has occurred since 2011 with the discovery of a new Nepalese paper manuscript of the text by Prof. Michio Yano. Based on this, and also making use of better copies of

[53] H. Falk, "The Yuga of Sphujidhvaja and the Era of the Kuśanas", *Silk Road Art and Archaeology*, 7 (2001), pp. 121–136.

Table 1: Corrections carried out by Shukla.

Verse	Pingree's edition/translation	Shukla's corrected edition/translation
5	*Tithi* $= (1 - \frac{1}{64})$ civil day Civil day $= (1 + \frac{1}{60})$ *tithi* No. of *tithis* in a *yuga* $= 990 \times 62 =$ 61,380	*Tithi* $= (1 - \frac{1}{64})$ civil day Civil day $= (1 + \frac{1}{63})$ *tithi* No. of omitted *tithis* in a *yuga* $= 958$
6	No. of civil days in a *yuga* = 60,265	No. of *tithis* in a *yuga* = 61,230
7	No. of tithis in a *yuga* = 61,230	No. of civil days in a *yuga* = 60,272
11	Solar month $= 30;26,9,52,4$ days	Solar month $= 30;26,25,27,16$ days $= 30.44040$ days
12	Synodic month $= 30;3,55,34$ days	Synodic month $= 29;31,50,14,24$ days $= 29.53062$ days
13	Sidereal month $= 27;17,10,34$ days	Sidereal month $= 27;19,18,39$ days $= 27.32185$ days
19	Intercalary days in a solar year $= 11;11$	Intercalary days in a solar year $= 11\frac{1}{11}$
28, 29	1 *Nāḍikā* = 30 *Kalās*	1 *Nāḍikā* = 10 *Kalās*
34	Solar year $= 365;14,47$ days	Solar year $= 365 + \frac{47}{165}$ days $= 365;17,5,27,16$ days $= 365.28485$ days

the earlier manuscripts, Bill Mak has published a new critical edition of the Chapter 79 of *Yavanajātaka* with translation and notes.[54]

Bill Mak's new edition and translation has added fresh evidence in support of all the corrections—to the text as well as translation of the ten verses, and the corresponding *yuga* parameters—carried out by Shukla in his pioneering study of 1989.[55] These were indeed remarkable corrections carried out merely on the basis of the apparatus supplied by Pingree in his edition. Just to illustrate this, we shall here present a brief extract from the new edition of the corrected text of verse 6, along with the translations of Mak (M) and Pingree (P) and the notes provided by Mak:[56]

> **क्रमेण चन्द्रक्षयवृद्धिलक्ष्यः तिथिश्चतुर्मानविधानजीवम् ।**
> **षद्व्यञ्चकाग्रा द्विशती सहस्रं तेषां युगे विद्ध्ययुतानि षट् च ॥**
>
> M: The *tithi*, which is to be defined by the gradual waning or waxing of the Moon, is the soul of the principles of the four (systems of time-) measurement. Know that there are 60,000 plus 1000 plus 200 and 65 (i.e. 61,230) of them (i.e., *tithis*) in a *yuga*.
> P: The Moon is to be characterized by waning and waxing in order. The *tithi* possesses the seed of the principles of the four (systems of time-) measurement. There are 60,265 (days) in a *yuga*.
>
> ... The main problem of Pingree's reading of this particular verse lies on the fact that he assumed the *teṣām* in *pāda* d to refer to *dina* as opposed to *tithi*, leading to his suggestion that 'a more logical order might be achieved by interchanging 6c–d with 7c–d'. As Shukla pointed out, the verse concerns entirely the number of *tithis* in a *yuga* and the numbers in *pādas* c, d require no emendation. Pingree's fantastic emendation[57] of *binduyutāni ṣaṭ* to mean 60 leads also to his audacious and subsequently highly misleading statement—'If my restoration ... is correct, this is the earliest reference known to the decimal place-value system with a symbol for zero (*bindu*) in India. The extreme clumsiness with which Sphujidhvaja expresses numbers is a reflection of the fact

[54] B. Mak, "The Last Chapter of Sphujidhvaja's *Yavanajātaka* Critically Edited With Notes", *Sources and Commentaries in Exact Sciences*, 13 (2013), pp. 59–148.

[55] The new edition has also reconfirmed the correction of the epoch of the text carried out by Falk.

[56] B. Mak, *Ibid.* pp. 90–92.

[57] One of Pingree's claims has been that the *Yavanajātaka* presents us with the earliest evidence of use of *bhūtasaṅkhyā* and the place value system with a symbol for zero. The dates 149 CE for Yavaneśvara and 269 CE for Sphujidhvaja and some of the emended parameters in Chapter 79 were arrived at on the basis of this supposition. The new edition of Mak shows that, none of this is really attested to in the manuscripts and thus there is no basis for the claim made by Pingree (Mak, ibid., pp. 68–71).

> that a satisfactory and consistent method of versifying them had not yet been devised in the late third century.'[58] This remark is problematic because elsewhere the author of this chapter had no problem expressing himself mathematically without the use of zero or the explicit reference to a place-value system. Thus as Shukla pointed out, Pingree's reading 60,265 is completely wrong and the correct reading is in fact given in his own apparatus. The last line should thus read 60,000 (*ayutāni ṣaṭ*) plus 1,000 (*sahasram*) plus 200 (*dviśatī*) plus 6×5 (*ṣaṭ pañcakāgrā*). ...
>
> Another noteworthy point about this verse is the emphasis on the *tithi* as the 'soul' (*jīva*) of the four calculations. The importance of *tithi* may be summarized by the words of Sastry in the notes to his new reading of *Pañcasiddhānitkā* I.4, where *tithi* was unwarrantedly emended to *kṛta* by Thibaut/Dvivedi and to *stvatha* by Neugebauer/Pingree: "... [the *tithi*] is the chief of the five *aṅgas*, viz. *tithi*, *vāra*, *nakṣatra*, *yoga* and *karaṇa* ... [it] is most useful not only for religious but also civil purposes, ... [it is] the sine qua non of all astronomical computation".[59] The number of *tithis* is first stated here as the basis of some of the remaining calculations. The use of *tithi* is not attested in any Greek work extant and the importance given to it in this work suggests this formulation of the 'best of the Greeks' may be the work of the Greek community long settled in India with great familiarity with the indigenous systems, rather than a translation of a 'lost work composed in Alexandria' with sporadic Indian flavors as Pingree suggested.

On the basis of his critical study of the new manuscript (and fresh copies of the older ones), Mak has indeed provided incontrovertible evidence to overturn many of the claims by Pingree in his edition, claims which have had a major impact on the historiography of astronomy and astrology in India in relation to developments in Mesopotamia and Greece.[60] His new edition of the Chapter 79 also shows that what may be needed is perhaps a new edition of the entire text. When, thirty years ago, Prof. Shukla presented his review of a section of Chapter 79 of Pingree's edition of *Yavanajātaka*, indeed few would have imagined that it would lead to a denouement such as this.

[58] The reference is to D. Pingree, *The Yavanajātaka of Sphujidhvaja*, Vol. II, Harvard University Press, Cambridge (1978), pp. 406–7.

[59] The reference is to *Pañcasiddhāntikā of Varāhamihira*, ed. and tr. with notes by T. S. Kuppanna Sastry and K. V. Sarma, PPST Foundation, Madras 1993, p. 5.

[60] For further details see B. Mak (2013), cited above. Also, B. Mak, "The Date and Nature of Sphujidhvaja's *Yavanajātaka* reconsidered in the light of some newly discovered materials", *History of Science in South Asia*, I (2013), pp. 1–20.

5 Publication of Part III of “History of Hindu Mathematics: A Source Book” by Datta and Singh

The two parts of the famous “History of Hindu Mathematics A Source Book” by Bibhutibhusan Datta (1888–1958) and Avadhesh Narayan Singh (1905–1954) were published in 1935 and 1938. They dealt with Arithmetic and Algebra, respectively. In their preface to the first part (dated July 1935), the authors mention that they had prepared a third part also:[61]

> It has been decided to publish the book in three parts. The first part deals with the history of numerical notation and arithmetic. The second is devoted to algebra, a science in which the ancient Hindus made remarkable progress. The third part contains the history of geometry, trigonometry, calculus and various other topics such as magic squares, theory of series and permutations and combinations.

Datta had resigned from the Calcutta University in 1929 itself. He returned to the University in 1931 to deliver his famous lectures on The Science of Śulba, which got published in a book form in 1932. He finally retired from the University in 1933 and took Sanyāsa in 1938 (the year in which the second part of the Datta and Singh book appeared) and became Swami Vidyāraṇya. He spent much of his later life at Pushkar.

As regards the Part III, R. C. Gupta mentions the following in his biographical essay of 1980 on Datta:[62]

> Part III (Geometry, Trigonometry, Calculus, etc.) of the History of Hindu Mathematics by Datta and A. N. Singh (died 1954) has never been published although more than 40 years have passed since the appearance of Part II. The information given by the late Binod Bihari Dutt [brother of Bibhutibhusan Datta] in a personal communication dated September 11, 1966 . . . that Part II has been lost, turned out to be wrong. Manuscripts of Part III exist at Lucknow with Dr. K. S. Shukla . . . and with the writer (R. C. G.) of the present article who received it from (and due to kindness of) Dr. S. N. Singh (son of A. N. Singh). It is unfortunate that the authors (particularly A. N. S.) could not ensure the publication of Part III . . ., although they lived long enough after the appearance of Part II to have perhaps done so. It is also unfortunate that when

[61] Bibhutibhusan Datta and Avadhesh Narayan Singh, *History of Hindu Mathematics: A Source Book*, Part I Motilal Banarsi Das, Lahore 1935, p. ix.

[62] R. C. Gupta, “Bibhutibhusan Datta (1999–1958), Historian of Indian Mathematics”, *Historia Mathematica*, 7 (1980), pp. 126–133.

> Parts I and II were reprinted [in 1962], no attempt was made to bring the work up to date. Part III is expected to appear shortly, in a serialised form, in the *Indian Journal of History of Science*.

On the history of publication of Part III, Sukomal Dutt notes the following in his 1988 article on Datta:[63]

> Manuscript of Part III of the book was traced by the writer in 1979, 41 years after Part II in a miraculous way. Though strange and unbelievable it may sound to others, he was guided by the Holy Spirit, Swami Vidyaranyaji; after a year's intensive prayer to him. Only then, Dr. K. S. Shukla retired professor of mathematics of Lucknow University, kindly took upon hand its publication serially in the '*Indian Journal of History of Science*'. According to his statement Swamiji himself handed over the manuscript to him after death of Dr. A. N. Singh, which should have been before 1958. He did not take any action on it till the writer found him out and asked for the mss.

In this context, we may draw attention to the fact that Prof. Shukla himself has referred to his interaction with Bibhutibhusan Datta (Swami Vidyāraṇya) in 1954. In the preface to his 1976 edition and translation of *Āryabhaṭīya*, Shukla acknowledges the valuable suggestions made by Datta in 1954:[64]

> I wish to express my deep sense of gratitude to my teacher, the late Dr. A. N. Singh, and to the late Dr. Bibhutibhusan Datta, who, in 1954, had gone through the English translation and notes and had offered valuable suggestions for their improvement.

Since A. N. Singh also passed away around the same time, in 1954, that would have been the occasion when Datta had bequeathed the manuscript of Part III of their work to Shukla. In the same edition of *Āryabhaṭīya*, Shukla also refers to the manuscript of Part III, while citing the translation of Datta and Singh of Verse 12 of *Gaṇitapāda*.[65] Perhaps he had already made up his plans to publish a revised version of Part III after his retirement in 1979. This revised version was published in the form of the following eight articles which appeared in the *Indian Journal of History of Science* during 1980–1993:

[63] Sukomal Dutt, "Bibhuti Bhusan Datta (1888–1958) or Swami Vidyaranya", *Gaṇita-Bhāratī*, 10 (1988), pp. 3–15.

[64] *Āryabhaṭīya* 1976, p. lxxvii. It may be noted that in the preface to the 1960 edition of *Mahābhāskarīya*, Prof. Shukla makes a similar acknowledgement: "I am also under great obligation to the late Dr. Bibhutibhusan Datta (*alias* Swami Vidyaranya) who kindly went through the whole of this work and gave valuable suggestions and advice" (*Mahābhāskarīya* 1960, p. ix).

[65] *Āryabhaṭīya* 1976, p. 52.

1. "Hindu Geometry", *Indian Journal of History of Science*, 15 (1980), pp. 121–188.
2. "Hindu Trigonometry", *Indian Journal of History of Science*, 18 (1982), pp. 39–108.
3. "Use of Calculus in Hindu Mathematics", *Indian Journal of History of Science*, 19 (1984), pp. 95–104.
4. "Magic Squares in India", *Indian Journal of History of Science*, 27 (1992), pp. 51–120.
5. "Use of Permutations and Combinations in India", *Indian Journal of History of Science*, 27 (1992), pp. 231–249.
6. "Use of Series in India", *Indian Journal of History of Science*, 28 (1993), pp. 103–129.
7. "Surds in Hindu Mathematics", *Indian Journal of History of Science*, 28 (1993), pp. 253–264.
8. "Approximate values of Surds in Hindu Mathematics", Indian Journal of History of Science, 28 (1993), pp. 265–275.

Prof. K. S. Shukla has thus contributed immensely to our current understanding of the concepts, techniques and methodology of the Indian tradition of astronomy and mathematics, including its historical development. He has also left us with extremely readable books which can be profitably used by students who are keen to study this vast subject.

Part II

Studies in Indian Mathematics: Bhāskara I to Nārāyaṇa Paṇḍita

Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (I) *

This paper, being the first of the series, is introductory and deals with certain notable features of Bhāskara I's mathematics.

1 Introduction

Hindu works exclusively dealing with *Pāṭīgaṇita* (=arithmetic and mensuration) and *Bījagaṇita* (=algebra) written before the ninth century AD have not come down to us. Bhāskara I writing in 629 AD refers to the mathematicians Maskarī, Pūraṇa,[1] Mudgala, Patana, and others; and Pṛthūdaka writing in 860 AD refers to the mathematician Skandasena who was an author of a work on *Pāṭīgaṇita*.[2] But the writings of these mathematicians have not survived as they were replaced by the works of the subsequent writers such as Mahāvīra (850 AD), Śrīdhara (c. 900 AD), Bhāskara II (1150 AD), and others. Our knowledge regarding the growth and development of mathematics in the fifth, sixth, and seventh centuries AD is therefore based on the chapters dealing with *Pāṭīgaṇita* and *Bījagaṇita* occurring in the *Āryabhaṭīya* (499 AD) of Āryabhaṭa I and the *Brāhmasphuṭasiddhānta* (628 AD) of Brahmagupta. Although these works have thrown ample light on the nature and scope of arithmetic and algebra in those times, but, being essentially devoted to astronomy, their treatment of arithmetic and algebra is brief and confined to the statement of the important rules pertaining to those subjects. They do not go into the details and do not even give examples based on the various rules stated, as exercises for the student. It has therefore hitherto not been

* K. S. Shukla, *Gaṇita*, Vol. 22, No. 1 (June 1971), pp. 115–130.

[1] Maskarī Pūraṇa as one name is mentioned by Ācārya Śrutasāgara Sūri (1525 AD) in his commentary on the *Bodhaprābhṛta* (*Gāthā* 53) and the *Bhāvaprābhṛta* (*Gāthās* 84 and 135) of Ācārya Kunda Kunda (c. 450 AD). So it may be that Maskarī Pūraṇa is the name of one and the same person. I am indebted for this information to the late Swami Vidyāraṇya (formerly Dr Bibhutibhushan Datta).

[2] Reference to Skandasena is also made in another mathematical work called *Gaṇitāvalī*, where the work of Skandasena is described as "very obscure and very difficult to understand". A palm-leaf manuscript of the *Gaṇitāvalī* occurs in the collection of the Royal Asiatic Society of Bengal.

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possible to have a clear idea of the nature of the mathematical problems set to the students in those ancient times.

The discovery of the works of Bhāskara I has now enabled us to have some more light regarding the mathematical knowledge in India in the seventh century AD. Bhāskara I, who was a contemporary of Brahmagupta, wrote a commentary on the *Āryabhaṭīya* in 629 AD. In this commentary, he has not only given a detailed and exhaustive exposition of the mathematical rules stated in the *Gaṇita Section* of the *Āryabhaṭīya* but has also supplied illustrative examples with full solutions for each and every rule stated in that *Section*. These examples are the earliest on record, excepting those of the *Bakhshālī Manuscript* of uncertain date. Of these examples, the most interesting ones are the hawk and rat problems, the bamboo problems, the lotus problems, and the crane and fish problems, which are meant to illustrate the application of the following property of the circle: "If the diameter ABC and the chord LBN of a circle intersect at right angles, then $LB^2 = AB \times BC$". Bhāskara I has ascribed these problems to previous writers and they must have occurred in the works of earlier mathematicians referred to and consulted by him. Similar examples are found to occur in the writings of Mahāvīra (850 AD), Pṛthūdaka (860 AD), Bhāskara II (1150 AD), and Nārāyaṇa (1356 AD) also.

Other important examples relate to the theory of the pulverizer, i.e., the indeterminate equations of the first degree of the types:

$$\text{(i)} \quad M = ax \pm b = cx \pm d = \dots,$$

$$\text{(ii)} \quad \frac{ax \pm c}{b} = y,$$

where x and y are the unknown variables and M the unknown constant to be determined, and $a, b, c, d, \dots$ are the known constants given in the examples. Bhāskara I's examples constitute a good collection on the subject. In fact, a better collection is not to be found in any other work on Hindu mathematics. A number of these examples relate to the various applications of the theory of the pulverizer to problems in astronomy. In order to facilitate solution of such astronomical examples, Bhāskara I has appended a number of tables relating to the various planets or relevant astronomical constants. These, it may be noted, are the only tables of their kind available to us. Bhāskara I's exposition of Āryabhaṭa I's rules of the pulverizer is very detailed and exhaustive and it won for him a great name as a scholar. Devarāja, who wrote an independent work entitled "the Crest of the Pulverizer" (*Kuṭṭākāra-śiromaṇi*) in a concluding stanza of that work remarks:

> In this work I have exhibited the theory of the pulverizer by indications only. For every other matter that has remained from being included here, one should consult Bhāskara I's *Bhāṣya* on the *Āryabhaṭīya*, etc.

The scope of Bhāskara I's examples, it must be pointed out, is strictly limited to the rules of the *Gaṇita Section* of the *Āryabhaṭīya* and so they give only a partial view of the mathematical problems set to students in those times.

Besides giving the examples, Bhāskara I has quoted a number of passages from the then existing works on mathematics. Of these passages, some are taken from the works which were popular and studied as text-books in his time. In the case of such passages only the initial few words have been mentioned, the rest being assumed to be well known to the reader. Some passages have been cited to point out the approximate character of the rules contained in them so as to emphasise the superiority of the corresponding rules of the *Āryabhaṭīya* and some are quoted to find fault with them. A number of passages are in *Prākṛta Gāthās* and seem to have been derived from Jaina sources. From what Bhāskara I has written and from the quotations cited by him it appears that the mathematical works then in existence were generally of the same pattern and followed the same sequence of arrangement as the *Pāṭīgaṇita* or *Triśatikā* of Śrīdharācārya or the other later works on mathematics.

In the present series of papers we propose to throw light on certain notable features of Bhāskara I's mathematics and discuss the mathematical examples as well as the mathematical quotations occurring in Bhāskara I's commentary on the *Āryabhaṭīya*.

2 Notable features of Bhāskara I's mathematics

2.1 Use of numbers and symbolism

Integral and fractional numbers

Bhāskara I freely uses both integral and fractional numbers, and amply illustrates all arithmetical operations on integral as well as fractional numbers. The following results are known to him:

$$a \pm \left(\frac{b}{c}\right) = \frac{(ac \pm b)}{c},$$

$$\left(\frac{a}{b}\right) \pm \left(\frac{c}{d}\right) = \frac{(ad \pm bc)}{bd},$$

$$\left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{ac}{bd},$$

$$\left(\frac{a}{b}\right) \div \left(\frac{c}{d}\right) = \frac{ad}{bc},$$

$$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2},$$

$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3},$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$
$$\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$$

He writes integral numbers within square or circular cells, and also sometimes without making use of any cells. Thus, the numbers 4, 6, and 7 are written in any one of the following ways :

$$\begin{array}{|c|c|c|}\hline 4 & 6 & 7 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|} 4 & 6 & 7 \\ \hline\end{array} \qquad \textcircled{4} \quad \textcircled{6} \quad \textcircled{7} \qquad 4\text{—}6\text{—}7$$

For writing fractional numbers also, no uniformity is maintained. Sometimes they are enclosed within rectangular cells or brackets, sometimes not. The dividing line is never used. Thus, the number $4\frac{5}{6}$ is written in any one of the following ways:

$$\begin{array}{|c|}\hline 4 \\ 5 \\ 6 \\ \hline\end{array} \qquad \begin{array}{|c|} 4 \\ 5 \\ 6 \\ \hline\end{array} \qquad \underbrace{\begin{array}{|c|} 4 \\ 5 \\ 6 \end{array}} \qquad \begin{array}{|c|} 4 \\ 5 \\ 6 \end{array} \qquad \left(\begin{array}{c} 4 \\ 5 \\ 6 \end{array}\right) \qquad \begin{array}{c} 4 \\ 5 \\ 6 \end{array}$$

The enclosures were probably meant to avoid confusion and also to ensure that the numbers enclosed were not mixed up with the written matter.

Surds

Use is also made by Bhāskara I of surds. He knows that

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab},$$
$$a \times \sqrt{b} = \sqrt{a^2 b},$$
$$\sqrt{ab} + \sqrt{bc} = \sqrt{b}(\sqrt{a} + \sqrt{c}), \text{ etc.}$$

He has also quoted rules on surds from earlier works.

Symbol for the minus sign

For the minus sign, Bhāskara I makes use of a little circle (°) on the right of the number to be subtracted. For example, $\left(\frac{1}{2}\right) - \left(\frac{1}{6}\right)$ is written as

$$\begin{array}{|cc|} 1 & 1 \\ 2 & 6^\circ \\ \hline\end{array}$$

In later works, the little circle is generally replaced by a dot.[3]

[3] In the *Bakhshālī Manuscript*, the symbol used for the minus sign is the modern *plus* sign (+). In another manuscript acquired from Kashmir, containing an anonymous commentary on the *Pāṭīgaṇita* of Śrīdhara, the symbols used for the negative sign are both + and

Negative numbers

Bhāskara I has called a negative number by the term *ṛṇa*.[4] The following results are known to him:

$$\begin{aligned} b - a &= -(a - b), \quad a > b, \\ (-a) - (-b) &= -(a - b), \\ a - (-b) &= a + b, \\ (-a) - b &= -(a + b). \end{aligned}$$

Symbols of operation

Mathematical operations are sometimes indicated by placing the tachygraphic abbreviations after the quantities affected. Thus, the operation of addition is indicated by *kṣe* (from *kṣepa*), subtraction by *a* (from *antara*), multiplication by *gu* (from *guṇakāra*), and division by *hā* (from *hāra*). Similar abbreviations are found to be used in the *Bakhshālī Manuscript* also.

Symbols for unknowns

Quantities of unknown value are called *yāvattāvat* (meaning "as many as", or "as much as"), or *gulikā*. The latter term was used earlier by Āryabhaṭa I, and is interpreted by Bhāskara I as follows:

> By the term *gulikā* is expressed a thing of unknown value.[5]

Gulikā and *yāvattāvat* are used as synonyms. Bhāskara I writes:

> These very *gulikās* of unknown value are called *yāvattāvat*.[6]

In arithmetical problems, the unknown or missing quantities have been denoted by the zero symbol, as in the *Bakhshālī Manuscript* and other Hindu works on arithmetic.

×, which are sometimes written to the right and sometimes to the left of the numbers affected. The use of + above the number affected is found in early Jaina literature. For example, in the commentary, entitled *Dhavalā*, on the *Ṣaṭkhaṇḍāgama*, Vol. 10, p. 151, the commentator writes −1 as $\boxed{\overset{+}{1}}$. The letter *ri*, the first letter of the word *riṇa*, is written in the Brāhmī script as . It may be that this letter was originally used to denote the negative sign. Subsequently,  changed into +. How the little circle came to be used for the negative sign is not very clear.

[4] In the above-mentioned manuscript, acquired from Kashmir, free use is made of negative numbers. The number −2, for example, is written as +2 or ×2.

[5] गुलिकाशब्देनाविज्ञातमूल्यवस्तु त्वभिधीयते। Comm. on *Ā* (=*Āryabhaṭīya*), ii. 30.

[6] एत एव गुलिका अविज्ञातप्रमाणा यावत्तावन्त उच्यन्ते। Comm. on *Ā*. ii. 30.

2.2 Classification of mathematics

First classification

Bhāskara I writes:

> This mathematics (*gaṇita*) is (fundamentally) of two kinds, which permeate the four (fundamental operations). These two kinds are increase and decrease. Addition is increase, and subtraction is decrease. These two varieties permeate the whole of mathematics. So has been said: 'Multiplication and involution are the kinds of addition, and division and evolution, of subtraction. Seeing that the science (of mathematics) is permeated by increase and decrease, this science is indeed of two kinds.[7]

To emphasise the above dictum, he raises the following doubts:

> If it is so, how will the operations (of mathematics) be performed? For, when we multiply $\frac{1}{4}$ by $\frac{1}{5}$, we get $\frac{1}{20}$. But multiplication has been defined (above) as a kind of addition, and here it has turned out to be a kind of subtraction. Similarly, when we divide $\frac{1}{20}$ by $\frac{1}{4}$, we get $\frac{1}{5}$. So here (division) which has been defined as a kind of subtraction has turned out to be a kind of addition.[8]

He resolves the doubts as follows:

> In both the cases, the doubts are removed as follows: In a square field with unity as length and breadth, there are twenty rectangular fields. Each one of them has $\frac{1}{5}$ for its length, and $\frac{1}{4}$ for its breadth. Their product $\frac{1}{20}$ is the area of the (rectangular) field. So there is no fault (fallacy) if $\frac{1}{20}$ divided by $\frac{1}{4}$ comes out to be $\frac{1}{5}$. This is how the above doubts are removed geometrically. In order to remove them symbolically attempts may be made.[9]

The last passage is of special significance to historians of Hindu mathematics. It proves that in the first half of the seventh century (if not earlier) use was made of two methods of demonstration in mathematics: (i) geometrical, and (ii) symbolical. In this connection we quote the following lines from Datta and Singh's *History of Hindu mathematics*:[10]

> The method of demonstration has been stated to be 'always of two kinds: one geometrical (*kṣetragata*) and the other symbolical

[7] See the beginning of the comm. on $\bar{A}$. ii.
[8] *l.c.*
[9] See the beginning of the commentary on $\bar{A}$, ii.
[10] Part II, pp. 3–4.

> (*rāśigata*)'.[11] We do not know who was the first in India to use geometrical methods in demonstrating algebraical rules. Bhāskara II ascribes it to 'ancient teachers'.[12]

Second classification

Bhāskara I informs us that certain scholars classified mathematics under the two heads, *kṣetragaṇita* (geometrical mathematics) and *rāśigaṇita* (symbolical mathematics). "Other teachers say," writes he, "that mathematics is of two kinds—symbolical (*rāśi*) and geometrical (*kṣetra*)."[13] According to this divisions, says he, proportion and indeterminate analysis of the first degree, etc., fall under the former, and series, problems on shadow, etc., fall under the latter. The mathematics of surds (*karaṇī-parikrama*) formed part of both of them. For, a surd quantity was both a number and a line (represented by the hypotenuse of a right-angled triangle).

It may be asked: On what grounds were series classified under geometrical mathematics? To a student of modern mathematics, who recognises series as part of algebra, the question is quite relevant. But nowhere in his commentary has Bhāskara I made an attempt to throw light on this point. The mathematics of series has special reference to the area of a ladder. The word *śreḍhī*, which is used to denote a series in Hindu mathematics, means a ladder; the word *pada* or *gaccha*, which is used to mean the number of terms in a series, means the steps of a ladder; and the word *śreḍhīphala*, which is used for the sum of a series, means the area of a ladder. This clearly shows why in Hindu mathematics series were called by the name *śreḍhī* (ladder). The above explanation is confirmed by the writings of later Hindu mathematicians. For example, the celebrated Śrīdharācārya in his *Pāṭīgaṇita*, describes the series-figure as follows:

> I shall now describe the method for finding the lengths of the base (i.e., lower side) and the face (i.e., upper side) of the (ladder-like) series-figure (corresponding to the first term of the series).
>
> The number of terms, i.e., one, is the altitude of the (corresponding) series-figure; the first term of the series as diminished by half the common difference of the series is the base; and that (base) increased by the common difference of the series is the face ...
>
> Having constructed the series-figure (for altitude unity) in this manner, one should determine the face for the desired altitude

[11] *BBi* (=Bhāskara II's *Bījagaṇita*, Benaras Sanskrit Series), p. 125.
[12] *BBi*, p. 127.
[13] See the beginning of the commentary on $\bar{A}$, ii.

> (i.e., for the desired number of terms of the series) (by the following rule):
>
> The face (for altitude unity) minus the base (for altitude unity), multiplied by the desired altitude, and then increased by the base (for altitude unity), gives the face (for the desired altitude).[14]

So has also been stated by Nārāyaṇa in his *Gaṇitakaumudī*.[15] Moreover, some of the problems set by Nārāyaṇa are based on ladder-like figures; and in the solutions supplied to those problems, Nārāyaṇa has actually drawn such figures. Pṛthūdaka also in his commentary on the *Brāhmasphuṭasiddhānta* makes a similar remark. He writes:

> The *saṅkalita* (i.e., the sum of a series), which has been exhibited by Ācārya Skandasena on the analogy of a ladder, is meant to demonstrate it by means of a figure.[16]

The ladder-like figure representing a series had a smaller base and a larger top, so it looked like a drinking glass. Śrīdhara has, therefore, compared a series-figure with a drinking glass. Writes he:

> As in the case of an earthen drinking pot (*śarāva*) the width at the base is smaller and at the top greater, so also is the case with a series-figure.[17]

It is thus clear why in early days series were looked upon as part of geometrical mathematics, not of algebra as in modern mathematics.

2.3 The four *bījas* of *Gaṇita* and their nomenclature.

Bhāskara I refers to the four *bījas* of *Gaṇita*, and calls them *prathama* (first), *dvitīya* (second), *tṛtīya* (third), and *caturtha* (fourth), or *yāvattāvat*, *vargāvarga*, *ghanāghana*, and *viṣama*.[18] *Bīja* means "method of analysis". It is stated to be of four kinds, because in Hindu Mathematics equations are classified into four varieties.[19] Each class of equations has its own method of analysis. The *yāvattāvat bīja* is the "method of solving simple equations", the *vargāvarga bīja* is the "method of solving quadratic equations", the *ghanāghana bīja* is the "method of solving cubic equations", and the *viṣama bīja* is the "method of solving equations in more than one unknown".

[14] For details see K.S. Shukla, *Pāṭīgaṇita*, English translation, pp. 66–68.
[15] *GK* (=*Gaṇitakaumudī*), rule 73–74.
[16] Quoted by Sudhākara Dvivedi in his comm. on *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xii. 2.
[17] *PG* (=*Pāṭīgaṇita*), English translation, p. 66.
[18] See Bhāskara I's comm. on *Ā*, i. 1.
[19] *Cf.* B. Datta and A.N. Singh, *History of Hindu Mathematics*, Part II, p. 6.

The above nomenclature of the four *bījas* has not been found in any other known work on Hindu mathematics. In an anonymous commentary on the *Kuṭṭakādhyāya* (a chapter of the *Brāhmasphuṭasiddhānta*),[20] the quartet of the four *bījas* is said to consist of (i) the theory of solving simple equations (*ekavarṇa-samīkaraṇa*), (ii) the elimination of the middle term (*madhyamā-haraṇa*), i.e., the theory of solving quadratic equations, (iii) the theory of solving equations involving several unknowns (*anekavarṇa-samīkaraṇa*), and (iv) the theory of solving equations of the type $axy = bx + cy + d$ (called *bhāvita*). This quartet of the four *bījas* is also mentioned by Bhāskara II.[21]

2.4 Evidence of the use of symbolic algebra before the time of Bhāskara I

We have seen above that Bhāskara I in his commentary makes use of the unknown quantities *yāvattāvat* and *gulikā.* The commentary due to its limited scope does not throw much light on the contemporary algebra, but there are reasons to believe that symbolic algebra had very much developed by that time. In this connection we will draw the attention of historians of mathematics to a very significant term mentioned by Bhāskara I. This is *yāvakaraṇa.* Bhāskara I writes : "*varga, karaṇī, kṛti, vargaṇā*, and *yāvakaraṇa* are synonyms."[22] We thus see that, according to Bhāskara I, the word *yāvakaraṇa* means "squaring". The literal meaning of that word is "making *yāva*".[23] But what is that *yāva*? According to V.S. Apte's *Sanskrit-English Dictionary*, the word *yāva* means (i) food prepared from barley, or (ii) red dye. Etymologically, that word may mean "to mix" or "to separate" (*yu*+*ghañ*). If these are the possible meanings of the word *yāva*, how is it that Bhāskara I takes *yāvakaraṇa* as a synonym of *varga* (or squaring)? The word *yāvakaraṇa* owes for its origin to algebraic symbolism. In the commentary of Pṛthūdaka on the *Brāhmasphuṭasiddhānta*, the equation $10x - 8 = x^2 + 1$ is written as

$$\begin{array}{llllll} \textit{yāva} & 0 & \textit{yā} & 10 & \textit{rū} & \dot{8} \\ \textit{yāva} & 1 & \textit{yā} & 0 & \textit{rū} & 1 \end{array}$$

This is the standard Hindu symbolism, and was always used in analysis. It occurs in all Hindu works on algebra, esp. the commentaries on algebraical works. In this symbolism, *yā* is used as an abbreviation of *yāvattāvat* (the unknown quantity, i.e., x), and *yāva* as an abbreviation of *yāvattāvadvarga* (the square of the unknown quantity, i.e., x^2); *rū* stands for *rūpa* (absolute term). Thus, we see that, according to the algebraic symbolism of the Hindus,

[20] A micro-film copy of this comm. is in our possession.
[21] See *BBi*, *ekavarṇa-samīkaraṇa*, 1–3 (comm.).
[22] Comm. on *Ā*, ii, 3 (i).
[23] *Cf. samakaraṇa*, meaning "making equal", or "equating", or "equation" (*sama*=equal, *karaṇa*=making).

yāva stands for *yāvattāvadvarga* (the square of any quantity whatever). *Yāvakaraṇa*, therefore, means "making the square of any quantity", i.e., "squaring a quantity", or simply "squaring".

The term *yāvakaraṇa* was evidently coined after the symbolism on which it is based was developed in India. Bhāskara I mentions that word as one of the synonyms of *varga* (squaring), but nowhere in his commentary has he used that term. It is probable that it was handed down to him by tradition.

2.5 Use of unusual or special terms.

2.5.1 The term *udvartanā* (meaning "multiplication")

Bhāskara I writes: "*saṃvarga*, *ghāta*, *guṇanā*, *hatiḥ* and *udvartanā* are synonyms".[24] The term *saṃvarga* is used but rarely, but the term *udvartanā* is rather unusual, as it is not found to occur in any other work. It is similar to the term *apavartana* (meaning "division") and is evidently its antonym.

The word *abhyāsa* is also used in the sense of multiplication.

2.5.2 Terms for the surd

The usual Hindu term for the surd is *karaṇī*. Bhāskara I, in addition to this term, has also used the term *karaṇi*.[25] Both these terms are also found to occur in the *gāthās* quoted by Bhāskara I. So it seems that both these forms were used in early times.

It is interesting to note that the term *karaṇī*, or *karaṇi*, when operating on a number (> 1), is generally used in its plural form. That is to say, instead for writing *karaṇī* 216, it is written as *karaṇyaḥ* 216. Still more interesting is the method of writing the *karaṇī* of a compound fraction. For example, Bhāskara I writes

$$\sqrt{31\frac{42683983}{1953125000}}$$

in the following way:

$$\textit{karaṇyaḥ}\ 31,\ \textit{karaṇībhhāgāśca}\ \left|\begin{array}{c}42683983\\1953125000\end{array}\right|$$

2.5.3 Terms for "power" and "root"

We have seen above that the terms *abhyāsa*, *saṃvarga*, *ghāta*, *guṇanā*, *hatiḥ*, and *udvartanā* have been used by Bhāskara I in the sense of multiplication. More particularly, these terms have been used in the sense of "multiplication of unequal quantities." For the multiplication of equal quantities, Bhāskara I

[24]Comm. on *Ā*, ii. 3 (i).

[25]Opening lines of the comm. on *Ā*, ii. and comm. on *Ā*, ii, 7 and 10.

uses a special term, *"gata"*. "*Guṇanā* is the multiplication *(abhyāsa)* of unequal quantities, and *gata*," says he, "is the multiplication of equal quantities."[26] The term *dvigata*, according to him, means "square"; *trigata* means "cube"; and so on. The *dvigata* of 4 is the product of 4 and 4, i.e., 4^2; the *trigata* of 4 is the continued product of 4 and 4 and 4, i.e., 4^3; and so on. According to terminology, m^n will be expressed by saying "n^{th} *gata* of m", which corresponds to our present day expression "n^{th} power of m". Following the same terminology, the roots have been called *gatamūla*. Thus 4 is the *dvigatamūla* of 4^2, the *trigatamūla* of 4^3, and so on. In general, m is the "n^{th} *gatamūla* of m^n". This, too, corresponds to the modern expression "n^{th} root of m^n".

The credit of this scientific terminology is given to Brahmagupta.[27] But it was devised by some earlier Hindu mathematician, as both those terms, *gata* and *gatamūla*, are found to be used in the same sense in a stanza quoted by Bhāskara I from some anterior work.

The term *bhāvitaka* (or *bhāvita*), which Brahmagupta uses in the sense of "the product of two dissimilar quantities", does not occur in the commentary. Brahmagupta writes:

> The product of two equal quantities is called *varga* (square); the product of three or more equal quantities is called "the *gata* of that quantity"; and the product of (two) dissimilar quantities is called *bhāvitaka*.[28]

2.5.4 Other notable terms

The following unusual terms used by Bhāskara I also deserve notice:

1. *adhyardhāśrikṣetra* (= a right-angled triangle).
2. *saṅkalanā* (= the sum of natural numbers).[29]
3. *vargasaṅkalanā* (= the sum of the series of squares of natural numbers).[30]
4. *ghanasaṅkalanā* (= the sum of the series of cubes of natural numbers).[31]
5. *saṅkalanā-saṅkalanā* (= the sum of the series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$).[32]

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[26] असदृशयो राश्योरभ्यासं गुणना, गतं सदृशाभ्यासः। See the opening lines of the commentary on *Ā*, ii.
[27] See B. Datta and A.N. Singh, *History of Hindu Mathematics*, Part II, p. 10.
[28] *BrSpSi*, xviii. 42(43).
[29] The usual term is *saṅkalita*.
[30] The usual term is *vargasaṅkalita*.
[31] The usual term is *ghanasaṅkalita*.
[32] The usual term is *saṅkalita-saṅkalita*.

2.6 Weights and measures

The weights and measures, used by Bhāskara I, and their relations may be stated in the tabular form as follows:

(i) Measures of gold, saffron, etc.:[33]

5 *guñjās* = 1 *māṣa*,
16 *māṣās* = 1 *karṣa*,
4 *karṣas* = 1 *pala*,
2000 *palas* = 1 *bhāra*.

(ii) Measures of grain, etc.:[34]

4 *mānakas* = 1 *setikā*,
4 *setikās* = 1 *kuḍuba*,
4 *kuḍubas* = 1 *prastha*.

(iii) Money measures:[35]

1 *rūpaka* = 20 *viṁśopakas*

Other measures used by Bhāskara I are the same as stated by Āryabhaṭa I .

2.7 Classification of the pulverizer (*kuṭṭākāra*).

Bhāskara I is the first to classify mathematical problems based on the indeterminate equation of the first degree called pulverizer (*kuṭṭākāra*) into two types: (i) residual pulverizer (*sāgra-kuṭṭākāra*) and (ii) non-residual pulverizer (*niragra-kuṭṭākāra*). These types may be illustrated by means of the following examples:

> Residual Pulverizer : Find what is that number which leaves 1 as remainder when divided by 5, and 2 (as remainder) when divided by 7.
>
> Non-residual Pulverizer : 8 is multiplied by some number and the product is increased by 6, and that sum is then divided by 13. If the division be exact, what is the unknown multiplier and what is the resulting quotient?

[33] *Cf. PG* (= Śrīdhara's *Pāṭīgaṇita*), definition, 10; *Triś* (=*Triśatikā*), def. 5; *Kauṭilya's Arthaśāstra*, ii. ch. xix; *Abhidhānappadīpikā*, *gāthās* 479–80; *GK* (=*Gaṇita-kaumudī*, def. 5; *L* (=*Līlāvatī*), def. 4; Raghunātha-rāja's comm. on *Ā*, ii. 2.

[34] *Cf. GT* (=*Gaṇitatilaka*), def. 7; *Anuyogadvārasūtra*. The latter is quoted by H. R. Kapāḍiyā in the introduction (p. xxxvii) to his edition of the *GT*.

[35] This relation is unusual.

An astronomical problem based on the indeterminate equation of the first degree is called planetary pulverizer (*graha-kuṭṭākāra*). Bhāskara I in his commentary illustrates numerous types of such problems. Two types, which deserve particular notice, may be mentioned here. One is called "week-day pulverizer" (*vāra-kuṭṭākāra*) and the other is called "time pulverizer" (*velā-kuṭṭākāra*). Examples of these types are:

> Week-day Pulverizer: The mean (position) of the Sun (for sunrise) on a Wednesday is stated to be 8 signs, 25 degrees, 36 minutes, and 10 seconds. Say correctly after how much time (since the beginning of *kaliyuga*) will the Sun again assume the same position (at sunrise) on a Thursday, a Friday, and a Wednesday.
>
> Time Pulverizer: The revolutions, etc., of the Sun's mean longitude, calculated from an *ahargaṇa* plus a few *nāḍīs* elapsed, have now been destroyed by the wind; 71 minutes are seen by me to remain intact. Say the *ahargaṇa*, the Sun's (mean) longitude, and the correct value of the *nāḍīs* (used in the calculation).

2.8 Bhāskara I's examples illustrating Āryabhaṭa I's rules

These examples, as mentioned earlier, form one of the most notable features of Bhāskara I's mathematics. The set of Bhāskara I's examples consists of as many as 132 problems which are the earliest on record excepting those of the *Bakhshālī Manuscript* of uncertain date.

Some of the methods employed by Bhāskara I are also worthy of note. Mention may, for example, be made of his ingenious method for finding the Sun's longitude from the residue of the omitted lunar days (*avamaśeṣa*).[36]

2.9 Bhāskara I's tables giving the least integral solutions of the equation $ax - 1 = by$ corresponding to all sets of values of a and b that may arise in astronomical problems based on the pulverizer

These tables are meant to facilitate the solution of the astronomical problems based on theory of the pulverizer and form a unique feature of Bhāskara I's mathematics as tables of this kind are not to be met with in any other known work on Hindu mathematics.[37]

We conclude this paper by giving Bhāskara I's views regarding Āryabhaṭa I's mathematics as set forth in the *Gaṇita Section* of the *Āryabhaṭīya*.

[36] *Vide* infra, Ex. 129.

[37] Bhāskara I's examples and tables will be displayed in the second and third papers of the series to be published in *Gaṇita*, Vol 22, No. 2 and *Gaṇita*, Vol 23, No. 1. respectively.

2.10 Bhāskara I's views on Āryabhaṭa I's mathematics and reference to the detailed works of Professors Maskarī, Pūraṇa, and Mudgala, etc.

According to Bhāskara I, the mathematics dealt with in the *Gaṇita Section* of the *Āryabhaṭīya* is not the true representative of the mathematical knowledge in his time. He calls it only "a bit of mathematics". Writes he:

> In the *Gaṇita-pāda* (= *Gaṇita Section*) the Ācārya (i.e., Ācārya Āryabhaṭa I) has dealt with the subject of mathematics (*Gaṇita*) by indications only, whereas in the *Kālakriyā-pāda* and *Gola-pāda* he has discussed "reckoning with time" and "spherical astronomy" in detail. So by the word *Gaṇita* (used by Ācārya Āryabhaṭa I) one must understand "a bit of mathematics." Otherwise, the subject of mathematics is vast. There are eight *vyavahāras* (determinations), viz. *miśraka* (mixtures), *śreḍhī* (series), *kṣetra* (plane figures), *khāta* (excavations), *citi* (piles of bricks), *krākacika* (saw problems), *rāśi*, and *chāyā* (shadow). The *miśraka* is that which involves a mixture of several things. The *śreḍhī* is that which has a beginning (i.e., the first term), and an increase (i e., common difference). The *kṣetra* tells us how to calculate the area of a figure having several angles. The *khāta* enables us to know the volumes of excavations. The *citi* tells us the measure of a brick pile in terms of bricks. The *krākacika*: The *krākaca* (saw) is a tool which saws timber; that which relates to the sawing of timber, i.e., that which tells the measure of the timber sawn, is called *krākacika* (*vyavahāra*). The *rāśi* tells us the amount of a heap of grain, etc. The *chāyā* tells us the time from a shadow of a gnomon, etc. Of the *vyavahāra-gaṇita* (practical or commercial mathematics, i.e., *Pāṭīgaṇita*), which is thus of eight varieties, there are four *bījas*, viz. first, second, third and fourth, i.e., *yāvattāvat* ("theory of simple equations"), *vargāvarga* ("theory of quadratic equations"), *ghanāghana* ("theory of cubic equations"), and *viṣama* ("theory of equations with several unknowns"). Rules and examples pertaining to each one of these have been compiled (in independent works) by Professors Maskarī, Pūraṇa, Mudgala and others. How can that be stated by Ācārya (Āryabhaṭa I) in a small work (like the *Āryabhaṭīya*)? So we have rightly said "a bit of mathematics".[38]

Reference to the eight *vyavahāras* and a brief treatment of each of them under separate heads is also found to occur in the *Brāhmasphuṭasiddhānta* of

[38] See Bhāskara I's comm. on the opening verse of the *Āryabhaṭīya*.

his contemporary Brahmagupta. As regards the *bījas*, Brahmagupta has classified them in his own way under three heads, viz. *Ekavarṇa-samīkaraṇa Bīja* (theory of equations with one unknown), *Anekavarṇa-samīkaraṇa Bīja* (theory of equations with several unknown), and *Bhāvita Bīja* (theory of equations involving the product of different unknowns). He has included the theory of quadratic equations in the first, but other mathematicians generally treat that topic separately.

From the passage quoted above it is evident that in the time of Bhāskara I and Brahmagupta, both arithmetic and algebra were in a fully developed form and a number of works written exclusively on these subjects by Professors Maskarī, Pūraṇa, and Mudgala, etc. were in existence.[39]

[39]Passages quoted from the earlier works by Bhāskara I will be displayed in the fourth paper of the series to be published in *Gaṇita*, Vol. 23, No. 2.

Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (II) *

This paper is the second of the series and deals with the mathematical examples set by Bhāskara I in illustration of the rules given in the *Gaṇita* Section of the *Āryabhaṭīya*.

3 Bhāskara I's examples

Below are given the examples set by Bhāskara I in illustration of the various rules of the *Āryabhaṭīya*. The rule under which the particular example occurs is given within square brackets after the statement of the example.

3.1 Examples on arithmetic and mensuration

On the squaring of integral numbers

Ex. 1. "Separately tell (me) the squares of (integral numbers) beginning with 1 and ending in 9, and also the square of 25 and of 100 plus 25."[1]

[$\bar{A}$,ii. 3 (i)]

On the squaring of fractional numbers

Ex. 2. "Tell me the squares of 6 plus $\frac{1}{4}$, 1 plus $\frac{1}{5}$, and 2 minus $\frac{1}{9}$."

[$\bar{A}$, ii. 3 (i)]

On the cubing of integral numbers[2]

Ex. 3. "Tell me separately the cubes of integral numbers beginning with 1 and ending 9, and also the cubes of $(8 \times 8)^2$ and $(25^2)^2$."

[$\bar{A}$, ii. 3 (i)]

* K. S. Shukla, *Gaṇita*, Vol. 22, No. 2 (December 1971), pp. 61–78.

[1] Ex. 1 reappears it Yallaya's commentary on $\bar{A}$, ii. 3.

[2] Exs. 3 and 4 reappear in Yallaya's comm. on $\bar{A}$, ii. 3.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_7

On the cubing of fractional numbers

Ex. 4. "If you have clear understanding of cubing a number, say correctly the cubes of 6, 15, and 8 as respectively diminished by $\frac{1}{6}$, $\frac{1}{15}$, and $\frac{1}{8}$ (i.e., the cubes of 6 minus $\frac{1}{6}$, 15 minus $\frac{1}{15}$, and 8 minus $\frac{1}{8}$)." [*Ā*, ii. 3 (ii)]

On extracting the square root of integral numbers

Ex. 5. "I want to know, O friend, the square root of the (square) numbers 1, etc., previously determined, and also of the square number 625."

[*Ā*, ii. 4]

On extracting the square root of fractional numbers

Ex. 6. "Calculate, in accordance with the arithmetic of (Ārya)bhaṭa, the square root of 6 plus $\frac{1}{4}$ and of 13 plus $\frac{4}{9}$ and state the two results."

[*Ā*, ii. 4]

On extracting the cube root of integral numbers

Ex. 7. "Tell me separately the cube roots of the cube numbers 1, etc. Also quickly calculate the cube root of 1728."[3] [*Ā*, ii. 5]

Ex. 8. "Correctly state, in accordance with the rules prescribed in the *Bhaṭa-śāstra* (i.e., *Āryabhaṭīya*), the cube root of 8291469824."[4] [*Ā*, ii. 5]

On extracting the cube root of fractional numbers

Ex. 9. "Correctly calculate in accordance with the arithmetic of Āryabhaṭa, the fractional (cube) root of 13 plus $\frac{103}{125}$." [*Ā*, ii. 5]

On the determination of the area of triangles

Ex. 10. "Tell (me), O friend, the areas of the (three) equilateral triangles whose sides are 7, 8, and 9 (units) respectively, and also the area of the isosceles triangle whose base is 6 (units) and the lateral sides each 5 (units)."[5] [*Ā*, ii. 6 (i)]

Ex. 11. "Carefully say the area of the isosceles triangle in which the two lateral sides are each stated to be 10 (units) and the base is given to be 16 (units)." [*Ā*, ii. 6 (i)]

[3]Exs. 7 and 8 reappear in Yallaya's comm. on *Ā*, ii. 5.
[4]See footnote 3.
[5]Ex. 10 reappears in Yallaya's comm. on *Ā*, ii. 6.

Ex. 12. "O friend, what is the area of the scalene triangle in which one lateral side is 13 (units), the other (lateral side) 15 (units), and the base 14 (units)?"[6] [$\bar{A}$, ii. 6 (i)]

Ex. 13. "Say what is the area of the scalene triangle in which the base is 51 (units), one lateral side is 37 (units), and the other lateral side is stated to be 20 (units)." [$\bar{A}$, ii. 6 (i)]

For finding the area of a triangle, Āryabhaṭa I states the general formula: Area $= \frac{1}{2}$ base $\times$ altitude. This formula is not directly applicable to finding the areas of triangles in which the three sides are given. In order to make use of that formula it is necessary to find the altitude. In the case of equilateral and isosceles triangles, in which the altitude bisects the base, the altitude is easily obtained by the formula:

$$(\text{altitude})^2 = (\text{lateral side})^2 - \left(\frac{\text{base}}{2}\right)^2.$$

In case of scalene triangles, Bhāskara I makes use of the following result:

If a be base and b and c the lateral sides of a triangle, then

$$(\text{altitude})^2 = b^2 - x^2 \text{ or } c^2 - (a-x)^2,$$
$$\text{where} \quad x = \frac{1}{2}\left[a + \frac{(b^2 - c^2)}{a}\right],$$
$$\text{and} \quad a - x = \frac{1}{2}\left[a - \frac{(b^2 - c^2)}{a}\right].$$

This rule occurs in the *Brāhmasphuṭasiddhānta* (xii. 22) also. Brahmagupta has also given the formula:[7]

$$\text{area} = \sqrt{(s-a)(s-b)(s-c)},$$
$$\text{where } 2s = a + b + c,$$

but Bhāskara I has not used this, perhaps because it was irrelevant to him. It must be borne in mind that Bhāskara I aims at illustrating the rules given by Āryabhaṭa I only.

On the determination of the volume of a triangular pyramid

Ex. 14. "Quickly tell me the more accurate volume and also the measure of the altitude of the solid of the shape of a trapa in which each edge is 12 (units)." [$\bar{A}$, ii. 6 (ii)]

[6] Ex. 12 appears twice in *Gaṇitasārasaṅgraha*. See *GSS*, vii. 10 and 53. It occurs also in the *Triśatikā* of Śrīdharācārya and the *Līlāvatī* (p. 154) of Bhāskara II.

[7] See *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xii. 21.

Ex. 15. "The length of each edge of a trapa is given to be 18 (units). I want to know, O friend, the altitude and the volume thereof." [$\bar{A}$, ii. 6 (ii)]

Āryabhaṭa I's formula for the volume of a pyramid is

$$\text{volume} = \frac{1}{2}\ (\text{area of base}) \times (\text{altitude}).$$

Bhāskara I has made little improvement in this result. His contemporary Brahmagupta has, however, given the correct formula for the volume of a cone.[8]

On the determination of the circumference and area of a circle

Ex. 16. "The diameter (of three circles) are accurately determined by me to be 8, 12, and 6 (units) respectively. Tell me separately the circumference and area of each of these circles." [$\bar{A}$, ii. 7 (i)]

On the volume of a sphere

Ex. 17. "The diameters of (three) spheres are to be known as 2, 5, and 10 (units) respectively. I want to know their volumes briefly." [$\bar{A}$, ii. 7 (ii)]

Āryabhaṭa I's formula for the volume of a sphere is

$$\text{volume} = (\text{area of central circle})^{\frac{3}{2}}.$$

Āryabhaṭa I writes that this is the accurate value for the volume of a sphere. Bhāskara I too holds the same view. In fact, that value is not only inaccurate but also wrong. The correct formula was given by Bhāskara II.[9]

On the determination of the junction-lines[10] and the area of a trapezium

Ex. 18. "(In a trapezium) the base is 14 (units), the face (i.e., the upper side) is 4 units and the lateral sides each 13 (units). Give out the junction-lines and the area."[11] [$\bar{A}$, ii. 8]

Ex. 19. "(In a trapezium) the base, the lateral sides and the face are stated to be 21 (units), 10 (units) each, and 9 (units) respectively. Give out the area and the junction–lines." [$\bar{A}$, ii. 8]

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[8] See *BrSpSi*, xii. 44.

[9] See *Līlāvatī* (*Ānandāśrama* Sanskrit Series), p. 201, stanza 201.

[10] By the junction-lines are meant the segments of the altitude through the intersection of the diagonals.

[11] Ex. 18 reappears in the commentaries of Sūryadeva, Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 8.

Ex. 20. "(In a trapezium) the base is 33 (units), and the other sides are each stated to be 17 (units). What is the area thereof and what are the junction-lines?" [*Ā*, ii. 8]

Ex. 21. "(In a trapezium) having 25 (units) for the face, the base is stated to be 60 (units); the lateral sides are 13 (units) multiplied by 4 and 3 respectively. (Find the area and the junction-lines)."[12]

Ex. 22. "(In a trapezium) the altitude is stated to be 12 (units), the base 19 (units) and the face 5 (units). The lateral sides of that are given to be 10 (units) as severally increased by 5 and 3 (units). I want to know the area and the junction-lines correctly."[13] [*Ā*, ii. 8]

On the determination of the area of a rectangle, etc.

Ex. 23. "(Of three rectangles) the breadths are 8, 5, and 10 (units); and the lengths of these are 16, 12, and 14 (units) (respectively). What are the areas of rectangles?" [*Ā*, ii. 9]

Ex. 24. "How will the verification be made in the case of all the areas of triangles, quadrilaterals, and circles which have been determined by theoretical calculation?"[14] [*Ā*, ii. 9]

Ex. 25. "(In a trapezium) one face (i.e., side) is seen to be 11 (units), the opposite (parallel) face is stated to be 9 (units), and the length (=distance) (between them) is 20 (units). What, O mathematician, is the area of that figure?"[15] [*Ā*, ii. 9]

On the determination of the area of a figure resembling the drum-shaped musical instrument *Paṇava*

Ex. 26. "The two (parallel) faces of (a figure resembling) a *Paṇava* are each 8 (units), the central width is 2 (units), and the length (between the faces) is 16 (units). Say what is the area of this figure resembling the (musical instrument) *Paṇava*."[16] [*Ā*, ii. 9]

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[12] Ex. 21 reappears in Yallaya's comm. on *Ā*, ii. 8.

[13] This is an example of a trapezium in which the lateral sides are unequal. In such a trapezium, the area and the junction-lines are determined if, besides the sides, the altitude is also known.

[14] According to Bhāskara I, the first half of *Ā*, ii. 9 relates to the verification of areas of rectilinear figures. What is meant is that the given figure should be deformed into a rectangle and then the area should be obtained by multiplying the length of the rectangle by its breadth. A rectangle is chosen because its area is well known. In this connection Bhāskara I has quoted a passage from some unknown mathematical work.

[15] Exs. 25 and 26 reappear in Raghunātha Rāja's comm. on *Ā*, ii. 9.

[16] See footnote 15.

The figure contemplated is a double trapezium obtained by placing two equal trapeziums in juxtaposition in such a way that the smaller of the two parallel sides of the trapeziums forms the central width of the double trapezium. The formula used by Bhāskara I for the area of this figure is

$$\text{area} = \frac{1}{2}\left(\frac{a+b}{2} + c\right) \times l$$

where a, b are the lengths of the parallel faces, l the distance between them, and c the central width.

On the determination of the area of a figure resembling the tusk of an elephant

Ex. 27. "The width (at the base) is stated to be 5 (units), the belly (i.e., inner curved side) is 9 (units), and the back (i.e., outer curved side) is 15 (units). Say, what is the area of this (figure resembling the) tusk of an elephant."[17] [*Ā*, ii. 9]

The figure envisaged is a curvilinear triangle, bounded by a straight base and two curved sides curved in the same direction. The formula used by Bhāskara I for the area of such a figure is

$$\text{area} = \frac{a}{2} \times \frac{b+c}{2},$$

where a is the base and b, c the curved sides.

On the area of a circle

Ex. 28. "Calculate, O friend, according to the *Gaṇita* (of Āryabhaṭa), the nearest approximations to the areas of the circles whose diameters are 2, 4, 7, and 8 respectively." [*Ā*, ii. 10]

On the determination of the diameter of a circle from the given circumference

Ex. 29. "Calculate and tell me the diameters of the circles whose peripheries are 3299 minus $\frac{8}{25}$ and 216000 respectively." [*Ā*, ii. 10]

On the determination of the local latitude from the midday shadow of the gnomon

Ex. 30. "When at an equinox the Sun is on the meridian, the shadow of a gnomom, divided into 12 units, on level ground is seen to be 5, 9, and $3\frac{1}{2}$ (units at three different places). (Find the latitudes of those places)."

[17] Ex. 27 reappears in Raghunātha Rāja's comm. on *Ā*, ii. 9.

[*Ā*, ii. 14]

Ex. 31. "The shadow of the gnomon of 15 *aṅgulas* at midday on an equinox is (seen to be) 6 plus $\frac{1}{4}$ *aṅgulas*. Give out the Rsines of the latitude and the co-latitude." [*Ā*, ii. 14]

Ex. 32. "Say what is the distance of the Sun, whose rays are (profusely) spread all round, from the zenith, when the shadow of a gnomon of 30 *aṅgulas* is observed to be 16 *aṅgulas*."[18] [*Ā*, ii. 14]

On the shadow of a gnomon due to a lamp-post

Ex. 33. "Tell (me the length of) the shadow situated at a distance of 80 (*aṅgulas*) from the foot of the lamp-post of height 72 (*aṅgulas*); and also that of another gnomon situated at a distance of 20 (*aṅgulas*) from a lamp-post of height 30 (*aṅgulas*)."[19] [*Ā*, ii. 15]

Ex. 34. "Say what is the distance of the foot of the lamp-post of height 72 (*aṅgulas*) from the gnomon of 12 (*aṅgulas*) if the shadow (cast by the gnomon) is 16 (*aṅgulas*)."[20] [*Ā*, ii. 15]

Ex. 35. "The shadow of a gnomon, situated at a distance of 50 (*aṅgulas*) from the foot of a lamp-post, is 10 (*aṅgulas*). Say what is the height of the lamp."[21] [*Ā*, ii. 15]

Ex. 36. "(The lengths of) the shadows of two equal gnomons (of 12 *aṅgulas*) are seen to be 10 and 16 (*aṅgulas*) respectively; the distance between the shadow-ends is seen to be 30 (*aṅgulas*). Give out the upright and the base for each (gnomon)."[22] [*Ā*, ii. 16]

The "base" means "the height of the lamp-post" and the "upright" means "the distance of the shadow-end from the foot of the lamp-post". The two gnomons are assumed to be in the same line as seen from the lamp-post.

Ex. 37. "(The lengths of) the shadows of two equal gnomons (of 12 *aṅgulas*) are stated to be 5 and 7 (*aṅgulas*) respectively. The distance between the shadow-ends is observed to be 8 (*aṅgulas*). Give out the base and the upright." [*Ā*, ii. 16]

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[18] By saying that the rays of the Sun are profusely spread it is stated that it is midday.

[19] Ex. 33 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on *Ā*, ii. 15.

[20] Ex. 34 reappears in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 15.

[21] Ex. 35 reappears in the commentary of Raghunātha Rāja on *Ā*, ii. 16.

[22] Ex. 36 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on *Ā*, ii. 16.

On the so called Pythagoras' theorem

Ex. 38. "Give out the hypotenuses (for three right-angled triangles) where the bases and the uprights are 3 and 4, 6 and 8, and 12 and 9 (units) respectively." [$\bar{A}$, ii. 17 (i)]

On the following property of the circle: "If the diameter ABC and the chord LBM of a circle intersect at right angles, then $LB^2 = AB \times BC$," AB and BC being called the arrows and LB the Rsine

Ex. 39. "In a circle of diameter 10 (units), the arrows (i.e., segments of a diameter) are seen by me to be 2 and 8 (units); in the same circle, another set of arrows are 9 and 1 (units). Tell (me) the corresponding Rsines."[23] [$\bar{A}$, ii. 17 (ii)]

The Hawk-and-Rat problems

Ex. 40. "A hawk is siting at the top of a rampart whose height is 12 cubits. The hawk sees a rat at a distance of 24 cubits away from the foot of the rampart; the rat, too, sees the hawk. Thereupon the rat, for fear of him, hastens to his own dwelling situated at (the foot of) the rampart but is killed in between by the hawk who came along a hypotenuse (i.e., along an oblique path). I want to know the distance traversed by the rat and also the (horizontal) motion of the hawk (the speeds of the two being the same)."[24] [$\bar{A}$, ii. 17 (ii)]

Ex. 41. "A hawk is sitting on a pole whose height is 18 (cubits). A rat, who has gone out of his dwelling (at the foot of the pole) to a distance of 81 (cubits), while returning towards his dwelling, afraid of the hawk, is killed by the cruel (bird) on the way. Say how far has he gone towards his hole, and also the (horizontal) motion of the hawk (the speeds of the rat and the hawk being the same)."[25] [$\bar{A}$, ii. 17 (ii)]

The above two examples (Exs. 40 and 41) have been called the "hawk-and-rat problems". Bhāskara I ascribes such problems to previous writers. He writes: "At this very place they narrate the hawk-and-rat problems."

[23] Ex. 39 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 17.

[24] Ex. 40 reappears in Raghunātha Rāja's comm. on $\bar{A}$, ii. 17. A similar example occurs in Pṛthūdaka's comm. on *BrSpSi*, xii. 41. See H.T. Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanskrit of Brahmegupta and Bhāscara*, London (1817), p. 309, footnote.

[25] Ex. 41 reappears in Yallaya's comm. on $\bar{A}$, ii. 17.

The Hindu method for solving such problems has been explained by Bhāskara I in detail. Following that method, Ex. 41 may be solved as follows:

Draw a circle with centre at O. Let $ABOC$ be the horizontal diameter and LBM a vertical chord intersecting the diameter at B. Imagine that BL is the pole and BC the track of the rat. The hawk is sitting at L and the rat is at C. They see each other. The rat then runs to his hole at B but is killed by the hawk at O, the distance traversed by the hawk (i.e., LO) and by the rat (i.e., CO) being the same.

It is given that $LB = 18$ cubits, and $BC = 81$ cubits. Since $LB^2 = AB \times BC$, therefore $AB = 4$ cubits. Therefore,

$$BO = \frac{1}{2}(BC - AB) = 38\frac{1}{2} \text{ cubits,}$$
$$\text{and} \quad CO = \frac{1}{2}(BC + AB) = 42\frac{1}{2} \text{ cubits.}$$

Hence, the distance traversed by the rat is $42\frac{1}{2}$ cubits and the horizontal motion of the hawk is $38\frac{1}{2}$ cubits.

It is interesting to note that Yallaya and Raghunātha Rāja have prescribed the same method for solving the hawk-and-rat problems as described above. The peacock-and-serpent problems given by Bhāskara II, Yallaya, and Raghunātha Rāja are similar to the hawk-and-rat problems.

The Bamboo problems

Ex. 42. "A bamboo of height 18 (cubits) is felled by the wind. It falls at a distance of 6 (cubits) from the root, thus forming a (right-angled) triangle. Where is the break?"[26] [*Ā*, ii. 17 (ii)]

Ex. 43. "A bamboo of 16 cubits is felled by the wind; it falls at a distance of 8 cubits from its root. Say where has it been broken by the wind."

[*Ā*, ii. 17 (ii)]

In the case of the bamboo problems like Exs. 42 and 43, BC (in the figure of Ex. 41) is taken to represent the bamboo which breaks at O and reaches the ground (BL) at L. To find the height of the break, we have to obtain the length BO. As before $BO = \frac{1}{2}(BC - AB)$, where $AB = \frac{LB^2}{BC}$.

Ex. 42 is found to occur in Pṛthūdaka's commentary on the *Brāhmasphuṭasiddhānta* of Brahmagupta.[27] His method of solution is the same as used by

[26] Ex. 42 reappears in Pṛthūdaka's comm. on *BrSpSi*, xii. 41 and in Raghunātha Rāja's comm. on *Ā*, ii. 17.

[27] See H.T. Colebrooke, *l.c.*, p. 309, footnote.

Bhāskara I.[28] Similar problems are also found to occur in the *Gaṇitasāra-saṅgraha*[29] of Mahāvīra, the *Līlāvatī*[30] and the *Bījagaṇita*[31] of Bhāskara II, and the *Gaṇitakaumudī*[32] of Nārāyaṇa.

The Lotus problems

Ex. 44. "A full blown lotus of 8 *aṅgulas* is seen (just) above the water. Being carried away by the wind it just submerges at a distance of one cubit. Quickly say the height of the lotus plant and the depth of the water."[33]

[$\bar{A}$, ii. 17 (ii)]

Ex. 45. "A lotus flower of 6 *aṅgulas* just dips (into the water) when it advances through a distance of 2 cubits. I want to know the height of the lotus plant and the depth of the water."[34] [$\bar{A}$, ii. 17 (ii)]

Consider a circle with centre at O. Let $ABOC$ be its vertical diameter and LBM a horizontal chord intersecting the vertical diameter at B.

In the case of the lotus problems, the horizontal diameter of the circle is supposed to denote the mud-level; the chord LBM the water-level; O is supposed to be the root of the lotus plant, OB the lotus stalk, AB the lotus flower, and L and M the points where the lotus flower just dips into the water. Then,

$$OA \text{ (i.e., height of lotus plant)} = \frac{1}{2}(BC + AB),$$

where

$$BC = \frac{LB^2}{AB}; \quad \text{and} \quad OB \text{ (i.e., depth of water)} = \frac{1}{2}(BC - AB).$$

The Crane-and-Fish Problems

Ex. 46. "There is a reservoir of water of dimensions 6 × 12. At the east-north corner thereof there is a fish; and at the west-north corner there

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[28] See B. Datta, "On the supposed indebtedness of Brahmagupta to *Chiu-chang Suan-shu*," *Bull. Cal. Math. Soc.*, vol. xxii, p. 41.

[29] vii. $191\frac{1}{2}$–$192\frac{1}{2}$.

[30] See *L* (*Ānandāśrama* Sanskrit Series) p. 141.

[31] See *Bījagaṇita*, ed. by Sudhakara Dvivedi and Muralidhara Jha, Banaras (1927), p. 57.

[32] *Kṣetra-vyavahāra*, Ex. 26.

[33] Ex. 44 reappears in Pṛthūdaka's comm. on *BrSpSi*, xii. 41. (Colebrooke, *l.c.*, p. 309, footnote), and in the comm. of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 17.

[34] Similar examples occur in the works of Bhāskara II (*L*, Ex. 155, p. 145; *BBi*, Ex. 112) and Nārāyaṇa (*GK, Kṣetra-vyavahāra*, Ex. 28).

Problems similar to Exs. 44 and 45 are reported to occur in a Chinese work called *Chiu-chang Suan-shu*, but the Chinese solution to those problems is quite different from that of Bhāskara I. The Hindu solution is based on the property of right-angled triangles which was known in India as early as the Vedic period.

is a crane. For fear of him (i.e., of the crane) the fish, crossing the reservoir, hurriedly went towards the south in an oblique direction but was killed by the crane who came along the sides of the reservoir. Give out the distances travelled by them (assuming that their speeds are the same)."[35] [$\bar{A}$, ii. 17 (ii)]

Ex. 47. "There is a reservoir of water of dimensions 12×10. At the east-south corner there is a crane and at the east-north corner there is a fish. (The crane walks along the sides of the reservoir and the fish swims obliquely). Say, on reaching which point of the western side of the reservoir is the fish killed by the crane."[36] [$\bar{A}$, ii. 17 (ii)]

Following the method of Bhāskara I, the first of the above two examples (i.e., Ex. 46) may be solved as follows:

Let $LBQP$ be the reservoir in which $BQ = LP = 12$, and $LB = PQ = 6$. Also suppose that LB is the east side, PQ the west side, LP the north side, and BQ the south side of reservoir. Initially the fish is at L and the crane at P. After some time the fish swimming along LO reaches O, a point in BQ. In the same time the crane, walking along PQ and then along QB, also reaches O and kills the fish. The speeds of the fish and the crane being the same, $LO = PQ + QO$. Let OC (along OQ produced) be equal to OL. Then the circle drawn with O as centre and OL as radius must pass through C, and we have

$$BC = BQ + PQ = 12 + 6 = 18.$$

If CB produced intersects the circle at A, then

$$AB = \frac{LB^2}{BC} = \frac{36}{18} = 2.$$

Hence, $AC = AB + BC = 20$ giving $OL = 10$. Therefore, the distances traversed by the fish and the crane are each equal to 10.

Proceeding as above, it can be shown that the point required in Ex. 47 divides the western side of the reservoir in the ratio $8\frac{8}{11} : 3\frac{3}{11}$.

An example similar to the above two occurs in the *Gaṇitakaumudī* of Nārāyaṇa. See *kṣetra-vyavahāra*, pp. 38–39, Ex. 29–31.

On the determination of the arrows of the intersecting arcs of the Moon and the shadow when the portion eclipsed is given

Ex. 48. "When 8 out of 32 of (the diameter of) the Moon are eclipsed by the shadow of diameter 80, I want to know then what are the arrows of (the

[35] Ex. 46 reappears in the comm. of Raghunātha Rāja on $\bar{A}$, ii. 17 (ii). A similar example occurs in the comm. of Yallaya also.

[36] Ex. 47 reappears in Raghunātha Rāja's comm. on $\bar{A}$, ii. 17 (ii).

intersecting arcs of) the shadow and the full Moon."[37] [$\bar{A}$, ii. 18]

On the determination of the middle term and the sum of a series in A.P.

Ex. 49. "In a series (in A.P.) the first term is seen to be 2; the successive increase is stated to be 3; and the number of terms is stated to be 5. Tell (me) the middle term and the sum of the series."[38] [$\bar{A}$, ii. 19]

Ex. 50. "In a series (in A.P.) in which the first term is 8, the successive increase is stated to be 5 and the number of terms is seen to be 18. Give out the middle terms and the sum of the series." [$\bar{A}$, ii. 19]

On the determination of the desired term of a series in A.P.

Ex. 51. "(In a series in A.P.) in which the successive increase is 11 and the first term 7, the number of terms is 25. Quickly say the ultimate and penultimate terms of that series and also say what is the twentieth term."[39] [$\bar{A}$, ii. 19]

On the determination of partial sums of a series in A.P.

Ex. 52. "In the month of *Kārtika* a certain king daily gives away some money (in charity) starting with 2 on the first day (of the month) and increasing that by 3 per day. Fifteen days having passed away, there arrived a Brāhmana well-versed in the Vedas. The amount for the next ten days was given to him; that for the (remaining) five days (of the month), to someone else. Say what do the last two persons get." [$\bar{A}$, ii. 19]

Ex. 53. "(In a series in A.P.) in which the first term is 15, the successive increase is stated to be 18 and the number of terms 30. Quickly calculate the sum of the ten middle terms (of that series)." [$\bar{A}$, ii. 19]

On the determination of the sum of a series in A.P. when the first term, the last term, and the number of terms are given

Ex. 54. "(Of 11 conch-shells which are arranged in the increasing order of their prices which are in A.P.) the first conch-shell is acquired for 5

[37] Ex. 48 reappears in Mahāvīra's *Gaṇitasārasaṅgraha*. See *GSS*, vii. $232\frac{1}{2}$. A similar example occurs also in the commentary of Sūryadeva on $\bar{A}$, ii. 18.

[38] Exs. 49 and 51 reappear in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on $\bar{A}$, ii. 19.

[39] See footnote 38.

and the last for 95. Say what is the price of all the 11 conch-shells."[40]
[*Ā*, ii. 19]

Ex. 55. "(In an arithmetic series) the first term is stated to be 1. The last term is declared by the learned to be 100; the same is also stated to be the number of terms. What is the sum of all the terms (of that series)?"
[*Ā*, ii. 19]

On the determination of the number of terms of an arithmetic series when the first term, the common difference, and the sum of the series are given

Ex. 56. "In a series (in A.P.) the first term is stated to be 5; the successive increase is 7 and the sum 95. Say what is the number of terms thereof."
[*Ā*, ii. 20]

Ex. 57. "(In an arithmetic series) in which the successive increase and the first term are 9 and 8 respectively, the sum is stated to be 583. Tell (me) the number of terms found by you." [*Ā*, ii. 20]

On the sum of the series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$

Ex. 58. "There are (three pyramidal) piles (of balls) having respectively 5, 8, and 14 layers which are triangular. Tell me the number of units (balls) (in each of them)."[41] [*Ā*, ii. 21]

In the topmost layer of the pyramidal piles, there is 1 ball; in the second layer from the top, there are $1 + 2 = 3$ balls; in the third layer, there are $1 + 2 + 3 = 6$ balls; in the fourth layer, there are $1 + 2 + 3 + 4 = 10$ balls; and so on. Every layer is in the form of a triangle.

The number of balls in the first pile having five layers

$$= 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + 4 + 5)$$
$$= \frac{5 \times 6 \times 7}{6} \quad \text{or} \quad 35.$$

Similarly, the number of balls in the other two piles are 120 and 560 respectively.

The series $1 + 2 + 3 + \ldots$ has been called by Āryabhaṭa I by the name *citi* or *upaciti* and the series $1 + (1 + 2) + (1 + 2 + 3) + \ldots$ by the name *citighana*. Their sums are also called by the same terms. Bhāskara I calls the

[40] Ex. 54 reappears in Yallaya's comm. on *Ā*, ii. 19.

[41] Ex. 58 reappears in the commentaries of Sūryadeva, Yallaya, and Raghunātha Rāja on *Ā*, ii. 21. Also see *GSS*, *miśraka-vyavahāra*, Ex. $331\frac{1}{2}$.

sum of the former by the term *saṅkalanā* and that of the latter by the term *saṅkalanā-saṅkalanā*. In the *Brāhmasphuṭasiddhānta* (xii. 19) and other later works, they are called a *saṅkalita* and *saṅkalita-saṅkalita* respectively.

On the determination of the sum of the series $1^2 + 2^2 + 3^2 + \dots$ to any number of terms

Ex. 59. "There are (three pyramidal) piles on square bases having 7, 8, and 17 layers which are also squares. Say the number of units therein (i.e., the number of bricks of unit size used in each of them)."[42] [$\bar{A}$, ii. 22]

In the topmost layer there is one brick, in the next there are four, in the next nine, and so on. The number of bricks used in the three piles are 140, 204, and 1785 respectively.

The sum of the series $1^2 + 2^2 + \cdots + n^2$ has been called *vargacitighana* by Āryabhaṭa I. Bhāskara I calls it *vargasaṅkalanā*. It is generally known as *vargasaṅkalita*.[43]

On the determination of the sum of the series $1^3 + 2^3 + 3^3 + \cdots + n^3$

Ex. 60. "There are (three pyramidal) piles having 5, 4, and 6 cubodial layers. They are constructed of cubodial bricks (of unit dimensions) with one brick in the topmost layer. (Find the number of bricks used in each of them)."[44] [$\bar{A}$, ii. 22]

There is 1^3 brick in the topmost layer, 2^3 bricks in the next layer, 3^3 bricks in the next, and so on. The number of bricks in the three piles are 225, 100, and 441 respectively.

The sum of a series of cubes of natural numbers has been called *ghanacitighana* by Āryabhaṭa I. Bhāskara I calls it *ghanasaṅkalanā*. In later works it is called *ghanasaṅkalita*. The same term has been used by Brahmagupta.[45]

On finding the product of two given numbers by the formula

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$$

Ex. 61. "What are the products of 5 and 4, of 7 and 9, and of 8 and 10? Quickly say separately." [$\bar{A}$, ii. 23]

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[42] Ex. 59 reappears in Sūryadeva's comm. on $\bar{A}$, ii. 22.
[43] See e.g. *BrSpSi*, xii. 20.
[44] Ex. 60 reappears in Yallaya's comm. on $\bar{A}$, ii. 22.
[45] See *BrSpSi*, xii. 20.

On the determination of two numbers whose difference and product are known

Ex. 62. "The product (of two numbers) is clearly seen to be 8; their difference is 2. (Of other two numbers) the product being 18, the difference is 7. Tell (me) the numbers multiplied in the two cases." [*Ā*, ii. 24]

On interest

Ex. 63. "I do not know the interest on 100, but I do know that the interest plus interest on interest accruing on 100 in 4 months is 6. Give out the monthly interest on 100."[46] [*Ā*, ii. 25]

Ex. 64. "I do not know the monthly interest on 25 (*rūpas*). But the monthly interest on 25 (*rūpas*) lent out at the same rate (of interest) is seen to amount to 3 (*rūpas*) minus $\frac{1}{5}$ in 5 months. I want to know the monthly interest on 25 (*rūpas*) as also the interest for 5 months on the interest of 25 (*rūpas*)." [*Ā*, ii. 25]

Ex. 65. "The monthly interest on 100 (*rūpas*) is not known, but the interest on 100 (*rūpas*) lent out elsewhere (at the same rate of interest) is seen to amount with interest thereon to 15 (*rūpas*) in 5 months. I want to know — what is the interest on 100 (*rūpas*) as also what is the interest that accrued in 5 months on 100 (*rūpas*)?" [*Ā*, ii. 25]

On the rule of three

Ex. 66. "5 *palas* of sandalwood are purchased by me for 9 *rūpakas*. How much of sandalwood will, then, be purchased for one *rūpaka*?"[47]

[*Ā*, ii. 26–27 (i)]

Ex. 67. "If one *bhāra* of ginger is sold for 10 plus $\frac{1}{5}$ (*rūpakas*), tell me quickly the price of 100 plus $\frac{1}{2}$ *palas* of ginger."[48] [*Ā*, ii. 26–27 (i)]

Ex. 68. "$1\frac{1}{2}$ *palas* of musk are had for 8 plus $\frac{1}{3}$ (*rūpakas*). Let Kṛtavīrya find out how much of musk will be had for 1 plus $\frac{1}{5}$ (*rūpakas*)."[49]

[*Ā*, ii. 26–27 (i)]

Ex. 69. "A serpent of 20 cubits in length enters into a hole, moving forward at the rate of $\frac{1}{2}$ of an *aṅgula* per *muhūrta*[50] and backward at the rate of

[46] Ex. 63 reappears in the commentaries on *Ā*, ii. 25 of Yallaya and Raghunātha Rāja.
[47] Ex. 66 reappears in Yallaya's comm. on *Ā* ii. 26–27 (i).
[48] 1 *bhāra*=2000 *palas*.
[49] Ex. 68 reappears in Yallaya's comm. on *Ā*, ii. 26–27 (i).
[50] 1 *muhūrta* = 48 minutes.

$\frac{1}{5}$ of an *aṅgula* (per *muhūrta*): in how many days does he get into the hole completely?"[51] [*Ā*, ii. 26–27 (i)]

On proportion and partnership

Ex. 70. "(Out of 11 cattle) 8 are tamed and 3 untamed — so are the cattle described. Out of 1001 cattle, then, how many are tamed and how many untamed?"[52] [*Ā*, ii. 26–27 (i)]

Ex. 71. "15 merchants collaborate (in a business); the capitals invested by them are in A.P. with 1 as the first capital and also 1 as the successive increase. The profit that accrued (on the whole capital) amounts to 1000. Say what should be given to whom." [*Ā*, ii. 26–27 (i)]

Ex. 72. "The combined profit of three merchants, whose investments are in the ratio of $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{8}$ respectively, amounts to 70 minus 1. What is whose profit (individually)?"[53] [*Ā*, ii. 26–27 (i)]

On the rule of five

Ex. 73. "Given that 100 increases by 5 in a month, say, if you are versed in (Ārya)bhaṭa's *Gaṇita*, by how much will 20 increase in 6 months."[54].

[*Ā*, ii. 26–27 (i)]

Ex. 74. "100 invested for two months increases by 5; by how much will 25 invested for two months increase?" [*Ā*, ii. 26–27 (i)]

Ex. 75. "If $4\frac{1}{2}$ *rūpakas* be the increase (interest) on 100 (*rūpakas*) for $3\frac{1}{2}$ months, what will be the increase on 50 (*rūpakas*) for 10 months?"[55]

[*Ā*, ii. 26–27 (i)]

Ex. 76. "A sum of 20 plus $\frac{1}{2}$ (*rūpakas*) increases by 1 plus $\frac{1}{3}$ *rūpakas* in 1 plus $\frac{1}{5}$ months. (Say) after carefully understanding "the method of elimination of divisors" from the aphorism of the (*Ārya*)*bhaṭa-tantra*,

[51] Ex. 69 reappears in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26. Raghunātha Rāja has, however, put the example in a slightly different form. A similar example is found to occur in the *Bakhshālī Manuscript*. *Cf.* G.R. Kaye, *Bakhshālī Manuscript*, Arch. Survey of India, New Imperial Series, Vol. XLIII, Parts I and II, 1927, Ex. 99, p. 51.

[52] Exs. 70 and 71 reappear in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26.

[53] Ex. 72 reappears in Yallaya's comm. on *Ā*, ii. 26.

[54] Ex. 73 reappears in the commentaries of Yallaya and Raghunātha Rāja on *Ā*, ii. 26.

[55] Ex. 75 reappears in Yallaya's comm. on *Ā*, ii. 26.

what will be the increase of 7 minus $\frac{1}{4}$ (*rūpakas*) in 6 plus $\frac{1}{10}$ months."[56]
[$\bar{A}$, ii. 26–27 (i)]

On the rule of seven

Ex. 77. "If 9 *kuḍavas* of pure parched and flattened rice are obtained daily for an elephant whose height is 7 (cubits), periphery 30 (cubits), and length 9 (cubits), say how much of parched and flattened rice will be obtained for an elephant whose height is 5 (cubits), length 7 (cubits), and periphery 28 (cubits)."[57] [$\bar{A}$, ii. 26–27 (i)]

Ex. 78. "If 2 and a half *kuḍavas* of kidney beans (*māṣa*) are obtained for an excellent elephant whose height is 4 cubits, length 6 (cubits), and breadth 5 (cubits), how much should be obtained for an elephant whose height is 3 (cubits), length 5 (cubits), and breadth $4\frac{1}{2}$ (cubits)?[58]
[$\bar{A}$, ii. 26–27 (i)]

On inverse proportion

Ex. 79. "When one *pala* is equivalent to 5 *suvarṇas*, a certain quantity of gold weighs 16 *palas*, what will the same gold weigh when one *pala* is equivalent to 4 *suvarṇas*?"[59] [$\bar{A}$, ii. 26–27 (i)]

Ex. 80. "8 baskets are seen (to contain the whole grain) when each (basket) contains 14 *prasṛtis*[60] (of grain); say how many baskets would be (required) when each (basket) can contain 8 *prasṛtis* (of grain) (only)."[61]
[$\bar{A}$, ii. 26–27 (i)]

On the simplification of fractions

Ex. 81. "$\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, and $\frac{1}{4}$ being respectively added together (two at a time), say what is the aggregate."[62] [$\bar{A}$, ii. 27(ii)]

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[56] Exs. 76 and 77 reappear in the commentaries of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 26.

[57] See footnote 56.

[58] After solving this example, Bhāskara I adds: "Similarly, (the rules of Āryabhaṭa I) should be applied to problems involving nine quantities or more." This shows that the so called rules of nine and eleven, etc. were well known in the time of Bhāskara I.

[59] Ex. 79 reappears in Yallaya's comm. on $\bar{A}$, ii. 26.

[60] *Prasṛti* is a measure of grain, equivalent to one handful. According to *Anuyogadvārasūtra*, 2 *prasṛtis* are equivalent to 1 *setikā*. See Section 2 (6) of this paper. (**ed.** in Part I of this paper.)

[61] Ex. 80 reappears in Yallaya's comm. on $\bar{A}$, ii. 26.

[62] Ex. 81, in different words, is found to occur in Pṛthūdaka's comm. on *BrSpSi*, xii. 8.

Ex. 82. "What are the sums of $\frac{1}{2}$, $\frac{1}{6}$ and $\frac{1}{3}$, and $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{20}$ and $\frac{1}{5}$?" [$\bar{A}$, ii. 27(ii)]

Ex. 83. "Calculate, O mathematician, what the following sums amount to: $\frac{1}{2}$ minus $\frac{1}{6}$; $\frac{1}{5}$ minus $\frac{1}{7}$; and $\frac{1}{3}$ minus $\frac{1}{4}$." [$\bar{A}$, ii. 27(ii)]

On the method of inversion

Ex. 84. "A number is multiplied by 2; then increased by 1; then divided by 5; then multiplied by 3; then diminished by 2; and then divided by 7: the result (thus obtained) is 1. Say what is the initial number."[63] [$\bar{A}$, ii. 28]

Ex. 85. "What is that number which when multiplied by 3, then diminished by 1, then halved, then increased by 2, then divided by 3 and finally diminished by 2, yields 1?" [$\bar{A}$, ii. 28]

3.2 Examples on Algebra

On simultaneous linear equations

Ex. 86. "In a forest there are (four) herds of elephants consisting (severally) of elephants in rut, elephants not in rut, female elephants, and young elephants. The sums of the elephants in the four herds excepting one (herd) at a time are known to be 30, 36, 49, and 50 (respectively). Correctly state the total number of elephants and also the number in each herd separately."[64] [$\bar{A}$, ii. 29]

Ex. 87. "The Sums of the numbers of elephants, horses, goats, asses, camels, mules, and cows neglecting one of those animals at a time, are respectively 28, and the same number (i.e. 28) successively diminished by 1, the last number (thus obtained) being further diminished by 1. If you have read the whole of the (chapter on) *Gaṇita* composed by Āryabhaṭa from a teacher, correctly state the total number of the animals and also the numbers of the different animals separately."[65] [$\bar{A}$, ii. 29]

[63] Ex. 84 reappears in the commentaries of Yallaya and Raghunātha Rāja on $\bar{A}$, ii. 28.

[64] Ex. 86 reappears in the commentaries of Sūryadeva and Raghunātha Rāja on $\bar{A}$, ii. 29. It requires the solution of the simultaneous equations: $x_2+x_3+x_4=30$, $x_3+x_4+x_1=36$, $x_4+x_1+x_2=49$, $x_1+x_2+x_3=50$, where x_1, x_2, x_3, and x_4 denote the numbers of animals in the four herds. See B. Datta and A. N. Singh, *History of Hindu Mathematics*, Part II, pp. 47 ff.

[65] If x_1, x_2, x_3, x_4, x_5, x_6, and x_7 are the numbers of the various animals and S their sum, then we have to solve the simultaneous equations: $S-x_1=28$, $S-x_2=27$, $S-x_3=26$, $S-x_4=25$, $S-x_5=24$, $S-x_6=23$, $S-x_7=21$.

On simple equations

Ex. 88. "(There are two merchants.) With the first merchant are seen by me 7 stout horses bearing auspicious marks and money amounting to 100 (*rūpakas*) in hand; with the second (merchant) there are 9 horses and money amounting to 80 (*rūpakas*). If the two merchants be equally rich and the price of each horse be the same, tell (me) the price of one horse and also the equal wealth (with them)."[66] [*Ā*, ii. 30]

Ex. 89. "A certain person has 8 *palas* of saffron and money amounting to 90 *rūpakas*; another person possesses 12 *palas* of saffron and 30 *rūpakas*; (and the two persons are equally rich). If the two persons have bought the saffron at the same rate per *pala*, I want to know the price of one *pala* (of saffron) and also the equal wealth of the two." [*Ā*, ii. 30]

Ex. 90. "7 *yāvattāvat* + 7 *rūpaka* = 2 *yāvattāvat* + 12 *rūpaka*. What is the value of 1 *yāvattāvat*?" [*Ā*, ii. 30]

Ex. 91. "9 *gulikā* + 7 *rūpaka* = 3 *gulikā* + 3 *rūpaka*. What is the price of 1 *gulikā*?" [*Ā*, ii. 30]

Ex. 92. "9 *gulikā* − 24 *rūpaka* = 2 *gulikā* + 18 *rūpaka*. Say what is the price of 1 *gulikā*." [*Ā*, ii. 30]

Ex. 93. "One (man) goes from Valabhī at the speed of $1\frac{1}{2}$ *yojanas* a day; another (man) comes (along the same route) from Harukaccha at the speed of $1\frac{1}{4}$ *yojanas* a day. The distance between the two (places) is known to be 18 *yojanas*. Say, O mathematician, after how much time (since start) they meet each other."[67] [*Ā*, ii. 31]

Ex. 94. "One man goes from Valabhī to the Ganges at the speed of $1\frac{1}{2}$ *yojanas* a day, and another from Śivabhāgapura at the speed of $\frac{2}{3}$ *yojanas* a day. The distance between the two (places) has been stated by the learned to be 24 *yojanas*. If they travel along the same route, after how much time will they meet (each other)?"[68] [*Ā*, ii. 31]

[66] Similar examples occur in Raghunātha Rāja's comm. on *Ā*, ii. 30.

[67] If t denotes the required time in days, then $1\frac{1}{2}t + 1\frac{1}{4}t = 18$, giving $t = 6\frac{6}{11}$ days.

[68] If t denotes the required time in days, then

$$1\frac{1}{2}t - \frac{2}{3}t = 24,$$

giving $t = 28\frac{4}{5}$ days. Exs. 93 and 94 reappear in Raghunāntha Rāja's comm. on *Ā*, ii. 31.

Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (III) *

This paper is the third of the series and deals with the examples set by Bhāskara I in illustration of Āryabhaṭa I's rules on the indeterminate analysis of the first degree, given in the last two stanzas of the *Āryabhaṭīya*. Bhāskara I's tables stating the least integral solutions of the planetary pulverizers are also given.

4 Bhāskara I's Examples (continued)

Below are given the examples set by Bhāskara I in illustration of Āryabhaṭa I's rules on the indeterminate analysis of the first degree known as the pulverizer (*kuṭṭaka* or *kuṭṭākāra*).

4.1 Mathematical Examples on the Pulverizer

Ex. 95. "A number leaves 1 as the remainder when divided by 5, and 2 (as the remainder) when divided by 7. Calculate what is that number."

Solution. Let the desired number be N. Then $N = 5x + 1 = 7y + 2$, whence $\frac{(7y+1)}{5} = x$. Solving this, we get $x = 3$, $y = 2$ as the least integral solution. Therefore $N = 16$.

Ex. 96. "A number yields 5 when divided by 12, and the same number is again seen by me to yield 7 when divided by 31. What is that number?"[1]

Ex. 97. "Calculate what is that number which is said to yield 5 as the remainder when divided by 8, 4 when divided by 9, and 1 when divided by 7."

Solution. Let the desired number be N. Then $N = 8x+5 = 9y+4 = 7z+1$. We first solve $N = 8x + 5 = 9y + 4$ or $\frac{(9y-1)}{8} = x$. This gives $x = 1 + 9t$, $y = 1 + 8t$ as the general solution, so that $N = 72t + 13$. Now we solve

* K. S. Shukla, *Gaṇita*, Vol. 23, No. 1 (June 1972), pp. 57–59.

[1] Answer 317.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
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$N = 72t + 13 = 7z + 1$ or $\frac{(72t+12)}{7} = z$ or $\frac{(2t+5)}{7} = z'$. This gives $t = 1$, as the least integral value of t. Therefore $N = 85$.

Ex. 98. "Quickly say, O mathematician, what is that number which when divided by the numbers beginning with 2 and ending in 6 (in each case) leaves 1 as the remainder, and is exactly divisible by 7."[2]

Ex. 99. "8 is multiplied by some number and the product is increased by 6 and that sum is then divided by 13. If the division be exact, what is the (unknown) multiplier and what is the resulting quotient?"

Solution. Let the multiplier be x and the quotient y. Then we have to solve the equation $\frac{(8x+6)}{13} = y$. Solving this, we get $x = 9$, $y = 6$.

Ex. 100. "11 is multiplied by a certain number, the product is diminished by 3, and the difference (thus obtained) being divided by 23 is (found to be) exactly divisible. Tell me the quotient and the multiplier."[3]

Exs. 95 to 98 are illustrations of the residual pulverizer (*sāgra-kuṭṭākāra*) and Exs. 99 and 100 are illustrations of the non-residual pulverizer (*niragra-kuṭṭakāra*). Classification of the pulverizer (*kuṭṭākāra*) into the residual (*sāgra*) and non residual (*niragra*) varieties is probably due to Bhāskara I. Such classification is not found to occur in the *Brāhmasphuṭasiddhānta* of Brahmagupta, who was a contemporary of Bhāskara I. Bhāskara I has shown that Āryabhaṭa I's rule (*Ā*, ii. 32–33) is applicable to both the residual and non-residual pulverizers.

Examples like 95 to 98 are now known as "the Chinese problems of remainders". One such example occurs in the Chinese arithmetical work, the *Sun-Tsū Suan-ching*, written about the last quarter of the first century AD. Sun Tsū, the author of the work, was able to get only a single solution of his problem. A general solution of the indeterminate equation of the first degree was not known in China even in the sixth and seventh centuries.

> "By that time, an indeterminate problem was attacked by three successive Chinese mathematicians of note and they obtained only three tentative solutions.[4] The Chinese indeterminate analysis, called *t'ai-yen-shu* or *t'ai-yan-ch'iu-i-shu* ("great extension method of finding unity") was materially developed by the Buddhist priest I-hsing in 727 AD and later on by Ch'iu Chiu-shao in 1247 AD.[5]

[2] Answer 721. This example reappears in the commentaries of Sūryadeva and Raghunātha Rāja on *Ā*, ii. 32–33.

[3] Answer 8, 17.

[4] *Toung Pao*, Vol. xiv (1913), p. 203.

[5] *Cf.* Yoshio Mikami, "The Development of Mathematics in China and Japan," Leipzig (1913), pp. 58, 63 et seq. Also *cf.* N. K. Majumdar, "On Chinese Indeterminate Analysis," *Bull. Calcutta Math. Soc.*, Vol. 5, pp. 9–11.

Now I-hsing was a Sanskrit scholar. He came to India in 673 AD and learnt, amongst various other things, the ingenious device of solving astronomical problems with the help of indeterminate analysis which seems to have been a favourite subject of study with the learned Hindu scholars of the time. On return to his native land, I-hsing availed himself of this helpful device in composing a new calendrical system for the Chinese and for so doing he was once accused of too much Hindu bias by the native Chinese calendar-makers. Professor Mikami has pointed out that the Chinese interest in indeterminate analysis grew after their contact with the Hindu culture and he seems to be of further opinion that it did so, indeed, under the influence of the latter.[6] It is, however, noteworthy that the interest of Chinese in indeterminate analysis always remained confined amongst the astronomers.[7]"[8]

4.2 Astronomical Examples on the pulverizer[9]

Ex. 101. "The mean (position) of the Sun has been observed by me at sunrise to be in the sign Leo in the middle of the *navamāṃśa* Sagittarius.[10] Calculate the *ahargaṇa* (i.e., the number of days elapsed since the beginning of *Kaliyuga* when the longitude of the planets was zero) according to the *(Ārya)bhaṭa-śāstra*, and also the revolutions performed by the Sun since the beginning of *Kaliyuga*."[11]

[6] *Cf.* Mikami, *l.c.*, p. 58.
[7] *Cf.* Mikami, *l.c.*, p. 65.
[8] *Cf.* B. Datta, "The Hindu Contributions to Mathematics — Presidential Address at the Annual Meeting of the Association," *Bull. Math. Association University of Allahabad*, Vol. II, 1928–29, pp. 9–10.
[9] The early Hindu theory of the planetary pulverizer is fully discussed in the *Laghubhāskarīya* (pp. 103–114) and the *Mahābhāskarīya* (pp. 29–46, 219–224) edited by me.
[10] The *navamāṃśa* Sagittarius of the sign Leo is the ninth *navamāṃśa* (=ninth part) of that sign and extends from $146°40'$ to $150°$ of longitude. The longitude of the middle point of that *navamāṃśa* is thus $148°20'$.
[11] For a solution of this example, see *MBh* (=*Mahābhāskarīya*) edited by me, English Translation, p. 34. This example has also been solved by Govinda-svāmī and Parameśvara in their commentaries on *MBh*, i. 56.

Hindu mathematics in the seventh century as found in Bhāskara I's commentary on the *Āryabhaṭīya* (IV) *

This paper is the fourth and the last of the series and deals with quotations from the earlier mathematical works occurring in Bhāskara I's commentary on the *Āryabhaṭīya*. Of these quotations, some are taken from such works as were popularly used in the time of Bhāskara I, some are mentioned to point out the approximate character of rules contained in them so as to emphasise the superiority of the corresponding rules given in the *Āryabhaṭīya*, and some are quoted to find fault with them. Some of these quotations are in Prākṛta *gāthās* and seem to have been taken from Jaina sources. These quotations show that in the seventh century when Bhāskara I wrote his commentary on the *Āryabhaṭīya* there existed works on mathematics which were written not only in Sanskrit but also in Prākṛta. These works were probably of the same nature as the *Pāṭīgaṇita* and the *Triśatikā* and were probably written by Maskarī Pūraṇa, Mudgala, and Patana etc. whose names have been mentioned by Bhāskara I.

5 Passages quoted by Bhāskara I from mathematical works

5.1 Quotation 1: Proclaiming twofold nature of mathematics

आह च–
संयोगभेदा गुणना गतानि शुद्धेश्च भागो गतमूलयुक्तः ।
व्याप्तं समीक्ष्योपचयक्षयाभ्यां विद्यादिदं द्व्यात्मकमेव शास्त्रम् ॥ [$\bar{A}$, ii. intro]

Multiplication and involution are the kinds of addition, and division and evolution, of subtraction. Seeing that the science of mathematics is permeated by increase and decrease, this science is indeed of two kinds.

This passage seems to have been taken from the introductory verses of a certain work on *Pāṭīgaṇita*.

* K. S. Shukla, *Ganita*, Vol. 23, No. 2 (December 1972), pp. 41–50.

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5.2 Quotation 2: Giving a rule for squaring a number

In the *Āryabhaṭīya* there is no rule for squaring a number, so Bhāskara I quotes the following rule:

> अन्त्यपदस्य च वर्गं कृत्वा द्विगुणं तदेव चान्त्यपदम् ।
> शेषपदैराहन्याद् उत्सार्योत्सार्य वर्गविधौ ॥[1] [*Ā*, ii. 3 (i)]
>
> Having squared the last digit (on the left) (and then having written that square underneath the last digit), multiply twice of that last digit by the remaining digits of the number, and write the products successively one place ahead (to the right). This is the procedure (to be adopted) in squaring a number.[2]

It is interesting to note that Bhāskara I refers to the above rule as *lakṣaṇa-sūtra*. In all later works on mathematics a rule has been called *karaṇa-sūtra*. Possibly, in the work from which the above passage was taken the rules were called *lakṣaṇa-sūtra*.

5.3 Quotation 3: Giving a formula for the simplification of a fraction of the type $a + \frac{b}{c}$

While solving Ex. 2,[3] Bhāskara I quotes (under *karaṇa*) the following formulatory sentence which probably formed part of some rule given by an earlier author:

> छेदगुणं सांशम् ।
>
> Multiply (the whole number) by the denominator and add the numerator.[4]

---

[1]Similar rules occur in *Triśatikā* (Rule 10, p. 3) and *Gaṇitasārasaṅgraha* (Rule 31, p. 13).

[2]To square 34, for example, the following procedure will be adopted:

		3	4		
Square of the last digit		9			
Product of 2 times 3 multiplied by the next digit 4 written one place ahead		2	4		(One round of the operation is over and the rule is repeated)
Square of the last but one digit i.e., 4, written one place ahead			1	6	
Addition gives	1	1	5	6	which is the required number.

In the actual Hindu process of working, the numbers were not allowed to accumulate. Addition was performed after every step and only one number was allowed to remain on the writing board.

[3]Vide supra 3.1.

[4]That is to say, $a + \frac{b}{c}$ is equal to $\frac{(ac+b)}{c}$.

The similarity of this formula with the corresponding formula "*chedasaṅguṇaṃ sāṃśam*" of Śrīdhara is noteworthy.[5]

5.4 Quotation 4: Giving a rule for cubing a number

On the cubing of a number also there is no specific rule in the *Āryabhaṭīya*. Bhāskara I refers to the following rule:

> "अन्त्यपदस्य घनं स्या"दित्यादि लक्षणसूत्रम् । [*Ā*, ii. 3 (ii)]
>
> "Obtain the cube of the last digit (on the left) etc." is the rule (for the purpose).

Like the rule of squaring, this rule has also been called by the name *lakṣaṇa-sūtra*, which seems to suggest that both the rules have been extracted from the same source. The striking similarity in the phraseology is noteworthy. The reference of the above rule by indicating its beginning alone proves the popularity of the work from which it has been taken.

5.5 Quotation 5: On the position of the altitude of an equilateral triangles as a line of symmetry

> करणम् – समत्र्यश्रिक्षेत्रे समैवावलम्बकस्थितिरिति । [*Ā*, ii. 6, Ex. 10]
>
> *Process.* (Applying the rule) "In an equilateral triangle, the position of the altitude is that of (the line of) symmetry," (we have that, etc.).

5.6 Quotation 6: Giving an approximate rule for the area of a circle

In order to emphasise the accuracy of Āryabhaṭa I's rule (*Ā*, ii. 7), Bhāskara I quotes the following popular rule and points out that it is only approximate, not accurate. In fact, he says, there is no other accurate rule.

> व्यासार्धकृतिस्त्रिसंगुणा गणितम् । [*Ā*, ii. 7 (i)]
>
> The square of the radius multiplied by 3 is the area (of a circle).

5.7 Quotation 7: Giving an approximate rule for the volume of a sphere

According to Bhāskara I, the rule (*Ā*, ii. 7) of Āryabhaṭa I gives the accurate value of the volume of a sphere.[6] All other rules, he tells us, are only approximate. Of these approximate rules he quotes the following:

[5] *Cf. Triśatikā*, edited by S. Dvivedi, Banaras (1899), Rule 24, p. 10.
[6] In fact, however, Āryabhaṭa I's rule stated in *Ā*, ii. 7 is wrong.

व्यासार्धघनं भित्वा नवगुणितमयोगुडस्य घनगणितम् । [$\bar{A}$, ii. 7 (ii)]

The cube of the radius divided by two and multiplied by nine is the volume of an iron ball.[7]

It is strange that the accurate formula for the volume of a sphere was not known in India. This seems to suggest that Greek Geometry was not known at all in India and that all geometrical results and mensuration formulae were independently discovered by Hindu mathematicians.

5.8 Quotation 8: On the verification of areas

Bhāskara I is of the opinion that the rule stated in $\bar{A}$, ii. 9 (i) relates to the verification of the area of any rectilinear figure by deforming it into a rectangle. In support of his statement, he cites the following passages:

उक्तञ्च –
करणैरुक्तैर्नित्यं फलमनुगम्यायते तु विज्ञेयम् ।
प्रत्ययकरणं क्षेत्रे व्यक्तं फलमायते यस्मात् ॥ [$\bar{A}$, ii. 9 (i)]

So has it been stated too. Having determined the area in accordance with the prescribed rules, demonstration (verification) should always be made by (deforming the figure into) a rectangle because it is the rectangle only of which the area is obvious.

5.9 Quotation 9: Giving an approximate formula for the area of a plane figure resembling the tusk of an elephant

पृष्ठोदरसमासार्धं विस्तारार्धगुणं फलम् । [$\bar{A}$, ii. 9 (ii)]

Half the sum of the back and the belly multiplied by half the width (at the base) is the area (of the figure that resembles the tusk of an elephant).[8]

5.10 Quotation 10: One-half of a *gāthā* giving a rule for finding the circumference of a circle

विक्खम्भवग्गदहगुणकरणी वट्टस्य परिरओ होदि । [$\bar{A}$, ii. 10]
(विष्कम्भवर्गदशगुणकरणी वृत्तस्य परिणाहः भवति ॥)

The square root of ten times the square of the diameter is the circumference of a circle.

[7] *Cf. Gaṇitasārasaṅgraha* viii. $28\frac{1}{2}$ (i). Also *Cf. Trilokasāra*, *gāthā* 19 (i).
[8] *Vide supra*, 3.1, Ex. 27.

The above *gāthā* is almost the same as *gāthā* 185 of the Jaina work, the *Jyotiṣakaraṇḍaka*.[9] The latter runs as follows:

> विक्खम्भवग्गदहगुणकरणी वट्टस्स परिरओ होइ ।

This *gāthā* is also found to occur in the *Trilokasāra*[10] and the *Kṣetrasamāsa*, and has been quoted by Malayagiri in his commentary on the *Jīvābhigama* (*Sūtra* 109).

Bhāskara I has quoted the above *gāthā* as giving the so called accurate value of π, viz. $\sqrt{10}$. Bhāskara I has taken pains to demonstrate that this value was far from being accurate. In this connection he makes use of the rules given in Quotations 11, 12, 13, and 14. In the end he concludes in derision:

> So I bow to $\sqrt{10}$ whose grace is not well conceived.[11]

5.11 Quotation 11: One *gāthā* giving a rule for getting the value of the chord of a circle

> ओगाहूणं विक्कम्भ एगाहेण संगुणं कुर्यात् ।
> चउगुणियस्सतु मूउं सा जीवा सघ्वखत्ताणम् ॥ (MS T)
> ओगाहूणं विक्खम्भ एगाहेण संगुणं कुर्यात् ।
> चउगुणियः स तु (मू) उं सा जीवा सव्वखत्ताणम् ॥ (MS I) [$\bar{A}$, ii. 10]

Probable Sanskrit version:

> अवगाहोनं विष्कम्भमवगाहेण सङ्गुणं कुर्यात् ।
> चतुर्गुणितस्य तु मूलं सा जीवा सर्वक्षेत्राणाम् ॥

> Multiply the diameter as diminished by the depth (of the chord) by the depth. The square root of four times the (product) is (the length of) the chord of any (circular) figure.

In the adjoining figure (**ed.** see Figure 1), let AB be a chord of the circle ACB and let CD be the perpendicular diameter intersecting the chord at X. Then CX ($< XD$) denotes the depth of the chord.

If R be taken to denote the radius of the circle, then the rule stated above may be written as

$$\text{chord} = \sqrt{4 \times \text{depth} \times (2R - \text{depth})}.$$

[9] *Cf.* Rājeśvarīdatta Miśra, "Vṛttakṣetra kā gaṇita: Jaina tathā Jainetara ācāryon ke siddhānta," *Jaina-siddhānta-bhāskara* (Jaina Antiquary), Part XV, vol. 2, pp. 105 ff.

[10] *Gāthā* 96.

[11] अतोऽस्यै अविचारितमनोहरायै नभोऽस्तु दशकरण्यै ।

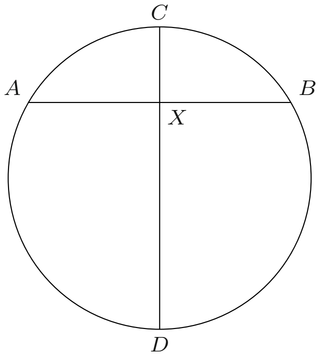

Figure 1

This formula is also mentioned in the *Tatvārthādhigamasūtrabhāṣya* of Umāsvātī.[12] Ācārya Yativṛṣbha puts it in the form[13]

$$\text{chord} = \sqrt{4\{R^2 - (R - \text{depth})^2\}}.$$

5.12 Quotation 12: One *gāthā* giving a rule for the area of a segment of a circle

इसुपायगुणा जीवा दसिकरणि भवेद् विगणिय पदम् ।
धनुपट्ट अम्मिखत्ते एदं करणं तुदा अव्वम् ॥ (MS T)
इषुपायगुणा जीवा दशकरणि भवेद् विगणिय पदम् ।
धनुवट्ट अम्मिखत्ते एदं करणं तुदा अप्पम् ॥[14] (MS I) [$\bar{A}$, ii. 10]

This *gāthā* states that the area of a segment of a circle

$$= \sqrt{10}\left(\frac{1}{4}\right)(\text{arrow})(\text{bounding chord}),$$

[12] See B. Datta, The Jaina School of Mathematics, *Bull. Cal. Math. Soc.*, Vol. XXI, 1929. Umāsvātī, according to the tradition of the Śvetāmbara Jainas, lived about 150 BC. According to the Digambara tradition he is sometimes called Umāsvāmī and is said to have lived between 135 AD and 219 AD. Satischandra Vidyabhushan thinks that he lived in the first century AD. See Datta, *l.c.*

[13] *Cf. Tiloya-paṇṇattī* (*Triloka-prajñapti*), edited by A. N. Upadhye and Hiralal Jain, and published by Jaina Saṃrakṣaka Saṃgha, Sholapur (1943), Part I, iv. 180.

According to A. N. Upadhye and Hiralal Jain, Ācārya Yativṛṣbha lived sometime between 473 AD and 609 AD. See *Tiloya-paṇṇattī*, Part II, edited by Hiralal Jain and A. N. Upadhye, and published by Jaina Saṃrakṣaka Saṃgha, Sholapur (1951), p. 7.

[14] The following is the probable Sanskrit version:

इषुपादगुणा जीवा दशकरणीभिर्भवेद् विगुण्य फलम् ।
धनुपट्टेऽस्मिन् क्षेत्रे एतत्करणं तु ज्ञातव्यम् ॥

the arrow being the depth of the chord (explained above).

The occurrence of the above *gāthā* in Bhāskara I's commentary shows that rules for obtaining the area of a segment of a circle were devised in India prior to the seventh century AD. Datta once remarked:

> We do not find amongst the Hindus, as far as is known, any expression for the area of a segment of a circle before the time of Śrīdhara (c. 750) though it was known in Greece and China long before.[15]

The rule described in the above *gāthā* is also found to occur in the *Gaṇitasāra-saṅgraha*[16] of Mahāvīra (850) and in the *Laghu-kṣetra-samāsa* of Ratneśvara Sūri (1440 AD).

The rule, Datta rightly observes,[17] is incorrect and was probably derived by analogy from the rule for the area of a circle.

5.13 Quotation 13: One *gāthā* giving a rule for the the addition of two surds

> औवट्टि अदस्सकेण इमूलसमासस्समोत्थवत् ।
> ओं पट्टणाय गुणियं करणिसमासं तुणा अव्वम् ॥ (MS T)
> औवट्टि असस्सकेण इमूलसमासः समोत्थवत् ।
> ओवट्टणाय गुणियं करणिसमासं तुणा अव्वम् ॥[18] (MS I) [$\bar{A}$, ii. 7]

A. N. Singh has given the following translation:[19]

> Reducing them (i.e. the two surds) by some suitable number, add the square roots of the quotients: the square of the result multiplied by the reducer should be known as the sum of the surds.

In other words,

$$\sqrt{\alpha} + \sqrt{\beta} = \sqrt{c\left(\sqrt{\frac{\alpha}{c}} + \sqrt{\frac{\beta}{c}}\right)^2}$$

where $\frac{\alpha}{c}$ and $\frac{\beta}{c}$ are assumed to be perfect squares. This rule is found to occur in several later works also.[20]

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[15] *Cf.* B. Datta, "The Jaina School of Mathematics."

[16] vii. $70\frac{1}{2}$.

[17] *l.c.*

[18] The following is the probable Sanskrit version:

> अपवर्त्याभीप्सितेन करणीमूलसमासोत्थवर्गो यः ।
> अपवर्तनेन गुणितः करणिसमासस्तु ज्ञातव्यः ॥

[19] *Cf.* A. N. Singh, "On the arithmetic of surds amongst the ancient Hindus," *Mathematica*, Vol. XII, p. 104.

[20] See, for instance, *Brāhmasphuṭasiddhānta*, xviii. 38 and *Gaṇitasārasaṅgraha*, vii. $88\frac{1}{2}$.

5.14 Quotation 14: Containing a rule for the length of an arc of a circle, when the arc is less than a semi-circle

ज्यापादशरार्धयुतिः स्वगुणा दशसङ्गुणा करण्यस्ताः । [$\bar{A}$, ii. 10]

The sum of one-fourth of the chord and one-half of the arrow, multiplied by itself, and then by 10: the square root of so much (is the length of the corresponding arc).

That is to say,

$$\text{arc} = \left(\frac{\text{chord}}{4} + \frac{\text{arrow}}{2}\right) \times \sqrt{10}. \qquad (1)$$

This formula has not been traced in any other Jaina or Hindu work although other formulae for the arc of a circle are found to occur elsewhere.

In the first century Umāsvāti in his *Tatvārthādhigamasūtrabhāṣya* gave the formula

$$\text{arc} = \sqrt{6(\text{arrow})^2 + (\text{chord})^2}. \qquad (2)$$

If in (1) and (2), we put arrow $= r$ and chord $= 2r$, ($r =$ radius), both (1) and (2) reduce to

$$\text{arc (of the semi-circle)} = r\sqrt{10}.$$

This shows that both (1) and (2) have been derived from the formula for the arc of a semi-circle, using $\pi = \sqrt{10}$. But the methods used in the derivations are obviously different.

Other formulae for an arc of a segment of a circle are known to occur in the *Gaṇitasārasaṅgraha*,[21] and in the *Mahāsiddhānta*[22] of Āryabhaṭa II (950). The formulae given in those works have been derived in the same way as (2) but with different values of π (viz. $\pi = 3$, $\sqrt{10}$, and $\frac{22}{7}$).

It is noteworthy that the term *jīvā*, which usually means a half-chord, has been used in the above quotations in the sense of a chord.

5.15 Quotation 15: On the arrangement of the three quantities in a problem on the Rule of Three

उक्तञ्च –
आद्यन्तयोस्तु सदृशौ विज्ञेयौ स्थापनासु राशीनाम् ।
असदृशराशिर्मध्ये त्रैराशिकसाधनाय बुधैः ॥ [$\bar{A}$, ii. 26–27 (i)]

So has been stated —

In order to solve a problem involving three quantities, the learned should note that of the three quantities the two of like denomination should be set down in the beginning and the end, and the

[21] vii. 43, $73\frac{1}{2}$.
[22] xv. 90, 94, 95.

third quantity of unlike denomination should be written in the middle (of those two).

For example in Ex. 66,[23] the quantities "9 *rūpakas*" and "one *rūpaka*" are of like denomination (because both of them are *rūpakas*) and the third quantity "5 *palas*" is of unlike denomination. These quantities are therefore to be written as

9 5 1

Similar direction regarding the arrangement (*sthāpana*) of the three quantities is given by Śrīdhara and others.

5.16 Quotation 16: One *gāthā* stating how to perform subtraction when the minuend and the subtrahend are both positive, both negative, or one positive and the other negative

सोज्झं भूणारधनं अणं अणदो न यमदो न यमदो सोज्झं ।
विपरीते साधण एषो झं वाकिवगुहोल ॥ (MS T)
सोज्झं भ्रूणारयणं भ्रणं अणदो न यमदो न यमदो सोज्झम् ।
विपरीते सायण एषो ज्झं वा किं व्व गुहोल ॥[24] (MS I) [*Ā*, ii. 30]

Both the readings are defective. From the context and the nature of the *gāthā* it is clear that it deals with the subtraction of positive and negative quantities. We are sure that the rules given therein are the same as stated in the following stanzas of the *Brāhmasphuṭasiddhānta*:

ऊनमधिकाद्विशोध्यं धनं धनादृणमृणादधिकमूनात् ।
व्यस्तं तदन्तरं स्यादृणं धनं धनमृणं भवति ॥
शून्यविहीनमृणमृणं धनं धनं भवति शून्यमाकाशम् ।
शोध्यं यदा धनमृणादृणं धनाद्वा तदा क्षेपम् ॥

From the greater should be subtracted the smaller; (the final result is) positive, if positive from positive, and negative, if negative from negative. If however, the greater is subtracted from the less, the difference is reversed (in sign), negative becomes positive and positive becomes negative. When positive is to be subtracted from negative or negative from positive, then they must be added together.[25]

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[23] Vide *Supra*, 3.1.

[24] The probable Sanskrit version is as follows:

शोध्यमृणादृणं धनं धनतः न धनतो न ऋणतः शोध्यम् ।
विपरीते शोधनमेव धनं न किमपि गूढमत्र ॥

[25] B. Datta and A. N. Singh, *History of Hindu Mathematics*, Part II, pp. 21–22. Also see, *Brāhmasphuṭasiddhānta* xviii. 31–32.

The *gāthā* mentioned above has been quoted by Bhāskara I in connection with the solution of the equation

$$9x - 24 = 2x + 18.$$

(Ex. 92). Application of Āryabhaṭa I's rule ($\bar{A}$, ii. 30) gives

$$x = \frac{18 - (-24)}{9 - 2}.$$

The *gāthā* is meant to show how -24 is to be subtracted from 18. Using the rule mentioned therein, one gets

$$x = \frac{18 + 24}{7} = 6.$$

The above *gāthā* clearly shows that negative numbers were introduced in analysis in India and methods were also devised for subtracting greater numbers from smaller ones or positive numbers from negative numbers or vice versa much before the time of Bhāskara I.

On Śrīdhara's rational solution of $Nx^2 + 1 = y^2$ *

1 Introduction

Śrīdhara is remembered as one of the greatest Hindu Mathematicians.[1] Unfortunately there is no definite evidence to show when and where he lived. Even his works are not all available. His very extensive (*ativistṛta*) treatise on algebra, which has been mentioned and quoted by the celebrated astronomer-mathematician Bhāskara II[2] (1150), is known only by name. Probably it is lost. Of his two works on arithmetic, known to us, the smaller one, known as *Pāṭīgaṇitasāra*, was edited by Sudhakara Dvivedi (1899) and published under the title of *Triśatikā*.[3] An English translation, with notes and introduction, of the rules occurring in this work has also appeared in the *Bibliotheca Mathematica*[4] under the joint authorship of N Ramanujacharia and G. R. Kaye. The bigger work on arithmetic, known as Śrīdhara's *Pāṭīgaṇita*, has not yet appeared in print. This work has been called *Navaśatī* and has been quoted by Makkibhaṭṭa in his commentary on the *Siddhāntaśekhara* of Śrīpati[5] (1039). An incomplete MS of this work is preserved in the Raghunatha Temple Library of His Highness the Maharaja of Jammu and Kashmir. It is a copy of an older MS, is written in modern Kashmīrī script and extends to 157 leaves with 9 lines to a page and about 44 letters to a line. Starting from the very beginning, it runs up to about the middle of the *kṣetra-vyavahāra* and is furnished with a commentary. The name of the commentator does not occur anywhere

* K. S. Shukla, *Ganita*, Vol. 1, No. 2 (1950), pp. 53–64.

[1] The following stanza, which occurs with the colophon at the end of Śrīdhara's *Pāṭīgaṇitasāra* in certain manuscripts, gives an idea of the highest position which Śrīdhara occupied in his time as a mathematician:

> उत्तरतः सुरनिलयं दक्षिणतो मलयपर्वतं यावत् ।
> प्रागपरोदधिमध्ये नो गणकः श्रीधरादन्यः ॥
>
> Up to the abode of the gods (i.e., the Himālayas) towards the north and up to the Malaya mountains towards the south and between the eastern and the western oceans, there is no mathematician (worth the name) except Śrīdhara.

[2] *Cf.* Bhāskara II's *Bījagaṇita*, conclusion. Also see *madhyamāharaṇa*, 1–3 (comm.).
[3] *Triśatikā* or *Triśatī* is another name of Śrīdhara's *Pāṭīgaṇitasāra*.
[4] Vol. XIII (1912–13), p. 203–217.
[5] *Cf.* *Siddhāntaśekhara*, i. 26 (comm.).

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_10

in the MS. A transcript copy of this, in Devanāgarī characters, exists in the Lucknow University Library.

This MS contains a rule for the rational solution of the equation

$$Nx^2 + 1 = y^2,$$

based on the rational solution of the equation $x^2 + y^2 = z^2$ and is fundamentally different from those given by Brahmagupta (628), Śrīpati (1039), and other Hindu mathematicians. The object of the present paper is to explain and illustrate Śrīdhara's method of obtaining this rule and to give analogous rules for the rational solution of certain other equations of a similar nature.

2 Śrīdhara's lemma

Śrīdhara starts with a lemma stating how to construct a rational rectangle (or right triangle). It runs:

> भुजस्य कृतिरिष्टस्य भक्तोनेष्टेन तद्दलम् ।
> कोटिरिष्टाधिका कर्णश्चतुरश्रायतस्य ते ॥[6]
>
> The square of the base (*bhuja*), chosen at pleasure, when divided and diminished by an arbitrary number and then halved gives the perpendicular (*koṭi*), and that increased by the (same) arbitrary number gives the hypotenuse (*karṇa*)—all of them of a rectangle.[7]

That is, if a number b be chosen for the base and ε for the arbitrary number, then there corresponds a rectangle of base b having the rational numbers

$$\frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} \quad \text{and} \quad \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} + \varepsilon$$

for its perpendicular and hypotenuse (or diagonal) respectively.

For example, choose 3 for the base and 1 for the arbitrary number. Then there corresponds a rectangle whose base is 3, perpendicular $\frac{1}{2}\left\{\frac{3^2}{1} - 1\right\}$ i.e., 4, and hypotenuse 5. Similarly, choosing 10 for the base and 2 for the arbitrary number, we obtain a rectangle whose base is 10, perpendicular $\frac{1}{2}\left\{\frac{10^2}{2} - 2\right\}$ i.e., 24, and hypotenuse 26; and so on.

[6] *Pāṭīgaṇita*, *kṣetra-vyavahāra*.

[7] This rule occurs elsewhere also. For example, see *Brāhmasphuṭasiddhānta* xii. 35; *Siddhāntaśekhara*, xiii. 41; *Gaṇitasārasaṅgraha*, vii. $97\frac{1}{2}$; *Līlāvatī*, *kṣetra-vyavahāra*, rule 5. The text of the above passage closely resembles that of the *Brāhmasphuṭasiddhānta*, xii. 35.

The rationale of this is as follows: Let b, k, and h be the base, perpendicular, and hypotenuse of a rectangle. Then

$$b^2 = h^2 - k^2 = (h - k)(h + k).$$

Let $h - k = \varepsilon$. Then $k = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\}$ and $h = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\} + \varepsilon$.

3 The equation $Nx^2 + 1 = y^2$

This equation has been called *vargaprakṛti* (square-nature) by Hindu mathematicians. The number N is called *prakṛti* or *guṇaka* (i.e., the co-efficient or multiplier of the square); x is called *ādyapada* (i.e., the first root) or *kaniṣṭha-pada* (i.e., the lesser root); y is called *anyapada* (i.e., the other root) or *jyeṣṭha-pada* (i.e., the greater root); and the absolute term 1 is called *kṣepa* (additive) or *śodhya* (subtractive) according as it is positive or negative.

4 Rational solution of $Nx^2 + 1 = y^2$

For the rational solution of $Nx^2 + 1 = y^2$, Śrīdhara gives the following rule, making use of the above lemma:[8]

> गुणके वर्गयोर्मध्ये तत्पदाधो भुजश्रुती ।
> केचित् प्राक्कथिते तत्र वज्रकेणाहती तयोः ॥
> अन्तरस्य कृतिः क्षेपः तत्कोटिः प्रथमं पदम् ।
> ऋजुहत्यन्तरं ज्येष्ठं रूपक्षेपेऽन्तरोद्धृते ॥ [9]
>
> The multiplier (*guṇaka*) having been expressed as a difference of two squares (set down their square-roots in the ascending order of magnitude and) below those square-roots set down any (set of the) base and hypotenuse stated there (in the lemma) before. Then obtain their cross-products. The square of the difference between those (cross-products) gives the additive (*kṣepa*); the perpendicular of that (above set) denotes the first square-root (corresponding to that additive) and the difference between their direct (vertical) products denotes the greater square-root. When these (first and greater square-roots) are divided by the difference (between the above cross-products), they correspond to the additive unity.

Let the multiplier N be expressed as the difference $A^2 - B^2$ $(A > B)$ of two squares[10] and let the base, perpendicular, and hypotenuse determined from the above lemma be b, k, and h respectively. Then setting down B and A in the ascending order of magnitude, we have

$$B \qquad\qquad A$$

[8] In fact, he gives this rule as an application of the above lemma.

[9] *Pāṭīgaṇita*, *l. c.*

[10] Making use of Śrīdhara's lemma, N can always be expressed as

$$\left\{\frac{1}{2}\left(\frac{N}{c}+c\right)\right\}^2 - \left\{\frac{1}{2}\left(\frac{N}{c}-c\right)\right\}^2.$$

and below them writing down the base b and the hypotenuse h respectively, we have

$$\begin{array}{cc} B & A \\ b & h \end{array}$$

Multiplying them across, we obtain

$$Bh \qquad \text{and} \qquad Ab;$$

and multiplying them directly, we have

$$Bb \qquad \text{and} \qquad Ah.$$

The square of the difference between the cross-products i.e., $(Bh \sim Ab)^2$, then, denotes the so called additive (*kṣepa*); and the perpendicular k and the difference $Ah - Bb$ of the direct products respectively denote the first and greater roots corresponding to that additive. These first and greater roots when divided by the difference $(Bh \sim Ab)$ of the cross-products give the corresponding quantities for additive unity.

In other words, Śrīdhara's rule amounts to saying that if

$$A^2 - B^2 = N \tag{1}$$

and

$$h^2 - b^2 = k^2 \tag{2}$$

then

$$Nk^2 + (Bh \sim Ab)^2 = (Ah - Bb)^2,$$ [11]

whence

$$N\left\{\frac{k}{Bh \sim Ab}\right\}^2 + 1 = \left\{\frac{Ah - Bb}{Bh \sim Ab}\right\}^2.$$

That is

$$x = \frac{k}{Bh \sim Ab}, \qquad y = \frac{Ah - Bb}{Bh \sim Ab}$$

is the rational solution of

$$Nx^2 + 1 = y^2.$$

[11] This result is easily obtained by multiplying (1) and (2) side by side and bringing the resulting product to the requisite form.

4.1 Illustration

The following example will illustrate the above rule.

Example What is that number whose square-root when multiplied by 55 and then increased by 1 becomes capable of yielding a square-root?

The resulting equation is

$$55x^2 + 1 = y^2.$$

One solution

We have

$$8^2 - 3^2 = 55,$$
$$\text{and} \qquad 5^2 - 3^2 = 4^2.$$

Therefore, setting down 3 and 8 and below them 3 and 5 respectively, we have[12]

$$\begin{array}{cc} 3 & 8 \\ 3 & 5 \end{array}$$

The difference between the cross-products is 9, the perpendicular is 4, and the difference between the direct products is 31. Therefore,

$$55(4)^2 + (9)^2 = (31)^2,$$
$$\text{whence} \qquad 55\left(\frac{4}{9}\right)^2 + 1 = \left(\frac{31}{9}\right)^2.$$

Hence $x = \frac{4}{9}$, $y = \frac{31}{9}$ is a rational solution of the above equation. Thus the required number is $\frac{4}{9}$.

Another solution

Again we have

$$8^2 - 3^2 = 55,$$
$$\text{and} \qquad 10^2 - 8^2 = 6^2.$$

Therefore, setting down 3 and 8 and below them 8 and 10 respectively, we have

$$\begin{array}{cc} 3 & 8 \\ 8 & 10 \end{array}$$

[12]The Hindu way of writing is $\begin{array}{|c|c|} 3 & 8 \\ 3 & 5 \\ \hline \end{array}$

The difference between the cross-products is 34, the perpendicular is 6, and difference between the direct products is 56. Therefore,

$$55(6)^2 + (34)^2 = (56)^2,$$
$$\text{or} \qquad 55\left(\frac{6}{34}\right)^2 + 1 = \left(\frac{56}{34}\right)^2.$$

Hence $x = \frac{3}{17}$, $y = \frac{28}{17}$ is another rational solution of the above equation. This gives the required number to be $\frac{3}{17}$.

Other solutions

Other rational solutions of the same equation will be obtained by altering the sides of the rational right triangle for every new solution.

5 Another form of rational solution of $Nx^2+1=y^2$

It will be easily seen that when

$$A^2 - B^2 = N$$
$$\text{and} \qquad h^2 - b^2 = k^2,$$
$$\text{then} \qquad Nk^2 + (Bh \overset{+}{\sim} Ab)^2 = (Ah \pm Bb)^2$$
$$\text{or} \qquad N\left\{\frac{k}{Bh \overset{+}{\sim} Ab}\right\}^2 + 1 = \left\{\frac{Ah \pm Bb}{Bh \overset{+}{\sim} Ab}\right\}^2.$$

That is, both

$$\left.\begin{aligned} x &= \frac{k}{Bh + Ab} \\ y &= \frac{Ah + Bb}{Bh + Ab} \end{aligned}\right\} \qquad \text{and} \qquad \left.\begin{aligned} x &= \frac{k}{Bh \sim Ab} \\ y &= \frac{Ah - Bb}{Bh \sim Ab} \end{aligned}\right\}$$

are the rational solutions of $Nx^2+1=y^2$.

In the foregoing rule, Śrīdhara gives the solution in the latter form.

5.1 Other forms

Putting

$$k = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} - \varepsilon\right\},$$
$$h = \frac{1}{2}\left\{\frac{b^2}{\varepsilon} + \varepsilon\right\},$$
$$B = \frac{1}{2}\left\{\frac{N}{c} - c\right\},$$
and
$$A = \frac{1}{2}\left\{\frac{N}{c} + c\right\},$$

Śrīdhara's solution may be stated as

$$x = \frac{2c(b^2-\varepsilon^2)}{N(b-\varepsilon)^2 \sim c^2(b+\varepsilon)^2}$$
$$y = \frac{N(b-\epsilon)^2 + c^2(b+\varepsilon)^2}{N(b-\varepsilon)^2 \sim c^2(b+\varepsilon)^2}$$

where b, ε, and c are any numbers.

This may be set in the form

$$x = \frac{4pq}{4p^2N \sim q^2}$$
$$y = \frac{4p^2N + q^2}{4p^2N \sim q^2}$$

in which it was discovered in Europe by John Wallis[13] (1657).

Further, putting $\frac{q}{2p} = r$, this may be written as

$$x = \frac{2r}{N \sim r^2}$$
$$y = \frac{N + r^2}{N \sim r^2}$$

in which form it was given by Śrīpati[14] (1039), Bhāskara II[15] (1150), Nārāyaṇa[16] (1356), Jñānarāja, and Kamalākara[17] (1658), and in Europe by W. Brouncker[18] (1657).

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[13] *Cf. Oeuvres de Fermat*, III, (1896), *Lettre ix*; John Wallis A Kenelm Digby, p. 417 ff. Also *Cf.* Dickson, L. E., *History of the Theory of Numbers*, II, p. 351.

[14] *Cf. Siddhāntaśekhara*, xiv. 33.

[15] *Cf.* His *Bījagaṇita*, *vargaprakṛti*, rule 6.

[16] *Cf.* His *Bījagaṇita*, *I*, rule 77 f.

[17] *Cf.* His *Siddhāntatattvaviveka*, xiii. 216.

[18] *Cf. Oeuvres de Fermat*, III, (1896), *Lettre ix*; John Wallis A Kenelm Digby, p. 417 ff. Also *Cf.* Dickson, L. E.,*History of the Theory of Numbers*, II, p. 351.

6 Rational solution of other equations

Śrīdhara has given simply the rational solution of $Nx^2 + 1 = y^2$. But the method used by him can be easily applied to the determination of the rational solutions of $Nx^2 - 1 = y^2$, $1 - Nx^2 = y^2$, or the general forms $Nx^2 \pm C = y^2$ and $C - Nx^2 = y^2$. In what follows, we propose to give the rational solutions of these equations in accordance with his method.

6.1 Rational solution of $Nx^2 - 1 = y^2$

In this case the rule may be stated as follows:

Rule—Express the multiplier (N) as a sum of two squares.[19] Set down the square-roots of those squares and underneath them the base and the perpendicular of any rectangle (determined form the lemma). Multiplying them across and directly, obtain the sum and the difference of the cross-products and of the direct products. Then corresponding to the square of the sum (or difference) of the cross-products as the subtractive, the hypotenuse of the rectangle (chosen above) is the first square-root and the difference (or sum) of the direct products is the other square-root; and corresponding to the square of the sum (or difference) of the direct products as the subtractive, the hypotenuse of the rectangle is the first square-root and the difference (or sum) of the cross-products is the other square-root. These first and other square-roots when divided by the square-roots of the corresponding subtractives give the first and other square-roots for the subtractive unity.

In other words, if

$$A^2 + B^2 = N$$

$$\text{and} \qquad b^2 + k^2 = h^2,$$

$$\text{then} \qquad Nh^2 - (Ak \overset{+}{\sim} Bb)^2 = (Ab \overset{\sim}{+} Bk)^2$$

$$\text{and} \qquad Nh^2 - (Ab \overset{+}{\sim} Bk)^2 = (Ak \overset{\sim}{+} Bb)^2.$$

Or,

$$N\left\{\frac{h}{Ak \overset{+}{\sim} Bb}\right\}^2 - 1 = \left\{\frac{Ab \overset{\sim}{+} Bk}{Ak \overset{+}{\sim} Bb}\right\}^2$$

$$\text{and} \qquad N\left\{\frac{h}{Ab \overset{+}{\sim} Bk}\right\}^2 - 1 = \left\{\frac{Ak \overset{\sim}{+} Bb}{Ab \overset{+}{\sim} Bk}\right\}^2.$$

[19] "When unity is the subtractive the solution of the problem is impossible unless the multiplier is the sum of two squares." (Bhāskara II).

"In the case of unity as the subtractive the multiplier must be the sum of two squares. Otherwise, the solution is impossible." (Nārāyaṇa).

That is,

$$\left.\begin{aligned} x &= \frac{h}{Ak+Bb} \\ y &= \frac{Ab \sim Bk}{Ak+Bb} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{h}{Ak \sim Bb} \\ y &= \frac{Ab+Bk}{Ak \sim Bb} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{h}{Ab+Bk} \\ y &= \frac{Ak \sim Bb}{Ab+Bk} \end{aligned}\right\}, \text{ and } \quad \left.\begin{aligned} x &= \frac{h}{Ab \sim Bk} \\ y &= \frac{Ak+Bb}{Ab \sim Bk} \end{aligned}\right\}$$

are the rational solutions of $Nx^2 - 1 = y^2$.

6.1.1 A particular solution

Choosing 1 for the base (b) and also 1 for the arbitrary number (ε) and applying Śrīdhara's lemma, we have

$$b = 1, \; k = 0, \text{ and } h = 1.$$

Substituting these values of b, k, and h in the above rational solutions of $Nx^2 - 1 = y^2$, we obtain

$$\left.\begin{aligned} x &= \frac{1}{B} \\ y &= \frac{A}{B} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{1}{A} \\ y &= \frac{B}{A} \end{aligned}\right\}$$

as the particular rational solutions of $Nx^2 - 1 = y^2$.

These particular solutions were mentioned by Bhāskara II[20] (1150).

6.2 Rational solution of $1 - Nx^2 = y^2$

In this case the rule may be stated as follows:

Rule—Express the multiplier (N) as a difference of two squares. Set down the square roots of those squares in the descending order of magnitude and below those square-roots set down the base and hypotenuse of any rectangle (determined from the lemma), in order. Next obtain the cross-products and the direct products. The perpendicular of the rectangle (chosen above) divided by the sum (or difference) of the cross-products, then, denotes the first square-root and the sum (or difference) of the direct products divided by the sum (or difference) of the cross-products denotes the other square-root.

In other words, if

$$A^2 - B^2 = N$$

$$\text{and} \qquad h^2 - b^2 = k^2,$$

$$\text{then} \qquad (Ah \pm Bb)^2 - Nk^2 = (Ab \overset{+}{\sim} Bh)^2$$

$$\text{or} \qquad 1 - N\left\{\frac{k}{Ah \pm Bb}\right\}^2 = \left\{\frac{Ab \overset{+}{\sim} Bh}{Ah \pm Bb}\right\}^2.$$

[20] *Bījagaṇita*, *cakravāla*, 5(ii)–6.

That is, both

$$\left.\begin{aligned} x &= \frac{k}{Ah+Bb} \\ y &= \frac{Ab+Bh}{Ah+Bb} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{k}{Ah-Bb} \\ y &= \frac{Ab\sim Bh}{Ah-Bb} \end{aligned}\right\}$$

are the rational solutions of $1 - Nx^2 = y^2$.

6.3 Rational solution of $Nx^2 \pm C = y^2$ and $C - Nx^2 = y^2$

These are the general forms of the equations considered above. When N and C are both non-square integers, Śrīdhara's method is not applicable to their solution. When, however, at least N or C is a perfect square, Śrīdhara's method may be used to obtain the rational solutions of the above forms. When C is a perfect square, the above forms easily reduce to the forms discussed above. It is sufficient, therefore, to consider only the two forms *viz.*

$$\begin{aligned} &\text{(i)} \quad a^2x^2 \pm C = y^2 \\ \text{and} \quad &\text{(ii)} \quad C - a^2x^2 = y^2. \end{aligned}$$

6.3.1 Rational Solution of $a^2x^2 \pm C = y^2$

Rule—Express the additive or subtractive as a difference of two squares. Set down the square-roots of these squares in the ascending or descending order of magnitude according as the *kṣepa* is additive or subtractive and below them set down the base and the hypotenuse of any rational rectangle in order. Obtain the cross-products and the direct products. The sum or difference of the cross-products divided by the product of the square-root of the multiplier and the perpendicular of the rectangle, then, gives the first square-root and the sum or difference of the direct products divided by the perpendicular of the rectangle gives the other square-root.

In other words, if

$$\begin{aligned} A^2 - B^2 &= C \\ \text{and} \quad h^2 - b^2 &= k^2, \end{aligned}$$

$$\text{then} \quad a^2\left\{\frac{Bh \overset{+}{\sim} Ab}{ak}\right\}^2 + C = \left\{\frac{Bb \overset{+}{\sim} Ah}{k}\right\}^2$$

$$\text{and} \quad a^2\left\{\frac{Ah \pm Bb}{ak}\right\}^2 - C = \left\{\frac{b \overset{+}{\sim} h}{k}\right\}^2.$$

That is

$$\left.\begin{aligned} x &= \frac{Bh+Ab}{ak} \\ y &= \frac{Bb+Ah}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Bh\sim Ab}{ak} \\ y &= \frac{Bb\sim Ah}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2+C=y^2$; and

$$\left.\begin{aligned} x &= \frac{Ah+Bb}{ak} \\ y &= \frac{Ab+Bh}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ah-Bb}{ak} \\ y &= \frac{Ab\sim Bh}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2-C=y^2$.

Another form

If

$$\begin{aligned} A^2-B^2 &= \pm C \\ \text{and} \qquad h^2-b^2 &= k^2, \end{aligned}$$

then, as shown above

$$\left.\begin{aligned} x &= \frac{Ab+Bh}{ak} \\ y &= \frac{Ah+Bb}{k} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ab\sim Bh}{ak} \\ y &= \frac{Ah-Bb}{k} \end{aligned}\right\}$$

are the rational solutions of $a^2x^2\pm C=y^2$.

Choosing

$$\begin{aligned} A &= \frac{\pm C+1}{2} \\ B &= \frac{\pm C-1}{2} \end{aligned}$$

and putting

$$\begin{aligned} k &= \frac{1}{2}\left\{\frac{b^2}{\varepsilon}-\varepsilon\right\} \\ h &= \frac{1}{2}\left\{\frac{b^2}{\varepsilon}+\varepsilon\right\} \end{aligned}$$

in one of these solutions, say in the first, we have

$$\begin{aligned} x &= \frac{1}{2a}\left\{\frac{\pm C(b+\varepsilon)}{b-\varepsilon}-\frac{b-\varepsilon}{b+\varepsilon}\right\} \\ y &= \frac{1}{2}\left\{\frac{\pm C(b+\varepsilon)}{b-\varepsilon}+\frac{b-\varepsilon}{b+\varepsilon}\right\} \end{aligned}$$

or, on setting $\frac{b-\varepsilon}{b+\varepsilon} = \lambda$,

$$\left.\begin{aligned} x &= \frac{1}{2a}\left\{\frac{\pm C}{\lambda} - \lambda\right\} \\ y &= \frac{1}{2}\left\{\frac{\pm C}{\lambda} + \lambda\right\} \end{aligned}\right\},^{21}$$

which is another form of the rational solution of $a^2x^2 \pm C = y$. This was given by Brahmagupta[22] (628), Bhāskara II[23] (1150), and Nārāyaṇa[24] (1356).

6.3.2 Rational solution of $C - a^2x^2 = y^2$

A rational solution of this equation is possible by the above method if C is capable of being expressed as a sum of two squares. In that case, we are led to the following rule for the rational solution of the above equation:

Rule—Express C as a sum of two squares, set down their square-roots and below them the base and the perpendicular of any rational rectangle. Obtain their cross-products and direct products. The sum or difference of the cross-products (or of the direct products) divided by the product of the square-root of the multiplier and the hypotenuse of the rectangle, then, gives the first square-root and the sum or difference of the direct products (or of the cross-products) divided by the hypotenuse of the rectangle gives the other square-root.

In other words, if

$$A^2 + B^2 = C$$

$$\text{and} \qquad b^2 + k^2 = h^2,$$

$$\text{then} \qquad C - a^2\left\{\frac{Ak \pm Bb}{ah}\right\}^2 = \left\{\frac{b \mp Bk}{h}\right\}$$

$$\text{and} \qquad C - a^2\left\{\frac{Ab \pm Bk}{ah}\right\}^2 = \left\{\frac{Ak \mp Bb}{h}\right\}^2.$$

---

[21] This solution can be easily derived from Śrīdhara's lemma. For, $\pm C$, $(ax)^2$, and $(y)^2$ may be algebraically treated as the squares of the base, perpendicular, and hypotenuse of a rectangle; so that, choosing an arbitrary number λ and making use of Śrīdhara's lemma we have

$$x = \frac{1}{2a}\left\{\frac{\pm C}{\lambda} - \lambda\right\}, \qquad y = \frac{1}{2}\left\{\frac{\pm C}{\lambda} - \lambda\right\}.$$

[22] *Cf.* *Brāhmasphuṭasiddhānta*, xviii. 69.

[23] *Cf.* His *Bījagaṇita*, *cakravāla*, rule 8.

[24] *Cf.* His *Bījagaṇita*, I, rule 85.

That is,

$$\left.\begin{aligned} x &= \frac{Ak+Bb}{ah} \\ y &= \frac{Ab+Bk}{h} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{Ak\sim Bb}{ah} \\ y &= \frac{Ah\sim Bk}{h} \end{aligned}\right\}, \quad \left.\begin{aligned} x &= \frac{Ab+Bk}{ah} \\ y &= \frac{Ak+Bb}{h} \end{aligned}\right\}, \quad \text{and} \quad \left.\begin{aligned} x &= \frac{Ab\sim Bk}{h} \\ y &= \frac{Ak\sim Bb}{h} \end{aligned}\right\}$$

are the rational solutions of $C - a^2x^2 = y^2$.

7 Conclusion

The rectangular method discussed above seems to be Śrīdhara's own contribution to the subject. This method occurs exclusively in Śrīdhara's *Pāṭī-gaṇita.* It is not found in any other Hindu mathematical work so far known nor has it been included among the various solutions of the equations of the form $Nx^2+1=y^2$ surveyed by L. E. Dickson (1920) in his *History of the Theory of Numbers, Vol. II.* It is fundamentally different from the tentative method stated by Datta and Singh (1938) in their *History of Hindu Mathematics, Part II*, and from the method given by Śrīpati (1039) and followed by Bhāskara II (1150), Nārāyaṇa (1356), Jñānarāja (1503), and Kamalākara (1658).

Śrīdhara's method is extremely general. It is applicable to the equations of the forms $Nx^2 \overset{+}{\sim} C = y^2$ for obtaining their rational solutions. These solutions, it may be added, are most general. In the above discussion we have shown that the rational solutions given by Brahmagupta (628), Śrīpati (1039), Bhāskara II (1150), Nārāyaṇa (1356), Jñānārāja (1503), and Kamalākara (1658), as also those given by John Wallis and W. Brouncker (1657) are deducible from the rational solutions obtained from Śrīdhara's method.

Ācārya Jayadeva, the mathematician *

1 Introduction

The object of the present paper is to invite attention of historians of science to an important Hindu algebraist, Ācārya Jayadeva, who lived and wrote in the early 11th century of the Christian era (or earlier). His name and quotations from his work on algebra are found to occur in the *Sundarī*, which is the name of Śrīmad Udayadivākara's commentary on the *Laghubhāskarīya* of Bhāskara I (629 AD). The *Sundarī* has not yet seen the light of day but manuscript copies of that work are preserved in H. H. the Maharajah's Palace Library, Trivandrum, and in the Curator's Office Library, Trivandrum. A transcript copy of that work has been very recently procured for our use from the former by the Tagore Library of the Lucknow University. The extracts from Ācārya Jayadeva's work, which have been quoted and explained with illustrations by the commentator, relate to the solution of the indeterminate equation of the second degree of the type $Nx^2+1 = y^2$. These extracts, it may be pointed out, are of immense historical interest as they include rules giving the well known cyclic method of finding the integral solution of the above-mentioned equation. The credit of the first inception of that ingenious method was hitherto given to the twelfth century mathematician Bhāskara II (1150 AD) who himself not only did not claim originality for that method but also ascribed it to earlier writers. The discovery of that method in an anterior work definitely proves that the cyclic method was invented in India much earlier. Jayadeva may or may not have been its inventor but quotations from his work in the *Sundarī* are the earliest sources of our information regarding that method. Another noteworthy feature of the references from Ācārya Jayadeva's composition is the solution of the equation $Nx^2 + C = y^2$, C being positive or negative. This method, though not superior to that suggested by Brahmagupta (628 AD), Bhāskara II, and Nārāyaṇa (1356 AD), is nevertheless different from the known methods. Incidentally we have also given Udayadivākara's method for the solution of the multiple equations, $x + y =$ a square, $x - y =$ a square, $xy + 1 =$ a square. This method, though inferior to those given by Brahmagupta and Nārāyaṇa, deserves attention because of the ingenuity displayed by the author. It also shows that Udayadivākara knew full well how

* K. S. Shukla, *Ganita*, Vol. 5, No. 1 (1954), pp. 1–20.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_11

to tackle and solve the general indeterminate equation of the second degree of the type $ax^2 + bx + c = y^2$. Equations of that type were hitherto found treated in the *Bījagaṇita* of Bhāskara II, though some of his examples relating to such equations prove his indebtedness to ancient authors.[1]

2 The *Sundarī*

The transcript of the *Sundarī*, which is available to us, is written in *Devanāgarī* characters on paper in foolscap size. It is scribed in good hand but there are the usual imperfections and omissions. The manuscript is practically complete and extends to 252 leaves written on one side only. There are 21 lines to a page and about 24 letters to a line. The beginning and end of the *Sundarī* are as follows:

Beginning

॥श्रीः॥
लघुभास्करीयम्।

उदयदिवाकरप्रणीतया सुन्दर्याख्यया व्याख्यया समेतम्।
नत्वा समस्तजगतामधिपं मुरारि-
माचार्यमार्यभटमप्यभिवन्द्य भक्त्या ।
यद्भास्करेण गुरुणा ग्रहतन्त्रमुक्तं
लघ्वस्य विस्तृततरां विवृतिं विधास्ये ॥

तत्र तावदाचार्यः प्रथममेव भास्करीयं नाम ग्रहकर्मनिबन्धनं प्रतिपाद्य तदेव पुनः संक्षिप्तं चिकीर्षुस्तद्विघ्नोपशान्तये भगवते भास्कराय प्रणाममाद्यश्लोकेनाचष्टे—

भास्कराय ...

Colophon

इति ज्योतिषिकभट्टश्रीमदुदयदिवाकरविरचितायां लघुभास्करीयविवृतौ सुन्दर्याभिधानायां मध्यगतिः प्रथमोऽध्यायः।

End

एवं पुनः पुनर्भावनयानीतज्येष्ठमूलेनैवान्यौ राशी स्यातामिति।

Colophon

इति लघुभास्करीयविवृतौ सुन्दर्याभिधानायां नक्षत्रध्रुवग्रहयोगाध्यायोऽष्टमः।

[1] See Datta, B., and Singh, A. N., *History of Hindu Mathematics*, Part II, p. 181.

From the colophons at the ends of the chapters it is clear that the *Sundarī* is a commentary on the *Laghubhāskarīya* and that the name of the commentator is Bhaṭṭa Udayadivākara. The former conclusion is confirmed by the contents of the work.

In the commentary there is no reference to the time of birth of the commentator or of writing the commentary. But at one place in the commentary[2] the commentator cites an example where he states the *ahargaṇa* (i.e., the number of days elapsed since the beginning of Kaliyuga) for Friday, the 10th lunar date, *Vaiśākha*, bright fortnight, Śaka year 995. This epoch corresponds to Friday, April 19, AD 1073. It is usual to give the *ahargaṇa* for the current day. So we infer that the *Sundarī* was written in the year 1073 of the Christian era.

As regards the authenticity of the *Sundarī* there is little doubt. Reference to that work has been made by Nīlakaṇṭha (1500 AD) who in his commentary on the *Āryabhaṭīya*[3] of Āryabhaṭa I (499 AD) mentions the name *Laghubhāskarīya-vyākhyā Sundarī* and quotes two stanzas from that work. Both of those stanzas are found to occur in the transcript of the *Sundarī* available to us. Moreover, five manuscripts of that work which seems to be derived from different sources are preserved in H. H. the Maharajah's Palace Library, Trivandrum[4] and two in the Curator's Office Library, Trivandrum.[5]

3 Reference to Ācārya Jayadeva

Reference to Ācārya Jayadeva is made in the *Sundarī* in connection with the solution of the indeterminate equation of the second degree, viz.

$$Nx^2 + 1 = y^2,$$

which in Hindu mathematics is called by the name *varga-prakṛti* (square-nature).

In verse 18 of the eighth chapter of the *Laghubhāskarīya* there is an astronomical problem whose solution depends upon the solution of the simultaneous equations

$$8x + 1 = y^2 \qquad (1)$$

$$7y^2 + 1 = z^2. \qquad (2)$$

As regards the solution of these equations, Udayadivākara tells us that the value of y should be determined from equation (2) by solving it by the method

[2] Comm. on ii. 29.

[3] ii. 17 (ii).

[4] See Descriptive Catalogue, Vol. IV, MSS. Nos. 942, 943, 944, 945, and 977.

[5] See Descriptive Catalogue, Vol. V, MSS. Nos. 761 and 762.

applicable to the *varga-prakṛti*; and then the value of the unknown quantity x should be determined from equation (1) by the method of inversion. In order to give a detailed working of the process, Udayadivākara mentions Ācārya Jayadeva and his rules. He writes:

> In order to demonstrate this (working), we here set forth with exposition and illustration the rules for the *varga-prakṛti*, which were composed by Ācārya Śrī Jayadeva.[6]

4 Quotations from Ācārya Jayadeva's work

Quotations from Ācārya Jayadeva's work comprise 20 stanzas. Below we translate and explain those stanzas.

4.1 Stanza 1. Origin of the name *varga-prakṛti*

> इष्टकृतिरिष्टगुणिताऽभीष्टेन युता विशोधितेष्टा वा ।
> वर्गस्य यतः प्रकृतिर्वर्गप्रकृतिस्ततोऽभिहिता ॥१॥
>
> As (in an equation of the type $Nx^2 \pm C = y^2$) the square of an optional number multiplied by a given number and then the product increased or decreased by another given number is of the nature of a square, so (such an equation) is called *varga-prakṛti* (square-nature).

This proves the significance of the name *varga-prakṛti.*

4.2 Stanza 2. Technical terms explained

> यस्याभीष्टेन कृतिर्विहन्यते तत्कनिष्ठमूलं स्यात् ।
> क्षेपयुताद्रहिताद्वा मूलज्येष्ठं भवति तन्मूलम् ॥२॥
>
> The number whose square is multiplied by the given number is called the lesser root; that product having been increased or decreased by the interpolator (*kṣepa*), the square root thereof is called the greater root.

That is to say, in the equation $Nx^2 \pm C = y^2$, x is the lesser root and y the greater root. N is called *prakṛti* and C the interpolator.

We will see presently that Ācārya Jayadeva calls the lesser and greater roots by the names first root and last root also.

[6] तत्प्रदर्शनायाचार्यजयदेवविरचितवर्गप्रकृतिकरणसूत्राणि सविवरणान्यालिख्यन्ते।

4.3 Stanza 3. Writing down an auxiliary equation

ईप्सितराशेर्वर्गे चोदितगुणकारताडिते चिन्त्यम् ।
युक्तेन कृतिः (कियता) कियद्वियुक्तेन वेति धिया ॥३॥

> The square of an optionally chosen number having been multiplied by the given multiplier, think out how much be added to or subtracted from that product that it may become a perfect square.

That is, first choose an arbitrary number α for x. Then find out a number k, positive or negative, such that $N\alpha^2 + k$ may become a perfect square, say β^2. Then

$$N\alpha^2 + k = \beta^2$$

is an auxiliary equation. We will see how this equation is helpful in finding a solution of

$$Nx^2 + 1 = y^2.$$

4.4 Stanza 4. *Bhāvanā*

अशेषकरणव्यापि भावनाकरणं द्विधा ।
तत्समासविशेषाभ्यां तुल्यातुल्यतयापि च ॥४॥

> The process of *bhāvanā*, which pervades all mathematical operations (dealing with the *varga-prakṛti*), is twofold—*samāsa-bhāvanā* and *viśeṣa-bhāvanā*, or *tulya-bhāvanā* and *atulya-bhāvanā*.

The word *bhāvanā* is a technical term. According to Udayadivākara, *bhāvanā* is multiplication.[7] According to B. Datta and A. N. Singh, it means lemma or composition. At any rate the process called *bhāvanākaraṇa* is a special mathematical operation in which multiplication is inherent. The process is described in the next two stanzas.

4.4.1 Stanza 5. *Samāsa-bhāvanā*

वज्राभ्याससमासात् प्रथमं प्रथमाहतिः प्रकृतिघातात् ।
अन्त्यपदाभ्यासयुतादितरन्मूलं हतिः क्षिप्तयोः ॥५॥

> Summing up the cross products (of the first and last roots) is obtained a (new) first root; multiplying the product of the first roots by the *prakṛti* and then increasing that by the product of the last roots is obtained a (new) last root; and the product of the interpolators (is the corresponding new interpolator).

[7] Compare the term *bhāvita*, which is the name given to an equation of the type $xy = c$ (involving the product of two unknown quantities).

That is to say, if

$$N\alpha^2 + k = \beta^2, \tag{3}$$
$$N\alpha_1^2 + k_1 = \beta_1^2, \tag{4}$$

then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Proof

The auxiliary equations (3) and (4) may be written as

$$N\alpha^2 - \beta^2 = -k,$$
$$N\alpha_1^2 - \beta_1^2 = -k_1.$$

Multiplying these equations side by side, we get

$$N^2\alpha^2\alpha_1^2 + \beta^2\beta_1^2 - N(\alpha^2\beta_1^2 + \alpha_1^2\beta^2) = kk_1,$$

which is the same as

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Actual working explained

Using 4.3 (**ed.** i.e. Section 4.3) we write down two auxiliary equations, say

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2.$$

Now we set down the *prakṛti* and then the lesser roots, the greater roots, and the interpolators corresponding to the two auxiliary equations one under the other as follows:

Prakṛti	Lesser root	Greater root	Interpolator
N	α	β	k
	α_1	β_1	k_1

Now we find out the cross products of the lesser and greater roots and put down their sum underneath the lesser root. Thereafter we obtain the products of the *prakṛti* and the lesser roots and of the greater roots and put down their sum underneath the greater root. And then we write down the product of the interpolators underneath the interpolator. Thus, we get

Prakṛti	Lesser root	Greater root	Interpolator
N	α	β	k
	α_1	β_1	k_1
	$\alpha\beta_1 + \alpha_1\beta$	$N\alpha\alpha_1 + \beta\beta_1$	kk_1

In this way we obtain another auxiliary equation, viz.

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

Repeating the above process over and over again, any number of auxiliary equations can be found out.

Note

The above process is called *samāsa-bhāvanā*. Also since the operation has been made on two different auxiliary equations, this may be called *atulya-bhāvanā* (or *atulya-samāsa-bhāvanā*). If everywhere in the above process, α_1 be replaced by α, β_1 by β, and k_1 by k, the above process will be called *tulya-bhāvanā*.

The result of *tulya-samāsa-bhāvanā* may be stated as follows:

$$\text{If} \qquad N\alpha^2 + k = \beta^2,$$
$$\text{then} \qquad N(2\alpha\beta)^2 + k^2 = (N\alpha^2 + \beta^2)^2.$$

Thus we see that the *tulya-bhāvanā* is a particular case of the *atulya-bhāvanā*.

4.4.2 Stanza 6. *Viśeṣa-bhāvanā*

वज्राभ्यासविशेषादादिममाद्याहतिः प्रकृतिघातात् ।
अन्त्यपदाभ्यासेन च विशेषितान्मूलमन्त्यं स्यात् ॥६॥

Taking the difference of the cross products (of the first and the last roots), we get a (fresh) first root; multiplying the product of the first roots by the *prakṛti* and then taking the difference of that and the product of the last roots, we get a (fresh) last root. (The corresponding interpolator is the product of the interpolators).

That is to say, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2,$$

then

$$N(\alpha\beta_1 - \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 - \beta\beta_1)^2.$$

The proof and working are as in the previous case.

The rules stated in stanzas 5 and 6 are known as Brahmagupta's lemmas. They occur for the first time in the *Brāhmasphuṭasiddhānta* of Brahmagupta. In Europe they were rediscovered by Euler in 1764 and by Lagrange in 1768.

4.5 Stanza 7. Rational solution of $Nx^2 + 1 = y^2$

> प्रक्षेपकसंवर्गो वर्गश्चेदस्य वर्गमूलेन ।
> मूले भाज्ये तद्भावने च रूपं भवेत् क्षेपः ॥७॥
>
> When (in the above process) the product of the interpolators becomes a perfect square, by the square root thereof divide the (lesser and greater) roots: then they correspond to the interpolator unity and so they continue to be even when the process of the (*tulya*)*bhāvanā* is applied thereafter.

From what has been said above, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2,$$

then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + kk_1 = (N\alpha\alpha_1 + \beta\beta_1)^2.$$

If $kk_1 = K^2$, then

$$N(\alpha\beta_1 + \alpha_1\beta)^2 + K^2 = (N\alpha\alpha_1 + \beta\beta_1)^2,$$

$$\text{i.e.,} \qquad N\left(\frac{\alpha\beta_1 + \alpha_1\beta}{K}\right)^2 + 1 = \left(\frac{N\alpha\alpha_1 + \beta\beta_1}{K}\right)^2.$$

In other words, if

$$N\alpha^2 + k = \beta^2,$$
$$N\alpha_1^2 + k_1 = \beta_1^2,$$

and $kk_1 = K^2$, then

$$x = \frac{(\alpha\beta_1 + \alpha_1\beta)}{K},$$
$$y = \frac{(N\alpha\alpha_1 + \beta\beta_1)}{K}$$

is a solution of $Nx^2 + 1 = y^2$. In particular, if

$$N\alpha^2 + k = \beta^2,$$

then

$$x = \frac{2\alpha\beta}{k},$$
$$y = \frac{(N\alpha^2 + \beta^2)}{k},$$

is a solution of $Nx^2 + 1 = y^2$.

Illustration

Solve $7x^2 + 1 = y^2$.

Let the auxiliary equation be

$$7(1)^2 + 2 = 3^2.$$

Then applying the process of *tulya-bhāvanā*, we have

Prakṛti	Lesser root	Greater root	Interpolator
7	1	3	2
	1	3	2
	6	16	4

Thus

$$7(6)^2 + 4 = 16^2,$$
$$\text{or} \quad 7(3)^2 + 1 = 8^2.$$

Hence $x = 3$, $y = 8$ is a solution of the given equation. To get another solution, we treat the equation

$$7(3)^2 + 1 = 8^2$$

as the auxiliary equation. Then applying the process of *tulya-bhāvanā*, we have

7	3	8	1
	3	8	1
	48	127	1

Hence $x = 48$, $y = 127$ is another solution of the same equation. To obtain still another solution, we treat the equations

$$7(3)^2 + 1 = 8^2,$$
$$7(48)^2 + 1 = 127^2,$$

as auxiliary equations. Then applying *samāsa-bhāvanā*, we have

7	3	8	1
	48	127	1
	765	2024	1

Hence $x = 765$, $y = 2024$ is another solution of the same equation. Proceeding like this, we can get any number of solutions.

4.6 Stanzas 8–15. Integral solution of $Nx^2 + 1 = y^2$. The *Cakravāla* or the Cyclic Method

ह्रस्वज्येष्ठक्षेपान् प्रतिराश्य क्षेपभक्तयोः क्षेपात् ।
कुट्टाकारे च कृते कियद्गुणं क्षेपकं क्षिप्त्वा ॥८॥
तावत्कृतेः प्रकृत्या हीने प्रक्षेपकेण संभक्ते।
स्वल्पतरावाप्तिः स्यादित्याकलितोऽपरः क्षेपः ॥९॥
प्रक्षिप्तप्रक्षेपककुट्टाकारे कनिष्ठमूलहते ।
सज्येष्ठपदे प्रक्षेप(के)ण लब्धं कनिष्ठपदम् ॥१०॥
क्षिप्तक्षेपककुट्टागुणितात्तस्मात्कनिष्ठमूलहतम् ।
पाश्चात्यं प्रक्षेपं विशोध्य शेषं महन्मूलम् ॥११॥
कुर्यात् कुट्टाकारं पुनरनयोः क्षेपभक्तयोः पदयोः ।
तत्सेष्टहतक्षेपे सदृशगुणेऽस्मिन् प्रकृतिहीने ॥१२॥
प्रक्षेपः क्षेपाप्ते प्रक्षिप्तक्षेपकाच्च गुणकारात् ।
अल्पघ्नात् सज्येष्ठात् क्षेपावाप्तं कनिष्ठपदम् ॥१३॥
एतत्क्षिप्तक्षेपककुट्टकघातादनन्तरक्षेपम् ।
हित्वाल्पहतं शेषं ज्येष्ठं तेभ्यश्च गुणकादि ॥१४॥
कुर्यात्तावद्यावत् षण्णामेकद्विचतुर्णां पतति ।
इति चक्रवालकरणेऽवसरप्राप्तानि योज्यानि ॥१५॥

Set down the lesser root, the greater root, and the interpolator at two places. (At one place divide the lesser and greater roots by the interpolator. Treating the remainder of the lesser root as the dividend, the remainder of the greater root as the addend, and the interpolator as the divisor of an indeterminate equation of the first degree (*kuṭṭākāra*), solve that equation.) The *kuṭṭākāra*[8] having been (thus) determined from those (lesser and greater roots) divided out by the interpolator and the interpolator, ascertain how many times the interpolator be added to it so that the square of that sum being diminished by the *prakṛti* and then divided by the interpolator may yield the least value. The least value thus obtained is the new interpolator. The *kuṭṭākāra* as increased by (the chosen multiple of) the interpolator when multiplied by the lesser root, then increased by the greater root, and then divided by the interpolator, the quotient is the new lesser root. That (new lesser root) should be multiplied by the *kuṭṭākāra* as increased by (the chosen multiple of) the interpolator and from the product should be subtracted the new interpolator as multiplied by the lesser root; the remainder (thus obtained) is the new greater root.

From these (new lesser and greater) roots divided out by the

[8] In the indeterminate equation of the first degree $\frac{(ax+c)}{b} = y$, a is called the dividend, b the divisor, c the addend, and x the *kuṭṭākāra*.

(new) interpolator again find out the *kuṭṭākāra* (as before). Increase it by the proper multiple of the interpolator: the square of that (sum) being diminished by the *prakṛti* and then divided by the interpolator, the quotient is the (fresh) interpolator. The *kuṭṭākāra* (*guṇakāra*) increased by the chosen multiple of the interpolator being multiplied by the lesser root and increased by the greater root and then divided by the interpolator, the quotient is the fresh lesser root. This (fresh lesser root) being multiplied by the *kuṭṭākāra*, to which the chosen multiple of the interpolator has been added, and the product being diminished by the product of the fresh interpolator and the lesser root, the remainder is the fresh greater root.

From them again calculate the *kuṭṭākāra* etc. and continue the process till the interpolator comes out to be one of the six numbers ± 1, ± 2, and ± 4.

(One of these numbers having been obtained as the interpolator) in the (above) cyclic process (*cakravāla*), necessary operations should be made (to get the integral solution for unit interpolator).

Lemma of the Cyclic Method

The above method is based on the following lemma:

$$\text{If} \qquad N\alpha^2 + k = b^2,$$

where a, b, k are integers, k being positive or negative, then

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2.$$

Proof

Treating

$$\begin{aligned} N\alpha^2 + k &= b^2, \\ \text{and} \qquad N(1)^2 + (t^2 - N) &= t^2, \end{aligned}$$

as auxiliary equations, and applying the process of *samāsa-bhāvanā*, we have

$$\begin{array}{cccc} N & a & b & k \\ & 1 & t & t^2 - N \\ \hline & at + b & Na + bt & k(t^2 - N) \end{array}$$

Therefore,

$$N(at+b)^2 + k(t^2 - N) = (Na + bt)^2$$

$$\text{or} \qquad N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left(\frac{Na+bt}{k}\right)^2,$$

which is the same as

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2.$$

The Cyclic Process explained

Suppose that an auxiliary equation is

$$Na^2 + k = b^2,$$

where a, b, and k are integers, k being positive or negative. Then, from the above lemma,

$$N\left(\frac{at+b}{k}\right)^2 + \frac{t^2-N}{k} = \left[t\left(\frac{at+b}{k}\right) - a\left(\frac{t^2-N}{k}\right)\right]^2. \tag{5}$$

Now we choose t such that $\frac{at+b}{k}$ is a whole number, and $\left|\frac{(t^2-N)}{k}\right|$ is as small as possible. Let that value be T. Then let

$$a_1 = \frac{aT+b}{k},$$

$$b_1 = T\left(\frac{aT+b}{k}\right) - a\left(\frac{T^2-N}{k}\right),$$

$$k_1 = \frac{T^2-N}{k}.$$

The numbers a_1, b_1, k_1 are all integral.[9] The equation (5) then becomes

$$Na_1^2 + k_1 = b_1^2. \tag{6}$$

[9]From the form of b_1 it is clear that it will be an integer provided a_1 and k_1 are integers. But a_1 is an integer by assumption. So we have only to show that k_1 is an integer. Now if we eliminate b between

$$a_1 = \frac{aT+b}{k}, \qquad \text{and} \qquad b_1 = \frac{bT+Na}{k},$$

we get

$$\frac{k}{a}(a_1T - b_1) = T^2 - N.$$

Since the right side is integral, therefore the left side is also so. But k and a are prime to each other. Therefore, $a_1T - b_1$ must be perfectly divisible by a. Hence

$$\frac{a_1T - b_1}{a} = \frac{T^2-N}{k} = k_1 = \text{an integer}.$$

Now treating this as the auxiliary equation, and proceeding as above, we derive from (6) a new equation of the same kind

$$Na_2^2 + k_2 = b_2^2,$$

where again a_2, b_2, k_2 are whole numbers. Successive repetition of this process would, according to Ācārya Jayadeva, lead us to an equation in which the interpolator k is ± 1, ± 2, ± 4, and in which a, b are integers. And by the process of *samāsa-bhāvanā* such an equation easily leads us to an equation of the type

$$N\alpha^2 + 1 = \beta^2,$$

where α, β are integers.[10] Thus we get $x = \alpha$, $y = \beta$ as an integral solution of $Nx^2 + 1 = y^2$.

Illustration

Find an integral solution of $7x^2 + 1 = y^2$.

Let the auxiliary equation be

$$7(1)^2 - 3 = 2^2.$$

The process of *tulya-bhāvanā* does not lead to an integral solution. So we apply the Cyclic Process.

From the auxiliary equation

$$\text{lesser root} = 1,$$
$$\text{greater root} = 2,$$
$$\text{interpolator} = -3.$$

Therefore solving the equation

$$\frac{(t+2)}{-3} = \text{a whole number},$$

we get $t = -3\lambda + 1$. Putting $\lambda = 0$, we get $t = 1$ which gives to $\left|\frac{(t^2-7)}{-3}\right|$ the smallest value 2. Therefore,

$$\text{new interpolator} = 2,$$
$$\text{new lesser root} = -1,$$
$$\text{new greater root} = -3.$$

Since the new interpolator is 2, therefore the cyclic process stops here. Applying the *tulya-bhāvanā*, we have

[10] This fact was known to Brahmagupta. For details see Datta, B., Singh, A. N., *History of Hindu Mathematics*, Part II, pp. 157 ff.

7	-1	-3	2
	-1	-3	2
	6	16	4
or	3	8	1

Hence one integral solution of the given equation is

$$x = 3, \quad y = 8.$$

4.7 Stanzas 16–20. Solution of $Nx^2 + C = y^2$, C being positive or negative

प्रकृतौ तावद्द्याद् यावति वर्गो भवेत् क्षेपात् ।
तावद् वर्गः शोध्यो यथाकृतयथोक्तयोरनयोः ॥१६॥
स्वक्षेपशोधनाभ्यां वज्राहतियोगतोऽपि वर्गः स्यात् ।
एवं मूलं कुर्यात् सक्षेपप्रकृतिमूलेन ॥१७॥
शोधनमूलगुणेनोनाधिकता ततश्च शेषाभ्याम् ।
प्रकृतिक्षेपेणाप्ते मूले लघुनी भवेतां द्वे ॥१८॥
संयुतिगुणमूलहते प्रतिराशि तयोस्तयोः क्षिपेदूने ।
क्षेपकशोधनमूलं विशोधयेदधिकतः क्रमशः ॥१९॥
मूले महती स्यातामतो महीयांसमायोज्यम् ।
प्रकटितमतिगहनमिदं मरुतिमुखे मक्षिकाकरणम् ॥२०॥

Add such a number to the *prakṛti* as makes the sum a perfect square. Then from the interpolator subtract a square number which is chosen in such a way that when the *prakṛti* and the interpolator, as obtained after the said addition and subtraction or as they are stated, are cross multiplied by the additive and the subtractive quantities, the sum of the cross products is again a square number. Then extract the square root of that (square number). Then by the product of the square root of the increased *prakṛti* and the square root of the subtractive (square number) (severally) decrease and increase that square root. The two numbers (thus obtained) being divided by the number added to the *prakṛti* become the two lesser roots. Set them down at two places and multiply both of them by the square-root of the increased *prakṛti*. Then respectively add the square-root of the number subtracted from the interpolator to the lesser one and subtract the same from the greater one. Then they become the two greater roots. A large number of lesser and greater roots may then be determined.

Thus we have revealed a determination which is as difficult as setting a fly against the wind.

Exposition

In order to solve the equation $Nx^2 + C = y^2$, choose a number a such that $N + a$ may become a perfect square. Then choose a number b such that the sum of the cross products of

$$\begin{matrix} N+a & c-b^2 \\ a & b^2 \end{matrix} \quad \text{or} \quad \begin{matrix} N & c \\ a & b^2 \end{matrix}$$

i.e., $(N+a)b^2 + (c-b^2)a$ or $Nb^2 + Ca$ also may become a perfect square. Let

$$N + a = P^2,$$
$$\text{and} \quad Nb^2 + Ca = Q^2.$$

Then according to the rule the two lesser roots are

$$\frac{(Q-Pb)}{a} \quad \text{and} \quad \frac{(Q+Pb)}{a};$$

and the corresponding greater roots are

$$\frac{P(Q-Pb)}{a} + b \quad \text{and} \quad \frac{P(Q+Pb)}{a} - b.$$

That is to say, the two solutions of $Nx^2 + C = y^2$ are

$$\left.\begin{aligned} x &= \frac{(Q-Pb)}{a} \\ y &= \frac{P(Q-Pb)}{a} + b \end{aligned}\right\}, \qquad \left.\begin{aligned} x &= \frac{(Q+Pb)}{a} \\ y &= \frac{P(Q+Pb)}{a} - b \end{aligned}\right\}.$$

Rationale

Let $N + a$ be equal to P^2. Then

$$\begin{aligned} Nx^2 + C &\equiv (N+a)x^2 + (C - b^2 - ax^2) + b^2 \\ &\equiv (Px)^2 + (C - b^2 - ax^2) + b^2. \end{aligned} \qquad (7)$$

Therefore, let

$$Nx^2 + C = (Px \pm b)^2. \qquad (8)$$

Then from (7) and (8), we have

$$(Px \pm b)^2 = (Px)^2 + (C - b^2 - ax^2) + b^2$$
$$\text{or} \quad ax^2 \pm 2Pbx = (C - b^2)$$
$$\text{or} \quad (ax)^2 \pm 2Pbax = a(C - b^2).$$

Adding $(Pb)^2$ to both the sides, we have

$$\begin{aligned}(ax \pm Pb)^2 &= a(C - b^2) + (Pb)^2\\ &= a(C - b^2) + (N + a)b^2\\ &= Nb^2 + Ca\\ &= Q^2, \text{ say.}\end{aligned}$$

$$\therefore \qquad ax \pm Pb = Q$$

$$\text{or} \qquad x = \frac{(Q - Pb)}{a} \quad \text{or} \quad \frac{(Q + Pb)}{a}.$$

Consequently,

$$\begin{aligned}y &= Px + b \ \text{ or } \ Px - b \ \text{ respectively}\\ &= \frac{P(Q - Pb)}{a} + b \ \text{ or } \ \frac{P(Q + Pb)}{a} - b.\end{aligned}$$

Hence the rule.

Alternative rationale

Let

$$N(1)^2 + a = P^2,$$
$$N(b)^2 + Ca = Q^2.$$

Now treating these as auxiliary equations and applying the process of *samāsa-bhāvanā*, we have

$$\begin{array}{cccc} N & 1 & P & a\\ & b & Q & Ca\\ \hline & Q + Pb & Nb + PQ & Ca^2\\ \text{or} & \left(\dfrac{Q + Pb}{a}\right) & \left(\dfrac{Nb + PQ}{a}\right) & C \end{array}$$

Therefore, one solution of $Nx^2 + C = y^2$ is

$$x = \frac{Q + Pb}{a}$$
$$y = \frac{(Nb + PQ)}{a} \quad \text{i.e.,} \quad \frac{P(Q + Pb)}{a} - b,$$

Similarly, applying the process of *viśeṣa-bhāvanā*, we get

$$x = \frac{(Q - Pb)}{a},$$
$$y = \frac{P(Q - Pb)}{a} + b,$$

as another solution of the same equation.

Illustration

> If you know (the method for solving) the *vargaprakṛti*, say what is that number whose square being multiplied by 60 and then increased by 8 times 20 is again a perfect square.

Here we have to solve the equation

$$60x^2 + 160 = y^2.$$

Obviously, a is 4, so that $P = 8$. Now b is to be chosen in such a way that $Nb^2 + Ca$ i.e., $60b^2 + 640$ may be a perfect square. By trial, we get $b = 4$, so that $Q = 40$.

Hence two solutions of the above equation are

$$x = 2,\ y = 20; \qquad x = 18,\ y = 140.$$

To get another solution, we may proceed as follows. Taking

$$60(18)^2 + 160 = 140^2$$

as an auxiliary equation and applying the process of *samāsa-bhāvanā*, we have

60	18	140	160
	18	140	160
	5040	39040	25600
or	$\frac{63}{2}$	244	1

Now taking

$$60(18)^2 + 160 = 140^2, \qquad \text{and} \qquad 60\left(\frac{63}{2}\right)^2 + 1 = 244^2$$

as auxiliary equations, and applying the process of *samāsa-bhāvanā* we have

60	18	140	160
	63/2	244	1
	8802	68180	160

Therefore, $x = 8802$, $y = 68180$ is another solution of the same equation.

Similarly, any number of solutions may be written down.

N. B.—The solution $x = 18$, $y = 140$ may also be derived from $x = 2$, $y = 20$ in the same way as $x = 8802$, $y = 68180$ has been derived from $x = 18$, $y = 140$.

5 Udayadivākara's procedure for solving the multiple equations

$$\begin{aligned} x + y &= \text{a perfect square,} \\ x - y &= \text{a perfect square,} \\ xy + 1 &= \text{a perfect square.} \end{aligned}$$

[Udayadivākara works out these equations because their solution is required in a problem set in the *Laghubhāskarīya* (viii. 17). His method does not give the general solution of the problem but it certainly throws light on the technique employed by early Hindu astronomers in solving algebraical equations. Udayadivākara's method under consideration deserves mention here because it is based on a previous rule of Ācārya Jayadeva.]

To begin with, Udayadivākara assumes that

$$xy + 1 = (2y + 1)^2,$$

so that he gets

$$x = 4y + 4.$$

Thus

$$x - y = 3y + 4.$$

Udayadivākara, now assumes that

$$3y + 4 = (3z + 2)^2.$$

Thus he gets

$$\begin{aligned} y &= 3z^2 + 4z, \\ \text{making} \qquad x &= 12z^2 + 16z + 4. \end{aligned}$$

Therefore,

$$x + y = 15z^2 + 20z + 4.$$

To make $x + y$ a perfect square, Udayadivākara puts

$$15z^2 + 20z + 4 = u^2$$

which, after multiplication and transposition, he writes as

$$\begin{aligned} 900z^2 + 1200z + 400 &= 60u^2 + 160 \\ \text{or} \qquad (30z + 20)^2 &= 60u^2 + 160. \end{aligned}$$

This equation can be written as a pair of equations

$$60u^2 + 160 = \lambda^2, \tag{9}$$

$$30z + 20 = \lambda. \tag{10}$$

Udayadivākara solves equation (9) in the same way as we have solved it under Section 4.7 above. He gets the solutions

$$u = 2, \quad \lambda = 20$$
$$u = 18, \quad \lambda = 140$$
$$u = 8802, \quad \lambda = 68180.$$

Making use of the values $\lambda = 140$ and $\lambda = 68180$, he gets $z = 4$, and $z = 2272$. Likewise he obtains

$$x = 64, \quad y = 260$$
$$x = 15495040, \quad y = 61980164$$

as two solutions of the proposed multiple equations.

6 Conclusion

The most interesting feature of the stanzas discussed above is the cyclic method of finding the integral solution of the equation $Nx^2 + 1 = y^2$. That method has been called *cakravāla* and is the same as given by Bhāskara II[11] and Nārāyaṇa[12] (1356). As regards the cyclic method, H. Hankel has remarked:

> It is above all praise; it is certainly the finest thing which was achieved in the theory of numbers before Lagrange.[13]

As already mentioned the cyclic method was hitherto found to occur for the first time in the *Bījagaṇita* of Bhāskara II, so the credit of that method was attributed to him. But Bhāskara II ascribed the name *cakravāla* to previous writers[14] which shows that the cyclic method was not actually devised by him. The discovery of that method in a work written about a century earlier confirms his admissions and takes away the credit of that method from him. But who is to be given the credit of that method? In this connection we must quote the following stanza which occurs at the end of Bhāskara II's *Bījagaṇita*.

[11] *Cf. Bījagaṇita, cakravāla*, 1–4.
[12] *Cf. Gaṇitakaumudī, vargaprakṛti*, 8–11; *Bījagaṇita*, I, Rule 79–82.
[13] *Cf.* Zur Geschichte der Math. in Altertum und Mittelalter, Leipzig, 1874, pp. 203–204.
[14] The original text is "**चक्रवालमिदं जगुः**". The commentator Kṛṣṇa explains: "**आचार्या एतद्गणितं चक्रवालमिति जगुः**" i.e., "the learned professors call this method of calculation the *cakravāla*".

> As the works on algebra of Brahmagupta, Śrīdhara, and Padmanābha are very extensive, so for the satisfaction of the students I have taken the essence of those works and compiled this small work with demonstrations.

This stanza shows that the *Bījagaṇita* of Bhāskara II was drawn mostly from the works on algebra of Brahmagupta, Śrīdhara, and Padmanābha. The cyclic method does not occur in the works of Brahmagupta: it is likely that the work of Śrīdhara or Padmanābha or both contained that method.[15] There is no doubt that Bhāskara II got that method from some earlier work. If that was Jayadeva's work, Bhāskara II must have mentioned his name along with the other names mentioned by him. But he has not made even a single reference to Jayadeva. At the same time it cannot be said definitely whether the algebraical works of Śrīdhara and Padmanābha contained the cyclic method. In fact, we have absolutely no information about them. It is simply by chance that we have come across the name of and quotations from Ācārya Jayadeva who is otherwise unknown to us. Under these circumstances the question of this invention cannot be decided until we receive some more light in this direction.

[15] P. C. Sengupta (1944) expressed the hope

> that further researches may show that this achievement is to be ascribed to Padmanābha, if his work be ever brought to light.

See the Presidential Address delivered by him at the Technical Sciences Section of the Twelfth All India Oriental Conference held at Banaras, 1944. I fail to understand why Sri Sengupta has shown special favour to Padmanābha against Śrīdhara whose claims are equally good if not greater.

Series with fractional number of terms *

1 Introduction

Series of numbers with fractional number of terms have generally no meaning and so they are not treated in modern works on algebra. But such series are found to occur in ancient Indian works on arithmetic, where they have been assigned a geometrical or symbolical significance. Originally such series were interpreted with the help of figures resembling a ladder or a drinking glass, but in course of time an analytical meaning was also given to them. In doing so the Indian mathematicians were guided by certain problems that arose in everyday life. In this brief note we shall put forward the Indian stand-point with reference to arithmetic series having fractional number of terms.

2 Occurrence

Problems on series involving fractional number of terms seem to have attracted the Hindu mind from very early times. The following three problems are found to occur in the earliest Hindu treatise on mathematics, the Bakhshali Manuscript (c. 300 AD):

(1) There are two labourers of whom one earns 10 *māṣakas* per day and the other does work which brings him 2 *māṣakas* increasing by 3 *māṣakas* each day. In what time will they have earned an equal amount?

(2) Earnings of one man are in A.P., whose first term is 5 and common difference 6; those of another, also in A.P., with its first term equal to 10 and common difference equal to 3. When will they have an equal amount of money?

(3) One man walks 5 *yojanas* on the first day and 3 *yojanas* more on each successive day. Another man walks 7 *yojanas* each day, and he has already walked for 5 days. Say, O ! excellent mathematician, when they will meet.

* K. S. Shukla, in *Bhāratī Bhānam* (Dr. K. V. Sarma Felicitation Volume), Panjab University Indological Series - 26, Hoshiarpur (1980), pp. 475–481.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
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The following problem, occurring in Pṛthūdakasvāmin's commentary (860 AD) on the *Brāhmasphuṭasiddhānta*,[1] makes mention of the fractional number of terms directly:

(1) A king bestowed gold continually to venerable priests during 3 days and a ninth part, giving one and a half (*bhāras*), with a daily increase of a quarter. What are the mean and last terms and the total?

Ācārya Mahāvīra,[2] about the middle of the 9th century AD, gave numerous examples on arithmetic series of fractional numbers involving fractional number of terms. The following are the typical ones:

(1) $\frac{2}{3}$, $\frac{1}{6}$, and $\frac{3}{4}$ are (respectively) the first term, common difference, and the number of terms (of one series), and $\frac{2}{5}$, $\frac{3}{4}$, and $\frac{2}{3}$ those of another (series). Say what is the sum (of each of these series).

(2) Find the first term and common difference of the series whose number of terms are $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, and $\frac{11}{12}$, and whose sums are the squares and cubes of those numbers (respectively).

(3) In a series, whose first term is twice the common difference, the number of terms is $\frac{13}{18}$, and the sum is $\frac{67}{216}$. Find out the first term and the common difference.

(4) In relation to one series, the first term is $\frac{2}{5}$, the common difference is $\frac{3}{4}$, and the sum is $\frac{7}{54}$; again (in relation to another series), the common difference is $\frac{5}{8}$, the value of the first term is $\frac{3}{8}$, and the sum is $\frac{3}{40}$. In respect of these two (series), O! friend, give out the number of terms quickly.

(5) Give out the first term and the common difference (respectively) in relation to (the two series having) $\frac{31}{150}$ as the sum, and having $\frac{3}{4}$ (in one case) as the common difference and $\frac{4}{5}$ as the number of terms, and (in the other case) $\frac{1}{3}$ as the first term and $\frac{4}{5}$ as the number of terms.

(6) Of two series whose number of terms are 11 minus $\frac{2}{3}$ and 9 plus $\frac{1}{5}$, respectively, the sum of one is equal to the sum of the other as multiplied or divided by an integer 1, 2, 3, etc. If the first term and common difference of those series be mutually interchangeable, say, friend, what they are.

Ācārya Śrīdhara[3] classifies series into two categories, (A) series which admit of geometrical interpretation, and (B) series which admit of symbolical interpretation. Under the former he set the following problems:

[1]Chap. xii, (Banares, 1902).
[2]See *Gaṇitasārasaṅgraha*, Chap. 3 (Madras, 1912).
[3]See his *Pāṭīgaṇita*, *śreḍhī-vyavahāra* (Lucknow, 1959).

(1) What is the sum of 5 terms of the series whose first term is 2 and common difference 3? And what of one half of a term? (Also) say the sum of one-fifth of a term of a series whose common difference is 5 and the first term 2.

(2) In a leather bag full of oil there occurs a fine hole, and the oil leaks through it. The bag has to be carried to a distance of 3 *yojanas*. If the wages for the first *yojana* be 10 *paṇas* and for the subsequent *yojanas* successively less by 2 *paṇas*, what are the wages for a *krośa*? (1 *krośa* = $\frac{1}{4}$ of a *yojana*).

Under the latter he gives the following problems:

(3) One man gets 3, and the other men get 2 more in succession; say, what do (the first) $4\frac{1}{2}$ men get.

(4) If a labourer gets $\frac{1}{2}$ in the first month and $\frac{1}{3}$ more in succession in the following months, what will he get in (the first) $3\frac{1}{2}$ months?

3 Geometrical interpretation

The geometrical interpretation of an arithmetic series is met with in its fuller form in the *Pāṭīgaṇita*[4] of Ācārya Śrīdhara, who has compared it with the shape of a drinking glass. Writes he:

> As in the case of an earthen drinking glass the width at the bottom is the smaller and at the top greater, so also is the case with the figures of a series.
>
> The areas of the series-figure for the successive units of the altitude form a series which begins with the given first term and increases successively by the given common difference.
>
> I now describe the method for finding the base and the face (i.e., the top side) of the series-figure (corresponding to the first term).
>
> The number of terms, i.e., one, is the altitude of the series-figure; the first term of the series as diminished by half the common difference of the series is the base; and the base as increased by the common difference of the series is the face. All these should be shown by means of threads.
>
> Two threads should then be stretched out, one on either side, joining the extremities of the base and the face: these are the lateral sides of the series-figure.

[4] See his *Pāṭīgaṇita*, *śreḍhī-vyavahāra* (Lucknow, 1959).

> Having constructed the series-figure (for altitude unity) in this manner, one should determine the face for the desired altitude (i.e., for the desired number of terms of the series) (by the following rule):
>
> The face (for altitude unity) minus the base (for altitude unity), multiplied by the desired altitude, and then increased by the base (for altitude unity), gives the face (for the desired altitude).

That is to say, if we construct a symmetrical trapezium with

$$\text{base} = a - \frac{1}{2}d, \quad \text{face} = a + (n - \frac{1}{2})d, \quad \text{and altitude} = n,$$

and subdivide it in smaller trapeziums by drawing $(n-1)$ horizontal lines at equal distances, then the areas of these sub-trapeziums, taken from bottom to top, will severally correspond to the n terms of the series

$$a + (a+d) + (a+2d) + \cdots + \{a + (n-1)d\};$$

and the area of the whole trapezium will correspond to the sum of the n terms of the series.

For, the first trapezium from the bottom will have

$$\text{base} = a - \frac{1}{2}d, \quad \text{face} = a + \frac{1}{2}d, \quad \text{and altitude} = 1.$$

Therefore its area will be equal to a, which is the first term of the series; the second trapezium from the bottom will have

$$\text{base} = a + \frac{1}{2}d, \quad \text{face} = a + d, \quad \text{and altitude} = 1.$$

Therefore its area will be equal to $a + d$; and so on. The area of the whole trapezium is equal to $\frac{n}{2}\{2a + (n-1)d\}$.

Thus, according to the above interpretation, the series

$$a + (a+d) + (a+2d) + \ldots \text{ to } (n + p/q) \text{ terms}$$

stand for the area of the trapezium with

$$\text{base} = a - \frac{1}{2}d, \quad \text{face} = a + \left(n + \frac{p}{q} - \frac{1}{2}\right)d, \quad \text{and altitude} = n + \frac{p}{q}.$$

Since the area of this trapezium is equal to

$$\frac{1}{2}\left(n + \frac{p}{q}\right)\left\{2a + \left(n + \frac{p}{q} - 1\right)d\right\},$$

the sum of above series is also equal to that.

Hence Śrīdhara enunciates the following general formula for the sum of a series having integral or fractional number of terms:

> The common difference as multiplied by one half of the number of terms minus one, being increased by the first term and then multiplied by the number of terms, gives the sum of the series (in arithmetical progression). And the area of the (corresponding) series-figure is equal to the product of one-half of the sum of the base and the face, and the altitude.

3.1 A paradoxical situation

Now, we draw the attention of the reader to the third part of Śrīdhara's Problem One. It relates to finding the sum of one-fifth of a term of the arithmetic series whose first term is 2 and common difference 5. If we apply Śrīdhara's rule, we find that the sum comes out to be 0. This is indeed a very curious situation, for the sum of a series whose first term, common difference, and the number of terms are all positive comes out to be 0. The situation becomes still more curious if we find the sum of one-fifth of a term of the same series, for then we get a negative sum.

To resolve this difficultly, Śrīdhara says:

> When the base of the series-figure comes out to be negative, the two threads (joining the extremities of the base and face corresponding to the first term) should be stretched crosswise (so that in this case the series-figure corresponding to the first term will reduce to two triangles, one lying below the other). In the upper triangle, the altitude will be equal to the face divided by face minus base; and that subtracted from one will give the altitude in the lower triangle.

Thus, in the first case under consideration, the series-figure reduces to two triangles, the upper one having

$$\text{base} = \frac{1}{2}, \text{ and altitude} = \frac{1}{10},$$

and the lower having

$$\text{base} = -\frac{1}{2}, \text{ and altitude} = \frac{1}{10}.$$

Hence, the

$$\begin{aligned}\text{sum of the series} &= \text{area of the upper triangle} + \text{area of the lower triangle}\\ &= \frac{1}{40} - \frac{1}{40} = 0.\end{aligned}$$

In the second case, the upper triangle of the series-figure has

$$\text{base} = \frac{1}{3}, \text{ and altitude} = \frac{1}{15},$$

and the lower triangle has

$$\text{base} = \frac{1}{2}, \text{ and altitude} = \frac{1}{10},$$

so that the area of the series comes out to be equal to

$$\frac{1}{90} - \frac{1}{40}, \text{ i.e., } -\frac{1}{72}.$$

3.2 Note

The idea of interpreting series by means of geometrical figures is very old. For we learn from Bhāskara I (629 AD) that in his time certain astronomers regarded the subject of series as forming part of geometry and not of algebra. He says:

> Other mathematicians say that mathematics is of two types, symbolical and geometrical ... Proportion and pulverizer (i.e., indeterminate analysis of the first degree), etc., are stated to be parts of symbolical mathematics, and series and shadow, etc., of geometrical mathematics.

Pṛthūdakasvāmin (860 AD) has mentioned the name of an ancient Indian mathematician Skandasena who explained the sum of an arithmetic series by means of geometrical figures. Possibly his interpretation was the same as that of Śrīdhara. It is interesting to note that series-figures attracted the Hindu mind and appear in Indian works on arithmetic as late as the fourteenth century AD. Ācārya Nārāyaṇa (1356 AD) has discussed these figures in his *Gaṇitakaumudī* in the chapter on plane figures.

4 Symbolical interpretation

According to the symbolical interpretation, the series

$$a + (a+d) + (a+2d) + \dots \text{ to } (n + p/q) \text{ terms}$$

means the sum of n terms together with p/q^{th} part of the $(n+1)^{\text{th}}$ term. Thus the sum of the above series will be equal to

$$\frac{1}{2}n\{2a + (n-1)d\} + \frac{p}{q}(a + nd).$$

Hence Śridhara says:

> The common difference as multiplied by the integral part of the number of terms should be increased by the first term, and the

> result obtained should be kept undestroyed (at one place). The same result (written in another place) being increased by the first term, then diminished by the common difference, then multiplied by one-half of the integral part of the number of terms, and then added to the undestroyed result as multiplied by the fractional part of the number of terms, gives the sum of the series.

Śrīdhara has also given rules for finding the first term, common difference, and the number of terms when the other quantities are known.

5 Non-equivalence of interpretations

It is evident that, unless the series contains an integral number of terms, the two interpretations are non-equivalent, and would lead to different results. As to which interpretation is to be followed in a particular problem will depend on the nature of the problem. For instance, to solve Problem Two of Śrīdhara one must apply the geometrical interpretation, whereas to solve Problem Four one must apply the symbolical interpretation. But in Problem One of Śrīdhara both interpretations are equally good, and it would be difficult to accept one in preference to the other. Śrīdhara does not explicitly say as to which interpretation should be applied in such cases. But as he sets that problem under the geometrical interpretation, it means that he assumes that such problems are to be interpreted geometrically. Other Indian writers on the subject also seem to be of the same view.

Hindu methods for finding factors or divisors of a number *

1 Introduction

Factoring or finding divisors of a number does not appear as a subject of treatment in any early work on Hindu arithmetic. There are, however, reasons to believe that the ordinary method of factoring a number by successive division by 2, 3, 5, etc. was well known but being much too elementary was not considered suitable for inclusion in an arithmetical work. Even in Mahāvīrācārya's (850 AD) voluminous *Gaṇitasārasaṅgraha* where we have explicit references to factorisation no rule has been stated for the purpose. Śrīpati (1039 AD) is probably the first Hindu writer who has formally dealt with the subject of factoring a number in his *Siddhāntaśekhara*. Besides stating the ordinary method based on successive division, he gives an additional method for factoring a non-square number by expressing it as a difference of two squares. This latter method was subsequently stated in its complete form by another notable Hindu mathematician Nārāyaṇa (1356 AD) who, in his *Gaṇitakaumudī*, devoted a full chapter to the subject of factoring and finding all possible divisors of a number. It is interesting to note that the method of factoring a non-square number in its complete form in which it was stated by Nārāyaṇa, was rediscovered in Europe about two centuries later by the French mathematician Fermat. The object of the present paper is to throw light on the methods given by Śrīpati and Nārāyaṇa and to invite attention of historians of mathematics to them.

2 Srīpati's rule

Śrīpati[1] states his rules for factoring a number as follows:

> द्वाभ्यां द्वाभ्यां भाज्यराशिं समे तत् आदिस्थाने पञ्चके पञ्चकेन ।
> एवं कुर्याद्यावदोजं तु तावत् त्र्याद्यैर्हारैर्भाज्यराशिं भजेत्तु ॥

* K. S. Shukla, *Gaṇita*, Vol. 17, No. 2 (1966), pp. 109–117.

[1] Śrīpati, *Siddhāntaśekhara*, Part II, edited by Babuaji Misra, Calcutta (1947), ch. xiv, vv. 36–37.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_13

वर्गश्चेत्तन्मूलमेवास्य हारं नो चेदासन्नं पदं द्विघ्नमस्मिन् ।
रुपं युक्त्वा शेषहीने कृतिः स्यात् तन्मूलं तद्युक्तमूलं युतोने ॥

(Rule 1). So long as the dividend (i.e. the number to be factored) is even, it is to be divided out by 2 again and again; whenever 5 happens to occur in the unit's place, it should be divided by 5; this should be done until the dividend is reduced to an odd number (with unit's digit different from 5), and then (the prime numbers) 3 etc. should be tried as divisors.

(Rule 2). If the dividend is a perfect square, its square root itself is a divisor; if not, its nearest square root should be multiplied by 2, then increased by 1, and then diminished by the residue of the square root; if the resulting number is a perfect square, find the square root of this perfect square and also the square root of the dividend as increased by this perfect square, and take their sum and difference. (Thus are obtained the two factors of the dividend).

Rule 1 gives the ordinary method of factoring a number. Rule 2 may be symbolically expressed as follows:

Let N be a non-square number, equal to $a^2 + r$ say. Then if

$$2a + 1 - r = b^2,$$

we have

$$N + b^2 = a^2 + r + (2a + 1 - r) = (a + 1)^2,$$

so that

$$\begin{aligned} N &= (a+1)^2 - b^2 \\ &= (a+b+1)(a-b+1). \end{aligned}$$

3 Nārāyaṇa's rules

Nārāyaṇa[2] states his rules as follows:

असकृद् विभजेद् द्वाभ्यां समराशिं यावदेति वैषम्यम् ।
सत्सु प्रथमस्थाने पञ्चसु भाज्ये च पञ्चभिश्छिन्द्यात् ॥
न समो भाज्यः प्रथमः तस्मिन् यदि पञ्चकं स्थाने ।
अच्छेद्याः कल्प्यन्ते त्रिसप्तकैकादशादयश्छेदाः ॥
यावच्छेदप्राप्तिस्तावद् हरसाधनं क्रियते ।
भाज्यो वर्गश्चेत् तन्मूलं छेदो द्विधा भवति ॥

[2] Nārāyaṇa, *Gaṇitakaumudī*, Part II, edited by Padmakara Dvivedi, Benaras (1942), ch. xi, rules 2–9(i).

अपदप्रदस्तु भाज्यः कयेष्टकृत्या युतात् पदं भाज्यात् ।
पदयोः संयुतिवियुती हारौ परिकल्पितौ भाज्यौ ॥
राश्योस्तु तयोः प्राग्वत् कुर्वीतच्छेदशोधनं सुधिया ।
अपदप्रदस्य राशेः पदमासन्नं द्विसङ्गुणं सैकम् ॥
मूलावशेषहीनं वर्गश्चेत् क्षेपकश्च कृतिसिद्ध्यै ।
वर्गो न भवेत् पूर्वासन्नपदं द्विगुणितं त्रिसंयुक्तम् ॥
आद्याद् द्युत्तरवृद्ध्या तावद् यावद् भवैद् वर्गः ।
असमानानां पूर्वहताः परे पुरःस्थास्तथा चान्ये ॥
तुल्यानां पूर्वघ्नः परः पृथक् तेऽन्यहरनिघ्नाः ।

(Rule 1). If the number is even, divide it by 2 again and again until it becomes odd; if there is 5 in the unit's place, divide by 5. If the number to be divided is neither even, nor there is 5 in the unit's place, the prime numbers 3, 7, 11, etc. should by tried as divisors. One should find out the divisors until it is possible to do so.

(Rule 2). In case the number to be divided is a perfect square, its square root is a twice repeated factor. If the number to be divided is a non-square number, find an optional square number which added to the dividend gives a perfect square. The sum and the difference of their square-roots are the divisors of that (non-square) number, which are to be treated as dividends. The intelligent should now proceed, as before, to find out the divisors of those numbers.

(The method of finding the optional square number contemplated above is as follows): Find out the nearest square root of the non-square number, multiply that by 2, and then subtract therefrom the residue of the square root: in case it is a perfect square, it is to be taken as the number to be added to the dividend to make it a perfect square. In case the above number is not a perfect square, it should be further increased by the successive terms of the arithmetic series whose first term is twice the nearest square root (of the non-square number) plus 3, and common difference 2, until it becomes a perfect square.

(Rule 3). In the case of unequal divisors (thus obtained) (proceed as follows): (Having set down the divisors one after another) multiply the succeeding divisors by the preceding ones and by the product of the preceding ones taken two, three, ..., all at a time, and set down the products (thus obtained) ahead of those divisors.

In the case of equal divisors (proceed as follows): (Having set down the divisors one after another) multiply each of the succeeding divisors by (the product of) the preceding ones.

Each of these should be severally multiplied by the other numbers (i.e. by those due to unequal divisors).

(Thus are obtained all the divisors of the given number).

Rule 2 above may be symbolically expressed as follows:
Let N be a non-square number, equal to $a^2 + r$ say. If

$$2a + 1 - r = b^2,$$

then

$$N + b^2 = a^2 + r + (2a + 1 - r) = (a + 1)^2,$$

so that

$$\begin{aligned} N &= (a + 1)^2 - b^2 \\ &= (a + b + 1)(a - b + 1). \end{aligned}$$

If, however,

$$2a + 1 - r = c$$

where c is not a perfect square, then we add to c as many terms of the series

$$(2a + 3) + (2a + 5) + (2a + 7) + \ldots$$

as are necessary to make the resulting sum a perfect square. Let r terms of the series be added and we have

$$c + [(2a + 3) + (2a + 5) + \cdots + (2a + 2r + 1)] = k^2.$$

Then

$$N + k^2 = (a + r + 1)^2,$$

so that

$$N = (a + r + 1)^2 - k^2 = (a + r + k + 1)(a + r - k + 1).$$

Rule 3 gives the method for writing down all possible divisors of the given number N. Let

$$N = a \times a \times a \times b \times c \times d.$$

Then all the divisors of N are:

(1) b, c, d, bc, bd, cd, bcd,

(2) a, a^2, a^3,

(3) ba, ba^2, ba^3; ca, ca^2, ca^3; da, da^2, da^3;
bca, bca^2, bca^3; bda, bda^2, bda^3; cda, cda^2, cda^3;
$bcda$, $bcda^2$, $bcda^3$;

3.1 Examples

Nārāyaṇa[3] illustrates the above rules by the following examples:

Ex. 1. Mathematician, quickly tell me the numbers by which the number 2048 is exactly divisible; also those numbers by which the number 3125 is exactly divisible.

We have

$$2048 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2,$$

so that all possible divisors of 2048 are 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024 and 2048.

Also

$$3125 = 5 \times 5 \times 5 \times 5 \times 5,$$

so that all possible divisors of 3125 are 5, 25, 125, 625, 3125.

Ex. 2. Tell me if you know the numbers by which 7520 is exactly divisible; also the numbers by which 10201 is exactly divisible.

We have

$$7520 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 47.$$

Therefore all possible divisors of 7520 are

(1) 5, 47, 235

(2) 2, 4, 8, 16, 32

(3) 10, 20, 40, 80, 160; 94, 188, 376, 752, 1504; 470, 940, 1880, 3760, 7520.

Also

$$10201 = 101 \times 101.$$

Therefore the divisors of 10201 are 101 and 10201 only.

Ex. 3. O you, proficient in mathematics, tell me, if you know the subject of finding divisors, the numbers by which the number 1161 is exactly divisible.

We have

$$1161 = 34^2 + 5$$

and

$$2 \times 34 + 1 - 5 = 8^2,$$

[3] *Ibid*, ch. xi, Exs. 1 to 6.

therefore

$$1161 + 8^2 = 35^2,$$

so that

$$1161 = 35^2 - 8^2 = 43 \times 27 = 3 \times 3 \times 3 \times 43.$$

Therefore, all possible divisors of 1161 are 3, 9, 27, 43, 129, 387, 1161.

Ex. 4. If you are fully proficient in the subject of finding divisors, quickly tell me the numbers by which 1001 is exactly divisible.

Because

$$1001 = 31^2 + 40,$$

and

$$2 \times 31 + 1 = 63,$$

and

$$63 - 40 + 65 + 67 + \ldots + 89 = 32^2,$$

therefore

$$1001 + 32^2 = 45^2.$$

Therefore,

$$1001 = 45^2 - 32^2 = 77 \times 13 = 7 \times 11 \times 13.$$

Therefore, all possible divisors of 1001 are 7, 11, 13, 77, 91, 143, 1001.

Ex. 5. Friend, if you are fully proficient in mathematics, quickly tell me, the numbers by which 4620 is exactly divisible.

Proceeding as before, we have

$$4620 = 2 \times 2 \times 3 \times 5 \times 7 \times 11.$$

Therefore, all the divisors of 4620 are

(1) 3, 5, 7, 11, 15, 21, 33, 35, 55, 77, 105, 165, 231, 385, 1155

(2) 2, 4

(3) 6, 10, 14, 22, 30, 42, 66, 70, 110, 154, 210, 330, 462, 770, 2310, 12, 20, 28, 44, 60, 84, 132, 140, 220, 308, 420, 660, 924, 1540, 4620.

Ex. 6. Friend, if you are versed in mathematics, quickly tell me the numbers by which 3927 is exactly divisible.

Proceeding as before, we have $3972 = 3 \times 7 \times 11 \times 17$, so that all possible divisors of 3927 are 3, 7, 11, 17, 21, 33, 51, 77, 119, 187, 231, 357, 561, 1309, 3927.

4 Nārāyaṇa's rule rediscovered by Fermat

Rule 2 of Nārāyaṇa was rediscovered by the French mathematician Fermat about 1643 AD. In a letter written about that time, Fermat explains his method as follows:[4]

> An odd number not a square can be expressed as the difference of two squares in as many ways as it is the product of two factors, and if the squares are relatively prime the factors are. But if the squares have a common divisor d, the given number is divisible by $\sqrt{d}$. Given a number n, for examples 2027651281, to find if it be prime or composite and the factors in the latter case. Extract the square root of n. I get $r = 45029$, with the remainder 40440. Subtracting the latter from $2r + 1$, I get 49619, which is not a square in view of the ending 19. Hence I add $90061 = 2 + 2r + 1$ to it. Since the sum 139680 is not a square, as seen by the final digits, I again add to it the same number increased by 2, i.e., 90063, and I continue until the sum becomes a square. This does not happen until we reach 1040400, the square of 1020. For by an inspection of the sums mentioned it is easy to see that the final one is the only square (by their endings except for 499944). To find the factors of n, I subtract the first number added, 90061, from the last, 90081. To half the difference add 2. There results 12. The sum of 12 and the root r is 45041. Adding and subtracting the root 1020 of the final sum 1040400, we get 46061 and 44021, which are the two numbers nearest to r whose product is n. They are the only factors since they are primes. Instead of 11 additions, the ordinary method of factoring would require the division by all the numbers from 7 to 44021.

It may be added that at the time of writing his letter to Mersenne, December 26, 1638, Fermat had no such method.[5] This shows that the method was well known in India long before Fermat rediscovered it. Although Nārāyaṇa was the first to state the method in its complete form, the credit of the first inception of the method is indeed due to Śrīpati.

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[4] *Cf.* Dickson, L.E., "History of the Theory of Numbers," Vol. I, p. 357.

[5] *Ibid*, p. 357, footnote.

5 Nārāyaṇa's alternative rule

Nārāyaṇa[6] gives the following as an alternative method for finding the divisors of a number:

> इष्टोनासन्नपदं हारः स्यादिष्टवर्गशेषयुतिः ॥
> हारहृता चेच्छुध्यति तेनावश्यं हृतो भाज्यः ।
> न विशुध्यति चेदिष्टं स्वधिया परिकल्पयेदन्यत् ॥
>
> The nearest square root (of the given number) as diminished by an optional number is the "divisor". If the optional number squared plus the residue of the square root is exactly divisible by the 'divisor' the given number shall be exactly divisible by the same. If not, one should apply one's intellect to choose another (appropriate) optional number.

Let $N = a^2 + r$. Then choosing λ as an optional number, we can write

$$\begin{aligned} N &= a^2 - \lambda^2 + \lambda^2 + r \\ &= (a+\lambda)(a-\lambda) + \lambda^2 + r \\ \therefore \frac{N}{a-\lambda} &= a + \lambda + \frac{\lambda^2 + r}{a-\lambda}. \end{aligned}$$

Therefore, if $\lambda^2 + r$ is exactly divisible by $a - \lambda$, then N is also exactly divisible by $a - \lambda$. Hence the rule.

Ex. 7. Quickly tell me, proficient in mathematics, the numbers by which 120 is exactly divisible, and also those by which 231 is exactly divisible.[7]

Here $120 = 10^2 + 20$. Choosing $\lambda = 2$, we see that $a - \lambda = 8$ is an exact divisor of 120. Thus

$$120 = 8 \times 15$$

so that

$$120 = 2 \times 2 \times 2 \times 3 \times 5.$$

All possible divisors of 120 are therefore

(1) 3, 5, 15

(2) 2, 4, 8

(3) 6, 12, 24; 10, 20, 40; 30, 60, 120.

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[6] Nārāyaṇa, *Gaṇitakaumudī*, Part II, edited by Padmakara Dvivedi, Benaras (1942), ch. xi, rule 9(ii)–10.
[7] *Ibid*, ch. xi, Ex. 7.

Again $231 = 15^2 + 6$. Choosing $\lambda = 4$, we see that $a - \lambda = 11$ is an exact divisor of 231. Thus

$$231 = 11 \times 21$$

so that

$$231 = 3 \times 7 \times 11.$$

Hence all possible divisors of 231 are 3, 7, 11, 21, 33, 77, 231.

Magic squares in Indian mathematics *

The following is a magic square:

4	9	2
3	5	7
8	1	6

In this the sum of numbers in each row is 15. The sum in each column is 15; and the numbers in each diagonal also sum up to 15. The above magic square was known to all ancient people. It is the simplest magic square consisting of 3×3 $(= 9)$ cells. The problem of arranging numbers in squares containing more cells is not easy. Magic squares were used in India about first century AD as charms and reference to them is found in the earliest tantric literature.

The construction of magic squares requires ingenuity because no definite method can be given, at least in some cases. For this reason, the subject has aroused the interest of mathematicians. The Hindu mathematician Nārāyaṇa, who flourished in the fourteenth century AD has included in his work on arithmetic, called *Gaṇitakaumudī*, a chapter on the construction of magic squares and other allied figures. In this respect the *Gaṇitakaumudī* is unique as no other treatise on arithmetic known to us deals with the construction of magic squares. From Nārāyaṇa's work we find that methods of construction of all types of magic squares were known in India.

Methods of constructing magic squares have been given by European and American mathematicians from the 17th century onwards but the Hindu methods appear to be the simplest, although some of the methods recently developed in the west are more general. We propose to give here some simple methods of constructing magic squares and allied figures mostly from Hindu sources.

Classification of magic squares

Magic squares, in accordance with their methods of construction, are divided into three classes:

* K. S. Shukla, in *Interaction between Indian and Central Asian Science and Technology in Medieval Times*, Vol. 1, Indo-Soviet Joint Monograph Series, INSA (1990), pp. 249–270.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_14

(1) those having an odd number of cells in each row

(2) those having $4n$ cells in each row, and

(3) those having $4n + 2$ cells in each row.

1 Odd magic squares

The construction of odd magic squares is the easiest. We give below two methods of constructing such magic squares, These methods have been given in the *Gaṇitakaumudī* of Nārāyaṇa.

1.1 First method

This method is illustrated by the following four squares (**ed.** see Figure 1) in which we will use the natural numbers 1, 2, 3, etc. in succession.

Rule for filling

Begin with the middle cell of the top row and write 1 in the cell. Then proceed along the outward drawn diagonal of the cell. (The direction of the diagonal has been indicated in each figure by arrow-heads). This leads you beyond the square. Now, if the square were wrapped around a cylinder, you would get into the lowest cell of the next column. Therefore, write 2 in it. Then proceed again in the direction of the above outward drawn diagonal and write the next successive numbers 3, 4, 5, etc. in the cells thus encountered. Thus you again reach a cell which leads you beyond the square. Again imagine that the square is wrapped around a cylinder. Thus you reach the first cell of the next row, write the next number in the cell. Then proceed again in the direction of the above outward drawn diagonal and continue the above process till all the cells are filled. Whenever the above process leads you to a cell which is already occupied by a number, write the next number in the cell below and proceed in the direction of the diagonal.

It will be easily seen that the above squares may also be filled by proceeding in the direction of the other outward drawn diagonal of the middle cell of the top row. Similarly, the above process may also be started with the middle cell of the bottom row or with the middle cell of the first or last column. Thus an odd square can be filled up in 8 ways to form a magic square.

The magic squares constructed by the above method are such that the sum of any two numbers which are equidistant from the centre is equal to twice the number at the centre. Such squares have been called perfect by W. S. Andrews (Magic square and cubes, Chicago, 1908).

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

30	39	49	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Figure 1: Odd magic squares: First method.

Note

The four magic squares, which have been shown above (**ed.** see Figure 1), have been filled by the natural numbers 1, 2, 3, etc. But that is not necessary. A $n \times n$ magic square with total $\underline{S}$ may, in general, be filled either by $\underline{n} \times \underline{n}$ numbers in arithmetical progression and having $n \times S$ for their sum, or else, by $\underline{n}$ sets each of $\underline{n}$ numbers belonging to the same arithmetical progression and having $\underline{n} \times \underline{S}$ for their sum. Thus, for example, a 3×3 magic square with total 33 may be constructed by the numbers

(a) 7, 8, 9, 10, 11, 12, 13, 14, and 15, or
(b) 2, 4, 6, 9, 11, 13, 16, 18, and 20

taken in succession. The series (a) give the magic square (i); and the series (b) give the magic square (ii):

(i)

14	7	12
9	11	13
10	15	8

(ii)

18	2	13
6	11	16
9	20	4

1.2 Second method (superposition)

According to this method, a 3×3 magic square with total 21, say, is constructed as follows:

Take two sets of numbers each containing 3 numbers in arithmetical progression, say, 1, 5, 9 and 0, 1, 2. Multiply the numbers in the second set by

$$\frac{\text{given total} - \text{sum of the first set}}{\text{sum of the second set}} = \frac{21 - 15}{3} = 2.$$

Thus we get 0, 2, 4 as the second set.

Now construct two squares having 3×3 cells. In the cells of the middle row or column of the first square fill the numbers 1, 5, 9 of the first set, in the remaining cells fill the same numbers in a cyclic order as below. Similarly, in the second square fill the numbers 0, 2, 4 of the second set. Thus are obtained the skeleton squares (i) and (ii) below:

(i)

9	1	5
1	5	9
5	9	1

(ii)

4	0	2
0	2	4
2	4	0

Now fold the paper in such a way that the second square may fall on the first and then add the numbers which fall on each other. The gives

11	1	9
5	7	9
5	13	3

which is the required magic square with total 21.

This method of superposition was called by Nārāyaṇa *chādya-chādaka-vidhi*. The method was rediscovered in Europe by M. de la Hire in the beginning of the 18th century.

2 $4\underline{n} \times 4\underline{n}$ magic squares

The simplest squares having $4\underline{n}$ cells in a row are 4×4 magic squares. These are also constructed by means of numbers forming an arithmetical progression, or by 4 sets of 4 contiguous numbers belonging to the same arithmetical progression. We give below three methods for constructing magic squares.

2.1 First method

This method is given by Nārāyaṇa in his *Gaṇitakaumudī*. We illustrate it by constructing 4×4 magic squares with the help of the natural numbers 1, 2, 3 etc. The total in the resulting magic squares will be 34.

Rule

Arrange the above 16 numbers in their order as follows:

$$1, 2, 3, 4, 5, 6, 7, 8;\ \ 9, 10, 11, 12, 13, 14, 15, 16.$$

Now write the numbers in a square directly and inversely as in (a) below. In the upper half of the square (a) interchange the numbers in the last two columns and in the lower half interchange the numbers in the first two columns. This would give the square (b).

(a)

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

(b)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Now fill the numbers in the first and second row of the square (b) in a 4×4 square by knight's move as in chess, starting with 1 in the first cell and with 8 either in a contiguous cell or in the last cell. The numbers in the second half of the square (b) are filled similarly, so that the numbers 25 and 10 occupy cells opposite to 1 and 8 respectively. Thus we can obtain the following four magic squares.

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1	14	4	15
8	11	5	10
13	2	16	3
12	7	9	6

1	12	13	8
14	7	2	11
4	9	16	5
15	6	3	10

1	14	4	15
12	7	9	6
13	2	16	3
8	11	5	10

To obtain another set of four squares, we interchange the numbers in the second and fourth columns in the upper half of the square (b) and the first and third columns in the lower half. This gives the square (c).

The numbers in the square (c) are filled, as before, to give another set of 4 magic squares.

(c)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

The above process may also be started with any one of the following arrangements of the 16 numbers 1, 2, 3 etc.

$$1, 3, 5, 7, 2, 4, 6, 8, 9, 11, 13, 15, 10, 12, 14, 16,$$

or

$$1, 5, 2, 6, 3, 7, 4, 8, 9, 13, 10, 14, 11, 15, 12, 16.$$

Each of these would give 8 magic squares as above.

The total number of magic squares obtained by Nārāyaṇa's method is thus twenty-four.

2.2 Second method

First fill the numbers

Nil, 1, Nil, 8, Nil, 9, Nil, 2, 6, Nil, 3, Nil, 4, Nil, 7, Nil

in the cells of a 4×4 square as below:

(S)

	1		8
	9		2
6		3	
4		7	

Even Total

If the total is even, say $2\underline{n}$, fill the remaining cells in such a way that in every diagonal the sum of the alternate numbers is equal to $\underline{n}$.

Odd total

If the total is odd, say $(2n + 1)$, fill every blank cell, which is diagonally alternate to the cell containing 1, 2, 3, 4 by $\underline{n}$ minus the diagonally alternate number; and fill the remaining blank cells by $(\underline{n} + 1)$ minus the diagonally alternate number.

$n-3$	1	$n-6$	8
$n-7$	9	$n-4$	2
6	$n-8$	3	$n-1$
4	$n-2$	7	$n-9$

Total $2\underline{n}$

$n-3$	1	$n-5$	8
$n-6$	9	$n-4$	2
6	$n-7$	3	$n-1$
4	$n-2$	7	$n-8$

Total $2\underline{n}+1$

The skeleton square ($\underline{\text{S}}$) may, in general, be constructed by filling 8 cells of a 4×4 square by any 8 numbers forming an arithmetical progression or by two set of 4 contiguous numbers belonging to the same arithmetical progression in accordance with the first method above.

2.3 Third method (superposition)

We illustrate this method by constructing a magic square with total 40.

Take two sets of numbers each containing 4 numbers in arithmetical progression, say,

(i) 1, 2, 3, 4; and

(ii) 0, 1, 2, 3.

Multiply the numbers of the second set by

$$\frac{\text{given total} - \text{sum of the first set}}{\text{sum of the second set}} = \frac{40-10}{6} = 5.$$

Thus we obtain 0, 5, 10, 15 as second set.

Now fill the numbers of the first set in a square as in (i) and the numbers of the second set in another square as in (ii):

(i)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

(ii)

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Fold the paper in such a way that the square (ii) may cover the square (i) and add the numbers which fall on each other. This gives the following magic square.

17	13	2	8
1	9	16	14
18	12	3	7
4	6	19	11

3 $(4\underline{n}+2) \times (4\underline{n}+2)$ magic squares

The construction of $(4\underline{n}+2) \times (4\underline{n}+2)$ magic squares is comparatively more difficult. Nārāyaṇa has suggested two methods which are helpful in constructing such squares. We illustrate one of these methods by constructing a 6×6 magic square with the help of the natural numbers 1, 2, 3, . . . , 36.

3.1 Nārāyaṇa's method

Fill the numbers 1, 2, 3, etc. in the direct and inverse order as in the square (i) below:

(i)

1	2*	3	4	5*	6
12*	11	10	9	8	7*
13*	14	15	16	17	18*
24*	23	22	21	20	19*
25*	26	27	28	29	30*
36	35*	34	33	32*	31

where the cells marked (*) are known as *śliṣṭa*.

Interchange the numbers 18 and 19 lying in the last two *śliṣṭa* cells of the two middle rows with the corresponding numbers 15 and 22 in the left vertical half of the square; then interchange the numbers in the *śliṣṭa* cells lying in the upper half of the square with those symmetrically lying in the lower half. Then rotate the rectangles containing the numbers 3 and 10, 4 and 9, and 28 and 33 about the centre of the square in the anticlockwise direction and bring each of them to the position of the next rectangle. This gives the following magic square:

1	35	4	33	32	6
25	11	9	28	8	30
24	14	18	16	17	22
13	23	19	21	20	15
12	26	27	10	29	7
36	2	34	3	5	31

where the numbers enclosed within circles do not undergo any change.

4 Other magic figures

4.1 8×4 magic rectangle

This is

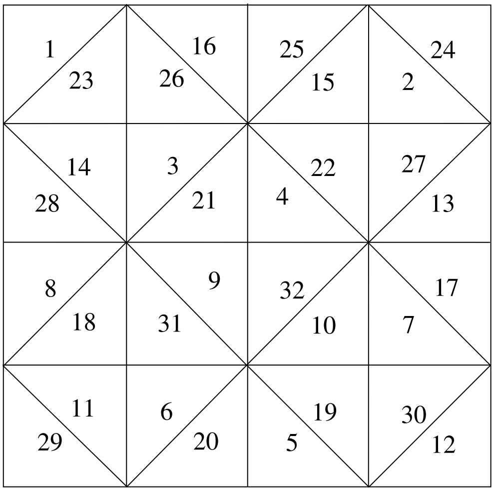
1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Total: rows = 132, columns = 66.

The following magic figures are based on this rectangle:

4.1.1 *Vitana* (canopy)

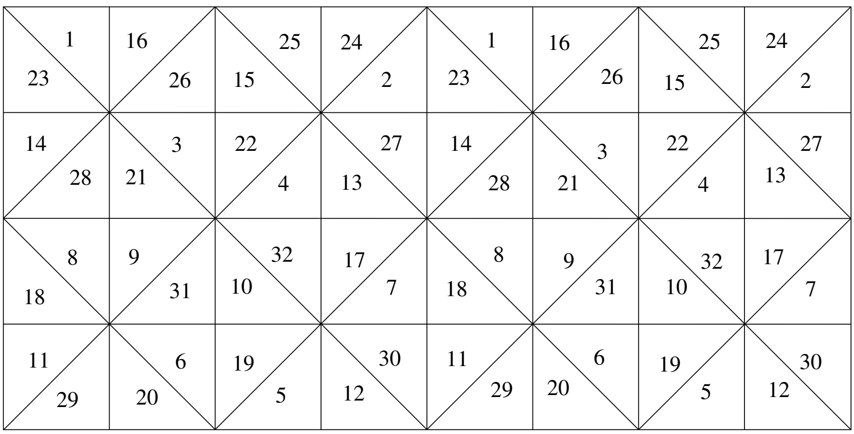
The figure is



Here the sum of 8 numbers taken symmetrically, either horizontally or vertically or diagonally or in a circle or in a square is the same (132). Also the sum of groups of four numbers taken symmetrically is 66.

4.1.2 *Maṇḍapa* (altar)

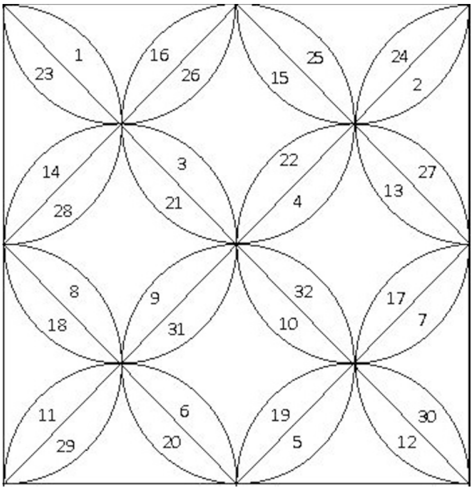
The figure is



Here any set of 8 numbers occurring together, horizontally, vertically or diagonally sum up to 132. The 8 numbers lying in a square also have the same total. There is cylindrical symmetry i.e., if the figure be rolled on a cylinder any continuous 8 numbers or those lying in a square have the total 132. It is easy to find 26 sets of 8 numbers having the same total 132.

4.1.3 *Padma* (Lotus)

The figure is



Here any set of 8 numbers taken vertically, horizontally or in any 4 leaves symmetrically situated gives the same total 132. There is cylindrical symmetry. In this case 32 sets of 8 numbers having the same total can be easily picked up.

4.2 12×4 magic rectangle

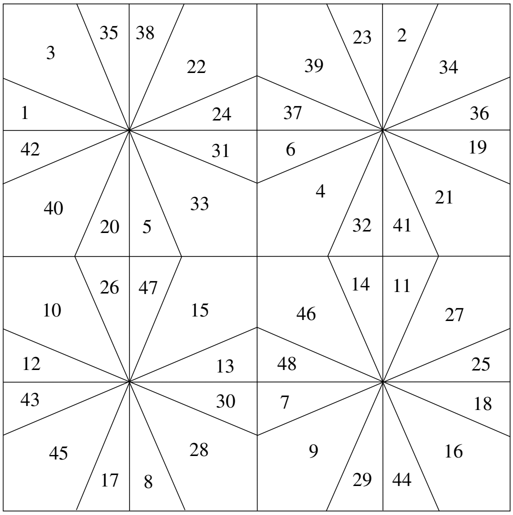
This is

1	24	37	36	2	23	38	35	3	22	39	34
42	31	6	19	41	32	5	20	40	33	4	21
12	13	48	25	11	14	47	26	10	15	46	27
43	30	7	18	44	29	8	17	45	28	9	16

The following magic figures are based on this rectangle:

4.2.1 *Dvādaśakara* (twelve hands)

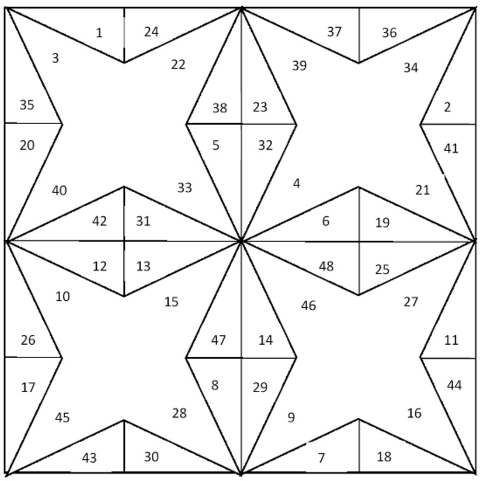
The figure is



Here all groups of 12, of 8 or of 4 numbers have even totals 294, 196 and 98 respectively.

4.2.2 *Vajra Padma*

The figure is



Here every group of 4 numbers occurring in a line or in cell has the total 98;

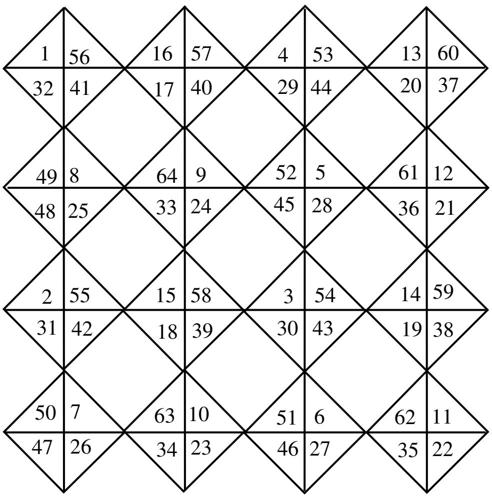
every group of 8 numbers has the total 196 and every group of 12 numbers taken horizontally, vertically or in a circle has the total 294.

4.3 8 × 8 magic square

This is

1	56	16	57	4	53	13	60
38	41	17	40	29	44	20	37
49	8	64	9	52	5	61	12
48	25	33	24	45	28	36	21
2	55	15	58	3	54	14	59
31	42	18	39	30	43	19	38
50	7	63	10	51	6	62	11
47	26	34	23	46	27	35	22

On this is based the following magic figure, square called the *sarvatobhadra*:



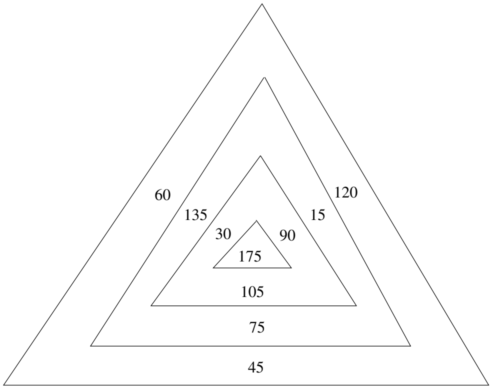
In this figure the totals of all four, eight and sixteen numbers are 130, 260 and 520 respectively.

4.4 Magic triangle

The following is a magic triangle together with its key square.

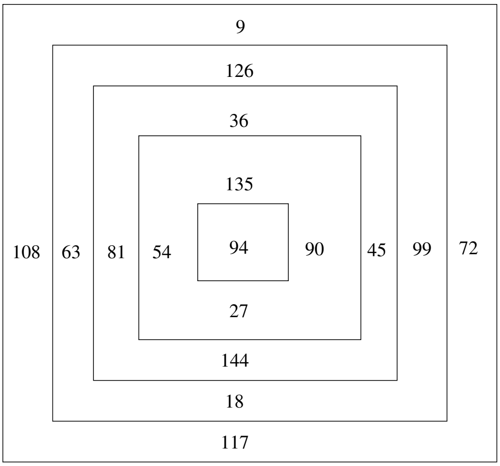
120	45	60
15	75	135
90	105	30

Total 225



4.5 Magic cross

This figure is



(Total 400)

This is based on the following 4×4 magic square:

9	72	117	108
126	99	18	63
36	45	144	81
135	90	27	54

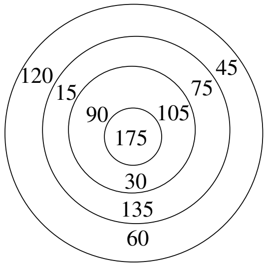
(Total 306)

4.6 Magic circle

Nārāyaṇa has given a number of magic circles. We give below one of these together with the key square:

120	45	60
15	75	135
90	105	30

Total 225



Total 400

5 Magic squares used by Muslims

We shall now give a brief account of the magic squares which were used by the Muslims in India in medieval times as amulets and charms. These magic squares include the following varieties: *Dūpāyā* ("two-legged"), *Sulsī* (3 × 3), *Rubā'ī* (4×4), *Khamsī* (5×5), *Musaddas* (6×6), *Musabba'* (7×7), *Muṣamman* (8 × 8), *Mustassa* (9 × 9), and *Ma'shsher* (10 × 10). Sometimes they were so large as to have 100 cells in a row or column.

The following is a *Dūpāyā* square with total 12:

3	8	1
2	4	6
7		5

To construct a *Dūpāyā* square, divide the given total by 12. Fill the cells with the quotient increasing by itself in every next cell as you proceed (see the above figure). In case the division by 10 yields any (non-zero) remainder it is to be added to the number in the sixth cell. For example, let the total be 786. Division by 12 yields 65 as the quotient and 6 as the remainder. This gives the following square:

195	526	55
130	260	396
461		325

The following is the *Sulsī* square with total 15:

4	9	2
3	5	7
8	1	6

To construct a *Sulsī* square, subtract 12 from the given total, then divide the remainder by 3 and with the quotient fill up the 9 cells of 3 × 3 square until the whole square is filled up. The filling may be started from the central cell of a bordering row or column depending on the elements earth, water, air, or fire, thus:

Air

2	7	6
9	5	1
4	3	8

Fire

4	9	2
3	5	7
8	1	6

Earth

6	7	2
1	5	9
8	3	4

Water

6	1	8
7	5	3
2	9	4

To construct a *Rubāʿī* square deduct 30 from the given total, then divide the remainder by 4, and with the quotient fill up the 16 cells of 4 × 4 square. If 1 remains over, add one to the 13th cell; if 2, add 1 to the 9th cell; if 3, add 1 to the 5th.

With total 34, there will result 4 *Rubāʿī* squares depending on the elements, viz.

Earth

8	11	14	1
13	2	7	12
3	16	9	6
10	6	4	15

Water

14	4	1	15
7	9	12	6
11	5	8	10
2	16	13	3

Air

15	1	4	14
10	8	5	11
6	12	9	7
3	13	16	2

Fire

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

To construct a *Khamsī* square, subtract 60 from the given total, then divide the remainder by 5 and with the quotient fill up the 25 cells of 5 × 5 square. If 1 remains over, 1 is to be added to the 21st cell; if 2, to the 16th ; if 3, to the 11th; if 4, to the 6th.

7	13	19	25	1
20	21	2	8	14
3	9	15	16	22
11	17	23	4	10
24	5	6	12	18

Khamsī square with total 65

To construct a *Musaddas* square, deduct 105 from the given total, then divide by 6 and with the quotient fill up the square. If 1 remains over, add 1 to the 31st cell; if 2, to the 35th; if 3, to the 19th; if 4, to the 13th; if 5, to the 7th.

36	18	30	19	7	1
13	26	2	34	24	12
5	9	22	29	15	31
25	6	14	8	35	23
21	32	10	17	3	28
11	20	33	4	27	16

Musaddas square with total 111

To construct a *Musabba*‘ square, deduct 168 from the given total, then divide by 7 and with the quotient fill up 7 × 7 squares. If from 1 to 5 remain as the remainder, add 1 to the 43rd cell.

40	23	13	45	35	18	1
32	15	5	37	27	10	49
24	14	46	29	19	2	41
16	6	38	28	11	43	33
8	47	30	20	3	42	25
7	39	22	12	44	34	17
48	31	21	4	36	26	9

Musabba square with total 175

To construct a *Muṣamman* square, subtract 252 from the given total, then divide by 8 and with the quotient fill up 8 × 8 square. If 1 to 7 are obtained as the remainder, add 1 to the number in the 75th cell.

36	43	35	32	27	60	26	1
41	4	49	59	21	17	45	24
37	15	11	10	58	51	50	28
23	47	57	52	12	9	18	42
3	46	8	13	53	56	19	62
25	63	54	55	7	14	2	40
31	20	16	6	44	48	61	34
64	22	30	33	38	5	39	29

Muṣamman square with total 260

To construct a *Mustassa* square, subtract 360 from the given total, then divide by 9 with the quotient fill up 9 × 9 square. If 1 to 8 are obtained as the remainder, add 1 to the 73rd cell.

70	59	27	16	76	55	43	22	1
50	39	28	6	66	54	33	12	81
40	18	7	67	56	34	13	73	61
60	29	17	77	46	44	23	2	71
20	19	78	57	45	24	3	72	51
30	8	68	47	25	14	74	62	41
9	79	58	37	35	4	64	52	31
10	69	48	36	15	75	53	42	21
80	49	38	26	5	65	63	32	11

Mustassa square with total 369

To construct a *Ma‘shshar* square, subtract 495 from the given total, then divide by 10 and with the quotient fill up 10 × 10 square. If 1 to 9 remain as the remainder, add 1 to the 91st cell.

28	60	42	61	39	70	98	72	34	1
33	4	26	74	76	95	84	24	21	68
69	83	13	92	10	90	86	12	18	32
2	79	14	50	53	56	43	87	22	99
71	96	85	55	44	49	54	16	5	30
66	19	8	45	58	51	48	93	82	35
36	20	94	52	47	46	57	7	81	65
37	23	89	9	91	11	15	88	78	64
63	80	75	27	25	6	17	77	97	38
100	41	59	40	62	31	3	29	67	73

Ma‘shshar square with total 505

For further information regarding Muslim magic squares, see "*Islam in India* or *Qānūn-i-Islām*" by Sa‘far Sharīf, translated by G. A. Herklots.

Part III

Revised version of the Manuscript of the Third Volume of Datta and Singh

Hindu geometry *

1 General survey

1.1 Origin of Hindu geometry

The Hindu Geometry originated in a very remote age in connection with the construction of the altars for the Vedic sacrifices. The sacrifices, as described in the Vedic literature of the Hindus, were of various kinds. The performance of some of them was obligatory upon every Vedic Hindu, and hence they were known as *nitya* (or "obligatory", "indispensable"). Other sacrifices were to be performed each with the purpose of achieving some special object. Those who did not aim at the attainment of any such object had no need to perform any of them. These sacrifices were classed as *kāmya* (or "optional", "intentional"). According to the strict injunctions of the Hindu scriptures, each sacrifice must be made in an altar of prescribed shape and size. It was emphasised that even a slight irregularity and variation in the form and size of the altar would nullify the object of the whole ritual and might even lead to an adverse effect. So the greatest care had to be taken to secure the right shape and size of the altar. In this way there arose in ancient India problems of geometry and also of arithmetic and algebra. There were multitudes of altars. Let us take for instance the three primary ones, viz. the *Gārhapatya*, *Āhavanīya* and *Dakṣiṇa*, in which every Vedic Hindu had to offer sacrifices daily. The *Gārhapatya* altar was prescribed to be of the form of a square, according to one school, and of a circle according to another. The *Āhavanīya* altar had always to be square and the *Dakṣiṇa* altar semi-circular. But the area of each had to be the same and equal to one square *vyāma*.[1] So the construction of these three altars involved three geometrical operations: (i) to construct a square on a given straight line; (ii) to circle a square and vice versa; and (iii) to double a circle. The last problem is the same as the evaluation of the surd $\sqrt{2}$. Or it may be considered as a case of doubling a square and then circling it. There were altars of the shape of a falcon with straight or bent wings, of a square, an equilateral triangle, an isosceles trapezium, a circle, a wheel (with or without spokes), a

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* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 15, No. 2 (1980), pp. 121–188.

[1] 1 *vyāma* = 96 *aṅgulis* (or "finger breadths") = 2 yards.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_15

tortoise, a trough and of other complex forms all having the same area. Again at the second and each subsequent construction of an altar, it was necessary to increase its size by a specified amount, usually one square *puruṣa*,[2] but the shape was always kept similar to that of the first construction. Thus there arose problems of equivalent areas and transformation of areas. The Vedic geometers also treated problems of 'application of areas'.

1.2 Different early schools of geometry

In the course of time, Hindu geometry grew beyond its original sacrificial purpose or the bounds of practical utility and began to be cultivated as a science for its own sake. This happened in the Vedic age when different schools of geometry were founded. More notable ones amongst them were the schools of Baudhāyana, Āpastamba and Kātyāyana. Though the geometrical propositions treated in all of them were almost the same, and there were many things common in the methods of their solution, still there were other things to distinguish one school from another. Even in the solution of elementary propositions such as the construction of a square, rectangle or an isosceles trapezium, different schools had preferential liking for differential methods. The difference appears most marked in the solution of the problems of the division of figures. The large altars, of which the fundamental one was of the shape of a falcon, had to be built with 200 bricks. Geometrically, it was a case of division of a figure into 200 parts. We have described before how the different Vedic Schools of Geometry did this in different ways.

1.3 Intuitive and demonstrative geometry

Early Hindu geometers did not describe proofs of the propositions discovered by them. Only the bare results were recorded and those too in a language as concise as possible, sometimes even to the fault of ellipticity. This was, of course, in keeping with the characteristic of the Hindu race and was manifested in all their early works. Indeed the character of all the sciences of all the early nations is found to be more or less intuitive. Still the Vedic Geometry, as found in the manuals of the *Śulba*, was not wholly intuitional without any semblance of demonstration. In fact we find a kind of proof in case of certain propositions of the *Śulba*. For instance, how to find the area of a trapezium, has been demonstrated by Āpastamba in the course of the mensuration of the *Mahāvedi* which is of the shape of an isosceles trapezium whose altitude, face and base are respectively 36, 24 and 30 *padas* (or *prakramas*). He says:

> The *Mahāvedi* measures (in area) one thousand less twenty-eight (square) *padas*. Draw a straight line from the south-eastern corner

[2] 1 *puruṣa* = 120 *aṅgulis* = $2\frac{1}{2}$ yards.

of the *vedi* to a point 12 *padas* towards the south-western corner. Place the portion thus cut off on the other (i.e. the northern) side of the *vedi* after inverting it. It (the *Mahāvedi*) will then become a rectangle. After that construction the area will be apparent.[3]

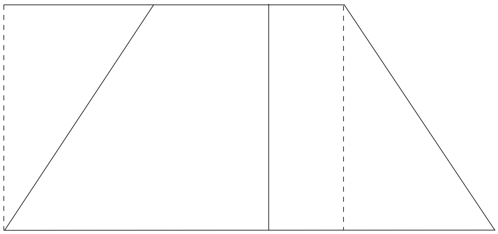

Figure 1

After the general enunciation of the theorem of the square of the diagonal, the so-called Pythagorean theorem, *Baudhāyana* observes that the truth of it will be "realised" in case of certain rational rectangles enumerated. This is an attempt for a kind of demonstration. After describing the constructions necessary in a proposition, the early Hindu geometers are found to have remarked *sa samādhiḥ* (or "This is the construction"). The significance of such an observation is obvious. It emphasises that the construction which was required to be made, has thus been actually made, and indeed corresponds to the expression *Quod Erat Faciendum* (or "what was required to do") occurring at the end of a proposition of Euclid's *Elements*. Further it discloses a rational and demonstrative attitude of the mind of the early Hindu geometers.[4]

1.4 Post-vedic geometry

The Hindu geometry which started in a brilliant way not only by going much in advance of the ancient Egyptian or Chinese geometry but also by anticipating some of the notable discoveries of the posterior Greek geometry, did not make much progress in the post-Vedic period as it ought to have done. In this period there was renaissance of Hindu Mathematics.[5] But compared with arithmetic and algebra, geometry seems to have received little impetus for further development. It will not be true to think that the study of geometry was completely neglected by the Hindus of the early renaissance period. On the other hand, it is found to have become widespread and came to be regarded as a part of general education of the people. In an early Jaina canonical

[3] *Āpastamba Śulba*, v. 7.

[4] See Datta, B., *The Science of the Śulba*, pp. 50f.

[5] See Datta, Bibhutibhusan, "The Scope and Development of the Hindu *Gaṇita*", *Ind. His. Quart.*, V, (1929), pp. 479 ff. We have drawn here heavily on this article.

work, composed circa 300 BC we find the remark, “Geometry is the lotus in Mathematics, ... and the rest is inferior.”[6] But it appears strange that we do not find evidence of much progress and improvement in geometry. The notable contributions of this period to geometry are, however, the discovery of the ellipse, elliptic cylinder, the value $\pi = \sqrt{10}$ and certain formulae for the mensuration of the segment of a circle. The value $\pi = \sqrt{10}$, though not a fairly accurate one, is an improvement upon the *Śulba* value. It occurs as early as in the *Sūryaprajñapti* (c. 500 BC).[7] The ellipse is called *viṣama-cakravāla*, in contradistinction to *cakravāla*, meaning “circle” in the *Sūryaprajñapti*,[8] and *parimaṇḍala* in the *Dhammasaṅgani* (before 350 BC)[9] and *Bhagavatī-sūtra* (c. 300 BC).[10] In the last mentioned work its form has been described as the *yavamadhya-vṛttasaṃsthāna* or “the circular figure resembling the middle (longitudinal section) of a barley corn.”[11] It seems to have been known that the ellipse is symmetrical about its either axis.[12] The mention of the elliptic cylinder, called *ghana-parimaṇḍala* (or “solid ellipse”) in contradistinction to *pratara-parimaṇḍala* (“plane ellipse”) occurs in the *Bhagavatī-sūtra*.[13]

1.5 Later Hindu geometry

Later Hindu geometry consists mainly of some mensuration formulae and solution of certain rectilinear figures such as triangles and quadrilaterals of different varieties. In some of these the Hindus undoubtedly showed considerable proficiency and indeed they obtained some remarkable results, e.g. a new proof of the Pythagorean theorem, formulae for the area and diagonals of an inscribed convex quadrilateral and rational solution of triangles and cyclic quadrilaterals. But on the whole their geometry remained empirical. There were no definitions, no postulates, no axioms, no proofs of theorems, in short, no scientific treatment of the subject. It is perhaps noteworthy that the later Hindus included geometry in their treatises of arithmetic (*pāṭīgaṇita*) more particularly in the sections on *kṣetra* (“plane figures”), *khāta* (“excavations”), *citi* (“piles of bricks”), *rāśi* (“maunds of grain”) and *krākacika* (“saw”). The last four topics are pertaining to solid figures.

[6] *Sūtrakṛtāṅga-sūtra*, 2nd *Śrutaskanda*, ch. 1, verse 154.
[7] *Sūtra* 20.
[8] *Sūtra* 19, 25, 100. See Weber, *Indische Studien*, X, p. 274.
[9] Sec. 617.
[10] *Sūtra* 726–7.
[11] *Bhagavatī-sūtra*, *Sūtra* 725. Bhuddhaghosa (350) describes it as *kukkuṭāṇda-saṃsthāna* (or “a figure of the shape of an egg of a hen”) and the *Petavattu* commentary as the *āyatavṛtta* (or “the elongated circle”).
[12] Compare *Bhagavatī-sūtra, Sūtra* 726.
[13] *Sūtra* 726.

1.6 Euclid's *Elements* in India

Though Hindu geometry is not connected with Euclid's *Elements* in any way, whether directly or indirectly, it will be interesting to know when and how it came to India. The earliest attempt, as far as known, to introduce Euclid's *Elements* into India, in the garb of Sanskrit verses, was made by the eminent Persian mathematician and traveller, Al-Bīrūnī (b. 973). But that attempt did not succeed. With the establishment of Muhammadan supremacy in India towards the close of the twelfth century of the Christian era, Arabic and Persian works on mathematics began to be brought into this country. There were very likely amongst them Arabic versions of the *Elements.* King Firuz Shah Bahmani (1397–1422), we are informed by Ferishta, was used to hear on three days in a week, lectures on botany, geometry and logic.[14] A son of Daud Shah was very fond of *Taḥrīr-u-Uqlidas* (Euclid's *Elements*) and used to teach it regularly to his students.[15] Akbar (1575) included it into the course of study for the school boys.[16] In his *Ain-i-Akbari*, Abul Fazl (1590) has referred to a few propositions of the *Elements* in a way which shows his thorough acquaintance with the work. The work, however, remained confined to the circle of Moslem schools in India. We do not find any trace of its influence in any work of a Hindu writer before the middle of the seventeenth century. In 1658 AD Kamalākara, the court-astronomer of the Emperor Jahangir of Delhi, wrote a voluminous treatise on astronomy entitled *Siddhānta-tattva-viveka.* Certain passages in this work can be easily recognised to have been adapted from Euclid's *Elements.*[17] The first complete translation of the work in Sanskrit was made in 1718 AD under the title *Rekhāgaṇita* ("Mathematics of lines") by Samrāṭa Jagannātha, by the order of his patron King Jaya Siṃha of Jaipur. Another Sanskrit version is known as the *Siddhānta-Cūḍāmaṇi.* The author of this version is still unknown.

2 Hindu names for geometry

The Hindu name for the science of geometry has varied from time to time.[18] The earliest name was *śulba.* It is at least as old as the *Śrautasūtra* of Āpastamba (c. 1000 BC). Geometry was then sometimes also called *rajju*, as is evident from the opening aphorism of the *Śulba* of Kātyāyana, "I shall speak of the collection of (rules regarding) the *rajju*". In the *Mānava Śulba* and

[14] Law, N. N., *Promotion of Learning in India during Muhammadan Rule* (by Muhammadans), 1916, p. 84.
[15] *Ibid*, p. 81, footnote 1.
[16] Abul Fazl's *Ain-i-Akbari*, English translation by Blockmann, p. 279.
[17] See *Siddhānta-tattva-viveka*, iii. 22 ff.
[18] Datta, Bibhutibhusan, "Origin and history of the Hindu names for Geometry", *Quellen und. Studien z. Gesh. d. Math.*, Ab. B; Bd, I, pp. 113–9.

Maitrāyaṇīya Śulba we get the name *Śulba-vijñāna* ("The Science of the *Śulba*") for the science of geometry. In the early canonical works of the Jainas (500–300 BC) the more common name for geometry is found to be *rajju*.

The Sanskrit words *śulba* and *rajju* have the identical significance, which is ordinarily "a rope", "a cord". The word *śulba* (or *śulva*) is derived from the root *śulb* (or *śulv*) meaning "to measure" and hence its etymological significance is "measuring" or "act of measurement". From that it came to denote "a thing measured" and consequently " a line (or surface)" as well as "an instrument of measurement" or "the unit of measurement". Thus the terms *śulba* and *rajju* have four meanings: (i) mensuration—the act and process of measuring; (ii) line (or surface)—the result obtained by measuring; (iii) a measure—the instrument of measuring; and (iv) geometry—the science of measurement. Mention of a linear measure, called *rajju* is found in the *Āpastamba-śulba*, *Mānava-śulba*, *Arthaśāstra* of Kauṭilya and later on in the *Śilpa-śāstra*. In fact in ancient India, there were three kinds of measures—linear, superficial and voluminal—having the same epithet *rajju*. In the Jaina canonical works they are sometimes distinguished as *sūcī-rajju* ("needle-like or linear *rajju*"), *pratara-rajju* ("superficial *rajju*") and *ghana-rajju* ("cubic *rajju*"). In the *Arthaśāstra* of Kauṭilya the superficial unit of *rajju* is called *pardeśa* and the cubical unit *nivartana*. In the works on the *Śulba*, we find the use of the word *rajju* in the sense of a measuring tape as also of a line.

In later times, geometry was called by the Hindus *kṣetra-gaṇita* ("Mathematics of the *kṣetra*"). This term appears in the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In this work the term *kṣetra* denotes a plane figure. In the mathematical treatises of Brahmagupta (628), Śrīdhara (900) and Bhāskara II (1150), the section devoted to the treatment of plane figures is called *kṣetra-vyavahāra* ("Treatment of plane figures"). The epithet *kṣetra-gaṇita* occurs as early as the works of Siddhasena Gaṇi (550). There the term *kṣetra* has a wider connotation so as to include both areas and volumes. In the same significance it appears in the title of the Jaina cosmographical works called *kṣetra-samāsa*. We think that the term *kṣetra-gaṇita* had a wider connotation in the beginning so as to include the geometry of plane as well as solid figures. But in later times, when the two branches of geometry began to be treated separately, the old name was reserved only for the geometry of plane figures.

Jagannātha (1718) called his version of Euclid's *Elements* the *Rekhāgaṇita* ("Mathematics of lines"). Bāpūdeva Śāstri preferred the name *kṣetra-miti* ("Measurement of areas and volumes"). He seems to have intended an accurate translation of the Greek name, but it is less scientific. For the Greek science is indeed the geometry of lines, but not the geometry of areas and volumes. Jagannātha's epithet is more in keeping with the spirit of the Greek geometry. He had probably discarded the Greek epithet intentionally as it is a misnomer.

In some of the modern vernacular tongues of India, geometry is now more commonly known as *kṣetra-tattva* ("Principles of areas and volumes") or *jyāmiti.* This latter term is highly interesting because it is very alike the Greek term "geometry", not only phonetically but also in significance, and at the same time it is not a hinduised Greek word. The word *jyāmiti* is a compound of pure Sanskrit origin derived from *jyā*, meaning 'earth' and *miti*, meaning "measure". Hence its literal significance is "earth-measurement". It is thus clearly a translation of the Greek name.

One who was well versed in the science of geometry was called in ancient India as *saṃkhyājña* ('the expert in numbers'), *parimāṇajña* ('the expert in measuring'), *sama-sūtra-nirañchaka* ('uniform-rope-stretcher'), *śulba-vid* ('the expert in the *śulba'*) and *śulba-paripṛcchaka* ('the inquirer into the *śulba*'). In the *Śilpa-śāstra*, he is spoken of as the *sūtra-grāhī* or *sūtra-dhāra* ('rope-holder') and he is further described as an expert in alignment (*rekhājña*, lit. 'one who knows the line'). In the early *Pāli* literature we find the terms *rajjuka* and *rajju-grāhaka* ('rope-holder') for the king's land-surveyor. The first of these terms appears copiously in its various case-endings, in the inscriptions of the Emperor Aśoka (250 BC).

3 Technical terms

3.1 Line

The history of a few technical terms of Hindu geometry will be considered here. There is no attempt to define those terms in any early work. Only in a work of the seventeenth century, *Siddhānta-tattva-viveka* of Kamalākara (1658), we come across some definitions but, as already stated, it was influenced by Euclid's *Elements.* The line is called in the *śulba*, *rekhā* or *lekhā*, both the terms being identical as, according to the rules of Sanskrit grammar, the letters *r* and *l* can replace each other. In posterior geometry we, however, commonly meet with the term *rekhā* only. A straight line is distinguished with the help of the qualifying adjective *ṛju* or *sarala*, meaning "straight".

3.2 Rectilinear figures

In Hindu geometry, we find two different systems of nomenclature for the rectilinear geometrical figures.[19] In one system the naming is according to the number of sides of the figures and the names are formed by juxtaposition of the number names with *bhuja*, meaning "arm", "side"; e.g. *tribhuja* ('trilateral'), *catur-bhuja* ('quadrilateral'), *pañca-bhuja* ('pentalateral'), *ṣaḍ-bhuja* ('hexa-lateral'). In the other, the naming is based on the number of angles

[19]The subject has been treated fully in an article of Datta, B. *JASB* (new series), Vol. XXVI (1930), pp. 283–299; see also his *Śulba*, pp. 221–6.

and corners in the figures, and the names are compounds of number names with *karṇa* or *koṇa*. The Sanskrit word *karṇa* means the ear. Applied to geometrical figures, it implies, the angle.[20] In the *Katyāyana Śulba*[21] (c. 500 BC), we find the terms *trikarṇa* ('triangle'), *pañca-karṇa* ('pentangle'). The word *karṇa* degenerated into *koṇa* in the *Prākṛta* languages.[22] So in the Ardha-Māgadhī work, *Sūryaprajñapti*[23] (c. 500 BC), we get *tri-koṇa* ('trigonon'), *catuṣkoṇa* ('tetragonon'), *pañca-koṇa* ('pentagon'), etc. These terms are, however, accepted in posterior Sanskrit literature.[24] The oldest Hindu compound name for rectilinear figures ending with *srakti* meaning the angle or corner, is *catuḥsrakti* ('quadrangle') which occurs in the *Saṃhitās* and the *Brāhmaṇas* (c. 2000 BC). In the time of the *Śrauta-sūtra* (c. 2000–1500 BC), was introduced another kind of name consisting of compounds of number names with *aśra* or *asra*, e.g. *tryasra*, *caturasra*, etc. Though these words *aśra* and *asra*, ordinarily mean "corner" or "angle", in compound names for rectilinear figures, they are sometimes found to denote "side". It is perhaps noteworthy that like the early Hindus, the early Greeks also followed the usage of naming the rectilinear figures according to the number of sides as well as of angles.[25] But while with the Hindus the angle-nomenclature is older than the side-nomenclature, with the Greeks quite the contrary is the case.[26]

Triangles are classified according to the sides: *sama-tribhuja* ('equilateral triangle'), *dvisama-tribhuja* ('isosceles triangle') and *viṣama-tribhuja* ('scalene triangle'). The classification according to the angles is not found here. Only the right-angled triangle is called by the name *jātya-tribhuja* by Brahmagupta and others.[27] The oblique triangles are grouped according as the perpendicular (*lamba*) from the vertex on the base falls inside or outside the figure, viz. *antarlamba* ('in-perpendicular') and *bahir-lamba* ('out-perpendicular'). In the *Taittirīya Saṃhitā* (c. 3000 BC), the *Brāhmaṇa* (c. 2000 BC) and the *Śulba*, an isosceles triangle is called *prauga*, derived probably from *pra* + *yuga*, meaning "the fore part of the shafts of a chariot". A rhombus is similarly called

[20]The term *karṇa* is used to denote the hypotenuse of a right-angled triangle (*vide infra*).

[21]iv. 7–8.

[22]Some writers are of opinion that the word *koṇa* is derived from Greek sources, but we do not think so.

[23]*Sūtra* 19, 25.

[24]See for instance, *Pariśiṣṭas of the Atharva-Veda*, xxiii. 1; 5; xxv, 1, 3, 6, 7, etc.; *Artha-śāstra* of Kauṭilya, ii. 11, 29.

[25]Tropfke, J. *Geschichte der Elementar-Mathematik*, (1923), Bd. IV. pp. 60–1.

[26]The conjecture of S. Gandz that "the observation of the corners and angles and the classification according to their number seem to be distinctly Greek, a specific invention of the Greek science, based upon the introduction of angle-geometry" is erroneous. *Vide* his article on "The origin of angle-geometry" in *Isis*, XII, pp. 452–481; more particularly p. 473.

[27]The Sanskrit word *jātya* means "noble", "well-born", "genuine". The name *jātya-tribhuja* for the right-angled triangle seems to imply that all other triangles are derived from it.

ubhayataḥ prauga (‘*prauga* on both sides’).[28]

In the *Śulba*[29] the diagonal of a rectilinear figure is called the *akṣṇa* or *akṣṇayā* (‘that which goes across or transversely’, i.e. ‘the cross line’); also *karṇa*, meaning ‘the line going across the *karṇa* or angle’, or ‘the line going across from corner to corner’. Referring to the instrument of measurement, it is sometimes termed the *akṣṇayā-veṇu* (‘diagonal bamboo-rod’) or *akṣṇayā-rajju* (‘diagonal cord’). Out of all these only the term *karṇa* has survived, others have become obsolete.

The classification of quadrilaterals according to the sides as well as the angles is found as early as the *Sūryaprajñapti.* There are generally distinguished five kinds of quadrilaterals; *sama-caturbhuja* (‘square’), *āyata-caturbuja* (‘rectangle’), *dvisama-caturbhuja* (‘isosceles trapzium’), *trisama-caturbhuja* (‘equitrilateral trapezium’), and *viṣama-caturbhuja* (‘quadrilateral of unequal sides’). Similarly we have the *sama-caturasra*, *āyata-caturasra*, *dvisama-caturasra* and *viṣama-caturasra* for those figures (*caturbhuja* = *caturasra* = quadrilateral.) In the *Śulba*, the square is generally called *sama-caturasra* and the rectangle *dīrgha-caturasra* (‘longish quadrilateral’).

3.3 Circle

In early geometry, the circle was termed *maṇḍala* (‘round’) or *pari-maṇḍala* (‘round on all sides’); the circumference, *pariṇāha* (‘surrounding boundary line’); the diameter, *viṣkambha* or *vyāsa* (‘breadth’); and the centre, *madhya* (‘middle’). The last term had, however, wider use so as to denote the middle most point of a square, rectangle or line. So also the terms *viṣkambha* and *vyāsa.* In Prākṛta works of the fourth century before the Christian era, the term *pari-maṇḍala* is used to denote the ellipse.[30] In later geometry, the term for the circle is *vṛtta*[31] and for the centre *kendra.*[32] The significance of the terms *vyāsa* and *viṣkambha* has now become fixed for the diameter of a circle. The radius is called *vyāsārdha* or *viṣkambhārdha* (‘semi-diameter’). These terms occur as early as the works of Umāsvāti (c. 150).[33] Still earlier in the *Āpastamba Śulba*, we find the term *ardha-vyāyāma*, having the identical significance.

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[28] Datta, *Śulba*, pp. 223f.
[29] *Ibid*, pp. 224f.
[30] *Dhammasaṅgani* 617; *Bhagavatī-sūtra, Sūtra* 724–6. See Datta, *Hindu Contribution to Mathematics*, p. 8.
[31] See *Bhagavatī-sūtra, Sūtra* 724–6.
[32] In Hindu astronomy the term *kendra* is used to signify the anomaly.
[33] See his *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14; *Jambūdvīpa-samāsa*, ch. iv.

3.4 Surface and area

In the early Hindu geometry, a plane surface bounded by a figure was called by the term *kṣetra* and its area by *bhūmi*. Occasionally, however, the term *kṣetra* was employed also to signify area. In the canonical works of the Jainas (500–300 BC), a plane surface is termed *pratara* ('expanse'), and it is defined as that which is obtained by multiplying line by line. In posterior geometry, the *bhūmi*, together with its synonyms *bhū*, *mahī*, etc., signifying earth, denotes the ground or base of a plane figure; the area is called *kṣetraphala*, *kṣetra-gaṇita* or simply *phala*, or *gaṇita*. These terms carry the concept of specific operations of mensuration by breaking up the figure into smaller portions and calculating them so that the area is what is obtained as the result (*phala*) of such calculation (*gaṇanā*). Another term is more explicit. It is *sama-koṣṭha-miti* ('the measure of like compartments' or 'the measure of the number of equal squares'). A curved surface or surface of a solid is called its *pṛṣṭha* ('back'), from *dharā-pṛṣṭha* (or 'the back of the earth') which is rounded. The term for the superficial area of a solid is *pṛṣṭha-phala*.

4 Typical propositions of early geometry[34]

The *Śulba-sūtras*, which form a part of the Vedic literature of the Hindus, deal with the construction of fire altars for sacrificial purposes. At present we know of seven *Śulba-sūtras*, although it is quite likely that many more such works existed in ancient times. According to European scholars, these *Sūtras* were composed in the period 800 to 500 BC, but they are probably much older. The *vedīs* ('altars') dealt with in these *sūtras* are of various forms. Their construction requires a knowledge of the properties of the square, the rectangle, the rhombus, the trapezium, the triangle and the circle. The geometrical propositions involved in the constructions are the following.

4.1 Constructions

1. To divide a line into any number of equal parts.[35]
2. To divide a circle into any number of equal areas by drawing diameters.[36]
3. To divide a triangle into a number of equal and similar areas.[37]

[34] For details consult Datta, B., *The Science of the Śulba*, Calcutta, (1932).
[35] The knowledge of this construction is throughout assumed. It was probably done by drawing parallels, as in Euclid. The following construction shows this surmise to be correct.
[36] *BŚl*, ii. 73–4; *ĀpŚl*, vii. 13–14.
[37] *BŚl*, iii. 256; See Datta, *Śulba*, p. 46.

4. To draw a straight line at right angles to a given line.[38]
5. To draw a straight line at right angles to a given straight line from a given point on it.[39]
6. To construct a square on a given side.[40]
7. To construct a rectangle of given sides.[41]
8. To construct an isosceles trapezium of given altitude, face and base.[42]
9. To construct a parallelogram having given sides at a given inclination.[43]
10. To construct a square equal to the sum of two different squares.[44]
11. To construct a square equivalent to two given triangles.[45]
12. To construct square equivalent to two given pentagons.[46]
13. To construct a square equal to a given rectangle.[47]
14. To construct a rectangle having a given side and equivalent to a given square.[48]
15. To construct an isosceles trapezium having a given face and equivalent to a given square or rectangle.[49]
16. To construct a triangle equivalent to a given square.[50]
17. To construct a square equivalent to a given isosceles triangle.[51]
18. To construct a rhombus equivalent to a given square or rectangle.[52]
19. To construct a square equivalent to a given rhombus.[53]

[38] *KŚl*, i. 3.
[39] *Ibid.*
[40] *ĀpŚl*, viii. 8–10; xi. 1; i. 7; i. 2; *BŚl*, i. 22–28, 29–35, 42–44; iii. 13. *TS*, v, 2.5.1.; ff. *MaiS*, iii. 2.4; *KṭS*, xx. 3.4; *KapS*, xxxii. 5.6; *ŚBr* x. 2.3.8 (2000 BC), etc.
[41] *BŚl*, i. 36–40.
[42] *BŚl*, i. 41; *ĀpŚl*, v. 2–5.
[43] *ĀpŚl*, xix. 5.
[44] *BŚl*, i. 51–52; *ĀpŚl*, ii. 4–6; *KŚl*, ii. 22, iii. 1.
[45] This follows from the above.
[46] *BŚl*, iii. 68, 288; *KŚl*, iv. 8.
[47] *BŚl*, i. 58, *ĀpŚl*, ii. 7; *KŚl*, iii. 2, 3.
[48] *ĀpŚl*, iii. 1, *BŚl*, i. 53.
[49] *BŚl*, i. 55; *ŚBr*, x. 2.1.4.
[50] *BŚl*, i. 56.
[51] *KŚl*, iv. 5.
[52] *BŚl*, i. 57; *ĀpŚl*, xii. 9; *KŚl*, iv. 4.
[53] *KŚl*, iv. 6.

4.2 Theorems

The following theorems are either expressly stated or the results are implied in the methods of construction of the altars of different shapes and sizes:

1. The diagonals of a rectangle bisect each other. They divide the rectangle into four parts, two and two (vertically opposite) of which are equal in all respects.[54]

2. The diagonals of a rhombus bisect each other at right angles.

3. An isosceles triangle is divided into two equal halves by the line joining the vertex to the middle point of the base.[55]

4. The area of a square formed by joining the middle points of the sides of a square is half that of the original one.

5. A quadrilateral formed by the lines joining the middle points of the sides of a rectangle is a rhombus whose area is half that of the rectangle.

6. A parallelogram and rectangle on the same base and within the same parallels have the same area.

7. The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

8. If the sum of the squares on two sides of a triangle be equal to the square on the third side, then the triangle is right-angled.

4.3 The Baudhāyana theorem

Theorem 7 given above has been stated by Baudhāyana (c. 800 BC) in the following words:

> The diagonal of a rectangle produces both areas which its length and breadth produce separately.[56]

Āpastamba[57] and Kātyāyana[58] give the above theorem in almost identical terms. The theorem is now universally associated with the name of the Greek Pythagoras (c. 540 BC) though "no really trustworthy proof exists that it was actually discovered by him".[59] The Chinese knew the numerical relation for

[54] Implied in *BŚl*, iii. 168–9, 178.
[55] *BŚl*, iii. 256.
[56] *BŚl*, i. 48: दीर्घचतुरस्रस्याक्ष्णयारज्जुः पार्श्वमानी तिर्यङ्मानी च यत्पृथग्भूते कुरुतस्तदुभयं करोति।
[57] *ĀpŚl*, i. 4.
[58] *KŚl*, ii. 11.
[59] Heath, *Greek Math.*, Vol. I, p. 144f.

the particular case $3^2 + 4^2 = 5^2$ probably in the time of Chou-Kong (d. 1105 BC).[60] The *Kahun* Papyrus (c. 2000 BC) contains four similar numerical relations, all of which can be derived from the above one.[61] As for the Hindus, one instance of that kind, $39^2 = 36^2 + 15^2$, occurs in the *Taittirīya Saṃhitā*[62] (before 2000 BC). It should be noted that this instance is different from that known to other early nations.

Although particular instances of the theorem are found amongst several ancient nations, the first enunciation of the theorem in its general form is found in India. It cannot be said what made Baudhāyana give the theorem in the general form. It is not improbable that he possessed a proof of the theorem. But what this proof was will never be known with certainty. Bürk, Hankel, Thibaut and Datta are of opinion that Baudhāyana knew a proof of the theorem.[63] It is conjectured that this proof may have been one of the following.

4.4 Hindu proofs

(i) Let $ABCD$ be a given square. Draw the diagonal AC; produce AB and cut off AE equal to AC (Figure 2). Construct the square $AEFG$ on AE. Join DE and on it construct the square $DHME$. Complete the construction as indicated in Figure 2. Now the square $DHME$ is seen to be comprised of four right-angled triangles each equal to DAE and the small square $ANPQ$. This square will be easily recognised to be equal to the square $CRFS$ and triangles equal to the rectangles $AERD$ and $ABSG$. Therefore, the square $DHME$ is equal to the sum of the squares $ABCD$ and $AEFG$. Hence the theorem.

It might be mentioned that constructions like the above are necessary in the usual course in the *Śulba*.

(ii) Let ABC be a right-angled triangle (**ed.** see Figure 3) of which the angle C is a right-angle. From C draw the perpendicular CD on AB. Then the triangle ABC, ACD and CBD are similar. Therefore,

$$AB : AC :: AC : AD,$$

or $AC^2 = AB \times AD$. Similarly, $CB^2 = AB \times DB$. Adding we get

$$AC^2 + CB^2 = AB^2.$$

[60] Mikami. Y., *The Development of Mathematics in China and Japan*, Leipzig (1913), p. 7.
[61] These are $1^2 + \left(\frac{3}{4}\right)^2 = \left(1\frac{1}{4}\right)^2$, $2^2 + \left(1\frac{1}{2}\right)^2 = \left(2\frac{1}{2}\right)^2$, $8^2 + 6^2 = 10^2$, $16^2 + 12^2 = 20^2$.
[62] vi. 2.4.6.; It also occurs in the *Śatapatha Brāhmaṇa*, x. 2.3.4.
[63] Datta, *The Science of the Śulba*, ch. ix.

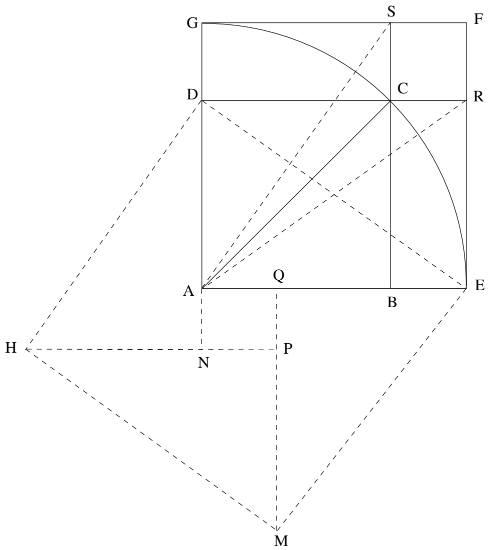

Figure 2

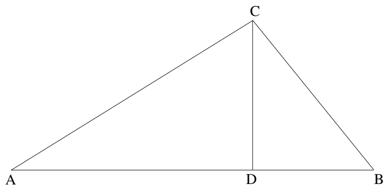

Figure 3

This proof is given by Bhāskarācārya,[64] and does not occur in the west until 1693 when it was rediscovered in Europe by Wallis.

(iii) Let a, b, c be the sides of a right-angled triangle. Taking four such triangles they are arranged as in Figure 4a, inside a square whose side is equal to the hypotenuse of the given triangle. Obviously then,

$$c^2 = 4\left(\frac{ab}{2}\right) + (b-a)^2 = a^2 + b^2.$$

This proof was anticipated by the Chinese by several centuries.[65]

The technique employed in this proof was used by Āpastamba for the enlargement of a square. Thus to construct a square whose side will

[64] *Cf.* Colebrooke, *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, London, 1817, pp. 221–2.
[65] Mikami, *l. c.*, p. 5.

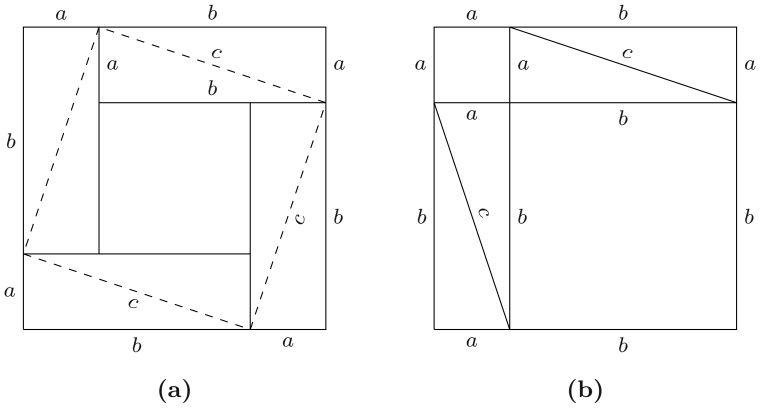

Figure 4

exceed a side b of a given square by a, add, says Āpastamba, on the two sides of the given square two rectangles whose lengths are equal to b and breadths to a; then add on the corner a square whose sides are equal to the increment a. Thus will be obtained a square with a side equal to $a + b$ (Figure 4b). A similar method was taught by Baudhāyana.[66]

4.5 Particular case

The particular case of the above theorem relating to the diagonal of a square has been stated thus:

> The diagonal of a square produces an area twice as much.

The statement is given in all the *Śulbasūtras*[67] and the theorem has been used for "doubling the square" at several places. Instances of its use are found in the *Taittirīya* (before 2000 BC) and other *Saṃhitās*, and can be traced back to the *Ṛgveda* (before 3000 BC).

Thibaut says:

> The authors of the *sūtras* do not give us any hint as to the way in which they found their proposition regarding the diagonal of a square; but we suppose that they, too, were observant of the fact that the square of the diagonal is divided by its diagonals into four triangles, one of which is equal to half the first square (Figure 5). This is at the same time an immediately convincing proof of the

[66] See Datta, *The Science of the Śulba*, p. 117.
[67] *BŚl*, i. 45; *ĀpŚl*, i. 5; *KŚl*, ii. 12; etc.

> Pythagorean proposition as far as squares or equilateral rectangular triangles are concerned.[68]

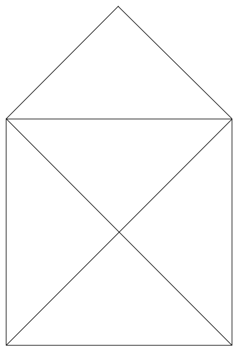

Figure 5

5 Measurement of triangles

5.1 Area of a triangle

The method for finding the area of a triangle that was known in the *Śulba*[69] was

$$\text{Area} = \frac{1}{2}(\text{base} \times \text{altitude}),$$

and that was one of the methods followed in later times. Āryabhaṭa I says:

> The area of a triangle is the product of the perpendicular and half the base.[70]

According to Brahmagupta:

> The product of half the sums of the sides and counter-sides of a triangle or a quadrilateral is the rough value of its area. Half the sum of the sides is severally lessened by the three or four sides, the square-root of the product of the remainders is the exact area.[71]

That is to say, if a, b, c, d, be the four sides of a quadrilateral taken in

[68] Thibaut, *Śulbasūtras*, p. 8.
[69] See Datta, *The Science of the Śulba*, p. 96.
[70] *Ā*, i. 6.
[71] *BrSpSi*, xii. 21.

order, we have

$$\text{Area} = \frac{c+d}{2} \times \frac{a+b}{2}, \text{ roughly;}$$
$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \text{ exactly,}$$

where

$$s = \frac{1}{2}(a+b+c+d).$$

In case of a triangle $d = 0$; so that we get

$$\triangle = \frac{c}{2} \times \frac{a+b}{2}, \text{ roughly;}$$
$$\triangle = \sqrt{s(s-a)(s-b)(s-c)} \text{ exactly.}$$

The second formula was given before by the Greek Heron of Alexandria (c. 200).[72] Pṛthūdakasvāmi calculates by these methods the area of the triangle (14, 15, 13) to be 98 roughly, 84 exactly.

Śridhara says that the exact value of a triangle will be given by the formulae[73]

$$\triangle = \frac{1}{2} (\text{base} \times \text{altitude}),$$
$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Mahāvīra,[74] Āryabhaṭa II,[75] and Śrīpati[76] teach both these accurate methods as well as the rough one of Brahmagupta. Bhāskara II[77] adopts the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

5.2 Segments and altitudes

Bhāskara I (629) writes:

> In a triangle the difference of the squares of the two sides or the product of their sum and difference is equal to the product of the sum and difference of the segments of the base. So divide it by the base or the sum of the segments; add and subtract the quotient to and from the base and then halve, according to the rule of concurrence. Thus will be obtained the values of the two

[72] Heath, *History of Greek Mathematics*, II, p. 321.
[73] *Triś*, R. 43.
[74] *GSS*, vii. 7, 50.
[75] *MSi*, xv. 66, 69, 78.
[76] *SiŚe*, xiii. 30.
[77] *L*, p. 41.

> segments. From the segments of the base of a scalene triangle, can be found its altitude.[78]

That is to say

$$a^2 - b^2 = (a+b)(a-b) = c_1^2 - c_2^2 = (c_1 + c_2)(c_1 - c_2),$$

also

$$c_1 + c_2 = c.$$

Therefore

$$c_1 - c_2 = \frac{a^2 - b^2}{c}.$$

Hence

$$c_1 = \frac{1}{2}\left(c + \frac{a^2 - b^2}{c}\right),$$
$$c_2 = \frac{1}{2}\left(c - \frac{a^2 - b^2}{c}\right),$$
$$h = \sqrt{a^2 - c_1^2} = \sqrt{b^2 - c_2^2}.$$

By means of these formulae Bhāskara I finds the segments (9, 5; 35, 16) of the bases (14, 51), altitudes (12, 12) and areas (84, 306) of the scalene triangles (13, 15, 14) and (20, 37, 51).

Brahmagupta (628) gives the same set of formulae. He says:

> The difference of the squares of the two sides being divided by the base, the quotient is added to and subtracted from the base; the results, divided by two, are the segments of the base. The square-root of the square of a side as diminished by the square of the corresponding segment is the altitude.[79]

Pṛthūdakasvāmi proves these formulae in the same way as Bhāskara I and also applies them to the latter's first example (13, 15, 14).

Śrīdhara first finds the area of the triangle by means of the formula

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}$$

and then deduces the segments and perpendicular. His rules are:

> Twice the area of the triangle divided by the base is the altitude. (Then there will be two right-angled triangles of which) the uprights are equal to that altitude, bases are the segments and hypotenuses, the two sides (of the given triangle).[80]

[78] *Vide* his commentary on *Ā*, ii. 6.
[79] *BrSpSi*, xii. 22.
[80] *Triś*, R. 50.

Mahāvīra says:

> Divide the difference between the squares of the two sides by the base. From this quotient and the base, by the rule of concurrence, will be obtained the values of the two segments (of the base) of the triangle; the square-root of the difference of the squares of a segment and its corresponding side is the altitude: so say the learned teachers.[81]

Āryabhaṭa II writes:

> In a triangle, divide the product of the sum and difference of the two sides by the base. Add and subtract the quotient to and from the base and then halve. The results will be the segments corresponding to the greater and smaller sides respectively. The segment corresponding to the smaller side should be considered negative, if it lies outside the figure. The square-root of the difference of the squares of a segment and its corresponding side is the perpendicular.[82]

Similar rules are given by Śrīpati[83] and Bhāskara II.[84] The latter gives in illustration a case of a scalene triangle whose have is 9, and sides 10, and 17. There the segments are 6 and 15, and perpendicular 8.

5.3 Circumscribed circle

Brahmagupta says:

> The product of the two sides of a triangle divided by twice the altitude is the heart-line (*hṛdaya-rajju*). Twice it is the diameter of the circle passing through the corners of the triangle and quadrilateral.[85]

Pṛthūdakasvāmi proves it substantially as follows:

Let ABC be a scalene triangle (**ed.** see Figure 6). Draw AD perpendicular to BC. Produce it to A' making $A'D = AD$. Let O be the centre of the circle circumscribing the triangle ABC. Join OA, OC. Triangles BAA' and OAC are similar. Therefore,

$$AB : OA :: AA' : AC.$$

[81] *GSS*, vii. 49.
[82] *MSi*, xv. 76–7.
[83] *SiŚe*, xii. 29.
[84] *L*, p. 40.
[85] *BrSpSi*, xii. 27.

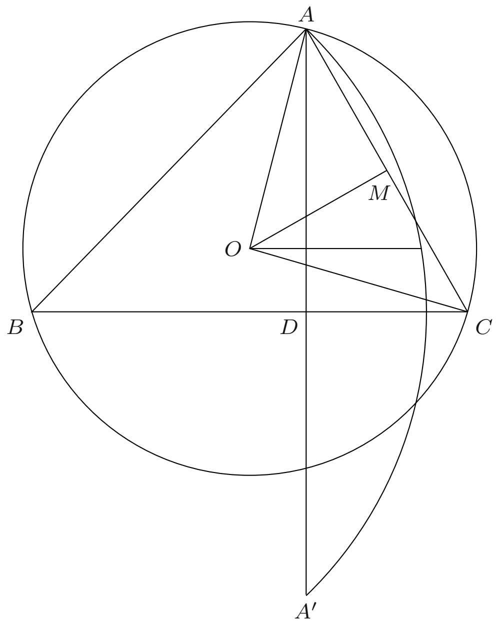

Figure 6

Hence,

$$OA = \frac{AB \times AC}{AA'},$$

or,

$$R = \frac{cb}{2h},$$

where R denotes the radius of the circumscribed circle.

Mahāvīra writes:

> In a triangle, the product of the two sides divided by the altitude is the diameter of the circumscribed circle.[86]

Example:[87] The circum-diameter of the triangle (14, 13, 15) is $16\frac{1}{4}$.

Śrīpati states:

> Half the product of the two sides divided by the altitude is the heart-line.[88]

[86] *GSS*, vii. $213\frac{1}{2}$.
[87] *GSS*, vii. $219\frac{1}{2}$.
[88] *SiŚe*, xiii. 31.

5.4 Inscribed circle

To find the radius of a circle inscribed in a triangle (or quadrilateral, when possible) whose area as well as perimeter are known, Mahāvīra gives the following rule:

> Divide the precise area of a figure other than a rectangle by one-fourth of its perimeter; the quotient is stated to be the diameter of the inscribed circle.[89]

That is to say, if r denote the radius of the circle inscribed within the triangle $(a,\ b,\ c)$, we shall have

$$r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)},$$

where

$$2s = a + b + c.$$

5.5 Similar triangles

The properties of similar triangles and parallel lines were known to the ancient Hindus.[90] For example, take the case of the Mount Meru or Mandara. It has been described in the early canonical works of the Jainas as follows:

> At the centre of Jambūdvīpa, there is known to be a mountain, Mandara by name, whose height above (the earth) is 99000 *yojanas*, whose depth below is 1000 *yojanas*, its diameter at the base is $10090\frac{10}{11}$ *yojanas*, at the ground 10000 *yojanas*. Then (its diameter) diminishes by degrees until at the top it is 1000 *yojanas*. Its circumference at the base is $31910\frac{3}{11}$ *yojanas*, at the ground 31623 *yojanas*, and at the top a little over 3162 *yojanas*. It is broader at the base, contracted at the middle and (still) shorter at the top and is of the form of a cow's tail (i.e. a truncated right cone).[91]

To find the diameter of any other section parallel to the base, Jinabhadra Gaṇi (c. 560) gives the following rule:

> Wherever is wanted the diameter (of the Mandara): the descent from the top of the Mandara divided by eleven and then added to a thousand will give the diameter. The ascent from the bottom should be similarly (divided by eleven) and the quotient subtracted

[89] *GSS*, vii. $223\frac{1}{2}$.

[90] See Datta, Bibhutibhusan "Geometry in the Jaina Cosmography", *Quellen und Studien z. Gesch. d. Math.*, Ab. B, Bd. 1., 1930, pp. 249ff.

[91] *Jambūdvīpa-prajñapti*, *Sūtra* 103.

> from the diameter of the base: what remains will be the diameter there (i.e. at that height) of that (Mandara).[92]

It is stated further:

> Half the difference of the diameters at the top and the base should be divided by the height; that (will give) the rate of increase or decrease on one side; that multiplied by two will be the rate of increase or decrease on both sides; in going from either end of the mountain.
>
> Subtract from the diameter of the base of the mountain the diameter at any desired place: what remains when multiplied by the denominator (meaning eleven) will be the height (of that place).[93]

All these rules will follow at once from the following general formulae (**ed.** see Figure 7):

$$
\begin{aligned}
a &= \frac{D-d}{2h}x,\\
\delta &= a + \frac{D-d}{h}x,\\
y &= (D-\delta')\frac{h}{D-d},\\
b &= \frac{D-d}{2h}y,\\
\delta' &= D - \frac{D-d}{h}y.
\end{aligned}
$$

Rules similar to those stated above and hence the general properties leading to them, were known to the people long before Jinabhadra Gaṇi. For as early as the second century before the Christian era (or after) Umāsvāti correctly observed that in case of the Mount Meru, "for every ascent of 11000 *yojanas*, the diameter diminishes by 1000 *yojanas*."[94]

Again, "Half the difference between the breadths at the source and the mouth being divided by 45000 *yojanas*, and the quotient multiplied by two will give the rate of increase (of the breadth) on both sides, in case of rivers."[95] (45000 *yojanas* is the length of a river).

They are found even in the early canonical works (500–300 BC). According to the Jaina cosmography, the Salt Ocean is annular in shape, having a breadth of 200000 *yojanas*. In the undisturbed state its height as well as

[92] *Vṛhat Kṣetra-samāsa*, i. 307–8.
[93] *Ibid*, i. 309–11.
[94] *Tattvārthādhigama-sūtra-bhāṣya*, iii. 9.
[95] *Jambūdvīpa-samāsa*, ch, iv.

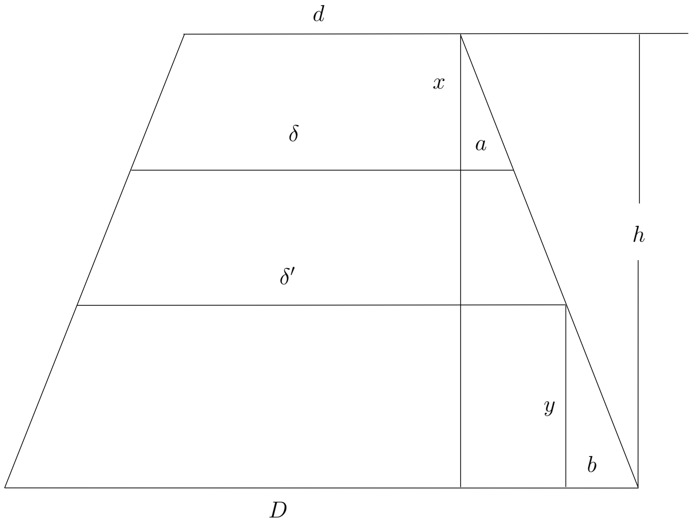

Figure 7

depth are said to be varying continuously from its either banks till at distances of 95000 *yojanas* from the banks where the height is 16000 *yojanas* and the depth 1000 *yojanas*. The radial section of the Salt Ocean in the calm state will be represented by Figure 8, where

$$AE = A'E' = 95000 \textit{ yojanas},$$
$$CE = C'E' = 16000 \textit{ yojanas},$$
$$ED = E'D' = 1000 \textit{ yojanas},$$
$$\text{and} \qquad EE' = 10000 \textit{ yojanas}.$$

It is described in the *Jivābhigama-sūtra* that "from either bank of the Salt Ocean, for proceeding every 95 *padas*, the height is known to be increased by 16 *padas* and so on, until on proceeding to 95000 *yojanas*, the height is known to be increased to 16000 *yojanas*".[96]

These can be easily verified thus:

From the properties of similar triangles

$$QR = \frac{ED \times AR}{AE} = \frac{1}{95} AR,$$
$$PR = \frac{EC \times AR}{AE} = \frac{16}{95} AR.$$

If $AR = 95x$, where x is any unit of measurement, then $QR = x$, $PR = 16x$.

[96] *Jivābhigama-sūtra*, *Sūtra* 172.

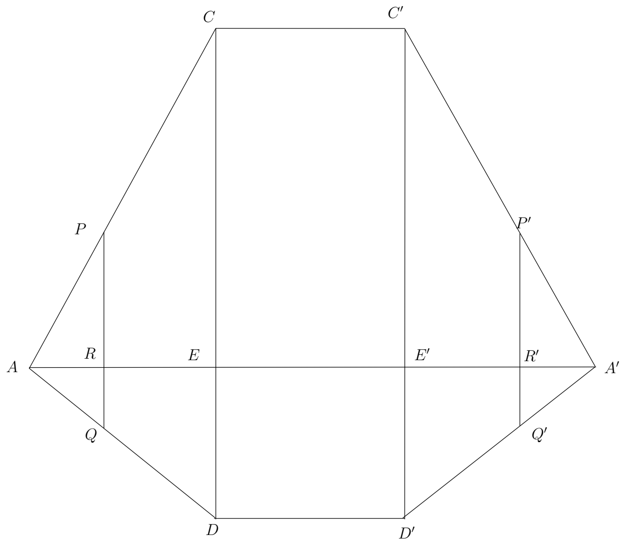

Figure 8

Again it is stated in the *Jambūdvīpa-prajñapti*[97] that at a height of 500 *yojanas* above the ground the breadth of the Mount Mandara is $9954\frac{6}{11}$ *yojanas*, while at 63000 *yojanas* above it is $4272\frac{8}{11}$ *yojanas*. These values, as can be easily verified, tally with the general formulae.

6 Measurement of quadrilaterals

6.1 Area

It should be noted at the outset that four sides alone are not sufficient to determine the true shape of a quadrilateral and consequently its size. For, there can be formed various quadrilaterals with the same four sides. Hence in order to make a quadrilateral determinate we must know, besides the sides, another element such as a diagonal, the altitude of a corner, or an angle. Thus Āryabhaṭa II remarks:

> The mathematician who wishes to tell of the area or the altitudes of a quadrilateral without knowing a diagonal, must be a fool or a blunderer.[98]

Bhāskara II writes:

[97] *Sūtra* 104–5.
[98] *MSi*, xii. 70.

> The diagonals of a quadrilateral (whose four sides are given) are uncertain. How can, then, the area be determinate? The diagonals as calculated by previous teachers will be true only in case of quadrilaterals (of a particular kind) contemplated by them, but not in case of others. For with the same (four) sides, there can be various other pairs of diagonals and consequently the area also is manifold. In a quadrilateral, when two opposite corners are so drawn as to bring the sides contiguous to them inwards, the diagonal joining them is shortened, while the other two corners bulge outwards and consequently their diagonal is lengthened. So it has been stated (just before) that with the same sides there can be other pairs of diagonals. Without specifying one of the altitudes or diagonals, how can one ask to find the other of them and also the area, as these are truly indeterminate? The questioner who does not know the indeterminate nature of a quadrilateral must be a blunderer; still more so is he, who answers such a problem.[99]

6.2 Brahmagupta's formula

To find the area (A) of an inscribed convex quadrilateral whose sides are a, b, c, d, Brahmagupta (628) gives the following formula:[100]

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where

$$2s = a + b + c + d.$$

This formula has been reproduced by Śrīdhara[101] (900), Mahāvīra[102] (850) and Śrīpati[103] (1039). None of these writers has expressly mentioned the limitation that it holds only for an inscribed figure. Still it seems to have been implied by them. So this appears from the particular remark of Bhāskara II that the formula holds only in case of a special kind of quadrilateral contemplated by them. Further we find that the examples of quadrilaterals, viz. (4, 13, 14, 13), (25, 25, 39, 25) and (25, 39, 60, 52) given by Śrīdhara[104] and Pṛthūdakasvāmi[105] and those, namely (14, 36, 61, 36), (169, 169, 407, 169)

[99] *L.* p. 44.
[100] *BrSpSi*, xii. 21.
[101] *Triś*, R. 43.
[102] *GSS*, vii. 50.
[103] *SiŚe*, xiii. 28.
[104] *Triś*, Ex. 78, 79, 80.
[105] *Vide* his commentary on *BrSpSi*, xii. 21. Elsewhere (xii. 26) he finds the circum-radii of these quadrilaterals.

and (125, 195, 300, 260) given by Mahāvīra,[106] in illustration of the above formula, are all of the cyclic variety. Bhāskara II has shown that in the other cases, the above formula gives only an approximate value of the area of a quadrilateral.[107]

6.3 Diagonal, altitude and segment

Āryabhaṭa I says (**ed.** see Figure 9):

> The two sides (severally) multiplied by the altitude and divided by their sum will give the perpendiculars let fall on them from the point of intersection of the diagonals. Half the sum of the two sides multiplied by the altitude should be known as the area.[108]

$$h_1 = \frac{ah}{a+c},$$
$$h_2 = \frac{ch}{a+c},$$
$$\text{Area} = \frac{1}{2}h(a+c).$$

Brahmagupta writes:

> In an isosceles trapezium[109] the square-root of the sum of the products of the sides and counter-sides is the diagonal. The square-root of the square of the diagonal as diminished by the square of half the sum of the face and base, is the altitude.[110]

$$d = \sqrt{ac+b^2}, \quad h = \sqrt{d^2 - \left(\frac{a+c}{2}\right)^2}.$$

> The upper and lower portions of the diagonal or the altitude at the junction of the two diagonals or of a diagonal and an altitude, will be given by the corresponding segments of the base divided by their sum and multiplied again by the diagonal or altitude, as the case may be.[111]

[106] *GSS*, vii. 57, 58, 59. Compare also vii. $215\frac{1}{2}$, $216\frac{1}{2}$, $217\frac{1}{2}$ where it is required to find the diameters of the circles circumscribing these very quadrilaterals.

[107] *L*, p. 41.

[108] *Ā*, ii. 8.

[109] The Sanskrit term is *aviṣama-caturasra*, meaning literally "the quadrilateral not of unequal sides". Brahmagupta classifies quadrilaterals (*caturasra*, *caturbhuja*) into five varieties: *sama-caturasra* (square), *āyata-caturasra* (rectangle), *dvisama caturasra* (isosceles trapezium), *trisama caturasra* (trapezium with three equal sides) and *viṣama caturasra* (quadrilateral of unequal sides). Hence *aviṣama caturasra* must mean all except those of the last class. But here more particularly the isosceles trapezium is meant.

[110] *BrSpSi*, xii. 23.

[111] *BrSpSi*, xii. 25.

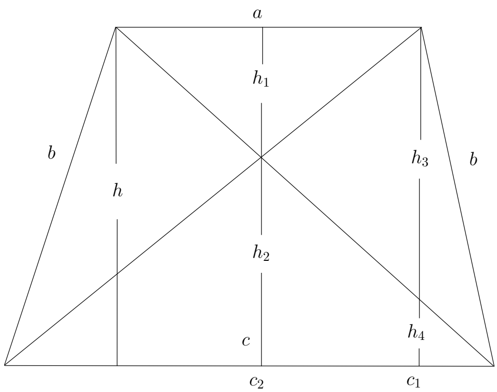

Figure 9

$$h_3 = \frac{c_2 h}{c_1 + c_2}, \quad d_1 = \frac{c_2 d}{c_1 + c_2},$$
$$h_4 = \frac{c_1 h}{c_1 + c_2}, \quad d_2 = \frac{c_1 d}{c_1 + c_2}.$$

For quadrilaterals other than isosceles trapeziums, Brahmagupta gives the following rules:

> Considering two scalene triangles within the quadrilaterals[112] by means of the two diagonals, find separately the segments of the base in them by the method taught before; and thence the two altitudes.[113]
>
> Supposing two scalene triangles within the quadrilateral, with the diagonals as bases, find in each of them separately the segments of the base. They will be the portions of the diagonals above and below their point of intersection. The lower portions of the diagonals are taken to be the sides of another triangle whose base is the same as that of the given quadrilateral. Its altitude is the lower portion of the perpendicular (to the base through the junction of the diagonals). The upper portions of it will be obtained by subtracting this portion from half the sum of the two altitudes.[114]
>
> At the intersection of the diagonals and perpendiculars, the lower segment of a diagonal and of a perpendicular can be found by proportion. On subtracting these segments from the whole, the

[112] The Sanskrit term is *viṣama caturasra*. As pointed out just before, it denotes "a quadrilateral of unequal sides" including a trapezium.
[113] *BrSpSi*, xii. 29.
[114] *BrSpSi*, xii. 30–31.

> upper portions will be found. Such is (the method) also in the needle (i.e. the intersection of two opposite sides produced) and the intersection (of a prolonged side and perpendicular).[115]

Śrīdhara states:

> To find the altitude of a trapezium,[116] suppose a triangle whose base is the difference of the base and face of the trapezium and whose sides are the same as those at the flanks of the given figure; (and then proceed as in the case of finding the altitude of a triangle).[117]

Mahāvīra's rule will be clear from the following problem with reference to which it has been defined (**ed.** see Figure 10):

AB, CD are two vertical pillars. AE, CF are two strings joining the tops A and C of the these pillars to points E and F on the ground. PQ is the perpendicular from the point of intersection of the strings. It has been named "the inner perpendicular."

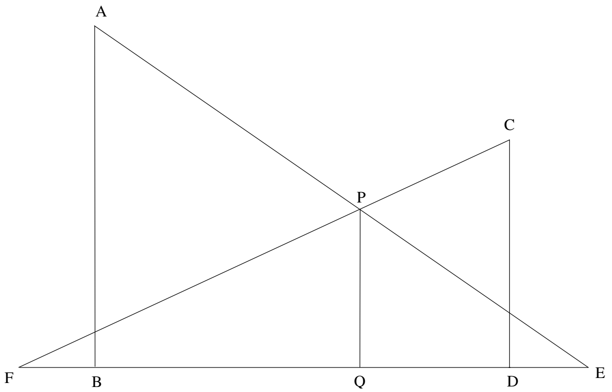

Figure 10

Mahāvīra says:

> Divide each pillar by its distance from (the farthest point of contact of) the string (with the ground), divide again the quotients by their sum and then multiply by the (whole) base. The results are the segments (of the base by the inner perpendicular). These being multiplied by the (first) quotients in the inverse order give the inner perpendicular.[118]

[115] *BrSpSi*, xii. 32.
[116] The Sanskrit term is *ṛjuvadana-caturbhuja* of "the quadrilateral with parallel face."
[117] *Triś*, R. 49.
[118] *GSS*, vii. $180\frac{1}{2}$.

That is to say, we have

$$QF = \frac{\frac{AB}{BE} \times FE}{\frac{AB}{BE} + \frac{CD}{DF}} = \frac{AB \times DF \times FE}{AB \times DF + CD \times BE},$$
$$QE = \frac{\frac{CD}{DF} \times FE}{\frac{AB}{BE} - \frac{CD}{DF}} = \frac{CD \times BE \times FE}{AB \times DF - CD \times BE},$$
$$PQ = \frac{AB}{BE} \times QE = \frac{CD}{DF} \times QF.$$

Example from Mahāvīra:[119] Find the inner perpendicular and the segments of the base caused by it in the quadrilateral (7, 15, 21, 3).

Śrīpati says:

> In an isosceles trapezium, the square-root of the sum of the products of opposite sides is the diagonal. Next I shall speak of quadrilaterals of unequal sides.[120]

Bhāskara II gives several rules. Of them we note the following:

> In a quadrilateral, assume the value of one diagonal. Then in the two triangles lying on either sides of this diagonal, it will be the base and others (i.e. the given sides of the quadrilateral) sides. Now find the perpendiculars and segments (in these triangles). Then the square of the difference of the two segments lying on the same side (i.e. taken from the same corner) being added to the square of the sum of the perpendiculars, the square-root of the resulting sum will be the second diagonal in all quadrilaterals.[121]

Ganeśa has demonstrated the rule substantially as follows (**ed.** see Figure 11):

Let $ABCD$ be a quadrilateral whose diagonal AC as well as the sides are known. Draw BN, DM perpendiculars to AC. Produce BN and draw DP perpendicular to it. Join DB. Then

$$DB^2 = BP^2 + DP^2,$$
$$= (BN + DM)^2 + (AN - AM)^2.$$

> Suppose a triangle whose base is equal to the difference of the face and base of a trapezium, and whose sides are the flank sides of the latter; then as in case of a triangle, find its altitude and segments of the base. Subtract from the base of the given trapezium one

[119] *GSS*, vii. $187\frac{1}{2}$.
[120] *SiŚe*, xiii. 33.
[121] *L*, p. 47f.

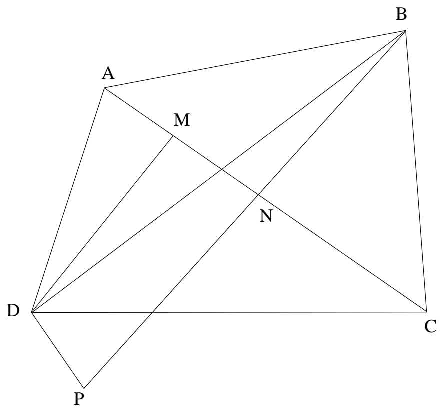

Figure 11

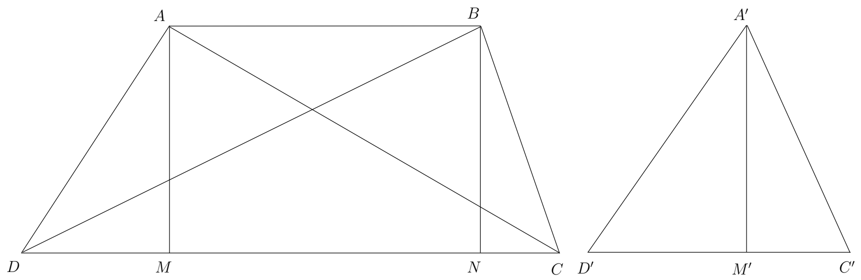

Figure 12

> of the segments. The square of the remainder being added to the square of the perpendicular, the square-root of the sum is the diagonal. In a trapezium, the sum of the base and smaller flank side is greater than the sum of the face and the other flank.[122]

Gaṇeśa's Proof (**ed.** see Figure 12): Let $ABCD$ be a trapezium. Draw the perpendiculars AM, BN. Combine the two triangles ADM and BCN into one triangle $A'C'D'$. Then the altitude $A'M'$ of the new triangle is equal to the altitude of the trapezium.

Join AC and BD. Then

$$AC^2 = AM^2 + MC^2 = A'M'^2 + (DC - D'M')^2,$$
$$BD^2 = BN^2 + DN^2 = A'M'^2 + (DC - C'M')^2.$$

Again

$$A'D' - A'C' < D'C' = DC - AB.$$

[122] *L.* p. 48f.

Therefore

$$DC + A'C' > AD + AB.$$

6.4 Circumscribed circle

To find the radius of the circle described round a quadrilateral, Brahmagupta gives the following rule:

> The diagonal of an isosceles trapezium being multiplied by its flank side and divided by twice its altitude gives its heart line: in case of a quadrilateral of unequal sides it is half the square-root of the sum of the squares of the opposite sides.[123]

Now it has been given by Brahmagupta that

$$h^2 = d^2 - \left(\frac{a+c}{2}\right)^2.$$

Substituting the value of $d^2 = ac + b^2$, we get

$$h = \sqrt{(s-a)(s-c)}.$$

Hence according to the above, the radius of the circle described round the isosceles trapezium (a, b, c, b) is

$$\frac{1}{2}b\sqrt{\frac{ac+b^2}{(s-a)(s-c)}}.$$

In case of a quadrilateral of unequal sides the circum-radius is

$$= \frac{1}{2}\sqrt{a^2+c^2} = \frac{1}{2}\sqrt{b^2+d^2}.$$

This formula holds only in that kind of inscribed convex quadrilaterals in which the diagonals are at right angles.

Mahāvīra says:

> In a quadrilateral, the diagonal divided by the perpendicular and multiplied by the flank side, gives the diameter of the circumscribed circle.[124]

Śrīpati states all the above formulae. He says:

> In a quadrilateral, half the product of a diagonal and flank side divided by the altitude, gives the radius of the circumscribed circle. In a quadrilateral of unequal sides, half the square-root of the sum of the squares of the opposite sides is stated to be the radius and twice it the diameter of the circumscribed circle.[125]

[123] *BrSpSi*, xii. 26.
[124] *GSS*, vii. $213\frac{1}{2}$
[125] *SiŚe*, xiii. 31–2.

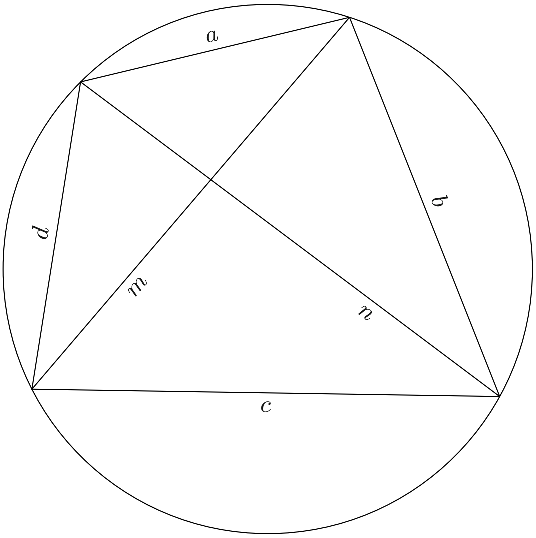

Figure 13

6.5 Inscribed circle

We have already cited Mahāvīra's formula for the diameter of the inscribed circle.

$$\text{Diameter} = \text{Area} \div \frac{\text{Perimeter}}{4}.$$

6.6 Theorems for diagonals

Brahmagupta (628) gives two remarkable theorems for the lengths of the diagonals of an inscribed convex quadrilateral. He says (**ed.** see Figure 13):

> Divide mutually the sums of the products of the sides attached to both the diagonals and then multiply the quotients by the sum of the products of the opposite sides: the square-roots of the results are the diagonals of the quadrilateral.[126]

$$m = \sqrt{\frac{(ab+cd)(ac+bd)}{(ad+bc)}},$$

$$n = \sqrt{\frac{(ad+bc)(ac+bd)}{(ab+cd)}}.$$

Mahāvīra (850) writes:

[126] *BrSpSi*, xii. 28.

> The two flank sides multiplied by the base are added (respectively) to those sides (taken reversely) multiplied by the face. Make the sums (thus obtained respectively) the multiplier and divisor, again the divisor and multiplier of the sum of the products of the opposite sides. The square-roots of the results are the diagonals.[127]

Śrīpati's (1039) enunciation[128] of the theorems is nearly the same as that of Brahmagupta.

It will be noticed that neither Brahmagupta, nor any of the posterior writers mentioned above, has expressly stated the limitation that the theorems hold only in case of inscribed convex quadrilaterals. Did they at all know it will be the question that will be naturally asked. Looking at the context, we think, it will have to be answered in the affirmative. For in the two rules just preceding the one in question, Brahmagupta teaches how to find the radii of the circles circumscribed about a quadrilateral and a triangle respectively. So in the present rule too he has in view a quadrilateral of the type which can be circumscribed by a circle. Illustrative examples given by the commentator Pṛthūdakasvāmi, as also by Mahāvīra, are all of quadrilaterals of that kind. Further Bhāskara II observed in connection with these theorems that they hold in case of quadrilaterals contemplated to be of a particular kind by their author.

7 Squaring the circle

7.1 Origin of the problem

The problem of 'squaring the circle', or what was more fundamental in India, the problem of 'circling the square', originated and acquired special importance in connexion with the Vedic sacrifices, before the earliest hymns of the *Ṛgveda* were composed (before 3000 BC). The three primarily essential sacrificial altars of the Vedic Hindus, namely the *Gārhapatya*, *Āhavanīya* and *Dakṣiṇa*, were constructed so as to be of the same area, but of different shapes, square, circular and semi-circular. Again in constructing the fire-altars called the *Rathacakra-citi*, *Samuhya-citi* and *Paricāyya-citi*, which are mentioned in the *Taittirīya Saṃhitā* (c. 3000 BC) and other works, one had to draw in each case at first a square equal in area to that of the *Śyena-citi*, viz. $7\frac{1}{2}$ square *puruṣas*, and then to transform it into a circle. We find also other instances in the early Hindu works requiring the solution of the problem of circling the square and its converse.[129]

[127] *GSS*, vii. 54.
[128] *SiŚe*, xiii. 34.
[129] See Datta, Bibhutibhusan, *Śulba*, ch. xi, for further informations on the problem.

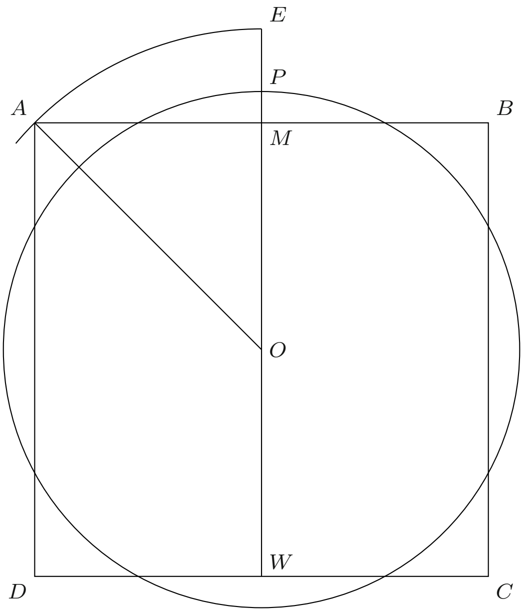

Figure 14

7.2 Circling the square

Baudhāyana writes:

> If you wish to circle a square, draw half its diagonal about the centre towards the east-west line; then describe a circle together with the third part of that which lies outside (the square).[130]

The same method is taught in different words also by Āpastamba[131] and Kātyāyana.[132]

Let $ABCD$ be the square which is to be transformed into a circle (**ed.** see Figure 14). Let O be the central point of the square. Join OA. With centre O and radius OA, describe a circle intersecting the east-west line EW at E. Divide EM at P, such that $EP = 2\,PM$. Then with centre O and radius OP describe a circle. This circle is roughly equal in area to the square $ABCD$.

Let $2a$ denote a side of the given square and r the radius of the circle equivalent to it. Then

$$OA = a\sqrt{2}, \quad ME = (\sqrt{2}-1)a.$$

[130] *BŚl*, i. 58.
[131] *ĀpŚl*, iii. 2.
[132] *KŚl*. iii. 13.

Hence

$$r = a + \frac{a}{3}\left(\sqrt{2}-1\right) = \frac{a}{3}\left(2+\sqrt{2}\right).$$

Āpastamba observes that the circle thus constructed will be inexact (*anitya*). Now, according to the *Śulba*,

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.$$

Therefore

$$r = a \times 1.1380718\ldots$$

7.3 Squaring the circle

Baudhāyana says:

> If you wish to square a circle, divide its diameter into eight parts; then divide one part into twenty-nine parts and leave out twenty-eight of these, and also the sixth part (of the preceding sub-division) less the eighth part (of the last).[133]

That is to say, if $2a$ be the side of a square equivalent to a circle of diameter d, then

$$2a = \frac{7d}{a} + \left\{\frac{d}{a} - \left(\frac{28d}{8 \times 29} + \frac{d}{8 \times 29 \times 6} - \frac{d}{8 \times 29 \times 6 \times 8}\right)\right\},$$

or putting $d = 2r$,

$$a = r - \frac{r}{8} + \frac{r}{8 \times 29} - \frac{r}{8 \times 29 \times 6} + \frac{r}{8 \times 29 \times 6 \times 8}.$$

Baudhāyana further teaches a still rough method of squaring the circle:

> Or else divide (the diameter) into fifteen parts and remove two (of them). This is the gross (value of the) side of the (equivalent) square.[134]

This method is described also by Āpastambā[135] and Kātyāyāna.[136] According to it

$$a = r - \frac{2r}{15}.$$

According to Manu a square of two by two cubits is equivalent to a circle of radius 1 cubit and 3 *aṅgulis*.[137]

[133] *BŚl*, i. 59.
[134] *BŚl*, i. 60.
[135] *ĀpŚl*, iii. 3.
[136] *KŚl*, iii. 14.
[137] *MāŚl*, i. 27.

Dvārakānātha's corrections

Dvārakānātha Yajvā, a commentator of the *Baudhāyana Śulba*, proposed a correction to the above formula for the transformation of a square into a circle. According to him

$$r = \left\{a + \frac{a}{3}\left(\sqrt{2} - 1\right)\right\} \times \left\{1 - \frac{1}{118}\right\},$$

or

$$r = a \times 1.1284272\ldots$$

Similarly he improves the formula for the reverse operation:

$$a = r\left(1 - \frac{1}{8} + \frac{1}{8 \times 29} + \frac{1}{8 \times 29 \times 6} - \frac{1}{8 \times 29 \times 6 \times 8}\right)\left(1 + \frac{1}{2} \times \frac{3}{133}\right).$$

7.4 Later formulae

In the Jaina cosmography, the earth is supposed to be a flat plane divided into successive regions of land and water by a system of concentric circles. The innermost region is one of land and is called Jambūdvīpa. It is a circle of diameter 100000 *yojanas*. Its circumference is given as a little over 316227 *yojanas* 3 *gavyūtis* 128 *dhanus* $13\frac{1}{2}$ *aṅgulas* and its area as 7905694150 *yojanas* 1 *gavyūti* 1515 *dhanus* 60 *aṅgulas*.[138] It will be seen that in calculating these values of the circumference and area from the assumed value of the diameter, the following two formulae have been employed:

$$C = \sqrt{10d^2}, \qquad A = \frac{1}{4}Cd,$$

where d = the diameter of a circle, C = its circumference and A = its area.

Umāsvāti (c. 150 BC or AD) writes:

> The square-root of ten times the square of the diameter of a circle is its circumference. That (circumference) multiplied by a quarter of the diameter (gives) the area.[139]

So does also Jinabhadra Gaṇi (529–589).[140]

Āryabhaṭa I says:

[138] See *Jambūdvīpa-prajñapti*, *Sūtra* 3; *Jīvābhigama-sūtra*, *Sūtra* 82, 124; *Anuyogadvāra-sūtra*, *Sūtra* 146. Compare also *Sūryaprajñapti*, *Sūtra* 20.

[139] *Tattvārthādhigama-sūtra* with the *Bhāṣya* of Umāsvāti, edited by K. P. Mody, Calcutta, 1903, iii. 11 (gloss); *Jambūdvīpa-samāsa*, ch. iv. The latter work of Umāsvāti has been published in the Appendix C of Mody's edition of the former.

[140] *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, Bhavanagara, 1919, i. 7.

> Half the circumference multiplied by the semi-diameter certainly gives the area of a circle.[141]

Brahmagupta:

> Three times the diameter and the square of the semi-diameter give the practical values of the circumference and area (respectively). The square roots of ten times the squares of them are the neat values.[142]

Śrīdhara:

> The square-root of the square of the diameter of a circle as multiplied by ten is its circumference. The square-root of ten times the square of the square of the semi-diameter is the area.[143]

Mahāvīra:

> Thrice the diameter is the circumference. Thrice the square of the semi-diameter is the area ... So said the teachers.[144]
>
> The diameter of a circle multiplied by the square-root of ten, becomes the circumference. The circumference multiplied by the fourth part of the diameter gives the area.[145]

Āryabhaṭa II:

> The square-root of the square of the diameter of a circle as multiplied by ten is the circumference. The fourth part of the square of the diameter being squared and multiplied by ten, the square-root of the product is the area.[146]
>
> The diameter multiplied by 22 and divided by 7 will become nearly equal to the circumference. If the square of the semi-diameter be so treated, the result will be the value of the area as precise as that of the circumference.[147]
>
> Twice the sine of three signs of the zodiac (i.e. 3438) is the diameter and the circumference is then 21600. Multiply the circumference by 191 and divide by 600; the quotient is the diameter.[148]

[141] *A*, ii. 7.
[142] *BrSpSi*, xii. 40.
[143] *Triś*, R. 45.
[144] *GSS*, vii. 19.
[145] *GSS*, vii. 60.
[146] *MSi*, xv. 88.
[147] *MSi*, xv. 92f.
[148] *MSi*, xvi. 37.

Śrīpati's rule is the same as the first one of Āryabhaṭa the Younger. Bhāskara II writes:

> When the diameter is multiplied by 3927 and divided by 1250, the result is the nearly precise value of the circumference; but when multiplied by 22 and divided by 7, it is the gross circumference which can be adopted for practical purposes.[149]
>
> In a circle, the one-fourth of the diameter multiplied by the circumference gives the area.[150]
>
> The square of the diameter being multiplied by 3927 and divided by 5000 gives the nearly precise value of the area; or being multiplied by 11 and divided by 14 gives the gross area which can be applied in rough works.[151]

7.5 Values of π

The formulae of Baudhāyana, noted above, yield the following values of π:

$$\pi = \frac{4}{\left\{1 + \frac{1}{3}\left(\sqrt{2} - 1\right)\right\}^2} = 3.0883\ldots$$

$$\pi = 4\left(1 - \frac{1}{8} + \frac{1}{8 \times 29} - \frac{1}{8 \times 29 \times 6} + \frac{1}{8 \times 29 \times 6 \times 8}\right) = 3.0885\ldots$$

$$\pi = 4\left(1 - \frac{2}{15}\right)^2 = 3.004.$$

Baudhāyana has once employed the very rough value, 3. From the rule of Manu, we get

$$\pi = 4\left(\frac{8}{9}\right)^2 = 3.16049\ldots$$

With the corrections of Dvārakānātha, we have

$$\pi = 3.141109\ldots, \quad 3.157991\ldots$$

In the early canonical works of the Jainas (500–300 BC) is employed the value $\pi = \sqrt{10}$.[152] This value has been adopted by Umāsvāti, Varāhamihira (505), Brahmagupta (628), Śrīdhara (c. 900) and others. It is stated in the *Jīvābhigama-sūtra*,[153] that for an increment of 100 *yojanas* in the diameter,

[149] *L*, p. 54.
[150] *L*, p. 55.
[151] *L*, p. 56f.
[152] See Datta, Bibhutibhusan, "The Jaina School of Mathematics", *BCMS* xxi (1929), p. 13; "Hindu Values of π", *JASB*, xxii (1926), pp. 25–43. The latter article given fuller information on the subject.
[153] *Sūtra* 112.

the circumference increases by 316 *yojanas.* Here has been used the value $\pi = 3.16$.

Āryabhaṭa the Elder (499) gives a remarkably accurate value. His rule is:

> 100 plus 4, multiplied by 8, and added to 62000: this will be the nearly approximate (*āsanna*) value of the circumference of a circle of diameter 20000.[154]

That is to say, we have

$$\pi = \frac{62832}{20000} = \frac{3927}{1250} = 3.1416.$$

This value appears in the works of Lalla[155] (c. 749), Bhaṭṭotpala[156] (966), Bhāskara II and others. We have it on the authority of a writer of the sixteenth century who was in possession of the larger treatise of arithmetic by Śrīdhara that this value of π was adopted there.

The value

$$\pi = \frac{21600}{6876} = \frac{600}{191} = 3.14136\ldots$$

introduced first by Āryabhaṭa the Younger (950) is undoubtedly derived from the value of the Elder Āryabhaṭa. For if the circumference of a circle measures 21600, its diameter will be

$$21600 \times \frac{1250}{3927} = 6875\frac{625}{1309}.$$

Āryabhaṭa takes the value of the diameter to be 6876 in round numbers.[157] This relation (21600 : 6876) between the circumference and diameter of a circle was, however, worked out before by Bhāskara I (629).[158] The value $\pi = \frac{600}{191}$ appears also in the treatises of arithmetic by Gaṇeśa II (c. 1550) and Munīśvara (1656).

It should be particularly noted that the Greek value, $\pi = \frac{22}{7}$, is found in India first in the work of Āryabhaṭa the Younger.[159] Bhāskara II (1150) employs it as a rough approximation suitable for practical purposes.

7.6 Later approximations of π

Later Hindu writers found much closer approximations to the value of π. Nārāyaṇa, a priest of Travancore, gave in 1426, the following rule to construct a temple of circular shape having a given perimeter:

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[154] *Ā*, ii. 10.
[155] *ŚiDVṛ*, i. 1, 2; ii. 3; etc.
[156] See his commentary on *Bṛhat Saṃhitā*, p. 53.
[157] *MSi*, xv. 88.
[158] *Vide* his commentary on *Ā*, ii. 10.
[159] *MSi*, xv. 92f

> Divide the given perimeter into 710 parts; with 113 of them as the radius describe a circle and thus construct the circular temple.[160]

Hence he has employed $\pi = \frac{355}{113}$, the Chinese value.

Śaṅkara Vāriyar (c. 1500–60) says:

> The value of the given diameter being multiplied by 104348 and divided by 33215, becomes the accurate value of the circumference. Again from the circumference can be obtained the correct value of the diameter by proceeding reversely; that is, by multiplying the value of the circumference by 33215 and then dividing by 104348, or by multiplying by 113 and dividing by 355.[161]

$$\pi = \frac{104348}{33215} = 3.14159265391\ldots$$
$$\pi = \frac{355}{113} = 3.1415929\ldots$$

The first value is correct up to the ninth place of decimals, the tenth being too large, and the second up to the sixth place of decimals, the seventh being too large.

Mādhava (of Saṅgamagrāma) writes:

> It has been stated by learned men that the value of the circumference of diameter 900000000000 in length is 2827433388233.[162]

Therefore we have

$$\pi = \frac{2827433388233}{900000000000} = 3.141592653592\ldots$$

correct up to the tenth place of decimals, the eleventh being too large.

Putumana Somayājī (c. 1660–1740), the author of the *Karaṇa-paddhati*, observes:

> When the value of the circumference of a circle is multiplied by 10000000000 and divided by 31415926536, the quotient is the value of the diameter. Half that is the radius.[163]

Śaṅkaravarman (1800–38) says:

[160] Nārāyaṇa, *Tantra-samuccaya*, edited by T. Ganapati Sastri, Trivandrum Sanskrit Series, 1919, ii. 65.
[161] *Tantra-saṃgraha*, (commentary in verse, edited by K. V. Sarma), p. 103, vss. 298–9.
[162] Quoted by Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* (ii. 10) edited by K. Sambasiva Sastri, Trivandrum Sanskrit Series, 1930.
[163] *Karaṇa-paddhati*, vi. 7.

> In this way, if the diameter of a great circle measure one *parārdha* (i.e. 10^{17}), its circumference will be 314159265358979324.[164]

Here we have a value of π, 3.14159265358979324, which is correct up to 17 places of decimals.

7.7 Values in series

Śaṅkara Vāriyar (c. 1500–60) gave certain interesting approximations in series for the value of the circumference of a circle in terms of its diameter. He says:

> Multiply the diameter by four and divide by one; subtract from and add to the result alternately the successive quotients of four times the diameter divided severally by the odd numbers 3, 5, etc. Take the even number next to that odd number on division by which this operation is stopped; then as before multiply four times the diameter by the half of that and divide by its square plus unity. Add the quotient thus obtained to the series in case its last term is negative; or subtract if the last term be positive. The result will be very accurate if the division be continued to many terms.[165]

That is to say, if C denotes the circumference and d the diameter, then we shall have

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d(n+1)}{(2n+2)^2+1},$$

where $n = 1, 2, 3 \ldots$

He then continues:

> Now I shall write of certain other correction more accurate than this: In the last term the multiplier should be the square of half the even number together with one, and the divisor four times that, added by unity, and then multiplied by half the even number. After division by the odd numbers 3, 5, etc., the final operation must be made as just indicated.[166]

$$C = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots + (-1)^n \frac{4d}{2n+1} - (-1)^n \frac{4d(n^2+2n+2)}{(n+1)(4n^2+8n+9)}.$$

The author seems to have realised the slow convergence of the above infinite series; so in order to get a closer approximation to its value after retaining a

[164] *Sadratna-mālā*, iv. 2.

[165] *Tantra-saṃgraha*, (commentary in verse), p. 101, vss. 271–4. This rule is really that of Mādhava. See *Kriyākramakarī* (Śaṅkara Vāriyar's commentary on *Līlāvatī*), p. 379.

[166] *Tantra-saṃgraha*, (commentary in verse), p. 103, vss. 295–296.

sufficient number of terms, modified the next one in the way described above and then neglected the rest. This series, without the correction in any form, is found also in the *Karaṇa-paddhati*, as follows:

> Divide four times the diameter many times severally by the odd numbers 3, 5, 7, etc. Subtract and add successive quotients alternately from and to four times the diameter. The result is an accurate value of the circumference.[167]

It was rediscovered in Europe two centuries later by Liebnitz (1673) and De Lagney (1682).

Śaṅkara Vāriyar (c. 1500–60) says:

> The square-root of twelve times the square of the diameter is the first result. Divide this by three; again the quotient by three; and so on continuously up to as many times as desired. Then divide the results successively by the odd numbers 1, 3, etc. Of the quotients thus obtained the sum of the odd ones (i.e. 1st, 3rd, etc.) diminished by the sum of the even ones (i.e. 2nd, 4th, etc.) will be the value of the circumference.[168]

That is to say, we shall have

$$C = \sqrt{12d^2}\left(1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \dots\right).$$

The same series is described in slightly difference words in the *Sadratnamālā*.[169] It is also given by Abraham Sharp (c. 1717), who used it for calculating the value of π up to 72 places of decimals.

Śaṅkara Vāriyar writes:

> The fifth powers of the odd numbers 1, 3, etc. are increased by four times their respective roots. Divide sixteen times a given diameter severally by the sums thus obtained and subtract the sum of the even quotients from that of the old ones. The remainder will be the circumference.[170]

That is

$$C = 16d\left(\frac{1}{1^5 + 4 \times 1} - \frac{1}{3^5 + 4 \times 3} + \frac{1}{5^5 + 4 \times 5} - \frac{1}{7^5 + 4 \times 7} + \dots\right).$$

[167] *Karaṇa-paddhati*, vi. 1.
[168] *Tantra-saṃgraha* (commentary in verse), p. 96, vss. 212(c–d)–214(a–b).
[169] *Sadratnamālā*, iv. 2.
[170] *Tantra-saṃgraha* (commentary in verse), p. 102, vss. 287–8.

> Or divide four times the diameter severally by the cubes of the odd numbers beginning with 3, after diminishing each by its respective root; add and subtract the successive quotients alternately to and from thrice the diameter. Hence deduce the value of the circumference also in this way.[171]

$$C = 3d + 4d\left(\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \dots\right).$$

This infinite series is stated also in the *Karaṇa-paddhati*.[172]

> Or the squares of the even numbers 2, etc. each diminished by unity, are the several denominators. Add and subtract the quotients alternately to and from twice the diameter. Take the odd number next to last even number (at which the series is stopped). The square of it added by two and then multiplied by two should be taken as the divisor at the end.[173]

$$C = 2d + 4d\left\{\frac{1}{2^2-1} - \frac{1}{4^2-1} + \dots + (-1)^{n-1}\,\frac{1}{(2n)^2-1} - (-1)^{n-1}\,\frac{1}{2(2n+1)^2+2}\right\}.$$

> Squares of the numbers beginning with two or four and increasing by four, diminished each by unity, are the several denominators; and the numerator in each case is eight times the given diameter. The value of the circumference of the circle is equal in the first case to the sum of the quotients and in the second to half the numerator minus the quotients.[174]

$$C = \left(\frac{8d}{2^2-1} + \frac{8d}{6^2-1} + \frac{8d}{10^2-1} + \dots\right),$$
$$C = 4d - \left(\frac{8d}{4^2-1} + \frac{8d}{8^2-1} + \frac{8d}{12^2-1} + \dots\right).$$

The *Karaṇa-paddhati* adds a new series. It says:

> Or divide six times the diameter by squares of twice the squares of even numbers minus unity as diminished by the squares of the respective even numbers. Thrice the diameter added by these quotients is the value of the circumference.[175]

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[171] *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 290.
[172] *Karaṇa-paddhati*, vi. 2.
[173] *Tantra-saṃgraha* (commentary in verse), p. 103, vs. 292.
[174] *Tantra-saṃgraha* (commentary in verse), p. 103, vss. 293–4.
[175] *Karaṇa-paddhati*, vi. 4.

$$C = 3d + \frac{6d}{(2\times 2^2-1)^2-2^2} + \frac{6d}{(2\times 4^2-1)^2-4^2} + \frac{6d}{(2\times 6^2-1)^2-6^2} + \dots$$

Or,

$$C = 3d + 6d\left(\frac{1}{1\times 3\times 3\times 5} + \frac{1}{3\times 5\times 7\times 9} + \frac{1}{5\times 7\times 11\times 13} + \dots\right).$$

Śaṅkaravarman gives another:

> Take the square-root of twelve times the square of the diameter and also its third part. Divide these continuously by nine. Again divide the quotients (thus obtained) respectively by twice the odd numbers 1, etc. (in the former case) and by twice the even numbers 2, etc. (in the latter case), each as diminished by unity. The difference of the two sums of the final quotients is the value of the circumference of the circle.[176]

$$C = \sqrt{12d^2}\left\{\frac{1}{9(2\times 1-1)} + \frac{1}{9^2(2\times 3-1)} + \frac{1}{9^3(2\times 5-1)} + \dots\right\} - \frac{\sqrt{12d^2}}{3}\left\{\frac{1}{9(2\times 2-1)} + \frac{1}{9^2(2\times 4-1)} + \frac{1}{9^3(2\times 6-1)} + \dots\right\}.$$

8 Measurement of segment of circle

8.1 Data in Jaina canonical works

In the early cosmographical works of the Jainas, we find certain interesting and valuable data relating to the mensuration of a segment of a circle.[177] Jainas suppose that Jambūdvīpa, which has been described before to be a circle of diameter 100000 *yojanas*; is divided into seven *varṣas* ("countries") by a system of six parallel mountain ranges running due East-to-West. The southern region of it is called Bhāratavarṣa. Dimensions of this segment, in

[176] *Sadratnamālā*, iv. 1.
[177] See the article of Datta, Bibhutibhusan, on "Geometry in the Jaina Cosmography" in *Quellen und Studien zur Gesch. d. Math.* Ab. B, Bd. 1, 1930 pp. 245–254, from which extracts are here made.

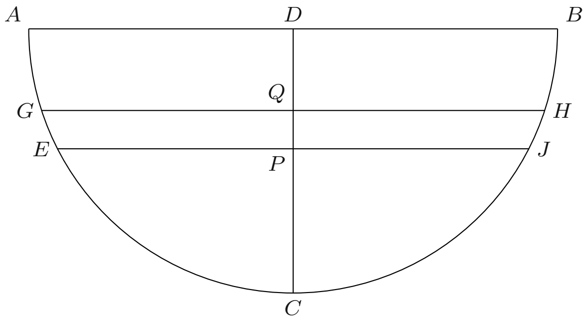

Figure 15

terms of *yojanas*, are as follows (**ed.** see Figure 15):

$$AB = 1447\frac{6}{19} \text{ (a little less)}, \qquad ACB = 14528\frac{11}{19},$$
$$PQ = 50, \qquad GCH = 10743\frac{15}{19},$$
$$CD = 526\frac{6}{19}, \qquad ECJ = 9766\frac{1}{19} \text{ (a little over)},$$
$$CP = QD = 238\frac{3}{19}, \qquad AG = BH = 1892\frac{7}{19} + \frac{1}{33},$$
$$EJ = 9748\frac{12}{19}, \qquad EG = JH = 488\frac{16}{19} + \frac{1}{33},$$
$$GH = 10720\frac{12}{19}.$$

These numerical data will be found to conform to the following formulae for the mensuration of a segment of a circle:

$$c = \sqrt{4h(d-h)},$$
$$d = \frac{c^2}{4h} + h,$$
$$a = \sqrt{6h^2 + c^2},$$
$$a' = \frac{1}{2}\{(\text{bigger arc}) - (\text{smaller arc})\},$$
$$h = \frac{1}{2}(d - \sqrt{d^2 - c^2}),$$
$$\text{or} \qquad h = \sqrt{\frac{(a^2 - c^2)}{6}},$$

where d = the diameter of the circle, c = a chord of it, a = an arc cut off by that chord, h = height of the segment or its arrow and a' = an arc of the circle lying between two parallel chords.

These formulae are not found clearly defined in abstract in any of the early canonical works, though they state in minute details some of the above numerical data.[178]

8.2 Umāsvāti's rules

In his gloss on his own treatise *Tattvārthādhigama-sūtra*, Umāsvāti (c. 150 BC or AD) says:

> The square-root of four times the product of an arbitrary depth and the diameter diminished by that depth is the chord. The square-root of the difference of the squares of the diameter and chord should be subtracted from the diameter: half of the remainder is the arrow. The square-root of six times the square of the arrow added to the square of the chord (gives) the arc. The square of the arrow plus the one-fourth of the square of the chord is divided by the arrow: the quotient is the diameter. From the northern (meaning the bigger) arc should be subtracted the southern (meaning the smaller) arc: half of the remainder is the side (arc).[179]

All these rules have been restated by Umāsvāti in another work, *Jambūdvīpa-samāsa* by name.[180] But there the formula for the arrow is different:

> The square-root of one-sixth of the difference between the squares of the arc and the chord is the arrow.

It is clearly approximate.

8.3 Āryabhaṭa I and Brahmagupta

Āryabhaṭa I writes:

> In a circle, the product of the two arrows is the square of the semi-chord of the two arcs.[181]

Brahmagupta says:

> In a circle, the diameter should be diminished and then multiplied by the arrow; then the result is multiplied by four: the square root of the product is the chord. Divide the square of the chord

[178] For instance see *Jambūdvīpa-prajñapti*, *Sūtra* 3, 10–15; *Jīvābhigama-sūtra*, *Sūtra* 82, 124; *Sūtrakṛtāṅga-sūtra*, *Sūtra* 12.
[179] *Tattvārthādhigama-sūtra*, iii. 11 (gloss).
[180] *Jambūdvīpa-samāsa*, ch. iv.
[181] *Ā*, ii. 17.

> by four times the arrow and then add the arrow to the quotient: the result is the diameter. Half the difference of diameter and the square-root of the difference between the squares of the diameter and chord, is the smaller arrow.[182]

8.4 Jinabhadra Gaṇi's rules

Jinabhadra Gaṇi (529–589) writes:

> Multiply by the depth, the diameter as diminished by the depth: the square-root of four times the product is the chord of the circle.[183]
>
> Divide the square of the chord by the arrow multiplied by four; the quotient together with the arrow should be known certainly as the diameter of the circle. The square of the arrow multiplied by six should be added to the square of the chord; the square-root of the sum should be known to be the arc. Subtract the square of the chord certainly from the square of the arc; the square-root of the sixth part of the remainder is the arrow. Subtract from the diameter the square-root of the difference of the squares of the diameter and chord; half the remainder should be known to be the arrow.[184]
>
> Subtract the smaller arc from the bigger arc; half the remainder should be known to be the side arc. Or add the square of half the difference of the two chords to the square of the perpendicular; the square-root of the sum will be the side arc.[185]

Jinabhadra Gaṇi next cites two formulae for finding the area of a segment of a circle cut off by two parallel chords.

> For the area of the figure, multiply half the sum of its greater and smaller chords by its breadth.[186]

or

> Sum up the squares of its greater and smaller chords; the square-root of the half of that (sum) will be the 'side'. That multiplied by the breadth will be its area.[187]

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[182] *BrSpSi*, xii. 41f.
[183] *Vṛhat Kṣetra-samāsa* of Jinabhadra Gaṇi, i. 36.
[184] *Vṛhat Kṣetra-samāsa*, i. 38–41.
[185] *Vṛhat Kṣetra-samāsa*, i. 46–7.
[186] *Ibid*, i. 64.
[187] *Vṛhat Kṣetra-samāsa*, i. 122.

That is to say, if c_1, c_2 be the lengths of the two parallel chords and h, the perpendicular distance between them, then the area of the segment will be given by

$$\text{(i)}\quad \text{Area} = \frac{1}{2}(c_1 + c_2)h,$$

$$\text{(ii)}\quad \text{Area} = \sqrt{\frac{1}{2}(c_1^2 + c_2^2)} \times h.$$

Neither of these formulae, the author thinks, will be available for finding the area of the Southern Bhāratavarṣa which, as has been described before, has only a single chord. So he gives a third formula as follows:

> In case of the Southern Bhāratavarṣa, multiply the arrow by the chord and then divide by four; then square and multiply by ten: the square-root (of the result) will be its area.[188]

$$\text{(iii)}\quad \text{Area} = \sqrt{10\left(\frac{ch}{4}\right)^2}.$$

None of the above formulae will give the desired result to a fair degree of accuracy. Formula (i) indeed gives the area of the isosceles trapezium of which the two parallel chords form the two parallel sides. The result obtained by it will therefore be approximately correct only when the breadth is small. Otherwise as has been observed by the commentator Malayagiri (c. 1200), the formula will give only a wrong result. Jinabhadra Gaṇi seems to have been aware of this limitation of the formula. For he has not followed it in practice. The rationale of formula (ii) which has been followed by our author, cannot be easily determined. Formula (iii) seems to have been derived by analogy with the formula for the finding the area of a semi-circle.

8.5 Śrīdhara's rule

In his smaller treatise or arithmetic, Śrīdhara (c. 900) includes a formula for finding the area of a segment of a circle. He says:

> Multiply half the sum of the chord and arrow by the arrow; multiply the square of the product by ten and then divide by nine. The square-root of the result will be the area of the segment.[189]

$$\text{Area} = \sqrt{\frac{10}{9}\left\{h\left(\frac{c+h}{2}\right)\right\}^2}.$$

[188] *Ibid.*, i. 122.
[189] *Triś*, R. 47.

8.6 Mahāvīra's rules

For the mensuration of a segment of a circle, Mahāvīra (850) gives two sets of formulae; the first set gives results serving all practical purposes (*vyāvahārika phala*), while the second set yields nearly precise results (*sūkṣma phala*). He says:

> Multiply the sum of the arrow and chord by the half of the arrow: the product is the area of the segment. The square-root of the square of the arrow as multiplied by five and added by the square of the chord is the arc.[190]
>
> The square-root of the difference between the squares of the arc and chord, as divided by five, is stated to be the arrow. The square-root of the square of the arc minus five times the square of the arrow is the chord.[191]

Thus the rough formulae are:

$$\text{Area} = \frac{1}{2}h(c+h),$$
$$h = \sqrt{\frac{a^2-c^2}{5}},$$
$$c = \sqrt{a^2-5h^2},$$
$$a = \sqrt{5h^2+c^2}.$$

For calculation of nearly precise results his rules are as follow:

> In case of a figure of the shape of (the longitudinal section of) a barley and a segment of a circle, the chord multiplied by one-fourth the arrow and also by the square-root of ten becomes, it should be known, the area.[192]
>
> The square of the arrow is multiplied by six and then added by the square of the chord; the square-root of the result is the arc. For finding the arrow and the chord the process is the reverse of this. The square-root of the difference of the squares of the arc and chord, as divided by six, is stated to be the arrow. The square-root of the square of the arc minus six times the square of the arrow is the chord.[193]

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[190] *GSS*, vii. 43.
[191] *GSS*, vii. 45.
[192] *GSS*, vii. $70\frac{1}{2}$.
[193] *GSS*, vii. $74\frac{1}{2}$.

Thus the nearly precise formulae of Mahāvīra are:

$$\text{Area} = \frac{\sqrt{10}}{4} ch,$$
$$h = \sqrt{\frac{a^2 - c^2}{6}},$$
$$a = \sqrt{6h^2 + c^2},$$
$$c = \sqrt{a^2 - 6h^2}.$$

8.7 Āryabhaṭa II's rules

Like Mahāvīra, Āryabhaṭa II (950) too gives two sets of formulae, rough (*sthūla*) as well as nearly precise (*sūkṣma*) for the mensuration of a segment of a circle. But it will be noticed that the rough formulae are the same as the nearly precise ones of his predecessor: one about the area yields distinctly better results. Āryabhaṭa II writes:

> The product of the arrow and half the sum of the chord and arrow is multiplied by itself; the square-root of the result increased by its one-ninth is the rough value of the area of the segment. The square-root of the square of the arrow multiplied by six and added by the square of the chord is the arc. The square-root of the difference of the square of the arc and chord as divided by six, is the arrow. The square-root of the remainder left on subtracting six times the square of the arrow from the square of the arc, is the chord. The half of the arc multiplied by itself is diminished by the square of the arrow; on dividing the remainder by twice the arrow, the quotient will be the value of the diameter.[194]

That is to say, the rough formulae are:

$$\text{Area} = \sqrt{\left(1 + \frac{1}{9}\right)\left\{h\left(\frac{c+h}{2}\right)\right\}^2},$$
$$a = \sqrt{6h^2 + c^2},$$
$$h = \sqrt{\frac{a^2 - c^2}{6}},$$
$$c = \sqrt{a^2 - 6h^2},$$
$$d = \frac{1}{2h}\left(\frac{1}{2}a^2 - h^2\right).$$

Āryabhaṭa II then continues:

[194] *MSi*, xv. 89–92.

> On dividing by 21 the product of half the sum of the chord and arrow, as multiplied by the arrow and again by 22, the quotient will be the nearly precise value of the area of the segment. The square of the arrow being multiplied by 288 and divided by 49, is increased by the square of the chord; the square-root of the result is the near value of the arc. The square-root of the difference of the squares of the arc and chord, as multiplied by 49 and divided by 288, is the arrow. The square-root of what is left on subtracting from the square of the arc, the square of the arrow multiplied by 288 and divided by 49 will be the chord. Multiply the square of the arc by 245 and then divide by 484; divide the quotient as diminished by the square of the arrow, by twice the arrow: the quotient will be the diameter. Similarly the chord will be the square-root of the diameter as diminished by the arrow and then multiplied by four times the arrow. The square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. The square of the semi-chord being added with the square of the arrow, the quotient of the sum divided by the arrow is the diameter.[195]

Hence

$$\begin{aligned}
\text{Area} &= \frac{22}{21}h\left(\frac{c+h}{2}\right),\\
a &= \sqrt{\frac{288}{49}h^2 + c^2},\\
h &= \sqrt{\frac{49}{288}\left(a^2 - c^2\right)},\\
c &= \sqrt{a^2 - \frac{288}{49}h^2},\\
d &= \frac{1}{2h}\left(\frac{245}{484}a^2 - h^2\right),\\
e &= \sqrt{4h(d-h)},\\
h &= \frac{1}{2}\left\{d - \sqrt{d^2 - c^2}\right\},\\
d &= \frac{1}{h}\left\{\left(\frac{c}{2}\right)^2 + h^2\right\}.
\end{aligned}$$

It should perhaps be noted that the last three formulae are exact, while others are approximate.

[195] *MSi*, xv. 93–99.

8.8 Śrīpati's rules

Śrīpati (c. 1039) states:

The diameter of a circle is diminished by the given arrow and then multiplied by it and also by four: the square-root of the result is the chord. In a circle, the square-root of the difference of the squares of the diameter and chord being subtracted from the diameter, half the remainder is the arrow. In a circle, the square of the semi-chord being added to the square of the arrow and then divided by the arrow, the result is stated to be the diameter ... Six times the square of the arrow being added to the square of the chord, the square-root of the sum is the arc here. The difference of the squares of the arc and chord being divided by six, the square-root of the quotient is the value of the arrow. From the square of the arc being subtracted the square of the arrow as multiplied by six, the square-root of the remainder is the chord. Twice the square of the arrow being subtracted from the square of the arc, the remainder divided by four times the arrow, is the diameter.[196]

8.9 Bhāskara II's rules

Bhāskara II (1150) does not mention the formulae for the calculation of approximate results, but gives only the exact ones. He writes:

> Find the square-root of the product of the sum and difference of the diameter and chord, and subtract it from the diameter: half the remainder is the arrow. The diameter being diminished and then multiplied by the arrow, twice the square-root of the result is the chord. In a circle, the square of the semi-chord being divided and then increased by the arrow, the result is stated to be the diameter.[197]

These rules have been reproduced by Munīśvara.[198]

8.10 Sūryadāsa's proof

Sūryadāsa (born 1508) proves the formulae for the arrow and diameter as follows (**ed.** see Figure 16):

Let AB be a chord of the circle having its centre at O and CH the arrow of the segment ABC. Join BO and produce it to meet the circumference in

[196] *SiŚe*, xiii. 37–40.
[197] *L*, p. 58.
[198] *Pāṭīsāra*, R. 220–1.

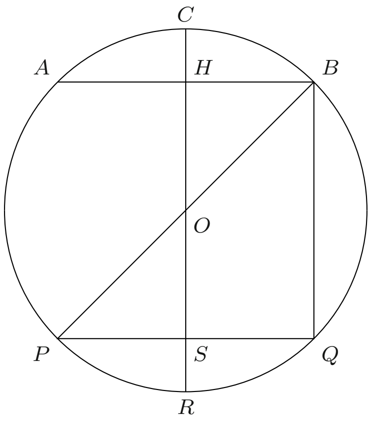

Figure 16

P. Draw PSQ parallel to AB. Join BQ. Then clearly

$$\begin{aligned} CH &= \frac{1}{2}(CR - HS), \\ &= \frac{1}{2}(CR - BQ), \\ &= \frac{1}{2}(CR - \sqrt{BP^2 - PQ^2}). \end{aligned}$$

Hence

$$CH = \frac{1}{2}(CR - \sqrt{CR^2 - AB^2}).$$

Again, since

$$HB^2 = CH \times HR,$$

we get

$$HR = \frac{HB^2}{CH}.$$

Therefore

$$CR = \frac{HB^2}{CH} + CH.$$

8.11 Other formulae for area

For the area of a segment of a circle, Viṣṇu Paṇḍita (c. 1410) and Keśava II (1496) gave the formula:

$$\text{Area} = \left(1 + \frac{1}{20}\right)\left\{h\left(\frac{h+c}{2}\right)\right\}.$$

Gaṇeśa (1545) and Rāmakṛṣṇadeva state:

$$\begin{aligned}\text{Area} &= (\text{area of the sector}) - (\text{area of the triangle})\\ &= \frac{1}{4}ad - \frac{1}{2}c\left(\frac{1}{2}d - h\right).\end{aligned}$$

8.12 Intersection of two circles

When two circles intersect, the common portion cut off is called the *grāsa* ("the erosion"). The origin of the term seems to be connected with the eclipse of the moon (or the sun) which is narrated in the popular mythology of the early Hindus as being caused by the dragon *Rāhu* (earth's shadow) swallowing the moon. The portion swallowed up is the *grāsa.* In fact, the geometrical theorem, just to be described, had its application in the calculation of the eclipse. The common portion is also called *matsya* (fish) as it resembles a fish. (**ed.** see Figure 17.)

Āryabhaṭa I writes:

> (The diameters of) the two circles being severally diminished and then multiplied by (the breadth of) the erosion, the products divided severally by the sum of the diameters (each) as diminished by the erosion, will be the two arrows lying within the erosion.[199]

This rule is nearly reproduced by Mahāvīra.[200]

$$AP \times PA' = PB^2 = DP \times PD',$$

or

$$(AA' - A'P)\ A'P = (DD' - D'P)D'P,$$

or

$$\begin{aligned}AA' \times A'P - DD' \times D'P &= A'P^2 - D'P^2\\ &= (A'P + D'P)(A'P - D'P)\\ &= A'D'(A'P - D'P),\end{aligned}$$

or

$$(AA' - A'D')A'P = (DD' - A'D')D'P.$$

Hence

$$\begin{aligned}\frac{A'P}{DD' - A'D'} &= \frac{D'P}{AA' - A'D'}\\ &= \frac{A'D'}{(DD' - A'D') + (AA' - A'D')}.\end{aligned}$$

[199] $\bar{A}$, ii. 18.

[200] *GSS*, vii. $231\frac{1}{2}$.

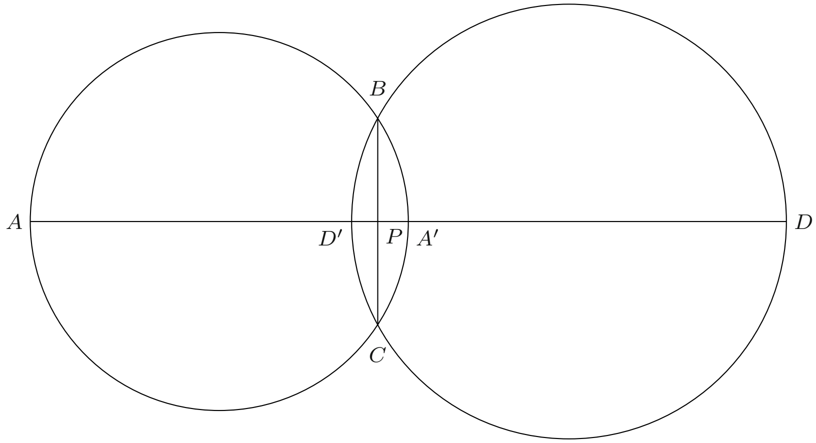

Figure 17

Therefore

$$A'P = \frac{A'D'(DD' - A'D')}{DD' + AA' - 2A'D'},$$
$$D'P = \frac{A'D'(AA' - A'D')}{DD' + AA' - 2A'D'}.$$

Brahmagupta says:

> The erosion being subtracted (severally) from the two diameters, the remainders, multiplied by the erosion and divided by the sum of the remainders, are the arrows.[201]
>
> The square of half the (common) chord being divided severally by the two given arrows, the quotients added with the respective arrows give the two diameters. The sum of the two arrows is the erosion; and that of the quotients is the sum of the diameters minus the erosion.[202]

9 Miscellaneous figures

9.1 Miscellaneous figures

Śrīdhara, Mahāvīra and Āryabhaṭa II have treated the mensuration of certain other plane figures such as of the shape of a barley corn (*yava*), drum (*muraja*, *mṛdaṅga*), elephant's tusk (*gajadanta*), crescent moon (*bālendu*), felloe (*nemi*),

[201] *BrSpSi*, xii. 42.
[202] *BrSpSi*, xii. 43.

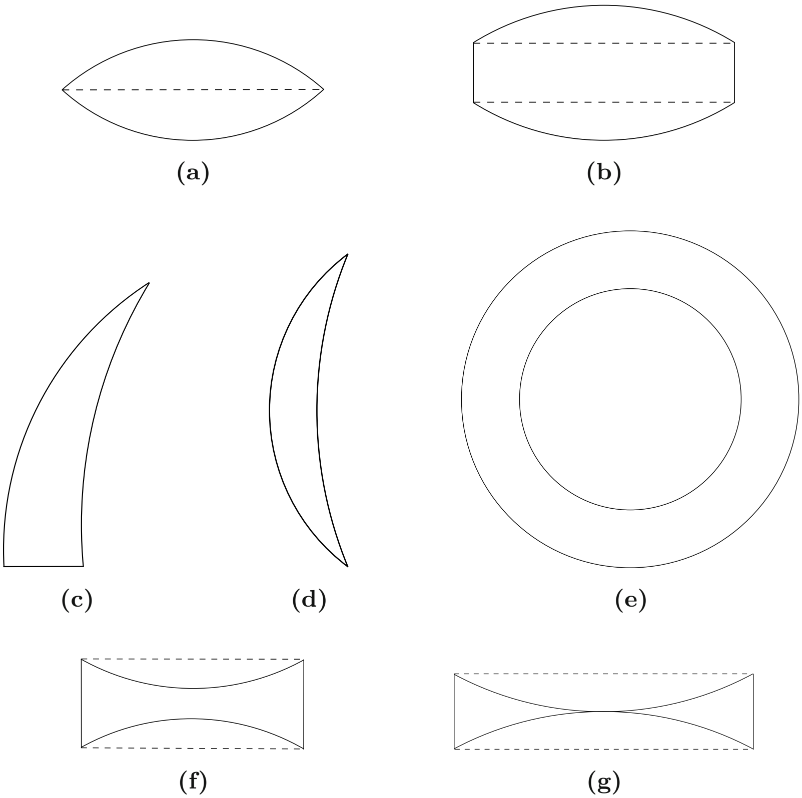

Figure 18: (a) barley corn, (b) drum, (c) elephant's tusk, (d) crescent, (e) felloe, (f) *vajra* (after Śrīdhara and Āryabhaṭa II) or *paṇava* (after Mahāvīra), (g) *vajra* (after Mahāvīra).

thunder-bolt (*vajra*) etc. The formulae described in case of most of them are only roughly approximate and some of them are deduced easily from the results already obtained. It was probably from the point of view of some practical utility that all the results have been stated separately.

9.2 Śrīdhara's rules

Śrīdhara says:

> A figure of the shape of an elephant tusk (may be considered) as a triangle, of a felloe as a quadrilateral, of a crescent moon as two triangles and of a thunderbolt as two quadrilaterals.[203]

[203] *Triś*, R. 44.

> A figure of the shape of a drum, should be supposed as consisting of two segments of a circle with a rectangle intervening; and a barley corn only of two segments of a circle.[204]

9.3 Mahāvīra's rules

For finding the gross value of the areas of above figures Mahāvīra gives the following rules:

> In a figure of the shape of a felloe, the area is the product of the breadth and half the sum of the two edges. Half that area will be the area of a crescent moon here.[205]
>
> The diameter increased by the breadth of the annulus and then multiplied by three and also by the breadth gives the area of the outlying annulus. The area of an inlying annulus (will be obtained in the same way) after subtracting the breadth from the diameter.[206]
>
> In case of a figure of the shape of a barley corn, drum, *paṇava*, or thunderbolt, the area will be equal to half the sum of the extreme and middle measures multiplied by the length.[207]

For finding the neat values of the areas of them, Mahāvīra has the following rules:

> The diameter added with the breadth of the annulus being multiplied by $\sqrt{10}$ and the breadth gives the area of the outlying annulus. The area of the inlying annulus (will be obtained from the same operations) after subtracting the breadth from the diameter.[208]
>
> Find the area by multiplying the face by the length. That added with the areas of the two segments of the circle associated with it will give the area of a drum-shaped figure. That diminished by the areas of the two associated segments of the circle will be the area in case of a figure of the shape of a *paṇava* as well as of a *vajra*.[209]
>
> In case of a felloe-shaped figure, the area is equal to the sum of the outer and inner edges as divided by six and multiplied by the

[204] *Triś*, R. 48.
[205] *GSS*, vii. 7. The formula for the area of the felloe yields, indeed, the accurate value of it.
[206] *GSS*, vii. 28.
[207] *GSS*, vii. 32.
[208] *GSS*, vii. $67\frac{1}{2}$.
[209] *GSS*, vii. $76\frac{1}{2}$.

> breadth and $\sqrt{10}$. The area of a crescent moon or elephant's tusk is half that.[210]

9.4 Āryabhaṭa II's rules

Āryabhaṭa II writes:

> In (a figure of the shape of) the crescent moon, there are two triangles and in an elephant's tusk only one triangle; a barley corn may be looked upon as consisting of two segments of a circle or two triangles.[211]
>
> In a drum, there are two segments of a circle outside and a rectangle inside; in a thunderbolt, are present two segments of two circles and two quadrilaterals.[212]

9.5 Polygons

According to Śrīdhara, regular polygons may be treated as being composed of triangles.[213] Mahāvīra says:

> One-third of the square of half the perimeter being divided by the number of sides and multiplied by that number as diminished by unity will give the (gross) area of all rectilinear figures. One-fourth of that will be the area of a figure enclosed by circles mutually in contact.[214]

That is to say, if $2s$ denote the perimeter of a polygon of n sides, whether regular or otherwise, but without a re-entrant angle, then its area will be roughly given by the formula

$$\text{Area} = \frac{(n-1)s^2}{3n}.$$

Mahāvīra has treated some very particular cases of polygons with re-entrant angles. He says:

> The product of the length and the breadth minus the product of the length and half the breadth is the area of a di-deficient figure; by subtracting half the latter (product form the former) is obtained the area of a uni-deficient figure.[215]

---

[210] *GSS*, vii. $80\frac{1}{2}$.
[211] *MSi*, xv. 101.
[212] *MSi*, xv. 103.
[213] *Triś*, R. 48.
[214] *GSS*, vii. 39.
[215] *GSS*, vii. 37.

The figures contemplated in this rule are those formed by leaving out two vertically opposite ones or any one of the four portions into which a rectangle is divided by its two diagonals. In the first case, the figure is technically called the *ubhaya-niṣedha-kṣetra* ("di-deficient figure") and in the other the *eka-niṣedha-kṣetra* (uni-deficient figure).

Mahāvīra further says:

> On subtracting the accurate value of the area of one of the circles from the square of a diameter, will be obtained the (neat) value of the area of the space lying between four equal circles (touching each other).[216]
>
> The accurate value of the area of an equilateral triangle each side of which is equal to a diameter, being diminished by half the area of a circle, will yield the area of the space bounded by three equal circles (touching each other).[217]
>
> A side of a regular hexagon, its square and its biquadrate being multiplied respectively by 2, 3, and 3 will give in order the value of its diagonal, the square of the altitude, and the square of the area.[218]

Āryabhaṭa II observes:

> A pentagon is composed of a triangle and a trapezium, a hexagon of two trapeziums; in a lotus-shaped figure there is a central circle and the rest are triangles.[219]

9.6 Ellipse

Though the ellipse was known to the Hindus as early as circa 400 BC, we do not find any formula for its mensuration in any of their works on mathematics, except the *Gaṇita-sāra-saṃgraha* of Mahāvīra (850). In the latter again, we have only roughly approximate results. Mahāvīra says:

> The length of an ellipse being added by half its breadth and multiplied by two, gives the gross value of its circumference. The circumference multiplied by one-fourth the breadth becomes the gross value of the area.[220]

[216] *GSS*, vii. $82\frac{1}{2}$.
[217] *GSS*, vii. $84\frac{1}{2}$.
[218] *GSS*, vii. $86\frac{1}{2}$.
[219] *MSi*, xv. 102.
[220] *GSS*, vii. 21.

> The square-root of six times the square of the breadth added with the square of twice the length, will be the neat value of the circumference of an ellipse. That multiplied by one-fourth the breadth will become the neat value of the area.[221]

That is to say if $2a$ be the longer diameter of an ellipse and $2b$ its shorter diameter, then, according to Mahāvīra,

$$\begin{aligned}\text{Circumference } (\textit{Gross}) &= 2(2a+b),\\ \text{Circumference } (\textit{Neat}) &= \sqrt{16a^2+24b^2},\\ \text{Area } (\textit{Gross}) &= b(2a+b),\\ \text{Area } (\textit{Neat}) &= \frac{1}{2}\, b\, \sqrt{16a^2+24b^2}.\end{aligned}$$

10 Measurement of volumes

10.1 Solids considered

Things in everyday life of the ancient Vedic Hindus which led them to develop formulae for the measurement of volumes were fire-altars and excavations. Amongst the fire-altars described in the extant works on the *Śulba*, we find that some are right prisms of various cross-sections, and others are right circular cylinders. Only in one case, namely, the fire-altar of the shape of the cemetery, the solid considered resembles a frustum of a pyramid. For the measurement of the latter, the Hindus developed an approximate formula. Though we meet with copious descriptions of pits, caves and mountains etc., of the shape of truncated cones and pyramids, in the early canonical works of the Jainas, there is nothing to indicate that the mensuration of those solids was known to them. In later Hindu treatises of arithmetic, solids generally treated are excavations, mounds of grains and piles of bricks.

10.2 Prism and cylinder

The formula for calculating the volumes of prisms and cylinders is found in the *Śulba*.[222]

$$\text{Volume of a prism or cylinder} = (\text{base}) \times (\text{height}).$$

The same formula is stated in later works.[223]

[221] *GSS*, vii. 63.
[222] Datta, *Śulba*, p. 101. See also *Jaina Math., Quel, und Stud. z. Gesch. d. Math.* Bd. I. (1930), p. 253.
[223] *BrSpSi*, xii. 44; *Triś*, R. 53; *GSS*, viii. 4; etc.

It may be noted that in later treatises of arithmetic, an excavation (*khāta*) whose depth is uniform is called the *sama-khāta*. The section of the base may be of any form, as it has not been particularly mentioned. The word *sama* (equal) implies that all sections parallel to the face or base are equal.

In the *Veda* and *Saṃhitā*, the prisms whose sections are regular polygons, were named according to the number of edges. Thus in the *Ṛgveda* (c. 3000 BC), the triangular prism is called *trirasri* (three-edged solid; *tri* = three, *asri* = edge), a quadrangular prism *caturasri* (= four edged solid) and so on.[224] But these terms do not seem to have been completely standardised. For in comparatively later times, a cube was called *dvādasāsrika* (= twelve-edged solid).

10.3 Cone and pyramid

The Hindus do not always distinguish between a cone and a pyramid. They include both under a generic name *sūcī*, which means literally "a needle", "a sharp pointed object", and hence, "a solid of the form of the needle", "a sharp pointed solid". Thus the term generally denotes a pyramid with a base of any form; as the base may be a circle it includes a cone as well. A triangular pyramid is, however, distinguished as the *ghana-ṣaḍasri* or simply *ṣaḍasri* (literally, "six-edged solid").

Āryabhaṭa I says:

> Half the product of this area (of the triangular base) and the height is the volume of the six-edged solid.[225]

This formula for the volume of the triangular pyramid is wrong. The correct formula is found in the works of Brahmagupta. He states:

> The volume of the uniform excavation divided by three is the volume of the needle-shaped solid.[226]

That is to say, we shall have

$$\text{Volume of a cone or pyramid} = \frac{1}{3}(\text{base}) \times (\text{height}).$$

This formula reappears in the works of Āryabhaṭa II,[227] Nemicandra,[228] Śrīpati[229] and Bhāskara II.[230]

[224]Datta, "On the Hindu names for the rectilinear geometrical figures", *loc. cit*, pp. 284f.
[225]*Ā*, ii. 6.
[226]*BrSpSi*, xii. 44.
[227]*MSi*, xv. 105.
[228]*Trilokasāra*, *Gāthā* 19.
[229]*SiŚe*, xiii. 44.
[230]*L*, p. 62

For measuring the mounds of grains which approximate to the form of a right circular cone, the Hindus ordinarily employed a rough formula. In such cases, they further assume the height of the mound to be equal to the circumference of the base divided by 9, 10 or 11 according to the kind of grain of which the mound is composed. Thus Brahmagupta says:

> In case of *śuki* grains one-ninth, in case of course grains one-tenth and in case of fine grains one-eleventh of the circumference (of the base) is the height; that multiplied by the square of the sixth part of the circumference will be the volume.[231]

Śrīpati writes:

> Of a heap of grains standing on the plane surface of the earth, the square of one-sixth the circumference multiplied by the height is the volume in terms of *Māgadha Khārikā*. In case of grains known as *syāmāka*, *śāli*, *tila*, *sarṣapa*, etc., the circumference is nine times the height; in case of *godhūma*, *mudga*, *yava*, *dhānyaka*, etc., it is ten times; and in case of *vadara*, *kaṅgu*, *kulattha*, etc., eleven times.[232]

The rough formula was obtained probably thus:

$$\text{Volume of a cone} = \frac{1}{3}(\text{base}) \times (\text{height}).$$

If r denote the radius of the base, we have

$$\text{Base} = \pi r^2 = \frac{2\pi r \times 2\pi r}{4\pi} = \frac{(\text{circumference})^2}{4\pi}.$$

Hence

$$\text{Volume of a cone} = \frac{1}{12\pi}(\text{circumference})^2 \times (\text{height}).$$

Now putting $\pi = 3$ roughly we get,

$$\text{Volume} = \left(\frac{\text{circumference}}{6}\right)^2 \times (\text{height}).$$

This approximate formula is stated also by Srīdhara,[233] Āryabhaṭa II,[234] Nemicandra[235] and Bhāskara II.[236] The ancient commentators have observed that it was intended only for "rough calculation".

[231] *BrSpSi*, xii. 50.
[232] *SiŚe*, xiii. 50–1.
[233] *Triś*, R. 61.
[234] *MSi*, xv. 115.
[235] *Trilokasāra*, *Gāthā*, 22, 23.
[236] *L*, pp. 69f.

10.4 Frustum of a cone

To find the volume of a frustum of a right circular cone, Śrīdhara gives the following formula:

> The square-root of ten times the square of the sum of the squares of the diameters of the face, base and of their sum, being multiplied by the height and divided by twenty-four, will be the volume of a well.[237]

That is to say, if d, d' denote the diameters of the upper and lower faces of the frustum of a right circular cone and h its height, then its volume V will be given by

$$V = \frac{h}{24}\sqrt{10\left\{d^2 + d'^2 + (d + d')^2\right\}^2},$$

or

$$V = \frac{\pi}{3}\left(r^2 + r'^2 + rr'\right)h,$$

where r, r' denote the radii of the upper and lower faces and $\pi = \sqrt{10}$, the value adopted by Śrīdhara. Other writers have included the treatment of the frustum of a cone in that of a more general kind of obelisk.

Example from Śrīdhara:

> The diameter of the top of a well is 16 cubits, and of the bottom 4 cubits; its depth is 12 cubits. Find, O learned man, its volume.[238]

10.5 Obelisk

An approximate formula for calculating the volume of a frustum of a pyramid on a rectangular base is found as early as the works on the *Śulba* by Baudhāyana (800 BC) and others.[239] If (a, b) be the length and breadth of the base of the solid, (a', b') the corresponding sides of the face parallel to it and h the height, then

$$\text{Volume of the frustum} = \left(\frac{a + a'}{2}\right)\left(\frac{b + b'}{2}\right) \times h.$$

In later treatises of arithmetic we find the accurate formula for the same. Thus Brahmagupta says:

> The area from half the sum of (the edges of) the face and base, being multiplied by the depth gives *vyāvahārika* volume; half the sum of the areas of the face and base being multiplied by the depth

[237] *Triś*, R. 54.
[238] *Triś*, Ex. 91.
[239] Datta, *Śulba*, p. 103.

> will be the *autra* volume. Subtract the *vyāvahārika* volume from the *autra* volume and divide the remainder by three; the quotient added with the *vyāvahārika* volume will become the truly accurate volume.[240]

It is noteworthy that Brahmagupta does not specify the shape of the face and base of the excavation contemplated by him. His text is *mukhatalayuti-dalagaṇitam* etc., or "the area from the half the sum of the face and base," etc. If we, however, suppose them to be rectangular, then according to the rule, we shall have,

$$\begin{aligned} V' &= \left(\frac{a+a'}{2}\right)\left(\frac{b+b'}{2}\right) h, \\ A &= \frac{1}{2}\left(ab+a'b'\right) h, \\ V &= \frac{1}{3}\left(A-V'\right)+V', \end{aligned}$$

where V', A and V denote respectively the *vyāvahārika*, *autra* and accurate volumes of the obelisk. Substituting the values in the last formula, we get

$$V = \frac{h}{6}\left\{(a+a')(b+b')+ab+a'b'\right\}.$$

If the face and base be circular, and if their radii be r' and r respectively, then by the rule

$$\begin{aligned} V' &= \pi\left(\frac{r+r'}{2}\right)^2 h = \frac{\pi}{4}(r+r')^2 h, \\ A &= \left(\frac{\pi r^2+\pi r'^2}{2}\right) h. \end{aligned}$$

Hence

$$\begin{aligned} V &= \frac{h}{3}\left\{\frac{\pi}{2}\left(r^2+r'^2\right)-\frac{\pi}{4}(r+r')^2\right\}+\frac{\pi}{4}(r+r')^2 h, \\ &= \frac{1}{3}\pi h\left(r^2+r'^2+rr'\right). \end{aligned}$$

10.6 Particular cases

(i) Put $a' = 0 = b'$; then we get

$$\text{Volume of a cone or pyramid} = \frac{1}{3}(\text{base})(\text{height}).$$

[240] *BrSpSi*, xii. 45–6.

(ii) Let $b' = 0$;

$$\text{Volume of a wedge} = \frac{h}{6}(2ab + a'b).$$

(iii) Suppose $a = b$, $a' = b'$; then

$$\text{Volume of a truncated square pyramid} = \frac{h}{3}(a^2 + a'^2 + aa').$$

Pṛthūdakasvāmi has worked out the following example in illustration of the above rule of Brahmagupta:

> There is a square tank whose each side is 10 cubits long at the face and 6 cubits long at the base; it is excavated so as to have a depth of 30 cubits. Tell me its *vyāvahārika*, *autra* and truly accurate volumes.

This example has misled some of the modern historians of mathematics to presume that Brahmagupta's rule was meant for the measurement of the volume of a truncated pyramid on a square base only.[241] But, as already pointed out, there is nothing in the definition of the rule to warrant such a limited application of it.[242]

Mahāvīra writes:

> Of the outer (i.e. at the ground) and various inner sections (of the excavation) the sides of the ground section are added by all the corresponding sides of the other sections and divided (by the number of sections). Multiply these sides (of the average section) mutually in accordance with the method of finding the area of a figure of that shape; the result (thus obtained) multiplied by the depth will be the *karmāntika* volume. Find the areas of those sections (severally), add them together and then divide by the number of sectional areas; the quotient multiplied by the depth will be the *aundra* volume. One third the difference of those two volumes added with the *karmāntika* volume will be the truly accurate volume.[243]

It will be noticed that in finding the average volumes, Mahāvīra takes into consideration several parallel sections of the solid, instead of only two, the

[241] Such is the opinion of Cantor, followed by J. Tropfke and D. E. Smith.
[242] See also the article of Datta, Bibhutibhusan, "On the supposed indebtedness of Brahmagupta to Chiu-chang Suan-Shu" in the *BCMS*, xiii (1930), pp. 39–51; more particularly pp. 45 ff.
[243] *GSS*, viii. 9–11$\frac{1}{2}$.

face and base.[244] In three of the illustrative examples,[245] he actually states three sections of the solid. If however, we take into consideration only the top and base, the formula obtained will be the same as that of Brahmagupta.

In illustration of his rule, Mahāvīra gives examples of excavations of various kinds, which are indeed inverted cases of truncated pyramids on square, rectangular, or equilateral triangular bases, and truncated cones. There is an instance of a truncated wedge:

> (In a well with rectangular sections), the lengths at the top, middle and base are 90, 80 and 70 respectively; and the breadths are 22, 16 and 10. Its depth is 7. (Calculate its volume).[246]

Āryabhaṭa II says:

> Divide the sum of the areas of the face, base and that arising from the sum of (the dimensions of) them by six; the quotient multiplied by the height will be the volume of an excavation such as a well and tank.[247]

That is to say,

$$V = \frac{h}{6}\left\{(a + a')\,(b + b') + ab + a'b'\right\}.$$

This formula reappears in the works of Śrīpati and Bhāskara II. The former says:

> The sum of the areas of the face, base and that arising from the sum of their sides, being divided by six and multiplied by the depth, will be the truly accurate value of the volume.[248]

Bhāskara II writes:

> The sum of the areas from (the linear dimensions of) the face, base and their sums, divided by six gives the area of the equivalent prism (*samaṃ kṣetraphalam*) (of the same height). That multiplied by the depth is the true volume.[249]

---

[244] Hence Raṅgācārya is wrong in supposing that the rule contemplates only the face and base.
[245] *GSS*, xiii. $16\frac{1}{2}$–$18\frac{1}{2}$.
[246] *GSS*, vii. $16\frac{1}{2}$. In the printed text 22 is wrongly stated as 32.
[247] *MSi*, xv. 106.
[248] *SiŚe*, xiii. 49.
[249] *L*, p. 65.

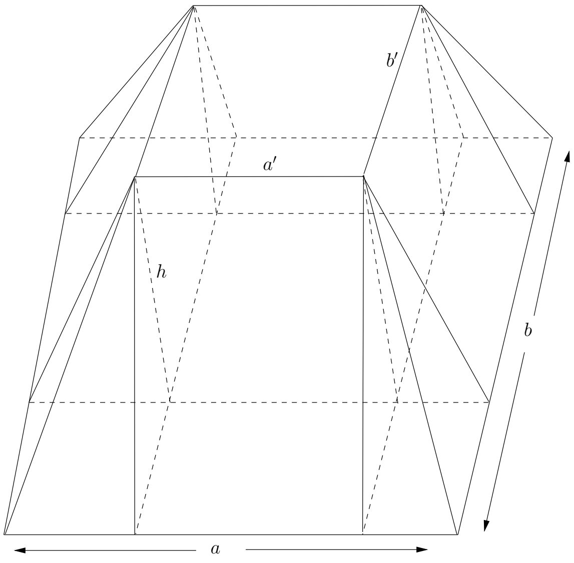

Figure 19

Gaṇeśa's proof

Gaṇeśa demonstrates this formula substantially as follows:

Suppose (a, b) and (a', b') denote the length and breadth of the base and face of the solid respectively. Let its height be h. Then it is clear from the figure (**ed.** see Figure 19) that

$$\begin{aligned} \text{Volume of the obelisk} &= \text{volume of the prism at the centre} \\ &\quad + \text{volumes of four pyramids at the corners} \\ &\quad + \text{volumes of four prisms on four sides.} \end{aligned}$$

Now the four pyramids at the corners can be combined into one of base $(a - a')$ by $(b - b')$ and height h. Hence its volume is

$$\frac{h}{3}(a - a')(b - b').$$

The four side prisms can be combined into two others: (1) one on a triangle of base $(b - b')$ and altitude h, its height being a'; and (2) the other on a triangle of base $(a - a')$ and altitude h; its height will be b'. Therefore their volumes are together equal to

$$\frac{1}{2}(b - b')ha' + \frac{1}{2}(a - a')hb'.$$

Therefore

Volume of the obelisk

$$= a'b'h + \frac{h}{3}\,(a-a')\,(b-b') + \frac{1}{2}\,(b-b')\,ha' + \frac{1}{2}\,(a-a')\,hb'.$$
$$= \frac{h}{6}\,(2ab + 2a'b' + a'b + ab')$$
$$= \frac{h}{6}\,\{(a+a')\,(b+b') + ab + a'b'\}\,.$$

Mahāvīra has treated a problem like this : A fort wall of height h and length l, whose extremities are vertical, has its base b in breadth and face a. Its upper portion is blown off by cyclone, obliquely. It is required to calculate the volume of the portion still intact.[250]

Another problem runs as follows:

> The heights (of a certain construction) are 12, 16, and 20 cubits (at one end, middle and other end respectively); the breadths (at those points) are respectively 7, 6 and 5 cubits at the base and 4, 3 and 2 cubits at the top; the length is 24 cubits. (Find the number of bricks employed in the construction.)[251]

10.7 Surface of a sphere

The earliest reference to a formula for the surface of a sphere occurs, so far as known, in the treatise on arithmetic by Lalla (c. 749). That work is now lost. But the relevant passage has survived in a citation by Bhāskara II.[252] It is as follows:

> The area of the circle (of a diametral section) multiplied by its circumference will be equal to the area of the surface of a sphere.

If r be the radius of a sphere, then according to this rule, its surface S will be

$$S = \pi r^2 \times 2\pi r = 2\pi^2 r^3.$$

This formula is clearly inaccurate. So it has been adversely criticised and discarded by Bhāskara II.[253]

Āryabhaṭa II was undoubtedly aware of a formula for the surface of a sphere, though he has not expressly defined it anywhere. For he says, "the diameter

[250] *GSS*, viii. $52\frac{1}{2}$–$54\frac{1}{2}$.
[251] *GSS*, viii. $51\frac{1}{2}$.
[252] *SiŚi*, *Gola*, iii. 57 (*vāsanā*).
[253] *SiŚi*, *Gola*, iii. 53.

of the earth is a little less than 2109; its circumference is 6625; and the area of its surface is 13971849."[254]

Now according to Āryabhaṭa II, $\pi = \frac{21600}{6876}$. Then

$$\text{Diameter of earth} = \frac{6876}{21600} \times (\text{circumference of the earth}) = \frac{6876}{21600} \times 6625.$$

$$\text{Surface} = 6625\left(2109 - \frac{1}{24}\right) = 13971849 - \frac{1}{24}.$$

Thus it seems that Āryabhaṭa II employed the formula

$$\text{Surface of a sphere} = (\text{circumference}) \times (\text{diameter}).$$

This formula is, however, expressly stated by Bhāskara II.[255] He further says:

> That (the area of a diametral section) multiplied by four is the net lying all over a round ball (i.e., the area of the surface of a sphere).[256]

$$S = 4\pi r^2.$$

Bhāskara II has given a demonstration of this formula by means of the method of infinitesimals. We shall describe it later on.

10.8 Volume of a sphere

Āryabhaṭa I writes:

> That (the area of a diametral section) multiplied by its own square-root is the exact volume of a sphere.[257]

That is to say, if r be the radius of a sphere, then according to Āryabhaṭa I,

$$\text{Volume of a sphere} = \pi r^2\sqrt{\pi r^2}.$$

This formula is inaccurate. Śrīdhara says:

> Half the cube of the diameter of a sphere, then added with its eighteenth part, will give the volume.[258]

$$\text{Volume} = \frac{19 \times (\text{diameter})^3}{18 \times 2}.$$

Mahāvīra writes:

[254] *MSi*, xvi. 35–6.
[255] *SiŚi*, *Gola*, iii. 52, 61.
[256] *L*, p. 55.
[257] *Ā*, ii. 7.
[258] *Triś*, R. 56.

> Nine times the half of the cube of the semi-diameter is the *vyāvahārika* volume of a sphere. Nine-tenth of that will be the very accurate volume.[259]

Āryabhaṭa II:

> The cube of the diameter of a sphere being halved and then added with its eighteenth part, will give its volume in cubic cubits: such is the formula taught (by the ancient teachers).[260]

This formula was given before by Śridhara. It reappears also in the works of Śrīpati.[261] All the above-mentioned formulae for the volume of a sphere are more or less approximate. The truly accurate formula is, however, given by Bhāskara II. He says:

> That area of the surface multiplied by the diameter and divided by six, will be the accurate value of the volume of a sphere.[262]

That is to say, we shall have

$$\text{Volume} = \frac{1}{6}(\text{surface}) \times (\text{diameter}).$$

Now according to Bhāskara II,

$$\begin{aligned} \text{Surface} &= 4 \text{ (area of a diametral section)}, \\ \text{Area of a diametral section} &= \frac{1}{4}(\text{circumference}) \times (\text{diameter}), \\ \text{Circumference} &= \frac{22}{7}(\text{diameter}). \end{aligned}$$

Therefore

$$\begin{aligned} \text{Volume} &= \frac{22}{42}D^3, \\ &= \left(1 + \frac{1}{21}\right) \frac{D^3}{2}. \end{aligned}$$

Hence Bhāskara II writes:

> Half the cube of the diameter being added with its twenty-oneth part becomes the volume of a sphere.[263]

He has further observed that the volume of a sphere obtained by this formula is "rough" (*sthūla*). This is clearly so because that formula is derived with the rough value $\frac{22}{7}$ of π instead of its accurate value $\frac{3927}{1250}$.

[259] *GSS*, viii. $28\frac{1}{2}$.
[260] *MSi*, xvi. 108.
[261] *SiŚe*, xiii. 46.
[262] *L*, p. 55.
[263] *L*, p. 57.

10.9 Average value

In measuring the volume of an excavation whose length, breadth or depth is different at different portions, the other two dimensions remaining the same, the Hindus take for all practical purposes the arithmetic mean of the varying elements. This mean value is technically called *sama-rajju* ("mean measure") by Brahmagupta, *samīkaraṇa* ("equalising value") by Mahāvīra, *sāmya* ("equability", i.e. "equivalent value") by Śrīpati and *samamiti* ("average value") by Bhāskara II.

Brahmagupta says:

> In an excavation having the same breadth at the face and bottom, the aggregates (of the partial products of lengths and depths) divided by the total (length) will be the mean measure (*sama-rajju*) of the depth.[264]

Example from Pṛthūdakasvāmi:

> A tank 30 cubits in length and 8 cubits in breadth contains within it five different excavations which subdivide the length into five portions of lengths four, five etc. (cubits). The depths (of these portions) are successively 9, 7, 7, 3 and 2. Tell at once what is the mean depth of the excavation.

$$\text{Mean depth} = \frac{4\times 9+5\times 7+6\times 7+7\times 3+8\times 2}{4+5+6+7+8} = \frac{150}{30} = 5.$$

Therefore the volume of the tank $= 8 \times 30 \times 5 = 1200$.

Mahāvīra writes:

> Find the half of the top and bottom dimensions; the sum of all the halves divided by the number of them will be the equivalent value.[265]
>
> The sum of the depths (measured at different places) divided by the number of places will be the average depth.[266]

According to Bhāskara II,

> Calculate the breadth at several places: the sum of them divided by the number of places is the average value. Do in the same way in case of the length and depth.[267]

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[264] *BrSpSi*, xii. 44.
[265] *GSS*, viii. 4.
[266] *GSS*, viii. $23\frac{1}{2}$.
[267] *L*, p. 64.

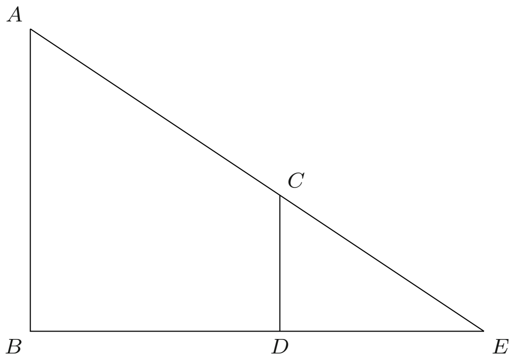

Figure 20

11 Measurement of heights and distances

11.1 Shadow reckoning

The *chāyā*, meaning literally "shadow", but implying truly the measurement by means of shadow of a gnomon, is a common topic for discussion in the Hindu treatises of mathematics. It is applied for measurement of time as well as of heights and distances. We shall, however, notice here only those rules which are related to its application in this latter aspect.[268]

Āryabhaṭa I says:

> Multiply the distance between the gnomon and the lamp-post[269] by the length of the gnomon and divide by the difference between the lengths of the gnomon and the lamp-post. The result will be the length of the shadow of the gnomon measured from its base.[270]

(**ed.** In Figure 20:)

$$AB = \text{the lamp-post},$$
$$CD = \text{the gnomon},$$
$$DE = \text{the shadow of the gnomon},$$
$$DE = \frac{BD \times DC}{AB - CD}.$$

[268]The measurement of time by means of a gnomon is more fully treated in treatises on astronomy.

[269]The Sanskrit original is *bhuja*. Ordinarily the term denotes a side of a triangle (or any rectilinear figure). All the commentators agree in interpreting it as implying here the lamp-post. Latter rules are quite explicit.

[270]$\bar{A}$, ii. 15.

Similar rules are given by Brahmagupta,[271] Mahāvīra,[272] Śrīpati[273] and Bhāskara II.[274] Some later writers[275] have described separately the formulae for calculating the height of the lamp from the length of the shadow and the distance of the gnomon, and the distance from the height of the lamp and the length of the shadow, though the same follow at once from the formula stated above.

11.2 Heights and distances

Another and more useful problem is to find the height and distance of a far off object. By way of illustration of the method employed a high light-post is generally taken into consideration. Then two gnomons of equal heights or the same gnomon successively, being set up in two places at a known distance apart, the two shadows are measured.

Āryabhaṭa I writes:

> The distance between the tips of the two shadows being multiplied by the length of a shadow and divided by the difference between the lengths of the two shadows gives the *koṭi*. That *koṭi* multiplied by the length of the gnomon and divided by the length of the shadow corresponding to it will be the height of the lamp-post.[276]

AB is the lamp-post to be measured (**ed.** see Figure 21); CD, $C'D'$ = the gnomon in its two positions; and DE, $D'E'$ = the shadows respectively. Then the rule says:

$$BE = \frac{EE' \times DE}{D'E' - DE}, \quad BE' = \frac{EE' \times D'E'}{D'E' - DE},$$
$$AB = \frac{BE \times CD}{DE} = \frac{BE' \times CD}{D'E'}.$$

These formulae are stated also by Brahmagupta[277] and Bhāskara II.[278]

11.3 Brahmagupta's rules

The procedure to be adopted in actual practice in measuring the height of a distant object has been indicated by Brahmagupta as follows:

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[271] *BrSpSi*, xii. 53.
[272] *GSS*, ix. $40\frac{1}{2}$.
[273] *SiŚe*, xiii. 54.
[274] *L*, p. 73.
[275] See *GSS*, viii. 43, 45; *SiŚe*, xiii. 55; *L*, p. 74.
[276] *Ā*, ii. 16.
[277] *BrSpSi*, xii. 54.
[278] *L*, p. 75.

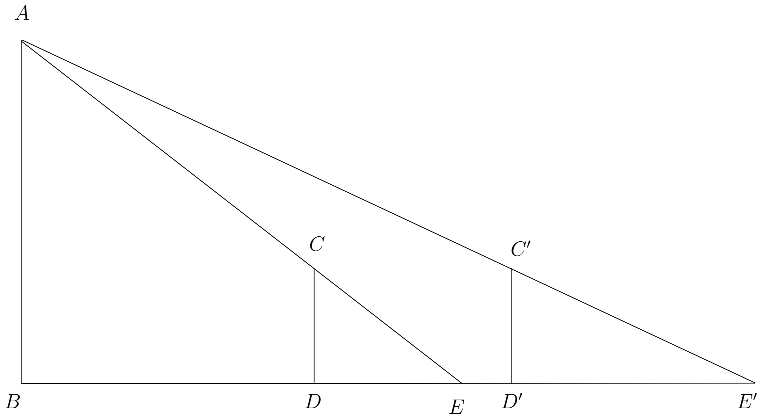


Figure 21

1. Selecting a plane ground, the gnomon is fixed vertically in the position CD (**ed.** see Figure 21). Now the eye is put at the level of the ground at such a place E that E, C and A are in the same straight line. Then the distance DE of the eye from the gnomon is measured. It is called as *dṛṣṭi* (sight) by Brahmagupta. Similar observations are taken with the gnomon in a different position $C'D'$ and the eye E'. The formulae to be applied then are the same as those stated above.

   Brahmagupta re-describes them as follows:

   > The displacement (of the eye) multiplied by a *dṛṣṭi* and divided by the difference of the two *dṛṣṭis* will give the distance of the base. The distance of the base multiplied by the length of the gnomon and divided by its own *dṛṣṭi* will give the height.[279]

2. Observations may also be taken, thinks Brahmagupta, by placing the gnomon horizontally on the level ground (**ed.** see Figure 22). In this case a graduated rod CR is fixed vertically at the extremity C of the gnomon CD nearer to the object to be measured. Then placing the eye at the other end D, the graduation P which is in a straight line with the tip of the object is noted. This gives the altitude CP. Brahmagupta calls it by the term *śalākā* (rod). Observations are taken again with the gnomon in the position $C'D'$.

   Then Brahmagupta says:

   > The displacement (of the gnomon) multiplied by the other *śalākā* and divided by the difference of the two *śalākās* will give the

[279] *BrSpSi*, xxii. 33.

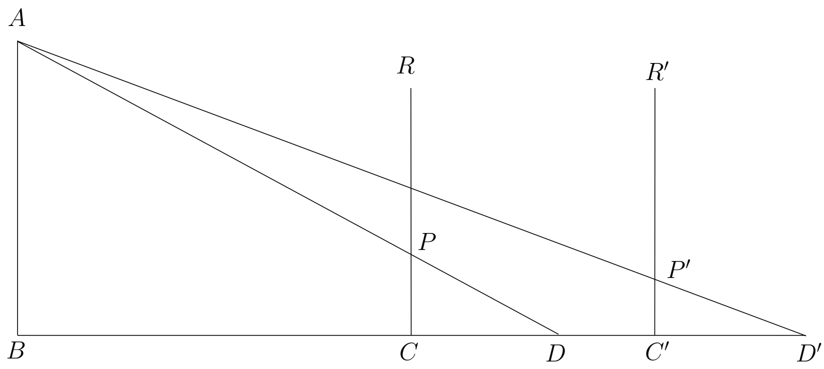


Figure 22

> distance of the base. The distance of the base multiplied by the *śalākā* corresponding to it and divided by the length of the gnomon will give the height of the house etc.[280]

$$BD = \frac{DD' \times C'P'}{CP - C'P'}, \quad BD' = \frac{DD' \times CP}{CP - C'P'},$$
$$AB = \frac{BD \times CP}{CD} = \frac{BD' \times C'P'}{CD}.$$

3. Brahmagupta then gives a different method (**ed.** see Figure 23): Placing the eye at E, the gnomon is first directed towards the base B of the object and then towards its tip A. From the front extremities G, G' of the gnomon in the two positions draw the perpendiculars GN, $G'N'$ to the ground. Also draw the perpendicular EM. Measure the distances MN, MN'.

   Now it can be proved easily that

$$BM = \frac{ME \times MN}{ME - GN},$$

   and

$$AB = ME + \frac{BM(G'N' - ME)}{MN'} \text{ in Figure 23a,}$$

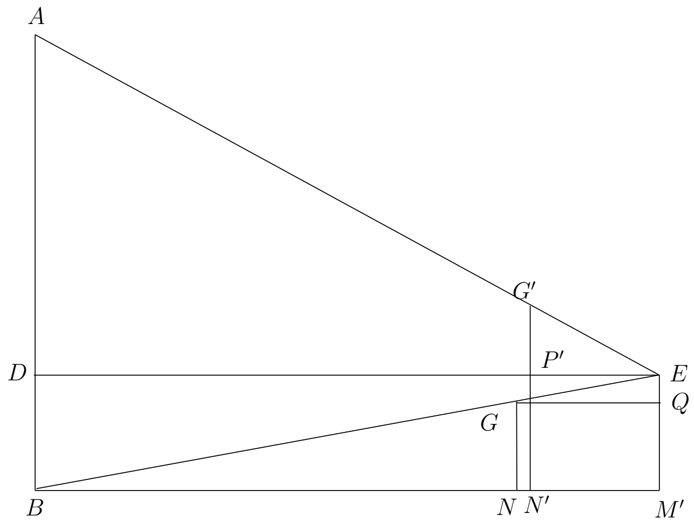
   or

$$AB = ME - \frac{BM(ME - G'N')}{MN'} \text{ in Figure 23b.}$$

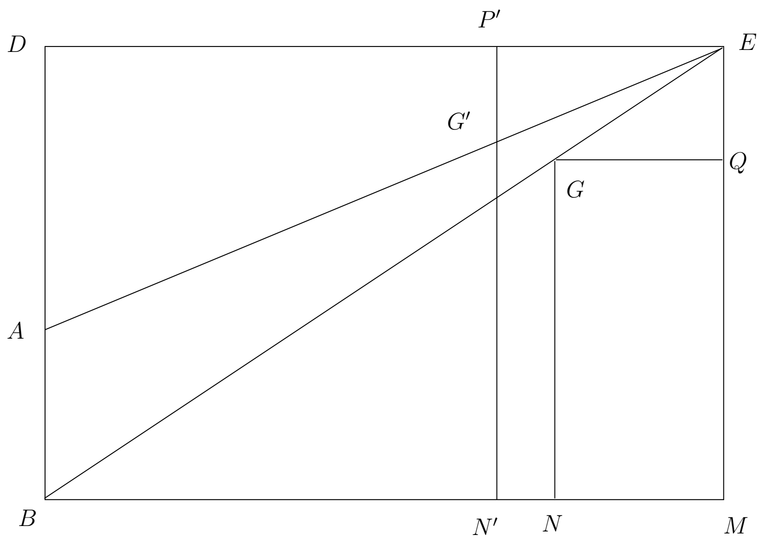
   Hence Brahmagupta says:

   > The distance between the feet of the altitudes (of the eye and the front extremity of the gnomon in the first observation) being

[280] *BrSpSi*, xxii. 32.



(a)



(b)

Figure 23

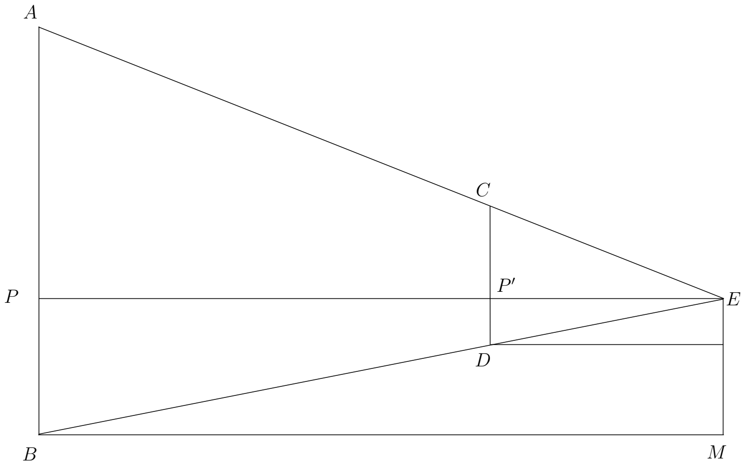


Figure 24

> divided by the difference between the altitudes and multiplied by the greater (altitude) gives the distance of the base. Multiply the distance of the base by the difference between the altitudes (of the eye and the front extremity of the gnomon in the second observation) and divide by the distance between the feet of these altitudes. Then subtract the quotient from the altitude of the eye, if the altitude of the front extremity of the gnomon (in the second observation) be less than the altitude of the eye; or add, if greater. The result gives the height of the house. (Thus the height and distance of an object can be determined) by means of observations of its base and tip.[281]

4. Another method of Brahmagupta is as follows: Placing the eye at E, at an altitude ME over the ground, then fix the gnomon CD in front in such a position that its lower end D will be in the line of sight of the bottom of the object AB and its upper end C in the line of sight of the top of the object (Figure 24). Also note the portion DP' of the gnomon below EP, the horizontal line of sight and the distance EP' of the eye from the gnomon. Then, says Brahmagupta:

> The distance of the eye from the gnomon multiplied by the altitude of the eye and divided by the portion of the gnomon below (the horizontal line of sight) will be the distance of the base. The distance of the base multiplied by the whole gnomon and divided

[281] *BrSpSi*, xxii. 34–5.

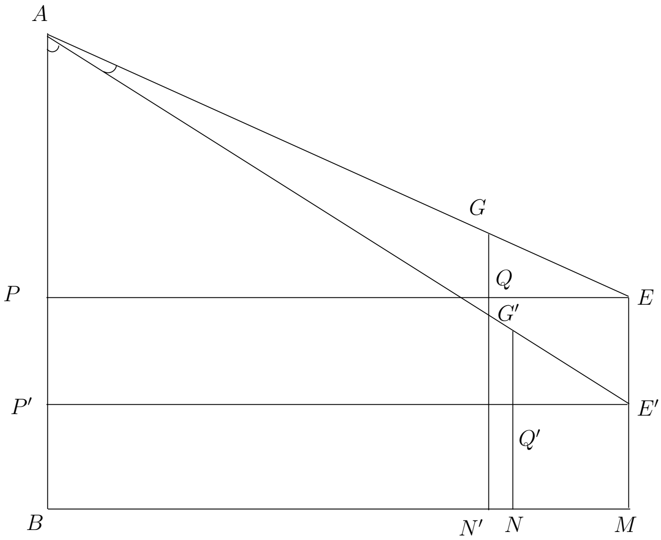


Figure 25

by the distance of the eye from the gnomon will be the height.[282]

$$BM = \frac{EP' \times ME}{DP'}, \quad AB = \frac{BM \times CD}{EP'}.$$

11.4 Bhāskara II

For measuring the heights and distances of far-off objects, Bhāskara II gives two methods, one of which is taken from Brahmagupta. He remarks in general that observations should be made on a plane horizontal ground. Directing the gnomon towards the distant object perpendiculars are drawn from its two extremities on the plane of observation. The horizontal distance between them is the base (*bhuja*), the difference between them is the upright (*koṭi*) and the gnomon itself is the hypotenuse (*karṇa*) of the triangle of observation, says Bhāskara.

> (a) Thus observing the bottom of the bamboo, multiply the base (of the triangle of observation) by the altitude of the eye and divide by the upright: the result is the horizontal distance between the self and the bamboo. Then observing the top of the bamboo, multiply the horizontal distance by the upright the divide by the base; the result added with the altitude of the eye is the height of the bamboo.[283] (Figure 25.)

[282] *BrSpSi*, xxii. 36.
[283] *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 43–4.

(b) Observe the top (of the bamboo) first in the standing posture and then again in the sitting posture. Divide each altitude by its base. The difference of the altitudes of the eye divided by the difference of those quotients gives the horizontal distance. The height of the bamboo can then be determined separately as before.[284]

$$PE = \frac{ME - ME'}{\frac{G'Q'}{E'Q'} - \frac{GQ}{EQ}},$$
$$AB = ME + \frac{PE \times GQ}{EQ} = ME' + \frac{PE \times G'Q'}{E'Q'}.$$

Abbreviations

Ā	*Āryabhaṭīya*	*MaiS*	*Maitrāyāṇīya Saṃhitā*
ĀpŚl	*Āpastamba Śulba*	*MāŚl*	*Mānava Śulba*
BrSpSi	*Brāhmasphuṭasiddhānta*	*MSi*	*Mahā-siddhānta*
BŚl	*Baudhāyana Śulba*	*ŚBr*	*Śatapatha Brāhmaṇa*
GSS	*Gaṇita-sāra-saṃgraha*	*ŚiDVṛ*	*Śiṣyadhī-vṛddhida*
KapS	*Kapisthala Saṃhitā*	*SiŚe*	*Siddhāntaśekhara*
KŚl	*Kātyāyana Śulba*	*SiŚi*	*Siddhāntaśiromaṇi*
KṭS	*Kāṭhaka Saṃhitā*	*Triś*	*Triśatikā*
L	*Līlāvatī*	*TS*	*Tantrasaṃgraha*

[284] *SiŚi*, *Golādhyāya*, *Yantrādhyāya*, 45–6.

Hindu trigonometry *

1 Trigonometrical functions. Definitions.

The Hindu name for the science of trigonometry is *jyotpatti-gaṇita* or "the science of calculation for the construction of the sine".[1] It is found as early as in the *Brāhmasphuṭasiddhānta* of Brahmagupta (628).[2] Sometimes that name is simplified into *jyā-gaṇita* (or "the science of calculation of the sines").[3] In very recent years there has appeared the name *trikoṇamiti*,[4] which is a literal as well as phonetic rendering of the Greek name for the science.

The Hindus introduced and usually employed three trigonometrical functions, namely *jyā*, *koṭi-jyā*, and *utkrama-jyā*. It should be noted that they are functions of an arc of a circle, but not of an angle. If AP be an arc of a circle with centre at O (**ed.** see Figure 1), then its *jyā* $= PM$, *koṭi-jyā* $= OM$, and *utkrama-jyā* $= OA - OM = AM$. Hence their relation with modern trigonometrical functions will be *jyā* $AP = R\sin\theta$, *koṭi-jyā* $AP = R\cos\theta$, *utkrama-jyā* $AP = R - R\cos\theta = R\,\mathrm{versin}\,\theta$, where R is the radius of the circle and θ the angle subtended at the centre by the arc AP. Thus the values of the Hindu trigonometrical functions vary with the radius chosen. The earliest Hindu treatise in which the above trigonometrical functions are now found recorded is the *Sūryasiddhānta*.

Jyā

The Sanskrit word *jyā* means "a bow-string"; and hence "the chord of an arc", for the arc is called "a bow" (*dhanu*, *cāpa*). Its synonyms are *jīvā*, *siñjinī*,[5] *guṇa*, *maurvī*, etc. This trigonometrical function is also called *ardha-jyā*[6]

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 18, No. 1 (1983), pp. 39–108.

[1] *Jyā* ("sine") + *utpatti* ("construction", "generating") + *gaṇita* ("the science of calculation").

[2] *Brāhmasphuṭasiddhānta*, xii. 66.

[3] Compare *Siddhāntatattvaviveka*, ii. 1.

[4] *Trikoṇa* ("triangle") + *miti* ("measure").

[5] *Śiṣyadhīvṛddhida*, ii. 9; *Mahāsiddhānta*, iii. 2.

[6] *Āryabhaṭīya*, i. 10; *Brāhmasphuṭasiddhānta*, ii. 2.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_16

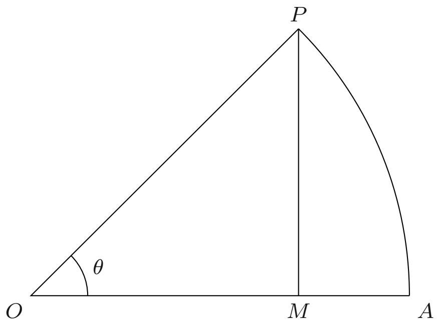


Figure 1

("half-chord") or *jyārdha*[7] ("chord-half"). Thus Bhāskara II (1150) explicitly observes, "It should be known that *ardha-jyā* is here called *jyā*".[8] Parameśvara (1430) remarks:

> A part of a circle is of the form of a bow, so it is called the "bow" (*dhanu*). The straight line joining its two extremities is the "bow-string" (*jīvā*); it is really the "full-chord" (*samasta-jyā*). Half of it is here (called) the "half-chord" (*ardha-jyā*), and half that arc is called the "bow" of that half-chord. In fact the *R*sine (*jyā*) and *R*cosine (*koṭi-jyā*) of that bow are always half-chords.[9]

Kamalākara (1658) is more explicit. "Having seen the brevity", says he, "the half-chords are called *jyā* by mathematicians in this (branch of) mathematics and are used accordingly."[10] The function *jyā* is sometimes distinguished as *krama-jyā*[11] or *kramārdha-jyā*,[12] from *krama*, "regular" or "direct" meaning "direct sine" or "direct half-chord".

It may be noted that the modern term sine is derived from the Hindu name. The Sanskrit term *jīvā* was adopted by the early Arab mathematicians but was pronounced as *jība*. It was subsequently corrupted in their tongue into *jaib*. The latter word was confused by the early Latin translators of the Arabic works such as Gherardo of Cremona (c. 1150 AD) with a pure Arabic word of alike phonetism but meaning differently "bosom" or "bay" and was rendered as *sinus*, which also signifies "bosom" or "bay".[13]

[7] *Sūryasiddhānta*, ii. 15; *Āryabhaṭīya*, ii. 11, 12; *Brāhmasphuṭasiddhānta*, xxi. 17, 22.
[8] *Siddhāntaśiromaṇi*, *Graha*, iii. 2.
[9] *Āryabhaṭīya*, ii. 11 (comm.).
[10] *Siddhāntatattvaviveka*, ii. 52.
[11] *Pañcasiddhāntikā*, iv. 28 (*kramaśo-jyā*); *Brāhmasphuṭasiddhānta*, ii. 15; vii. 12.
[12] *Śiṣyadhīvṛddhida*, ii. 1.
[13] *Cf. Nouv. Ann. Math.*, XIII (1854), p. 393; Smith, *History II*, p. 616.

The degeneration and variations of the term *kramajyā* are still more interesting. In the Arabic tongue it was corrupted into *karaja* or *kardaja*. According to *Fihrist*, the title of a work of Ya'kūb ibn Ṭārik (c. 770 AD) is "On the table of *kardaja*". This table was copied from the *Brāhmasphuṭasiddhānta* of Brahmagupta. In the same connection, al-Khowārizmī (825 AD) used the variant *karaja*. In the Latin translations of the term we find several variants such as *kardaga*, *karkaya*, *gardaga*, or *cardaga*. These terms had in foreign lands also the restricted uses for the arc of $3°45'$, sometimes of $15°$.[14]

Koṭi-jyā

The Sanskrit word *koṭi* means, amongst others "the curved end of a bow" or "the end or extremity in general"; hence in trigonometry it came to denote "the complement of an arc to 90°."[15] So the radical significance of the term *koṭi-jyā* is "the *jyā* of the complementary arc". But it began early to be used as an independent technical term.[16] The modern term cosine appears to be connected with *koṭijyā*, for in Hindu works, particularly in the commentaries *koṭijyā* is often abbreviated into *kojyā*. When *jyā* became *sinus*, *kojyā* naturally became *ko-sinus* or *co-sinus*.

Utkrama-jyā

Utkrama means "reversed", "going out", or "exceeding". Hence the term *utkrama-jyā* literally means "reversed sine". This function is so called in contradistinction to *krama-jyā*, for it is, rather its tabular values are, derived from the tabular values of the latter by subtracting the elements from the radius in the reversed order. Or in other words it is the exceeding portion of the *krama-jyā* taken into consideration in the reversed order. Thus it is stated:

> The (tabular) versed sines are obtained by subtracting from the radius the (tabular) sines in the reversed order.[17]
>
> They (*jyārdha*), (being subtracted from the radius), in the reversed order beginning from the end, will certainly give the versed sines, that is, the arrows."[18]

Again, it is noteworthy that from a table of differences of sines, the successive sines are obtained by adding the differences in the direct order (from

[14] Woepeke, F. "Sur le mot *kardaga* et sur une méthode indienne pour calcul les sinus", *Nouv. Ann. Math.*, XIII (1854), pp. 386–393; Braunmühl, A. *Geschichte der Trigonometrie*, 2 vols., Leipzig, 1900, 1903 (hereafter referred to as Braunmühl, *Geschichte*); Vol. I, pp. 44, 45, 78, 102, 110, 120; *vide* also Sarton's note on the point in *Isis*, xiv (1930), pp. 421f.

[15] In Hindu mathematics, the term *koṭi* also denotes "the side of a right-angled triangle".

[16] Compare *Śiṣyadhīvṛddhida*, ii. 30 (infra p. 10).

[17] *Sūryasiddhānta*, ii. 22.

[18] *Brāhmasphuṭasiddhānta*, xxi. 18; Compare also *Mahāsiddhānta*, iii. 3.

the top) whereas the corresponding versed sines will be found by adding the elements in the reversed order (from the end). This fact has been particularly noted by Sūryadeva Yajvā (born 1191 AD) and Śrīpati (1039 AD). The former observes:

> In order to get the direct sines (*krama-jyā*), these (tabular) differences of sines (*khaṇḍa-jyā*) should be added regularly from the beginning; and in order to determine the reversed sines (*utkrama-jyā*), they should be added in the reversed order from the end.[19]

Śrīpati says:

> The difference of sines are called *jyākhaṇḍa* (tabular "difference of sines"); (adding them) in the reversed way beginning from the end will be obtained the versed sines (*vyasta-jyā*) of the half-arcs equal to the 96th parts of the celestial circle.[20]

This function is also called *vyasta-jyā*[21] (from *vyasta*, "cast or thrown asunder", "reversed") or *viloma-jyā*[22] (from *viloma*, "reverse"). Occasionally it is termed *utkrama-jyārdha*.[23] Another name for it is "arrow" (*iṣu*, *bāṇa*).[24] Bhāskara II observes:

> What is really the arrow between the bow and the bowstring is known amongst the scholars here (i.e. in trigonometry) as the versed sine.[25]

So also says Kamalākara (1658):

> What lies between the chord and the arc, like the arrow, is the versed sine.[26]

Tangent and Secant

The Hindus approached very near the tangent and secant functions and actually employed them in astronomical calculations, though they did not expressly recognise them as separate functions. The *Sūryasiddhānta* gives the following rule for calculating the equinoctial midday shadow of the gnomon at a station:

[19] *Āryabhaṭīya*, i. 10 (comm.).
[20] *Siddhāntaśekhara*, xvi. 10.
[21] *Brāhmasphuṭasiddhānta*, ii. 5; *Mahāsiddhānta*, iii. 3, 6.
[22] *Śiṣyadhīvṛddhida*, I, ii. 5.
[23] *Sūryasiddhānta*, ii. 22, 27.
[24] *Brāhmasphuṭasiddhānta*, xxi. 18.
[25] *Siddhāntaśiromaṇi*, *Gola*, xiv. 5; Compare also *Graha*, ii. 20 (gloss).
[26] *Siddhāntatattvaviveka*, ii. 58.

> The sine of the latitude (of the station) multiplied by 12 and divided by the cosine of the latitude gives the equinoctial mid-day shadow.[27]

Here 12 is the usual height of a Hindu gnomon. So that

$$S = \frac{jy\bar{a}\ \phi \times h}{kojy\bar{a}\ \phi},$$

where ϕ denotes the latitude of the place, S is the equinoctial mid-day shadow, and h is the gnomon. This is equivalent to

$$S = h \tan \theta.$$

Again to find the mid-day shadow (s) of the gnomon (h) and the hypotenuse (d), having known the meridian zenith distance (z) of the sun, we have the rules:[28]

$$s = h \tan z, \qquad d = h \sec z.$$

Similar rules occur in other astronomical works also.[29] In the *Gaṇita-sāra-saṅgraha* of Mahāvīra (850) by the term "shadow" of a gnomon is sometimes meant the ratio of the actual shadow to the height of the gnomon.[30] This ratio, as has been just stated, is equal to the tangent of the zenith distance of the sun.

Quadrants

A circle is ordinarily divided into four equal parts, called *vṛtta-pāda*, by two perpendicular lines, usually the east-to-west line and the north-to-south line. The quadrants are again classified into odd (*ayugma*, *viṣama*) and even (*yugma*, *sama*). Earlier Hindu writers do not explain this fact fully and particularly. Thus Bhāskara I (629) simply observes: "Three signs form a quadrant".[31] Lalla writes:

> Three anomalistic signs form a quadrant. The quadrants are successively distinguished as odd and even.[32]

But the description of Bhāskara II (1150) is very full. He says:

> Three signs together form a quadrant. In a circle there will be four such; and they should be successively called odd and even.[33]

[27] *Sūryasiddhānta*, iii. 16.
[28] *Sūryasiddhānta*, iii. 21.
[29] *Pañcasiddhāntikā*, iv. 22.
[30] *Gaṇita-sāra-saṅgraha*, ix. $8\frac{1}{2}$.
[31] *Mahābhāskarīya*, iv. 1; *Laghubhāskarīya*, ii. 1.
[32] *Śiṣyadhīvṛddhida*, ii. 10.
[33] *Siddhāntaśiromaṇi*, *Graha*, ii. 19.

He then explains it further thus:

> On a plane surface describe a circle of any specified radius with a pair of compasses. Mark on its circumference 360 degrees. Draw the east-to-west and north-to-south lines through its centre. These lines will divide the circle into quadrants, which should be taken into consideration in the left-wise manner (*savya-krama*, that is 'anti-clockwise')[34] proceeding from the east-point (*prācī*); They should be called odd and even (quadrants) successively.[35]

Variation in value

As regards the variation in the value of a trigonometrical function as its argument changes, Bhāskara II observes as follows:

> In the first quadrant, mark a point on the circumference of the circle at any optional distance from the east point. The perpendicular distance of that point from the east-to-west line is called the Rsine (*doḥ-jyā*); and its distance from the north-to-south line is the Rcosine (*koṭi-jyā*). The corresponding arcs are called *bhuja* and *koṭi*. (Starting from the east point) as the point gradually moves forward in the same way (i.e. anti-clockwise), the Rsine increases and the Rcosine decreases. When the point arrives at the end of the quadrant, the Rcosine vanishes and the Rsine is equal to the radius. Then in the second quadrant, the Rcosine increases; at the end of that quadrant the Rcosine is maximum (irrespective of sign) and the Rsine vanishes.[36]

One fact perhaps deserves a particular notice here. It is that in Hindu trigonometry the *jyā* of an arc of 90° in a circle is equal to the radius of that circle. On account of that, the radius is called in Hindu mathematics by the terms *tri-jyā*, *tri-bha-jyā*, *tribhavana-jyā*, etc., every one of which literally means the "sine of three signs". The radius is also called *viṣkambhārdha*, *vyāsārdha*, or *ardha-vyāyāma* meaning the "semi-diameter". All these terms are very old.[37]

Functions of a complement or supplement

Sūryasiddhānta says:

[34] The Sanskrit term *savya-krama* ordinarily signifies the "clockwise direction"; but it may also denote the "anti-clockwise direction".

[35] *Siddhāntaśiromaṇi*, *Graha*, ii. 19 (gloss).

[36] *Ibid*, ii. 20 (gloss). The Sanskrit terms *jyā* and *kojyā* have been translated as Rsine and Rcosine because they are equal to $R \times$ sine and $R \times$ cosine respectively.

[37] Compare *Āpastamba-Śulbasūtra*, vii. 11 (*ardha-vyāyāma*); *Jambūdvīpasamāsa* of Umāsvāti, iv (*vyasārdha*); *Tattvārthādhigama-sūtra-bhāṣya*, iv. 14 (*viṣkambhārdha*).

> In odd quadrants, the arc passed over gives the *R*sine, while the arc to be passed over gives the *R*cosine; and in the even quadrants, the arc to be passed over gives the *R*sine and that passed over gives the *R*cosine.[38]

Bhāskara I writes:

> In the odd quadrants the arc described and that to be described should respectively be known as the *bhuja* and *koṭi*; but in the even quadrants they are respectively the *koṭi* and *bhuja*; this is the fact.[39]

Lalla remarks:

> When (the anomaly[40] is) greater than 90°, it is subtracted from the semi-circle (i.e. 180°); when greater than the semi-circle, 180° is subtracted from it; when greater than 270°, it is subtracted from the complete circle (i.e. 360°); the remainder is called the (corresponding) *bhuja* by the expert in the subject.[41]

In the words of Brahmagupta:

> The *R*sine and *R*cosine (are obtained) in the odd quadrants from the arc passed over and to be passed over (respectively); and in the even quadrants in the reverse way.[42]

Or,

> In the odd quadrants (the *R*sine is determined) from the arc described and in the even quadrants from the arc to be described.[43]
>
> (For the determination of) the *R*sine (proceed with the anomaly as it is) when the anomaly is less than three signs (i.e. 90°); when greater than three signs subtract it from six signs; when greater than six signs, subtract six signs (from it); when greater than nine signs, subtract it from the complete circle.[44]

Mañjula (932) says:

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[38] *Sūryasiddhānta*, ii. 30.

[39] *Laghubhāskarīya*, ii. 1–2; compare also *Mahābhāskarīya*, iv. 8–9.

[40] It is in connection with the treatment of the anomaly that the remark of Lalla, as of several other Hindu mathematicians, occurs.

[41] *Śiṣyadhīvṛddhida*, ii. 10–11.

[42] *Brāhmasphuṭasiddhānta*, ii. 12.

[43] *Khaṇḍakhādyaka* (Bina Chatterjee's edition), I, i. 16.

[44] *Khaṇḍakhādyaka*, I, i. 16.

> In the odd quadrants, the *bhuja* and *koṭi* are (to be calculated) from the arc described and that to be described (respectively); but in the even quadrants in the contrary way.[45]

His commentator and younger contemporary Praśastidhara (962) dilates upon this point thus:

> In the odd quadrant, where the anomaly is less than three signs (i.e. 90°), the *R*sine should be calculated from it and the *R*cosine should be calculated after subtracting that from 90°. In the even quadrant, where the anomaly exceeds 90° but is less than 180°; in that case the *R*sine should be taken after subtracting it from 180° and the cosine after subtracting 90° from it. In the odd quadrant, where the anomaly is greater than 180°, but less than 270°, the *R*sine should be calculated after subtracting 180° from it and the *R*cosine after subtracting it from 270°. In the even quadrant when the anomaly exceeds 270°, but is less than 360°, the *R*sine is determined after subtracting it from 360°, and the *R*cosine after subtracting 270° from it.[46]

Śrīpati (c. 1039) remarks:

> In the odd and even quadrants, the arc passed over and to be passed over (respectively) is the *bhuja* and the *koṭi* is otherwise. Or, as the learned have said, the *R*sine of 90° minus the anomaly is the *R*cosine (of the anomaly).[47]

And Bhāskara II:

> In the odd quadrants, the arc passed over and in the even quadrants the arc to be passed is the *bhuja*. Ninety degrees minus the *bhuja* is said to be the *koṭi*.[48]

The above results can be represented graphically as shown in Figures 2 and 3.

Relation between functions

Varāhamihira says:

> The *R*sine of 90° minus latitude is the *R*cosine of the latitude.[49]

[45] *Laghumānasa*, ii. 2.
[46] Commentary on the same.
[47] *Siddhāntaśekhara*, iii. 13.
[48] *Siddhāntaśiromaṇi*, *Graha*, ii. 19.
[49] *Pañcasiddhāntikā*, iv. 28.

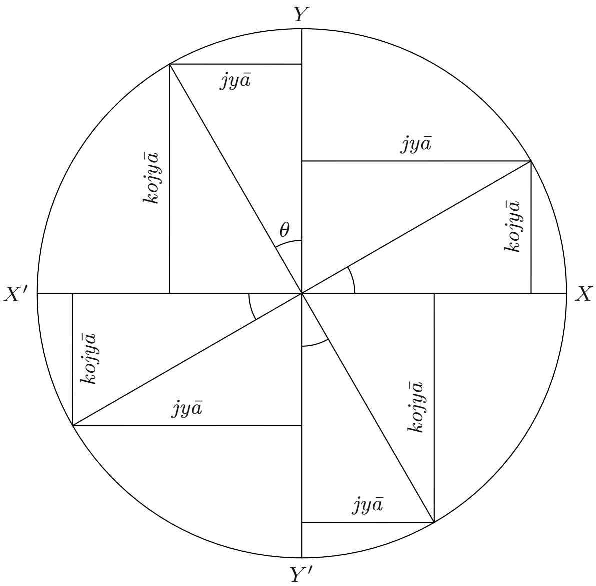

Figure 2

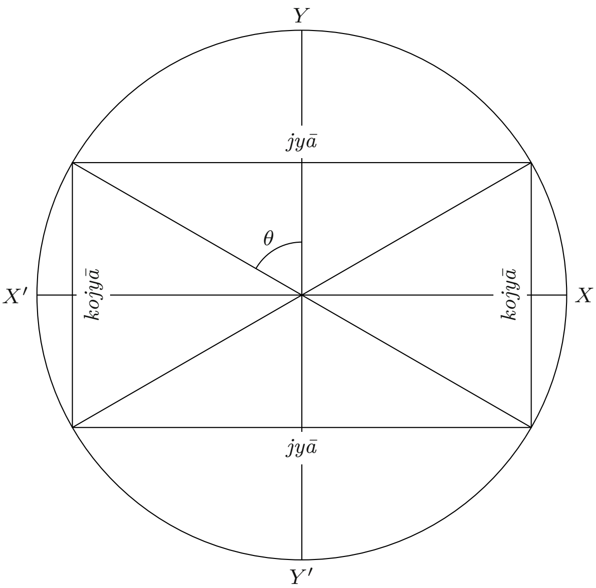

Figure 3

Lalla:

> The square of the base-sine (*bhuja-jyā*) is subtracted from the square of the radius; the square root of the remainder is the *R*cosine; or it is the *R*sine of 90° minus the *bhuja* arc.[50]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = kojy\bar{a}\,\alpha$$
$$\text{or,} \qquad kojy\bar{a}\,\alpha = jy\bar{a}\,(90^\circ - \alpha),$$

where *kojyā* is the usual Hindu symbol for *koṭi-jyā*.

Brahmagupta says:

> The radius diminished by the versed *R*sine of an arc or of its complement will give the *R*sine of the other. The square-root of the difference of the square of the radius and that of the *R*sine of an arc or of its complement will be the *R*sine of the other.[51]

$$R - utjy\bar{a}\,\alpha = jy\bar{a}\,(90^\circ - \alpha),$$
$$R - utjy\bar{a}\,(90^\circ - \alpha) = jy\bar{a}\,\alpha,$$
$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = jy\bar{a}\,(90^\circ - \alpha),$$
$$\sqrt{R^2 - \{jy\bar{a}\,(90^\circ - \alpha)\}^2} = jy\bar{a}\,\alpha,$$

where *utjyā* is the usual abbreviation for *utkrama-jyā*a

> The direct *R*sine of the excess of an arc over 90° added to the radius will give versed *R*sine of that arc.[52]

$$R + jy\bar{a}\,(\alpha - 90^\circ) = utjy\bar{a}\,\alpha,$$

where $\alpha > 90^\circ$.

Śrīpati writes:

> The square of the radius is diminished by the square of the *R*sine; the square root of the remainder will be the *R*cosine. Again the square-root of the square of the radius minus the square of the *R*cosine will be the *R*sine. The radius minus the versed *R*sine of the complement of an arc is equal to the *R*sine of the arc, and minus the versed *R*sine of the arc becomes the *R*sine of the other (i.e. complement).[53]

[50] *Śiṣyadhīvṛddhida*, ii. 30.
[51] *Brāhmasphuṭasiddhānta*, xiv. 7.
[52] *Ibid*, vii. 12.
[53] *Siddhāntaśekhara*, iii. 14.

The treatment of Bhāskara II is exhaustive. He says:

> Subtract from the radius the direct *R*sine of an arc and of its complement; the results will be the versed *R* sines of the complement and the arc (respectively). Subtract from the radius the versed *R*sine of an arc and of its complement; the remainders will be the direct *R*sines of the complement and the arc (respectively).[54]

$$\begin{aligned} R - jy\bar{a}\,\alpha &= utjy\bar{a}\,(90^\circ - \alpha), \\ R - jy\bar{a}\,(90^\circ - \alpha) &= utjy\bar{a}\,\alpha, \\ R - utjy\bar{a}\,\alpha &= jy\bar{a}\,(90^\circ - \alpha), \\ R - utjy\bar{a}\,(90^\circ - \alpha) &= jy\bar{a}\,\alpha. \end{aligned}$$

> The square of the *R*sine of an arc and of its complement are (severally) subtracted from the square of the radius, the square-roots of the results are (respectively) the *R*sines of the complement and of the arc.[55]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = jy\bar{a}\,(90^\circ - \alpha); \quad \sqrt{R^2 - \{jy\bar{a}\,(90^\circ - \alpha)\}^2} = jy\bar{a}\,\alpha.$$

> The square of the radius is diminished by the square of the *R*sine of an arc; the square-root of the result is the *R*cosine of the arc.[56]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = kojy\bar{a}\,\alpha.$$

Kamalākara writes:

> The square-root of the square of the radius diminished by the square of the *R*sine of an arc, is the *R*cosine of the arc; similarly, the square-root of the square of the radius diminished by the *R*cosine of an arc, is the *R*sine of the arc. Again, the *R*sines of an arc and its complement when subtracted from the radius will give the versed *R*sines of the complement and the arc (respectively).[57]

$$\sqrt{R^2 - (jy\bar{a}\,\alpha)^2} = kojy\bar{a}\,\alpha, \qquad \sqrt{R^2 - (kojy\bar{a}\,\alpha)^2} = jy\bar{a}\,\alpha,$$
$$R - jy\bar{a}\,\alpha = utjy\bar{a}\,(90^\circ - \alpha), \qquad R - jy\bar{a}\,(90^\circ - \alpha) = utjy\bar{a}\,\alpha.$$

[54] *Siddhāntaśiromaṇi, Graha*, ii. 20; also *Gola*, v. 2; xiv. 5.
[55] *Siddhāntaśiromaṇi*, *Graha*, ii. 21.
[56] *Siddhāntaśiromaṇi*, *Gola*, v. 2; xiv. 4.
[57] *Siddhāntatattvaviveka*, ii. 56–7.

Change of sign of a function

The Hindus were fully aware of the changes of sign of a trigonometrical function according as its argument lies in different quadrants. Though nowhere do we find any systematic treatment of this principle in any Hindu work there are ample concrete instances of its application in almost all their important astronomical treatises. Thus it is stated in the *Sūryasiddhānta*:

> The *śīghra-koṭiphala* is positive, when the *kendra* (mean anomaly) lies in a position beginning with the Capricorn; and it is to be subtracted from the radius in a position beginning with the Cancer.[58]

Now according to the *Sūryasiddhānta* and other Hindu astronomical works, the *śīghra-koṭiphala* (the result derived from the complement of the distance from the conjunction) is given by $D\cos\theta$, where θ is the *śīghra-kendra*[59] (the distance of the mean planet from its apex of swiftest motion; hence mean *śīghra* anomaly) and D, a certain known constant. The Cancer is the fourth sign of the Zodiac and Capricorn is the tenth sign. Again the motion of the mean planet is anti-clockwise. Hence it is clear from the above rule that the author was aware that the cosine of an angle lying between 0° and 90° or between 270° and 360° is positive and that it is negative when the angle lies between 90° and 270°.

Again it has been said:

> In case of the *manda* and *śīghra* corrections of all planets, the *phala* (equation) will be positive, if the *kendra* lies in the six signs beginning with the Aries and it will be negative in the six signs beginning with the Libra.[60]

Now the *phala* is defined as arc $(D'\sin\theta)$, where D' does not change sign. Hence clearly the author knows that the sign is positive in the first two quadrants and negative in the other two quadrants.

Similar rules are found in other treatises of astronomy.[61] The statement of Mañjula (932) is more explicit and fuller. He says:

> The (mean) planet when diminished by its apogee or aphelion is the *kendra* (mean anomaly). Its *R*sine is positive or negative in the upper or lower halves (of the quadrants); and its *R*cosine is

[58] *Sūryasiddhānta*, ii. 40.

[59] "Subtract the longitude of a planet from that of its apex of slowest motion (*mandocca*); so also subtract it from that of its apex of swiftest motion (conjunction); the result (in either case) is its *kendra*." *Sūryasiddhānta*, ii. 29.

[60] *Sūryasiddhānta*, ii. 45.

[61] For instance *Āryabhaṭīya*, iii. 22; *Mahābhāskarīya*, iv. 5, 9; *Laghubhāskarīya*, ii. 6; *Śiṣyadhīvṛddhida*, ii. 32; *Brāhmasphuṭasiddhānta*, ii. 14ff.

> positive, negative, negative, and positive (respectively) according to the (successive) quadrants.[62]

Thus the Hindus knew very early what in modern trigonometrical notations will be expressed as

$$\begin{aligned}
&\sin(\pi \mp \theta) = \pm \sin\theta, && \cos(\pi \mp \theta) = -\cos\theta,\\
&\sin(2\pi - \theta) = -\sin\theta, && \cos(2\pi - \theta) = +\cos\theta,\\
&\sin\left(\frac{\pi}{2} \mp \theta\right) = +\cos\theta, && \cos\left(\frac{\pi}{2} \mp \theta\right) = \pm\sin\theta,\\
&\sin\left(\frac{3\pi}{2} \mp \theta\right) = -\cos\theta, && \cos\left(\frac{3\pi}{2} \mp \theta\right) = \mp\sin\theta.
\end{aligned}$$

Again it has been stated before that according to a rule of Brahmagupta

$$R + jy\bar{a}\ (\alpha - 90^\circ) = utjy\bar{a}\ \alpha.$$

But by definition,

$$utjy\bar{a}\ \alpha = R - kojy\bar{a}\ \alpha = R - jy\bar{a}\ (90^\circ - \alpha).$$

These clearly show that the author knows that the value of the sine function changes sign along with its argument. Or symbolically

$$\sin(\pm\theta) = \pm\sin\theta.$$

2 Trigonometrical formulae

(1) $\sin^2\theta + \cos^2\theta = 1.$

It has been stated before that according to Hindu astronomers, if α be an arc of a circle of radius R

$$\sqrt{R^2 - (jy\bar{a}\ \alpha)^2} = kojy\bar{a}\ \alpha, \quad \sqrt{R^2 - (kojy\bar{a}\ \alpha)^2} = jy\bar{a}\ \alpha.$$

These are of course equivalent to the modern formulae

$$\sqrt{1 - \sin^2\theta} = \cos\theta, \quad \sqrt{1 - \cos^2\theta} = \sin\theta$$

or

$$\sin^2\theta + \cos^2\theta = 1,$$

where θ is the angle subtended at the centre of the circle by the arc α.

(2) $4\sin^2\dfrac{\theta}{2} = \sin^2\theta + \mathrm{versin}^2\,\theta.$

This formulae has been stated first by Varāhamihira (505). He says:

[62] *Laghumānasa*, ii. 1.

> To find the *R*sine of any other desired arc, double the arc and subtract from the quarter of a circle; diminish the radius by the *R*sine of the remainder. The square of half the result is added to the square of half the *R*sine of the double arc. The square-root of the sum is the desired *R*sine.[63]

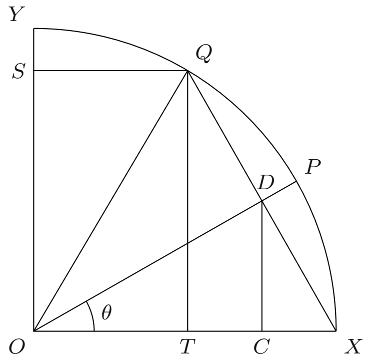

Figure 4

(**ed.** In Figure 4) Let the arc XP = arc $PQ = \alpha$; then arc $QY = 90° - 2\alpha$. Now

$$XQ^2 = QT^2 + TX^2$$
$$\text{or}\quad 4XD^2 = QT^2 + TX^2$$
$$\text{or}\quad XD^2 = \left(\frac{QT}{2}\right)^2 + \left(\frac{TX}{2}\right)^2.$$

Hence

$$(jy\bar{a}\ \alpha)^2 = \left(\frac{jy\bar{a}\ 2\alpha}{2}\right)^2 + \left(\frac{R - jy\bar{a}\ (90° - 2\alpha)}{2}\right)^2;$$

which is equivalent to

$$4\sin^2\theta = \sin^2 2\theta + \text{versin}^2\, 2\theta.$$

Āryabhaṭa I (499) seems to have been aware of this formula before Varāhamihira. It reappears also in later works.

Brahmagupta says:

> The sum of the squares of the *R*sine and versed *R*sine of the same arc is divided by four; subtract this quotient from the square of the radius. Take the square-root of the two results. The former will be

[63] *Pañcasiddhāntikā*, iv. 2f.

> the *R*sine of half that arc, and the other the *R*sine of the arc equal to the quarter circle less that half.[64]

The formula has been described almost similarly by Śrīpati (1039).[65] Bhāskara II (1150) writes very briefly thus:

> Half the square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc, will be the *R*sine of half that arc.[66]

Parameśvara (1430) says:

> The square-root of the sum of the square of the *R*sine and of the versed *R*sine of an arc is the 'whole chord' (*samasta-jyā*) of that arc. Half that is the half-chord (i.e. the *R*sine) of half that arc.[67]

(3) $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta.$

This is given first by Varāhamihira. He says:

> Twice any desired arc is subtracted from three signs (i.e. 90°), the *R*sine of the remainder is subtracted from the *R*sine of three signs. The result multiplied by sixty is the square of the *R*sine of that arc.[68]

In the Figure 4, since the triangles XCD and XDO are similar, we have:

$$XD : XC :: XO : XD$$
$$\therefore XD^2 = XO \times XC = \frac{1}{2}XO \times XT.$$

Hence

$$(jy\bar{a}\ \alpha)^2 = \frac{1}{2}R\left\{R - jy\bar{a}\ (90^\circ - 2\alpha)\right\}.$$

The factor $\frac{1}{2}R$ on the right-hand side has been stated by Varāhamihira as 60 since he has taken the value of the radius to be equal to 120. In modern notations, the above formula becomes

$$\sin^2\theta = \frac{1}{2}\,(1 - \cos 2\theta).$$

This also follows easily from the preceding formula.

Brahmagupta says:

> The square-root of the fourth part of the versed *R*sine of an arc multiplied by the diameter is the *R*sine of half that arc.[69]

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[64] *Brāhmasphuṭasiddhānta*, xxi. 20f.
[65] *Siddhāntaśekhara*, xvi. 14–5.
[66] *Siddhāntaśiromaṇi*, *Gola*, v. 4; xiv. 10.
[67] Quoted in his commentary of *Āryabhaṭīya*, ii. 11.
[68] *Pañcasiddhāntikā*, iv. 5.
[69] *Brāhmasphuṭasiddhānta*, xxi. 23.

Bhāskara II writes:

> Or, the square-root of half the product of the radius and the versed *R*sine of an arc, will be the *R*sine of half that arc.[70]

He has further given the following proof of it.[71]

Since

$$kojy\bar{a}\,\alpha = R - utjy\bar{a}\,\alpha,$$

so that squaring

$$(kojy\bar{a}\,\alpha)^2 = R^2 + (utjy\bar{a}\,\alpha)^2 - 2R \times utjy\bar{a}\,\alpha.$$

Therefore

$$R^2 - (kojy\bar{a}\,\alpha)^2 = 2R \times utjy\bar{a}\,\alpha - (utjy\bar{a})^2.$$

Or

$$(jy\bar{a}\,\alpha)^2 = 2R \times utjy\bar{a}\,\alpha - (utjy\bar{a}\,\alpha)^2$$
$$(jy\bar{a}\,\alpha)^2 + (utjy\bar{a}\,\alpha)^2 = 2R \times utjy\bar{a}\,\alpha.$$

But by the formula (2), the right-hand side is equal to

$$4\left(jy\bar{a}\,\frac{\alpha}{2}\right)^2.$$

Hence,

$$jy\bar{a}\,\frac{\alpha}{2} = \sqrt{\frac{1}{2}R \times utjy\bar{a}}.$$

This rule of Bhāskara II together with his proof has been reproduced by Kamalākara.[72]

(4) $\sin \frac{1}{2}\,(90° \pm \theta) = \sqrt{\frac{1}{2}(1 \pm \sin\theta)}.$

This formula first appears in the works of Āryabhaṭa II (950). He says:

> The *R*sine of any arc multiplied by the radius is subtracted from or added to the square of the maximum value of the *R*sine; the square-root of half the results are extracted. These will be the *R*sine of 45° decreased or increased by half that arc.[73]

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[70] *Siddhāntaśiromaṇi*, *Gola*, v. 5; xiv. 10.
[71] *Siddhāntaśiromaṇi*, *Gola*, (gloss).
[72] *Siddhāntatattvaviveka*, ii. 78 and its commentary.
[73] *Mahāsiddhānta*, iii. 2.

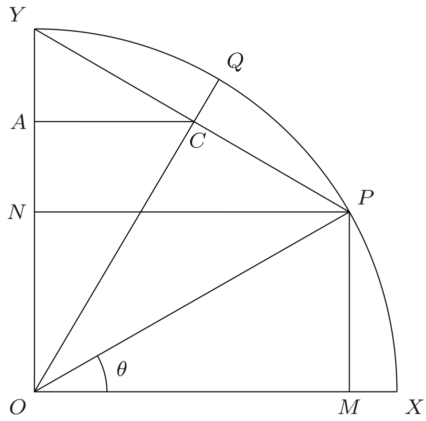

Figure 5

(**ed.** In Figure 5) Let the arc XP be denoted by α. Bisect the complementary arc YP at Q. Then

$$\begin{aligned} YP^2 &= YN^2 + NP^2 \\ &= (OY - PM)^2 + PN^2 \\ &= OY^2 + PM^2 + OM^2 - 2\,OY \times PM. \end{aligned}$$

Therefore,

$$4PC^2 = 2(OP^2 - OP \times PM).$$

Hence,

$$jy\bar{a}\,\frac{1}{2}\,(90^\circ - \alpha) = \sqrt{\frac{1}{2}\,(R^2 - R \times jy\bar{a}\,\alpha)}.$$

Similarly it can be proved that

$$jy\bar{a}\,\frac{1}{2}\,(90^\circ + \alpha) = \sqrt{\frac{1}{2}\,(R^2 + R \times jy\bar{a}\,\alpha)}.$$

These are of course equivalent to

$$\sin\frac{1}{2}(90^\circ \pm \theta) = \sqrt{\frac{1}{2}(1 \pm \sin\theta)}.$$

Bhāskara II (1150) writes:

> The square of the radius is diminished or increased by the product of the radius and the *R*sine of an arc; the square-root of half the results will be the *R*sine of the half of 90° minus or plus that arc.[74]

[74] *Siddhāntaśiromaṇi*, *Gola*, xiv. 12.

Kamalākara defines:

> The product of the radius and the Rsine of an arc is added to or subtracted from the square of the radius. The square-root of the half of the results are taken. They will respectively be the Rsine of the half of three signs plus or minus the arc.[75]

He adduces the following proof of it:[76]

$$R \pm jy\bar{a}\ \alpha = utjy\bar{a}\ \ (90^\circ \pm \alpha).$$

Squaring and adding $\{jy\bar{a}\ (90^\circ \pm a)\}^2$ to both the sides, we get

$$R^2 + (jy\bar{a}\ \alpha)^2 + \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 \pm 2R\ jy\bar{a}\ \alpha = \{jy\bar{a}\ (90^\circ \pm \alpha)\}^2 + \{utjy\bar{a}\ (90^\circ \pm \alpha)\}^2$$

or,

$$2(R^2\ \pm R\, jy\bar{a}\ \alpha) = 4\left\{jy\bar{a}\frac{1}{2}\ (90^\circ \pm \alpha)\right\}^2,$$

by formulae (1) and (2).

(5) $2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta.$

Bhāskara II remarks that if the arc α in the formula

$$jy\bar{a}\ \frac{1}{2}\ (90^\circ \pm \alpha) = \sqrt{\frac{1}{2}(R^2 + R\, jy\bar{a}\ (90^\circ - \alpha)}$$

be substituted by its complement $90^\circ - \alpha$, it will still be true.[77] So that,

$$jy\bar{a}\ \frac{1}{2}(90^\circ \pm \{90^\circ - \alpha\}) = \sqrt{\frac{1}{2}\ \{R^2 \pm R\, jy\bar{a}\ (90^\circ - \alpha)\}}$$

which leads to,

$$2\cos^2\frac{\theta}{2} = 1 + \cos\theta, \quad 2\sin^2\frac{\theta}{2} = 1 - \cos\theta.$$

Kamalākara says:

> Half the Rcosine of an arc is added to the Rsine of one sign (i.e. 30°) and the sum is multiplied by the radius; the square-root of the product should be known by the intelligent as the Rcosine of half that arc.[78]

[75] *Siddhāntatattvaviveka*, ii. 93.
[76] *Siddhāntatattvaviveka*, (gloss).
[77] *Siddhāntaśiromaṇi*, *Gola*, xiv. 12.
[78] *Siddhāntatattvaviveka*, ii. 91.

$$kojy\bar{a}\,\frac{\alpha}{2} = \sqrt{R\,(jy\bar{a}\;30^\circ + \frac{1}{2}kojy\bar{a}\;\alpha)}$$

or,

$$\cos^2\frac{\theta}{2} = \sin 30^\circ + \frac{1}{2}\cos\theta = \frac{1}{2}\,(1+\cos\theta).$$

(6) $\sin^2(45^\circ - \theta) = \frac{1}{2}(\cos\theta - \sin\theta)^2.$

Bhāskara II says:

> The square of the difference of the *R*sine and *R*cosine of an arc is halved; the square-root of the result is equal to the *R*sine of half the difference between that arc and its complement.[79]

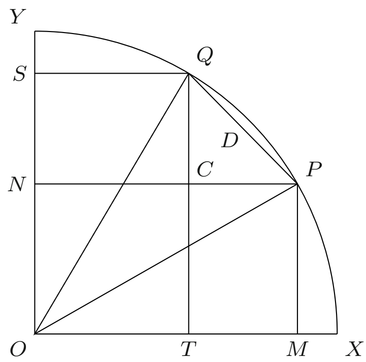

Figure 6

(**ed.** In Figure 6) Denote the arc XP by α; cut off the arc YQ equal to the arc XP. Bisect the chord PQ by the point D. Then,

$$\begin{aligned} CP &= PN - CN = PN - QS \\ &= kojy\bar{a}\;\alpha - jy\bar{a}\;\alpha \\ &= QT - PM = CQ. \end{aligned}$$

Therefore,

$$PQ^2 = 2CP^2$$
$$PD^2 = \frac{1}{2}CP^2$$

or,

$$jy\bar{a}\,\frac{1}{2}\{(90^\circ - \alpha) - \alpha\} = \sqrt{\frac{1}{2}(kojy\bar{a}\;\alpha - jy\bar{a}\;\alpha)^2}$$

[79] *Siddhāntaśiromaṇi*, *Gola*, xiv. 14.

which is equivalent to

$$\sin(45^\circ - \theta) - \sqrt{\frac{1}{2}(\cos\theta - \sin\theta)^2}.$$

Kamalākara writes:

> The *R*sine of half the difference between an arc and its complement should be known by the intelligent in this (science) as equal to the square-root of half the square of the difference of the *R*sine of the arc and of its complement.[80]

His proof of the formula is substantially the same as that stated above.

(7) $\cos 2\theta = 1 - 2\sin^2\theta$.

Bhāskara II gives:

> The square of the *R*sine of an arc is divided by half the radius; the difference between this quotient and the radius is equal to the *R*sine of the difference between that arc and its complement.[81]

In Figure 4,

$$\begin{aligned} QX^2 &= QT^2 + TX^2 = QT^2 + (OX - OT)^2 \\ &= QT^2 + OT^2 + OX^2 - 2\,OX \times OT. \\ \text{or} \qquad 4\,XD^2 &= 2\,OX^2 - 2\,OX \times QS. \end{aligned}$$

Hence,

$$QS = OX - \frac{XD^2}{OX/2}.$$

So that,

$$jy\bar{a}\ (90^\circ - 2\alpha) = R - \frac{(jy\bar{a}\ \alpha)^2}{\frac{R}{2}}$$

which is the same as

$$\cos 2\theta = 1 - 2\sin^2\theta.$$

This formula is practically the same as (3). In the words of Kamalākara:

> Twice the square of the *R*sine of an arc is divided by the radius, the quotient is subtracted from the radius; the remainder will be the *R*sine of the difference of the arc and its complement.[82]

(8) $\sin^2\theta + \text{versin}^2\theta = 2\,\text{versin}\,\theta$.

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[80] *Siddhāntatattvaviveka*, ii. 95.
[81] *Siddhāntaśiromaṇi*, *Gola*, xiv. 15.
[82] *Siddhāntatattvaviveka*, ii. 96.

(9) $2\sin\theta\cos\theta + [\text{versin}\,\theta - \text{versin}(90^\circ - \theta)]^2 = 1.$

(10) $(1+\sin\theta)\times\text{versin}(90^\circ - \theta) = \cos^2\theta.$

(11) $2\sin\theta \pm [\text{versin}\,\theta \sim \text{versin}(90^\circ - \theta)] = \sqrt{2 - [\text{versin}\,\theta \sim \text{versin}(90^\circ - \theta)]^2},$ according as $\sin\theta \lesseqgtr \cos\theta$.

(12) $(\cos\theta + \sin\theta)^2 + [\text{versin}\,\theta \sim \text{versin}(90^\circ - \theta)]^2 = 2.$

Formulae (8) to (12) and similar others occur in the *Vaṭeśvara-siddhānta* of Vaṭeśvara (904).

3 Addition and subtraction theorems

Bhāskara II (1150) says:

> The Rsines of any two arcs of a circle are reciprocally multiplied by their Rcosines; the products are then divided by the radius; the sum of the quotients is equal to the Rsine of the sum of the two arcs; and their difference is the Rsine of the difference of the arcs.[83]

If α and β be any two arcs, then the rule says:

$$jy\bar{a}\ (\alpha \pm \beta) = \frac{jy\bar{a}\ \alpha \times kojy\bar{a}\ \beta}{R} \pm \frac{kojy\bar{a}\ \alpha \times jy\bar{a}\ \beta}{R}$$

which is equivalent to

$$\sin\,(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi.$$

In the words of Kamalākara (1658):

> The quotients of the Rsines of any two arcs of a circle divided by its radius are reciprocally multiplied by their Rcosines; the sum and difference of them (the products) are equal to the Rsine of the sum and difference respectively of the two arcs.[84]

The rule for finding the Rcosine of the sum and difference of two arcs of a circle is enunciated by Kamalākara thus:

> The product of the Rcosines and of the Rsines of two arcs of a circle are divided by its radius; the difference and sum of them (the quotients) are equal to the Rcosine of the sum and difference (respectively) of the two arcs.[85]

[83] *Siddhāntaśiromaṇi*, *Gola*, xiv. 21f.
[84] *Siddhāntatattvaviveka*, ii. 68.
[85] *Ibid*, ii. 69.

$$kojy\bar{a}\ (\alpha \pm \beta) = \frac{kojy\bar{a}\ \alpha \times kojy\bar{a}\ \beta}{R} \pm \frac{jy\bar{a}\ \alpha \times jy\bar{a}\ \beta}{R}$$

which is equivalent to

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \pm \sin\theta\sin\phi.$$

Though we do not find this Rcosine theorem in the printed editions of the works of Bhāskara II, we are quite sure that it was known to him. For it has been attributed to him by his most relentless critic Kamalākara[86] as well as by his commentator Munīśvara.

The above theorems can be proved by methods algebraical as well as geometrical. Several such proofs were given by previous writers, observes Kamalākara[87] (1658). Unfortunately we have not been able to trace them as yet. The following two geometrical proofs are found in the *Siddhāntatattvaviveka*[88] of Kamalākara.

First proof

(**ed.** In Figure 7) Let the arc $YP = \beta$, and arc $YQ = \alpha$; α being greater than β. Join OP, OQ. Draw PN, PM perpendicular to OY, OX respectively. Also draw QS perpendicular to OY, and produce it to meet the circle again at Q'. Draw QT, $Q'T'$ perpendicular to OP. Then $PN = jy\bar{a}\ \beta$, $ON = kojy\bar{a}\ \beta$, $QS = jy\bar{a}\ \alpha$, $OS = kojy\bar{a}\ \alpha$; $PG = kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha$, $QG = jy\bar{a}\ \alpha + jy\bar{a}\ \beta$, $QT = jy\bar{a}\ (\alpha + \beta)$, $PT = R - kojy\bar{a}\ (\alpha + \beta)$.

Now $PG^2 + QG^2 = QP^2 = QT^2 + PT^2$. Therefore, substituting the values

$$(kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha)^2 + (jy\bar{a}\ \alpha + jy\bar{a}\ \beta)^2 = \{jy\bar{a}\ (\alpha + \beta)\}^2 + \{R - kojy\bar{a}\ (\alpha + \beta)\}^2.$$

---

[86] Kamalākara remarks:

> एवमानयनं चक्रे पूर्वं स्वीयशिरोमणौ ।
> भावनाभ्यामतिस्पष्टं संयगार्योऽपि भास्करः ॥
>
> This theorem, which is evident from the two *bhāvanās*, was stated before also by the highly respected Bhāskara in his (*Siddhānta-*)*śiromaṇi*.
>
> — *Siddhāntatattvaviveka*, ii. 70.

[87]

> तस्य चानयनस्यार्यैः सिद्धान्तज्ञैः पुरोदिता ।
> वासना बहुभिः स्वस्वबुद्विवैचित्र्यतः स्फुटाः ॥
>
> Many correct proofs of this theorem were given before by the learned authors of the *siddhāntas* according to the manifoldness of their intelligence.
>
> — *Siddhāntatattvaviveka*, ii. 71.

[88] ii. 68–9 (gloss).

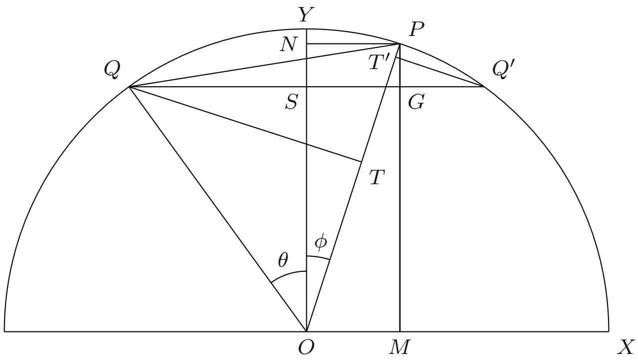


Figure 7

Simplifying we get

$$kojy\bar{a}\,(\alpha+\beta) = \frac{1}{R}(kojy\bar{a}\,\alpha \times kojy\bar{a}\,\beta - jy\bar{a}\,\alpha \times jy\bar{a}\,\beta)$$

which is equivalent to

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Again

$$\begin{aligned} &R^2 - \left\{kojy\bar{a}\,(\alpha+\beta)\right\}^2 \\ &\quad = \frac{1}{R^2}\left\{R^4 - (kojy\bar{a}\,\alpha \times kojy\bar{a}\,\beta - jy\bar{a}\,\alpha \times jy\bar{a}\,\beta)^2\right\} \\ &\quad = \frac{1}{R^2}\left[\left\{(jy\bar{a}\,\alpha)^2 + (kojy\bar{a}\,\alpha)^2\right\} \times \left\{(jy\bar{a}\,\beta)^2 + (kojy\bar{a}\,\beta)^2\right\} - \right. \\ &\qquad\qquad \left.(kojy\bar{a}\,\alpha \times kojy\bar{a}\,\beta - jy\bar{a}\,\alpha \times jy\bar{a}\,\beta)^2\right] \end{aligned}$$

or,

$$jy\bar{a}\,(\alpha+\beta) = \frac{1}{R}(jy\bar{a}\,\alpha \times kojy\bar{a}\,\beta + kojy\bar{a}\,\alpha \times jy\bar{a}\,\beta),$$

which is

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi.$$

Since

$$PG^2 + Q'G^2 = Q'P^2 = Q'T'^2 + PT'^2,$$

we have

$$\begin{aligned} &(kojy\bar{a}\,\beta - kojy\bar{a}\,\alpha)^2 + (jy\bar{a}\,\alpha - jy\bar{a}\,\beta)^2 = \\ &\qquad\qquad \left\{jy\bar{a}\,(\alpha-\beta)\right\}^2 + \left\{R - kojy\bar{a}(\alpha-\beta)\right\}^2. \end{aligned}$$

Therefore,

$$kojy\bar{a}(\alpha-\beta)=\frac{1}{R}(kojy\bar{a}\,\alpha\times kojy\bar{a}\,\beta+jy\bar{a}\,\alpha\times jy\bar{a}\,\beta),$$

which is

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi,$$

whence, proceeding as before, we get

$$jy\bar{a}\,(\alpha-\beta)=\frac{1}{R}\,(jy\bar{a}\,\alpha\times kojy\bar{a}\,\beta-kojy\bar{a}\,\alpha\times jy\bar{a}\,\beta)$$

or

$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi.$$

3.1 Alternative proof

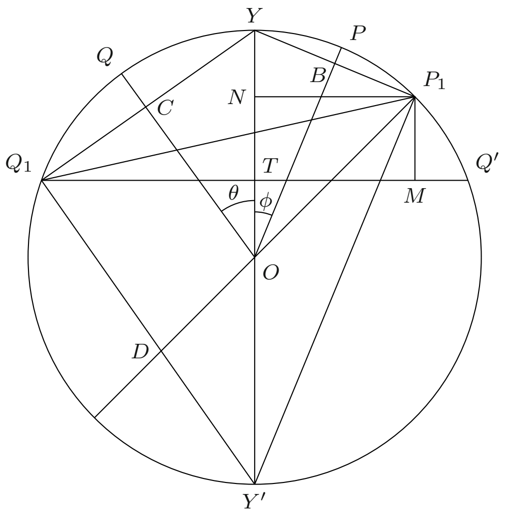


Figure 8

(**ed.** In Figure 8) Let the arc YP = arc PP_1 = β and the arc YQ = arc $QQ_1=\alpha$. Then it is obvious from the figure that

$$YP_1=2\;jy\bar{a}\;\beta,\qquad Y'P_1=2\;kojy\bar{a}\;\beta,$$
$$YQ_1=2\;jy\bar{a}\;\alpha,\qquad Y'Q_1=2\;kojy\bar{a}\;\alpha.$$

Also

$$Q_1P_1^2=Q_1D^2+DP_1^2=\{jy\bar{a}\;(2\alpha+2\beta)\}^2+\{(utjy\bar{a}\;(2\alpha+2\beta)\}^2.$$

Therefore,

$$Q_1P_1 = 2\, jy\bar{a}\ (\alpha + \beta).$$

Similarly

$$Q_1'P_1 = 2\, jy\bar{a}\ (\alpha - \beta).$$

By the geometrical rules for finding the height and the segments of the base of a triangle whose sides are known, it can be easily proved that

$$Y'N = \frac{2}{R}(kojy\bar{a}\ \beta)^2, \quad YN = \frac{2}{R}(jy\bar{a}\ \beta)^2, \quad P_1N = \frac{2}{R}\, jy\bar{a}\ \beta \times kojy\bar{a}\ \beta,$$
$$Y'T = \frac{2}{R}(kojy\bar{a}\ \alpha)^2, \quad YT = \frac{2}{R}(jy\bar{a}\ \alpha)^2, \quad QT = \frac{2}{R}\, jy\bar{a}\ \alpha \times kojy\bar{a}\ \alpha.$$

Now

$$\begin{aligned} Q_1P_1^2 &= Q_1M^2 + P_1M^2 \\ &= (Q_1T + P_1N)^2 + (YT - YN)^2 \\ &= \frac{4}{R^2}(jy\bar{a}\ \alpha \times kojy\bar{a}\ \alpha + jy\bar{a}\ \beta \times kojy\bar{a}\ \beta)^2 + \frac{4}{R^2}\left\{(jy\bar{a}\ \alpha)^2 - (jy\bar{a}\ \beta)^2\right\}^2 \\ &= \frac{4}{R^2}(jy\bar{a}\ \alpha \times kojy\bar{a}\ \beta + kojy\bar{a}\ \alpha \times jy\bar{a}\ \beta)^2. \end{aligned}$$

Therefore,

$$jy\bar{a}\ (\alpha + \beta) = \frac{1}{R}(jy\bar{a}\ \alpha \times kojy\bar{a}\ \beta + kojy\bar{a}\ \alpha \times\ jy\bar{a}\ \beta),$$

which is equivalent to

$$\sin(\theta + \phi) = \sin\theta \cos\phi + \cos\theta \sin\phi.$$

Again

$$\begin{aligned} Q'P_1^2 &= Q'M^2 + MP_1^2 \\ &= (Q_1T - P_1N)^2 + (YT - YN)^2 \\ &= \frac{4}{R^2}(jy\bar{a}\ \alpha.\ kojy\bar{a}\ \alpha - jy\bar{a}\ \beta \times kojy\bar{a}\ \beta)^2 + \frac{4}{R^2}\left\{(jy\bar{a}\ \alpha)^2 - (jy\bar{a}\ \beta)^2\right\}^2 \end{aligned}$$

whence

$$jy\bar{a}\ (\alpha - \beta) = \frac{1}{R}(jy\bar{a}\ \alpha \times kojy\bar{a}\ \beta - kojy\bar{a}\ \alpha \times jy\bar{a}\ \beta),$$

which is equivalent to

$$\sin\,(\theta - \phi) = \sin\theta \cos\phi - \cos\theta \sin\phi.$$

The above theorems are called *bhāvanā* ("demonstration" or "proof" meaning "any thing demonstrated or proved", hence "theorem").

They are again divided into *samāsa-bhāvanā* or *yoga-bhāvanā* ("Addition Theorem") and *antara-bhāvanā* or *viyoga-bhāvanā* ("Subtraction Theorem").[89]

In the proofs given above the arcs α and β have been tacitly assumed to be each less than 90°. But the theorems are quite general and hold true even when the arcs are greater than 90°.

Thus Kamalākara observes:

> Even when the two arcs go beyond 90° to any even or odd quadrant (the theorems) will remain the same, not otherwise. That is the opinion of those who are aware of the true facts.[90]

Functions of multiple angles

As corollaries to the general case of the theorems for expanding $\sin(\theta \pm \phi)$ and $\cos(\theta \pm \phi)$, Bhāskara II (1150) indicates how to derive the functions of multiple angles. He observes:

> This being proved, it becomes an argument for determining the values of other functions. For example, take the case of the combination of functions of equal arcs; by combining the functions of any arc with those of itself, we get the functions of twice that arc; by combining the functions of twice the arc with those of twice the arc, we get functions of four times that arc; and so on. Next take the case of combination of functions of unequal arcs; on combining the functions of twice an arc with those of thrice that arc, by the addition theorem we get the functions of five times that arc; but by the subtraction theorem, we get the functions of one time that arc; and so on.[91]

The theorems meant here are clearly these:

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta,\\
\cos 2\theta &= \cos^2\theta - \sin^2\theta,\\
\sin 4\theta &= 2\sin 2\theta\cos 2\theta,\\
&= 4\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta),\\
\cos 4\theta &= \cos^2 2\theta - \sin^2 2\theta,\\
&= \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta,\\
\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta,\\
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta,
\end{aligned}
$$

[89] *Siddhāntaśiromaṇi*, *Gola*, xiv. 21 (gloss); *Siddhāntatattvaviveka*, ii. 65.
[90] *Siddhāntatattvaviveka*, ii. 66f.
[91] *Siddhāntaśiromaṇi*, *Gola*, xiv. 21–2 (gloss).

$$\sin 5\theta = \sin 2\theta \cos 3\theta + \cos 2\theta \sin 3\theta,$$
$$\cos 5\theta = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta,$$
$$\sin \theta = \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta,$$
$$\cos \theta = \cos 3\theta \cos 2\theta + \sin 3\theta \sin 2\theta.$$

All these theorems have been expressly stated by Kamalākara (1658). He says:

> Hereafter I shall describe how to find the *R*sine of twice, thrice, four times or five times an arc, having known the *R*sine of the sum of two arcs. The product of the *R*sine and *R*cosine of an arc is multiplied by 2 and divided by the radius; the result is the *R*sine of twice that arc.[92]
>
> The difference of the squares of the *R*sine and *R*cosine of an arc is divided by the radius; the quotient is certainly the *R*cosine of twice that arc.[93]

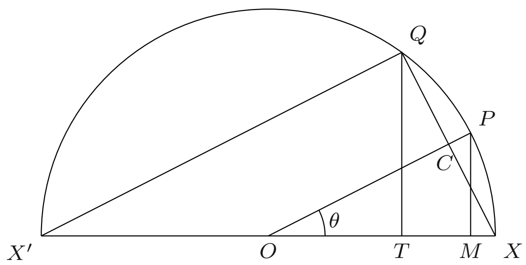


Figure 9

He has given the following proof of the above two formulae.[94]

(**ed.** In Figure 9) Let the arc $XP =$ arc PQ, then $X'Q = 2\,OC = 2$ *kojyā* α. Now from the right angled triangles OPM, $X'QT$, we have

$$PO : PM :: X'Q : QT, \qquad \text{and} \qquad OP : OM :: X'Q : X'T.$$

Therefore

$$OP \times QT = PM \times X'Q = 2\,PM \times OC,$$
$$\text{and} \qquad OP \times X'T = OM \times X'Q,$$

[92] *Siddhāntatattvaviveka*, ii. 73.
[93] *Ibid.*, ii. 90.
[94] See his own gloss on the preceding rules.

or

$$OP(X'O + OT) = OM \times 2\,OC = 2\,OM^2, \qquad \text{because } OC = OM$$

or

$$OP(OP + OT) = 2\,OM^2,$$

or

$$OP \times OT = 2\,OM^2 - OP^2 = OM^2 - PM^2.$$

Therefore,

$$QT = \frac{2PM \times OC}{R},$$
$$OT = \frac{OM^2 - PM^2}{R}.$$

Hence,

$$jy\bar{a}\ 2a = \frac{2\ jy\bar{a}\ \alpha \times kojy\bar{a}\ \alpha}{R},$$
$$kojy\bar{a}\ 2\alpha = \frac{(kojy\bar{a}\ \alpha)^2 - (jy\bar{a}\ \alpha)^2}{R}.$$

That is,

$$\sin 2\theta = 2 \sin\theta \cos\theta,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

It has been further observed that these results can be easily deduced from the Addition Theorem by putting $\phi = \theta$.

> The sine of an arc is divided by the sine of one sign (i.e. 30°); the square of the quotient is subtracted from 3 and the remainder is multiplied by the R sine of the arc; the result is the Rsine of thrice that arc.[95]

$$jy\bar{a}\ 3\alpha = jy\bar{a}\ \alpha \left\{ 3 - \left(\frac{jy\bar{a}\ \alpha}{jy\bar{a}\ 30^\circ} \right)^2 \right\}.$$

That is,

$$\sin 3\theta = \sin\theta \left(3 - \frac{\sin^2\theta}{\sin^2 30^\circ} \right).$$

[95] *Siddhāntatattvaviveka*, ii. 74; also the gloss.

By the successive application of the Addition Theorems, Kamalākara obtains the formulae:[96]

$$\begin{aligned}
jy\bar{a}\,3\alpha &= \frac{\left\{3R^2\, jy\bar{a}\,\alpha - 4\,(jy\bar{a}\,\alpha)^3\right\}}{R^2},\\
kojy\bar{a}\,3\alpha &= \frac{\left\{4\,(kojy\bar{a}\,\alpha)^3 - 3R^2\, kojy\bar{a}\,\alpha\right\}}{R^2},\\
jy\bar{a}\,4\alpha &= \frac{4\left\{(kojy\bar{a}\,\alpha)^3 jy\bar{a}\,\alpha - (jy\bar{a}\,\alpha)^3 kojy\bar{a}\,\alpha\right\}}{R^3},\\
kojy\bar{a}\,4\alpha &= \frac{\left\{(kojy\bar{a}\,\alpha)^4 - 6\,(kojy\bar{a}\,\alpha)^2(jy\bar{a}\,\alpha)^2 + (jy\bar{a}\,\alpha)^4\right\}}{R^3},\\
jy\bar{a}\,5\alpha &= \frac{\left\{(jy\bar{a}\,\alpha)^5 - 10\,(jy\bar{a}\,\alpha)^3(kojy\bar{a}\,\alpha)^2 + 5\,(jy\bar{a}\,\alpha)(kojy\bar{a}\,\alpha)^4\right\}}{R^4},\\
kojy\bar{a}\,5\alpha &= \frac{\left\{(kojy\bar{a}\,\alpha)^5 - 10\,(kojy\bar{a}\,\alpha)^3(jy\bar{a}\,\alpha)^2 + 5\,(kojy\bar{a}\,\alpha)(jy\bar{a}\,\alpha)^4\right\}}{R^4};
\end{aligned}$$

which are of course equivalent to

$$\begin{aligned}
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta,\\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta,\\
\sin 4\theta &= 4(\cos^3\theta\sin\theta - \sin^3\theta\cos\theta),\\
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta,\\
\sin 5\theta &= \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta,\\
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.
\end{aligned}$$

Functions of sub-multiple angles

It has been stated before that the following two formulae for the sine of half an angle were known to almost all the Hindu astronomers:

$$\begin{aligned}
\sin\frac{\theta}{2} &= \frac{1}{2}\sqrt{\sin^2\theta + \operatorname{versin}^2\theta},\\
\sin\frac{\theta}{2} &= \sqrt{\frac{1}{2}(1-\cos\theta)}.
\end{aligned}$$

Besides these[97] Kamalākara has given formulae for the functions of the third, fourth and fifth parts of an arc.

> Find the cube of one-third the *R*sine of an arc; divide it by the square of the radius; the quotient is added to its one-third and the

[96] *Siddhāntatattvaviveka*, ii. 75–7 and also the gloss on them.
[97] *Siddhāntatattvaviveka*, ii. 78f.

> sum again to one-third the *R*sine of the arc; the result is nearly the *R*sine of one-third that arc. From the cube of this again further accurate values can be obtained.[98]

$$jy\bar{a}\,\frac{\alpha}{3} = \frac{1}{3}\,jy\bar{a}\,\alpha + \frac{4}{3R^2}\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3.$$

The rationale of this formula has been stated to be this: As has been proved before

$$jy\bar{a}\,3\beta = 3\,jy\bar{a}\,\beta - \frac{4}{R^2}(jy\bar{a}\,\beta)^3.$$

Put $3\beta = \alpha$; then this formula will become

$$jy\bar{a}\frac{\alpha}{3} = \frac{1}{3}\,jy\bar{a}\,\alpha + \frac{4}{3R^2}\left(jy\bar{a}\,\frac{\alpha}{3}\right)^3. \tag{13}$$

Now $jy\bar{a}\,\frac{\alpha}{3}$ can be taken, says Kamalākara, as a rough approximation (*sthūla*) to be equal to $\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3$. So that approximately

$$jy\bar{a}\,\frac{\alpha}{3} = \frac{1}{3}\,jy\bar{a}\,\alpha + \frac{4}{3R^2}\left(\frac{jy\bar{a}\,\alpha}{3}\right)^3, \tag{14}$$

as stated in the rule. Very nearer approximation (*sūkṣmāsanna*) to the value of $jy\bar{a}\frac{\alpha}{3}$ can be found by substituting the cube of this value in the last term of (13) and by repeating similar operations.

The form (14) is equivalent to

$$\sin\frac{\theta}{3} = \frac{1}{3}\sin\theta + \frac{4}{81}\sin^3\theta.$$

> From the known value of the *R*sine of an arc, first calculate the value of the *R*sine of half that arc; the *R*sine of the arc is divided by that and multiplied by the square of the radius; the result is subtracted from twice the square of the radius. Half the square-root of the remainder is the value of the *R*sine of one-fourth that arc.[99]

$$jy\bar{a}\,\frac{\alpha}{4} = \frac{1}{2}\sqrt{2R^2 - R^2\frac{jy\bar{a}\,\alpha}{jy\bar{a}\,\left(\frac{\alpha}{2}\right)}}.$$

The rationale of this formula is given thus: It is known that

$$\begin{aligned} jy\bar{a}\,4\beta &= \frac{4}{R^3}\left\{(kojy\bar{a}\,\beta)^3 jy\bar{a}\,\beta - (jy\bar{a}\,\beta)^3 kojy\bar{a}\,\beta\right\}, \\ &= \frac{4}{R^3}\left\{R^2 jy\bar{a}\,\beta\;kojy\bar{a}\,\beta - 2\,(jy\bar{a}\,\beta)^3 kojy\bar{a}\,\beta\right\}. \end{aligned}$$

[98] *Ibid*, ii. 81.
[99] *Siddhāntatattvaviveka*, ii. 82–83.

Putting α for 4β, we get

$$R^3\, jy\bar{a}\,\alpha = 4\, jy\bar{a}\,\frac{\alpha}{4}\, kojy\bar{a}\,\frac{\alpha}{4}\left\{R^2 - 2\left(jy\bar{a}\,\frac{\alpha}{4}\right)^2\right\},$$
$$= 2R\, jy\bar{a}\,\frac{\alpha}{2}\left\{R^2 - 2\left(jy\bar{a}\,\frac{\alpha}{4}\right)^2\right\};$$

whence

$$jy\bar{a}\,\frac{\alpha}{4} = \frac{1}{2}\sqrt{2R^2 - \frac{R^2(jy\bar{a}\,\alpha)}{jy\bar{a}\,\frac{\alpha}{2}}},$$

or

$$\sin\frac{\theta}{4} = \frac{1}{2}\sqrt{2 - \frac{\sin\theta}{\sin\left(\frac{\theta}{2}\right)}}.$$

> The intelligent should first find the one-fifth of the *R*sine of the given arc; divide four times the cube of that by the square of the radius; the quotient should be called the "first". Multiply the "first" by the square of the fifth part of the *R*sine and divide the product by the square of the radius; lessen this quotient by its fifth part and mark the remainder as the "second". One-fifth of the *R*sine of the arc added with the "first" and diminished by the "second", will be clearly the value of the *R*sine of the fifth part of the arc. Finding the value of the "first" again from this, further approximate value to the *R*sine of one-fifth the arc can be found. Still closer approximations can be obtained by repeating the process stated above.[100]

$$jy\bar{a}\,\frac{\alpha}{5} = \frac{1}{5}\, jy\bar{a}\,\alpha + \frac{4}{R^2}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^3 - \frac{16}{5R^4}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^5.$$

The rationale is stated to be this: It has been established before that

$$R^4\, jy\bar{a}\,5\beta = (jy\bar{a}\,\beta)^5 - 10\,(jy\bar{a}\,\beta)^3(kojy\bar{a}\,\beta)^2 + 5\,(jy\bar{a}\,\beta)(kojy\bar{a}\,\beta)^4.$$

Substituting the value $R^2 - (jy\bar{a}\,\beta)^2$ for $(kojy\bar{a}\,\beta)^2$ in this, we get

$$R^4\, jy\bar{a}\,5\beta = 16\,(jy\bar{a}\,\beta)^5 - 20R^2\,(jy\bar{a}\,\beta)^3 + 5R^4\, jy\bar{a}\,\beta.$$

Putting α for 5β,

$$jy\bar{a}\,\frac{\alpha}{5} = \frac{1}{5}\, jy\bar{a}\,\alpha + \frac{4}{R^2}\left(jy\bar{a}\,\frac{\alpha^3}{5}\right) - \frac{61}{5R^4}\left(jy\bar{a}\,\frac{\alpha}{5}\right)^5. \qquad (15)$$

[100] *Siddhāntatattvaviveka*, ii. 84–87.

In the last two terms on the right hand side, one may take as a rough approximation

$$jy\bar{a}\,\frac{\alpha}{5} = \frac{1}{5}\,jy\bar{a}\,\alpha;$$

so that

$$jy\bar{a}\,\frac{\alpha}{5} = \frac{1}{5}\,jy\bar{a}\,\alpha + \frac{4}{R^2}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^3 - \frac{16}{5R^4}\left(\frac{jy\bar{a}\,\alpha}{5}\right)^5. \qquad (16)$$

Again substituting this value of $jy\bar{a}\,\frac{\alpha}{5}$ in the last two terms of (15) and repeating similar operations, closer approximations to the value of $jy\bar{a}\,\frac{\alpha}{5}$ can be obtained.

The formula (16) is equivalent to

$$\sin\frac{\theta}{5} = \frac{1}{5}\sin\theta + 4\left(\frac{\sin\theta}{5}\right)^3 - \frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5.$$

Kamalākara then observes that "in this way, the *R*sines of other desired submultiples of an arc should be obtained."[101]

$\sin\dfrac{(\theta-\phi)}{2}$

Bhāskara II says:

> Find the difference of the *R*sines of two arcs and also of their *R*cosines; then find the square-root of the sum of the squares of the two results; half this root will be the *R*sine of half the difference of the two arcs.[102]

That is,

$$jy\bar{a}\,\frac{1}{2}(\alpha-\beta) = \frac{1}{2}\left\{(jy\bar{a}\,\alpha - jy\bar{a}\,\beta)^2 + (kojy\bar{a}\,\alpha - kojy\bar{a}\,\beta)^2\right\}^{\frac{1}{2}}.$$

or, in modern notations,

$$\sin\frac{1}{2}(\theta-\phi) - \frac{1}{2}\left\{(\sin\theta-\sin\phi)^2 + (\cos\theta-\cos\phi)^2\right\}^{\frac{1}{2}}.$$

Kamalākara writes:

> Half the square-root of the sum of the squares of the differences of *R*sines and *R*cosines of two arcs is certainly equal to the *R*sine of half the difference of the two arcs.[103]

[101] *Siddhāntatattvaviveka*, ii. 87 (c–d).
[102] *Siddhāntaśiromaṇi*, *Gola*, xiv. 13.
[103] *Siddhāntatattvaviveka*, ii. 94.

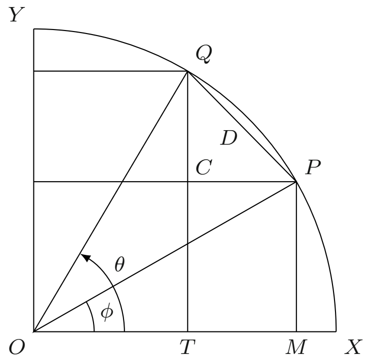


Figure 10

The latter has given the following proof of it.[104] (**ed.** In Figure 10) Let the arc XP be denoted by β and the arc XQ by α; then

$$QC = QT - PM = jy\bar{a}\,\alpha - jy\bar{a}\,\beta,$$
$$PC = OM - OT = kojy\bar{a}\,\beta - kojy\bar{a}\,\alpha.$$

Now,

$$PQ^2 = QC^2 + PC^2.$$

Hence,

$$jy\bar{a}\,\frac{1}{2}(\alpha - \beta) = \frac{1}{2}\left\{(jy\bar{a}\,\alpha - jy\bar{a}\,\beta)^2 + (kojy\bar{a}\,\alpha - kojy\bar{a}\,\beta)^2\right\}^{\frac{1}{2}};$$

which is equivalent to

$$\sin\frac{1}{2}(\theta - \phi) = \frac{1}{2}\left\{(\sin\theta - \sin\phi)^2 + (\cos\theta - \cos\phi)^2\right\}^{\frac{1}{2}}.$$

Theorem of sines

Brahmagupta[105] has made use of the important relation

$$\frac{a}{jy\bar{a}\;A} = \frac{b}{jy\bar{a}\;B} = \frac{c}{jy\bar{a}\;C}$$

which is of course equivalent to

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

between the sides (a, b, c) and angles (A, B, C) of a plane triangle.

[104] *Ibid*, (gloss).

[105] *Khaṇḍakhādyaka*, Part I, viii. 2. Our attention to this was first drawn by Professor P. C. Sengupta, who was then preparing a new edition of *Khaṇḍakhādyaka* with English translation and critical notes. This rule occurs in the works of other Hindu astronomers also.

4 Functions of particular angles

Sine of 30°, 45° and 60°

Preliminary to the calculation of tables of trigonometrical functions almost all the Hindu writers have stated the values of the Rsines of 30°, 45° and 60°:

$$jy\bar{a}\ 30^\circ = \sqrt{\frac{R^2}{4}}, \qquad jy\bar{a}\ 45^\circ = \sqrt{\frac{R^2}{2}}, \qquad jy\bar{a}\ 60^\circ = \sqrt{\frac{3R^2}{4}}$$

or, in modern notation

$$\sin 30^\circ = \frac{1}{2}, \qquad \sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

Śrīpati indicates the proof thus:

> The experts in spherics say that the circum-radius of a regular hexagon is equal to a side. So it will be perceived that the chord of the sixth part of the circumference of a circle is equal to its semi-diameter. The hypotenuse arising from the base and perpendicular (of a right-angled triangle) each equal to the semi-diameter is the chord of the fourth part of the circumference. Half those (chords) will be the Rsines of half those arcs.[106]

The same proof is also given by Bhāskara II:[107]

> The side of a regular hexagon inscribed in a circle is equal to its radius; this is well known and has also been stated in (my) Arithmetic. Hence follows that the Rsine of 30° is half the radius.
>
> Suppose a right-angled triangle whose base and perpendicular are each equal to the radius; the square-root of the sum of the squares of these will be equal to the side of a square inscribed in that circle and it is again the chord of 90°. Take the half of that. Hence the sum of the squares (of the sides) is divided by four; and the result is half the square of the radius. The square-root of that, it thus follows, is the Rsine of 45°.
>
> The Rsine of 60° is equal to the Rcosine of 30°, the Rsine of which is equal to half the semi-diameter.

Sine of 18° and 36°

Bhāskara II says:

[106] *Siddhāntaśekhara*, xvi. 11–2.
[107] *Siddhāntaśiromaṇi*, *Gola*, v. 3–4 (gloss).

The square-root of five times the square of the radius is diminished by the radius and the remainder is divided by four; the result is the exact value of the *R*sine of 18°.[108]

$$\begin{aligned} jy\bar{a}\ 18^\circ &= \frac{1}{4}(\sqrt{5R^2} - R); \\ \text{or} \qquad \sin 18^\circ &= \frac{1}{4}(\sqrt{5} - 1). \end{aligned}$$

The square-root of five times the square of the square of the radius is subtracted from five times the square of the radius and the remainder is divided by eight; the square-root of the quotient is the *R*sine of 36°.

Or the radius multiplied by 5878 and divided by 10000, is the *R*sine of 36°. The *R*cosine of that is the *R*sine of 54°.[109]

$$jy\bar{a}\ 36^\circ = \sqrt{\frac{1}{8}(5R^2 - \sqrt{5R^4})} = \frac{5878R}{10000}.$$

That is,

$$\sin 36^\circ = \sqrt{\frac{1}{8}(5 - \sqrt{5})} = \frac{5878}{10000}.$$

Since $\sqrt{5} = 2.236068$ approximately

$$\therefore \quad 5 - \sqrt{5} = 2.763932\ldots$$

$$\therefore \quad \sqrt{\frac{1}{8}(5 - \sqrt{5})} - \sqrt{.345492\ldots} = 0.5878 \text{ approximately.}$$

Kamalākara proved the results thus:
Let x denote *jyā* 18°; then

$$\begin{aligned} \frac{1}{2}R(R - x) &= \frac{1}{2}R \times utjy\bar{a}\ 72^\circ = (jy\bar{a}\ 36^\circ)^2; \\ \frac{2x^2}{R} &= R - kojy\bar{a}\ 36^\circ = utjy\bar{a}\ 36^\circ; \end{aligned}$$

$$\begin{aligned} \frac{1}{2}R(R - x) + \left(\frac{2x^2}{R}\right)^2 &= (jy\bar{a}\ 36^\circ)^2 + (utjy\bar{a}\ 36^\circ)^2 \\ &= 4\ (jy\bar{a}\ 18^\circ)^2 \\ &= 4x^2, \end{aligned}$$

[108] *Ibid*, *Gola*, xiv. 9.
[109] *Ibid*, *Gola*, xiv. 7–8.

or

$$8x^2R^2 = 8x^4 - R^3x + R^4,$$

or, multiplying by 8 and arranging,

$$16R^2x^2 + 8R^3x + R^4 = 9R^4 - 48R^2x^2 + 64x^4,$$

whence taking the positive square roots of the two sides, we get

$$\begin{aligned} & 4Rx + R^2 = 3R^2 - 8x^2, \\ \text{or} \qquad & (4x + R)^2 = 5R^2. \end{aligned}$$

Therefore $x = \frac{1}{4}(\sqrt{5R^2} - R)$; the other sign is neglected since x must be less than R.

Again

$$\begin{aligned} (jy\bar{a}\ 36°)^2 &= \frac{1}{2}R(R - x), \\ &= \frac{R}{8}(5R - \sqrt{5R^2}), \\ \therefore \quad jy\bar{a}\ 36° &= \sqrt{\frac{1}{8}(5R^2 - \sqrt{5R^4})}. \end{aligned}$$

Sin $\frac{\pi}{N}$

In his treatise on arithmetic, Bhāskara II has given a rule which yields the *R*sine of certain particular angles to a very fair degree of approximation.

> Multiply the diameter of a circle by 103923, 84853, 70534, 60000, 52055, 45922 and 41031 severally and divide the products by 120000; the quotients will be the sides of regular polygons inscribed in the circle from the triangle to the enneagon respectively.[110]

If S_n be a side of a regular polygon of n sides inscribed in a circle of diameter

[110] *Līlāvatī* (Ānandāśrama edition), vss. 206–7, p. 207.

D, then according to Bhāskara II,

$$S_3 = D\frac{103923}{120000} = D \times 0.866025$$
$$S_4 = D\frac{84853}{120000} = D \times 0.707108\dot{3}$$
$$S_5 = D\frac{70534}{120000} = D \times 0.58778\dot{3}$$
$$S_6 = D\frac{60000}{120000} = D \times 0.5$$
$$S_7 = D\frac{52055}{120000} = D \times 0.433791\dot{6}$$
$$S_8 = D\frac{45922}{120000} = D \times 0.38268\dot{3}$$
$$S_9 = D\frac{41031}{120000} = D \times 0.341925$$

where are given the formulae of Bhāskara II first in their original forms and then in decimals. Now, we know that

$$S_n = D \sin\frac{\pi}{n}.$$

Hence it is found that

$$\sin 60^\circ = 0.866025 \qquad \sin\frac{\pi}{7} = 0.433791\dot{6}$$
$$\sin 45^\circ = 0.707108\dot{3} \qquad \sin\frac{\pi}{8} = 0.38268\dot{3}$$
$$\sin 36^\circ = 0.58778\dot{3} \qquad \sin\frac{\pi}{9} = 0.341925$$

According to modern computation

$$\sin 60^\circ = 0.8660254\ldots \qquad \sin\frac{\pi}{7} = 0.4338819$$
$$\sin 45^\circ = 0.7071067\ldots \qquad \sin\frac{\pi}{8} = 0.3826834$$
$$\sin 36^\circ = 0.5877853 \qquad \sin\frac{\pi}{9} = 0.3420201$$

Comparing the two tables we find that except in case of $\sin\frac{\pi}{7}$ and $\sin\frac{\pi}{9}$ Bhāskara's approximations are correct up to five places of decimals; in these two latter cases the results are near enough.

Approximate formula of Bhāskara I

Bhāskara I (629) has given the following rule for the calculation of the Rsine and Rcosine of an arc without the help of a table.

> Subtract the arc in degrees from the degrees of the semi-circumference and multiplying the arc by the remainder, put down (the result) at two places, (at one place) subtract (the quantity) from 40500; by one-fourth of the remainder divide the quantity (at the second place) multiplied by the maximum value of the function; thus the value of the direct or reversed *R*sine of an arc and its complement is obtained wholly.[111]

If α be an arc of a circle of radius R in terms of degrees, then

$$jy\bar{a}\ \alpha = \frac{R\left(\frac{C}{2} - \alpha\right)\alpha}{\left\{40500 - \left(\frac{C}{2} - \alpha\right)\alpha\right\}/4},$$

where C denotes the circumference of the circle in terms of degrees. Since $40500 = \left(\frac{5}{4}\right) \times 180 \times 180$, we can write the formula in the form

$$jy\bar{a}\ \alpha = \frac{R\left(\frac{C}{2} - \alpha\right)\alpha}{\frac{5}{4}\left(\frac{C}{2}\right)^2 - \left(\frac{C}{2} - \alpha\right)\alpha},$$

which is of course equivalent to

$$\sin\theta = \frac{4(\pi - \theta)\theta}{\left(\frac{5}{4}\right)\pi^2 - (\pi - \theta)\theta}.$$

From a statement of Bhāskara I it appears that this formula was known to Āryabhaṭa I.[112]

The above formula has been restated by Brahmagupta (628) thus:

> Subtract the degrees of an arc or its complement from the semicircle (i.e. 180) and multiply (the remainder) by that; subtract one-fourth the product from 10125; divide the product by the remainder and multiply by the semi-diameter; (the result) is the *R*sine of that (arc or its complement).[113]

$$jy\bar{a}\ \alpha = \frac{R(180 - \alpha)\alpha}{10125 - \frac{(180-\alpha)\alpha}{4}}.$$

Almost in the same way Śrīpati (1039) says:

> Subtract the degrees of an arc or its complement from 180 and multiply (the remainder) by that; subtract one-fourth the product from 10125; multiply the product by the semi-diameter and divide

[111] *Mahābhāskarīya*, vii. 17ff.
[112] Bhāskara I's comm. on *Āryabhaṭīya*, i. 11, p. 40.
[113] *Brāhmasphuṭasiddhānta*, xiv. 23.

> by this remainder; thus the *R*sine of an arc or its complement can be found even without (a table of *R*sines).[114]

Bhāskara II (1150) writes:

> Subtract an arc from the circumference and multiply (the remainder) by the arc; this product is called the 'first'. From five times the fourth part of the square of the circumference subtract the 'first', and by the remainder divide the 'first' multiplied by four times the diameter; the quotient will be the chord of the arc.[115]

If s denotes the chord of an arc β of a circle, then

$$s = \frac{8R(C-\beta)\beta}{\frac{5}{4}C^2 - (C-\beta)\beta}.$$

Now if $\beta = 2\alpha$, then $s = 2\ jy\bar{a}\ \alpha$. So that on making the substitutions this formula will easily reduce to that of the elder Bhāskara.

This formula has been used by Gaṇeśa (1545) in his *Grahalāghava*.[116] Though it gives only a roughly approximate (*sthūla*) value of the *R*sine of an arc, observes Bhāskara II, it simplifies operations.

On putting $\theta = \frac{\pi}{2} - \phi$, in the above approximate formula, it becomes

$$\begin{aligned}\cos\phi &= \frac{16\left(\frac{\pi}{2}+\phi\right)\left(\frac{\pi}{2}-\phi\right)}{5\pi^2 - 4\left(\frac{\pi}{2}+\phi\right)\left(\frac{\pi}{2}-\phi\right)} \\ &= \frac{\pi^2 - 4\phi^2}{\pi^2+\phi^2} \\ &= \left(1 - \frac{4\phi^2}{\pi^2}\right)\left(1 - \frac{\phi^2}{\pi^2} + \frac{\phi^4}{\pi^4}\right),\end{aligned}$$

neglecting higher powers. Therefore, to the same order of approximation,

$$\cos\phi = 1 - \frac{5\phi^2}{\pi^2} + \frac{5\phi^4}{\pi^4}.$$

If we put $\pi = \sqrt{10}$ approximately, we get

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{20}$$

nearly. According to modern trigonometry, to the same order of approximation,

$$\cos\phi = 1 - \frac{\phi^2}{2} + \frac{\phi^4}{24}.$$

[114] *Siddhāntaśekhara*, iii. 17.
[115] *Līlāvatī* (Ānandāśrama edition), vs. 210, p. 21e. Also see *GK*, part 2, pp. 80–81.
[116] *Grahalāghava*, ii. 2f.

Again putting $\phi = \dfrac{\pi}{n}$ in Bhāskara I's formula, where n is an integer, we get

$$\sin\frac{\pi}{n} = \frac{16(n-1)}{5n^2 - 4(n-1)}$$

whence we have

$$\sin\frac{\pi}{7} = 0.4343\ldots, \qquad \sin\frac{\pi}{8} = 0.3835\ldots, \qquad \sin\frac{\pi}{9} = 0.3431\ldots,$$

which are correct up to two places of decimals, the third figure in every case being too large.

Inverse formula of Brahmagupta

Brahmagupta gave the following rule for finding approximately the arc corresponding to a given *R*sine function:

> Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; subtracting the quotient from the square of 90, extract the square-root and subtract (the root) from 90; the remainder will be in degrees and minutes; thus will be found the arc of the given *R*sine without the table of *R*sines.[117]

If α be the arc corresponding to the given *R*sine function m, then the rule says that

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{\left(\frac{m}{4} + r\right)}}.$$

This result follows easily on reversing the approximate formula for the *R*sine and was very likely obtained in the same way.

$$m = jy\bar{a}\ \alpha = \frac{R(180-\alpha)\alpha}{10125 - \frac{(180-\alpha)\alpha}{4}}.$$

Then

$$\alpha^2 - 180\alpha + \frac{10125m}{\left(\frac{m}{4} + r\right)} = 0.$$

Therefore

$$\alpha = 90 - \sqrt{8100 - \frac{10125m}{\left(\frac{m}{4} + r\right)}}.$$

The negative sign of the radical being retained, since α is supposed to be less than $90°$.

Śrīpati describes the inverse formula thus:

[117] *Brāhmasphuṭasiddhānta*, xiv. 25–6.

> Multiply 10125 by the given *R*sine and divide by the quarter of the given *R*sine plus the radius; then subtract the quotient from the square of 90; ninety degrees lessened by the square root (of the remainder) will be the arc (determined) without the table of *R*sines.[118]

Bhāskara II writes:

> By four times the diameter added with the chord divide the square of the circumference multiplied by five times a quarter of the chord; the quotient being subtracted from the fourth part of the square of the circumference, and the square-root of the remainder being diminished from half the circumference, the result will be the arc.[119]

That is:

$$\beta = \frac{C}{2} - \left\{ \frac{C^2}{4} - \frac{5sC^2}{4(8R+s)} \right\}^{\frac{1}{2}}$$

which follows at once from his form of the approximate formula for the chord s.

5 Trigonometrical tables

Twenty-four sines

The Hindus generally calculate tables of trigonometrical functions for every arc of $3^\circ 45'$, or what they call twenty-four *R*sines in a quadrant. In the choice of 24, they seem to have been led by an ancient observation that "the ninety-sixth part of a circle looks (straight) like a rod." Thus Balabhadra (c. 700 AD) observes, "If anybody asks the reason of this, he must know that each of these *kardajat* is $\frac{1}{96}$ of the circle =225 minutes (= $3\frac{3}{4}$ degrees). And if we reckon its *R*sine, we find it also to be 225 minutes."[120]

The origin of this idea again lies in the impression that the human eye-sight reaches to a distance of $\frac{1}{96}$th part of the circumference of the earth which appears flat.[121] A more plausible hypothesis about the choice of $3^\circ 45'$ as the unit will be this: Having determined previously a very accurate value of π it

118 *Siddhāntaśekhara*, iii. 18.

119 *Līlāvatī* (Ānandāśrama edition), vs. 212, p. 216.

120 Quoted by Al-Bīrūnī in his *India* (Sachau, *Alberuni's India*, I, p. 275). Balabhadra's works are now lost. According to Chambers' *Mathematical Tables*, we find $\sin(3^\circ 45') = 0.0654031$, $\tan(3^\circ 45') = 0.0655435$, radian$(3^\circ 45') = 0.0654498$, so that the assumption is fairly accurate.

121 Balabhadra says, "Human eyesight reaches to a point distant from the earth and its rotundity the 96th part of 5000 *yojana*, i.e. 52 *yojana* (exactly $52\frac{1}{12}$). Therefore a man does not observe its rotundity, and hence the discrepancy of opinions on the subject." This remark of Balabhadra has been quoted by Al-Bīrūnī (*India*, I, p. 273).

was simply natural for the Hindus to choose the radius of the circle of reference to be $3438'$. They also knew that $R\sin 30° = \text{semi-radius} = 1719'$. Starting with this they began to calculate the function of the semi arcs $15°, 7°30', 3°45'$ with the help of the well known formulae and in so doing they soon found that $3°45'$ is the first whose *R*sine contains the same number of minutes as the arc. So they chose this arc.[122]

Sūryasiddhānta

The earliest known Hindu work to contain a table of trigonometrical functions is the *Sūryasiddhānta* (c. 300 AD). It has a table of *R*sines and versed *R*sines for every arc of $3°45'$ of a circle of radius $3438'$. The method of computation has been indicated to be as follows:

> The eighth part of the number of minutes in a sign (i.e. $225'$) is the first *R*sine. It is divided by itself and then diminished by the quotient; the remainder added with the first *R*sine gives the second *R*sine.
>
> (Any) *R*sine is divided by the first *R*sine and then diminished by the quotient. The remainder added to the difference of that *R*sine and the preceding *R*sine will give the next *R*sine. Thus can be obtained the 24 *R*sines, which are as follows.[123]

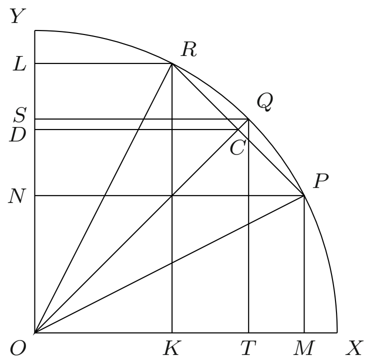


Figure 11

(**ed.** In Figure 11) Let the arc $PQ = \text{arc QR} = \alpha$. Then,

$$RK = OL = OS + SL = QT + SL,$$

$$\text{and} \quad NS - SL = 2DS = 2\frac{OS}{OQ} \times QC.$$

[122] *Cf. Nouv. Ann. Math.*, xiii (1854), p. 390.
[123] *Sūryasiddhānta*, ii. 15–6.

Now

$$CP^2 = QC(2OQ - QC);$$

$$\therefore \qquad QC = \frac{QP^2}{2OQ}.$$

Hence,

$$NS - SL = OS\left(\frac{QP}{OQ}\right)^2.$$

Or,

$$\begin{aligned} SL &= NS - OS\left(\frac{QP}{OQ}\right)^2 \\ &= (QT - PM) - OS\left(\frac{QP}{OQ}\right)^2. \end{aligned}$$

Therefore,

$$RK = QT + (QT - PM) - QT\left(\frac{QP}{OQ}\right)^2.$$

Now suppose the arc $XQ = n\alpha$; then arc $XP = (n-1)\alpha$; $XR = (n+1)\alpha$; further $QP = 2\ jy\bar{a}\ \frac{\alpha}{2}$. Hence

$$jy\bar{a}\ (n+1)\alpha = jy\bar{a}\ n\alpha + \{jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha\} - jy\bar{a}\ n\alpha \times \left(\frac{2\ jy\bar{a}\frac{\alpha}{2}}{R}\right)^2,$$

which is equivalent to

$$\sin\ (n+1)\theta = \sin\ n\theta + \{\sin\ n\theta - \sin\ (n-1)\theta\} - \sin\ n\theta\left(2\sin\frac{\theta}{2}\right)^2.$$

It is also probable that the formula was obtained trigonometrically thus:

$$jy\bar{a}\ (\xi \pm \eta) = \frac{1}{R}(jy\bar{a}\ \xi\ kojy\bar{a}\ \eta \pm kojy\bar{a}\ \xi\ jy\bar{a}\ \eta).$$

Then,

$$jy\bar{a}\ (\xi + \eta) - jy\bar{a}\ \xi = \frac{1}{R}(kojy\bar{a}\ \xi\ jy\bar{a}\ \eta - jy\bar{a}\ \xi\ utjy\bar{a}\ n),$$

and,

$$jy\bar{a}\ \xi - jy\bar{a}\ (\xi - \eta) = \frac{1}{R}(kojy\bar{a}\ \xi\ jy\bar{a}\ \eta + jy\bar{a}\ \xi\ utjy\bar{a}\ n).$$

Hence,

$$\begin{aligned} jy\bar{a}\ (\xi + \eta) - jy\bar{a}\ \xi &= jy\bar{a}\ \xi - jy\bar{a}\ (\xi - \eta) - \frac{2\ jy\bar{a}\ \xi\ utjy\bar{a}\ \eta}{R} \\ &= jy\bar{a}\ \xi - jy\bar{a}\ (\xi - \eta) - jy\bar{a}\ \xi\left(\frac{2\ jy\bar{a}\ \frac{\eta}{2}}{R}\right)^2. \end{aligned}$$

Now put $\eta = \alpha,\ \xi = n\alpha$; so that the formula becomes

$$jy\bar{a}\ (n+1)\alpha - jy\bar{a}\ n\alpha = jy\bar{a}\ n\alpha - jy\bar{a}\ (n-1)\alpha - jy\bar{a}\ n\alpha \left(\frac{2\ jy\bar{a}\ \frac{\alpha}{2}}{R}\right)^2.$$

So far the formula is mathematically accurate. According to the *Sūrya-siddhānta*

$$\alpha = 3°45' = 225', \qquad jy\bar{a}\ \alpha = 225', \qquad R = 3438'.$$

Therefore

$$\left(\frac{2\ jy\bar{a}\ \frac{\alpha}{2}}{R}\right)^2 = \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2 \text{ approximately}$$
$$= \left(\frac{225}{3438}\right)^2 = \left(\frac{1}{15.28}\right)^2 = \frac{1}{225} \text{ approximately.}$$

Hence we get

$$\sin(n+1)\theta = \sin\ n\theta + \{\sin\ n\theta - \sin(n-1)\theta\} - \frac{\sin n\theta}{225},$$

where

$$\theta = 3°45' \text{ and } n = 1, 2, \ldots, 24.$$

According to modern calculation, the divisor in the last term will be slightly different. For

$$\left(2\sin\frac{\theta}{2}\right)^2 = (2\sin 1°52'30'')^2 = 0.00428255 = \frac{1}{233.506}, \text{ nearly.}$$

This little discrepancy, however, does not make much difference in the values of the Rsine functions calculated on the basis of that formula. They are indeed fairly accurate even according to modern calculations except in a few instances.[124]

About this method of constructing the tables of Rsines, Delambre remarks:

> The method is curious; it indicates a method of calculating the table of sines by means of their second differences.[125]

He then goes on:

> This differential process has not up to now been employed except by Briggs who himself did not know that the constant factor was the square of the chord ΔA $(= 3°45')$ or of the interval, and who

[124] *Vide infra.*

[125] Delambre, *Histoire de l'Astronomie Ancienne*, t. 1. Paris (1817), p. 457.

> could not obtain it except by comparing the second differences obtained in a different manner. The Indians also have probably done the same; they obtained the method of differences only from a table calculated previously by a geometric process. Here then is a method which the Indians possessed and which is found neither amongst the Greeks, nor amongst the Arabs.[126]

We do not understand what valid grounds were there for Delambre to suppose that the Hindus discovered the above theorem of *R*sines by inspection after having calculated the table of *R*sines by a different method. For there is absolutely no doubt that the ancient Hindus were in possession of necessary and sufficient equipment to derive it in either of the ways indicated above. It is noteworthy that that theorem has an important geometrical foundation. If there be three arcs of a circle in arithmetical progression the sum of the sines of the two extreme arcs is to the sine of the middle arc as the sine of twice the common difference is to the sine of that difference. For

$$\begin{aligned} jy\bar{a}\ (\xi+\eta)\ jy\bar{a}\ (\xi-\eta) &\equiv 2\ jy\bar{a}\ \xi - \frac{2\ jy\bar{a}\ \xi\ utjy\bar{a}\ \eta}{R} \\ &= \frac{2\ jy\bar{a}\ \xi\ kojy\bar{a}\ \eta}{R}. \end{aligned}$$

Hence,

$$\frac{jy\bar{a}\ (\xi+\eta)+jy\bar{a}\ (\xi-\eta)}{jy\bar{a}\ \xi} = \frac{2\ kojy\bar{a}\ \eta}{R} = \frac{jy\bar{a}\ 2\eta}{jy\bar{a}\ \eta}.$$

This very remarkable property of the circle was discovered in Europe by Vieta (1580).[127]

Āryabhaṭa

The trigonometrical table of Āryabhaṭa I (499) contains the differences between the successive *R*sines for arcs of every $3°45'$ of a circle of radius $3438'$.[128] His first method of computing it, which is rather cryptic, seems to be the same as that followed by Varāhamihira (*infra*). The other is practically the same as that of the *Sūryasiddhānta*, though put in a different form. He says:

> Divide a quarter of the circumference of a plane circle (into as many equal parts as desired). From (right) triangles and quadrilaterals (can be obtained) the *R*sines of equal arcs, as many as desired, for (any given) radius.[129]

[126] *Ibid*, p. 459f.
[127] Playfair, J. "Observations on the Trigonometrical Tables of the Brahmins," *Trans. Roy. Soc. Edin.*, iv (1798), pp. 83–106; compare also *Asiatic Researches*, iv. p. 165.
[128] *Āryabhaṭīya*, i. 12.
[129] *Ibid*, ii. 11.

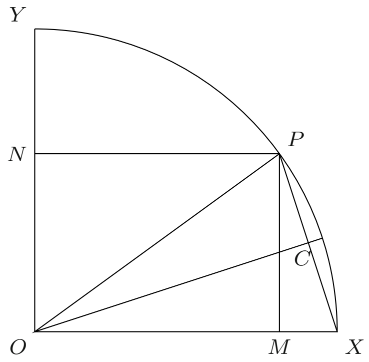


Figure 12

What is meant by the author is very probably this: If P (**ed.** see Figure 12) be any point on the arc of the quadrant, draw the perpendiculars PM and PN; also join PX. So that corresponding to P we have a rectangle $PMON$ and a right-angled triangle PMX. Now having given the *R*sine (PM) of the arc $XP(= \alpha)$, we can determine from the rectangle $PMON$ the side PN which is the sine of the arc $(90° - \alpha)$. Having found PN, we can calculate MX, which is equal to $R -$ *jyā* $(90° - \alpha)$. Then in the right-angled triangle PMX, we can determine the chord PX. Half of this is *jyā* $\dfrac{\alpha}{2}$. Again from a similar set of a rectangle and a right-angled triangle corresponding to the half arc, we can calculate *jyā* $(90° - \dfrac{\alpha}{2})$ and *jyā* $\dfrac{\alpha}{4}$. Proceeding thus we can compute the *R*sines of as many equal arcs as we please and it is clear that in so doing the quadrant will be broken up into a system of right-angled triangles and rectangles, as contemplated in the rule.

This is the interpretation of Āryabhaṭa's rule by his ancient commentators, like Sūryadeva Yajvā and Parameśvara (1430). Another interpretation will be this: The quadrant is trisected by the inscribed equilateral triangle and bisected by the inscribed square. The length of the arc between these points is $15°(= 45° - 30°)$. One-fourth of this is $3°45'$. So that the rule under discussion indicates how to divide the quarter of the circumference into portions of $3°45'$ each. If this interpretation is right,[130] which is rather forced, then it will have to be said that Āryabhaṭa I gave only one method of computing the trigonometrical table.[131]

The second method of Āryabhaṭa I is this:

> The first *R*sine divided by itself and then diminished by the quo-

[130] This interpretation has been suggested by Rodet, Kaye and Sengupta.

[131] In this connection, the reader is referred to "*Āryabhaṭīya* of Āryabhaṭa," edited with English translation by K. S. Shukla and K. V. Sarma, INSA, New Delhi, 1976, pp. 45–51.

> tient will give the second difference (of tabular *R*sines). For computing any other difference, (the sum of) all the preceding differences is divided by the first *R*sine and the quotient is subtracted from the preceding difference. Thus, all the remaining differences (can be calculated).[132]

Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ denote successive differences of the tabular *R*sines, such that, α being equal to $3°45'$,

$$\begin{aligned}\Delta_1 &= jy\bar{a}\,\alpha - jy\bar{a}\,0,\\ \Delta_2 &= jy\bar{a}\,2\alpha - jy\bar{a}\,\alpha,\\ &\vdots\\ \Delta_n &= jy\bar{a}\,n\alpha - jy\bar{a}\,(n-1)\alpha.\end{aligned}$$

Then $jy\bar{a}\,n\alpha = \Delta_1 + \Delta_2 + \ldots + \Delta_n$.

The rule says:

$$\Delta_{n+1} = \Delta_n - \frac{\Delta_1 + \Delta_2 + \ldots + \Delta_n}{jy\bar{a}\,\alpha}.$$

On substituting the values, this formula will be found to be equivalent to

$$\{jy\bar{a}\,(n+1)\alpha - jy\bar{a}\,n\alpha\} = \{jy\bar{a}\,n\alpha - jy\bar{a}\,(n-1)\alpha\} - \frac{jy\bar{a}\,n\alpha}{jy\bar{a}\,\alpha}.$$

It is also noteworthy that the text also admits of the following interpretation:

> The first *R*sine is divided by itself and then diminished by the quotient; the result with the first *R*sine will give the second *R*sine. For (computing), any of the remaining *R*sines, the sum of all the *R*sines preceding it is divided by the first *R*sine and the quotient is subtracted from the first *R*sine, and the result added to the preceding *R*sine.

$$jy\bar{a}\,(n+1)\alpha = jy\bar{a}\,n\alpha + jy\bar{a}\,\alpha - \frac{(jy\bar{a}\,\alpha + jy\bar{a}\,2\alpha + \ldots + jy\bar{a}\,n\alpha)}{jy\bar{a}\,\alpha}.$$

If $\Delta_1, \Delta_2, \ldots$ be the tabular differences as before, then

$$\begin{aligned}\Delta_1 - \Delta_2 &= \frac{2\,jy\bar{a}\,\alpha\,(R - kojy\bar{a}\,\alpha)}{R}\\ \Delta_2 - \Delta_3 &= \frac{2\,jy\bar{a}\,2\alpha\,(R - kojy\bar{a}\,\alpha)}{R}\\ &\vdots\\ \Delta_n - \Delta_{(n+1)} &= \frac{2\,jy\bar{a}\,n\alpha\,(R - kojy\bar{a}\,\alpha)}{R}.\end{aligned}$$

[132] *Āryabhaṭīya*, ii. 12.

Adding up, we get

$$\Delta_1 - \Delta_{n+1} = \frac{2(R - kojy\bar{a}\ \alpha)}{R}(jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots + jy\bar{a}\ n\alpha).$$

Now

$$\begin{aligned}\frac{2(R - kojy\bar{a}\ \alpha)}{R} &= \left(\frac{2\ jy\bar{a}\ \frac{\alpha}{2}}{R}\right)^2, \\ &= \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2 \quad \text{approximately,} \\ &= \frac{1}{225} \quad \text{approximately.}\end{aligned}$$

Therefore

$$jy\bar{a}\ (n+1)\alpha = jy\bar{a}\ n\alpha + jy\bar{a}\ \alpha - \frac{(jy\bar{a}\ \alpha + jy\bar{a}\ 2\alpha + \ldots + jy\bar{a}\ n\alpha)}{225}.$$

Also

$$\begin{aligned}\Delta_{n+1} &= \Delta_n - \frac{jy\bar{a}\ n\alpha}{225}, \\ &= \Delta_n - \frac{\Delta_1 + \Delta_2 + \ldots + \Delta_n}{225}.\end{aligned}$$

Of these two interpretations the first has been given by the commentator Parameśvara and the second by the commentators Prabhākara, Sūryadeva (b. 1191), Yallaya (1480) and Raghunātharāja (1597).

It should be observed that Āryabhaṭa I does not appear to have used this formula consistently to calculate the whole table. For as will be found from Table 1, certain values actually recorded by Āryabhaṭa differ from the values calculated by the formula. Probably he corrected the calculated values in those cases by comparison with the known values of the sines of 30°, 45°, 60°; or what is much more likely employed the formula only to calculate the *R*sines of intermediate arcs. Other plausible explanations of the discrepancy have been furnished by Krishnaswami Ayyangar[133] and Naraharayya.[134]

Varāhamihira and Lalla

Varāhamihira's (d. 587) table contains the *R*sines for every 3°45′ and the successive differences of the tabular *R*sines for the radius 60.[135] His method of

133 *Journal of the Indian Mathematical Society*, xv (1924), pp. 121–6.
134 *Ibid*, pp. 105–13 of "Notes and Questions."
135 *Pañcasiddhāntikā*, iv. 6–11, 12–15.

Table 1

Differences Δ_n, $n =$	Calculated according to the formula	Recorded by Āryabhaṭa	Calculated according to the modern method
1	225	225	224.856
2	224	224	223.893
3	222.005	222	221.971
4	219.018	219	219.100
5	215.045	215	215.289
6	210.089	210	210.557
7	204.156	205	204.923
8	198.245	199	198.411
9	191.36	191	191.050
10	182.512	183	182.872
11	173.694	174	173.909
12	163.245	164	164.202
13	153.196	154	153.792
14	142.512	143	142.724
15	130.876	131	131.043
16	118.294	119	118.803
17	105.745	106	106.053
18	92.289	93	92.850
19	78.88	79	79.248
20	64.527	65	65.307
21	50.240	51	51.087
22	36.014	37	36.648
23	21.849	22	22.051
24	6.752	7	7.361

computation is this:[136] Starting with the known values of $R\sin 30°$, $R\sin 45°$ and $R\sin 60°$, by the repeated and proper application of the formulae

$$\sin\frac{\theta}{2} = \frac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta}$$
$$\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}\text{versin}\ \theta},$$

says he, the other *R*sines may be computed. Lalla[137] gives a table of *R*sines and versed *R*sines for the radius 3438′. His method of computation is the same as that of Āryabhaṭa I and the *Sūryasiddhānta*. He has also a shorter table of *R*sines and their differences for intervals of 10° of arcs of a circle of radius 150.[138]

Brahmagupta

Brahmagupta (628) takes the radius quite arbitrarily to be 3270. His explanation[139] for this departure from the usual practice is unsatisfactory.[140] He has, however, indicated two methods of computation.[141] One is *graphic* and the other *mathematical*.

Graphic Method

> Starting from the joint of two quadrants, mark off successively (on either directions) portions of arcs equivalent to the eighth part of a sign (30°). Join two and two of these marks by threads. Half of them (lengths of threads) will be the *R*sines.[142]

Mathematical Method[143]

In this method Brahmagupta employs the trigonometrical formulae

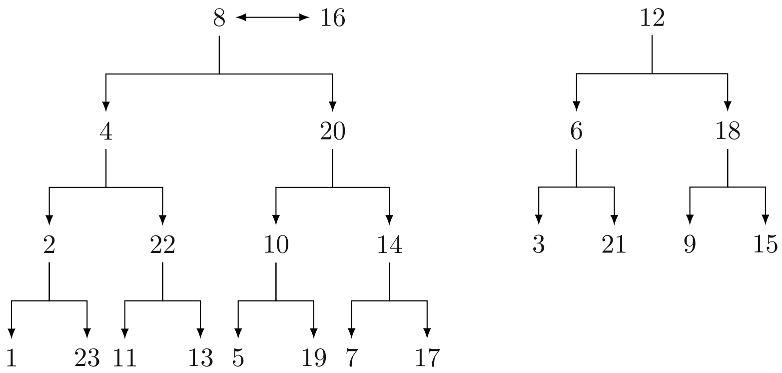
$$\sin\frac{\theta}{2} = \frac{1}{2}\sqrt{\sin^2\theta + \text{versin}^2\theta} \tag{17}$$
$$\sin\left(90° - \frac{\theta}{2}\right) = \sqrt{1 - \sin^2\frac{\theta}{2}}. \tag{18}$$

From the known value of the *R*sine of 8α, that is, of 30°, α being equal to 3°45′, we can calculate, by (17), the *R*sines of 4α, 2α, α. Then by (18) will be

[136] *Ibid*, iv. 2–5.
[137] *Śiṣyadhīvṛddhida*, ii. 1–8.
[138] *Śiṣyadhīvṛddhida*, xiii. 2–3.
[139] *Brāhmasphuṭasiddhānta*, xxi. 16.
[140] Datta, Bibhutibhusan, "Hindu Values of π", *JASB*, N. S., Vol. 22, (1926), pp. 25–42; see particularly p. 32, footnote 1.
[141] His table will be found in *Brāhmasphuṭasiddhānta*, ii. 2–9.
[142] *Brāhmasphuṭasiddhānta*, xxi. 17.
[143] *Ibid*, xxi. 20–21; compare also the verse 23.

obtained the Rsines of 20α, 22α, 23α. Again from the first two of the latter results, we shall obtain, by (17), the Rsines of 10α and 11α; and thence by (18) the Rsines of 14α and 13α. Continuing similar operations, we can compute the Rsines of 5α and 19α, 7α and 17α. Again starting with the Rsine of 12α, we shall obtain on proceeding in the same way, successively the values of the Rsines of 6α and 18α; 3α and 21α; 9α and 15α. Thus the values of all the twenty-four Rsines are computed.

It is perhaps noteworthy that $R \sin n\alpha$ is called by Brahmagupta as the nth Rsine. The successive order in which the various Rsines have been obtained above can be exhibited as follows:



Brahmagupta then observes:

> In this way (can be computed) the Rsines in greater or smaller numbers, having known first the Rsines of the sixth, fourth and third parts of the circumference of the circle.[144]

He further remarks that the Rsine of the semi-arc can be more easily calculated by the second formula of Varāhamihira.[145] Brahmagupta has also another table giving differences of Rsines for every 15° of a circle of radius 150.[146]

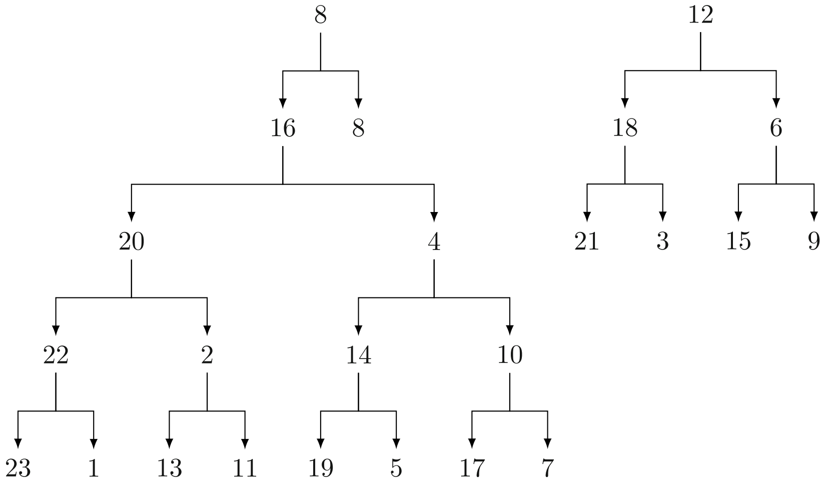
Āryabhaṭa II and Śrīpati

Āryabhaṭa II (950) gives the same table as that of the *Sūryasiddhānta*.[147] But his method of computation is entirely different.[148] He takes recourse to the formulae

$$\sin \frac{1}{2}(90^\circ \pm \theta) = \sqrt{\frac{1}{2}(1 \pm \sin \theta)}.$$

[144] *Brāhmasphuṭasiddhānta*, xxi. 22.
[145] *Ibid*, xxi. 23.
[146] *Khaṇḍakhādyaka*, Part I, iii. 6; *Dhyānagrahopadeśa*, 16.
[147] *Mahāsiddhānta*, iii. 4–8.
[148] *Mahāsiddhānta*, iii. 1–3.

Beginning with the known values of $R \sin 30°$ and $R \sin 45°$, like Brahmagupta, the successive order in which the *R*sines will come out in the course of computation, can be best exhibited thus:



The table of Śrīpati (c. 1039) gives the *R*sines and versed *R*sines for every 3°45′ of a circle of radius 3415.[149] His *first* method of computing it is the same as the graphic method of Brahmagupta. He says:

> Place marks at the eighth parts of a sign (30°); then (starting) from the joint of two quadrants, following up these marks, join two and two of them successively by means of threads; half of them will be the *R*sines.[150]

The *second* method followed by Śrīpati is identical with the mathematical method of Brahmagupta.[151]

Bhāskara II

The table of Bhāskara II (1150) contains the *R*sines and versed *R*sines as well as their differences for every 3°45′ of a circle of radius 3438′. He has indicated several methods of computing it. The *first* is practically the same as Brahmagupta's *graphic method*. He says:

> For computing the *R*sines, take any optional radius. On a plane ground describe a circle by means of a piece of thread equal to

[149] *Siddhāntaśekhara*, iii. 3–10.
[150] *Siddhāntaśekhara*, xvi. 9.
[151] *Siddhāntaśekhara*, xvi. 14ff.

> that radius. On it mark the cardinal points and 360 degrees; so in each quadrant of the circle there will be 90 degrees. Then divide every quadrant into as many equal parts as the number of *R*sines to be computed and put marks of these divisions. For instance, if it be required to calculate 24 *R*sines, there will be 24 marks. Then beginning from any of the cardinal points, and proceeding either ways, the threads connecting the successive points will be the chords. There will be thus 24 chords. Halves of these will be the *R*sines (required). So these half-chords should be measured and the results taken as the *R*sines.[152]

The *second* is again a reproduction of Brahmagupta's *theoretical method:*

> When twenty-four *R*sines are required (to be computed), the *R*sine of 30° is the eighth element; its *R*cosine is the sixteenth; and $R\sin 45°$ is the twelfth. From these three elements, twenty-four elements can be computed in the way indicated. From the eighth we get the *R*sine of its half, that is, the fourth (element), its *R*cosine is the twentieth. Similarly from the fourth, the second and the twenty-second; from the second, the first and the twenty-third. In the same way from the eighth are obtained the tenth and fourteenth, fifth and nineteenth, seventh and seventeenth, eleventh and thirteenth. Again from the twelfth follow the sixth and eighteenth, third and twenty-first, ninth and fifteenth. The radius is the twenty-fourth *R*sine.[153]

The *third* method of computing trigonometrical tables described by Bhāskara II is the same as that of Āryabhaṭa II. The speciality of this method, as also of the two following, is, says Bhāskara II, that it does not employ the versed *R*sine function. As for the successive order of derivation, he points out that "from the eighth *R*sine (will be obtained) the sixteenth; from the sixteenth, the fourth and the twentieth; from the fourth, the tenth and fourteenth. In this way all the rest may be deduced.[154]

The *fourth* method of Bhāskara II is based on the application of the formula

$$R\sin\frac{1}{2}(\theta-\phi)=\frac{1}{2}\left\{(R\sin\theta-R\sin\phi)^2+(R\cos\theta-R\cos\phi)^2\right\}^{\frac{1}{2}},$$

"so that knowing any two *R*sines others may be derived. For instance, let one be the fourth *R*sine and the other eighth *R*sine. From them is derived the second *R*sine. From the second and fourth, the first; and so on."[155]

[152] *Siddhāntaśiromaṇi*, *Gola*, v. 2–6 (gloss).
[153] *Siddhāntaśiromaṇi*, *Gola*, v. 2–6 (gloss); xiv. 10–11 (gloss).
[154] *Siddhāntaśiromaṇi*, *Gola*, xiv. 12 (gloss).
[155] *Siddhāntaśiromaṇi*, *Gola*, xv. 13 (gloss).

The *fifth* method depends on the formula

$$R\sin(45^\circ - \theta) = \sqrt{\frac{1}{2}(R\cos\theta - R\sin\theta)^2}.$$

"Thus, for instance, take the eighth *R*sine; its *R*cosine is the sixteenth *R*sine. From these the fourth is derived; and so on."[156]

All the theoretical methods described above require the extraction of the square-root. So Bhāskara II propounds a new method (the *sixth*) in which that will not be necessary. It is based on the employment of the formula

$$R\cos 2\theta = R - \frac{2(R\sin\theta)^2}{R},$$
$$\text{or} \qquad \cos 2\theta = 1 - 2\sin^2\theta.$$

But this method is defective in as much as "only certain elements of a table of *R*sines can be calculated thus,"[157] but not the whole table. This defect is present in a sense in the previous methods, for no one of the trigonometrical formulae employed in them suffices alone for the computation of a table containing more *R*sines (*vide infra*).

The *seventh* method of Bhāskara II for calculating a table of twenty-four *R*sines, has been described thus:

> Multiply the *R*cosine by 100 and divide by 1529; diminish the *R*sine by its $\frac{1}{467}$ part. The sum of these two results will give the next *R*sine and their difference the previous *R*sine. Here 225 less $\frac{1}{7}$ is the first *R*sine. And by this rule can be successively calculated the twenty-four *R*sines.[158]

$$jy\bar{a}\ (n\alpha \pm \alpha) = \left(jy\bar{a}\ n\alpha - \frac{jy\bar{a}\ n\alpha}{467}\right) \pm \frac{100}{1529}\ kojy\bar{a}\ n\alpha,$$

where $n = 1, 2, \ldots, 24$; $\alpha = 3^\circ 45'$; and $jy\bar{a}\ \alpha = 225 - \frac{1}{7}$.

The rationale of this formula is as follows:

By the Addition and Subtraction Theorems,

$$\begin{aligned} jy\bar{a}\ (n\alpha \pm \alpha) &= \frac{1}{R}(jy\bar{a}\ n\alpha \times kojy\bar{a}\ \alpha \pm kojy\bar{a}\ n\alpha \times jy\bar{a}\ \alpha), \\ &= jy\bar{a}\ n\alpha \times \frac{kojy\bar{a}\ \alpha}{R} \pm kojy\bar{a}\ n\alpha \times \frac{jy\bar{a}\ \alpha}{R}. \end{aligned}$$

[156] *Siddhāntaśiromaṇi*, *Gola*, xiv. 14 (gloss).
[157] *Siddhāntaśiromaṇi*, *Gola*, xiv. 15 (gloss).
[158] *Siddhāntaśiromaṇi*, *Gola*, xiv. 18–20.

Now

$$\frac{1}{R}\, jy\bar{a}\ \alpha = \frac{1}{3438}\left(225 - \frac{1}{7}\right) = \frac{787}{12033} = \frac{1}{15.289707\ldots}$$
$$= \frac{100}{1528.9707\ldots} = \frac{100}{1529} \text{ nearly,}$$

and

$$\frac{1}{R}\, kojy\bar{a}\ \alpha = \frac{1}{R}\sqrt{R^2 - (jy\bar{a}\ \alpha)^2} = \sqrt{1 - \left(\frac{jy\bar{a}\ \alpha}{R}\right)^2}$$
$$= \sqrt{1 - \frac{1}{233.775\ldots}} = 1 - \frac{1}{467.550\ldots}$$
$$= 1 - \frac{1}{467} \text{ nearly}$$

and hence the rule. This formula is very nearly accurate. For according to the modern values

$$jy\bar{a}\ (3°45') = 224.856\ldots$$

Therefore

$$\frac{1}{R}\, jy\bar{a}\ (3°45') = \frac{224.856}{3438} = \frac{1}{15.28978\ldots} = \frac{100}{1528.978\ldots}.$$

Bhāskara II has indicated how to compute a table of Rsines for every 3° of a circle of radius 3438′. He writes:

> For instance if (it be required to compute) thirty Rsines in a quadrant, half the radius is the tenth Rsine, its Rcosine is the twentieth Rsine. $R\sin 45°$ is the fifteenth Rsine; $R\sin 36°$ is the twelfth and $R\cos 36°$ the eighteenth. The Rsine of 18° is the sixth and its Rcosine is the twenty-fourth. Then by the rule for deriving the Rsine of the half arc from the square-root of the sum of the squares of the Rsine and versed Rsine of an arc, as stated before, from the tenth (is derived) the fifth; its Rcosine is the twenty-fifth. In that way from the twelfth (is calculated) the sixth and twenty-fourth; from the sixth, the third and twenty-seventh; from the eighteenth, the ninth and twenty-first. These are the only elements (of the table) of Rsines which can be calculated in this way. So it has been observed that ‘only certain elements etc.’ Next the formula for the Rsine of half the difference of two arcs should be employed. Let the fifth be the one Rsine and the ninth the other. From them will follow the second; its Rcosine is the twenty-eight Rsine. From these two again by employing the (previous) rule for the Rsine of semi-arcs from the square-root of the sum of the squares of the

> *R*sine and versed *R*sine, the first and fourteenth (are obtained). The remaining fourteen *R*sines can also be computed in the same way.[159]

Bhāskara II has further given a rule for computing a trigonometrical table for every degree. So it is called *pratibhāgika-jyakā-vidhi* ("The rule for the *R*sine of every degree").

> Deduct from the *R*sine of any arc its 6567th part; multiply its *R*cosine by 10 and then divide by 573. The sum of these two results is the next *R*sine and their difference the preceding *R*sine. Here the first *R*sine (i.e. $R\sin 1°$) will be $60'$ and other *R*sines may be successively found. Thus in a circle of radius equal to $3438'$, will be found 90 *R*sines.[160]

$$jy\bar{a}(\theta \pm 1°) = \left(jy\bar{a}\,\theta - \frac{jy\bar{a}\,\theta}{6567}\right) \pm \frac{10}{573}\,kojy\bar{a}\,\theta,$$

where $\theta = 1°, 2°, \ldots, 89°$; given $jy\bar{a}\ 1° = 60'$.

The rationale of this rule can be easily found: For by the Addition and Subtraction Theorems,

$$jy\bar{a}\ (\theta \pm 1°) = \frac{1}{R}\ (jy\bar{a}\ \theta \times kojy\bar{a}\ 1° \pm kojy\bar{a}\ \theta \times jy\bar{a}\ 1°).$$

Now it is stated that $R = 3438'$ and $jy\bar{a}\ 1° = 60'$. Therefore

$$\begin{aligned}
\frac{1}{R}\ jy\bar{a}\ 1° &= \frac{60}{3438} = \frac{10}{573}\\
\frac{1}{R}\ kojy\bar{a}\ 1° &= \sqrt{1 - \left(\frac{jy\bar{a}\ 1°}{R}\right)^2} = \left\{1 - \left(\frac{10}{573}\right)^2\right\}^{\frac{1}{2}}\\
&= 1 - \frac{100}{2 \times 328329} = 1 - \frac{1}{6566.58}\\
&= 1 - \frac{1}{6567}\\
&= 0.999847723\ldots
\end{aligned}$$

The denominator wrongly appears as 6569 in Bapu Deva's edition of the *Siddhāntaśiromaṇi*.[161]

The short table of Bhāskara II contains differences of *R*sines for intervals of $10°$ in a circle of radius 120.[162]

[159] *Siddhāntaśiromaṇi*, *Gola*, xiv. 15 (gloss).
[160] *Siddhāntaśiromaṇi*, *Gola*, xiv. 16–8.
[161] The text given by Bapu Deva runs as "*Svago'ṅgeṣusaḍaṃśena...*". It will be "*Svāgāṅgeṣu-ṣaḍaṃśena...*".
[162] *Siddhāntaśiromaṇi*, *Graha*, ii. 13.

Posterior Writers

Amongst the writers posterior to Bhāskara II (1150) who have given tables of trigonometrical functions, the most notable are Mahendra Sūri (1370) and Kamalākara (1658). The latter has a table of *R*sines and their differences for every degree of arc of a circle of radius 60, while the former gives tables of *R*sines and versed *R*sines together with their differences for every degree of the arc of a circle of radius 3600. Mahendra Sūri has furnished also some other tables for ready reckoning in Astronomy. It is noteworthy that Mahendra Sūri's work, *Yantrarāja*,[163] is admittedly based upon some Arabic work. We are informed by his commentator Malayendu Sūri, a direct disciple of the author and who wrote his commentary only 12 years after the text, that the author was the court astrologer of some potentate of the name of Firoz, who is probably the famous Sultan Firoz Shah Tughluk of Delhi (1351–88 AD). The illustrations chosen will agree with this date.

Table 2 gives the relevant details of the various *R*sine-Tables constructed in India from time to time.

6 Interpolation

Function of any arc

For finding the trigonometrical functions of an arc, other than those whose values have been tabulated, the Hindus generally follow the principle of proportional increase. Thus the *Sūryasiddhānta* says:

> Divide the minutes (into which the given arc is first reduced) by 225; the quotient will indicate the number of tabular *R*sines exceeded; (the remainder) is multiplied by the difference between the (tabular) *R*sine exceeded and that which is till to be reached and then divide by 225. The result thus obtained should be added to the exceeded tabular *R*sine; (the sum) will be the (required) direct *R*sine. This rule is applicable also to the case of (determining) the versed *R*sine.[164]

Brahmagupta states:

> Divide the minutes by 225, the quotient (will indicate) the number of tabular *R*sines (exceeded); the remainder is multiplied by the

[163] Mss. of the text with the commentary of *Yantrarāja* are available in the libraries of India Office, London (Nos. 2906-8), Benares Sanskrit College (No. 2905), Bikaner Palace (No. 760) and also at other places. Our copy has been procured from Benares.

[164] *Sūryasiddhānta*, ii. 31–2.

Table 2

Constructor of Table	Radius chosen	Interval taken	Sexagesimal places calculated
Author of *Sūryasiddhānta*[1]	3438′	225′	1 (minutes only)
Āryabhaṭa[2]	3438′	225′	1 (same Table as in *Sūryasiddhānta*)
Varāhamihira[3]	120	225′	2 (minutes and seconds)
Brahmagupta (1)[4]	3270	225′	1
Brahmagupta (2)[5]	150	15°	1
Deva[6]	300	10°	1
Lalla (1)[7]	3438′	225′	1 (same Table as in *Āryabhaṭīya*)
Lalla (2)[8]	150	10°	1
Sumati[9]	3438′	1°	1
Govindasvāmi[10]	3437′44″19‴	225′	3
Vaṭeśvara[11]	3437′44″	56′15″	2
Mañjula[12]	8°8′	30°	2 (degrees and minutes)
Āryabhaṭa II[13]	3438′	225′	1
Śrīpati[14]	3415′	225′	1
Udayadivākara[15]	12375859‴ or 3437′44″19‴	225′	1 (thirds only)
Bhāskara II (1)[16]	3438′	225′	1
Bhāskara II (2)	120	10°	1
Brahmadeva[17]	120	15°	1
Vṛddha Vaśiṣṭha[18]	1000	10°	1
Malayendu Sūri[19]	3600	1°	2
Madanapāla[20]	21600	1°	2
Mādhava[21]	3437′44″48‴	225′	3
Parameśvara[22]	3437′44″	225′	2
Munīśvara[23]	191	1°	4
Kṛṣṇa-daivajña[24]	500	3°	1
Kamalākara[25]	60	1°	5
Jagannātha Samrāṭa[26]	60	30′	5

[1] *Sūryasiddhānta*, ii. 17–22(a–b). [2] *Āryabhaṭīya*, i. 12.
[3] *Pañcasiddhāntikā*, iv. 6–12. [4] *Brāhmasphuṭasiddhānta*, ii. 2–5.
[5] *Khaṇḍakhādyaka*, Part I, iii. 6; *Dhyānagrahopadeśa*, 16.
[6] *Karaṇa-ratna.* i. 23. [7] *Śiṣyadhīvṛdhhida*, I, ii. 1–8. [8] *Ibid.*, xiii. 3. [9] *Sumati-mahā-tantra* and *Sumati-karaṇa.* [10] His comm. on *Mahābhāskarīya*, iv. 22.
[11] *Vaṭeśvara-siddhānta*, II, i. 2–26. [12] *Laghumānasa*, ii. 2(c–d).
[13] *Mahāsiddhānta*, iii. 4–6(a–b). [14] *Siddhāntaśekhara*, iii. 3–6. [15] His comm. on *Laghubhāskarīya*, ii. 2–3. [16] *Siddhāntaśiromaṇi*, *Gaṇita*, ii. 3–6; 13.
[17] *Karaṇa-prakāśa*, ii. 1. [18] *Vṛddha-Vaśiṣṭha-siddhānta*, ii. 10–11. [19] *Yantrarāja*, i. 5, commentary. [20] His comm. on *Sūryasiddhānta*, xii. 83. [21] See Nīlakaṇṭha's comm. on *Āryabhaṭīya*, ii. 12. [22] His comm. on *Laghubhāskarīya*, ii. 2(c–d)–3(a–b).
[23] *Siddhānta-sārvabhauma*, ii. 3–18. [24] *Karaṇa-kaustubha*, ii. 4–5.
[25] *Siddhāntatattvaviveka*, ii. 244-5 (Lucknow Edition).
[26] *Siddhānta-samrāṭ*, ii. beginning.

> (next) difference of *R*sines and divided by the square of 15; the result is added to the (tabular) *R*sine corresponding to the quotient. Such (is the method) for finding the *R*sine.[165]

Such rules appear also in other astronomical works.[166]

The method of Mañjula (932) is very simple though it yields results only roughly approximate. He says:

> The sum of the signs (in the given arc successively) multiplied by 4, 3 and 1 will give the degrees in the *R*sines and *R*cosines (to be found); such are the minutes.[167]

This rule though it appears to be cryptic has been fully explained by the commentators; Praśastidhara (968), Parameśvara (1430) and Yallaya (1482). We shall explain it with the help of a simple illustrative example: Suppose it is necessary to find the *R*sine of the angle $76°30'$. This angle can be written as 2 signs $16°30'$. Now the rule says that for the first sign take 4 and for the second sign 3. For the third sign of 30° we are to take 1, so for a portion $16°30'$ of that sign we should take $\frac{(16\frac{1}{2}\times 1)}{30}$. The sum of these 4, 3 and $\frac{33}{60}$ will be the degrees in the required *R*sine; and they will also be the minutes of the required *R*sine. Therefore

$$\begin{aligned} jy\bar{a}\ (76°30') &= \left(4+3+\frac{33}{60}\right) \text{ degrees} + \left(4+3+\frac{33}{60}\right) \text{ minutes} \\ &= 7°40'33''. \end{aligned}$$

The rationale of this rule which has also been given by the earlier commentators, is this: Mañjula considers the circle of reference to be of radius $488'$ or in sexagesimal notation $8°8'$; and his Table (**ed.** see Table 3) is very short. To find the *R*sine of any intermediate arc he applies the principle of proportional increase. And this at once leads to the rule.

Arcs of functions

The Hindus employed the principle of proportional increase also for the inverse problem of finding the arc which has a given trigonometrical function different from those tabulated. The *Sūryasiddhānta* says:

> Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine) multiply the remainder by 225 and divide by that difference (i.e.

[165] *Brāhmasphuṭasiddhānta*, ii. 10; compare also *Khaṇḍakhādyaka*, Part I, iii. 6.

[166] *Mahābhāskarīya*, iv. 3–4; *Laghubhāskarīya*, ii. 2–3; *Śiṣyadhīvṛddhida*, ii. 12; *Mahāsiddhānta*, iii. $10\frac{1}{2}$; *Siddhāntaśiromaṇi*, *Graha*, ii. $10\frac{1}{2}$; *Siddhāntatattvaviveka*, ii. 171; *Siddhāntaśekhara*, iii. 15.

[167] चतुस्त्रेकघ्नराश्यैक्यं दोःकोट्योरंशकाः कलाः।—*Laghumānasa*, ii. 2.

Table 3

Arcs	*R*sines		Differences	
0°	0°	0′		
			4°	4′
30°	4°	4′		
			3°	3′
60°	7°	7′[1]		
			1°	1′
90°	8°	8′		

[1] Accurately speaking

$$jy\bar{a}\ 60^\circ = 488 \times \frac{\sqrt{3}}{2} = 7^\circ 2'.608\ldots;$$

Mañjula takes the value to be 7°7′ obviously with the purpose of simplifying his rule.

> the difference corresponding to the interval in which the given *R*sine lies); the quotient added to the number (corresponding to the tabular *R*sine subtracted) multiplied by 225 will give the arc (required).[168]

Brahmagupta writes:

> Subtract the (nearest smaller tabular) *R*sine (from the given *R*sine); the remainder is multiplied by 225 and divided by the (tabular) difference of *R*sines; the result should then be added to the product of the number corresponding to the subtracted *R*sine and the square of 15: (this will be) the arc (required).[169]

Similarly in other works.[170]

Second difference

The process explained above for calculating the trigonometrical functions of a given arc or the arc having given trigonometrical functions of a given arc or the arc having given trigonometrical functions, will yield results correct only to a first degree of approximation in as much as the first difference alone of the

[168] *Sūryasiddhānta*, ii. 33.
[169] *Brāhmasphuṭasiddhānta*, ii. 11; compare also *Khaṇḍakhādyaka*, iii. 12; *Dhyānagrahopadeśa*, 70.
[170] *Mahābhāskarīya*, viii. 6; *Śiṣyadhīvṛddhida*, ii. 13; *Mahāsiddhānta*, iii. 12; *Siddhāntaśekhara*, iii. 16; *Siddhāntaśiromaṇi*, *Graha*, ii. 11f; *Siddhāntatattvaviveka*, ii. 172–3.

Table 4

Values of the argument α	Values of the function $f(\alpha)$	Differences of functions
α_1	f_1	
		Δ_1
α_2	f_2	
		Δ_2
α_3	f_3	

tabular *R*sines has been employed.[171] More accurate results will be obtained by taking into consideration also the second (and higher) differences. The earliest Hindu writer to do so was Brahmagupta. It is perhaps noteworthy that this more correct method of interpolation does not occur in his bigger work, *Brāhmasphuṭasiddhānta*, which was composed in 628 AD, but in his earlier monograph *Dhyānagrahopadeśa* as well as in his later work *Khaṇḍakhādyaka* written in 665 AD. These latter works, as has been stated before, contain a table of differences of *R*sines for every arc of 15° in a circle of radius equal to 150. He says:

> Half the difference between the (tabular) difference passed over and that to be passed is multiplied by (residual) minutes and divided by 900; half the sum of those differences plus or minus that quotient according as it is less or greater than the (tabular) difference to be passed, will be the (corrected) value of the difference to be passed over.[172]

Suppose it is required to calculate the function—*R*sine, *R*cosine or versed *R*sine—of an arc α'. Let α_1, α_2, α_3 be the three consecutive values of the argument in the table (**ed.** see Table 4) such that $\alpha_3 > \alpha_2 > \alpha_1$.

Now if $\alpha_3 > \alpha' > \alpha_2$ for the calculation of $f(\alpha')$, f_2 will be technically called "the function exceeded", f_3 "the function to be reached", Δ_1 "the difference passed over" and Δ_2 "the difference to be passed". Let $\alpha' - \alpha_2 = r$.

Now suppose that $\alpha_3 - \alpha_2 = \alpha_2 - \alpha_1 = h$, say. Then according to the rules

[171] The roughness of the result is due also to other causes. Bhāskara II observes: "As much large the radius of the circle is and into as many large number of (equal) arcs its quadrant is divided, so much accurate will be the *R*sines (calculated). Otherwise (the result) will be rough (*sthūla*)."—*Siddhāntaśiromaṇi*, *Graha*, ii. 15 (gloss).

[172] गतभोग्यखन्डकान्तरदलविकलवधाच्छतैर्नवभिराप्त्या ।
तोद्युतिदलं युतोनं भोग्यादूनाधिकं भोग्यम् ॥
Khaṇḍakhādyaka, Part 2, i, 4; *Dhyānagrahopadeśa*, 17. See Sengupta, P. C., "Brahmagupta on Interpolation", *Bulletin of Calcutta Mathematical Society*, xxii, 1931.

stated by all Hindu astronomers,

$$f(\alpha') = f_2 + \frac{r}{h}\Delta_2;$$

which is correct up to the first order of approximation. To get more accurate results, says Brahmagupta

$$\frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h}\left(\frac{\Delta_1 \sim \Delta_2}{2}\right)$$

should be taken as the value of “the difference to be passed”, instead of Δ_2; the positive or negative sign being taken, according as

$$\frac{\Delta_1 + \Delta_2}{2} < \text{ or } > \Delta_2.$$

Therefore, according to the method of Brahmagupta

$$f(\alpha') = f_2 + \frac{r}{h}\left\{\frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h}\left(\frac{\Delta_1 \sim \Delta_2}{2}\right)\right\}.$$

In the rule h is stated to be 900, as it was formulated with a view at the table of the *Khaṇḍakhādyaka*, in which the interval between the consecutive values of the argument, is 15° or 900′. This equation can be written in the form

$$f(\alpha') = f_2 + \frac{r}{h}\,\Delta_2 + \frac{r}{2h}\left(1 \pm \frac{r}{h}\right)(\Delta_1 \sim \Delta_2);$$

which agrees with the formula method of interpolation, correct up to the second degree.[173]

As has been observed by Bhāskara II[174] in the above formula one has to take the negative sign in calculating the Rsine functions and the positive sign for the versed Rsines. For in case of Rsine functions, the first difference continuously decreases as the argument increases, while contrary is the case with the versed Rsine functions. Therefore the mean value of any two differences

[173] It may be mentioned here that the formula

$$f(\alpha') = f_2 + \frac{r}{h}\Delta_1 - \frac{r}{2h}\left(1 + \frac{r}{h}\right)(\Delta_1 - \Delta_2)$$

has been stated by Vaṭeśvara (904) in his *Siddhānta* (Ch. 2, sec. 1, vss. 65–6) and the formula

$$f(\alpha') = f_2 + \frac{r}{h}\Delta_2 + \frac{r}{2h}\left(1 - \frac{r}{h}\right)(\Delta_1 - \Delta_2)$$

by Govindasvāmi (8th century) in his commentary on *Mahābhāskarīya* (iv. 22) and by Parameśvara (1408) in his commentary on *Laghubhāskarīya* [ii. 2(c–d)–3(a–b)]. Govindasvāmi has, however, prescribed it for the second sign only.

[174] **...ऊनं क्रियते यतः क्रमज्याकरणे खण्डान्यपचयेन वर्तन्ते। उत्क्रमज्याकरणे तूपचयेनातस्तत्र युतमित्युपपन्नम्।**

—*Siddhāntaśiromaṇi*, *Graha*, ii. 16 (gloss).

i.e. $\left(\frac{\Delta_1+\Delta_2}{2}\right)$ is greater than the succeeding one (Δ_2) in case of *R*sine functions and less in case of versed *R*sine functions.

Brahmagupta's method of interpolation appears also in the works of Mañjula (932) thus:

> (Find) half the sum of the tabular difference passed over and that to be passed; half their difference is multiplied by the (remaining) degrees etc. and divided by 30; half the sum minus this quotient will be the corrected value of the difference of (tabular) *R*sines to be passed in the (calculations of the *Laghu*-) *Mānasa*.[175]

The divisor is stated to be 30 in this rule, as Mañjula's table of *R*sines contains values at intervals of 30° each.

Bhāskara II (1150) writes:

> The difference of the (tabular) difference passed over and that to be passed is multiplied by the remaining degrees and divided by 20; half the sum of the (tabular) difference passed over and that to be passed minus or plus that quotient will be the corrected value of the difference to be passed over in calculation here for *R*sines and versed *R*sines.[176]

In formulating this rule Bhāskara II had in view a table calculated at intervals of 10°. The rationale of the rule has been explained by him thus:

> Half the sum of the (tabular) difference passed over and that to be passed will be the difference at the middle of those differences. But the difference to be passed is at the end of that interval to be passed. Hence proportion (should be taken) with their difference: If for an interval of 10°, we obtain half the difference of them, then

[175] गतैष्यखण्डयोगार्धमन्तरार्धेन सङ्गुणात् ।
भागादेः खाग्निलब्धोनं भोग्यज्या मानसे स्फुटाः ॥
There is a bit of uncertainty about the authenticity of this verse. In the Calcutta University Collection, there are three manuscripts of the *Laghumānasa* and four commentaries which contain also the text. The commentary of Praśastidhara (958) is "copied from Ms. No. B 583 and compared with other Mss. in the Oriental Library, Mysore." That by Parameśvara (1430) is "copied from a palm leaf manuscript in Malayalam character belonging to the office of the curator for the publication of Sanskrit Mss., Trivandrum." The source of the commentary of Yallaya (1482) is not mentioned. The above verse appears in the commentaries of Praśastidhara and Yallaya. But in the latter it has been attributed to Mallikārjuna Sūri. Now this writer flourished about 1180 AD. Thus he is posterior to Praśastidhara by more than two centuries. So, it is not possible for the latter to borrow anything from the former. I think the mistake has been made by Yallaya. The verse in question seems to me to be due in fact to Māñjula, and is more particularly from his *Bṛhat-mānasa*, which is now lost. Praśastidhara has quoted copiously from that work in his commentary of the *Laghumānasa* without, however, expressly mentioning it.

[176] *Siddhāntaśiromaṇi*, *Graha*, ii. 16.

> what will be obtained for (an interval of) the remaining degrees? Thus by the rule of three, 20 will be the divisor of the product of the remaining degrees and the difference of the (tabular) difference passed over and that to be passed. By the quotient is then diminished half the sum of the (tabular) difference passed over and that to be passed; for in the calculations of *R*sines the differences are in the decreasing order. But in the calculations of versed *R*sines they are in the increasing order and hence the plus in this case. Thus (the rule) is proved.

This method of interpolation has been severely criticised by Kamalākara.[177] But he is wrong. Munīśvara (1646) attempted to modify this method by iterating the process but his process of iteration is incorrect as he has replaced the (tabular) difference to be passed by the instantaneous difference, at every stage.[178]

The rationale of the rule can be shown with the help of trigonometry to be as follows:

$$\Delta_1 = \sin\alpha_2 - \sin\alpha_1 = \sin\alpha_2 - \sin(\alpha_2 - h) = \sin\alpha_2(1-\cos h) + \cos\alpha_2 \sin h.$$

Since h is small we can expand $\cos h$ and $\sin h$ in powers of h; then neglecting powers higher than the second, we get

$$\Delta_1 = h\cos\alpha_2 + \frac{h^2}{2}\sin\alpha_2.$$

Similarly,

$$\Delta_2 = h\cos\alpha_2 - \frac{h^2}{2}\sin\alpha_2.$$

Therefore,

$$\frac{\Delta_1+\Delta_2}{2} = h\cos\alpha_2 \qquad \text{and} \qquad \frac{\Delta_1-\Delta_2}{2} = \frac{h^2}{2}\sin\alpha_2.$$

Now, if $\alpha' = \alpha_2 + r$, up to the second order of approximation, we have

$$\sin\alpha' = \sin(\alpha_2 + r) = \sin\alpha_2\left(1 - \frac{r^2}{2}\right) + r\cos\alpha_2.$$

Therefore,

$$\sin\alpha' = \sin\alpha_2 + \frac{r}{h}\left(\frac{\Delta_1+\Delta_2}{2} - \frac{r}{h}\times\frac{\Delta_1-\Delta_2}{2}\right).$$

[177] *Siddhāntatattvaviveka*, i, ii. 175–83.

[178] For Munīśvara's process of iteration, see Gupta, R. C., "Munīśvara's modification of Brahmagupta's rule for second order interpolation", *Indian Journal of History of Science*, Vol. 14, No. 1, 1979, pp. 66–72.

Evidently in this case,

$$\frac{\Delta_1 + \Delta_2}{2} > \Delta_2.$$

Hence,

$$\sin \alpha' = \sin \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1 + \Delta_2}{2} - \frac{r}{h} \times \frac{\Delta_1 \sim \Delta_2}{2}\right).$$

In case of versine functions

$$\begin{aligned}
\Delta_1 &= \operatorname{versin} \alpha_2 - \operatorname{versin}(\alpha_2 - h), \\
&= \cos(\alpha_2 - h) - \cos \alpha_2 = h \sin \alpha_2 - \frac{h^2}{2} \cos \alpha_2; \\
\Delta_2 &= \operatorname{versin}(\alpha_2 + h) - \operatorname{versin} \alpha_2, \\
&= \cos \alpha_2 - \cos(\alpha_2 + h) = h \sin \alpha_2 + \frac{h^2}{2} \cos \alpha_2.
\end{aligned}$$

Therefore,

$$\frac{\Delta_1 + \Delta_2}{2} < \Delta_2,$$

and

$$\frac{\Delta_1 + \Delta_2}{2} = h \sin \alpha_2, \qquad \frac{\Delta_1 - \Delta_2}{2} = -\frac{h^2}{2} \cos \alpha_2.$$

Now

$$\operatorname{versin} \alpha' = 1 - \cos \alpha' = 1 - \cos(\alpha_2 + r) = \operatorname{versin} \alpha_2 + r \sin \alpha_2 + \frac{r^2}{2} \cos \alpha_2.$$

Therefore,

$$\operatorname{versin} \alpha' = \operatorname{versin} \alpha_2 + \frac{r}{h}\left(\frac{\Delta_1 + \Delta_2}{2} + \frac{r}{h} \times \frac{\Delta_1 \sim \Delta_2}{2}\right).$$

Combining these two results we get Brahmagupta's formula.

This method of interpolation has been applied also to the inverse problem of finding the arc having a given trigonometrical function.

Brahmagupta says:

> To find the arc, multiply the residue (after subtracting as many *R*sines from the given quantity as possible) by 900 and divide by the difference to be passed after having determined that difference by repeated operations. The degrees in the quotient will be the arc of the residue. Subtract (as many possible) *R*sines (from the given quantity), multiply the residue by 900 and divide by the (next) difference not subtracted; the quotient will be the second residue; multiply it by half the difference between the (tabular) difference passed over and that to be passed and then divide by 900.

> With this quotient proceed as before for the (adjusted) value of the (tabular) difference to be passed. Repeat the same operations with the residue until the result is obtained finally.[179] (By "*R*sines" in this rule is meant "tabular *R*sine-differences.")

The latter portion of this rule has become rather cryptic, as all the successive operations have not been fully described. But it appears from the explanations of the commentator Bhaṭṭotpala (966) that Brahmagupta has intended the same formula as has been clearly described by Bhāskara II. The latter says:

> Subtract the (tabular) differences (as many as possible from the given value); multiply half the remainder by the difference of the (tabular) difference passed over and that to be passed, and then divide by the (tabular) difference to be be passed. Half the sum of the (tabular) difference passed over and that to be passed plus or minus the quotient is the adjusted value of the (tabular) difference to be passed, whence (will follow) the arc (required).[180]

He then remarks as before that the negative sign should be taken in calculating the *R*sines and the positive sign for the versed *R*sines.

The rationale of this will be clear from the previous formula on inversion. There we shall now have $f(\alpha')$ known and r $(= \alpha' - \alpha_2)$ as unknown.

$$f = f_2 + \frac{r}{h}\left(\frac{\Delta_1 + \Delta_2}{2} \pm \frac{r}{h} \times \frac{\Delta_1 - \Delta_2}{2}\right)$$

or

$$\frac{r}{h} \times \Delta_2 = f - f_2 + \frac{r}{h} \times \frac{\Delta_2 - \Delta_1}{2} \mp \frac{r^2}{h^2} \times \frac{\Delta_1 - \Delta_2}{2}.$$

Now let us take for the first approximation, as before

$$\frac{r}{h} = \frac{f - f_2}{\Delta_2}.$$

Substituting this value of $\frac{r}{h}$ in the neglected terms; we get for the second approximation

$$\begin{aligned} r &= (f - f_2) \times \frac{h}{\Delta_2} \times \left(1 + \frac{\Delta_2 - \Delta_1}{2\Delta_2} \mp \frac{f - f_2}{{\Delta_2}^2} \times \frac{\Delta_1 - \Delta_2}{2}\right) \\ &= (f - f_2) \times \frac{h}{\Delta}, \quad \text{say,} \end{aligned}$$

[179] *Khaṇḍakhādyaka*, Part 2, i. 12. The printed text gives only the earlier part of the rule. The remaining portion has been taken from the text of Bhaṭṭotpala.

[180] *Siddhāntaśiromaṇi*, *Graha*, ii. 17.

so that Δ will be adjusted value of the (tabular) difference to be passed. Then

$$\frac{1}{\Delta} = \frac{1}{\Delta_2}\left(1 + \frac{\Delta_2 - \Delta_1}{2\Delta_2} \mp \frac{f - f_2}{{\Delta_2}^2} \times \frac{\Delta_1 - \Delta_2}{2}\right).$$

Therefore,

$$\begin{aligned} \Delta &= \Delta_2\left(1 + \frac{\Delta_2 - \Delta_1}{2\Delta_2} \mp \frac{f - f_2}{{\Delta_2}^2} \times \frac{\Delta_1 - \Delta_2}{2}\right)^{-1} \\ &= \Delta_2\left(1 - \frac{\Delta_2 - \Delta_1}{2\Delta_2} \pm \frac{f - f_2}{{\Delta_2}^2} \times \frac{\Delta_1 - \Delta_2}{2}\right) \\ \text{or} \qquad \Delta &= \frac{\Delta_1 + \Delta_2}{2} \pm \frac{f - f_2}{2\Delta_2} \times (\Delta_1 - \Delta_2), \end{aligned}$$

or as stated in the rule. But the more accurate result by inversion would have been

$$\Delta = \frac{\Delta_1 + \Delta_2}{2} \pm \frac{f - f_2}{\Delta_1 + \Delta_2} \times (\Delta_1 - \Delta_2).$$

Bhāskara II was clearly aware of this. For he is found to have remarked:

> This improved formula for calculating the arc is a little rough (*sthūla*). Though rough, it has been adopted for its simplicity (*sukhārtha*). By other means such as finer calculations or repeated applications it can be made more accurate.[181]

Generalised formula

Brahmagupta has extended his formula of interpolation so as to be applicable also to the case when the intervals between the consecutive tabular values of the argument are not equal. He says:

> Multiply the increase of the *śīghra* anomaly to be passed by the degrees of the increase of the *śīghra* equation passed over and divide by the increase of the *śīghra* anomaly passed over; the quotient is the (adjusted) increase of the *śīghra* equation in degrees. Multiply half the difference of that and the increase of the *śīghra* equation to be passed by the residue of the anomaly and divide by the increase of the *śīghra* anomaly to be passed; half the sum of these equations is decreased or increased by the (last) quotient, according as it (half the sum) is greater or less than the *śīghra* equation

[181] इदं धनुःखण्डस्फुटीकरणं किञ्चित् स्थूलम्। स्थूलमपि सुखार्थमङ्गीकृतम्। अन्यथा बीजकर्मणाऽसकृत्कर्मणा वा स्फुटं कर्तुं युज्यते। —*Ibid* (gloss).

to be passed; the result will be corrected *śīghra* equation to be passed.[182]

That is, if $\alpha_3 - \alpha_2 \neq \alpha_2 - \alpha_1$, let $\alpha_3 - \alpha_2 = h_2$ and $\alpha_2 - \alpha_1 = h_1$. Then

$$f(\alpha') = f_2 + \frac{r}{h_2}\left\{\left(\frac{1}{2} \times \frac{\Delta_1 \times h_2}{h_1} + \Delta_2\right) \pm \frac{r}{h_2} \times \frac{1}{2}\left(\frac{\Delta_1 \times h_2}{h_1} - \Delta_2\right)\right\},$$

or

$$f(\alpha') = f_2 + \frac{r}{h_2} \times \Delta_2 + \frac{r}{2h_2}\left(1 \pm \frac{r}{h_2}\right)\left(\frac{\Delta_1 \times h_2}{h_1} - \Delta_2\right),$$

where the upper or lower sign is to be taken according as

$$\frac{1}{2}\left(\frac{\Delta_1 \times h_2}{h_1} + \Delta_2\right) < \text{ or } > \Delta_2.$$

7 Spherical trigonometry

Solution of spherical triangles

From the use in the treatment of astronomical problem, we find that the Hindus knew how to solve spherical triangles, of oblique as well as of right-angled varieties. They do not seem to possess a general method of solution in this matter unlike the Greeks who systematically followed the method of Ptolemy (c. 150) based on the well-known theorem of Menelaus (90). Still with the help of the properties of similar plane triangles and of the theorem of the square of the hypotenuse, they arrived at a set of accurate formulae sufficient for the purpose. As has been proved conclusively by Sengupta,[183] Braunmühl[184] was wrong in supposing that in the matter of solution of spherical triangles the Hindus utilised the method of projection contained in the Analemma of Ptolemy.

Right-angled spherical triangle

The Hindus obtained the following formulae for the right-angled spherical triangle, right-angled at C:

(i) $\sin a = \sin c \sin A$,

(ii) $\cos c = \cos a \cos b$,

[182] भुक्तगतिफलांशगुणा भोग्यगतिर्भुक्तगतिहृता लब्धं
भुक्तगतेः फलभागास्तद्भोग्यफलान्तरार्धहतम् ।
विकलं भोग्यगतिहृतं लब्धेनोनाधिकं फलैक्यार्धं
भोग्यफलादधिकोनं तद्भोग्यफलं स्फुटं भवति ॥ —*Khaṇḍakhādyaka*, Part 2, ii. 2–3.

[183] Sengupta, P. C., "Greek and Hindu Methods in Spherical Trigonometry", *Journal of Department Letters, Calcutta University*, Vol. xxi (1931).

[184] Braunmühl, *Geschichte*, pp. 38ff; compare also Heath, *Greek Math.*, II, p. 291.

(iii) $\sin c \cos A = \cos a \sin b$,

(iv) $\sin b = \tan a \cot A$,

(v) $\cos A = \tan b \cot c$.

It should be particularly noted that as the tangent and cotangent functions are not recognised in Hindu trigonometry, the formulae (iv) and (v) are ordinarily written in the forms

$$\text{(iv)}\ \sin b = \frac{\sin a \times \cos A}{\cos a \times \sin A}, \qquad \text{(v)}\ \cos A = \frac{\sin b \times \cos c}{\cos b \times \sin c}.$$

These formulae were obtained thus: (See Figure 13)

Let ABC be a spherical triangle right-angled at C, lying on a sphere whose centre is at O. Produce the sides AC and AB to P and Q respectively, such that the arc AP = arc $AQ = 90°$. Join PQ by an arc of the great circle on the sphere. Produce the arcs CB and PQ so as to meet at the point Z.

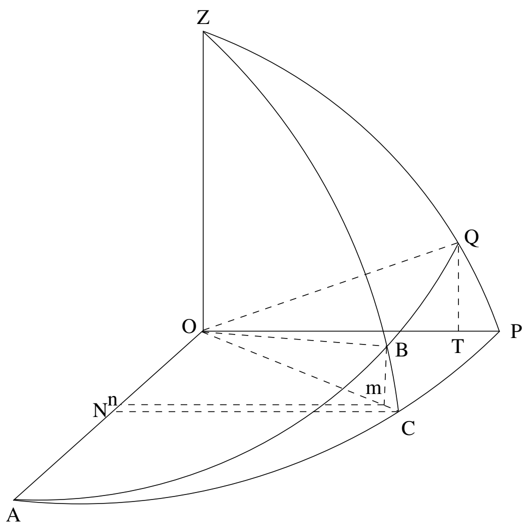


Figure 13

Then clearly Z is the pole of the circle ACP as A is of the circle PQZ. Join A, B, C, P, Q, Z, with O. Draw QT perpendicular to OP, Bm to OC, Bn and CN to OA. Join mn.

From the similar triangles OQT and nBm

$$\frac{nB}{OQ} = \frac{nm}{OT} = \frac{Bm}{QT} \tag{19}$$

and from the similar triangles Onm and ONC

$$\frac{On}{ON} = \frac{Om}{OC} = \frac{nm}{NC}. \tag{20}$$

Substituting the values in terms of trigonometrical functions, we get

$$\frac{jy\bar{a}\ c}{R} = \frac{nm}{kojy\bar{a}\ A} = \frac{jy\bar{a}\ a}{jy\bar{a}\ A}, \tag{21}$$

$$\frac{kojy\bar{a}\ c}{kojy\bar{a}\ b} = \frac{kojy\bar{a}\ a}{R} = \frac{nm}{jy\bar{a}\ b}. \tag{22}$$

Hence from (21), we get

$$jy\bar{a}\ a = \frac{jy\bar{a}\ c \times jy\bar{a}\ A}{R},$$

which is of course equivalent to

$$\sin a = \sin c \sin A. \tag{23}$$

Similarly from (22)

$$kojy\bar{a}\ c = \frac{kojy\bar{a}\ a \times kojy\bar{a}\ b}{R},$$

or

$$\cos\ c = \cos a \cos b. \tag{24}$$

Again equating the values of mn from (21) and (22), we get

$$jy\bar{a}\ c \times kojy\bar{a}\ A = kojy\bar{a}\ a \times jy\bar{a}\ b;$$

that is,

$$\sin c \cos A = \cos a \sin b. \tag{25}$$

Eliminating $\sin c$ between (23) and (25), we get

$$\sin b = \tan a \cot A;$$

and eliminating $\cos a$ between (24) and (25), we have

$$\cos A = \tan b \cot c. \tag{26}$$

As an illustration of the application of the above formulae let us take the problem of determination of the Sun's right ascension (α) when the Sun's longitude (λ) and declination (δ) are known. (**ed.** In Figure 14) Let γM be the equator, γS the ecliptic and S the Sun. Then $\lambda(= \gamma S)$ denotes the Sun's longitude, $\delta(= SM)$ the Sun's declination and $\alpha(= \gamma M)$ the Sun's right

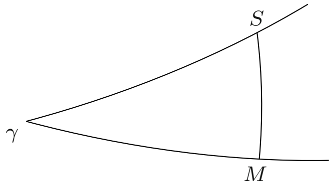


Figure 14

ascension. If ϵ denotes the obliquity of the ecliptic, then by the formula (iii), we get

$$\sin\alpha = \frac{\sin\lambda \times \cos\epsilon}{\cos\delta}.$$

This result is stated by Āryabhaṭa,[185] Brahmagupta[186] and others.[187] It is noteworthy that in this particular case the triangles Bnm and OQT are called technically *krānti-kṣetra* or "declination triangles" which shows definitely that they were actually drawn and the final result was actually obtained by the method stated above.[188]

Oblique spherical triangle

For the solution of an oblique spherical triangle the Hindus had equivalents of the following formulae:[189]

(i) $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

(ii) $\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A$,

(iii) $\dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B} = \dfrac{\sin c}{\sin C}$.

(**ed.** In Figure 15) Let ABC be a spherical triangle lying on a sphere whose centre is at O. Produce the arc AC to S and arc CB to Q, such that the arc CS = arc $CQ = 90°$. Then C will be the pole of the great circle SQN. Join OA, OS and ON. Through B draw the small circle RBR' perpendicular to OA, intersecting the great circle SQN at D and D'. Join DD' intersecting SON at V. Let the diameter RVR' of the small circle cut OA at O'. Again through O draw the straight line WOE parallel to DVD'. Draw the great

[185] *Āryabhaṭīya*, iv. 25.

[186] *Brāhmasphuṭasiddhānta*, iii. 15.

[187] *Sūryasiddhānta*, iii. 41–43; *Pañcasiddhāntikā*, iv. 29; *Siddhāntaśiromaṇi*, *Graha*, ii. 54–5 etc.

[188] Compare Sengupta, P. C., "*Papers on Hindu Mathematics and Astronomy*", Part I, Calcutta, 1916, pp. 46ff; *PS*, iv. 35 (comments).

[189] Braunmühl, *Geschichte*, I, p. 41; Kaye, G. R., "Ancient Hindu Spherical Astronomy", *JASB*, Vol. xv (1919), pp. 153ff; P. C. Sen Gupta, *Papers etc.*, pp. 57ff.

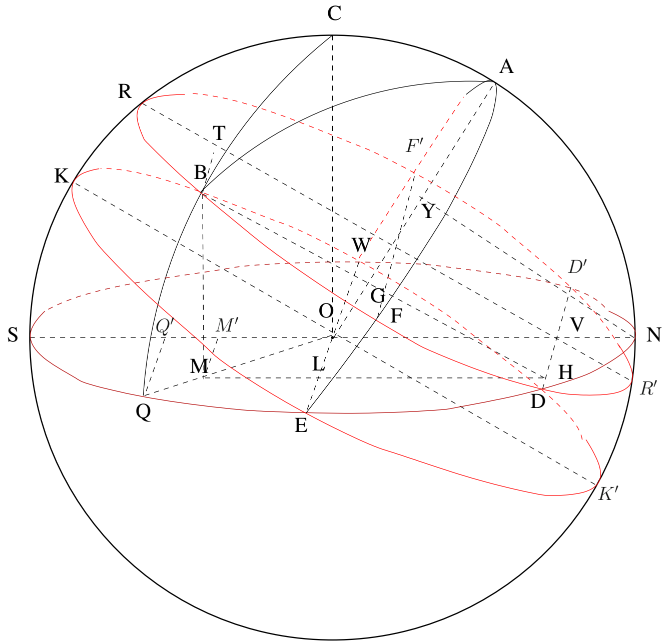


Figure 15

circle $KEK'W$ parallel to the small circle RBR' and the great circle EAW perpendicular to the latter and cutting it at F and F'. From B draw BT perpendicular to RVR', BH to DVD' and BM to OQ. Join MH cutting WOE at L. Draw NY perpendicular to OA, QQ' and MM' to OS. Let BH cut FF' at G.

From the similar triangles BMH and OYN, we get

$$\frac{BM}{OY} = \frac{MH}{NY} = \frac{HB}{NO}; \tag{27}$$

and from the similar triangles OVO' and ONY

$$\frac{OV}{ON} = \frac{VO'}{NY} = \frac{O'O}{YO}. \tag{28}$$

Hence substituting the values

$$\frac{kojy\bar{a}\ a}{kojy\bar{a}\ (90^\circ - b)} = \frac{MH}{jy\bar{a}\ (90^\circ - b)} = \frac{HB}{R},$$

and

$$\frac{OV}{R} = \frac{VO'}{jy\bar{a}\ (90^\circ - b)} = \frac{kojy\bar{a}\ c}{kojy\bar{a}\ (90^\circ - b)};$$

whence

$$HB = \frac{R\ kojy\bar{a}\ a}{jy\bar{a}\ b}, \qquad MH = \frac{kojy\bar{a}\ a \times kojy\bar{a}\ b}{jy\bar{a}\ b}, \tag{29}$$

and

$$OV = \frac{R\ kojy\bar{a}\ c}{jy\bar{a}\ b}, \qquad O'V = \frac{kojy\bar{a}\ b \times kojy\bar{a}\ c}{jy\bar{a}\ b}. \tag{30}$$

Further,

$$\left.\begin{aligned} O'R &= jy\bar{a}\ c, \\ RT &= \frac{jy\bar{a}\ c \times utjy\bar{a}\ A}{R}, \\ ML &= \frac{jy\bar{a}\ a \times kojy\bar{a}\ (C - 90°)}{R}. \end{aligned}\right\} \tag{31}$$

Now

$$HB = VT = RO' + O'V - RT.$$

Therefore substituting the values of the constituent elements on either sides of equations from (29), (30) and (31) we get

$$\begin{aligned} \frac{R\ kojy\bar{a}\ a}{jy\bar{a}\ b} &= jy\bar{a}\ c + \frac{kojy\bar{a}\ b \times kojy\bar{a}\ c}{jy\bar{a}\ b} - \frac{jy\bar{a}\ c \times utjy\bar{a}\ A}{R}, \\ &= \frac{kojy\bar{a}\ b \times kojy\bar{a}\ c}{jy\bar{a}\ b} + \frac{jy\bar{a}\ c \times kojy\bar{a}\ A}{R}; \end{aligned}$$

which is equivalent to

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Again

$$MH = ML + LH = ML + OV.$$

Therefore by (29), (30) and (31)

$$\frac{kojy\bar{a}\ a \times kojy\bar{a}\ b}{jy\bar{a}\ b} = \frac{jy\bar{a}\ a \times jy\bar{a}\ (C - 90°)}{R} + \frac{R\ kojy\bar{a}\ c}{jy\bar{a}\ b},$$

or

$$R^2\ kojy\bar{a}\ c = R\ kojy\bar{a}\ a \times kojy\bar{a}\ b + jy\bar{a}\ a \times jy\bar{a}\ b \times kojy\bar{a}\ C;$$

which is equivalent to

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{32}$$

a formula similar to (i).

From the similar triangles OQQ' and OMM', we have

$$\frac{OQ}{OM} = \frac{QQ'}{MM'} = \frac{Q'O}{M'O}. \tag{33}$$

Therefore

$$OM \times Q'O = OQ \times OM' = OQ(MH - OV)$$
$$= YN \times HB - OV \times R. \qquad \left[\because \frac{MH}{YN} = \frac{HB}{NO}\right]$$

Hence

$$OM \times Q'O = YN(RO' + O'V - RT) - OV \times R;$$

or

$$jy\bar{a}\ a \times jy\bar{a}\ (C - 90^\circ) =$$
$$jy\bar{a}\ (90^\circ - b)\left(jy\bar{a}\ c + \frac{kojy\bar{a}\ b \times kojy\bar{a}\ c}{jy\bar{a}\ b} - \frac{jy\bar{a}\ c \times utjy\bar{a}\ A}{R}\right) - \frac{R^2\ kojy\bar{a}\ c}{jy\bar{a}\ b};$$

or

$$jy\bar{a}\ a \times kojy\bar{a}\ C = \frac{kojy\bar{a}\ b \times jy\bar{a}\ c}{R}(R - utjy\bar{a}\ A) - \frac{kojy\bar{a}\ c}{jy\bar{a}\ b}\left[R^2 - (kojy\bar{a}\ b)^2\right],$$

or

$$jy\bar{a}\ a \times kojy\bar{a}\ C = kojy\bar{a}\ c \times jy\bar{a}\ b - jy\bar{a}\ c \times kojy\bar{a}\ b \times kojy\bar{a}\ A;$$

which is equivalent to

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A.$$

Since $MM' = BT$, we get from (33)

$$\frac{OM}{OQ} = \frac{BT}{QQ'}.$$

Hence

$$\frac{jy\bar{a}\ a}{R} = \frac{jy\bar{a}\ c \times jy\bar{a}\ A}{R\ jy\bar{a}\ C}$$

or

$$\frac{jy\bar{a}\ a}{jy\bar{a}\ A} = \frac{jy\bar{a}\ c}{jy\bar{a}\ C}.$$

Similarly it can be proved that

$$\frac{jy\bar{a}\ c}{jy\bar{a}\ C} = \frac{jy\bar{a}\ b}{jy\bar{a}\ B}.$$

These are of course equivalent to

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

As an illustration of the application of the above formulae we take up the problem of the determination of the relation between the zenith distance (z), azimuth (ψ) and hour angle (H) of a heavenly body of known declination (δ) at a station whose terrestrial latitude is ϕ. In Figure 15 let $NEQS$ denote the horizon, $NACS$ the meridian circle, KEK' the equator, RBR' the diurnal circle of the heavenly body (B), AFE the six o'clock circle, A the north pole and C the zenith. Then $a = z$, $b = 90° - \phi$, $c = 90° - \delta$, $\angle A = H$, $C = \psi$.

Substituting these values in the formulae (i), (32) and (ii), we obtain

$$\cos z = \sin\delta \sin\phi + \cos\delta\cos\phi\cos H,$$
$$\sin\delta = \cos z \sin\phi + \sin z\cos\phi\cos\psi,$$

and

$$\sin z\cos\psi = \sin\delta\cos\phi - \cos\delta\sin\phi\cos H.$$

These equations were obtained by most of the Hindu astronomers.[190] It should however be made clear that the final results were arrived at by successive stages. The straight line DVD', the line of intersection of the diurnal circle with the horizon, is technically called *udayāsta-sūtra* ("the thread through the rising and setting points"), BM is called *śaṅku* ("gnomon"), BH *cheda* or *iṣṭahṛti* ("optional divisor"), MH *śaṅkutala*, BG (= the *jyā* in the diurnal circle of the complement of the hour angle) *kalā*, GH (= the *jyā* of the arc of the diurnal circle intercepted between the horizon and the six o'clock circle) *kujyā* or *kṣitijyā* ("earth-sine"), HL (= *jyā* ED) *agrā*, ML *bāhu*, OM *dṛgjyā* ("*jyā* of the zenith distance"), the angle EAD *cara* ("the ascensional difference"). The existence of these technical terms proves conclusively that the Hindus actually made the constructions contemplated above. They recognise the angle MBH to be equal to the latitude of the observer's station.

The *Sūryasiddhānta* says:

> The *R*sine of the declination multiplied by the *palabhā* (= $12\tan\phi$) and divided by 12 gives the *kujyā* ("the earth-sine"); that multiplied by the radius and divided by the radius of the diurnal circle will give the *jyā* whose arc will be the *cara* ("ascensional difference").[191]

$$\textit{kujyā} = \frac{\textit{jyā}\ \delta \times 12\ \textit{jyā}\ \phi}{12 \times \textit{kojyā}\ \phi},$$
$$\textit{carajyā} = \frac{\textit{kujyā} \times R}{\textit{kojyā}\ \delta}.$$

Again

[190] *Pañcasiddhāntikā*, iv. 42–4; *Sūryasiddhānta*, iii. 28–31, 34–5; *Brāhmasphuṭasiddhānta*, iii. 25–40, 54–6; *Siddhāntaśiromaṇi*, *Graha*, iii. 50–52; etc.

[191] *Sūryasiddhānta*, ii. 61; also *Āryabhaṭīya*, iv. 26; *Śiṣyadhīvṛddhida*, ii. 17; *Brāhmasphuṭasiddhānta*, ii. 57–60; *Pañcasiddhāntikā*, iv. 26–7;*Siddhāntaśiromaṇi*, *Graha*, ii. 48.

> The radius plus the *carajyā* in the northern hemisphere, or minus it in the southern hemisphere is called *antyā*; subtract from it the versed *R*sine of the hour angle; (the remainder) multiplied by the radius of the diurnal circle and divided by the radius will be the *cheda*; that multiplied by the *R*sine of the co-latitude and divided by the radius will be the *śaṅku*; subtract the square of it from the square of the radius; the square-root of the remainder will be the *jyā* of the zenith distance.[192]

$$\begin{aligned} R \pm \textit{carajyā} &= \textit{antyā}, \\ \frac{(\textit{antyā} - \textit{utjyā}\ H) \times \textit{kojyā}\ \delta}{R} &= \textit{cheda}, \\ \frac{\textit{cheda} \times \textit{kojyā}\ \phi}{R} &= \textit{śaṅku}, \\ \text{and} \qquad \sqrt{R^2 - (\textit{śaṅku})^2} &= \textit{jyā}\ z. \end{aligned}$$

Therefore, in the northern hemisphere,

$$\begin{aligned} \textit{kojyā}\ z &= \textit{śaṅku}, \\ &= \frac{\textit{kojyā}\ \delta \times \textit{kojyā}\ \phi}{R^2} \left(R + R \frac{\textit{jyā}\ \delta \times \textit{jyā}\ \phi}{\textit{kojyā}\ \delta \times \textit{kojyā}\ \phi} - \textit{utjyā}\ H \right), \end{aligned}$$

or

$$R^2\ \textit{kojyā}\ z = R\ \textit{jyā}\ \delta \times \textit{jyā}\ \phi + \textit{kojyā}\ \delta \times \textit{kojyā}\ \phi \times \textit{kojyā}\ H;$$

which is of course equivalent to

$$\cos z = \sin\delta \sin\phi + \cos\delta \cos\phi \cos H.$$

Again it has been said that[193]

$$\textit{śaṅkutala} \mp \textit{bāhu} = \textit{agrā},$$

the negative or positive sign being taken according as the heavenly body is in the northern or southern hemisphere. Further[194]

$$\textit{agrā} = \frac{R\ \textit{jyā}\ \delta}{\textit{kojyā}\ \phi}, \qquad \text{and } \textit{bāhu} = -\frac{\textit{jyā}\ z \times \textit{kojyā}\ \psi}{R};$$

Also[195]

$$\textit{śaṅkutala} = \frac{\textit{śaṅku} \times \textit{jyā}\ \phi}{\textit{kojyā}\ \phi}.$$

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[192] *Sūryasiddhānta*, iii. 34–6.

[193] *Ibid*, iii. 23–4.

[194] *Sūryasiddhānta*, iii. 27; *Pañcasiddhāntikā*, iv. 39; *Āryabhaṭīya*, iv. 30; *Brāhmasphuṭasiddhānta*, xxi. 61.

[195] *Āryabhaṭīya*, iv. 28, 29; *Brāhmasphuṭasiddhānta*, iii. 65, xxi. 63.

Hence substituting the values

$$
\begin{aligned}
-\frac{jy\bar{a}\ z \times kojy\bar{a}\ \psi}{R} &= \frac{\acute{s}a\dot{n}ku \times jy\bar{a}\ \phi}{kojy\bar{a}\ \phi} - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi} \\
&= \frac{jy\bar{a}\ \phi \times kojy\bar{a}\ \delta}{R^2}\left\{R + R\frac{jy\bar{a}\ \phi \times jy\bar{a}\ \delta}{kojy\bar{a}\ \phi \times kojy\bar{a}\ \delta} - utjy\bar{a}\ H\right\} \\
&\qquad - \frac{R\ jy\bar{a}\ \delta}{kojy\bar{a}\ \phi} \\
&= \frac{jy\bar{a}\ \phi \times kojy\bar{a}\ \delta \times kojy\bar{a}\ H}{R^2} - \frac{jy\bar{a}\ \delta \times kojy\bar{a}\ \phi}{R},
\end{aligned}
$$

which is of course equivalent to

$$\sin z \cos\psi = \sin\delta\cos\phi - \cos\delta\sin\phi\cos H.$$

7.1 Expansion of trigonometrical functions

Remarkable work on the expansion of trigonometrical functions, $\sin\theta$, $\cos\theta$, $\tan^{-1}\theta$, etc., was done in India by the astronomers of Kerala in the fourteenth, fifteenth and sixteenth centuries AD. It will be discussed in another article which will be devoted to the “Calculus”.

Use of calculus in Hindu mathematics *

1 Differential calculus

1.1 A controversy

Attention was first drawn to the occurrence of the differential formula

$$\partial(\sin\theta) = \cos\theta\,\partial\theta$$

in Bhāskara II's (1150) *Siddhāntaśiromaṇi* by Pandit Bapu Deva Sastri[1] in 1858. The Pandit published a summarised translation of the passages which involve the use of the above formula. His summary was defective in so far as it did not bring into prominence the idea of the infinitesimal increment which underlies Bhāskara's analysis. Without making clear to his readers, the full significance of Bhāskara's result, the Pandit made the mistake of asserting—what was plain to him—that Bhāskara was fully acquainted with the principles of the differential calculus.

The Pandit was adversely criticised by Spotiswoode,[2] who without consulting the original on which the Pandit based his conclusions, remarked (1) that Bapu Deva Sastri had overstated his case in saying that Bhāskarācārya was fully acquainted with the principles of the differential calculus, (2) that there was no allusion to the most essential feature of the differential calculus, viz. the infinitesimal magnitudes of the intervals of time and space therein employed, and (3) that the approximative character of the result was not realised.

Since the above controversy took place no serious investigation of the subject seems to have been made by any scholar.[3] In order that the reader may be better able to judge the merit of the Hindu claim to the invention of the differential calculus, it is desirable that the problems which required the use of the above differential formula be stated first.

* Bhibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 19, No. 2 (1984), pp. 95–104.

[1] *JASB* (= *Journal of the Asiatic of Bengal*), Vol. 27, 1858, pp. 213–6.

[2] *JARS*, Vol. 17, 1860, pp. 21–2.

[3] Except for a paper by P. C. Sen Gupta in the *Journal of the Department of Letters, Calcutta University*, Vol. XXII (1931). Recently A. K. Bag has included this topic in his book "Mathematics in Ancient and Medieval India", Chaukhambha Orientalia, Varanasi, 1979.

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Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_17

1.2 Problems in astronomy

The calculation of eclipses is one of the most important problems of astronomy. In ancient days this problem was probably more important than it is now, because the exact time and duration of the eclipses could not be foretold on account of lack of the necessary mathematical equipment on the part of the astronomer. In India, the Hindus observed fast and performed various other religious rites on the occasion of eclipses. Thus their calculation was a matter of national importance. It afforded the Hindu astronomer a means of demonstrating the accuracy of his science and his own ability to the public who patronised him. The problem of the calculation of conjunction of planets and occultation of stars was equally important both from scientific as well as religious view points.

In problems of the above nature it is essential to determine the true instantaneous motion of a planet or star at any particular instant. This instantaneous motion was called by the Hindu astronomers *tātkālika-gati.* The formula giving the *tātkālika-gati* (instantaneous motion) is given by Āryabhaṭa and Brahmagupta in the following form:

$$u' - u = v' - v \pm e(\sin w' - \sin w) \tag{1}$$

where u, v, w are the true longitude, mean longitude, mean anomaly respectively at any particular time and u', v', w' the values of the respective quantities at a subsequent instant; and e is the eccentricity or the sine of the greatest equation of the orbit. The *tātkālika-gati* is the difference $u' - u$ between the true longitudes at the two positions under consideration. Āryabhaṭa and Brahmagupta used the sine table to find the value of $(\sin w' - \sin w)$. The sine table used by them was tabulated at intervals of $3°45'$ and thus was entirely unsuited for the purpose. To get the values of sines of angles not occurring in the table, recourse was taken to interpolation formulae, which were incorrect because the law of variation of the difference was not known.

1.3 A differential formula

Mañjula (932) was the first Hindu astronomer to state that the difference of the sines,

$$\sin w' - \sin w = (w' - w)\cos w,$$

where $(w' - w)$ is small.

He says:

> True motion in minutes[4] is equal to the cosine (of the mean anomaly) multiplied by the difference (of the mean anomalies) and

[4]This clearly shows that the formula is intended for use when difference is small, the result being expressible in minutes.

> divided by the *cheda*,[5] added or subtracted contrarily (to the mean motion).[6]

Thus according to Mañjula formula (1) becomes

$$u' - u = v' - v \pm e(w' - w)\cos w, \tag{2}$$

which, in the language of the differential calculus, may be written as

$$\partial u = \partial v \pm e \cos\theta \, \partial\theta.$$

We cannot say exactly what was the method employed by Mañjula to obtain formula (2). The formula occurs also in the works of Āryabhaṭa II (950),[7] Bhāskara II (1150),[8] and later writers. Bhāskara II indicates the method of obtaining the differential of sine θ, His method is probably the same as that employed by his predecessors.

1.4 Proof of the differential formula

Let a point P (See Figure 1) move on a circle. Let its position at two successive intervals be denoted by P and Q. Now, if P and Q are taken very near each other, the direction of motion in the interval PQ is the same as that of the tangent at P. Let PT be measured along the tangent at P equal to the arc PQ. Then PT would be the motion of the point P if its velocity at P had not changed direction.

Discussing the motion of planets, Bhāskarācārya says:

> The difference between the longitudes of a planet found at any time on a certain day and at the same time on the following day is called its (*sphuṭa*)*gati* (true rate of motion) for that interval of time.
>
> This is indeed rough motion (*sthūlagati*). I now describe the fine (*sūkṣma*) instantaneous (*tātkālika*) motion.[9] The *tātkālika-gati* (instantaneous motion) of a planet is the motion which it would have, had its velocity during any given interval of time remained uniform.

During the course of the above statement, Bhāskara II observes that the *tātkālika-gati* is *sūkṣma* ("fine" as opposed to rough), and for that the interval

[5] Here *cheda* (divisor) $= \frac{1}{e}$. According to Hindu astronomers $\frac{1}{e} = \frac{360}{P}$, where P is the periphery of the epicycle.

[6] *Laghumānasa*, ii. 7.

[7] *MSi* (=*Mahā-siddhānta*), iii. 15f.

[8] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36–7.

[9] *SiŚi* (=*Siddhāntaśiromaṇi*), *Gaṇitādhyāya*, *Spaṣṭādhikāra*, 36 (c–d).

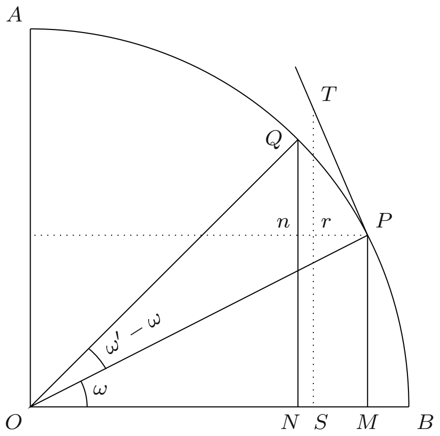

Figure 1

must be taken to be very small, so that the motion would be very small. This small interval of time has been said to be equivalent to a *kṣaṇa*[10] which according to the Hindus is an infinitesimal interval of time (immeasurably small).[11] It will be apparent from the above that Bhāskara did really employ the notion of the infinitesimal in his definition of *tātkālika-gati*.

But in actual practice, the intervals that are considered are not infinitesimal. How are we, then, to apply the notion of *tātkālika-gati* to actual problems? The answer to the above question is given by Bhāskara II as follows:

In equation (1) we have to consider the sine-difference $(\sin w' - \sin w)$. Let an arc of 90° be divided into n parts each equal to A, and let us consider the sine differences $R(\sin A - \sin O)$, $R(\sin 2A - \sin A)$, $R(\sin 3A - \sin 2A)$, etc. These differences are termed *bhogya-khaṇḍa*. Bhāskara II says:

> These are not equal to each other but gradually decrease, and consequently while the increase of the arc is uniform, the increment of the sine varies—on account of deflection of the arc.

In the figure given above, let the arc $PQ = A$. Then

$$R(\sin \angle BOQ - \sin \angle BOP) = QN - PM = Qn$$

which is the *bhogya-khaṇḍa*. Bhāskara introduces the notion of *tātkālika bhogya-khaṇḍa* (instantaneous sine difference) in order to find the variation

[10] The smallest unit of time, according to Bhāskara II is a *truṭi* (*SiŚi*, *Gaṇita*, *Madhyamādhikāra*, *Kālamānādhyāya*, 6), which is equivalent to $\frac{1}{33750}$ of a second. The *kṣaṇa* is smaller, in fact the smallest interval of time that can be imagined.

[11] These remarks are made with reference to the motion of the moon. As the motion of the moon is comparatively quicker, so the *tātkālika-gati* will not give correct result unless the time interval is taken small enough.

of the sine at P. According to him if the arc BP instead of being deflected towards Q, be increased in the direction of the tangent, so that $PT = PQ = A$, then $TS - PM = Tr$ is the *tātkālika bhogya-khaṇḍa* of the sine PT, i.e. the "instantaneous sine difference". By having recourse to this artifice Bhāskara II avoids the use of the infinitesimal in his analysis. It should be borne in mind that the "instantaneous sine difference" for a finite arc PQ, is a purely artificial quantity created with a special end in view, and is different from the actual "sine difference" $R(\sin BOQ - \sin BOP)$.

Now from the similar triangles PTr and PMO, we at once derive the proportion[12]

$$R : PT :: R \cos w : Tr. \tag{3}$$

Therefore $Tr = PT \cos w$. But

$$Tr = R(\sin w' - \sin w) \quad \text{and} \quad PT = R(w' - w).$$

Therefore

$$(\sin w' - \sin w) = (w' - w) \cos\ w.$$

Thus the *tātkālika bhogya-khaṇḍa* (the instantaneous sine difference) in modern notation is

$$\partial(\sin\ \theta) = \cos\theta\, \partial\theta.$$

This formula has been used by Bhāskara to calculate the *ayana-valana* ("angle of position").[13]

If the above were the only result occurring in Bhāskara II's work, one would be justified in not accepting the conclusions of Pandit Bapu Deva Sastri. There is however other evidence in Bhāskara II's work to show that he did actually know the principles of the differential calculus. This evidence consists partly in the occurrence of the two most important results of the differential calculus:

(i) He has shown that when a variable attains the maximum value its differential vanishes.

(ii) He shows that when a planet is either in apogee or in perigee the equation of the centre vanishes. Hence he concludes that for some intermediate position the increment of the equation of centre (i.e. the differential) also vanishes.[14]

[12] It should be noted that for the purpose of the following proof, it is immaterial, whether we take PQ small or not, because it is PT that we are considering and not PQ. Bhāskara actually takes the $\angle POQ = \left(3\frac{3}{4}\right)^\circ = 225'$ for exhibiting equation (3). The notion of the infinitesimal is here involved in the definition of *tātkālika bhogya-khaṇḍa*.

[13] *SiŚi*, *Golādhyāya*, *Grahaṇa*, *Grahaṇa-vāsanā*; see also Sen Gupta, *l.c.*, p. 11 ff.

[14] These results occur in the *Golādhyāya*, *Spaṣṭādhikāra vāsanā* of the *Siddhāntaśiromaṇi*, and were first noted by Sudhakara Dvivedi.

The second of the above results is the celebrated Rolle's Theorem, the mean value theorem of the differential calculus.

1.5 Remarks

The use of a formula involving differentials in the works of ancient Hindu mathematicians has been established beyond the possibility of any doubt. That the notion of instantaneous variation of motion entered into the Hindu idea of differentials as found in works of Mañjula, Āryabhaṭa II, and Bhāskara II is apparent from the epithet *tātkālika* (instantaneous) *gati* (motion) to denote these differentials. The main contribution of Bhāskara II to the theory of these differentials, which were already worked out by his predecessors, seems to be his proof of the formula by the rule of proportion without actually using the infinitesimal or varying quantities. He has, however, made it quite clear that the differentials give true results only when very small variations are concerned.

1.6 Nīlakaṇṭha's result

Nīlakaṇṭha (c. 1500) in his commentary on the *Āryabhaṭīya* has given proofs, on the theory of proportion (similar triangles) of the following results:

(i) The sine-difference $\sin(\theta+\partial\theta)-\sin\theta$ varies as the cosine and decreases as θ increases.

(ii) The cosine-difference $\cos(\theta+\partial\theta)-\cos\theta$ varies as the sine negatively and numerically increases as θ increases.

He has obtained the following formulae:

(i) $\sin(\theta+\partial\theta)-\sin\theta = 2\sin\frac{\partial\theta}{2}\ \cos\left(\theta+\frac{\partial\theta}{2}\right)$

(ii) $\cos(\theta+\partial\theta)-\cos\theta = -2\sin\frac{\partial\theta}{2}\sin\left(\theta+\frac{\partial\theta}{2}\right)$.

The above results are true for all values of $\partial\theta$ whether big or small. There is nothing new in the above results. They are simply expressions as products of sine and cosine differences.

But what is important in Nīlakaṇṭha's work is his study of the second differences. These are studied geometrically by the help of the property of the circle and of similar triangles. Denoting by $\Delta_2(\sin\theta)$, and $\Delta_2(cos\theta)$, the second differences of these functions, Nīlakaṇṭha's results may be stated as follows:

(i) The difference of the sine-difference varies as the sine negatively and increases (numerically) with the angle.

(ii) The difference of the cosine-difference varies as the cosine negatively and decreases (numerically) with the angle.

For $\Delta_2(\sin\theta)$, Nīlakaṇṭha[15] has obtained the following formula:

$$\Delta_2(\sin\theta) = -\sin\theta\left(2\sin\frac{\Delta\theta}{2}\right)^2.$$

Besides the above, Nīlakaṇṭha, has made use of a result involving the differential of an inverse sine function.[16] This result, expressed in modern notation, is

$$\partial(\sin^{-1} e\sin w) = \frac{e\cos w}{\sqrt{1-e^2\sin^2 w}}\partial w.$$

In the writings of Acyuta (1550–1621 AD) we find use of the differential of a quotient[17] also

$$\partial\left[\frac{e\sin w}{1\pm\cos w}\right] = \frac{e\cos w \pm [(e\sin w)^2/(1\pm e\cos w)]}{1\pm e\cos w}\partial w.$$

2 Method of infinitesimal-integration

2.1 Surface of the sphere

For calculating the area of the surface of a sphere Bhāskara II (1150) describes two methods which are almost the same as we usually employ now for the same purpose.

First method

> Make a spherical ball of clay or of wood. On it take a (vertical) circumference circle and divide this into 21600 parts. Mark a point on the top of it. With that point as the centre and with the radius equal to the 96th part of the circumference, i.e. to 225′, describe a circle. Again with same point as the centre with twice that arc as radius describe another circle; with thrice that another circle; and so on up to 24 times. Thus there will be 24 circles in all. The radii of these circles will be the *jyā* 225′ (= $R\sin 225'$), etc. From them the lengths of the circles can be determined by proportion. Now the length of the extreme circle is 21600′ and its radius is 3438′. Multiplying the Rsines (of 225′, 450′, etc.) by 21600 and dividing by 3438, we shall obtain the lengths of the circles. Between two

[15] This together with the results given above are proved by Nīlakaṇṭha in the commentary on the *Āryabhaṭīya*, ii. 12.

[16] *Cf. Tantrasaṅgraha*, ii. 53–4.

[17] *Cf. Sphuṭa-nirṇaya-tantra*, iii, 19–20; *Karaṇottama*, ii. 7.

> of these circles there lie annular strips and there are altogether 24 such. They will be many more in case of many *R*sines being taken into consideration (*bāhujyā-pakṣe-bahūni syuḥ*). In each annulus considering the larger circle at the lower end as the base and the smaller circle at the top as the face and 225′ as the altitude (of the trapezium), find its area by means of the rule 'half the sum of the base and the face multiplied by the altitude etc.'[18] Similarly the areas of all the annular figures severally can be found. The sum of all these areas is equal to the area of the surface of half of the sphere. So twice that is the area of the surface of the whole sphere. And that is equal to the product of the diameter and the circumference.[19]

In other words, if T_n denotes the nth *jyā* (or *R*sine), C_n the circumference of the corresponding circle, A_n the area of the nth annulus, and S the area of the surface of the sphere, then we shall have

$$\begin{aligned} C_n &= \frac{21600}{3438} \times T_n \\ A_n &= \frac{C_n + C_{n-1}}{2} \times 225 \\ &= \frac{225 \times 21600}{2 \times 3438}(T_n + T_{n+1}). \end{aligned}$$

Therefore,

$$\frac{1}{2}S = \sum A_n = \frac{225 \times 21600}{2 \times 3438} \sum (T_n + T_{n+1})$$

the summation being taken so as to include all the *R*sines in a quadrant of the circle. Since there are ordinarily 24 *R*sines in a Hindu trigonometrical table, we have

$$\begin{aligned} \frac{1}{2}S &= \frac{225 \times 21600}{3438} \sum (T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}) \\ &= \frac{21600 \times 225 \times 52513}{3438} \\ &= 21600 \times 3436.7 \ldots \end{aligned}$$

Hence approximately

$$S = 21600 \times 2 \times 3437.$$

Bhāskara II states:

$$\text{Area of the surface} = \text{circumference} \times \text{diameter}.$$

[18] The rule quoted here for finding the area of a trapezium is that given by Śrīdhara (*Triś*, R. 42). Bhāskara II's rule is defined slightly differently (*vide L*, p. 44).

[19] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 55–7 (gloss).

Second method

> Suppose the (horizontal) circumference-circle on the surface of the sphere to be divided into parts as many as four times the number of *R*sines (in a quadrant). As the surface of an emblic myrobalan is seen divided into *vapras* (i.e. lunes) by lines passing through its face (or top) and bottom, so the surface of the sphere should be divided into lunes by vertical circles as many as the parts of the above mentioned (horizontal) circumference-circle. Then the area of each lune should be determined by (breaking it up into) parts. And this area of a lune is equal to the sum of all the *R*sines diminished by half the radius and divided by the semi-radius. Since that is again equal to the diameter of the sphere, so it has been said that the area of the surface of a sphere is equal to the product of its circumference and diameter.[20]

The method has been further elucidated by him in his gloss thus:

> As many as are *R*sines in the table of any particular work selected, take four times that number, and suppose the (horizontal) circumference-circle on the sphere is to be divided into, as many parts. Like the natural lines seen on the surface of a round emblic myrobalan passing through its face and base and thus dividing it into lunes, draw circles on the surface of the given sphere, passing through its top and bottom and thereby dividing it into lunes as many as the number of parts into which the (horizontal) circumference-circle is divided. Next the area of each lune has to be determined. It can be done thus: For instance in the *Dhīvrddhida*,[21] there are 24 *R*sines. So suppose the (horizontal) circumference-circle measures 96 cubits. On drawing the vertical circles through every cubit, there will be as many lunes. Then the upper half of any one lune on drawing the transverse arcs at distances of every cubit, will be divided into portions equal to the number of *R*sines, that is, 24. The lengths of these transverse lines will be obtained by dividing the *R*sines severally by the radius. Of these the lowest line measures one cubit; but the upper and upper ones are a little smaller and smaller according to the *R*sines. But the altitude is all along one cubit in length. Now by finding the area of each portion in accordance with the rule, "half the sum of the top and the base multiplied by the altitude etc." they should be added together. This sum gives the area of half a lune; twice

[20] *Ibid*, verses 58–61.
[21] That is *Śiṣyadhīvṛddhida* of Lalla.

> that is the area of a lune. For the determination of that the rule is, "the sum of all the Rsines minus half the radius etc." Now the sum of all the Rsines, 225 etc., is 54233.[22] This diminished by the semi-radius becomes 52514. Dividing the result by the semi-radius we get the area of each lune as 30;33. Now 30;33 is equal to the diameter of a circle whose circumference measures 96. And as the number of lunes is equal to the number of portions of the circumference it is consequently proved that the area of the surface of a sphere is equal to the product of its circumference and diameter.

If l_n denotes the length of the nth transverse arc, we have

$$l_n = \frac{T_n \times 1}{R}.$$

Therefore,

$$\begin{aligned} \text{area of a lune} &= 2 \times \sum \frac{1}{2}(l_n + l_{n+1}) \times 1 \\ &= 2 \sum \frac{1}{2R}(T_n + T_{n+1}) \end{aligned}$$

the summation being taken so as to include all the Rsines. Hence

$$\begin{aligned} \text{area of a lune} &= 2 \times \frac{1}{R}\left(T_1 + T_2 + \cdots + T_{23} + \frac{1}{2}T_{24}\right) \\ &= \frac{1}{R/2}\left(T_1 + T_2 + \cdots + T_{24} - \frac{R}{2}\right) \\ &= 30; 32, 94 \ldots \end{aligned}$$

Now

$$96 \times \frac{1250}{3927} = 30; 33, 46 \ldots$$

Hence the area of a lune is numerically equal to the diameter of the sphere. As the number of lunes is equal to the number of parts of the circumference of the sphere, we get

$$\text{Area of the surface} = \text{circumference} \times \text{diameter}.$$

2.2 Volume of the sphere

To find the volume of a sphere Bhāskara II states the following method:

> Consider on the surface of the sphere pyramidal excavations, each of a base of a unit area having unit sides and of a depth equal to

[22] According to Lalla the sum of the Rsines is 54233.

> the radius, as many as the number of units of area in the surface. The apices of these pyramids meet at the centre of the sphere. The sum of the volumes of the pyramids is equal to the volume of the sphere. So it is proved (that the volume of a sphere is equal to the sixth part of the product of the surface area and diameter).[23]

The above results are the nearest approach to the method of the integral calculus in Hindu Mathematics. It will be observed that the modern idea of the "limit of a sum" is not present. This idea, however, is of comparatively recent origin so that credit must be given to Bhāskara II for having used the same method as that of the integral calculus, although in a crude form.

[23] *SiŚi*, *Gola*, *Bhuvanakośa*, verses 58–61, (gloss).

Use of permutations and combinations in India *

Interest of the Hindus in the subject of permutations and combinations originated in connection with the variation of the Vedic metres in a very early age. There are specific rules for the calculation of the variation of metres in the *Chandaḥ-sūtra* of Piṅgala (before 200 BC). Permutations and combinations seem to have been subjects of such a fascinating study for the Hindus that they applied the ideas about them in various other spheres of life, e. g. architecture, music, medicine, and astrology. Application of the principles of permutations and combinations is also found in the canonical literature of the Jainas in the study of philosophical categories. The present article aims at giving an account of the various uses of permutations and combinations in Indian literature.

1 Early interest in the subject

The Hindu interest in the subject of permutations and combinations began in a very early age, first probably in connection with the variation of the Vedic metres and philosophical categories.[1] In the *Chandaḥ-sūtra* ("Rules of the Metre") of Piṅgala, a work on Vedic metres, written before 200 BC, we find specific rules for computation of the possible number of variations of even, semi-even, and uneven metres in a group with a specified number of long and short syllables in a quarter of a verse. In the *Nāṭya-śāstra* of Bharata Muni[2] are stated the number of variations of even metres having six to 26 syllables in a quarter of a verse. In the classical treatise on Hindu medicine by Suśruta, called *Suśruta-saṃhitā*,[3] written about 600 BC, the total number of combinations that can be made out of six savours taking one, two, three, ..., five, and all at a time is found to have been correctly stated as 63. The early canonical literature of the Jaina's (500–300 BC)[4] abounds in

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 27, No. 3 (1992), pp. 231–249.

[1] See the article of Gurugovinda Chakravarti on the "Growth and development of permutations and combinations in India" in *BCMS*, XXIV (1932).

[2] See Ch. xiv, vv. 55–81.

[3] See Ch. lxiii.

[4] For instance see *Jambūdvīpa-prajñapti* xx. 4, 5; *Bhagavatī-sūtra, Sūtras* 314, 341, 371–4, etc.; *Anuyogadvāra-sūtra, Sūtras* 76, 92, 126. Compare Bibhutibhusan Datta, "The Jaina

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_18

instances of speculation about the different sub-categories that can arise out of a fixed number of fundamental philosophical categories by the combinations of one, two, or more of them at a time. There are also similar calculations of the groups that can be formed out of the different instruments of senses, of the selections that can be made out of a number of males, females, and eunuchs or of permutations and combinations of various other things. The principles of the subject seem to have appealed to the Hindu mind and are found to have been applied in various spheres, such as astrology, perfumery, architecture, and music, besides those mentioned above. Thus, Bhāskara II (1150) observes:

> It serves in prosody, for those versed therein, to find the variations of metres; in architecture to compute the changes in apertures, etc. (of a building); (in music), the scheme of musical permutations; and in medicine, the combination of different savours.[5]

2 Terminology

The oldest Hindu names for the subject of permutations and combinations are *vikalpa* (lit. "alternatives", "variations") and *bhaṅga* (lit. "poses"). Both these terms occur in the early canonical works of the Jainas (500–300 BC). The term *vikalpa* can be traced still earlier in the *Suśruta-saṃhitā* (c. 600 BC). Brahmagupta (628) calls it *chandaściti* ("piling of metres"),[6] obviously because it originated, as has been stated above, in connection with the variation of Vedic metres. This name appears in later works also. Mahāvīra (850) calls combinations by the term *yutibheda* ("variations of combinations")[7] and Śrīdhara (c. 750) and Bhāskara II (1150) by the term *bheda* or *vibheda* ("variation") only.[8] Bhāskara II introduces the names *aṅkapāśa* ("concatenation of numbers") and *gaṇita-pāśa*[9] for permutations. Nārāyaṇa (1356) has used the term *aṅkapāśa*[10] to denote the whole subject of permutations and combinations. The Hindu expressions corresponding to the modern "taken one at a time", "taken two at a time", etc., are *ekaka-saṃyoga* (lit. "one-combination"), *dvika-saṃyoga* ("two combination"), etc. These terms occur from the *Suśruta-saṃhitā* onwards. Other terms used in that sense are *eka-vikalpa* ("one variation"), *dvi-vikalpa* ("two variation") etc.

School of Mathematics", *BCMS*, XXI (1929), pp. 133 ff.

[5] See *L* (=*Līlāvatī*) p. 26 f. *Cf.* *GK* (= *Gaṇitakaumudī*), xiii. 2.

[6] See *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xx.

[7] See *GSS* (=*Gaṇitasārasaṅgraha*), Ch. vi.

[8] See *PG* (=*Pāṭīgaṇita*), p. 95.

[9] See *L*, p. 83.

[10] See *GK*, II, p. 286.

3 Suśruta's rules for combinations

Suśruta (c. 600 BC)[11] states that the number of combinations of six savours—sweet, acid, saline, pungent, bitter, and astringent—taken two at a time is 15. He seems to have arrived at it by writing down all the combinations exhaustively. For he observes:

> On making two combinations in successive way, those beginning with sweet are found to be 5 in number; those beginning with acid are 4; those with saline 3; those with pungent 2; bitter and astringent make 1 combination.

He then presents the actual combinations thus: sweet-acid, sweet-saline, sweet-pungent, sweet-bitter, sweet-astringent; acid-saline, acid-pungent, acid-bitter, acid-astringent; saline-pungent, saline-bitter, saline-astringent; pungent-bitter, pungent-astringent; and bitter-astringent. In the same way, Suśruta finds the number of 3-combinations to be 20; 4-combinations 15; 5-combinations 6; and 6-combinations 1. Thus, there are $6 + 15 + 20 + 15 + 6 + 1 = 63$ different combinations in all.

4 Jaina canonical works

In the early canonical works of the Jainas (500–300 BC), we find the results which correspond to

$$^{n}C_1 = n, \quad ^{n}C_2 = \frac{n(n-1)}{1 \times 2}, \quad ^{n}C_3 = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}, \ldots$$

After stating the results in case of $n = 1, 2, 3, 4$, the *Bhagavatī-sūtra* observes:

> And in this way 5, 6, 7, ..., 10, etc. numerable, innumerable, or infinite number of things may be mentioned. Forming one-combinations, two-combinations, three-combinations, and so on, ten-combinations, eleven-combinations, twelve-combinations, etc., as the successive combinations are formed, all of them should be considered.[12]

5 Varāhamihira's rule

To find the number of combinations of n unlike things taken 1, 2, 3, ... at a time successively, Varāhamihira (d. 587) gives the following rule:

[11] In *Suśruta-saṃhitā*, Ch. lxiii.
[12] *Bhagavatī-sūtra, Sūtra* 314.

> They say that the number (of combinations) is obtained by (writing down the natural numbers 1, 2, 3, etc. up to the total number of things, one above the other, and) adding the preceding number to the succeeding one (in succession) and rejecting the last number.[13]

The commentator Bhaṭṭotpala (966) has explained the process clearly by taking 16 different things. We reproduce from him the following scheme for it:

16			
15	120		
14	105	560	
13	91	455	1820
12	78	364	1365
11	66	286	1001
10	55	220	715
9	45	165	495
8	36	120	330
7	28	84	210
6	21	56	126
5	15	35	70
4	10	20	35
3	6	10	15
2	3	4	5
1	1	1	1

(The topmost number in the first column gives ${}^{16}C_1$, that in the second column gives ${}^{16}C_2$, that in the third column ${}^{16}C_3$, and that in the fourth column ${}^{16}C_4$. To get ${}^{16}C_5$ and others, the process of forming the successive columns should be continued further on.)

While dealing with the manufacture of perfumes in his *Bṛhatsaṃhitā*, Varāhamihira says:

> An immense number of perfumes can be made out of 16 ingredients, if every 4 of them are combined at will in one, two, three, and four proportions . . . . The total number of these perfumes will be 174720. Each substance (of a group of four) taken in one proportion being combined with the other three, taken in two, three, and four proportions, gives rise to 6 perfumes; and so it does, when taken in two, three or four proportions. One substance associated with a group of four substances (thus) gives rise to 24

[13] *Bṛhatsaṃhitā*, with the commentary of Bhaṭṭotpala, edited by Sudhākara Dvivedī, in two volumes, Benaras, 1897. lxxvi, 22.

> perfumes; and in the same way the remaining three substances (of that group) (also give rise to 24 perfumes). The total of all these is 96. Now when 16 substances are divided into separate groups of 4 each, there arise 1820 such groups. Since each group of four gives rise to 96 varieties (of perfumes), therefore that number (i.e., 1820) should be multiplied by 96. The number (resulting from this product) is the (total) number of perfumes.[14]

In another place, Varāhamihira states, "there are 31 varieties of *anaphā-yoga* and *sunaphā-yoga* each, and 180 of *durudharā-yoga*."[15] Now it has been defined that an *anaphā-yoga* occurs when one or more of the five planets, Mars, Mercury, Jupiter, Venus, and Saturn, occupy the twelfth house from the Moon; in *sunaphā-yoga*, a similar occurrence takes place in the second house from the Moon; and in the *durudharā-yoga*, the planets occupy both these houses. Hence, we get

$$^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$$
$$^5C_1({}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4) + {}^5C_2({}^3C_1 + {}^3C_2 + {}^3C_3) + {}^5C_3({}^2C_1 + {}^2C_2) + {}^5C_4({}^1C_1) = 75 + 70 + 30 + 5 = 180.$$

Brahmagupta (628) has devoted one full chapter (20th) of his treatise on astronomy, the *Brāhmasphuṭasiddhānta*, to the treatment of variation of metres. But on account of faulty readings, it has not been possible to make proper sense out of it.

6 Śrīdhara's rule

To find the number of combinations of the six savours, taken one, two, three, ..., five, and all at a time, Śrīdhara gives the following rule:

> Writing down the numbers beginning with one and increasing by one up to the (given) numbers of savours, in the inverse order, divide them by the numbers beginning with one and increasing by one in the regular order, and then multiply successively by the preceding (quotient) the succeeding one.[16]

[14] *Bṛhatsaṃhitā*, lxxvi, 13–21. See also lxxvi, 29–30. It will be noted that the total number of perfumes will be 24 × 1820, i.e., 43680, and not 174720, as stated by Varāhamihira. His commentator Bhaṭṭotpala rightly remarks: "This number (i.e. 174720) is obtained by taking all varieties subordinate to each ingredient, and not by taking the main varieties (which must be all different). Considering the main varieties only, the total number of perfumes comes to 43680, because a group of four yields only 24 varieties (of perfumes)."

[15] *Bṛhajjātaka*, edited by Sitaram Jha, with the commentary of Bhaṭṭotpala, Benaras, 1921, Ch. xiii, vs. 4.

[16] *PG*, Rule 72.

Thus, writing the numbers of the savours 1, 2, 3, 4, 5, 6 in the inverse order and dividing them by the same numbers in the regular order, we get

$$\frac{6}{1},\ \frac{5}{2},\ \frac{4}{3},\ \frac{3}{4},\ \frac{2}{5},\ \frac{1}{6}.$$

Performing the successive multiplication by the preceding quotient of the succeeding one, we get

$$\frac{6}{1},\ \frac{6}{1}\times\frac{5}{2},\ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3},\ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4},\ \frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5},$$
$$\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}\times\frac{1}{6}.$$

These are the values of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$, …, ${}^6C_6$ respectively.

7 Mahāvīra's rule

To find the number of combinations of unlike things, Mahāvīra gives the following general rule:

> Set down the numbers beginning with unity and increasing by one, up to the (given) number (of things) in the regular and inverse orders in upper and lower rows respectively. The product of the numbers (in the upper row) taken right-to-left-wise being divided by the product of the (corresponding) numbers (in the lower row) taken in the same way, the quotient gives the result.[17]

That is to say, if there be n things, we shall have the arrangement

$$\begin{array}{l} 1,\ 2,\ 3,\ \ldots,\ n-r,\ n-r+1,\ \ldots,\ n-2,\ n-1,\ n \\ n,\ n-1,\ n-2,\ \ldots,\ r+1,\ r,\ \ldots,\ 3,\ 2,\ 1. \end{array}$$

Then says Mahāvīra

$${}^nC_r = \frac{n\times(n-1)\times(n-2)\times\cdots\times(n-r+1)}{1\times2\times3\times\cdots\times r}.$$

It is perhaps noteworthy that one of the illustrative examples given by both Śrīdhara and Mahāvīra is the same as that given by Suśruta.[18] It appears also in Bhāskara II's *Līlāvatī*,[19] and Nārāyaṇa's *Gaṇitakaumudī*.[20]

[17] *GSS*, vi. 218.
[18] *PG*, Ex. 95, *GSS*, vi. 19.
[19] *L*, p. 27.
[20] *GK*, xiii. Ex. 22.

8 Śrīśaṅkara's rule

Bhaṭṭotpala (966) has quoted the following rule from another writer, probably Bhaṭṭa Śrīśaṅkara, of whom we know very little now:

> Write down (the natural numbers) in the reverse way and below them in the regular way. Multiply the numbers (in the two rows) taken in the regular way and divide the product from the upper row by that from the lower.[21]

So, the scheme in this case is

$$\begin{array}{l} n,\ n-1,\ n-2,\ \ldots,\ n-r+1,\ n-r,\ \ldots,\ 3,\ 2,\ 1 \\ 1,\ 2,\ 3,\ \ldots,\ r,\ r+1,\ \ldots,\ n-2,\ n-1,\ n. \end{array}$$

9 Bhāskara II's rule

Bhāskara II (1150) says:

> Divide the numbers from one upwards, increasing by unity, set down in the inverse order, by the same (arithmeticals) written in the regular order. The first quotient, the second multiplied by the first, the next multiplied by that, and so on, give the combinations by one, two, three, etc. This is the general rule.[22]

An example from Bhāskara II:

> A pleasant, spacious and elegant palace, constructed by a skilful architect for the landlord, has eight apertures in it. Tell me the number of combinations of them formed by taking one, two, three, etc. (at a time).

The total number of combinations

$$\begin{aligned} &= {}^8C_1 + {}^8C_2 + {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8 \\ &= 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1 \\ &= 255. \end{aligned}$$

10 Early rule for permutations

In the early canonical works of the Jainas, we find copious instances of calculation of permutations yielding results corresponding to the modern formulae.

$${}^nP_1 = n, \quad {}^nP_2 = n(n-1), \quad {}^nP_3 = n(n-1)(n-2), \quad \text{etc.}$$

---

21 *Bṛhajjātaka*, xii. 19 (comm).

22 *L*, p. 27.

But the earliest mention of a rule for finding the number of permutations of n things taken all at a time is found in the *Anuyogadvāra-sūtra*, a canonical work written before the beginning of the Christian era. It says:

> What is the direct arrangement? *Dharmāstikāya*, *Adharmāstikāya*, *Ākāśāstikāya*, *Jīvāstikāya*, *Pudgalāstikāya* and *Addhāsamaya*—this is the direct arrangement. What is the reverse arrangement? *Addhāsamaya*, *Pudgalāstikāya*, *Jīvāstikāya*, *Ākāśāstikāya*, *Adharmāstikāya*, and *Dharmāstikāya*—this is the reverse arrangement. What are the mixed arrangements? From the series of numbers beginning with one and increasing by one up to six terms. The mutual products of these minus 2 will give the number of mixed arrangements.[23]

We have similar rules for 7, 10, 16, 24 or any variable number (*asaṃkhyeya*) of unlike things.[24] Thus, it was known that the number of permutations of n unlike things taken all at a time is

$$1 \times 2 \times 3 \times \cdots \times (n-2) \times (n-1) \times n.$$

11 Jinabhadra Gaṇi's rule

Jinabhadra Gaṇi (529–589) says:

> Multiply mutually the numbers beginning with one and increasing by one up to the number of terms (i.e., unlike things); then the product (will give the number of permutations).[25]

A similar rule has been given by the commentator Śīlāṅka (862) from an unknown writer:

> Beginning with unity up to the number of terms, multiply continuously the (natural) numbers. The product should be known as the result (i.e., the total number) in the calculation of permutations (*vikalpagaṇita*).[26]

12 Bhāskara II's rules

To find the number of permutations of n unlike things taken all at a time. Bhāskara II (1150) gives a rule similar to those stated above:

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[23] *Anuyogadvāra-sūtra, Sūtra*, 97.
[24] *Ibid, Sūtras* 103, 114–9.
[25] *Viśeṣāvaśyaka-bhāṣya, Gāthā* 942.
[26] *Vide* Śīlāṅka's comm. on *Sūtrakṛtāṅga-sūtra*, *samayādhyayana*, *anuyogadvāra*, verse 28.

> The product of the numbers beginning with and increasing by unity and continued up to the number of places will be the number of different permutations with all of the specified things.[27]

He then gives a rule for finding the permutations of n unlike things taking any variable number of them at a time.

> The product of the numbers from the total number of places and decreasing by unity, continued up to the last of the (variable) places gives the number of permutations of unlike things.[28]

That is to say, the number of r permutations of n dissimilar things will be

$$n(n-1)(n-2)\ldots \text{up to } r \text{ factors.}$$

Similar rules are given by Nārāyaṇa.[29]

13 Permutations of things not all different

To find the number of ways in which n things may be arranged amongst themselves, taking all at a time, when some of the things are alike, Bhāskara II gives the following rule:

> Find separately the number of permutations for as many places as are occupied by like digits; then divide by that the number of permutations calculated before (on the supposition that all the digits are unlike): the quotient will be the (required) number of permutations.[30]

A similar rule is given by Nārāyaṇa:[31]

> That is to say, if p of the digits are alike of one kind, q of them are alike of a second kind, r of them are alike of a third kind, and the rest all different, then the number of permutations will be
>
> $$\frac{n!}{p!\; q!\; r!},$$
>
> n being the total number of places occupied by the digits (like and unlike).

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[27] *L*, p. 83.
[28] *L*, p. 84.
[29] *GK*, xiii. 45, 91.
[30] *L*, p. 84.
[31] *GK*, xiii. 55(c–d)–56(a–b).

13.1 Examples from Bhāskara II[32]

The different numbers that can be formed out of the digits 2, 2, 1, 1 are in all

$$\frac{4!}{2!\ 2!} = 6.$$

The various numbers that can be formed out of the digits 4, 8, 5, 5, 5 are altogether

$$\frac{5!}{3!} = 20.$$

Nārāyaṇa[33] states that when each of the n things is repeated, the number of r-permutations is n^r. As examples, he finds that with the digits 1 and 2 there can be formed as many as 2^6 or 64 numbers of six notational places each, and with the digits 1, 2, and 3 will be obtained 3^3 or 27 numbers of three notational places each.[34]

14 Sum of permutations

To find the sum of the numbers that can be formed by the permutations of some given digits, taken all at a time, Bhāskara II gives the following rule:

> That (the number of permutations) is divided by the number of digits and multiplied by their sum; the result being repeated according to the notational places (as many times as the number of digits) and added together will give the sum of the permuted numbers.[35]

This rule is equally applicable to both the cases when all the digits are unlike and when some of them are alike.[36]

14.1 Illustrative examples from Bhāskara II[37]

(1) The numbers that can be formed by permutation of the eight digits 2, 3, 4, 5, 6, 7, 8, 9 are altogether

$$= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320.$$

Now we have

$$\frac{40320}{8}(2+3+4+5+6+7+8+9) = 221760;$$

[32] *L*, p. 84.
[33] *GK*, xiii. 62 (a).
[34] *GK*, xiii. Ex. 27.
[35] *L*, p. 83.
[36] *L*, p. 84.
[37] *L*, pp. 83, 84.

also setting down 221760 eight times advanced forward one place each time and then adding together, we get

$$\begin{array}{ccccccccccccc}
2 & 2 & 1 & 7 & 6 & 0 & & & & & & & \\
 & 2 & 2 & 1 & 7 & 6 & 0 & & & & & & \\
 & & 2 & 2 & 1 & 7 & 6 & 0 & & & & & \\
 & & & 2 & 2 & 1 & 7 & 6 & 0 & & & & \\
 & & & & 2 & 2 & 1 & 7 & 6 & 0 & & & \\
 & & & & & 2 & 2 & 1 & 7 & 6 & 0 & & \\
 & & & & & & 2 & 2 & 1 & 7 & 6 & 0 & \\
 & & & & & & & 2 & 2 & 1 & 7 & 6 & 0 \\
\hline
2 & 4 & 6 & 3 & 9 & 9 & 9 & 9 & 7 & 5 & 3 & 6 & 0 \\
\hline
\end{array}$$

Hence, the sum of the numbers obtained by permutation is 2463999975360.

(2) The number that can be formed by the digits 2,2,1,1 has been found to be equal to 6 altogether. Now, we get

$$\frac{6}{4}(2+2+1+1) = 9;$$

and also

$$\begin{array}{cccc}
9 & & & \\
 & 9 & & \\
 & & 9 & \\
 & & & 9 \\
\hline
9 & 9 & 9 & 9 \\
\hline
\end{array}$$

Hence, the required sum is 9999.

The rationale of the rule is as follows.[38]

Case 1. Suppose there are n digits and all of them are unlike

The number of permutations that can be formed with these digits is $n!$. Now consider any of the digits, say a. In $(n-1)!$ of the numbers a will be in the units' place; in as many cases it will be in the tens' place; and so on. The sum arising from a alone, since there are n digits in all,

$$\begin{aligned}
&= (n-1)!(10^{n-1}a + 10^{n-2}a + \ldots + 10a + a) \\
&= \left(\frac{n!}{n}\right)(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)a.
\end{aligned}$$

[38] *Cf.* Haran Chandra Banerjee; *Līlāvatī*, Second edition, Calcutta (1927), pp. 192–195.

Proceeding in the same way with the other digits and adding up the partial sums, we get the sum of all the numbers resulting from permutations of the digits

$$= \left(\frac{n!}{n}\right)(10^{n-1} + 10^{n-2} + \ldots + 10 + 1) \times (\text{sum of the digits}).$$

Case 2. Suppose p of the digits to be alike and equal to k_1, q of them equal to k_2, r of them equal to k_3, and the rest unlike

The number of permutations that can be made with these digits is

$$\frac{n!}{p!\, q!\, r!}.$$

The number of cases in which k_1 is in the units' place is

$$\frac{(n-1)!}{(p-1)!\, q!\, r!}.$$

In as many cases it is in the tens' place; and so on. Hence, the partial sum arising out of k_1 is

$$\frac{(n-1)!}{(p-1)!\, q!\, r!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)k_1.$$

In the same way, the partial sums arising from k_2 and k_3 are respectively

$$\frac{(n-1)!}{(q-1)!\, p!\, r!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)k_2,$$
$$\frac{(n-1)!}{(r-1)!\, p!\, q!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)k_3,$$

and the partial sum due to the unlike digits k_4, k_5, ... is, by Case 1

$$\frac{(n-1)!}{p!\, q!\, r!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)(k_4 + k_5 + \ldots).$$

Hence, the required sum of all the numbers is

$$\frac{n!}{n\, p!\, q!\, r!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1)(pk_1 + qk_2 + rk_3 + k_4 + k_5 + \ldots)$$
$$= \frac{n!}{n\, p!\, q!\, r!}(10^{n-1} + 10^{n-2} + \ldots + 10 + 1) \times (\text{sum of all the digits}).$$

15 Bhāskara II's problem

Bhāskara II proposed an interesting problem: To find how many different numbers occupying a specified number of notational places can be formed out of digits having a definite sum. His solution is as follows:

> When the sum of the digits is fixed, divide the successive numbers beginning with that sum minus one, and decreasing by one, continued up to one less than the number of places, by one, two, etc. respectively. The variations of numbers will be equal to the product of those quotients. This rule is valid, it must be known, only when the sum of the digits is less than the specified number of notational places plus nine.[39]

15.1 Illustrative example from Bhāskara II [40]

The different numbers of 5 digits of sum 13 will be altogether

$$\frac{12}{1} \times \frac{11}{2} \times \frac{10}{3} \times \frac{9}{4} = 495.$$

16 Representation

It has been noted before that the interest of the ancient Hindus in the subject of permutations and combinations was not of theoretical origin, but grew out of a concrete purpose. For that it was essential not only to know the number of possible variations but also, and in a greater degree, to have the actual variations. So, we find that as early as the time of the Jaina canonical works, distinct consideration was being made between *bhaṅga-samutkīrṇatā* ("Telling permutations or combinations", that is, "Enumeration of possible variations") and *bhaṅga-pradarśanatā* ("Representation of permutations and combinations"). In the early state of the subject even the number of variations in any given case very probably used to be determined by writing them all down exhaustively. But the latter was obviously a laborious task and was often liable to be in error if all the operations be not carried out in a systematic way. Such a systematic scheme of operations is technically called the *Loṣṭa-prastāra* ("Spreading out of marked objects"), apparently because in the beginning the permutations or combinations used to be formed out of any given number of things by laying out objects, probably clay pieces, marked with the tachygraphic abbreviations of the names of the various things.

[39] *L*, p. 85.
[40] *L*, p. 85.

17 Representation of combinations

A scheme of writing down all the possible combinations formed out of a given number of unlike things is sufficiently clear from the descriptions of Suśruta. The same appears in the Jaina canonical works.[41] Varāhamihira's rule for that is as follows:

> Any one of the things taken optionally should be successively operated upon (by the rest); when that process is exhausted, the next (should be begun).[42]

The operations implied have been explained at length by Bhaṭṭotpala with the help of specific instances. In this connection he has quoted a rule from Bhaṭṭa Śrīśaṅkara.[43] Jinabhadra Gaṇi (c. 550) also has a rule for the same.[44]

18 Representation of permutations

Śīlāṅka (862)[45] has quoted a rule from an ancient writer who is not known now, describing a systematic scheme of forming all the possible permutations out of a given number of unlike things:

> The total number of permutations should be divided by the last term, then the quotient by the rest. They should be placed successively by the side of the initial term in the calculation of permutations.

The rule appears to be cryptic, but Śīlāṅka has explained it clearly with the help of an illustrative example: To find the numbers that can be formed by using the digits 1, 2, 3, 4, 5, 6. It is as follows:

Let there be n number of things $a_1, a_2, \ldots, a_n$. Then the total number of permutations that can be formed out of them will be $n!$. The number of permutations which can have any particular thing, say a_1, for its initial digit (*ādi*) will be $\frac{n!}{n}$, that is, $(n-1)!$. So, put a_1 in the beginning of $(n-1)!$ grooves and so on. Again amongst the first series of grooves, the number of sub-grooves that can have a_2 after a_1 will be $\frac{(n-1)!}{(n-1)}$ or $(n-2)!$. Place a_2 after a_1 in those sub-grooves. The number of sub-grooves that can have a_3 after a_1 will be $(n-2)!$ and put it after a_1 in those sub-grooves. Similarly, with $a_4, a_5, \ldots, a_n$. Again amongst the sub-grooves that can have any other particular thing in the third place will be $(n-3)!$ and it should be placed in

[41] *Bhagavatī-sūtra*, *Sūtra* 314.
[42] *Bṛhatsaṃhitā*, lxxvi. 22; *Bṛhajjātaka*, xiii. 4.
[43] *Vide* Bhaṭṭotpala's commentary on *Bṛhajjātaka*, xii. 19.
[44] *Viśeṣāvaśyaka-bhāṣya.*
[45] *loc. cit.*; *Cf.* B. Datta, *Jaina Math*, pp. 135 f.

those cases. Proceeding step by step in this way in a systematic manner, we can find out all the different permutations of things.

19 Piṅgala's rules

Piṅgala (before 200 BC) describes a scheme of forming all the permutations with a specified number of things when repetitions are allowed. As he was directly concerned with metres, he dealt with only two varieties of things, long and short syllables, which are represented respectively by the abbreviations g from *guru* ("long") and l from *laghu* ("short"). But the scheme is equally applicable to cases of more varieties. Piṅgala's scheme, described in short aphorisms,[46] will be clear from the following:

(i) Monosyllabic:

$$\begin{array}{ll}1. & g\\ 2. & l\end{array}$$

(ii) Disyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l}g\\ l\end{array}\right\} g\\ \left.\begin{array}{l}g\\ l\end{array}\right\} l\end{array}\right\} = \left\{\begin{array}{ll}1. & gg\\ 2. & lg\\ 3. & gl\\ 4. & ll\end{array}\right.$$

(iii) Trisyllabic:

$$\left.\begin{array}{l}\left.\begin{array}{l}gg\\ lg\\ gl\\ ll\end{array}\right\} g\\ \left.\begin{array}{l}gg\\ lg\\ gl\\ ll\end{array}\right\} l\end{array}\right\} = \left\{\begin{array}{ll}1. & ggg\\ 2. & lgg\\ 3. & glg\\ 4. & llg\\ 5. & ggl\\ 6. & lgl\\ 7. & gll\\ 8. & lll\end{array}\right.$$

and so on. Piṅgala states that the trisyllabics are 8 in number.[47] In general, a group of n syllables will have 2^n forms (*vide infra*).

[46] *Chandaḥ-sūtra* of Piṅgala, edited by Jīvānanda Vidyāsagara, with the commentary of Halāyudha, Calcutta, 1892; viii. 20–2.

[47] *Chandaḥ-sūtra* of Piṅgala, viii. 23.

The above systematic scheme of representation has the advantage that (a) we can easily find out the form of versification corresponding to a given serial number in it and vice versa, (b) we can allocate a given form of versification in its proper place in the scheme. Piṅgala's aphorisms for (a) are, "*l* when halved; *g* when added with one (and then halved)."[48] That is to say: Divide the given number successively by two; if at any step, the number obtained is not divisible by two, add one to it and then halve. Corresponding to each operation of exact division by two, set down *l*; and to that of halving after adding unity write down *g*. The operations are to be continued until the desired number of syllables in the group has been obtained. The operations for (b) are the reverse of these.[49] Taking unity, we shall have to double it successively as many times as there are syllables in the given form; but corresponding to each long syllable we shall have to subtract one from the corresponding product.

Piṅgala next gives a rule for finding the total number of variations without having recourse to writing them all down exhaustively according to the scheme described above. This method has already been described. It is found in later writings also.[50] By this rule, the total number of variations in a group of n syllables is found to be equal to 2^n.

Piṅgala has also an alternative method to find the total number of variations.[51] It is technically called *Meru-prastāra*, because the total is obtained by addition from numbers arranged in such a form as to present a fancied resemblance to the fabulous mountain Meru of the Hindu mythology. Piṅgala's aphorisms being too compressed and cryptic can be understood only with the help of a commentary. Halāyudha (10th century) has explained them as follows:

> Draw one square at the top, below it draw two squares, so that half of each of them lies beyond the former on either side of it. Below them in the same way draw three squares; then below them four; and so on up to as many rows as desired: this is the preliminary representation of the *Meru* (*Meru-prastāra*). Then putting down one in the first square, the marking should be started. In the next two squares write one in each. In the third row, put 1 in each of the two extreme squares and in the middle square, the sum of the two digits in the two squares of the second row. In the fourth row, put 1 in the two extreme squares; in an intermediate square put the sum of the digits in two squares of the previous row, which lie just above it. Putting down numbers in the other rows

[48] *Ibid*, viii. 24–5.
[49] *Ibid*, viii. 26–7.
[50] For instance, Mahāvīra (*GSS*, ii. 94). Pṛthudakasvāmī (*BrSpSi*, xii. 17. comm.), etc.
[51] *Chandaḥ-sūtra*, viii. 28–32.

> should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms: there are two forms, one consisting of a long and the other of a short syllable. The numbers in the third row give the disyllabic forms: in one form all syllables are long; in two forms one syllable is short; and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabics. There one form has all syllables long, three have one short syllable; three have two short syllables and one has all syllables short, and so on. In the fifth and succeeding rows also the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables.[52]

Thus according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows. (**ed.** See Figure 1. Caption added.)

Number of syllables														Total no. of variations
							1							
1						1		1						$2 = 2^1$
2					1		2		1					$4 = 2^2$
3				1		3		3		1				$8 = 2^3$
4			1		4		6		4		1			$16 = 2^4$
5		1		5		10		10		5		1		$32 = 2^5$
6	1		6		15		20		15		6		1	$64 = 2^6$

Figure 1: *Meru Prastāra*

From the above it is clear that Piṅgala knew the results:

1. ${}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} + {}^nC_n + 1 = 2^n$,
2. ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

Sanskrit prosody distinguishes three classes of metres: (1) even, in which the arrangement of syllables in all the quarters (*pādas*) is the same; (2) semi-even, in which the alternate quarters are alike; and (3) uneven, in which the quarters

[52] *Chandaḥ-sūtra*, viii. 33–4.

are all dissimilar. Now with a group of n syllables in a quarter, the total number of varieties of even metres will be, according to Piṅgala[53], 2^n; semi-even, $2^{2n} - 2^n$; and uneven, $2^{4n} - 2^{2n}$. The same formulae are stated also by Bhāskara II:

> The number of syllables in a quarter being taken for the period and the common ratio 2 the result from multiplication and squaring[54] will give the number of even metres. Its square, and square's square, minus their respective roots, will be the numbers of semi-even and uneven metres respectively.[55]

By way of illustration, Halāyudha[56] calculates that in the *Gāyatrī* metre, which has six syllables in a quarter, the number of even variations will be 64, semi-even 4032, and uneven 16773120. Bhāskara II[57]calculates that in the case of the *Anuṣṭubh* metre, which has 8 syllables in a quarter, even variations are 256, semi-even 65280, and uneven 4294901760.

20 Nemicandra's rules

We find in the works of Nemicandra, a Jaina philosophical writer of the tenth century (c. 975), certain interesting rules, some of which are akin to those of Piṅgala. According to the Jaina philosophy, there are 15 kinds of *pramāda* ("carelessness"), of which four belong to the category of *vikathā* ("wrong talk"), four to that of *kaṣāya* ("passion"), five to that of *indriya* ("sense"), and one each to those of *nidrā* ("sleep") and *praṇaya* ("attachment"). Combinations are made of five elements of carelessness, selecting only one element from each of the five categories. Again, they are formed by setting down the elements according to a systematic scheme and are marked serially. Hence, the problems that arise in this connection are, as enumerated by Nemicandra, to find: (i) the total number of combinations that can be made, (ii) a systematic scheme of laying out, (iii) the elements of a combination from its serial number, and (iv) the serial number of a particular combination.[58] Nemicandra has given rules for each.

(i) "All the combinations previously obtained combine with each element of the next category. Hence, the total number will be given by the multiplication (of the numbers of elements in the different categories)."[59]

[53] *Chandaḥ-sūtra*, v. 3–5.
[54] Reference here is to the operations described for finding the sum of a G.P. (*L*, p. 31).
[55] *L*, p. 31.
[56] See his commentary on *Chandaḥ-sūtra*, v. 3–5.
[57] *L*, p. 32.
[58] *Gommaṭasāra*, *Jīvakāṇḍa*, *Gāthā* 35.
[59] *Ibid*, *Gāthā* 36.

Thus, the total number of combinations that can be made out of the 15 elements of carelessness in the way described above is

$$4 \times 4 \times 5 \times 1 \times 1 = 80.$$

(ii) Nemicandra has described two schemes of representation of combinations: one is called (1) the *prastāra* and the other (2) the *parivartana*.

(1) "The distribution will be obtained thus: Write down severally the first element (of the first category) of carelessness and put over it each of the elements of the succeeding classes. When elements in the third category are exhausted, begin afresh with the second element of the second category (and so on). When all the elements of these two categories are thus distributed out, operations should be begun with (the second element of) the first category. (And so on)."[60]

If the four kinds of wrong talks be denoted by v_1, v_2, v_3, v_4; the four kinds of passions by k_1, k_2, k_3, k_4; and the five kinds of senses by i_1, i_2, i_3, i_4, i_5, the representation described here will be this:

$$\underbrace{i_1\ i_2\ i_3\ i_4\ i_5}_{\substack{k_1 \\ v_1}} \quad \underbrace{i_1\ i_2\ i_3\ i_4\ i_5}_{\substack{k_2 \\ v_1}} \quad \underbrace{i_1\ i_2\ i_3\ i_4\ i_5}_{\substack{k_3 \\ v_1}} \quad \underbrace{i_1\ i_2\ i_3\ i_4\ i_5}_{\substack{k_4 \\ v_1}}$$

and so on, v_2, v_3, v_4 coming successively in the place of v_1.

(2) "Write down the elements of the first category as many times as the number of elements in the second category; then put down over each group severally each of the elements of the second category; and proceed thus throughout. When all the elements of the first category are exhausted, begin afresh with the second category (and so on). When all the elements of these two categories are thus distributed out, the operation with the elements in third category begins."[61]

$$\overbrace{v_1\ v_2\ v_3\ v_4}^{\substack{i_1 \\ k_1}} \quad \overbrace{v_1\ v_2\ v_3\ v_4}^{\substack{i_1 \\ k_2}} \quad \overbrace{v_1\ v_2\ v_3\ v_4}^{\substack{i_1 \\ k_3}} \quad \overbrace{v_1\ v_2\ v_3\ v_4}^{\substack{i_1 \\ k_4}}$$

(iii) "Divide (the given serial number) successively by the number of elements in the different categories, adding each time unity to the quotient, except when the remainder is zero. The remainders determine the place of an element in its category; the zero remainder indicates the last element."[62]

[60] *Gommaṭasāra*, *Jīvakāṇḍa*, *Gāthās* 37, 39.
[61] *Ibid*, *Gāthās* 38, 40.
[62] *Ibid*, *Gāthā* 41.

For example, let us find the elements of the 13th combination.

First scheme: Dividing 13 by 5, we get 2 for the quotient and 3 as the remainder. So, the combination contains the element i_3. Adding 1 to the quotient 2, we have 3. Dividing 3 by 4, we get the quotient 0 and the remainder 3. Hence, there is the element k_3, $0+1=1$. On dividing 1 by 4, the remainder is 1. So, there is v_1. So, the 13th combination according to the first scheme contains i_3, k_3, and v_1, besides sleep and attachment.

Second scheme: On dividing 13 by 4, the quotient is 3 and the remainder 1. So, the combination contains v_1. Adding unity to the quotient 3, we get 4. On dividing 4 by 4, the quotient is 1 and the remainder 0. Hence, there is k_4.[63] Dividing 1 by 5, we get the remainder 1. So, there is i_1. Hence, the 13th combination according to the second scheme has v_1, k_4, i_1, besides sleep and attachment.

(iv) "Take unity. Multiply it by the total number of elements in a category beginning from the last and subtract from the product the number of elements there following the given element. Proceed in the same way throughout."[64]

For example, let us find the number of the combination $i_4k_3v_1$. Take 1. As there are 4 elements in the last category v, we multiply it by 4 and get $1 \times 4 = 4$. Since there are only 3 elements in that category after v_1, we subtract 3 from the product and get $4-3=1$. Next, we shall have to multiply the remainder 1 by 4, since there are 4 elements in the category of k and subtract from the result 1, since there lies only 1 element in the category after k_3. Thus, we get $1 \times 4 - 1 = 3$. Now we multiply 3 by 5, there being 5 elements in the category of i and then subtract 1, there being only one element after i_4. So, we get $3 \times 5 - 1 = 14$. Hence, the serial number of the combination $i_4k_3v_1$ is 14.

To get the same results as stipulated in rules (iii) and (iv) more easily and quickly, without going through the lengthy process of calculations described therein, Nemicandra gives two short tables. He says:

Table 1

"Place 1, 2, 3, 4, 5; 0, 5, 10, 15; 0, 20, 40 and 60 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations."[65]

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[63] As there is no element with zero suffix, the remainder gives k_4.
[64] *Ibid*, *Gāthā* 42.
[65] *Ibid*, *Gāthā* 43.

Table 1

i_1 1	i_2 2	i_3 3	i_4 4	i_5 5
k_1 0	k_2 5	k_3 10	k_4 15	
v_1 0	v_2 20	v_3 40	v_4 60	

Table: 2

"Set down 1, 2, 3, 4; 0, 4, 8, 12; 0, 16, 32, 48 and 64 in three rows (of cells) of the three categories of carelessness, and find the elements and the serial numbers of combinations."[66]

Table 2

v_1 1	v_2 2	v_3 3	v_4 4	
k_1 0	k_2 4	k_3 8	k_4 12	
i_1 0	i_2 16	i_3 32	i_4 48	i_5 64

Table 1 is to be used in case of distribution on the first scheme and Table 2 in that on the second scheme. To find the serial number of a given combination, we have simply to add together the figures placed in the cells of its elements in the tables. And to determine the elements occurring in a combination whose serial number is given, we shall have to break up that number into three parts picked up from three rows of cells in the tables and then write down in order the elements from those cells.

For example, since $13 = 3 + 10 + 0$ the 13th combination in the first scheme will be $i_3k_3v_1$ as determined before. According to the second scheme, it will be $v_1k_4i_1$, since $13 = 1 + 12 + 0$.

[66] *Ibid*, *Gāthā* 44.

Magic squares in India *

A square containing an equal number of cells in each row and each column is called a magic square, when the total of numbers in the cells of each row, each column and each diagonal happens to be the same. Magic squares have been known in India from very early times. It is believed that the subject of magic squares was first taught by Lord Śiva to the magician Maṇibhadra. Magic squares are said to have magical properties and were used in various ways by the Hindus as well as the Jainas. But the mathematics involved in the construction of magic squares and other magic figures was first systematically and elaborately discussed by the mathematician Nārāyaṇa (AD 1356) in his *Gaṇitakaumudī*. Some of his methods were unknown in the west and were recently discovered by the efforts of several scholars. The present article, besides giving a brief history of magic squares, explains the methods given by Nārāyaṇa and other Hindu writers for the construction of magic squares of various types.

1 Origin and early history

Little is known as regards the origin of magic squares and other figures. Hindu tradition assigns them to God Śiva. Nārāyaṇa (1356) says that the subject of progression, of which magic squares form a part, was taught by Śiva to Maṇibhadra,† the magician. The earliest unequivocal occurrence of magic squares is found in a work called *Kakṣapuṭa* composed by the celebrated alchemist and philosopher Nāgārjuna who flourished about the 1st century AD. One of the squares in this work is called *Nāgārjunīya* after him; so there can be no doubt that he did really construct some squares. The squares given by Nāgārjuna are all 4×4 squares, and some of these seem to have been known before him. The easier case of 3×3 square must have also been known earlier to Nāgārjuna. Another square is found in a work of Varāhamihira (d. 587 AD).

4×4 magic squares are considered to possess magical properties and are supposed to bring luck when worn as amulets. They are found on gates of buildings, on the walls where shopkeepers transact their business and on the

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 27, No. 1 (1992), pp. 51–120.

†Reference to Maṇibhadra Yakṣa occurs in the Buddhist work *Saṃyukta-nikāya* (i. 10, 4) and the Jaina work *Sūrya-prajñapti*. See D. N. Shukla, *Pratimā-Vijñāna* (in Hindi), p. 51.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
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covers of calendars used by astrologers even to this day. A 4×4 square occurs in a Jaina inscription of the 11th century, found in the ancient town of Khajuraho.

A systematic study of magic figures was taken up by Nārāyaṇa, who in his *Gaṇitakaumudī* (1356) gives general methods for the construction of all sorts of magic squares with the principles governing such constructions. He seems to have been the first to conceive of other figures in which numbers may be arranged so as to possess properties similar to those of magic squares. An account of the methods of constructing magic squares given by the authors mentioned above and also by other Hindu writers is given in this article.

It is the opinion of some historians of mathematics that magic squares first originated in China. This opinion is based on the occurrence of a square, filled with white and black dots, in the introduction of a Chinese work, the *I-king*. The square is called the *Loh Shu*, and is said to have come down to us from the time of the great emperor Yu (c. 2200 BC). According to Chinese tradition, the white dots denote odd numbers and the black dots even ones, and it has been conjectured that the *Loh Shu* represents the square shown in Figure 1.

But to consider the *Loh Shu* as a magic squares is to force upon it an interpretation which it originally did not possess. Arrangements of white and black dots in the figure of a square are met with elsewhere in the literature of the Chinese. One such arrangements represents the river Ho and has nothing to do with magic squares.[1]

The first unequivocal appearance of the *Loh Shu* in the form of a magic square is found in the writings of Tsai Yuan-Ting[2] who lived from 1135 to 1198 AD. Magic squares occur also in the writings of Hebrew[3] and Arab[4] scholars about the same period, while in India they were used much earlier. It would thus appear that the Chinese claim to the invention of magic squares is not well founded.[5]

[1]The *Loh Shu* and the map of the Ho are illustrated in *Magic Squares and Cubes* by W. S. Andrews, Chicago, 1908, p. 122.

[2]*Cf.* W. S. Andrews, *l.c.*, p. 123.

[3]In a work of Rabbi ben Ezra (c. 1140); *cf.* D. E. Smith, *History of Mathematics*, II, New York, 1923, p. 596.

[4]In the work of the Arab philosopher Gazzali, *cf.* Smith, D. E., *l.c.*, p. 597.

[5]It is said in the *Vedas* that the gods Indra and Viṣṇu divided 1000 into three. This incident is related in many works. (*Taittirīya Saṃhitā*, vii. 1.6.; iii. 2. 11; *Atharvaveda*, ii. 44.1.; *Taittirīya Brāhmaṇa*, i. 1.6.1; *Śatapatha Brāhmaṇa*, iii. 8.4.4. etc.). In the *Taittirīya Saṃhitā*, we have

Ye twain have conquered; ye are not conquered,
Neither of the two of them hath been defeated;
Indra and Viṣṇu when contended,
Ye did divide the thousand into three. (Keith)

The thousand is divided into three at the three-night festival; verily he makes

4	9	2
3	5	7
8	1	6

Figure 1

	1		8
	9		2
6		3	
4		7	

Figure 2

2 Nāgārjuna squares

In his *Kakṣapuṭa*, Nāgārjuna (100 AD) gives rules for the construction of 4×4 squares with even as well as odd totals.[6] These rules consist partly of mnemonic verses in which numbers are expressed in alphabetic notations. The general direction is

arka	*indunidhā*	*nāri*	*tena*	*lagna*	*vināsanam*
0 1	0 8 0 9	0 2	6 0	3 0	4 0 7 0

By inserting these values in the successive cells (of the 4×4 square) leaving blanks for zero, we get the primary skeleton. (**ed.** See Figure 2.)

The eight blank cells can be filled up in such a way as to give even as well as odd totals. But the methods of filling up differ slightly in the two cases.

Even total

In order to have an even total, fill up, says Nāgārjuna, the blank cells by writing the difference between half of that total and the number in the alternate cell in a diagonal direction from the cell to be filled up. This direction may be upwards or downwards, right or left.

Taking the total to be $2n$, where n is any integer, we thus get the complete magic square with even totals. (**ed.** See Figure 3.)

In this magic square, the totals of all the rows, horizontal, vertical, and diagonal, of every group of four forming a sub-square, and separated by such

her possessed of a thousand, he makes her the measure of a thousand.

In the above passages it is not clear what "dividing a thousand into three" means. As the problem was considered so difficult that only the gods could solve it, so it is certain that "division into three" did not mean division into three equal parts or into any three parts or into three parts in arithmetic progression, for division as above can be easily made by the use of ordinary fractions which were known in those times. The passage very probably refers to the construction of a magic square with 1000 as total, especially as it has been stated that it confers benefits acting as a charm if the operation is performed at the three-night festival.

But to produce this passage as an evidence of the existence of magic squares, without other corroborative facts would, in our opinion, be as unjustifiable as the use of the *Loh Shu* to establish the existence of the magic square in China in 2200 BC.

[6] See *Indian Antiquary*, XI, 1882, pp. 83f.

$n-3$	**1**	$n-6$	**8**
$n-7$	**9**	$n-4$	**2**
6	$n-8$	**3**	$n-1$
4	$n-2$	**7**	$n-9$

Figure 3: Total $= 2n$.

$n-2$ $2n-10$ 10 $n+2$	$n+2$ $2n-10$ 10 $n-2$
$n-2$ 10 $2n-10$ $n+2$	$n+2$ 10 $2n-10$ $n-2$

Figure 4

3	$n-1$	6	$n-8$	3	$n-1$	6	$n-8$
7	$n-9$	4	$n-2$	7	$n-9$	4	$n-2$
$n-6$	8	$\mathbf{n-3}$	**1**	$\mathbf{n-6}$	**8**	$n-3$	1
$n-4$	2	$\mathbf{n-7}$	**9**	$\mathbf{n-4}$	**2**	$n-7$	9
3	$n-1$	**6**	$\mathbf{n-8}$	**3**	$\mathbf{n-1}$	6	$n-8$
7	$n-9$	**4**	$\mathbf{n-2}$	**7**	$\mathbf{n-9}$	4	$n-2$
$n-6$	8	$n-3$	1	$n-6$	8	$n-3$	1
$n-4$	2	$n-7$	9	$n-4$	2	$n-7$	9

Figure 5

a sub-square, and of the corner four of the square and about a small square, are equal. Another noteworthy feature of it is that each of its four minor squares has relation to others, as may be seen in Figure 4.

The above square is "continuous" according to the definition of Paul Carus; that is, "It may vertically as well as horizontally be turned upon itself and the rule holds good that wherever we may start four consecutive numbers in whatever direction, backward or forward, upward or downward, in horizontal, vertical or slanting lines, always yield the same sum ... and so does any small square of 2×2 cells."[7] Since the square cannot be bent upon itself at once in two directions, the result is shown in Figure 5 by extending the square in each direction by half its own size.

Odd total

For an odd total, say $2n+1$, we are to fill up the blank cells by writing the difference between n and the number in the alternate cell in a diagonal

[7] Andrews W. S., *Magic Square and Cubes*, Chicago, 1908, p. 125f.

$n-3$	**1**	$n-5$	**8**
$n-6$	**9**	$n-4$	**2**
6	$n-7$	**3**	$n-1$
4	$n-2$	**7**	$n-8$

Figure 6: Total $= 2n+1$.

$n-2$ $2n-9$ 10 $n+3$	$n+3$ $2n-9$ 10 $n-2$
$n-1$ 10 $2n-9$ $n+2$	$n+2$ 10 $2n-9$ $n-1$

Figure 7

30	16	18	36
10	44	22	24
32	14	20	34
28	26	40	6

Figure 8: Total = 100.

direction from the cell to be filled up, when the latter number happens to be 1, 2, 3 or 4; or the difference between $n+1$ and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number be 6, 7, 8, or 9. This direction may be, as in the previous case, upwards or downwards, right or left. Proceeding in this way, we get the complete magic squares having an odd total. (**ed.** See Figure 6.)

In this case, the totals of all rows, horizontal, vertical and diagonal, of every group of four forming a square (except the group of the fifth, sixth, ninth and tenth cells, and that of the seventh, eighth, eleventh and twelfth cells), of the corner four of the square, and of the four about the corners of a small square are equal. The relation between the four minor squares in this case is not as complete as in the previous case. (**ed.** See Figure 7.)

It is not a perfectly continuous square. The odd totals cannot be less than 19 in any case, and not less than 37 if the same number is not to appear more than once in the square. (See Figure 6.)

A particular case of 4×4 squares with even total, 100, has been specially noted by Nāgārjuna. Its form differs from that which results on putting $n = 50$ in the above general case, and further it does not contain the numbers from 1 to 9, except 6. (**ed.** See Figure 8.)

This magic square has been called the *Nāgārjunīya*.[8] This special epithet

[8] To fill this square the mnemonic formula stated by Nāgārjuna is:

Nīlaṃ^{30} *cāpi*^{16} *dayā*^{18} *-calo*^{36} *naṭa*^{10} *-bhuvaṃ*^{44} *khārī*^{22} *-varaṃ*^{24} *rāginam*^{32} |
Bhūpo^{14} *nārī*^{20} *vago*^{34} *jarā*^{28} *cara*^{26} *-nibhaṃ*^{40} *tānaṃ*^{06} *śataṃ*^{100} *yojayet* ||

will lead one to presume that this particular square was constructed by Nāgārjuna, while others described by him were recapitulations of former accomplishments.

3 Varāhamihira square

Varāhamihira (d. 587 AD) gives a form of 4×4 magic squares,[9] in which the total is 18 (**ed.** see Figure 9). It is however, a particular case of the Figures 10 and 11.

Varāhamihira has called his square *sarvatobhadra* ("Magic in all respects") and what are implied by that name, i.e., the special features of the square, have been pointed out fully by his commentator, Bhaṭṭotpala (966). Indeed it has properties similar to those of squares with even totals described by Nāgārjuna. The method of filling up the blank cells in the primary skeleton is the same. (**ed.** See Figures 12 and 13.)

The blank cells can be filled up so as to yield also an odd total; write the difference between $n+1$ and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 1, 2, 3, or 4, or the difference between n and the number in the alternate cell in a diagonal direction from the cell to be filled up, if the latter number happens to be 5, 6, 7 or 8. (**ed.** See Figures 14 and 15.)

4 Jaina squares

In a Jaina inscription found amongst the ruins of the ancient town of Khajuraho occurs a magic square of 4×4 cells of which the total is 34 (Figure 16). It possesses all the special features of the Nāgārjuna squares. It belongs to the eleventh century of the Christian era. In the *Tijapapahutta Stotra* of the Jainas, we find another 4×4 magic square having a total of 170[10] (Figure 17).

The date of this square is uncertain. It is certainly not later than the fourteenth century, when a commentary on the above *stotra* (hymn) was written. It is probably a very old one. Its total 170 is closely connected with an ancient Jaina mythology about the appearance of their prophets.

5 Nārāyaṇa's results

As has been already pointed out, the only Hindu work, known to us which gives a systematic mathematical treatment of the construction of magic squares and

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[9] *Bṛhat Saṃhitā*, lxxvii. 23ff.

[10] The square as it actually occurs is interspersed with the *bīja* ("elements of a *mantra*").

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

Figure 9: Total = 18.

$n-7$	**3**	$n-4$	**8**
5	$n-1$	**2**	$n-6$
4	$n-8$	**7**	$n-3$
$n-2$	**6**	$n-5$	**1**

Figure 10: Total = $2n$.

2	$n-6$	**5**	$n-1$
$n-4$	**8**	$n-7$	**3**
$n-5$	**1**	$n-2$	**6**
7	$n-3$	**4**	$n-8$

Figure 11: Total = $2n$.

	3		8
5		2	
4		7	
	6		1

Figure 12

2		5	
	8		3
	1		6
7		4	

Figure 13

$n-7$	**3**	$n-3$	**8**
5	n	**2**	$n-6$
4	$n-8$	**7**	$n-2$
$n-1$	**6**	$n-5$	**1**

Figure 14: Total = $2n+1$.

2	$n-6$	**5**	n
$n-3$	**8**	$n-7$	**3**
$n-5$	**1**	$n-1$	**6**
7	$n-2$	**4**	$n-8$

Figure 15: Total = $2n+1$.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Figure 16: Total = 34.

25	80	15	50
20	45	30	75
70	35	60	5
55	10	65	40

Figure 17: Total = 170.

other figures is the *Gaṇitakaumudī* of Nārāyaṇa. Chapter XIV of the work is devoted to this subject, and we propose to give here a summarised version of this chapter, preserving the order of treatment and giving explanatory notes wherever necessary.

5.1 Summary

In order to bring into prominence the remarkable achievements of Nārāyaṇa in the theory of magic squares, it is thought desirable to state briefly some of his most important results before entering into details. These results are:

1. Magic squares are of three types: (a) those which have $4n$ cells in a row, (b) those which have $4n + 2$ cells in a row, and (c) those which have an odd number of cells in a row.

2. Series in arithmetical progression are used for the construction of these squares.

3. Magic squares can be made of as many series or groups of numbers as there are cells in a column.

4. Each series or group is composed of as many numbers as there are groups.

5. The common difference must be the same for each group.

6. The initial terms of the groups are themselves in A.P.

7. The numbers is a group, although belonging to an arithmetical progression, may be disarranged in various ways for the filling of the square.

8. The method of the knight's move for the construction of a $4n \times 4n$ square.

9. The method of superposition for the construction of $4n \times 4n$ squares.

10. The method of equi-spacing for the construction of $(4n + 2) \times (4n + 2)$ squares.

11. The method of superposition for odd squares.

12. A special method for odd squares.

13. The construction of a magic rectangle (*vitāna* or canopy).

14. The construction of magic circles, triangles, hexagons and various other figures, such as the altar, the diamond, etc.

5.2 Preliminary remarks

According to Nārāyaṇa, magic squares may be classified into three groups: (1) *samagarbha* (2) *viṣamagarbha*, and (3) *viṣama*. These terms are defined as follows:

> If on dividing the *bhadrāṅka*[11] ("number of cells in a line of the square") by four, the remainder is zero, the magic square is said to be *samagarbha*. If the remainder is two, it is called *viṣamagarbha*; and if the remainder is one or three, it is simply *viṣama*.[12]

After giving the above classification, Nārāyaṇa remarks:

> In the construction of magic squares, the arithmetical progression is used.[13] In relation to that (magic square) which is required to be constructed, first find the *initial term* and the *common-difference* (of a series in arithmetical progression, corresponding to the given sum and the number of cells).[14] The sum divided by the *bhadrāṅka* ("number of the square") gives the *phala* ("total"). The number of terms to be taken in the progression is the number of *gṛha* ("cells") in the square.[15] If the number of cells (*koṣṭha*) is a square number, its root is called the *caraṇa* ("foot" or "row"). Such are the technical terms used by Nārāyaṇa in his *bhadragaṇita* ("calculations relating to magic figures").[16]

The method of finding out the initial term and the common-difference of an arithmetical progression, given the sum and the number of terms, follows the above preliminary remarks.[17]

5.3 The 4×4 magic squares[18]

Assuming that the required arithmetical progression has been found out, Nārāyaṇa gives the following rule for filling the cells of the 4×4 square with the numbers occurring in the progression:

---

[11] Henceforth we shall translate this term by "number of the square".

[12] *Gaṇitakaumudī*, xiv. 4. All the references that follow are from chapter xiv. To avoid unnecessary repetition, the number of the rules only, as found in Padmakara Dvivedi's edition of the *Gaṇitakaumudī* will be given.

[13] This seems to be the first statement of the result that magic squares are made from numbers in A.P., a result on which the whole theory of magic squares is founded.

[14] Rule 5, the technical terms are: *mukha* = initial term and *pracaya* = common-difference.

[15] Rule 6.

[16] Rule 7.

[17] Rule 8. The problem is indeterminate. The initial term is arbitrarily assumed, and the common- difference is obtained from the equation $s - \frac{n(n-1)d}{2} = na$, where $s =$ the given sum, $n =$ the number of cells (or terms), a = initial term, and d = the common-difference.

[18] *Caturbhadra* ("four magic square" or "4×4 magic square").

> In the manner of the chess-board, place the numbers forming the progression, (taking them) two and two, in two connected cells as well as in alternate cells, in the direct and inverse order. (Then) by right and left knight's move fill the cells (of the square) with the numbers (taking them as they have been placed above). This method has also been stated by previous teachers for the construction of the *samagarbha* magic square of sixteen cells. The numbers, in the horizontal cells, in the vertical cells as well as in the diagonal cells, added separately, give rise to the same total.[19]

Example from Nārāyaṇa:

> Friend, tell me, how a 4×4 magic square be filled up with the numbers beginning with unity and successively increasing by one, so that the horizontal, vertical and diagonal cells shall have the same sum.

Here the sum of the natural numbers from one to sixteen is $\frac{(16\times17)}{2} = 136$. Therefore, the required total is $\frac{136}{4} = 34$.

The numbers when written, two and two, in connected cells as well as alternate cells give Figures 18 and 19. Placing two and two in the direct and inverse orders, we get Figures 20 and 21.

[Note that the numbers in the 4 cells on the right of the first two rows are reversed and the same is done with the numbers in the 4 cells on the left of the last two rows. This would probably be an easier method of stating the method.]

Taking the arrangement (a), and filling by right and left knight's move, the four squares depicted in Figures 22–25 are obtained. The arrangement (b) similarly gives the four squares as per Figures 26–29.

The filling in of the numbers begins by putting 1 in the first cell. After the first row of numbers is exhausted, we begin by putting the first number of the second row (i.e., 8) in a contiguous cell, as in (a_1) or (a_2). [In order to make up the desired total, 10 is made to correspond to 8 and 15 to 1, as in the illustration above.] If the square be considered to be wrapped round a cylinder, the fourth cell is also contiguous to the first cell; hence the squares (a_3) and (a_4). It may be noted that (a_2) may be obtained by turning (a_1) through a right angle, whilst (a_3) and (a_4) may be obtained by wrapping (a_1) and (a_2) round a cylinder.

[19] Rules 10–12. Nārāyaṇa ascribes the above method to previous writers. It cannot be said how old it is. The squares formed by the method are very popular among Hindu astrologers.

(a_0)

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

Figure 18

(b_0)

1	3	2	4
8	6	7	5
9	11	10	12
16	14	15	13

Figure 19

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Figure 20

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Figure 21

(a_1)

1	**8**	13	12
14	11	**2**	**7**
4	**5**	16	9
15	10	**3**	**6**

Figure 22

(a_2)

1	14	**4**	15
8	11	**5**	10
13	**2**	16	**3**
12	**7**	9	**6**

Figure 23

(a_3)

1	12	13	**8**
14	**7**	**2**	11
4	9	16	**5**
15	**6**	**3**	10

Figure 24

(a_4)

1	14	**4**	15
12	**7**	9	**6**
13	**2**	16	**3**
8	11	**5**	10

Figure 25

(b_1)

1	**8**	13	12
15	10	**3**	**6**
4	**5**	16	9
14	11	**2**	**7**

Figure 26

(b_2)

1	15	**4**	14
8	10	**5**	11
13	**3**	16	**2**
12	**6**	9	**7**

Figure 27

(b_3)

1	12	13	**8**
15	**6**	**3**	10
4	9	16	**5**
14	**7**	**2**	11

Figure 28

(b_4)

1	15	**4**	14
12	**6**	9	**7**
13	**3**	16	**2**
8	10	**5**	11

Figure 29

(a)

1	2	4	3
8	7	5	6
10	9	11	12
15	16	14	13

Figure 30

(b)

1	3	4	2
8	6	5	7
11	9	10	12
14	16	15	13

Figure 31

(a′)

1	5	7	3
8	4	2	6
10	14	16	12
15	11	9	13

Figure 32

(b′)

1	5	6	2
8	4	3	7
11	15	16	12
14	10	9	13

Figure 33

(a″)

1	2	6	5
8	7	3	4
10	9	13	14
15	16	12	11

Figure 34

(b″)

1	3	7	5
8	6	2	4
11	9	13	15
14	16	12	10

Figure 35

5.4 Varieties of 4×4 magic squares

Nārāyaṇa remarks:

> Here, other 4×4 squares may be produced from a 4×4 square by turning four cells to make the numbers inverse.[20]
>
> In the rest of the *caraṇa* ("row") following the first cell, (by turning) four numbers produced in two connected pairs of cells, there result twenty-four varieties. And the same numbers arise from others separately.[21]

Example from Nārāyaṇa:

> How many 4×4 squares can be formed out of the series of natural numbers from one to sixteen, and what are their forms?[22]

The numbers are placed according to the previous rule as per Figures 30 and 31. By turning four cells, i.e., the two connected pairs in the middle of the first two rows, and doing the same for the last two rows, we have Figures 32 and 33. Performing the same operation on the two connected pairs of cells at the end in (a) and (b), we have Figures 34 and 35.

The numbers in the arrangements (a′), (b′), (a″) and (b″) are filled in the 4×4 square in the same way as those of (a) or (b). Thus, there will be altogether 24 squares with 1 in the first cell. As there are sixteen numbers, so there can be 384 varieties of 4×4 squares, formed out of the series of natural numbers one to sixteen.

The twenty-four varieties with 1 in the first cell have been shown by Nārāyaṇa as per Figure 36.

[20] Rule 13(a).
[21] Rule 13(b)–14(a).
[22] Ex. 4.

[1]

1	**8**	13	12
14	11	**2**	**7**
4	**5**	16	9
15	10	**3**	**6**

[2]

1	14	**4**	15
8	11	**5**	10
13	**2**	16	**3**
12	**7**	9	**6**

[3]

1	12	13	**8**
14	**7**	**2**	11
4	9	16	**5**
15	**6**	**3**	10

[4]

1	14	**4**	15
12	**7**	9	**6**
13	**2**	16	**3**
8	11	**5**	10

[5]

1	**8**	13	12
15	10	**3**	**6**
4	**5**	16	9
14	11	**2**	**7**

[6]

1	15	**4**	14
8	10	**5**	11
13	**3**	16	**2**
12	**6**	9	**7**

[7]

1	12	13	**8**
15	**6**	**3**	10
4	9	16	**5**
14	**7**	**2**	11

[8]

1	15	**4**	14
12	**6**	9	**7**
13	**3**	16	**2**
8	10	**5**	11

[9]

1	**8**	10	15
14	11	**5**	**4**
7	**2**	16	9
12	13	**3**	**6**

[10]

1	14	**7**	12
8	11	**2**	13
10	**5**	16	**3**
15	**4**	9	**6**

[11]

1	15	10	**8**
14	**4**	**5**	11
7	9	16	**2**
12	**6**	**3**	13

[12]

1	14	**7**	12
15	**4**	9	**6**
10	**5**	16	**3**
8	11	**2**	13

1	**2**	**4**	**3**
8	**7**	**5**	**6**
10	9	11	12
15	16	14	13

1	**3**	**4**	**2**
8	**6**	**5**	**7**
11	9	10	12
14	16	15	13

1	**5**	**7**	**3**
8	**4**	**2**	**6**
13	9	11	15
12	16	14	10

(a) Squares 1–12.

[13]

1	**8**	11	14
15	10	**5**	**4**
6	**3**	16	9
12	13	**2**	**7**

[14]

1	15	**6**	12
8	10	**3**	13
11	**5**	16	**2**
14	**4**	9	**7**

[15]

1	14	11	**8**
15	**4**	**5**	10
6	9	16	**3**
12	**7**	**2**	13

[16]

1	15	**6**	12
14	**4**	9	**7**
11	**5**	16	**2**
8	10	**3**	13

[17]

1	**8**	11	14
12	13	**2**	**7**
6	**3**	16	9
15	10	**5**	**4**

[18]

1	12	**6**	15
8	13	**3**	10
11	**2**	16	**5**
14	**7**	9	**4**

[19]

1	14	11	**8**
12	**7**	**2**	13
6	9	16	**3**
15	**4**	**5**	10

[20]

1	12	**6**	15
14	**7**	9	**4**
11	**2**	16	**5**
8	13	**3**	10

[21]

1	**8**	10	15
12	13	**3**	**6**
7	**2**	16	9
14	11	**5**	**4**

[22]

1	12	**7**	14
8	13	**2**	11
10	**3**	16	**5**
15	**6**	9	**4**

[23]

1	15	10	**8**
12	**6**	**3**	13
7	9	16	**2**
14	**4**	**5**	11

[24]

1	12	**7**	14
15	**6**	9	**4**
10	**3**	16	**5**
8	13	**2**	11

1	**5**	**6**	**2**
8	**4**	**3**	**7**
13	9	10	14
12	16	15	11

1	**2**	**6**	**5**
8	**7**	**3**	**4**
10	9	13	14
15	16	12	11

1	**3**	**7**	**5**
8	**6**	**2**	**4**
11	9	13	15
14	16	12	10

(b) Squares 13–24.

Figure 36: Nārāyaṇa's squares.

Example:

> In a certain 4×4 square, the total (*phala*) is 40, find the initial term and the common-difference. Also find them when the total is 64.

The equations giving the initial term (a) and the common difference (d) are:

$$\text{(i)}\ \ 10 - \frac{15}{2}d = a, \text{ when the total is 40; and}$$

$$\text{(ii)}\ 16 - \frac{15}{2}d = a, \text{ when the total is 64.}$$

These give:

for case (i)	$a = -5, \ldots;$	$d = 2, \ldots$
and for case (ii)	$a = 1, -14, \ldots;$	$d = 2, 4, \ldots$

The squares constructed, according to the rule given above, with the above values of a and d are shown in Figures 37–39.

5.5 Use of irregular series

Instead of employing 16 numbers in arithmetical progression to fill up a 4×4 square, four different arithmetic series, with different initial terms but the same common-difference consisting of four terms each may be used.[23] Nārāyaṇa gives the following examples to illustrate this:[24]

(a) To construct 4×4 magic squares with total 40.

In this case, the *caraṇas* (rows, i.e., the arithmetic progressions of which each has as many terms as there are cells in a row) may be supposed to be:

(i)	1	2	3	4	(ii)	1	2	3	4	(iii)	2	3	4	5
	6	7	8	9		5	6	7	8		6	7	8	9
	11	12	13	14		12	13	14	15		11	12	13	14
	16	17	18	19		16	17	18	19		15	16	17	18

Now filling up the cells by the same method as before, we get Figures 40–42.

(b) To construct 4×4 squares with total 64

The initial terms of the *caraṇas* (rows) may be supposed to be (i) 7, 12, 17, 22 or (ii) 4, 11, 18, 25 or (iii) 1, 10, 19, 28, the common-difference being unity in each case. The corresponding squares are shown in Figures 43–45.

[23] Rule 14(b)–15.

[24] Examples 5. These are the first examples of squares constructed by a set of numbers not in a regular A.P.

−5	9	19	17
21	15	−3	7
1	3	25	11
23	13	−1	5

Figure 37: Total = 40.

1	15	25	23
27	21	3	13
7	9	31	17
29	19	5	11

Figure 38: Total = 64.

−14	14	34	30
38	26	−10	10
−2	2	46	18
42	22	−6	6

Figure 39: Total = 64.

1	9	16	14
17	13	2	8
4	6	19	11
18	12	3	7

Figure 40: Total = 40.

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Figure 41: Total = 40.

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Figure 42: Total = 40.

7	15	22	20
23	19	8	14
10	12	25	17
24	18	9	13

Figure 43: Total = 64.

4	14	25	21
26	20	5	13
7	11	28	18
27	19	6	12

Figure 44: Total = 64.

1	13	28	22
29	21	2	12
4	10	31	19
30	20	3	11

Figure 45: Total = 64.

5.6 Construction of irregular series

Method 1

Narāyaṇa gives the following rule for the determination of irregular series to be used for filling a square with a given total:

> For the determination of the *caraṇas* ("rows") assume the first term and the common-difference optionally. First write down the initial term and then add to it successively the product of the common-difference and the number of cells in a row, and do so as many times as the number of rows less one. The series thus formed is the *mukhapaṅkti* ("the optionally assumed series of initial terms"). To the last term of this series add the first term together with the product of the common-difference into the number of rows minus one, and multiply by half the number of rows: this is the *mukhaphala* ("the total corresponding to the assumed series"). The desired total minus the *mukhaphala* is the *kṣepaphala* ("the total for the numbers to be interpolated"). Now determine the first term and the common-difference of a series in A.P. whose number of terms is equal to the number of rows and whose sum is equal to the *kṣepaphala*. Add the successive terms of the series thus obtained to the corresponding terms of the *mukhapaṅkti* ("the optionally assumed series of initial terms"). Thus will be determined the *caraṇas* for all magic squares.[25]

[25] Rules 16–20 (a). If the number of *caraṇas* ("rows") be n, and if the first term be assumed to be a and the common difference d, the sum of n^2 terms divided by n, the number of rows, is the total (*mukhaphala*) of the $n \times n$ square that will be constructed with this series. If the terms are written in rows of n, the initial terms of the rows., i.e., the *mukhapaṅkti*, will be

$$a,\ (a+nd),\ (a+2nd),\ \ldots,\ [a+n(n-1)\ d].$$

Let the given total be T. The total corresponding to the *mukhapaṅkti* (i.e., *mukhaphala*) is

$$\begin{aligned}\frac{n^2\left\{a+\frac{(n^2-1)d}{2}\right\}}{n} &= \frac{n}{2}[2a+(n-1)(n+1)d] \\ &= \frac{n}{2}\left\{[a+n(n-1)d]+a+(n-1)d\right\}\end{aligned}$$

which is the form in which the total is expressed by Nārāyaṇa.

$$\begin{aligned}Kṣepaphala &= T-\frac{n}{2}\left\{[a+n(n-1)d]+a+(n-1)d\right\} \\ &= K,\ \text{(say)}.\end{aligned}$$

We have now to find an arithmetic series of n terms whose sum is equal to K. The terms of this series are added to the corresponding terms of the *mukhapaṅkti*. The rationale of the above result can be easily worked out. It can be easily seen that if A, D are the first

Examples from Nārāyaṇa:

(i) Determine the *caraṇas* for a 4×4 magic square with total 40.

Optionally assume a series whose first term is 1 and the common-difference is 1. When the terms are placed in rows of four, the initial terms of the successive rows (i.e., *mukhapaṅkti*) are: 1, 5, 9, 13. Since the number of *caraṇas* is 4,

$$\textit{mukhaphala} = \frac{4}{2}[13 + 1 + (4-1) \times 1] = 34,$$
$$\textit{kṣepaphala} = 40 - 34 = 6.$$

Now, if A be the first term, and D, the common-difference and 6 the sum of an A.P. of 4 terms, we must have

$$\frac{6 - \frac{4}{2}(4-1)D}{4} = A.$$

Therefore,

$$A = 0, -3, \ldots \qquad \text{and} \qquad D = 1, 3, \ldots.$$

For the solution ($A = 0,\ D = 1$), the series is: 0, 1, 2, 3. For the solution ($A = -3,\ D = 3$), the series is: -3, 0, 3, 6. Therefore, the initial terms of the required *caraṇas* ("rows") are (1, 6, 11, 16) or (-2, 5, 12, 19).

(ii) Determine the *caraṇas* for the 4×4 square whose total is 64.

In this case, the *kṣepaphala* is $64 - 34 = 30$, so that

$$\frac{30 - 6D}{4} = A.$$

That is,

$$A = 6, 3, 0, \ldots \qquad \text{and} \qquad D = 1, 3, 5, \ldots$$

For the first solution, the series is (6, 7, 8, 9), for the second (3, 6, 9, 12) and for the third (0, 5, 10, 15). Therefore, the initial terms of the *caraṇas* are (7, 12, 17, 22) or (4, 11, 18, 25) or (1, 10, 19, 28).

The squares may now be constructed by the method of the knight's move.

---

term and the common-difference of the series whose sum is K, then the initial terms of the *caraṇas* ("rows") are

$$[a + A],\ [(a + nd) + A + D],\ \ldots,\ [(a + n(n-1)d) + A + (n-1)D].$$

Method 2[26]

> Divide the *kṣepaphala* ("total of numbers to be interpolated") by the *caraṇa* ("number of cells in a row"). The quotient increased by unity becomes the "*gaccha*",[27] provided the remainder is zero or equal to half the *caraṇa*. If the remainder is otherwise, the magic square is not possible. Add to the first and the second halves of the *mukhapaṅkti* respectively zero and half the *kṣepaphala* or these increased and decreased by unity successively. Thus will be determined the initial terms of the *caraṇas* in the cases of *samagarbha* and *viṣamagarbha* squares.

Examples from Nārāyaṇa:

(i) To construct a 4×4 square with total 40.

Assuming the series of natural numbers, the *kṣepaphala* is $40 - 34 = 6$. This divided by the *caraṇa*, i.e., $6 \div 4$, gives the quotient 1 and remainder 2. The construction of the square is thus possible, and $1 + 1 = 2$ squares may be obtained. The *mukhapaṅkti* is 1, 5, 9, 13. Half of the *kṣepaphala* $= 3$.

The numbers to be interpolated are, therefore, 0 and 3, or adding and subtracting unity, 1 and 2. Thus, adding these to the respective halves of the *mukhapaṅkti*, we get:[28]

0	3	1	5	12	16
1	2	2	6	11	15
Interpolators		Initial terms of the rows			

Thus, the numbers to be filled in the square are:

1,	2,	3,	4	or	2,	3,	4,	5
5,	6,	7,	8		6,	7,	8,	9
12,	13,	14,	15		11,	12,	13,	14
16,	17,	18,	19		15,	16,	17	18

and the corresponding squares are as shown in Figures 46 and 47.

(ii) To construct a 4×4 square with total 64.

Here, as before the *kṣepaphala* = 64 - 34 = 30. This divided by the *caraṇa*, i.e., $30 \div 4$ gives the quotient 7 and remainder 2. Thus, the square is possible and $7 + 1 = 8$ different squares may be obtained.

[26]Rules 20(b)–23(a).

[27]Here, the term *gaccha* means the "number" of different sets of series that may be obtained for the filling of the square with the required total.

[28]Nārāyaṇa gives the initial terms only.

1	8	16	15
17	14	2	7
4	5	19	12
18	13	3	6

Figure 46: Total = 40.

2	9	15	14
16	13	3	8
5	6	18	11
17	12	4	7

Figure 47: Total = 40.

As before, the *mukhapaṅkti* is 1, 5, 9, 13. Half the *kṣepaphala* is 15. The numbers to be interpolated are 0, 15, or adding and subtracting unity successively to get 8 different pairs we have:

0	15
1	14
2	13
3	12
4	11
5	10
6	9
7	8

Adding these pairs to the respective halves of the *mukhapaṅkti* (1, 5, 9, 13), we get the following 8 sets for the initial terms of the rows:

1	5	24	28
2	6	23	27
3	7	22	26
4	8	21	25
5	9	20	24
6	10	19	23
7	11	18	22
8	12	17	21

Eight squares may now be constructed as before.

5.7 Change of squares

> Construct a magic square of the type desired. Subtract its total from the given total, and divide by the number of cells in a line. On adding the quotient to the numbers in the cells of that square will be obtained the required square.[29]

Thus, to transform the 4×4 magic square of Figure 22, with total 34, into another with total 100, one has simply to add $\frac{(100-34)}{4}$, i.e., $\frac{33}{2}$ to the numbers in the cells of that magic square.

[29]Rule 23(b)–24(a).

5.8 Construction by superposition

First method

> Construct two *samagarbha* squares, one called *chādaka* ("covering one") and the other called *chādya* ("one to be covered"). The superposition is to be made in the manner of folding the palms of the hands. Form a series with an optional first term and an optional common-difference and with as many terms as the "number"[30] of the square; this is the *mūlapaṅkti* ("basic series"). With another first term and common-difference form another series: this is called *parapaṅkti*. Multiplying the terms of the *parapaṅkti* by the quotient obtained on dividing the given total minus the sum of the *mūlapaṅkti* by the sum of the *parapaṅkti*, is produced the progression which is called *guṇapaṅkti* ("product-series"). Divide the *mūlapaṅkti* and the *guṇapaṅkti* by turning each upon itself, so that each part will have terms equivalent to half the "number" of the square. The numbers are written down vertically, one above the other, and directly in the *chādaka* ("covering one") and in another fashion (i.e., horizontally and inversely) in the *chādya* ("one to be covered"). In the first, fill thus successively half the rows and in the second half the columns.[31] Fill the other half of each square in the contrary way. This method of constructing magic squares by superposition is taught by the son of Nṛhari (i.e., by Nārāyaṇa).[32]

Examples from Nārāyaṇa:

(i) To construct a 4×4 magic square with total 40.

Assume the *mūlapaṅkti* to be 1, 2, 3, 4. Let the *parapaṅkti*[33] be 0, 1, 2, 3. The multiplier $= \frac{(40-10)}{6} = 5$. Therefore, the *gunapaṅkti* is 0, 5, 10, 15.

Writing the *mūlapaṅkti* and *guṇapaṅkti* by turning them upon themselves, we get

1	2	and	0	5
4	3		15	10

respectively. Taking the first set, placing it vertically and then filling with it the horizontal half of a 4×4 square, we get Figure 48.

---

[30] The "number" of the square is the number of cells in a row of the square.

[31] i.e., the upper horizontal half of the first square is filled first and then the lower half, and in the second square, the left vertical half is filled first.

[32] Rules 24(b)–29. The author Nārāyaṇa was the son of Nṛsiṃha or Nṛhari.

[33] It is convenient to take the *parapaṅkti* such that its sum is less than that of the *mūlapaṅkti*, but this is not essential.

Then filling the other half with the same numbers in the inverse order, we get the *chādaka* (Figure 49). In the same way, filling horizontally with the second set, we get the *chādya* (Figure 50). Then, folding (A) over (B) and adding the numbers, we get the required square with total 40 (Figure 51).

Aliter. Or, if we take the *mūlapaṅkti* as before but the *parapaṅkti* as 1, 2, 3, 4, the multiplier is $\frac{(40-10)}{10} = 3$, so that the *guṇapaṅkti* is 3, 6, 9, 12. Thus we get

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 3 & 6 \\ 12 & 9 \end{matrix}$$

and the corresponding squares are as in Figures 52–54.[34]

(ii) To construct a 4×4 square with total 64.

Here, taking the *mūlapaṅkti* as 1, 2, 3, 4, and the *parapaṅkti* as 0, 1, 2, 3. The multiplying factor is

$$\frac{64-(1+2+3+4)}{(0+1+2+3)} = 9.$$

Therefore, the *guṇapaṅkti* is 0, 9, 18, 27. Arranging, we have correspondingly

$$\begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 0 & 9 \\ 27 & 18 \end{matrix}$$

and the corresponding squares (Figures 55–57) as before.

The above method of constructing squares was rediscovered in Europe by M. de la Hire (1705), and is now attributed to him.

(iii) To construct a 8×8 square with total 260.

Let the *mūlapaṅkti* be 1, 2, 3, 4, 5, 6, 7, 8, and the *parapaṅkti* 0, 1, 2, 3, 4, 5, 6, 7. The multiplying factor is

$$\frac{260-\frac{1}{2}\times 8\times(8+1)}{\frac{1}{2}\times 7\times(7+1)} = 8.$$

The *guṇapaṅkti* is 0, 8, 16, 24, 32, 40, 48, 56.

Breaking up the *mūlapaṅkti* and *guṇapaṅkti* into halves and writing them by turning upon themselves we have

$$\overset{\text{(a)}}{\begin{matrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{matrix}} \qquad \text{and} \qquad \overset{\text{(b)}}{\begin{matrix} 0 & 8 & 16 & 24 \\ 56 & 48 & 40 & 32 \end{matrix}}$$

Hence, the preliminary squares are as shown in Figures 58 and 59. Superposing these two as in the hinge, we get Figure 60.[35]

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[34] In $(A'B')$, the numbers are repeated. This is due to the fact that the *mūlapaṅkti* and the *parapaṅkti* in this case are the same.

[35] This square is practically the same as Frost's "Nasik Square". (W. S. Andrews, *l.c.*, p. 175, Fig. 288).

2	3	2	3
1	4	1	4

Figure 48

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Figure 49: *Chādaka.*

(B)

5	0	10	15
10	15	5	0
5	0	10	15
10	15	5	0

Figure 50: *Chādya.*

(AB)

8	2	13	17
14	16	9	1
7	3	12	18
11	19	6	4

Figure 51: Total = 40.

(A')

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Figure 52

(B')

6	3	9	12
9	12	6	3
6	3	9	12
9	12	6	3

Figure 53

$(A'B')$

9	5	12	14
13	13	10	4
8	6	11	15
10	16	7	7

Figure 54

(A)

2	3	2	3
1	4	1	4
3	2	3	2
4	1	4	1

Figure 55

(B)

9	0	18	27
18	27	9	0
9	0	18	27
18	27	9	0

Figure 56

(AB)

12	2	21	29
22	28	13	1
11	3	20	30
19	31	10	4

Figure 57: Total = 64.

4	5	4	5	4	5	4	5
3	6	3	6	3	6	3	6
2	7	2	7	2	7	2	7
1	8	1	8	1	8	1	8
5	4	5	4	5	4	5	4
6	3	6	3	6	3	6	3
7	2	7	2	7	2	7	2
8	1	8	1	8	1	8	1

Figure 58

24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0
24	16	8	0	32	40	48	56
32	40	48	56	24	16	8	0

Figure 59

60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40
61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33

Figure 60: Total = 260.

Second method

> In as many 4×4 squares as are present in the *samagarbha* ($4n \times 4n$ square) such as 8×8 square, etc., write the numbers produced in the series, as in the method of the 4×4 square by right and left (knight's) moves. Thus is said the easy method of constructing *samagarbha* ($4n \times 4n$) squares such as 8×8 square, etc.[36]

Example from Nārāyaṇa:

(i) To construct a 8×8 square with total 260.

It is easily seen that the series of natural numbers from 1 to 64 is to be used. Writing the numbers 1 to 64 in groups of 4, we have

	1	8	9	16	48	41	40	33	
	2	7	10	15	47	42	39	34	
(I)	3	6	11	14	46	43	38	35	(III)
	4	5	12	13	45	44	37	36	
	32	25	24	17	49	56	57	64	
(II)	31	26	23	18	50	55	58	63	(IV)
	30	27	22	19	51	54	59	62	
	29	28	21	20	52	53	60	61	

Interchanging the figures in the third and fourth columns, as in the method of filling 4×4 squares, we get

	1	8	16	9	48	41	33	40	
I	2	7	15	10	47	42	34	39	III
	3	6	14	11	46	43	35	30	
	4	5	13	12	45	44	36	37	
	32	25	17	24	49	56	64	57	
	31	26	18	23	50	55	63	58	
II	30	27	19	22	51	54	62	59	IV
	29	28	20	21	52	53	61	60	

Taking the first rows of I and II to fill the first 4×4 square, the second rows to fill the second and so on, we get Figure 61.

[36] Rules 30–31.

1	**32**			**2**	**31**		
		8	**25**			**7**	**26**
16	**17**			**15**	**18**		
		9	**24**			**10**	**23**
4	**29**			**3**	**30**		
		5	**28**			**6**	**27**
13	**20**			**14**	**19**		
		12	**21**			**11**	**22**

Figure 61

1	**32**	49	48	**2**	**31**	50	47
56	41	**8**	**25**	55	42	**7**	**26**
16	**17**	64	33	**15**	**18**	63	34
57	40	**9**	**24**	58	39	**10**	**23**
4	**29**	52	45	**3**	**30**	51	46
53	44	**5**	**28**	54	43	**6**	**27**
13	**20**	61	36	**14**	**19**	62	35
60	37	**12**	**21**	59	38	**11**	**22**

Figure 62: Total = 260.

1	2*	3	4	5*	**6**
12*	**11**	10	9	**8**	7*
13*	**14**	15	**16**	**17**	18*
24*	**23**	22	**21**	**20**	19*
25*	**26**	**27**	28	**29**	30*
36	35*	**34**	33	32*	**31**

Figure 63

Then taking the first rows of III and IV to fill the remaining cells of the first 4×4 square, the second rows to fill the remaining cells of the second 4×4 square and so on, we get Figure 62.[37]

5.9 *Viṣamagarbha* squares

Nārāyaṇa gives two methods of construction of the $(4n+2) \times (4n+2)$ squares.

First method

This method is described by Nārāyaṇa thus:

> The measure of the *śliṣṭa*[38] cells in half of half the "number of the square" minus one. All over the square write down the numbers in connected cells in the direct and inverse order, one below the other (in rows). The numbers standing in the middle two columns above and below the middle two rows, excepting those in the last but one column below, should be interchanged (by one place anticlockwise turning). Then the two middle numbers in the extreme right of the right half of the square should be interchanged with the corresponding ones of the left half of the square, which lie attached to the diagonal. Finally the numbers of the *śliṣṭa* cells in the upper and lower halves of the square should be interchanged symmetrically. Such is the procedure of filling the cells with numbers by the method of *śliṣṭa* cells. The numbers in the cells attached to the diagonal in the right half of the square should be left as they are. Others may be interchanged if necessary to make up the total. This is the method of constructing the *viṣamagarbha-bhadra* taught by Nārāyaṇa.[39]

Examples from Nārāyaṇa:

(i) To construct a 6×6 square with total 111.

It is easily seen that the series of natural numbers from 1 to 36 is to be used.

The measure of the *śliṣṭa* cells $= \dfrac{(3-1)}{2} = 1$.

[37] It will observed that all groups of 4 cells have the same total, except the groups included within the thick lines. If we interchange the third and fourth 4×4 squares, we get an 8×8 square in which all groups of 4 cells excepting the centre group have the total 130.

[38] The *śliṣṭa* cells are cells not belonging to the diagonal and lying in the two vertical halves of the square. These cells are counted from the boundary inwards as will appear from the examples given. The number of such cells in a $(4n+2) \times (4n+2)$ square is n cells on the right and n cells on the left of each row.

[39] Rules 32–36.

The numbers 1 to 36 are placed in the 6×6 square in the direct and inverse order, as in Figure 63. There is only one *śliṣṭa* cell in each half row. These lie at the ends and are marked by asterisks.

The numbers in the two middle columns lying above and below the two middle rows excepting those in the last but one co-column below are interchanged (by one place anticlockwise turning) as shown below. The numbers in the extreme right cells of the two middle rows are interchanged with the corresponding ones of the left half of the square. This gives Figures 64 and 65.

The numbers standing in the *śliṣṭa* cells above are then interchanged with the corresponding ones below, giving Figure 66.

(ii) To fill a 10×10 square with the natural numbers 1 to 100.

In this case, the total $= \dfrac{100 \times 101}{2 \times 10} = 505$. Placing the numbers 1 to 100 in a 10×10 square, we get Figure 67.

Interchanging the numbers in the middle columns as directed and also those in the extreme right cells of the two middle rows with the corresponding ones of the left half of the square, we get Figure 68. Then interchanging the numbers in the *śliṣṭa* cells (marked by asterisks) as before we have the required square (Figure 69).

(iii) To construct a 14×14 square with the series of natural numbers.

The numbers filled continuously in a 14×14 square and then interchanged according to Nārāyaṇa's rule give Figures 70 and 71.

Remarks

It will be observed that when the series employed is in A.P., the squares are constructed by making the minimum interchanges expressly stated by Nārāyaṇa. That this is so in all cases is illustrated by the 18×18 squares constructed according to this method (Figures 72 and 73).

When, however, the series to be used is a broken series, other changes have to be made. For instance, to construct a 6×6 square with total 132, one may take the series of initial terms 2, 9, 16, 23, 30, 37 and common-difference 1. Proceeding according to the rule, we get Figure 74. But in this square, the third and fourth rows do not have the desired total. We, therefore, replace the initial numbers 28 and 16 of the third and fourth rows by 29 and 15 respectively and thus we get the magic square shown in Figure 75.

In this magic square, no number has been repeated. Replacement of the numbers 17 and 27 (standing in the third and fourth rows) by 18 and 26, or 20 and 24 by 21 and 23, or 26 and 18 by 27 and 17 will also yield magic squares with total 132, but there will be repetitions of two numbers.

Nārāyaṇa, however, gives Figure 74 as a 6×6 magic square with total 132. But as pointed out above, it is truly speaking not a magic square.

1	*	4	33	*	**6**
*	**11**	9	28	**8**	*
*	**14**	15	**16**	**17**	18*
*	**23**	22	**21**	**20**	19*
*	**26**	**27**	10	**29**	*
36	*	**34**	3	*	**31**

Figure 64

1	*	4	33	*	**6**
*	**11**	9	28	**8**	*
*	**14**	18	**16**	**17**	15*
*	**23**	19	**21**	**20**	22*
*	**26**	**27**	10	**29**	*
36	*	**34**	3	*	**31**

Figure 65

1	35	4	33	32	**6**
25	**11**	9	28	**8**	30
24	**14**	18	**16**	**17**	22
13	**23**	19	**21**	**20**	15
12	**26**	**27**	10	**29**	7
36	2	**34**	3	5	**31**

Figure 66: Total = 111.

1	2	3	**4**	5	6	**7**	8	9	**10**
20	**19**	18	**17**	16	15	**14**	13	**12**	11
21	22	**23**	**24**	25	26	**27**	**28**	29	30
40	39	**38**	**37**	36	35	**34**	**33**	32	31
41	42	**43**	**44**	45	**46**	**47**	**48**	49	50
60	59	**58**	**57**	56	**55**	**54**	**53**	52	51
61	62	**63**	**64**	**65**	66	**67**	**68**	69	70
80	79	**78**	**77**	**76**	75	**74**	**73**	72	71
81	**82**	83	**84**	**85**	86	**87**	88	**89**	90
100	99	98	**97**	**96**	95	**94**	93	92	**91**

Figure 67

1	*	*	**4**	6	95	**7**	*	*	**10**
*	**19**	*	**17**	15	86	**14**	*	**12**	*
*	*	**23**	**24**	26	75	**27**	**28**	*	*
*	*	**38**	**37**	35	66	**34**	**33**	*	*
*	*	**43**	**44**	50	**46**	**47**	**48**	*	45*
*	*	**58**	**57**	51	**55**	**54**	**53**	*	56*
*	*	**63**	**64**	**65**	36	**67**	**68**	*	*
*	*	**78**	**77**	**76**	25	**74**	**73**	*	*
*	**82**	*	**84**	**85**	16	**87**	*	**89**	*
100	*	*	**97**	**96**	5	**94**	*	*	**91**

Figure 68

1	99	98	**4**	6	95	**7**	93	92	**10**
81	**19**	83	**17**	15	86	**14**	88	**12**	90
80	79	**23**	**24**	26	75	**27**	**28**	72	71
61	62	**38**	**37**	35	66	**34**	**33**	69	70
60	59	**43**	**44**	50	**46**	**47**	**48**	52	56
41	42	**58**	**57**	51	**55**	**54**	**53**	49	45
40	39	**63**	**64**	**65**	36	**67**	**68**	32	31
21	22	**78**	**77**	**76**	25	**74**	**73**	29	30
20	**82**	18	**84**	**85**	16	**87**	13	**89**	11
100	2	3	**97**	**96**	5	**94**	8	9	**91**

Figure 69: Total = 505.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
28	27	26	25	24	23	22	21	20	19	18	17	16	15
29	30	31	32	33	34	35	36	37	38	39	40	41	42
56	55	54	53	52	51	50	49	48	47	46	45	44	43
57	58	59	60	61	62	63	64	65	66	67	68	69	70
84	83	82	81	80	79	78	77	76	75	74	73	72	71
85	86	87	88	89	90	91	92	93	94	95	96	97	98
112	111	110	109	108	107	106	105	104	103	102	101	100	99
113	114	115	116	117	118	119	120	121	122	123	124	125	126
140	139	138	137	136	135	134	133	132	131	130	129	128	127
141	142	143	144	145	146	147	148	149	150	151	152	153	154
168	167	166	165	164	163	162	161	160	159	158	157	156	155
169	170	171	172	173	174	175	176	177	178	179	180	181	182
196	195	194	193	192	191	190	189	188	187	186	185	184	183

Figure 70: Key-square.

1	195	194	193	**5**	**6**	8	189	**9**	**10**	186	185	184	**14**
169	**27**	171	172	**24**	**23**	21	176	**20**	**19**	179	180	**16**	182
168	167	**31**	165	**33**	**34**	36	161	**37**	**38**	158	**40**	156	155
141	142	143	**53**	**52**	**51**	49	148	**48**	**47**	**46**	152	153	154
140	139	138	**60**	**61**	**62**	64	133	**65**	**66**	**67**	129	128	127
113	114	115	**81**	**80**	**79**	77	120	**76**	**75**	**74**	124	125	126
112	111	110	**88**	**89**	**90**	98	**92**	**93**	**94**	**95**	101	100	106
85	86	87	**109**	**108**	**107**	99	**105**	**104**	**103**	**102**	96	97	91
84	83	82	**116**	**117**	**118**	**119**	78	**121**	**122**	**123**	73	72	71
57	58	59	**137**	**136**	**135**	**134**	63	**132**	**131**	**130**	68	69	70
56	55	54	**144**	**145**	**146**	**147**	50	**149**	**150**	**151**	45	44	43
29	30	**166**	32	**164**	**163**	**162**	35	**160**	**159**	39	**157**	41	42
28	**170**	26	25	**173**	**174**	**175**	22	**177**	**178**	18	17	**181**	15
196	2	3	4	**192**	**191**	**190**	7	**188**	**187**	11	12	13	**183**

Figure 71: Total = 1379.

If we use the series of initial terms 1, 7, 13, 26, 32, 38 and common-difference 1, and proceed as above, we shall get Figure 76.

Here also, the third and fourth rows do not have the desired total. But if we replace the numbers 31 and 13, in those rows, by 38 and 6, or 14 and 30 by 21 and 23, or 17 and 27 by 24 and 20, or 29 and 15 by 36 and 8 respectively, we shall get 4 magic squares with total 132. In two of these magic squares there will be no repetition of numbers, but in the other two there will be repetition of numbers.

Second method

> In the *viṣamagarbha* squares such as 6 × 6, etc. the two middle lines of cells (both horizontal and vertical) are called *pīṭha*. Fill the cells of the square with the numbers (of the given or chosen series) in the direct order. Reverse the number in the cells of each diagonal. Then interchange the numbers lying at the north-east corner between the diagonal and the *pīṭha* with the numbers in the (corresponding) opposite cells. Then interchange the two numbers at the south *pīṭha*, and also those at the west *pīṭha*. Thus will be obtained the desired total in the horizontal and vertical outskirts of the square. The interchange of the numbers in the other cells should be made as required in order to make up the total by noting the deficit or excess from it.[40]

Examples:

(i) To construct a 6 × 6 square with the series of natural numbers.

The numbers are placed in the square in the direct order as in Figure 77. In the above, the *pīṭhas* ("central rows and columns") are marked by thick lines. The directions are indicated by the letters E, N, W and S. The numbers in the diagonal cells are reversed. The numbers 2 and 7 lying at the north-east corner between the diagonal cells and the *pīṭha* cells are interchanged with the numbers 32 and 12, respectively, which lie in the corresponding opposite cells. Then the numbers 18 and 24 at the south *pīṭha* are interchanged; so also are interchanged the numbers 33 and 34 at the west *pīṭha*. Thus, we have Figure 78, in which the totals of the bounding rows and columns are as desired. The sums of the diagonal cells are also as desired. The other numbers should now be interchanged by trial to get the desired total 111. The squares shown in Figures 79 and 80 result.

(ii) To construct a 10 × 10 square with the series of natural numbers.

The above process gives the square shown in Figure 81.

[40] Rules 37–39.

1	2	3	4	5	**6**	**7**	**8**	9	10	**11**	**12**	**13**	14	15	16	17	**18**
36	**35**	34	33	32	**31**	**30**	**29**	28	27	**26**	**25**	**24**	23	22	21	**20**	19
37	38	**39**	40	41	**42**	**43**	**44**	45	46	**47**	**48**	**49**	50	51	**52**	53	54
72	71	70	**69**	68	**67**	**66**	**65**	64	63	**62**	**61**	**60**	59	**58**	57	56	55
73	74	75	76	**77**	**78**	**79**	**80**	81	82	**83**	**84**	**85**	**86**	87	88	89	90
108	107	106	105	**104**	**103**	**102**	**101**	100	99	**98**	**97**	**96**	**95**	94	93	92	91
109	110	111	112	**113**	**114**	**115**	**116**	117	118	**119**	**120**	**121**	**122**	123	124	125	126
144	143	142	141	**140**	**139**	**138**	**137**	136	135	**134**	**133**	**132**	**131**	130	129	128	127
145	146	147	148	**149**	**150**	**151**	**152**	153	**154**	**155**	**156**	**157**	**158**	159	160	161	162
180	179	178	177	**176**	**175**	**174**	**173**	172	**171**	**170**	**169**	**168**	**167**	166	165	164	163
181	182	183	184	**185**	**186**	**187**	**188**	**189**	190	**191**	**192**	**193**	**194**	195	196	197	198
216	215	214	213	**212**	**211**	**210**	**209**	**208**	207	**206**	**205**	**204**	**203**	202	201	200	199
217	218	219	220	**221**	**222**	**223**	**224**	**225**	226	**227**	**228**	**229**	**230**	231	232	233	234
252	251	250	249	**248**	**247**	**246**	**245**	**244**	243	**242**	**241**	**240**	**239**	238	237	236	235
253	254	255	**256**	257	**258**	**259**	**260**	**261**	262	**263**	**264**	**265**	266	**267**	268	269	270
288	287	**286**	285	284	**283**	**282**	**281**	**280**	279	**278**	**277**	**276**	275	274	**273**	272	271
289	**290**	291	292	293	**294**	**295**	**296**	**297**	298	**299**	**300**	**301**	302	303	304	**305**	306
324	323	322	321	320	**319**	**318**	**317**	**316**	315	**314**	**313**	**312**	311	310	309	308	**307**

Figure 72: Key-square.

1	323	322	321	320	**6**	**7**	**8**	10	315	**11**	**12**	**13**	311	310	309	308	**18**
289	**35**	291	292	293	**31**	**30**	**29**	27	298	**26**	**25**	**24**	302	303	304	**20**	306
288	287	**39**	285	284	**42**	**43**	**44**	46	279	**47**	**48**	**49**	275	274	**52**	272	271
253	254	255	**69**	257	**67**	**66**	**65**	63	262	**62**	**61**	**60**	266	**58**	268	269	270
252	251	250	249	**77**	**78**	**79**	**80**	82	243	**83**	**84**	**85**	**86**	238	237	236	235
217	218	219	220	**104**	**103**	**102**	**101**	99	226	**98**	**97**	**96**	**95**	231	232	233	234
216	215	214	213	**113**	**114**	**115**	**116**	118	207	**119**	**120**	**121**	**122**	202	201	200	199
181	182	183	184	**140**	**139**	**138**	**137**	135	190	**134**	**133**	**132**	**131**	195	196	197	198
180	179	178	177	**149**	**150**	**151**	**152**	162	**154**	**155**	**156**	**157**	**158**	166	165	164	172
145	146	147	148	**176**	**175**	**174**	**173**	163	**171**	**170**	**169**	**168**	**167**	159	160	161	153
144	143	142	141	**185**	**186**	**187**	**188**	**189**	136	**191**	**192**	**193**	**194**	130	129	128	127
109	110	111	112	**212**	**211**	**210**	**209**	**208**	117	**206**	**205**	**204**	**203**	123	124	125	126
108	107	106	105	**221**	**222**	**223**	**224**	**225**	100	**227**	**228**	**229**	**230**	94	93	92	91
73	74	75	76	**248**	**247**	**246**	**245**	**244**	81	**242**	**241**	**240**	**239**	87	88	89	90
72	71	70	**256**	68	**258**	**259**	**260**	**261**	64	**263**	**264**	**265**	59	**267**	57	56	55
37	38	**286**	40	41	**283**	**282**	**281**	**280**	45	**278**	**277**	**276**	50	51	**273**	53	54
36	**290**	34	33	32	**294**	**295**	**296**	**297**	28	**299**	**300**	**301**	23	22	21	**305**	19
324	2	3	4	5	**319**	**318**	**317**	**316**	9	**314**	**313**	**312**	14	15	16	17	**307**

Figure 73: Total = 2925.

2	41	5	39	38	7
30	13	11	33	10	35
28	17	21	19	20	26
16	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Figure 74

2	41	5	39	8	7
30	13	11	33	10	35
29	17	21	19	20	26
15	27	23	25	24	18
14	31	32	12	34	9
42	3	40	4	6	37

Figure 75: Total = 132.

1	42	4	40	39	6
32	11	9	35	8	37
31	14	18	16	17	29
13	30	26	28	27	15
12	33	34	10	36	7
43	2	41	3	5	38

Figure 76

5.10 *Viṣama* squares

Nārāyaṇa gives two methods for construction of the *viṣamabhadra* (“odd squares”). The first of these is Nārāyaṇa’s own method, the method of superposition, which was rediscovered in the west by M. de la Hire (1705). The second method seems to have been known in India before Nārāyaṇa.

First method

> Determine the *mūlapaṅkti* and *guṇapaṅkti* in the way indicated before. The first term of the former should be placed in the centre cell of the top row of the first of the (*chādya* and *chādaka*) squares. Beneath it should be written down vertically the successive terms of the series. The other columns should be filled similarly, so that the numbers in the top row are in order. In the same way, beginning with the first term of the second series fill up the second square. The method of superposition of the *chādya* (“one to be covered”) and *chādaka* (“covering one”) is as before.[41]

[41] Rules 41–42.

E

	1	2	3	4	5	6	
	7	8	9	10	11	12	
N	13	14	15	16	17	18	**S**
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	

W

Figure 77

36	32	3	4	5	31
12	29			26	7
13		22	21		24
19		16	15		18
25	11			8	30
6	2	34	33	35	1

Figure 78

36	32	3	4	5	31
12	29	27	10	26	7
13	17	22	21	14	24
19	20	16	15	23	18
25	11	9	28	8	30
6	2	34	33	35	1

Figure 79: Total = 111.

36	32	3	4	5	31
12	29	9	28	26	7
13	14	22	21	17	24
19	23	16	15	20	18
25	11	27	10	8	30
6	2	34	33	35	1

Figure 80: Total = 111.

100	92	93	94	5	6	7	8	9	91
20	89	88	14	16	15	87	83	82	11
30	29	78	77	75	26	74	73	22	21
40	39	63	67	65	66	64	38	32	31
41	49	48	54	56	55	57	43	42	60
51	52	53	47	46	45	44	58	59	50
61	62	33	37	35	36	34	68	69	70
71	72	28	27	25	76	24	23	79	80
81	19	18	84	86	85	17	13	12	90
10	2	3	4	96	95	97	98	99	1

Figure 81

Examples from Nārāyaṇa:

(i) To construct a 3×3 square with total 24.

Assume the *mūlapaṅkti* ("basic series") to be 1, 2, 3. Let the *parapaṅkti* ("second series") be 0, 1, 2. Then the multiplying factor is

$$\frac{24 - (1 + 2 + 3)}{(0 + 1 + 2)} = 6.$$

Therefore, the *guṇapaṅkti* is 0, 6, 12.

Now, filling the *chādaka* ("one to be covered") square with the *mūlapaṅkti* and *chādaka* ("covering one") with the *guṇapaṅkti* as directed in the rule, we get Figures 82 and 83. Superposing these as in a hinge, we have the required square (Figure 84).

(ii) To construct a 5×5 square with total 90.

Let the *mūlapaṅkti* be 1, 2, 3, 4, 5. Also let the *parapaṅkti* be 1, 2, 3, 4, 5. Then the multiplier is

$$\frac{90 - (1 + 2 + 3 + 4 + 5)}{1 + 2 + 3 + 4 + 5} = 5.$$

Therefore, the *guṇapaṅkti* is 5, 10, 15, 20, 25. Filling the squares as before, we have Figures 85–87.

(iii) To construct a 7×7 square with total 238.

Here, taking the *mūlapaṅkti* as 1, 2, 3, 4, 5, 6, 7, and the *parapaṅkti* as 0, 1, 2, 3, 4, 5, 6, the *guṇapaṅkti* is 0, 10, 20, 30, 40, 50, 60. The squares obtained as above are shown in Figures 88–90.

Second method

> In the first cell of a middle line (of cells) write the first term of the series of numbers, and in the cell beside the opposite cell of the same line (write) the next number. Then, in the cells lying along the shorter diagonal from that write the following numbers. (On reaching an extremity) continue the filling beginning with the cell of the opposite line which will be diagonally in front (considering the square to be rolled on a cylinder). When the next diagonal cell is found to be already filled up, begin from the cell behind and fill successively (in the same way). In the *viṣamabhadra* there will be eight varieties."[42]

Examples from Nārāyaṇa:

(i) To construct a 3×3 square with the series of natural numbers.

[42] Rules 43–45.

3	1	2
1	2	3
2	3	1

Figure 82

12	0	6
0	6	12
6	12	0

Figure 83

9	1	14
13	8	3
2	15	7

Figure 84: Total = 24.

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

Figure 85: *Chādya.*

20	25	5	10	15
25	5	10	15	20
5	10	15	20	25
10	15	20	25	5
15	20	25	5	10

Figure 86: *Chādaka.*

19	15	6	27	23
25	16	12	8	29
26	22	18	14	10
7	28	24	20	11
13	9	30	21	17

Figure 87: Total = 90.

5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3

Figure 88: *Chādya.*

40	50	60	0	10	20	30
50	60	0	10	20	30	40
60	0	10	20	30	40	50
0	10	20	30	40	50	60
10	20	30	40	50	60	0
20	30	40	50	60	0	10
30	40	50	60	0	10	20

Figure 89: *Chādaka.*

35	26	17	1	62	53	44
46	37	21	12	3	64	55
57	41	32	23	14	5	66
61	52	43	34	25	16	7
2	63	54	45	36	27	11
13	4	65	56	47	31	22
24	15	6	67	51	42	33

Figure 90: Total = 238.

8	1	6
3	5	7
4	9	2

Figure 91: Total = 15.

6	1	8
7	5	3
2	9	4

Figure 92: Total = 15.

7	5	12
13	8	3
4	11	9

Figure 93: Total = 24.

Writing 1 at the top of the middle line (column), 2 in the last cell of the next column and proceeding diagonally upwards we have Figure 91. In the above, whenever a block occurs, we begin with the cell underneath.

Another filling would be as per Figure 92. As the filling can be started by placing the first term in any one of the four centre cells of the outskirts, there will be altogether 8 different squares, as stated by Nārāyaṇa.

(ii) To construct a 3×3 square with total 24.

Nārāyaṇa uses an irregular series for filling up the square. According to the method for finding out such series, we get 3, 7, 11 as the initial terms of the *caraṇas*, the common difference being 1. The numbers to be filled are, therefore,

3, 4, 5
7, 8, 9
11, 12, 13

Hence the magic square is as in Figure 93.

Note: In this and the following squares, the filling begins from the extreme right cell of the middle row.

(iii) To construct a 5×5 square with total 90.

Here, the initial terms of the *caraṇas* are found to be 4, 10, 16, 22, 28, the common-difference being unity. The square is shown in Figure 94.

(iv) To construct a 7×7 square with total 238.

In this case, the initial terms of the *caraṇas* may be taken as 7, 15, 23, 31, 39, 47, 55, the common-difference being unity. The square is shown in Figure 95.

The magic squares constructed by the above method are such that the sum of any two numbers that are geometrically equidistant from the centre is equal to twice the centre number. Such squares are called perfect by W. S. Andrews.[43]

5.11 Other magic squares

Nārāyaṇa says:

> With the help of 4×4 magic squares filled by natural numbers 1, etc. construct a magic rectangle or $4n \times 4n$ magic square. From it

[43]See W. S. Andrews, *l.c.*, p. 1 ff.

16	14	7	30	23
24	17	10	8	31
32	25	18	11	4
5	28	26	19	12
13	6	29	22	20

Figure 94: Total = 90.

31	29	20	11	58	49	40
41	32	23	21	12	59	50
51	42	33	24	15	13	60
61	52	43	34	25	16	7
8	55	53	44	35	26	17
18	9	56	47	45	36	27
28	19	10	57	48	39	37

Figure 95: Total = 238.

one can always construct other magic figures. Lines drawn through the corners in any desired way so as always to keep the number of cells the same give rise to the figures of *vitāna* ("canopy"), *maṇḍapa* ("altar"), *vajra* ("diamond"), etc. Those are *saṅkīrṇa-bhadra* ("other magic figures"). By the meeting together of lines between two cells and two diagonals are produced bases and uprights of triangle-pairs in all directions. Here, the triangles are filled with the numbers of a magic rectangle produced by $4n \times 4n$ squares, first in the direct order and then in the inverse order and so on. Such is the method of filling magic figures.[44]

Besides the three types of magic figures mentioned above, Nārāyaṇa has given rules for the construction of many other types of figures with illustration. These figures will be given and their peculiarities pointed out. The rules regarding their constructions will not be given, as they are apparent from the figures.

Vitāna **("Canopy")**

The figure is as shown in Figure 96.

This is a rectangle constructed with the natural numbers 1 to 32 and consists of two 4×4 squares. The numbers are filled according to the method of $4n \times 4n$ squares given before.[45] It will be observed that the total of each row in the above is 132 and that of each column is 66.

For another magic rectangle constructed with the natural numbers 1 to 48 and consisting of three squares, see below (Figure 105).

Maṇḍapa **("Altar")**

The figure is as shown in Figure 97.

[44] Rules 46–49.
[45] Rules 30–31.

Here the numbers of the magic rectangle (Figure 96) have been used by taking them successively in rows. Here, any set of eight numbers occurring together,[46] horizontally, vertically or diagonally, gives the total 132. The eight numbers lying in a square have the same total 132. There is cylindrical symmetry, i.e., if the figure be rolled on a cylinder, any continuous eight numbers or those lying in a square give the total 132. It is easy to find 26 sets of eight numbers having the same total 132.

Vajra ("*Diamond*")

The figure (Figure 98) is constructed from the magic rectangle in Figure 96. Any eight numbers lying together in the same line, as well as the vertical diagonal, have the same sum 132. The sum of two horizontal rows, one in the upper half of the square and the other in the lower half, together containing eight numbers is 132. The sum of eight numbers lying in a small square is 132. In this case, it is easy to find 32 sets of eight numbers having the same total.

Padma ("*Lotus*")

The figure (Figure 99) is constructed from the rectangle in Figure 96. Any set of eight numbers taken vertically, horizontally (along lines side by side) or in any four leaves symmetrically situated give the same total 132. There is cylindrical symmetry. In this case, 32 sets of eight numbers having the same total can be easily picked out.

Vajra ("*Diamond*")

The *vajra* ("*diamond*") (Figure 100) uses the numbers of the 8×8 magic square in Figure 62. In the above groups of 16 numbers with the same total 520, groups of eight numbers with total 260 and also groups of four numbers with total 130 can be picked out easily. The sixteen numbers may be taken horizontally, vertically, and in two rings etc. Groups of eight may be taken horizontally, vertically, diagonally, in rings, etc. Groups of four may be taken horizontally or vertically, as half rows or columns, in small squares, etc.

Maṇḍapa

The following *maṇḍapa* ("*altar*") (**ed.** see Figure 101) is constructed by using the numbers of the 8×8 magic square in Figure 62. It has groups of sixteen, eight and four numbers having equal totals, as in the *vajra*.

[46] i.e., any two lines of numbers that are side by side.

1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Figure 96: *Vitāna* or Canopy.

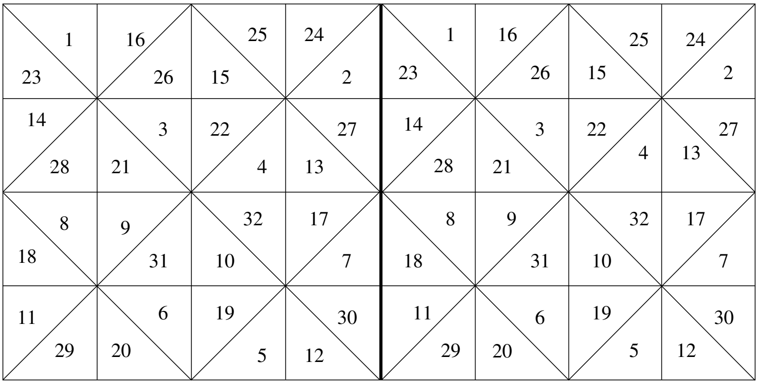

Figure 97: *Maṇḍapa* or Altar.

***Sarvatobhadra* ("*Perfect magic figure*")**

In this figure (Figures 102 and 103) constructed from the 8 × 8 magic square in Figure 62, the totals of all four, eight and sixteen numbers are 130, 260 and 520 respectively. The figure is perfectly continuous.

***Dvādaśakara* ("*Twelve hands*")**

The figure (Figure 104) is constructed by the numbers of the 12 × 4 rectangle (Figure 105) using the numbers 1 to 48. In the above figure (**ed.** see Figure 104) all groups of 12, of 8 or of 4 numbers have equal totals, 294, 196 and 98 respectively.

***Vajra Padma* ("*Diamond lotus*")**

The figure (Figure 106) is constructed with the numbers of the 12 × 4 rectangle given above (**ed.** see Figure 105). In this figure, every group of four numbers whether occurring in a line or cells has the total 98, every group of eight numbers has the total 196 and every group of 12 numbers taken horizontally, vertically or in a circle has total 294.

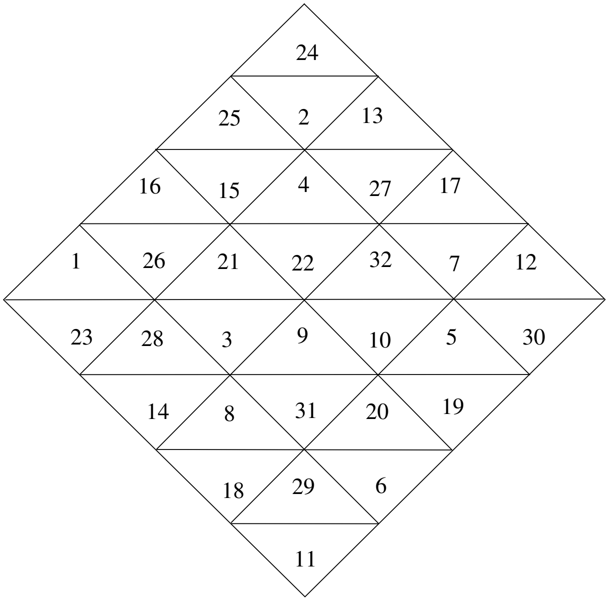

Figure 98: *Vajra* or Diamond.

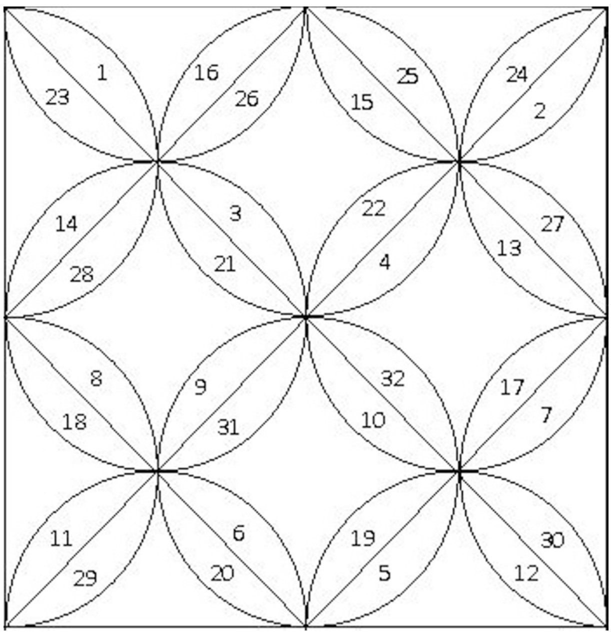

Figure 99: *Padma* or Lotus.

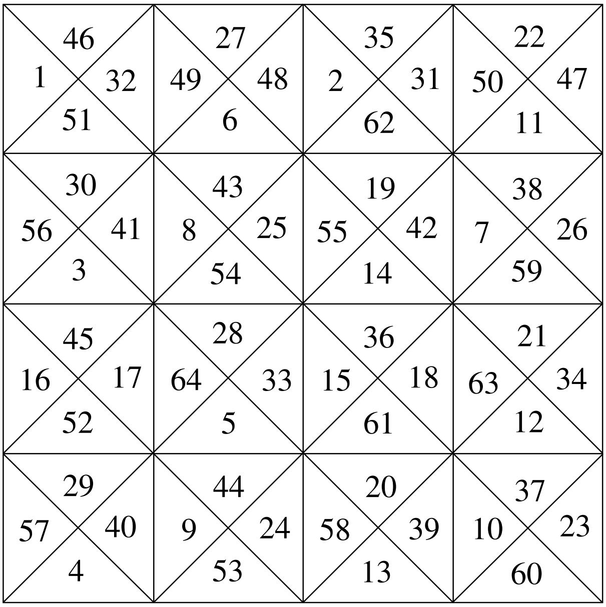

Figure 100: *Vajra* or Diamond.

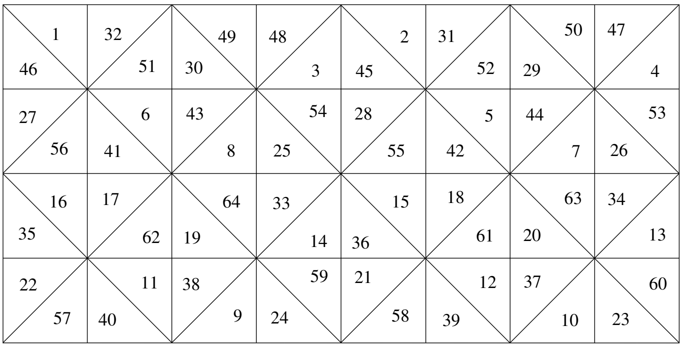

Figure 101: *Maṇḍapa.*

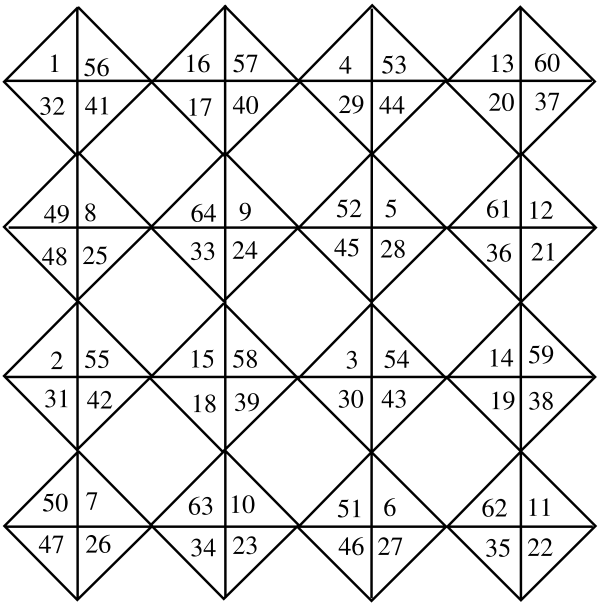

Figure 102: *Sarvatobhadra.*

1	56			16	57			4	53			13	60
32	41			17	40			29	44			20	37
49	8			64	9			52	5			61	12
48	25			33	24			45	28			36	21
2	55			15	58			3	54			14	59
31	42			18	39			30	43			19	38
50	7			63	10			51	6			62	11
47	26			34	23			47	27			35	22

Figure 103: *Sarvatobhadra.*

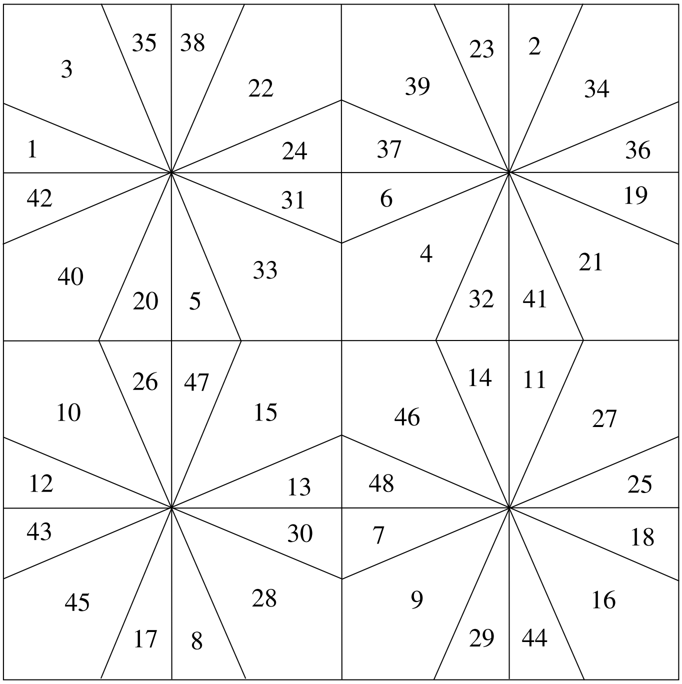

Figure 104: *Dvādaśakara.*

1	24	37	36	2	23	38	35	3	22	39	34
42	31	6	19	41	32	5	20	40	33	4	21
12	13	48	25	11	14	47	26	10	15	46	27
43	30	7	18	44	29	8	17	45	28	9	16

Figure 105

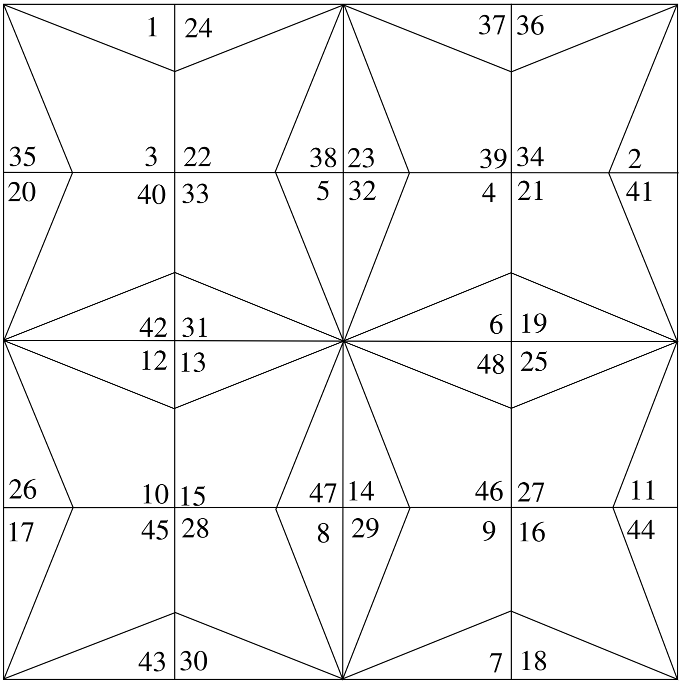

Figure 106: *Vajra Padma.*

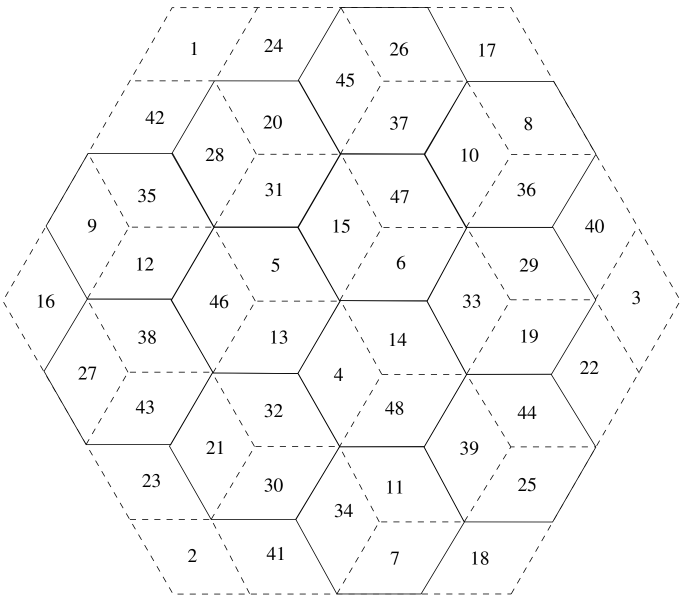

Figure 107: *Ṣaḍasra* or Hexagon.

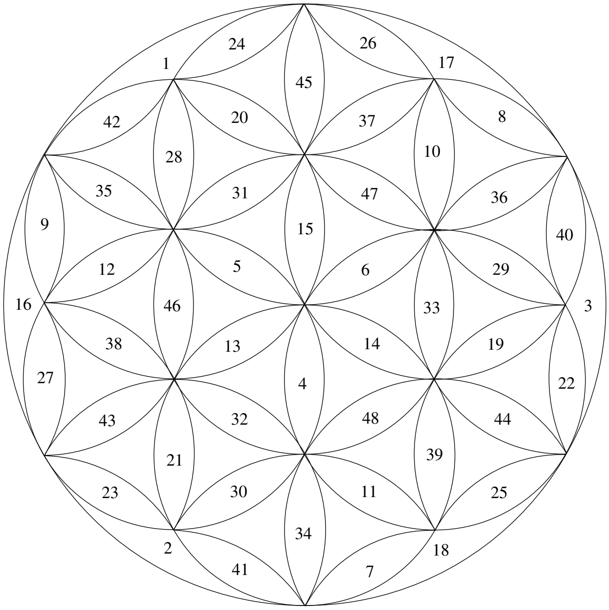

Figure 108: *Padmavṛtta* or Lotus circle.

Ṣaḍasra ("*Hexagon*")

The figure (Figure 107) is constructed with the numbers of the 12×4 rectangle (**ed.** see Figure 105). Every group of twelve numbers has the same sum 294.

Padma Vṛtta ("*Inscribed lotus*")

The figure (Figure 108) is constructed with the numbers of the 12×4 magic rectangle (**ed.** see Figure 105). Every group of twelve numbers has the same sum 294.

Magic triangle

Nārāyaṇa has proposed the problem of constructing a magic triangle (**ed.** see Figure 110) with total 400. His magic triangle is constructed with the help of the numbers of a magic square whose total is 225 (**ed.** see Figure 109).

The square is obtained by multiplying each of the numbers of a 3×3 square using the natural numbers by 15. It will be further observed that $(400-225) = 175$ is placed in the centre, so that the sum of each of the arms radiating from the centre may be 400.

120	15	90
45	75	105
60	135	30

Figure 109: Total = 225.

Magic cross

The figure of the magic cross given by Nārāyaṇa is shown in Figure 111. This cross has been made with the help of the numbers of the 4×4 square given in Figure 114. 94 has been placed in the centre to give the required total.

Magic circles

Nārāyaṇa has given a number of magic circles each with the total 400. These circles together with their key squares or rectangle are:

(i) Magic circle from a 3×3 square using the series whose first term is 15 and common-difference 15 (Figure 112 and 113).

(ii) Magic circle from a 4×4 square whose first term is 9 and common-difference 9 (Figures 114 and 115).

(iii) Magic circle from a 5×5 square whose first term is 4 and common-difference 4 (Figures 116 and 117).

(iv) Magic circle from a 6×6 square whose first term is 3 and common-difference 3 (Figures 118 and 119).

(v) Magic circle from a 8×4 rectangle using the natural numbers 1 to 32 (Figures 120 and 121).

6 Dharmanandana square

Dharmanandana, a Jaina scholar (circa fifteenth century) has given[47] the following 8×8 square[48] with total 260 (Figure 122).

The above square has been constructed by placing the natural numbers 1 to 64 in a 8×8 square in the direct order and then shifting the numbers so placed suitably. The square is divided into smaller squares of four cells each. The numbers in those squares that lie on the diagonals are unchanged, while

[47] The square occurs in the *catuḥṣaṣṭi-yoginī-maṇḍala-stuti* of Dharmanandana.
[48] This square is given in W. S. Andrews' book, *l.c.*, Figure 94, p. 43.

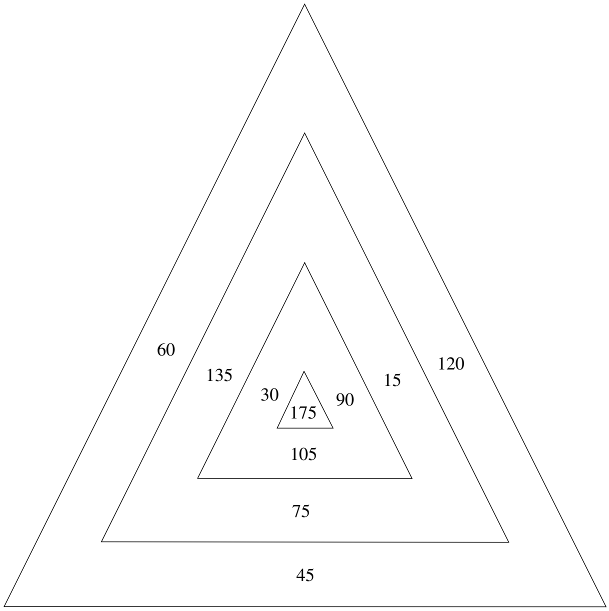

Figure 110: Total = 400.

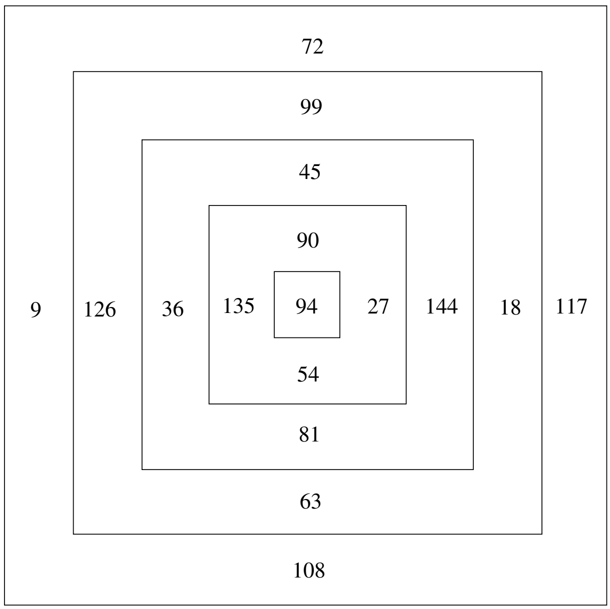

Figure 111: Total = 400.

120	45	60
15	75	135
90	105	30

Figure 112: Total = 225.

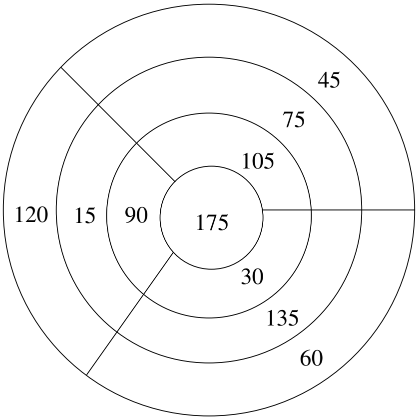

Figure 113: Total = 400.

9	72	117	108
126	99	18	63
36	45	144	81
135	90	27	54

Figure 114: Total = 306.

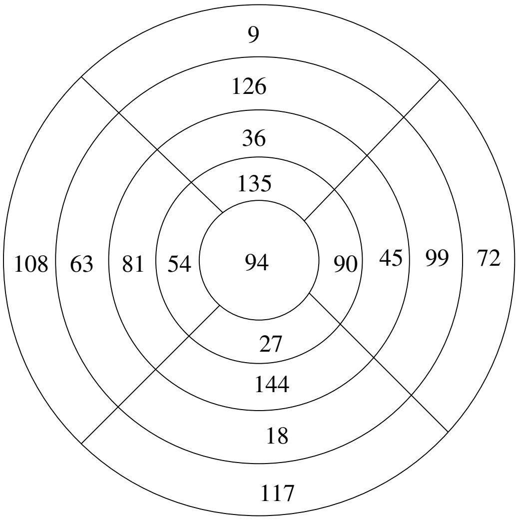

Figure 115: Total = 400.

68	92	16	40	44
96	20	24	48	72
4	28	52	76	100
32	56	80	84	8
60	64	88	12	36

Figure 116: Total = 260.

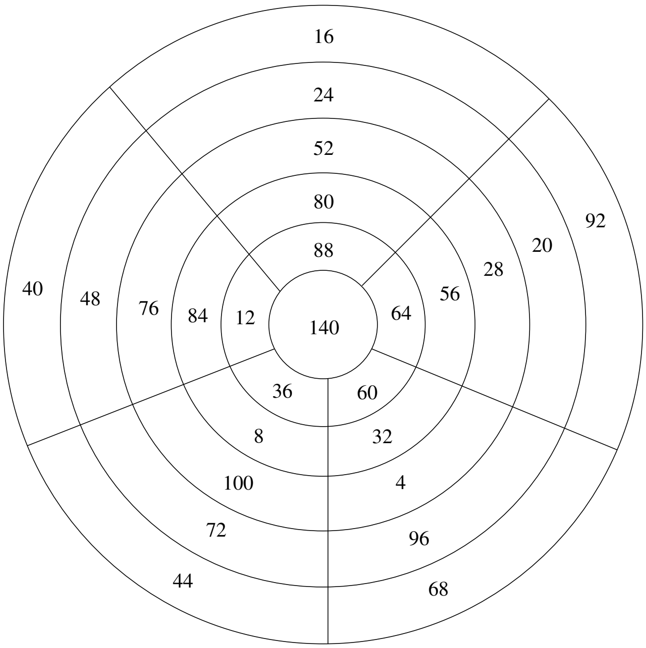

Figure 117: Total = 400.

3	105	12	99	96	18
75	33	27	84	24	90
72	42	54	48	51	66
39	69	57	63	60	45
36	78	81	30	87	21
108	6	102	9	15	93

Figure 118: Total = 333.

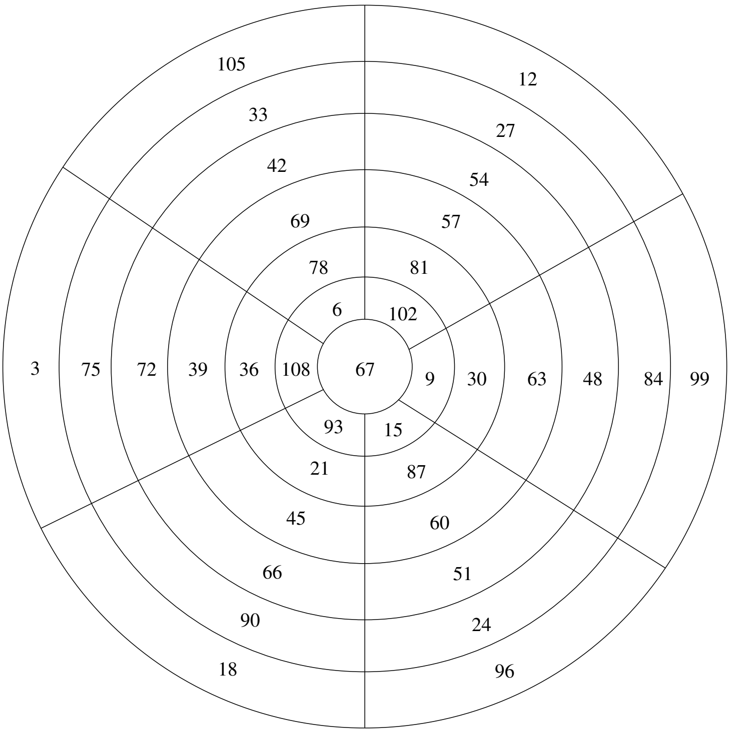

Figure 119: Total = 400.

1	16	25	24	2	15	26	23
28	21	4	13	27	22	3	14
8	9	32	17	7	10	31	18
29	20	5	12	30	19	6	11

Figure 120: Row total = 132, Column total = 66.

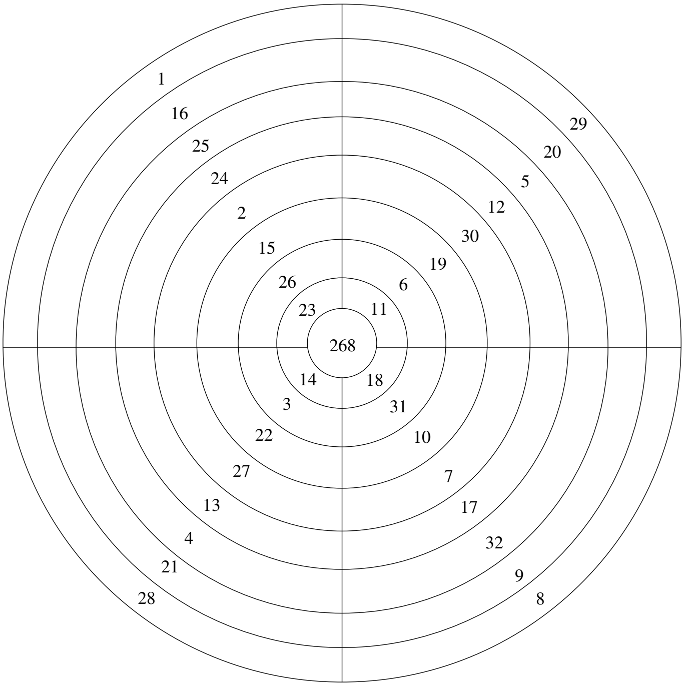

Figure 121: Total = 400.

8	**7**	59	60	61	62	**2**	**1**
16	**15**	51	52	53	54	**10**	**9**
41	42	**22**	**21**	**20**	**19**	47	48
33	34	**30**	**29**	**28**	**27**	39	40
25	26	**38**	**37**	**36**	**35**	31	32
17	18	**46**	**45**	**44**	**43**	23	24
56	**55**	11	12	13	14	**50**	**49**
64	**63**	3	4	5	6	**58**	**57**

Figure 122

8	**7**	6	5	4	3	**2**	**1**
16	**15**	14	13	12	11	**10**	**9**
24	23	**22**	**21**	**20**	**19**	18	17
32	31	**30**	**29**	**28**	**27**	26	25
40	39	**38**	**37**	**36**	**35**	34	33
48	47	**46**	**45**	**44**	**43**	42	41
56	**55**	54	53	52	51	**50**	**49**
64	**63**	62	61	60	59	**58**	**57**

Figure 123

1	**2**	**3**	4	5	6	7	8	9	**10**	**11**	**12**
13	**14**	**15**	16	17	18	19	20	21	**22**	**23**	**24**
25	**26**	**27**	28	29	30	31	32	33	**34**	**35**	**36**
37	38	39	**40**	**41**	**42**	**43**	**44**	**45**	46	47	48
49	50	51	**52**	**53**	**54**	**55**	**56**	**57**	58	59	60
61	62	63	**64**	**65**	**66**	**67**	**68**	**69**	70	71	72
73	74	75	**76**	**77**	**78**	**79**	**80**	**81**	82	83	84
85	86	87	**88**	**89**	**90**	**91**	**92**	**93**	94	95	96
97	98	99	**100**	**101**	**102**	**103**	**104**	**105**	106	107	108
109	**110**	**111**	112	113	114	115	116	117	**118**	**119**	**120**
121	**122**	**123**	124	125	126	127	128	129	**130**	**131**	**132**
133	**134**	**135**	136	137	138	139	140	141	**142**	**143**	**144**

Figure 124: Key-square.

1	**2**	**3**	141	140	139	138	137	136	**10**	**11**	**12**
13	**14**	**15**	129	128	127	126	125	124	**22**	**23**	**24**
25	**26**	**27**	117	116	115	114	113	112	**34**	**35**	**36**
108	107	106	**40**	**41**	**42**	**43**	**44**	**45**	99	98	97
96	95	94	**52**	**53**	**54**	**55**	**56**	**57**	87	86	85
84	83	82	**64**	**65**	**66**	**67**	**68**	**69**	75	74	73
72	71	70	**76**	**77**	**78**	**79**	**80**	**81**	63	62	61
60	59	58	**88**	**89**	**90**	**91**	**92**	**93**	51	50	49
48	47	46	**100**	**101**	**102**	**103**	**104**	**105**	39	38	37
109	**110**	**111**	33	32	31	30	29	28	**118**	**119**	**120**
121	**122**	**123**	21	20	19	18	17	16	**130**	**131**	**132**
133	**134**	**135**	9	8	7	6	5	4	**142**	**143**	**144**

Figure 125: Total = 870.

1	2	3	**4**
5	**6**	**7**	8
9	**10**	**11**	12
13	14	15	**16**

(a) Key-square.

1	15	14	**4**
12	**6**	**7**	9
8	**10**	**11**	5
13	3	2	**16**

(b) Total = 34.

Figure 126

4	9	2
3	5	7
8	1	6

Figure 127: Total = 15.

$n-8$	$n-1$	2	7
6	3	$n-4$	$n-5$
$n-2$	$n-7$	8	1
4	5	$n-6$	$n-3$

Figure 128: Total = $2n$.

8	15	2	7
6	3	12	11
14	9	8	1
4	5	10	13

Figure 129: Total = 32.

22	3	9	15	16
14	20	21	2	8
1	7	13	19	25
18	24	5	6	12
10	11	17	23	4

Figure 130: Total = 65.

those in the other squares are interchanged with the diagonally opposite ones. The manner of the change will be evident from the key square in Figure 123 in which the smaller squares that are not to be interchanged are marked by thick letters and thick boundaries.

Dharmanandana's method is quite general.[49] For instance, the 12×12 square shown in Figure 125 can be made by dividing the key square (Figure 124) into smaller squares of nine cells.

The 4×4 magic square (Figure 126) based on Dharmanandana's method is interesting as it is not included in Nārāyaṇa's squares (Figure 36).

7 Sundarasūri squares

Another Jaina scholar, Sundarasūri (circa fifteenth century), has given a number of interesting squares which have been constructed by novel methods.[50] An account of these squares is given below.

3×3 *square*: The filling of the 3×3 square as per Figure 127 is according to the traditional Hindu method, already noted in Nārāyaṇa's work.

4×4 *squares*: A 4×4 square with any desired even total may be constructed by giving particular values to n in Figure 128.

Sundarasūri exhibits the instance with total 32 as per Figure 129. In this figure, the number 8 occurs twice, because the total is less than 34, which is the least total for a 4×4 square constructed with a series of natural numbers.

Odd squares: Sundarasūri uses the elongated knight's move to obtain the 5×5 square shown in Figure 130.

[49]It is equally applicable to the smaller 4×4 square.

[50]These squares occur in a *stotra* by Sundarasūri

20	28	29	37	45	4	12
44	3	11	19	27	35	36
26	34	42	43	2	10	18
1	9	17	25	33	41	49
32	40	48	7	9	16	24
14	15	23	31	39	47	6
38	46	5	13	21	22	30

Figure 131: Total = 175.

The method of filling is: Put 1 in the extreme cell of the middle row; move two cells in front and one cell diagonally, and put down the next number 2 and so on. When a block occurs, put the next number in the adjoining cell in the direction of the move, and continue as before.[51]

The method can be easily generalised and is applicable to all odd squares. For filling up a $(2n + 1) \times (2n + 1)$ square the move to be used is n cells horizontally or vertically and one cell diagonally. When a block occurs, the next number is to be put down in front of the cell last filled in the direction of the move. By proceeding in this way, we obtain the required magic square. As an example, we give the 7×7 square as per Figure 131.

8×8 *square*: Sundarasūri gives the 8×8 square[52] shown in Figure 132.

It has been constructed by dividing symmetrically the following key-square into groups of four and two cells. The numbers that lie in groups standing on the diagonals remain unchanged, while those in the others are interchanged with the diagonally opposite ones. The method of division will be apparent from Figure 133 of the key-square.

Compound magic squares: Sundarasūri gives the 9×9 square shown in Figure 134.

The method of construction of the above square is apparent from the figure if we consider the square to be divided into nine smaller squares, as is done in the figure given above. It will be found that each of the smaller squares is a 3×3 magic square. Therefore, the method is: Divide the numbers 1–81 into 9 groups in order, and with these groups construct nine 3×3 squares. These nine squares, being numbered one to nine in order, are filled in the bigger square just as in the method of filling a 3×3 square with the numbers 1–9.[53]

[51] W. S. Andrews gives the above method (p. 4, Figure 5) and claims it as his own. He has been anticipated by Sundarasūri by several centuries.

[52] The same square has been given by W. S. Andrew, *l.c.*, Figure 53, p. 25. The square is perfect in all its characteristics. Sundarasūri's method can be generalised to obtain other squares.

[53] The above method is now attributed to Prof. Hermann Schubert (*cf.* W. S. Andrews, *l.c.*, Figure 96, p. 44). In India it was known several centuries earlier.

1	63	62	4	5	59	58	8
56	10	11	53	52	14	15	49
48	18	19	45	44	22	23	41
25	39	38	28	29	35	34	32
33	31	30	36	37	27	26	40
24	42	43	21	20	46	47	17
16	50	51	13	12	54	55	9
57	7	6	60	61	3	2	64

Figure 132: Total = 260.

1	2	3	**4**	**5**	6	7	**8**
9	**10**	**11**	12	13	**14**	**15**	16
17	**18**	**19**	20	21	**22**	**23**	24
25	26	27	**28**	**29**	30	31	**32**
33	34	35	**36**	**37**	38	39	**40**
41	**42**	**43**	44	45	**46**	**47**	48
49	**50**	**51**	52	53	**54**	**55**	56
57	58	59	**60**	**61**	62	63	**64**

Figure 133: Key-square.

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

Figure 134: Total = 369.

8 Concluding remarks

The foregoing pages would have shown to the reader that the Hindu achievements in the theory and construction of magic squares stand unsurpassed even up to the present day. The simplest square presenting any difficulty is the 4×4 square whose study began in India as early as the beginning of the Christian Era. The success obtained in constructing this square must have encouraged the consideration of larger squares. The construction of magic squares was not made a part of mathematics, as no theoretical treatment could be given in the earlier stages. There are, however, stray examples of the occurrence of magic squares from the beginning of the Christian Era right up to the time of Nārāyaṇa (1356).[54] A very elegant and satisfactory method for the construction of the 4×4 square was developed before the time of Nārāyaṇa. This method, which we may call the method of knight's move, gives us 384 magic squares, which are perfect and possess the characteristics of what are now called "Nasik squares". This method of construction will be new to western scholars of today.

Nārāyaṇa (1356), who undertook the study of these squares, obtained results which have been only recently found in the west by the efforts of several workers. Of his theoretical results, the most important is the demonstration of the fact that magic squares may be constructed with as many series or groups of numbers in A.P. as there are cells in a column. This result was first stated in the west by L. S. Frierson in the beginning of the present century.[55] Another very important feature of Nārāyaṇa's work is the division of magic squares into three types. In our opinion, the recent work done in the west suffers from considerable inelegance because of the absence of such classification.

Nārāyaṇa claims as his own the methods for construction of $4n \times 4n$ squares and odd squares by means of superposition, and also a method for the construction of $(4n+2) \times (4n+2)$ square. Methods for the construction of certain squares by means of superposition were devised by M de la Hire (1705).[56] Nārāyaṇa's methods, given more than six centuries earlier, are more elegant and practical, although theoretically there is little difference between the two. Nārāyaṇa's method for the construction $(4n + 2) \times (4n + 2)$ squares seems to be the only general method for the construction of such squares known up to the present.

[54] Magic squares were used as charms and the method of construction seems to have been kept secret by the astrologers who used them in their trade. Another reason for their not occurring more frequently is that they did not belong to any particular subject and so has no place in the literature of the land.

[55] Andrews, W. S., *l.c.*, pp. 62 and 151–152.

[56] *Memoires de l' Academie Royale* (1705). For a description of the method see also W. S. Andrews, *l.c.*

The squares given by the Jaina monks Dharmanandana and Sundarasūri have evidently been obtained by generalisation of Nārāyaṇa's methods and show that the study of magic squares engaged the attention of the Hindus up to the fifteenth century.

The history of the development of magic squares in India, detailed in the preceding pages, leads irresistibly to the conclusion that the magic square originated in India. The knowledge of these squares might have gone outside India at any time between the first century and the tenth century AD. But it appears to be most probable that the west as well as China got the magic squares from India through the Arabs about the tenth century. This would account for the simultaneous occurrence of the magic square in such far off places as China, Arabia and Western Europe.

Use of series in India *

Particular instances of arithmetic and geometric series have been found to occur in Vedic literature as early as 2000 BC. From Jaina literature it appears that the Hindus were in possession of the formulae for the sum of the arithmetic and the geometric series as early as the fourth century BC, or earlier. In the *Bakhshali Manuscript* and other works on *Pāṭīgaṇita*, series were treated as one of the major topics of study and a separate section was generally devoted to the rules and problems relating to series. In Europe, the series were looked upon as one of the fundamental operations, evidently due to Hindu influence through the Arabs. Besides the arithmetic and the geometric series, a number of other types of series, e.g., the series of sums, the series of squares or cubes of the natural numbers, the arithmetico-geometric series, the series of polygonal or figurate numbers, etc. occur in the works on *Pāṭīgaṇita*. There is, however, no mention of the harmonic series.

Evidence of the use of the infinite geometric series with common ratio less than unity is found in the ninth century. The formula for the sum of this series was known to the Jainas who used it to find the volume of the frustum of a cone. The Kerala mathematicians of the fifteenth century gave the expansions of $\sin x$, $\cos x$, $\tan x$ and π long before they were known in Europe or anywhere else.

The present article gives an account of the use of series in Indian literature.

1 Origin and early history

Series of numbers developing according to certain laws have attracted the attention of people in all times and climes. The Egyptians are known to have used the arithmetic series about 1550 BC.[1] Arithmetic as well as geometric series are found in the Vedic literature of the Hindus (c. 2000 BC). In the *Taittirīya-saṃhitā*[2] we find the series:

(i) 1, 3, 5, . . . , 19, 29, . . . , 99

(ii) 2, 4, 6, . . . , 20

(iii) 4, 8, 12, . . .

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 28, No. 2 (1993), pp. 103–129.

[1] In the *Ahmes Papyrus*. *Cf.* Peet, *Rhind Papyrus*, p. 78; Smith, *History*, II, p. 498.

[2] *TS*, vii. 2.12–17; iv. 3.10.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_20

(iv) 10, 20, 30, ...

(v) 1, 3, 5, ..., 33.

In the *Vājasaneyī-saṃhitā*[3], we have the *yugma* ("even") and the *ayugma* ("odd") series:

(vi) 4, 8, 12, 16, ..., 48

(vii) 1, 3, 5, 7, ..., 31.

The *Pañcaviṃśa-brāhmaṇa*[4] has the following geometric series:

(viii) 12, 24, 48, 96, ... , 196608, 393216.

Another geometric series occurs in the *Dīgha Nikāya*.[5] It is

(ix) 10, 20, 40, ..., 80000.

The Hindus must have obtained the formula for the sum of an arithmetic series at a very early date, but when exactly they did so cannot be said with certainty. It is, however, definite that in the 5th century BC, they were in possession of the formula for the sum of the series of natural numbers, for in the *Bṛhaddevatā* (500-400 BC)[6] we have the result

$$2 + 3 + 4 + \ldots + 1000 = 500499.$$

In the *Kalpa-sūtra* of Bhadrabāhu (c. 350 BC), we have the sum of the following geometric series

$$1 + 2 + 4 + \ldots + 8192 \text{ (i.e., to 14 terms)}$$

given correctly as 16383, showing that the Hindus possessed some method of finding the sum of the geometric series in the 4th century BC.

The following result occurs in the commentary, entitled, *Dhavalā*[7] by Vīrasena (c. 9th century AD) on the *Ṣaṭkhaṇḍāgama* of Puṣpadanta Bhūtabali:

$$49\frac{217}{452}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots \textit{ ad inf.}\right) = 65\frac{110}{113}.$$

This shows that the following formula giving the sum of the infinite geometric series was well known in India in the 9th century AD:

$$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}, \text{ when } r < 1.$$

[3] *VS*, xvii. 24.25.

[4] xviii. 3. Compare also *Lāṭyāyana Śrauta-sūtra*, viii. 10.1 et seq.; *Kātyāyana Śrauta-sūtra*, xxii. 9. 1–6.

[5] T. W. Rhys Davids, *Dialogues of the Buddha*, III, 1921, pp. 70–72.

[6] *Bṛhaddevatā* edited in original Sanskrit with English translation by A. Macdonell, Harvard, 1904.

[7] 1.3.2. Also see A. N. Singh, *History of India from Jaina Sources*, JA, Vol. xvi, Dec. 1950, No. 2, pp. 54–69.

2 Kinds of series

It thus appears that the Hindus studied the arithmetic and geometric series at a very early date. Āryabhaṭa I (499), Brahmagupta (628) and other posterior writers considered also the cases of the sums of the sums, the squares and the cubes of the natural numbers. Mahāvīra (850) gave a rule for the summation of an interesting arithmetico-geometric series, viz.

$$\sum_{1}^{n} t_m \quad \text{where } t_1 = a \text{ and } t_m = rt_{m-1} \pm b,\ m \geq 2;$$

and Nārāyaṇa (1356) considered the summation of the figurate numbers of higher orders.

3 Technical terms

The Sanskrit term for a series is *śreḍhī*, meaning literally "progression", "any set or succession of distinct things", or *śreṇī* (or *śreṇi*), literally "line", "row", "series", "succession"; hence in relation to mathematics it implies "a series or progression of numbers". Thus, it is clear that the modern terms progression and series are analogous to the Hindu terms and they seem to have been adopted in the West under Hindu influence, in preference to the Greek term $\epsilon\kappa\theta\epsilon\sigma\iota\varsigma$ (ekthesis) which literally means a setting forth. The Sanskrit name for a term of the series is *dhana*[8] (literally, "any valued object"). The first term is called *ādi-dhana* ("first term") and any other term *iṣṭa-dhana* ("desired term"). When the series is finite, its last term is called *antya-dhana* ("last term"), and the middle term *madhya-dhana* ("middle term"). Often, for the sake of abridgement, the second words of these compound names are deleted, so that we have the terms *ādi, iṣṭa, madhya* and *antya* in their places. The first term is also called *prabhava* ("initial term"), *mukha* ("face") or its synonyms. The technical names for the common difference in an arithmetic series are *caya* or *pracaya* (from the root *cay* "to go", hence meaning "that by which the terms go", that is, "increment"), *uttara* ("difference", "excess"), *vṛdhi* ("increment"), etc. The common ratio in a geometric series is technically called *guṇa* or *guṇaka* ("multiplier") and so this series is distinguished from the arithmetic series by the specific name *guṇa-śreḍhī*. The number of terms in a series is known as *pada* ("step", meaning "the number of steps in the sequence") or *gaccha* ("period"). The sum is called *sarva-dhana* ("total of all terms"), *średhī-phala* ("result of the progression"), *śreḍhī-gaṇita* (or simply *gaṇita*, because the sum of the series is obtained by computation), and *śreḍhī-saṃkalita* (or in short *saṃkalita*, "sum of the series").

[8] In mathematics *dhana* means an affirmative quantity or plus. This probably explains the use of this term to denote the elements of a series which have to be summed up.

The above-mentioned technical terms occur commonly in almost all the known Hindu treatises on arithmetic from the so-called Bakhshali treatise (c. 200) onwards. But in the latter, the series has been designated by *varga* meaning "group". Occasionally, we meet with the terms *paṅkti*[9] and *dhārā*,[10] which signify "continuous line or series". Nārāyaṇa (1356) has used also a special term, *āya* (literally, "income") for the sum of natural numbers.

4 Sum of an arithmetic progression

Problems on the summation of arithmetic series are met with in the earliest available Hindu work on mathematics, the *Bakhshali Manuscript*. The statement of the formula for the sum begins with the word *rūponā*, so that summation is indicated by the terms *rūponā karaṇena* ("by the operation *rūponā*, etc.") throughout the work. In the statement of the solution of problems, the first term, the common difference and the number of terms, are written together and the resulting sum after these, as follows:

ā	1	*u*	1	*pa*	19	*rūpoṇā karaṇena phalam*	190
	1		1		1		1

In the above, *ā* stands for *ādi* ("first term"), *u* for *uttara* ("common difference"), and *pa* for *pada* ("number of terms"). The above quotation may be translated thus: "the first term is $\frac{1}{1}$, the common difference is $\frac{1}{1}$, and the number of terms is $\frac{19}{1}$; therefore, performing *rūponā*, etc. the sum is $\frac{190}{1}$". [11]

Āryabhaṭa I (499) states the formulae for finding the arithmetic mean and the partial sum of a series in A. P. as follows:

> Diminish the given number of terms by one, then divide by 2, then increase by the number of the preceding terms (if any), then multiply by the common difference, and then increase by the first term of the (whole) series: the result is the arithmetic mean (of the given number of terms). This multiplied by the given number of terms is the sum of the given terms. Alternatively, multiply the sum of the first and last terms (of the series or partial series to be summed up) by half the number of terms.[12]

[9] See Chapter xiii of the *Gaṇitakaumudī* of Nārāyaṇa.

[10] For instance, see the *Triloka-sāra* of Nemicandra (c. 975).

[11] The denominator 1 is written in the case of all the integral quantities. This is to show that the quantities involved may have non-integral values also.

[12] *Ā*, ii. 19. The commentator Bhāskara I says that several formulae are set out here. For details see *Āryabhaṭīya*, edited and translated by K. S. Shukla in collaboration with K. V. Sarma, New Delhi (1976), pp. 62–63.

Let the series be

$$a + (a + d) + (a + 2d) + \dots$$

Then the rule says that:

(1) the arithmetic mean of the n terms

$$[a + pd] + [a + (p + 1)d] + \dots + [a + (p + n - 1)d] = a + \left(\frac{n-1}{2} + p\right) d;$$

(2) the sum of the n terms

$$[a+pd]+[a+(p+1)d]+\dots+[a+(p+n-1)d] = n\left[a + \left(\frac{n-1}{2} + p\right) d\right].$$

In particular (when $p = 0$)

(3) the arithmetic mean of the series

$$a + [a + d] + [a + 2d] + \dots + [a + (n - 1)d] = \left[a + \frac{n-1}{2} d\right];$$

(4) the sum of the series

$$a + [a + d] + [a + 2d] + \dots + [a + (n - 1)d] = n\left[a + \frac{n-1}{2} d\right].$$

Alternatively, the sum of n terms of an arithmetic series with A as the first term and L as the last term

$$= \frac{n}{2}\,(A + L),$$

where $\frac{1}{2}(A + L)$ is the arithmetic mean of the terms.

Brahmagupta says:

> The last term is equal to the number of terms minus one, multiplied by the common difference, (and then) added to the first term. The arithmetic mean (of the terms) is half the sum of the first and the last terms. This (arithmetic mean) multiplied by the number of terms is the sum.[13]

Similar statements occur in the works of Śrīdhara,[14] Āryabhaṭa II,[15] Bhāskara II[16] and others. Mahāvīra points out that the common difference may be a positive or negative quantity.[17]

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[13] *BrSpSi*, xii. 17.
[14] *Triś*, p. 28.
[15] *MSi*, xv. 47.
[16] *L*, p. 27.
[17] *GSS*, p. 102, (290).

The particular case

$$\sum_{1}^{n} r = \frac{n(n+1)}{2}$$

is mentioned in all the Hindu works.[18]

5 Ordinary problems on arithmetic progression

The problems of finding out (1) the first term or (2) the common difference or (3) the number of terms, are common to all Hindu works. They occur first in the *Bakhshali Manuscript*.[19] The problem of finding the number of terms requires the solution of a quadratic equation.[20] Some indeterminate problems in which more than the one of the above quantities are unknown also occur in the *Bakhshali Manuscript*, the *Gaṇitasārasaṅgraha* of Mahāvīra and the *Gaṇitakaumudī* of Nārāyaṇa. A typical example of such problems is the finding out of an arithmetic series that will have a given sum and a given number of terms.

As illustrations of some other types of Hindu problems of arithmetical progression may be mentioned the following:

(1) There were a number of *utpala* flowers representable as the sum of a series in arithmetical progression, whereof 2 is the first term and 3 the common difference. A number of women divided those flowers equally among them. Each woman had 8 for her share. How many were the women and how many the flowers?[21]

(2) A person travels with velocities beginning with 4 and increasing successively by the common difference 8. Again, a second person travels with velocities beginning with 10 and increasing successively by the common difference of 2. What is the time of their meeting?[22]

(3) The continued product of the first term, the number of terms and the common difference is 12. If the sum of the series is 10, find it.[23]

[18] It is sometimes mentioned in connection with addition, as in Śrīdhara's *Triśatikā* and Mahāvīra's *Gaṇitasārasaṅgraha*.

[19] See p. 25; p. 35 problem 9; and p. 36 problem 10. The solution of this problem is incorrectly printed.

[20] For the equation and its solution see the section on quadratic equations in the chapter on Algebra in Part II.

[21] *GSS*, vi. 295.

[22] *GSS*, vi. $323\frac{1}{2}$. A problem of the above type in which one of the men travels with a constant velocity occurs in the *Bakhshali Manuscript*, p. 37.

[23] *GK*, *Śreḍhī-vyavahāra*, Ex. under Rule 6.

(4) A man starts with a certain velocity and a certain acceleration per day. After 8 days, another man follows him with a different velocity and an acceleration of 2 per day. They meet twice on the way. After how many days do these meetings occur?[24]

6 Geometric series

Mahāvīra gives the formula:

$$S = \frac{a(r^n - 1)}{r - 1}$$

for the sum of a geometric series whose first term is a and common ratio r. He says:

> The first term when multiplied by the continued product of the common ratio, taken as many times as the number of terms, gives rise to the *guṇadhana*. And it has to be understood that this *guṇadhana*, when diminished by the first term and (then) divided by the common ratio lessened by 1, becomes the sum of the series in geometrical progression.[25]

The same result is stated by him in the following alternative form:

> In the process of successive halving of the number of terms, put zero or 1 according as the result is even or odd. (Whenever the result is odd subtract 1). Multiply by the common ratio when unity is subtracted and multiply so as to obtain square (when otherwise, i.e., when the half is even). When the result of this (operation) is diminished by 1 and is then multiplied by the first term and (is then) divided by the common ratio lessened by 1, it becomes the sum of the series.[26]

If n be the number of terms and r the common ratio, the first half of the above rules gives r^n. This process of finding the nth power of a number was known to Piṅgala (c. 200 BC), and has been used by him to find 2^n. The second half of the rule then gives

$$S = \frac{a(r^n - 1)}{r - 1}.$$

[24] *Ibid*, under rule 9.
[25] *GSS*, ii. 93.
[26] *GSS*, ii. 94; also vi. $311\frac{1}{2}$, where the rule is applied to the case in which the common ratio is a fraction.

The above formula for the sum is stated by Pṛthūdakasvāmi,[27] Āryabhaṭa II,[28] and Bhāskara II[29] in the second form which appears to be the traditional method of stating the result.

Mahāvīra has given rules for finding the first term, common ratio or number of terms, one of these being unknown and the others as well as the sum being given.[30]

As illustrations of problems on geometric series may be mentioned the following:

1. Having first obtained 2 golden coins in a certain city, a man goes on from city to city, earning everywhere three times of what he earned immediately before. Say how much he will make on the eighth day?[31]

2. When the first term is 3, the number of terms 6, and the sum of 4095, what is the value of the common ratio?[32]

3. The common ratio is 6, the number of terms is 5, and the sum is 3110. What is the first term here?[33]

4. How many terms are there in a geometric series whose first term is 3, the second ratio is 5, and the sum is 228881835937?[34]

7 Series of squares

The series whose terms are the squares of natural numbers seems to have attracted attention at a fairly early date in India. The formula

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

occurs in the *Āryabhaṭīya*[35] where it is stated in the following form:

> The sixth part of the product of the three quantities consisting of the number of terms, the number of terms plus 1, and twice the number of terms plus 1, is the sum of the squares.

[27] *BrSpSi*, xii. 17, quoted in the commentary.
[28] *MSi*, xv. 52–53.
[29] *L*, p. 31.
[30] *GSS*, ii. 97–103.
[31] *GSS*, ii. 96.
[32] *GSS*, ii. 102 (first half).
[33] *GSS*, ii. 102 (second half).
[34] *GSS*, ii. 105 (last half).
[35] *Ā*, ii. 22.

The formula occurs in all the known Hindu works.[36]

Mahāvīra (*GSS*, vi. 298, 299) gives the sum of a series whose terms are the squares of the terms of a given arithmetic series.

Let

$$a + [a+d] + \ldots + [a+(r-1)d] + \ldots + [a+(n-1)d]$$

be an arithmetic series. Then, according to him,

$$\begin{aligned} a^2 + [a+d]^2 + \ldots + [a+(r-1)d]^2 + \ldots + [a+(n-1)d]^2 \\ = n\left[\left(\frac{2n-1}{6}d^2 + ad\right)(n-1) + a^2\right] \\ = n\left[\frac{(2n-1)(n-1)d^2}{6} + a^2 + (n-1)ad\right]. \end{aligned}$$

Śrīdhara[37] and Nārāyaṇa[38] give the above result in the following form:

$$\sum_1^n [a+(r-1)d]^2 = a\sum_1^n [a+2(r-1)d] + d^2\sum_1^{n-1} r^2.$$

8 Series of cubes

Āryabhaṭa I states the formula giving the sum of the series formed by the cubes of natural numbers as follows:

> The square of the sum of the original series (of natural numbers) is the sum of the cubes.[39]

Thus, according to him,

$$\sum_1^n r^3 = \left(\sum_1^n r\right)^2 = \left[\frac{n(n+1)}{2}\right]^2.$$

The above formula occurs in all the Hindu works. The general case in which the terms of the series are cubes of the terms of a given arithmetic series, has been treated by Mahāvīra.[40]

Let

$$S = \sum_1^n \alpha_r$$

[36] Although this rule does not occur in the *Triśatikā*, it occurs in Śrīdhara's bigger work of which the *Triśatikā* is an abridgement. See *PG*, Rule 102.

[37] *PG*, Rule 105.

[38] *GK*, *Śreḍhī-vyavahāra*, $17\frac{1}{2}$ and the first half of 18.

[39] *Ā*, ii. 22.

[40] *GSS*, vi. 303.

be an arithmetic series whose first term is a, and common difference d. Then, according to Mahāvīra,

$$\sum_1^n \alpha_r^3 = d \times S^2 \pm Sa(a \sim d),$$

according as $a >$ or $< d$.

Śrīdhara[41] and Nārāyaṇa[42] have also given the above result in the same form as Mahāvīra.

9 Series of sums

Let

$$N_n = 1 + 2 + 3 + \ldots + n.$$

Then the series

$$\sum_1^n N_r$$

formed by taking successively the sums up to 1, 2, 3, ... terms of the series of natural numbers, is given in all the Hindu works,[43] beginning with that of Āryabhaṭa I, who says:

> In the case of an *upaciti* which has 1 for the first term and 1 for the common difference between the terms, the product of three terms having the number of terms (n) for the first term and 1 for the common difference, divided by six is the *citighana*. Or, the cube of the number of terms plus 1, minus the cube root of the cube,[44] divided by 6.[45]

The above rule states that

$$\sum_1^n N_r = \frac{n\,(n+1)\,(n+2)}{6}$$
$$= \frac{(n+1)^3 - (n+1)}{6}.$$

[41] *PG*, Rule 107.

[42] *GK*, *Śreḍhī-vyavahāra*, 18 (c–d) f.

[43] This rule does not occur in the *Triśatikā* of Śrīdhara, but it occurs in his *Pāṭīgaṇita*. See *PG*, Rule 103.

[44] This means $[(n+1)^3]^{\frac{1}{3}} = (n+1)$. Recourse is taken to this form of expression for the sake of meter.

[45] *Ā*, ii. 21.

The sum of the series $\sum_1^n N_r$ has been called by Āryabhaṭa I *citighana* which means “the solid content of a pile in the shape of pyramid on a triangular base”. The pyramid is constructed as follows:

Form a triangle with $\sum_1^n m$ things arranged as below:

$$\begin{array}{cccccccc} & & & 0 & 1 & & & \\ & & 0 & 0 & 2 & & & \\ & 0 & 0 & 0 & 3 & & & \\ & \dots & \dots & \dots & \dots & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & 0 & \dots & 0 & 0 & (n-1) & \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & n \end{array} \qquad \text{Total} = \frac{n(n+1)}{2}$$

Form a similar triangle with $\sum_1^{n-1} m$ things and place it on top of the first, then form another such triangle with $\sum^{n-2} m$ things and place it on top of the first two. Proceed as above till there is one thing at the top. The figure obtained in this manner will be a pyramid formed of n layers, such that the base layer consists of $\sum_1^n r$ things, the next higher layer consists of $\sum_1^{n-1} r$ things, and so on. The number of things in the solid pyramid *citighana* $= \sum_1^n N_r$, where

$$N_r = \sum_{m=1}^{m=r} m.$$

The base of the pyramid is called *upaciti*, so that

$$upaciti = \sum_{m=1}^{m=n} m.$$

The above *citighana* is the series of figurate numbers. The Hindus are known to have obtained the formula for the sum of the series of natural numbers as early as the fifth century BC. It cannot be said with certainty whether the Hindus in those times used the representation of the sum by triangles or not. The subject of piles of shots and other things has been given great importance in the Hindu works, all of which contain a section dealing with *citi* (“piles”). It will not be a matter of surprise if the geometrical representation of figurate numbers is traced to Hindu sources.

10 Mahāvīra’s series

Mahāvīra (850) has generalised the series of sums in the following manner:

Let

$$\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n$$

be a series in arithmetical progression, the first term being α_1, and the common difference β, so that

$$\alpha_r = \alpha_1 + (r-1)\beta.$$

Mahāvīra considers the following series

$$\sum_{r=1}^{r=n} \left(\sum_{m=1}^{m=\alpha_r} m \right)$$

and gives its sum as[46]

$$\frac{n}{2} \left[\left(\frac{(2n-1)\beta^2}{6} + \frac{\beta}{2} + \alpha_1\beta \right) (n-1) + \alpha_1(\alpha_1+1) \right].$$

Nārāyaṇa[47] gives the above result in another form. According to him

$$\sum_{r=1}^{r=n} \left(\sum_{m=1}^{m=\alpha_r} m \right) = \left(\sum_{1}^{\alpha_1+\beta} m - \sum_{1}^{\alpha_1} m \right) \sum_{1}^{n-1} m + n \sum_{1}^{\alpha_1} m + \beta^2 \sum_{1}^{n-2} \left(\sum_{1}^{r} m \right).$$

Denoting by N_r the sum of r terms of the series of natural numbers, Nārāyaṇa's result may be written in the form

$$\begin{aligned}\sum_{r=1}^{r=n} N_{\alpha r} &= (N_{\alpha 1+\beta} - N_{\alpha 1})N_{n-1} + nN_{\alpha 1} + \beta^2 \sum_{1}^{n-2} N_r \\ &= \left[\frac{(\alpha_1+\beta)(\alpha_1+\beta+1)}{2} - \frac{\alpha_1(\alpha_1+1)}{2} \right] \frac{n(n-1)}{2} + \frac{n_1\alpha_1(\alpha_1+1)}{2} \\ &\quad + \beta^2 \frac{(n-2)(n-1)n}{6}\end{aligned}$$

which can be reduced to Mahāvīra's form.

Śrīdhara[48] puts the result in the form

$$\sum_{r=1}^{r=n} \left(\sum_{m=1}^{m=\alpha_r} m \right) = \frac{1}{2} \left[\sum_{r=1}^{r=n} \alpha_r^2 + \sum_{r=1}^{r=n} \alpha_r \right].$$

[46] *GSS*, vi. 305–305$\frac{1}{2}$.
[47] *GK*, I, p. 117, lines 11–16.
[48] See *PG*, Rule 106.

11 Nārāyaṇa's series

Nārāyaṇa has given formulae for the sums of series whose terms are formed successively by taking the partial sums of other series in the following manner:

Let the symbol ${}^nV_1$ denote the arithmetic series of natural numbers up to n terms; i.e., let

$${}^nV_1 = 1 + 2 + 3 + \ldots + n.$$

Let ${}^nV_2$ denote the series formed by taking the partial sums of the series ${}^nV_1$. Then

$${}^nV_2 = \sum_{r=1}^{r=n} {}^rV_1.$$

Similarly, let

$${}^nV_3 = \sum_{r=1}^{r=n} {}^rV_2, \qquad {}^nV_4 = \sum_{r=1}^{r=n} {}^rV_3, \qquad \ldots, \qquad {}^nV_m = \sum_{r=1}^{r=n} {}^rV_{m-1}.$$

The series ${}^nV_m$ has been called by Nārāyaṇa as *m-vāra-saṅkalita* ("m-order-series") meaning thereby that the operation of forming a new series by taking the partial sums of a previous series has been repeated m times. The number m may be called the order (*vāra*) of the series.

Nārāyaṇa states the sum ${}^nV_m$ as follows:

> The terms of the sequence beginning with the *pada* (number of terms, i.e., n) and increasing by 1 taken up to the order (*vāra*) plus 1 times are successively the numerators and the terms of the sequence beginning with unity and increasing by 1 are respectively the denominators. The continued product of these (fractions) gives the *vāra-saṅkalita* ("sum of the iterated series of a given order").

Thus, according to the above, n being the number of terms of the iterated and m the order, we get the following sequence of numbers:

$$\frac{n}{1}, \frac{n+1}{2}, \frac{n+2}{3}, \ldots, \frac{n+m}{m+1}.$$

The sum of the series is the continued product of the above sequence, i.e.,

$${}^nV_m = \frac{n \times (n+1) \times (n+2) \times \cdots \times (n+m)}{1 \times 2 \times 3 \times \cdots \times (m+1)}.$$

Putting $m = 1, 2, 3, \ldots$, we get

$$
\begin{aligned}
{}^nV_1 &= \sum_{r=1}^{n} r \quad = \frac{n(n+1)}{1\times 2},\\
{}^nV_2 &= \sum_{r=1}^{n} rv_1 = \frac{n(n+1)(n+2)}{1\times 2\times 3},\\
{}^nV_3 &= \sum_{r=1}^{n} rv_2 = \frac{n(n+1)(n+2)(n+3)}{1\times 2\times 3\times 4},
\end{aligned}
$$

and so on.

Nārāyaṇa (1356) has made use of the numbers of the *vāra-saṅkalita* in the theory of combinations, in chapter xiii of his *Gaṇitakaumudī.* The series discussed above are now known as the series of figurate numbers. They seem to have been first studied in the west by Pascal (1665).

12 Generalisation

Nārāyaṇa has considered the more general series obtained in the same way as above form a given arithmetical progression.

Let

$$
{}^nS_1 = \sum_{1}^{n} \alpha_r = \alpha_1 + \alpha_2 + \ldots + \alpha_n,
$$

where $\sum_{1}^{n} \alpha_r$ is an arithmetic series whose first term is α_1 and common difference β. As above, let us define the iterated series ${}^nS_2$, ${}^nS_3$, …, ${}^nS_k$ as follows:

$$
{}^nS_2 = \sum_{r=1}^{r=n} {}^rS_1, \qquad {}^nS_3 = \sum_{r=1}^{r=n} {}^rS_2, \qquad \ldots, \qquad {}^nS_k = \sum_{r=1}^{r=n} {}^rS_{k-1}.
$$

Nārāyaṇa states the formula for the sum of the series ${}^nS_r$ thus:

> The sum of the iterated series of the given order derived from the natural numbers equal to the given number minus 1 is put down at two places. These become the multipliers. The order as increased by unity being divided by the given number of terms as diminished by unity is a multiplier of the first (of these multipliers). The first term and the common difference multiplied respectively by the two quantities and (the results) added together gives the required sum of the iterated series.

Suppose it be required to find ${}^{n}S_m$, where n is the *pada* ("number of terms") and m the *vāra* ("order") of the iterated series. Let, as before, ${}^{n}V_r$ denote the iterated series of the rth order derived from the series of n natural numbers. Then, taking ${}^{n-1}V_m$ as two places, and multiplying the first of these by $\frac{m+1}{n-1}$ as directed, we get

$$\frac{m+1}{n-1} \times {}^{n-1}V_m \quad \text{and} \quad {}^{n-1}V_m.$$

Multiplying the first term (α_1) and the common difference (β) by these two respectively and adding we get the required sum

$${}^{n}S_m = \alpha_1 \frac{m+1}{n-1} \times {}^{n-1}V_m + \beta \times {}^{n-1}V_m.$$

Rationale

The above formula has been evidently obtained by Nārāyaṇa as follows:

$$\begin{aligned}
{}^{n}S_1 &= \sum_1^n \alpha_r = \alpha_1 + [\alpha_1 + \beta] + \ldots + [\alpha_1 + (n-1)\beta] \\
&= n\left[\alpha_1 + \frac{n-1}{2}\beta\right] \\
{}^{n}S_2 &= \sum_1^n {}^{r}S_1 = \alpha_1 \sum_1^n r + \beta \sum_1^n \frac{r(r-1)}{2} \\
&= \alpha_1 \times {}^{n}V_1 + \beta \times {}^{n-1}V_2 \\
{}^{n}S_3 &= \sum_1^n {}^{n}S_2 = \alpha_1 \sum_1^n {}^{r}V_1 + \beta \sum_1^n {}^{r-1}V_2 \\
&= \alpha_1 \times {}^{n}V_2 + \beta \times {}^{n-1}V_3 \\
&\vdots \\
{}^{n}S_m &= \alpha_1 \times {}^{n}V_{m-1} + \beta \times {}^{n-1}V_m.
\end{aligned}$$

But

$${}^{n}V_{m-1} = \frac{m+1}{n-1} \times {}^{n-1}V_m.$$

Therefore

$${}^{n}S_m = \alpha_1 \frac{m+1}{n-1} \times {}^{n-1}V_m + \beta \times {}^{n-1}V_m.$$

13 Nārāyaṇa's problem

The above series have been investigated by Nārāyaṇa in order to solve the following type of problems:

A cow gives birth to one calf every year. The calves become young and themselves begin giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow.

Solution

(i) The number of calves produced during 20 years by the cow is 20.

(ii) The first calf becomes a cow in 3 years and begins giving birth to calves every year, so that the number of its progeny during the period under consideration is $(20-3) = 17$. Similarly, the second calf becoming a cow produces, during the period under consideration $(19 - 3) = 16$ calves, and so on. The total number of calves of the second generation is

$$\sum_{1}^{17} r = {}^{17}V_1.$$

(iii) The first calf of the eldest cow (of the group of 17) produces during the period under consideration $(17 - 3) = 14$ calves; the second calf of the same group produces 13 calves; and so on. The total progeny (of the second generation) of the group of 17 in (ii) is

$$14 + 13 + 12 + \ldots + 1 =^{14} V_1.$$

Similarly, the total progeny of 16 in (ii) is ${}^{13}V_1$ of the group of 15 in (ii) is ${}^{12}V_1$, and so on. Thus, the total progeny of the third generation is

$$\sum_{1}^{14} {}^{r}V_1 =^{14} V_2.$$

Similarly, the total progeny of the fourth generation is

$$\sum_{1}^{(14-3)} {}^{r}V_2 =^{11} V_3,$$

and so on.

The total number of cows and calves at the end of 20 years is

$$\begin{aligned} &1{+}20 + {}^{17}V_1 + {}^{14}V_2 + {}^{11}V_3 + {}^{8}V_4 + {}^{5}V_5 + {}^{2}V_6 \\ &= 1 + 20 + \frac{17 \times 18}{1 \times 2} + \frac{14 \times 15 \times 16}{1 \times 2 \times 3} + \frac{11 \times 12 \times 13 \times 14}{1 \times 2 \times 3 \times 4 \times 5} \\ &\quad + \frac{5 \times 6 \times 7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4 \times 5 \times 6} + \frac{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} \\ &= 1 + 20 + 153 + 560 + 1001 + 792 + 210 + 8 \\ &= 2745. \end{aligned}$$

After giving the solution of the problem Nārāyaṇa remarks:

> An alternative method of solution is by means of the *Meru* used in the theory of combination in connection with (the calculations regarding) metre. This I have given later on.

14 Miscellaneous results

The following results have been given by Śrīdhara, Mahāvīra, and Nārāyaṇa:

R_1:[49] $n^2 = 1 + 3 + 5 + \dots$ to n terms

R_2:[50] $n^3 = \sum_1^n [3r(r-1)+1] = 3\sum_1^n r(r-1) + n$

R_3:[51] $n^3 = n + 3n + 5n + \dots$ to n terms

R_4:[52] $n^3 = n^2(n-1) + \sum_{r=1}^n (2r-1)$

R_5:[53]
$$\left[(n+3)\frac{n}{4}+1\right](n^2+n) = \sum_1^n r + \sum_1^n r^2 + \sum_1^n r^3 + \sum_1^n \sum_1^r m = \sum_1^n r(1 + r + r^2 + \frac{r+1}{2})$$

R_6:[54]
$$\sum_1^n r + n^2 = 3\sum_1^n r - n; \quad \sum_1^n r + n^3 = \frac{(6n+1)\left(\sum_1^n r + n^2\right) + 4n}{9}$$

R_7:[55]
$$\sum_1^n r + n^2 + n^3 = \frac{n(n+1)(2n+1)}{2}$$

R_8:[56]
$$\sum_{m=1}^{m=n}\sum_{r=1}^{r=m} r + \sum_{r=1}^{r=n} r^2 + \sum_{r=1}^{r=n} r^3 = \frac{n(n+1)^2(n+2)}{4}$$

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[49] *Triś*, p. 5; *GSS*, ii. 29; *GK*, i. 18.
[50] *Triś*, p. 6; *GSS*, ii. 45; *GK*, i. 22.
[51] *GSS*, ii. 44; *GK*, *Śreḍhī-vyavahāra*, 10–11.
[52] *Ibid.*
[53] *GSS*, vii. 309$\frac{1}{2}$.
[54] (6) and (7) are given by Nārāyaṇa, *GK*, *l.c.*, Rules 11 and 12.
[55] *PG*, Rule 102; *GK*, *l.c.*, Rule 13 (a–b).
[56] *PG*, Rule 104.

R$_9$:[57] $$\sum_{r=1}^{a} r + \sum_{r=1}^{a+d} r + \sum_{r=1}^{a+2d} r + \ldots \text{ to } n \text{ terms} = \frac{1}{2}\left[\sum_{r=1}^{r=n}(a+(r-1)d)^2 + \sum_{r=1}^{r=n}(a+(r-1)d)\right]$$

R$_{10}$:[58] $$S \pm \left(\frac{S}{a} - n\right)\frac{m}{r-1} = a + (ar \pm m) + [(ar \pm m) + m] \pm [(ar \pm m)r \pm m]r \pm m + \ldots \text{ to } n \text{ terms,}$$
where $S = a + ar + ar^2 + \ldots$ to n terms.

15 Binomial series

The development of $(a+b)^n$ for integral values of n has been known in India from very early times. The case $n = 2$ was known to the authors of the *Śulba Sūtras* (1500-1000 BC). The series formed by the binomial coefficients

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

seems to have been studied at a very early date. Piṅgala (c. 200 BC), a writer on metrics, knew the sum of the above series [59] to be 2^n. This result is found also in the works of Mahāvīra (850),[60] Pṛthūdakasvāmī (860),[61] and all later writers.

16 Pascal triangle

The so-called Pascal triangle was known to Piṅgala, who explained the method of formation of the triangle in short aphorisms (*sūtra*). These aphorisms have been explained by the commentator Halāyudha thus:

> Draw one square at the top; below it draw two squares, so that half of each of them lies beyond the former on either side of it. Below them, in the same way, draw three squares; then below them four; and so on up to as many rows as are desired: this is the preliminary representation of the *Meru*. Then putting down 1 in the first square, the figuring should be started. In the next two squares put 1 in each. In the third row put 1 in each of the extreme squares, and in the middle square put the sum of the two numbers in the two squares of the second row. In the fourth row put 1 in

[57] *PG*, Rule 106.
[58] *GSS*, vi. 314.
[59] Piṅgala, *Chandaḥ Sūtra*, viii. 23–27.
[60] *GSS*, ii. 94.
[61] *BrSpSi*, xii. 17 comm.

Number of syllables		Total no. of variations
	1	
1...	1 | 1	...$2 = 2^1$
2...	1 | 2 | 1	...$4 = 2^2$
3...	1 | 3 | 3 | 1	...$8 = 2^3$
4...	1 | 4 | 6 | 4 | 1	...$16 = 2^4$
5...	1 | 5 | 10 | 10 | 5 | 1	...$32 = 2^5$
6...	1 | 6 | 15 | 20 | 15 | 6 | 1	...$64 = 2^6$

Figure 1: *Meru Prastāra*

each of the two extreme squares: in an intermediate square put the sum of the numbers in the two squares of the previous row which lie just above it. Putting down of the numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms: there are two forms, one consisting of one long and the other one short syllable. The numbers in the third row give the disyllabic forms: in one form all syllables are long, in two forms one syllable is short (and the other long), and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabic forms. There one form has all syllables long, three have one syllable short, three have two short syllables, and one has all syllables short. And so on in the fifth and succeeding rows; the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short, and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables.

Thus, according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* shown in Figure 1.

From the above it is clear that Piṅgala knew the result

$$^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n.$$

17 Infinite series

Early History

As already remarked, the formula for the sum of an infinite geometric series, with common ratio less than unity, was known to Jain mathematicians of the ninth century. Application of this formula was made to find the volume of the frustum of a cone in Vīrasena's commentary on the *Ṣaṭkhaṇḍāgama*, which was completed about 816 AD. The mathematicians of South India, especially those of Kerala, seem to have made notable contribution to the theory of infinite series. We find that in the first half of the fifteenth century they discovered what is now known as Gregory's series. Use of this series seems to have been made for the calculation of π, and in astronomy. As the works of this period are not available to us, it is not possible to trace the gradual evolution of the infinite series in India. Some of these series that are found to occur in the works of the Kerala mathematicians of the 16th, 17th, and 18th centuries are given below.

Series for the arc of a circle

Śaṅkara Vāriyar (1500–1560), the commentator of Nīlakaṇṭha Somayājī's *Tantra-saṅgraha*, gives an infinite series for the arc of a circle in terms of its sine and cosine and the radius of the circle. He says:

> By the method stated before for the calculation of the circle, the arc corresponding to a given value of the sine can be found. Multiply the given value (*iṣṭa*) of the sine (*jyā*) by the radius and divide by the cosine (*koṭijyā*). The result thus obtained is the first quotient. Then operating again and again with the square of the (given) sine as the multiplier and the square of the cosine as the divisor, obtain from the first quotient, other quotients. Divide the successive quotients by the odd numbers $1, 3$, etc., respectively. Now subtract the even order of quotients from the odd ones. The remainder is the arc required.[62]

That is to say, if R denotes the radius of a circle, α an arc of it, and θ the angle subtended at the centre by that arc, then

$$R\theta = \alpha = \frac{R\sin\theta}{1\times\cos\theta} - \frac{R\sin^3\theta}{3\times\cos^3\theta} + \frac{R\sin^5\theta}{5\times\cos^5\theta} - \frac{R\sin^7\theta}{7\times\cos^7\theta} + \dots$$

This series will be convergent if $\sin\theta < \cos\theta$, that is, if $\theta < \frac{\pi}{4}$. But if $\theta > \frac{\pi}{4}$, the series will be divergent and so the rule appears to fail. If in that case,

[62] Verses 206–208 of Śaṅkara Vāriyar's larger commentary on *TS* (=*Tantrasaṅgraha*), entitled *Yuktidīpikā*, ed. by K. V. Sarma, Hoshiarpur (1977).

however, we take $\sin(\frac{\pi}{2}-\theta)$ as given instead of $\sin\theta$, then in accordance with the rule, we shall get the series

$$\frac{R\pi}{2}-\alpha=\frac{R\sin(\frac{\pi}{2}-\theta)}{1\times\cos(\frac{\pi}{2}-\theta)}-\frac{R\sin^3(\frac{\pi}{2}-\theta)}{3\times\cos^3(\frac{\pi}{2}-\theta)}+\frac{R\sin^5(\frac{\pi}{2}-\theta)}{5\times\cos^5(\frac{\pi}{2}-\theta)}-\dots$$

$$\text{or}\quad \frac{R\pi}{2}-\alpha=\frac{R\cos\theta}{1\times\sin\theta}-\frac{R\cos^3\theta}{3\times\sin^3\theta}+\frac{R\cos^5\theta}{5\times\sin^5\theta}-\dots$$

which is convergent. Knowing the value of $\frac{R\pi}{2}-\alpha$, we can easily calculate the value of α. Thus, the rule will give the desired result even in the case $\theta>\frac{\pi}{4}$. Hence, the author remarks:

> Of the arc and its complement, one should take here (the sine of) the smaller as given (*iṣṭa*): this is what has been stated.[63]

The above series is stated also by Putumana Somayājī (c. 1660–1740) and Śaṅkaravarman (1800–38). The former writes:

> Find the first quotient by dividing by the cosine the given sine as multiplied by the radius. Then get the other quotients by multiplying the first and those successively resulting by the square of the sine and dividing them in the same way by the square of the cosine. Now dividing these quotients respectively by 1, 3, 5, etc. subtract the sum of even ones (in the series) from the sum of the odd ones. Thus, the sine will become the arc.[64]

Śaṅkaravarman says:

> Divide the product of the radius and the sine by the cosine. Divide this quotient and others resulting successively from it on repeated multiplication by the square of the sine and division by the square of the cosine by 1, 3, 5, etc., respectively. Then subtract the sum of the even quotients (in the series) from the sum of the odd ones. The remainder is the arc (required).[65]

Introducing the modern tangent function, the above series can be written as

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\frac{1}{7}\tan^7\theta+\dots$$

This series was rediscovered by James Gregory in 1671 and then by G. W. Leibnitz in 1673. It is now generally ascribed to the former. But rightly speaking, this series was first discovered in India, probably by the Kerala mathematician Mādhava, who lived about 1340–1425 AD.

[63] Verse 209 (a–b) of the commentary *Yuktidīpikā* on *TS*, ii.
[64] *Karaṇapaddhati*, vi. 18.
[65] *Sadratnamālā*, iv. 11.

For the case $\theta = \frac{\pi}{4}$, Jyeṣṭhadeva (c. 1500–1610), in his *Yuktibhāṣā*, gives three successively better approximations to $\frac{\pi}{4}$:[66]

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{1}{n+1} \tag{1}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{\frac{1}{2}(n+1)}{(n+1)^2+1} \tag{2}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{n} \pm \frac{[\frac{1}{2}(n+1)]^2+1}{\frac{1}{2}(n+1)[(n+1)^2+4+1]} \tag{3}$$

Saṅkara Vāriyar has also stated (2)[67] and (3)[68] and in addition the approximation[69]

$$\frac{1}{2} + \frac{1}{2^2-1} - \frac{1}{4^2-1} + \ldots \pm \frac{1}{n^2-1} \pm \frac{1}{2[(n+1)^2+2]}.$$

A number of infinite series expansions for π (circumference/diameter) occur in the works of Śaṅkara Vāriyar, Putumana Somayājī, and Śaṅkaravarman. Some of these are:

R_1: $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

R_2:[70] $$\pi = \sqrt{12}\left[\frac{1}{9(2\times1-1)} + \frac{1}{9^2(2\times3-1)} + \frac{1}{9^3(2\times5-1)} + \ldots\right] - \frac{\sqrt{12}}{3}\left[\frac{1}{9(2\times2-1)} + \frac{1}{9^2(2\times4-1)} + \frac{1}{9^3(2\times6-1)} + \ldots\right]$$

R_3:[71] $$\pi = \sqrt{12}\left[1 - \frac{1}{3\times3} + \frac{1}{5\times3^2} - \frac{1}{7\times3^3} + \ldots\right]$$

R_4:[72] $$\pi = 3 + 4\left[\frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \ldots\right]$$

R_5:[73] $$\pi = 16\left[\frac{1}{1^5+4\times1} - \frac{1}{3^5+4\times3} + \frac{1}{5^5+4\times5} - \ldots\right]$$

R_6:[74] $$\pi = 8\left[\frac{1}{2^2-1} + \frac{1}{6^2-1} + \frac{1}{10^2-1} + \ldots\right]$$

[66] C. T. Rajgopal and M. S. Rangachari, "On the Untapped Source of Medieval Keralese Mathematics", *Archives for History of Exact Sciences*, Vol. 18, No. 2, 1978.
[67] *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 271–274.
[68] *Ibid*, vss. 295–296.
[69] *Ibid*, vs. 292.
[70] *Sadratnamālā*, iv. 1.
[71] *Ibid*, iv. 2.
[72] *Karaṇapaddhati*, vi. 2.
[73] *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 287–288.
[74] *Ibid*, vss. 293–294.

R$_7$:[75] $$\pi = 4 - 8\left[\frac{1}{4^2-1} + \frac{1}{8^2-1} + \ldots\right]$$

R$_8$:[76] $$\pi = 3 + 6\left[\frac{1}{(2\times 2^2-1)^2-2^2} + \frac{1}{(2\times 4^2-1)^2-4^2} + \frac{1}{(2\times 6^2-1)^2-6^2} + \ldots\right].$$

Series for the sine and cosine of an arc

The Hindus discovered series also for the sine and cosine of an angle in powers of its circular measure. Putumana writes:

> In the series of quotients obtained by dividing an arc of a circle severally by 2, 3, etc., times the radius, multiply the arc by the first (term); the resulting product by the second (term); this product again by the third (term); and so on. Put down the even terms of the sequence so obtained after the arc and the odd ones after the radius, and subtract the alternative ones. The remainders will respectively be the *Jyā* and *Kojyā* of that arc.[77]

That is to say,

$$\textit{Jyā}\ \alpha = \alpha - \frac{\alpha^3}{3!\,R^2} + \frac{\alpha^5}{5!\,R^4} - \frac{\alpha^7}{7!\,R^6} + \ldots$$
$$\textit{Kojyā}\ \alpha = R - \frac{\alpha^2}{2!\,R} + \frac{\alpha^4}{4!\,R^3} - \frac{\alpha^6}{6!\,R^5} + \ldots$$

corresponding to our modern series

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

These series reappear in the works of Śaṇkaravarman.[78] When θ is small, we have the approximation

$$\sin\theta = \theta - \frac{1}{6}\theta^3.$$

Similarly

$$\theta = \sin\theta + \frac{1}{6}\sin^3\theta.$$

Thus, Puthumana says:

[75] *Ibid*, vss. 293–294.
[76] *Karaṇapaddhati*, vi. 4.
[77] *Karaṇapaddhati*, vi. 12f.
[78] *Sadratnamālā*, iv. 5.

> A small arc being diminished by the sixth part of its cube as divided by the square of the radius becomes the *Jyā*. A small *Jyā* being increased in the same way becomes the arc.[79]

So does Śaṅkaravarman.[80]

Śaṅkara Vāriyar has also given an infinite series expansion for $\sin^2\theta$. He says:

> (Repeatedly) multiply the square of the arc by the square of the arc and divide successively by the square of the radius as multiplied by the squares of 2, etc. diminished by half of their square roots. Write the square of the arc, and below it the successive results and (then starting from the lowest) subtract the lower from that above it. What is thus obtained is the square of the *Jyā*.[81]

That is to say,

$$\begin{aligned} Jy\bar{a}^2\ \alpha = \alpha^2 &- \frac{\alpha^4}{(2^2-\frac{2}{2})R^2} + \frac{\alpha^6}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})R^4} \\ &- \frac{\alpha^8}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})(4^2-\frac{4}{2})R^6} + \ldots \end{aligned}$$

or, in modern notation,

$$\begin{aligned} \sin^2\theta = \theta^2 &- \frac{\theta^4}{(2^2-\frac{2}{2})} + \frac{\theta^6}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})} \\ &- \frac{\theta^8}{(2^2-\frac{2}{2})(3^2-\frac{3}{2})(4^2-\frac{4}{2})} + \ldots \end{aligned}$$

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[79] *Karaṇapaddhati*, vi. 19.
[80] *Sadratnamālā*, iv. 12.
[81] *Tantrasaṅgrahavyākhyā Yuktidīpikā*, vss. 455–456.

Surds in Hindu mathematics*

Elementary treatment of surds, particularly their addition, multiplication and rationalisation, is found in the *Śulbasūtras*. Fuller treatment of this subject occurs in the works on Hindu algebra where rules for addition and subtraction, multiplication and involution, separation and extraction of square-root of surds and compound surds are given. The present article gives an account of the treatment of surds in Hindu mathematics.

The Sanskrit term for the surd is *karaṇī*. Śrīpati (1039) defines it as follows:

> The number whose square-root cannot be obtained (exactly) is said to form an irrational quantity *karaṇī*.[1]

Similar definitions are given by Nārāyaṇa (1356) and others.[2] Of course, the number is to be considered a surd when the business is with its square root. A surd number is indicated by putting down the tachygraphic abbreviation *ka* before the number affected. Thus, *ka* 8 means $\sqrt{8}$ and *ka* 450 means $\sqrt{450}$.

1 Origin of surds

The origin of the term *karaṇī* is interesting. Literally it means "making one" or "producing one". It seems to have been originally employed to denote the cord used for measuring (the side of) a square. It then meant the side of any square and was so called because it made a square (*caturaśra-karaṇī*). Hence, it came to denote the square-root of any number. As late as the second century of the Christian era, Umāsvātī (c. 150) treated the terms *mūla* ("root") and *karaṇī* as synonymous. In later times, however, the application of the term has been particularly restricted to its present significance as a surd. Nemicandra (c. 975)[3] has occasionally used the generic term *mūla* to signify a surd, e.g., *daśa-mūla* $= \sqrt{10}$.

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 28, No. 3 (1993), pp. 253–264.

[1] *SiŚe* (=*Siddhānta-śekhara*), xiv. 7.

[2] *NBi* (= Nārāyaṇa's *Bījagaṇita*) I, R. 25. See also the commentaries of Gaṇeśa and Kṛṣṇa on the *Bījagaṇita* of Bhāskara II.

[3] *Gommaṭa-sāra* of Nemicandra, *Jīvakāṇḍa*, *Gāthā* 170.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_21

2 Addition and subtraction

For addition of surds, we have the following ancient rule:

> Reducing them by some (suitable) number, add the square-roots of the quotients; the square of the result multiplied by the reducer should be known as the sum of the surds.

We do not know the name of the author of this rule. It is found to have been quoted by Bhāskara I (629) in his commentary on the *Āryabhaṭīya* (ii. 10). A similar rule is given by Brahmagupta (628):

> The surds being divided by a (suitable) optional number, the square of the sum of the square-roots of the quotients should be multiplied by that optional number (in case of addition); and the square of the difference (of the square-roots of the quotients being so treated will give the difference of the surds).[4]

Mahāvīra says:

> After reducing (the surd quantities) by an optional divisor, the square of the sum or difference of the square-roots of the quotients is multiplied by the optional divisor, the square-root (of the product) is the sum or difference of the square-root quantities. Know this to be the calculation of surds.[5]

Śrīpati writes:

> For addition or subtraction, the surds should be multiplied (by an optional number) intelligently (selected), so that they become squares. The square of the sum, or difference of their roots, should then be divided by that optional multiplier. Those surds which do not become squares on multiplication (by an optional number), should be put together (side by side).[6]

Bhāskara II says:

> Suppose the sum of the two numbers of the surds $(a+b)$ as the *mahatī* ('greater') and twice the square-root of their product $(2\sqrt{ab})$ as the *laghu* ('lesser'). The addition or subtraction of these like integers is so (of the original quantities).
>
> Multiply and divide as if a square number by a square number. In addition and subtraction, the square-root of the quotient of the

[4] *BrSpSi* (=*Brāhmasphuṭasiddhānta*), xviii. 38.
[5] *GSS* (=*Gaṇitasārasaṅgraha*), vii. $88\frac{1}{2}$.
[6] *ŚiŚe*, xiv. 8f.

> greater surd number divided by the smaller surd number should be increased or diminished by unity; the result multiplied by itself should be multiplied by the smaller surd number. The product (is the sum or difference of the two surds). If there be no (rational) root, the surds should be stated separately side by side.[7]

Nārāyaṇa says:

> Divide the two surds separately by the smaller or greater among them; add or subtract the square-roots of the quotients; then multiply the square of the result by that divisor. The product is the sum or difference.
>
> Or multiply the two surds by the smaller or greater one among them; add or subtract the square-roots of the products; then dividing the square of the result by that selected multiplier, the quotient is the sum or difference.
>
> Or divide the greater surd by the smaller one; add unity to or subtract unity from the square-root of the quotient; then multiply the result by itself and also by the smaller quantity. The result is the sum or difference (required). Or proceed in the same way with the greater surd.
>
> Or add twice the square-root of the product of the two surds, supposed as if rational, to or subtract that from their sum. The result is the sum or difference. If there be no rational root of the product, then the two surds should be stated severally.
>
> To add up several surds, divide them by an optional number and then take the sum of the square-root of the quotients. This sum multiplied by itself and also by that divisor will give the sum of them.[8]

Thus, we have the following methods for addition or subtraction of surds:

(i) $\sqrt{a} \pm \sqrt{b} = \sqrt{b\left(\sqrt{\frac{a}{b}} \pm 1\right)^2}$,

(ii) $\sqrt{a} \pm \sqrt{b} = \sqrt{\frac{1}{a}(a \pm \sqrt{ab})^2}$,

(iii) $\sqrt{a} \pm \sqrt{b} = \sqrt{c\left(\sqrt{\frac{a}{c}} \pm \sqrt{\frac{b}{c}}\right)^2}$,

(iv) $\sqrt{a} \pm \sqrt{b} = \sqrt{\frac{1}{c}(\sqrt{ac} \pm \sqrt{bc})^2}$,

[7] *BBi* (=*Bhāskara's Bījagaṇita*), pp. 12f.
[8] *NBi* (*Junction of a door*), I, R. 25–30.

(v) $\sqrt{a} \pm \sqrt{b} = \sqrt{(a+b) \pm 2\sqrt{ab}}$.

The optional number c is so chosen that (ac, bc) or $\left(\frac{a}{c}, \frac{b}{c}\right)$ become perfect squares.

Brahmagupta and Mahāvīra teach the method (iii), Śrīpati gives (iv). Bhāskara states (i) and (v). Nārāyaṇa gives (i), (ii), (iii), and (v).

3 Multiplication and involution

For the multiplication of surd expressions, the Hindu works give an algebraic method. Thus, Brahmagupta says:

> Put down the multiplicand horizontally below itself as many times as there are terms in the multiplier; then multiplying by the *khaṇḍa-guṇana* method (i.e., by the method of multiplication by component parts), add the (partial) products.[9]

Thus, to multiply $\sqrt{a}+\sqrt{b}$ by $\sqrt{c}+\sqrt{d}$, one should proceed as follows:

$$\begin{aligned}(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{d}) &= (\sqrt{a}+\sqrt{b}) \times \sqrt{c} + (\sqrt{a}+\sqrt{b}) \times \sqrt{d} \\ &= \sqrt{ac}+\sqrt{bc}+\sqrt{ad}+\sqrt{bd}.\end{aligned}$$

Brahmagupta further notes:

> The squaring of a surd is (finding) the product of two equal (surds).[10]

Śrīpati writes:

> Putting down the multiplicand and multiplier in the manner of the *kapāṭasandhi* multiply according to the method taught before. But those surds should be added, as before, in which the product yields a perfect square.[11]
>
> On multiplying two equal surd quantities, the square of that surd is obtained.[12]

Bhāskara II (1150) observes:

> For abridgement, multiplication or division of surd expressions should be proceeded with after addition (or subtraction) of two or more terms of the multiplier and multiplicand or of the divisor and dividend.[13]

A similar remark has been made by Nārāyaṇa.[14]

[9] *BrSpSi*, xviii. 38.
[10] *BrSpSi*, xviii. 39.
[11] *SiŚe*, xiv. 9.
[12] *SiŚe*, xiv. 11.
[13] *BBi*, p. 13.
[14] *NBi*, I, R. 31.

4 Division

Brahmagupta (628) teaches the following method of division of surds:

> Multiply the dividend and divisor separately by the divisor after making an optional term of it negative; then add up the terms. (Do this repeatedly until the divisor is reduced to a single term). Then divide the (modified) dividend by the divisor reduced to a single term.[15]

Śrīpati (1039) writes:

> Reversing the sign, negative or positive, of one of the surds occurring in the denominator, multiply by it both the numerator and the denominator separately and then add together the terms of the (respective) products. Repeat (the operations) until there is left only a single surd in the denominator. By it divide the dividend above. Such is the method of division of surds.[16]

This rule has been almost reproduced by Bhāskara II[17] (1150) and Nārāyaṇa[18] (1356). The latter delivers also another method similar to the division of one algebraic expression by another. He says:

> Multiply the divisor surd so as to make all or some of its terms square such that the sum of their square-roots will be equal to the rational term (in the dividend). Thus will be determined the multiplier surd. Subtract from the dividend the divisor multiplied by that. If there be left a remainder, the sum of the terms of the divisor multiplied by that multiplier should be subtracted from the terms of the dividend. In case of absence of a rational term (in the dividend), that by which the divisor is multiplied and then subtracted for the dividend so as to leave no remainder, will be the quotient.[19]

Example from Bhāskara II[20]

Divide $\sqrt{9}+\sqrt{450}+\sqrt{95}+\sqrt{45}$ by $\sqrt{25}+\sqrt{3}$.

---

[15] *BrSpSi*, xviii, 39.
[16] *SiŚe*, xiv. 11.
[17] *BBi*, p. 14.
[18] *NBi*, I, R. 37–8.
[19] *NBi*, I, R. 33–5.
[20] *BBi*, pp. 15–16.

$$
\begin{aligned}
\frac{\sqrt{9}+\sqrt{450}+\sqrt{95}+\sqrt{45}}{\sqrt{25}+\sqrt{3}} &= \frac{(\sqrt{9}+\sqrt{450}+\sqrt{95}+\sqrt{45})(\sqrt{25}-\sqrt{3})}{(\sqrt{25}+\sqrt{3})(\sqrt{25}-\sqrt{3})}\\
&= \frac{\sqrt{8712}+\sqrt{1452}}{\sqrt{484}}\\
&= \sqrt{18}+\sqrt{3}.
\end{aligned}
$$

Example from Nārāyaṇa[21]

First method:
Divide $5+\sqrt{90}+\sqrt{180}+\sqrt{648}$ by $\sqrt{5}+\sqrt{36}$.

$$
\begin{array}{ccc|ccccccc|ccc}
\sqrt{5} & + & \sqrt{36} & 5 & + & \sqrt{90} & + & \sqrt{180} & + & \sqrt{648} & \sqrt{5} & + & \sqrt{18}\\
 & & & 5 & & & + & \sqrt{180} & & & & & \\
\hline
 & & & & & \sqrt{90} & + & \sqrt{648} & & & & & \\
 & & & & & \sqrt{90} & + & \sqrt{648} & & & & &
\end{array}
$$

Second method:
Divide $\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60}$ by $\sqrt{5}+\sqrt{3}+\sqrt{2}$.

$$
\begin{aligned}
&\frac{\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60}}{\sqrt{5}+\sqrt{3}+\sqrt{2}}\\
&\quad= \frac{(\sqrt{175}+\sqrt{150}+\sqrt{105}+\sqrt{90}+\sqrt{70}+\sqrt{60})(\sqrt{5}+\sqrt{3}-\sqrt{2})}{(\sqrt{5}+\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{3}-\sqrt{2})}\\
&\quad= \frac{\sqrt{2100}+\sqrt{1800}+\sqrt{1260}+\sqrt{1080}}{\sqrt{60}+\sqrt{36}}\\
&\quad= \frac{(\sqrt{2100}+\sqrt{1800}+\sqrt{1260}+\sqrt{1080})(\sqrt{60}-\sqrt{36})}{(\sqrt{60}+\sqrt{36})(\sqrt{60}-\sqrt{36})}\\
&\quad= \frac{\sqrt{20160}+\sqrt{17280}}{\sqrt{576}}\\
&\quad= \sqrt{35}+\sqrt{30}.
\end{aligned}
$$

5 Rule of separation

Bhāskara II gives a rule for an operation converse to that of addition. He says:

> (Find) a square number by which the compound-surd will be exactly divisible. Breaking up the square-root of that (square-number) into parts at pleasure, multiply the square of the parts

[21] *NBi*, I, example on R. 33–5; also Ex. 18.

> of the previous quotient. These will be the several component surds.[22]

A similar rule is stated by Nārāyaṇa:

> Divide the compound-surd by the square of some number so as to leave no remainder. Parts of it multiplied by themselves and also by the quotient will be the (component) terms of the surd.[23]

That is to say, if $N = m^2k$ and $m = a + b + c + d$, then

$$\sqrt{N} = \sqrt{m^2k} = m\sqrt{k} = (a + b + c + d)\sqrt{k},$$
$$= (\sqrt{a^2k} + \sqrt{b^2k} + \sqrt{c^2k} + \sqrt{d^2k}).$$

6 Extraction of square-root

For the extraction of the square-root of a surd expression, Brahmagupta described the following method:

> The optionally chosen surds being subtracted from the square of the absolute (i.e. rational) term, the square-root of the remainder should be added to and subtracted from the rational term and halved; then the first is considered as a rational term and the second a surd different from the previous. (Such operations should be carried on) repeatedly (if necessary).[24]

Illustrative example from Pṛthūdakasvāmi (860)

To find the square-root of $16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24}$. It has been solved substantially as follows:

Subtract the surd numbers 120, 72, 48 from the square of the rational number, viz., 256; the remainder is $(256 - 120 - 72 - 48) = 16$. Its root is 4; $\frac{1}{2}(16 \pm 4) = 10, 6$. Now subtracting the surd numbers 60 and 24 from 10^2, we get 16; its root is 4; $\frac{1}{2}(10 \pm 4) = 7, 3$. Again subtracting the surd number 40 from 7^2, we have 9; its root is 3; and $\frac{1}{2}(7 \pm 3) = 5, 2$. Hence, the required square-root is $\sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}$.

The same method is taught by Śrīpati (c. 1039)[25] and Bhāskara II (1150). The latter says:

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[22] *BBi*, p. 15.
[23] *NBi*, I, R. 36.
[24] *BrSpSi*, xviii, 40.
[25] *SiŚe*, xiv. 12.

> From the square of the rational number in the (proposed) square-surd, subtract the rational equivalent to one or more of the surd numbers; the square-root of the remainder should be severally added to and subtracted from the rational number; halves of the results will be the two surds in the square-root. But if there be left any more surd term in the (proposed) square surd, the greater surd number amongst those two should again be regarded as a rational number (and the same operations should be repeated).[26]

The above example of Pṛthūdakasvāmi is solved by Bhāskara substantially thus:

$$\sqrt{16^2 - (48+40+24)} = \sqrt{144} = 12; \quad \frac{1}{2}(16 \pm 12) = 14, 2.$$
$$\sqrt{14^2 - (120+72)} = 2, \quad \frac{1}{2}(14 \pm 2) = 8, 6.$$
$$\sqrt{8^2 - 60} = 2, \quad \frac{1}{2}(8 \pm 2) = 5, 3.$$

Therefore,

$$(16 + \sqrt{120} + \sqrt{72} + \sqrt{60} + \sqrt{48} + \sqrt{40} + \sqrt{24})^{\frac{1}{2}} = \sqrt{6} + \sqrt{5} + \sqrt{3} + \sqrt{2}.$$

For the above rule, all the terms of the surd expression have been contemplated to be positive, as is also clear from the illustrative examples given. For the case in which there is a negative term, Bhāskara II lays down the following procedure:

> If there be a negative surd in the square (expression), the traction of roots should be proceeded with supposing it as if positive: but of the two surds deduced one, chosen at pleasure by the intelligent mathematician, should be taken as negative.[27]

Example from Bhāskara II[28]

To find the square-root of $10 + \sqrt{24} - \sqrt{40} - \sqrt{60}$.

The solution is given substantially as follows:

$$\sqrt{10^2 - (40+60)} = 0, \quad \frac{1}{2}(10 \pm 0) = 5, 5$$
$$\sqrt{5^2 - 24} = 1, \quad \frac{1}{2}(5 \pm 1) = 3, 2.$$

[26] *BBi*, pp. 17f.
[27] *BBi*, p. 19.
[28] *BBi*, pp. 19f.

Therefore, $(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{3}+\sqrt{2}-\sqrt{5}$,

$$\text{or,} \qquad \sqrt{10^2-(24+60)}=4, \qquad \frac{1}{2}(10\pm 4)=7,3.$$

The greater number, viz., 7 is considered as negative. Then

$$\sqrt{7^2-40}=3,\ \frac{1}{2}(7\pm 3)=5,2.$$

Hence,

$$(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{3}+\sqrt{2}-\sqrt{5}.$$

Also,

$$(10+\sqrt{24}-\sqrt{40}-\sqrt{60})^{\frac{1}{2}}=\sqrt{5}-\sqrt{3}-\sqrt{2}.$$

7 Limitation of the method

Bhāskara II indicates how to test whether a given multinominal surd has a square-root at all or not. "This matter has not been explained at length", observes he, "by previous writers. I do it for the instruction of the dull".[29] He then says:

> In a square surd, the number of irrational terms must be equal to a number same as the sum of the (natural) number 1 etc. In a square surd having three irrational terms, the rational number equal to two of the surd numbers; in a square surd having six irrational number terms, the rational equal to three of them in one of ten irrational terms, integers equal to four of them; and in one of fifteen irrational terms, integers equal to five of them; having been subtracted from the square of the rational term the square-root of the remainder should be extracted. If (done) otherwise (in any case), it will not be proper. The numbers to be subtracted from the square of the rational number (in extracting roots of a square-surd) should be exactly divisible by four times the smaller term in the resulting root-surd. The quotients obtained by this exact division will be the surd terms in the root. If they are not obtained by the last rule, then the (resulting) root is wrong.[30]

He has added the following explanatory notes to the above rule:

> In the square of an expression containing irrational terms, there must be a rational term. In the square of (an expression consisting

[29] *BBi*, p. 20.
[30] *BBi*, pp. 20ff.

of) a singe surd, there will be only a rational term; of two surds, one surd together with a rational term; of three surds, three irrational terms and a rational term; of four surds, six; of five surds, ten; of six surds, fifteen; and so on. Thus, in the square of surd expressions consisting of two or more irrational terms, the number of irrational terms will be equal to the sum of the natural numbers one, etc. respectively, besides the rational term. So if in an example (proposed), the number (of irrational terms present) be not such; then it must be considered as a compound surd. Break it up (into required number of component surds) and then extract the square-root. This is what has been implied. Thus will be clear the significance of the rule, "In a square surd having three irrational terms, the rational number equal to two of the surd numbers, etc."

Illustrative examples with solution from Bhāskara II

Example 1: Find the square-root of $10+\sqrt{32}+\sqrt{24}+\sqrt{8}$.

In this square there being three surd terms, a rational number equivalent to two of the surd numbers is first subtracted from the square of the rational term and the root (of the remainder) extracted. Then proceeding in the same way with (the remaining) one term, no root is found in this case. Hence this (i.e., the proposed expression) does not possess a root expressible in surd terms. If, however, we extract the root by subtracting, contrary to the rule, an integer equivalent to all the surd terms, we get $\sqrt{2}+\sqrt{8}$. But this is wrong as its square is 18.

Or on adding together the surds $\sqrt{32}$ and $\sqrt{8}$, (the expression becomes) $10+\sqrt{72}+\sqrt{24}$. Then (by the rule) we obtain $2+\sqrt{6}$. But that is also erroneous.[31]

Example 2: Find the square-root of $10+\sqrt{60}+\sqrt{52}+\sqrt{12}$.

Here in this square, are present three surd terms; so subtracting a rational number equal to two surd numbers, viz., 52 and 12, the two surd terms for the root are obtained as $\sqrt{8}$ and $\sqrt{2}$; of these the smaller one, namely, 2 multiplied by four, that is 8, does not exactly divide 52 and 12. So they should not be subtracted, for it has been stated, "The numbers to be subtracted from the square of the rational number (in extracting root of a square surd) should be exactly divisible by four times the smaller term in the resulting root-surd." Let it, however, be supposed that the mention of "the

[31] *BBi*, p. 23.

smaller term" here is metaphorical and may sometimes imply also "the greater term" and that it should be considered as "the greater term", if with that root-surd as the rational term other surd terms are deducible. Now on doing so we obtain for the root $\sqrt{2}+\sqrt{3}+\sqrt{5}$. But this is also wrong; for its square is $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$.[32]

Example 3: Extract the root of

$$13+\sqrt{48}+\sqrt{60}+\sqrt{20}+\sqrt{44}+\sqrt{32}+\sqrt{24}.$$

There being six surd terms in this, an integer equal to three of the surd terms should be first subtracted from the square of the rational term and the root (of the remainder) found; next an integer equal to two of the surd terms and then an integer equal to one surd term (should be subtracted). But on so doing, no root is found in this instance. If we, however, proceed in a different way and subtract from the square of the rational term first an integer equal to the first surd term, then an integer equal to the second and third terms and lastly an integer equal to the remaining surd terms, we get for the root $\sqrt{1}+\sqrt{2}+\sqrt{5}+\sqrt{5}$. But this is incorrect, since its square is $13+\sqrt{8}+\sqrt{80}+\sqrt{160}$.[33]

Bhāskara then observes in general:

This is certainly a defect of those (ancient writers) who have not defined the limitations of this method of extracting the square root of a surd. In case of such square surds, the roots should be found by taking the roots of the surd terms by the method for finding the approximate values of the roots and then combining them with the rational term.[34]

Further he says:

The mention of "the greater surd" is metaphorical, for sometimes it might imply the less.

Example from Bhāskara II[35] (1150)

To find the root of $17+\sqrt{40}+\sqrt{80}+\sqrt{200}$.

[32] *BBi*, p. 23.
[33] *BBi*, p. 24.
[34] *BBi*, p. 24.
[35] *BBi*, p. 24.

Here

$$\sqrt{17^2 - (80 + 200)} = 3, \; \frac{1}{2}(17 \pm 3) = 10, 7$$
$$\sqrt{7^2 - 40} = 3, \; \frac{1}{2}(7 - 3) = 5, 2.$$

Therefore,

$$(17 + \sqrt{40} + \sqrt{80} + \sqrt{200})^{\frac{1}{2}} = \sqrt{10} + \sqrt{5} + \sqrt{2}.$$

8 Nārāyaṇa's rules

For finding the square-root of a surd expression, Nārāyaṇa (1350) gives the following rules:

> The number of irrational terms in the square of a surd expression is equal to the sum of natural numbers: this is the usual rule. In the square of a single surd term, there is only a rational number. In the square of an expression consisting of two surds terms, there is one surd term together with a rational number; of three, three; of four, six; of five, ten; and in the square of an expression consisting of six surd terms, there will be as many as fifteen surd terms; so it should be known. In an expression having the number of surd terms equal to the sum of the natural numbers, subtract from the square of the rational term a rational number equal to the sum of that number of surd numbers and then extract the square-root of the remainder. Add and subtract this to the rational number and halve. The results are the two surd terms. If further terms remain to be operated upon, regard the greater of these two as a rational number and find the other terms (of the root) by proceeding as before. If the number of surd terms in any expression be not equal to the sum of the natural numbers, the (requisite) number should be made up by breaking up some of the terms and then the square-root should be extracted. If that is not possible, the problem is wrong.[36]
>
> Increase twice the number of surd terms (in a given expression) by one fourth and then extract the square-root. Subtract half from that. The residue will give the number of terms (the sum of which is to be subtracted from the square of the rational term).[37]
>
> Or divide all the surd numbers (present in an expression) by four and arrange the quotients in the descending order. Divide the

[36] *NBi*, I, R. 41–5.
[37] *NBi*, I, R. 50.

product of the two surds nearest to the first surd (in the series) by the latter. The square-root of the quotient will be a surd term (in the root). Those two surds divided by this root will give another two surd terms (of the root). By these (three surds) divide next (three) terms of the series and the quotient will be another surd of the root. Again by these should be divided the other terms and the quotient is another surd; and so on. If now the square of the sum of surd numbers (in the root) be subtracted from the rational term (in the given expression) no remainder will be left. If it be not so (i.e., if a remainder is left), then the (given) square expression is a compound surd and it should be broken up into other surds by the rule of separation.[38]

[38] *NBi*, I, R. 46–9.

Approximate values of surds in Hindu mathematics *

As has been shown in an earlier article, the Hindu interest in the mathematics of surds is very old. The ancient Hindus were interested not only in the operations of the surds but also in finding their approximate values. The present article gives an account of the methods used for this purpose.

1 Introduction

The method to find approximate values of surds is found as early as the time of the *Śulba*. Thus, Baudhāyana (800 BC) states:

> Increase the measure (of which the *dvi-karaṇī* is to be found) by its third part, and again by the fourth part (of this third part) less by the thirty-fourth part of itself (i.e., of this fourth part). (The value thus obtained is called) the *saviśeṣa* (approximate).[1]

That is to say, if d be the *dvi-karaṇī* of a, that is, if d be the side of a square whose area is double that of the square on a then we shall have:

$$d = a + \frac{a}{3} + \frac{a}{3 \times 4} - \frac{a}{3 \times 4 \times 34}, \quad \text{approx.},$$

whence, we get

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}, \quad \text{approx.}$$

Expressing in decimal fractions, we obtain $\sqrt{2} = 1.4142156\ldots$. According to modern calculation, $\sqrt{2} = 1.414213\ldots$. Thus, it is clear that the ancient Hindus attained a very remarkable degree of accuracy in calculating an approximate value of $\sqrt{2}$. There has been much speculation among modern writers about the method by which the Hindus arrived at this result.[2] The

* Bibhutibhusan Datta and Avadhesh Narayan Singh. Revised by K. S. Shukla. *Indian Journal of History of Science*, Vol. 28, No. 3 (1993), pp. 265–275.

[1] *Baudhāyana Śulba*, i. 61–2; see also *Āpastamba Śulba*, i. 6; *Kātyāyana Śulba*, ii. 13.

[2] Thibaut, *Śulvasūtras*, pp. 13 ff; C. Muller, "Die Mathematik der *Śulvasūtra*", *Abhand, a.d. Math. Sem.d. Hamburg University*, Bd. vii, 1929, pp. 173–204.

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most recent hypothesis is that of Bibhutibhusan Datta.[3] It is based on a simple and elegant geometrical procedure quite in keeping with the spirit of the early Hindu geometry and hence seems to be a very plausible one. According to Nīlakaṇṭha (c. 1500),[4] Baudhāyana supposed the side of a square to be 12 units in length, so that its diagonal would be

$$\sqrt{2 \times 12^2} = \sqrt{288} \ \text{ units.}$$

Now

$$\sqrt{288} = \sqrt{17^2 - 1} = 17 - \frac{1}{34}, \quad \text{nearly.}$$

Therefore

$$12\sqrt{2} = 17 - \frac{1}{34}, \quad \text{nearly.}$$

Hence,

$$\sqrt{2} = \frac{17}{12} - \frac{1}{12 \times 34},$$

or

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.$$

Other notable approximate values occurring in the *Śulba* are:[5]

$$\sqrt{2} = \frac{7}{5},\ 1\frac{11}{25}, \qquad \sqrt{29} = 5\frac{7}{18}, \qquad \sqrt{5} = 2\frac{2}{7}, \qquad \sqrt{61} = 7\frac{5}{6}.$$

Probably it was also known that[6]

$$\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{3 \times 5} - \frac{1}{3 \times 5 \times 52}.$$

In the early canonical works of the Jainas (500–300 BC),[7] we find applications of the formula

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a}.$$

This formula has been applied consistently by the Jaina writers even up to the middle ages.[8]

[3] Bibhutibhusan Datta, *Śulba*, pp. 192ff.

[4] *Vide* his commentary on the *Āryabhaṭīya*, ii. 4. His commentary has been published in the Trivandrum Sanskrit Series (Nos. 101, 110, and 185).

[5] Datta, *Śulba*, p. 205.

[6] For an elegant method of getting this approximate value see Datta, *Śulba*, pp. 194 ff.

[7] For instance, see *Jambūdvīpaprajñapti*, *Sūtra* 3, 10–16; *Jīvābhigamasūtra*, *Sūtra* 82, 124; *Sūtrakṛtāṅgasūtra*, *Sūtra* 12, etc.

[8] See the commentaries of Siddhasena Gani (c. 550), Malaya-giri (c. 1200) and others.

2 Bakhshālī formula

In the Bakhshālī treatise on arithmetic (c. 200), we have the following rule for determining the approximate root (*śliṣṭa-mūla*, literally "nearest root") of a non-square number:

> In case of a non-square number, subtract the nearest square number; divide the remainder by twice (the root of that number). Divide half the square of that (that is, the fraction just obtained) by the sum of the root and fraction and subtract. (This will be the approximate value of the root) less the square (of the last term).[9]

That is to say,

$$\sqrt{N} = \sqrt{a^2 + r} = a + \frac{r}{2a} - \frac{\left(\frac{r}{2a}\right)^2}{2\left(a + \frac{r}{2a}\right)}$$

approximately, the error being

$$\left[\frac{\left(\frac{r}{2a}\right)^2}{2\left(a + \frac{r}{2a}\right)}\right]^2.$$

Example from the work:

$$\sqrt{41} = 6 + \frac{5}{12} - \frac{\left(\frac{5}{12}\right)^2}{2\left(6 + \frac{5}{12}\right)},$$

$$\sqrt{339009} = 579 + \frac{384}{579} - \frac{\left(\frac{384}{579}\right)^2}{2\left(579 + \frac{384}{579}\right)}.$$

In applying this approximate formula to concrete examples, the Bakhshālī treatise exhibits an accurate method of calculating errors and an interesting process of reconciliation, the like of which are not met elsewhere.[10]

3 Lalla's formula

To find the square-root of a sexagesimal fraction Lalla gives the following rule:

> Find the square-root (of the integral part in minutes) by the method indicated before. Multiply by sixty the remainder plus unity and then add the seconds. The result divided by twice the root plus 2 will be the fractional part (of the square root in terms of seconds).[11]

[9] This rule is not preserved in its entirely at any place in the surviving portion of the Bakhshālī manuscript; but it can be easily restored from the cross-references, especially on the folios 56, recto and 57 verso. See Bibhutibhusan Datta, *Bakh. Math.*, pp. 11 ff.
[10] Datta, *Bakh. Math.*, pp. 14 ff.
[11] *ŚiDVṛ*, iii. 52.

That is if $\alpha = \beta^2 + \epsilon$, then we shall have

$$\sqrt{\alpha' r''} = \beta' + \left\{ \frac{60(\epsilon + 1) + r}{2(\beta + 1)} \right\}''$$

in sexagesimal fractions. The same formula appears in the *Rājamṛgāṅka* of Bhojarāja and the *Karaṇakutūhala* of Bhāskara II.[12] It is obviously based on the approximate formula:

$$\sqrt{a^2 + r} = a + \frac{r + 1}{2(a + 1)}.$$

4 Brahmagupta's formula

Brahmagupta (628) says:

> The integer (in degrees), multiplied by its sexagesimal fraction (in minutes) and divided by thirty is (approximately) the square due to the fraction which is to be added to the square of the integer.[13]

That is, we have

$$\begin{aligned}(\alpha^\circ \beta')^2 &= \left(\alpha + \frac{\beta}{60}\right)^2 \\ &= \alpha^2 + \frac{\alpha\beta}{30} + \left(\frac{\beta}{60}\right)^2 \\ &= \alpha^2 + \frac{\alpha\beta}{30}, \quad \text{nearly,}\end{aligned}$$

neglecting $\left(\frac{\beta}{60}\right)^2$ as being very small.

From the above rule, we easily obtain a formula for finding the approximate value of a non-square number. For if x be a small fraction compared with a, we have

$$(a + x)^2 = a^2 + 2ax.$$

Putting $2ax = r$, we get

$$x = \frac{r}{2a}.$$

Hence

$$\sqrt{a^2 + r} = a + \frac{r}{2a}.$$

Brahmagupta expressly states a formula very much akin to that found in the Bakhshālī treatise. To find the square-root of the sum or the difference of the squares of two numbers, the larger of which has a fractional part, he gives the following rule:

[12] *Rājamṛgāṅka*, vi. 26(c-d)–28(a-b); *Karaṇakutūhala*, *spaṣṭādhikāra*, vs. 14.

[13] *Brāhmasphuṭasiddhānta*, xii. 62.

> Divide the square of the given smaller number plus or minus the portion in the square of the other due to its fractional part by twice (the integral part of) the other (at one place) and (at a second place) by the latter plus or minus the quotient obtained at the other place. The (last) divisor being added or subtracted by the last quotient and halved gives the square-root of the sum or the difference of the two squares. Or it is the other number plus or minus that quotient.

That is, if $a > b$ and ϵ, a small fraction, then

$$\sqrt{(a+\epsilon)^2 \pm b^2} = \frac{1}{2}\left\{2a \pm \frac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a} \pm \frac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a \pm \dfrac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a}}\right\}, \quad (1)$$

or

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a \pm \dfrac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a}}. \quad (2)$$

The second formula gives an approximation by defect. The value

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a} \quad (3)$$

gives an approximation by excess. Taking the mean of (2) and (3), Brahmagupta finds the closer approximation given by (1).

On simplifying, we get from the formula (2):

$$\sqrt{(a+\epsilon)^2 \pm b^2} = a \pm \frac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a} \mp \frac{\left\{\dfrac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)^2}{2a}\right\}^2}{2a + \dfrac{b^2 \pm \left(2a\epsilon + \epsilon^2\right)}{2a}}.$$

Putting $\epsilon = 0$, $b^2 = r$, we have the formula

$$\sqrt{a^2 \pm r} = a \pm \frac{r}{2a} \pm \frac{\left(\dfrac{r}{2a}\right)^2}{2a \pm \dfrac{r}{2a}}.$$

5 Śrīdhara's formula

Śrīdhara (c. 750) gives the following rule for finding the approximate value of the square-root of a non-square number:

> Multiply the non-square number by some large square number; then take the square-root (of the product), neglecting the excess, and divide it by the root of the multiplier.[14]

$$\sqrt{N} = \frac{\sqrt{Nm^2}}{m} = \frac{R}{m}, \quad \text{nearly},$$

where m is an arbitrary large number and R is the nearest integral root of Nm^2. Śrīdhara gives two illustrative examples:

$$\sqrt{1000} = \frac{\sqrt{1000 \times 10000}}{100} = \frac{3162}{100} = 31\frac{31}{50},$$
$$\sqrt{6250} = \frac{\sqrt{6250 \times 10000}}{100} = \frac{7905}{100} = 79\frac{1}{20}.$$

There are found various other formulae based upon Śrīdhara's formula. Thus, Āryabhaṭa II (c. 950) gives:[15]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10000}}{b \times 100} = \frac{R}{b \times 100}.$$

Śrīpati (1039) has[16]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times m^2 \times 10000}}{b \times m \times 100} = \frac{R}{bm \times 100}.$$

Bhāskara II (1150) states the formula:[17]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{abm^2}}{bm} = \frac{R}{bm}.$$

Example from Bhāskara II:

$$\sqrt{\frac{169}{8}} = \frac{\sqrt{169 \times 8 \times 10000}}{800} = \frac{3677}{800} = 4\frac{477}{800}.$$

Munīśvara (1658) gives[18]

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{ab \times 10,000,000,000,000,000}}{b \times 100,000,000} = \frac{R}{b \times 100,000,000}.$$

Illustrative example from him:[19]

$$\sqrt{208} = \frac{\sqrt{2080,000,000,000,000,000}}{100,000,000} = \frac{1442220510}{100,000,000} = 14\frac{4222051}{10,000,000}.$$

[14] *Triś* (=*Triśatikā*), R. 46.
[15] *MSi* (= *Mahāsiddhānta*), xv. 55.
[16] *SiŚe* (= *Siddhāntaśekhara*), xiii. 36.
[17] *L* (= *Līlāvatī*), p. 34.
[18] *PāSā* (= *Pāṭīsāra*), R. 117.
[19] *PāSā*, R. 120.

6 Nārāyaṇa's method

Nārāyaṇa (1356) says:

> Obtain the roots (of a square-nature) having unity as the additive and the number whose square-root is to be determined (as the multiplier). Then the greater root divided by the lesser root will be the approximate value of the square-root.[20]

That is to say, to find the approximate value of the surd $\sqrt{N}$ we shall have to solve the quadratic indeterminate equation

$$Nx^2 + 1 = y^2.$$

If $x = \alpha$, $y = \beta$ be a solution of this equation, then, says Nārāyaṇa

$$\sqrt{N} = \frac{\beta}{\alpha}, \quad \text{approximately.}$$

In illustration of his method, Nārāyaṇa finds approximations to $\sqrt{10}$ and $\sqrt{\frac{1}{5}}$.[21] Since the roots of $10x^2 + 1 = y^2$ are (6, 19), (228, 721), (8658, 27379), etc., we have

$$\sqrt{10} = \frac{19}{6}, \ \frac{721}{228}, \ \frac{27379}{8658}, \ \text{etc.}$$

Again the values of (x, y) satisfying the equation

$$\sqrt{\frac{1}{5}x^2 + 1 = y^2}$$

are (20, 9), (360, 161), (6460, 2889), etc. Therefore

$$\sqrt{\frac{1}{5}} = \frac{9}{20}, \ \frac{161}{360}, \ \frac{2889}{6460}, \ \text{etc.}$$

7 Jñanarāja's method

Jñanarāja (1503) writes:

> Divide its square by the root of the nearest square number. The quotient together with that approximate root being halved will be a root more approximate than that. Values more and more accurate can certainly be found by proceeding in the same way repeatedly.[22]

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[20] *NBi* (= Nārāyaṇa's *Bījagaṇita*) I, R. 88. *Cf.* Bibhutibhusan Datta, "Nārāyaṇa's method for finding approximate value of a surd", *BCMS*, xxiii (1931), pp. 187–194.

[21] *NBi*, I, Ex. 45.

[22] आसन्नमूलेन हृतात् स्ववर्गाल्लब्धेन मूलं सहितं द्विभक्तम् ।
भवेदासन्नपदं ततोऽपि मुहुर्मुहुः स्यात् स्फुटमूलमेवम् ॥
—*Sundara-siddhānta*, *bījādhyāya*, 12(c–d)–13(a–b).

In other words, if a^2 be the square number nearest to the non-square number N, so that $N = a^2 \pm r$, then the first approximate value (α_1) of $\sqrt{N}$ will be, says Jñanarāja,

$$\frac{1}{2}\left(a + \frac{N}{a}\right).$$

The next approximation will be

$$\frac{1}{2}\left\{\frac{1}{2}\left(a + \frac{N}{a}\right) + \frac{N}{\frac{1}{2}\left(a + \frac{N}{a}\right)}\right\}$$

and so on. The following illustrative examples are given:

$$\sqrt{8} = \frac{1}{2}\left(3 + \frac{8}{3}\right) = \frac{17}{6} = 2^\circ 50' \text{ approximately,}$$

$$\text{also} \quad \sqrt{8} = \frac{1}{2}\left(\frac{17}{6} + \frac{8 \times 6}{17}\right) = \frac{577}{204} = 2^\circ 49' 42'', \quad \text{approximately.}$$

$$\therefore \quad \sqrt{8} = \frac{1}{2}\left(2^\circ 50' + 2^\circ 49' 42''\right)$$

$$= 2^\circ 49' 51'', \quad \text{approximately.}$$

$$\sqrt{20} = \frac{1}{2}\left(4 + \frac{20}{4}\right) = \frac{9}{2} = 4^\circ 30', \quad \text{approximately,}$$

$$\text{also} \quad \sqrt{20} = \frac{1}{2}\left(\frac{9}{2} + \frac{20 \times 2}{9}\right) = \frac{161}{36} = 4^\circ 28' 20'', \quad \text{approximately.}$$

$$\therefore \quad \sqrt{20} = \frac{1}{2}\left(4^\circ 30' + 4^\circ 28' 20''\right)$$

$$= 4^\circ 29' 10'', \quad \text{approximately.}$$

8 Formula of an anonymous writer

In his commentary on the *Līlāvatī* of Bhāskara II, Gaṇeśa (1545) has quoted a rule from a "previous writer" (*ādya*) for finding the approximate value of the square-root of a non-square number. It runs as:

> The residue of the root together with unity is multiplied by 60 and divided by twice the root plus 1. The sixtieth part of the root added with this fraction is (the required approximate value of) the root.

The process implied is clearly this:

$$\sqrt{N} = \frac{\sqrt{3600N}}{60}.$$

Now on finding the square-root of $3600N$ by the ordinary method for it, suppose the root comes out to be b and the residue in excess r. Then according to the rule

$$\sqrt{N} = \frac{1}{60}\left\{b + \frac{60(r+1)}{2(b+1)}\right\}$$

in sexagesimal fractions. It is obviously based on the approximate formula

$$\sqrt{a^2 + r} = a + \frac{r+1}{2(a+1)}.$$

9 Kamalākara

Kamalākara (1658) mentions all the formulae for finding the approximate value of a surd from that of Śrīdhara onwards.[23] But he has always employed the formula of Lalla. Its rationale has been given by him to be as follows.[24]

Suppose

$$\sqrt{b^2 + r} = b + \epsilon,$$

where ϵ is a small quantity. Then

$$b^2 + r = b^2 + 2b\epsilon + \epsilon^2,$$

or

$$\epsilon(2b + 2\epsilon) = r + \epsilon^2.$$

Therefore,

$$\epsilon = \frac{r + \epsilon^2}{2b + 2\epsilon} = \frac{r+1}{2b+2} \quad \text{approximately.}$$

Hence, we have the approximate formula

$$\sqrt{b^2 + r} = b + \frac{r+1}{2b+2},$$

or in sexagesimal fractions:

$$\sqrt{b^2 + r} = b + \frac{60(r+1)}{2(b+1)}.$$

Examples:

$$\sqrt{5} = 2^\circ 14' 10'', \qquad \sqrt{10} = 3^\circ 9' 44'' 12''', \qquad \sqrt{468^\circ 5'} = 21^\circ 28' 7''.$$

By the repeated application of the method, Kamalākara also finds the fourth root of numbers, e.g.,

$$10^{\frac{1}{4}} = 1^\circ 46' 41'' 36'''.$$

[23] *SiTVi*, iii. 10–19.
[24] *SiTVi*, xiv. 324 (comm.).

Part IV

Studies in Indian Astronomy: From Vedic Period to the Emergence of Siddhāntas

Astronomy in ancient and medieval India *

Glimpses of the ancient Hindu astronomy are found in the *Vedas* and the vedic literature. The *Vedāṅgajyotiṣa* (c. 500 BC), which exclusively deals with vedic astronomy, shows that the vedic seers were well versed in the motion of the Sun and the Moon and had developed a luni-solar calendar to regulate their activities. Further progress in the field of Hindu astronomy is recorded by the five well-known *siddhāntas* summarised by Varāhamihira in his *Pañcasiddhāntikā*. These *siddhāntas* were the result of the great Renaissance in Hindu *Gaṇita* which began some time before the beginning of the Christian era. Renaissance in Hindu astronomy which seems to have begun in the third or fourth century AD continued right up to the twelfth century AD. The *Āryabhaṭīya* of Āryabhaṭa I (b. 476 AD) is the earliest preserved work on astronomy written during this period. Of subsequent works, the notable ones are the *Brāhmasphuṭasiddhānta* of Brahmagupta (628 AD), the *Śiṣyadhīvṛddhida* of Lalla (c. 749 AD), the *Vaṭeśvarasiddhānta* of Vaṭeśvara (904 AD), the *Siddhāntaśekhara* of Śrīpati (c. 1039 AD) and the *Siddhāntaśiromaṇi* of Bhāskara II (1150 AD).

Astronomy has been studied in India from time immemorial. The earliest Indian astronomy is preserved in the *Vedas* and the *Vedāṅgajyotiṣa*. The time of composition of these works ranges from c. 2500 BC to c. 500 BC.[1] The *Ṛgveda* divides the Sun's yearly path into 12 and 360 divisions. The Moon's path was likewise divided into 27 parts and each part was called a *nakṣatra*. The stars lying near the Moon's path were also divided into 27 (or sometimes 28) groups and each of them was called a *nakṣatra* (asterism). The names of these *nakṣatras* are found to occur in the *Taittirīya-saṃhitā* of the *Black Yajurveda*. Some of them, viz. *Tiṣya* (i.e. *Puṣya*), *Aghā* (i.e. *Maghā*), *Arjunī* (i.e. *Phālgunī*), *Citrā*, and *Revatī*, are earlier mentioned in the *Ṛgveda*. The above 27 *nakṣatras* were utilised in the study of the position of the Sun and the Moon.

The culture of astronomy in vedic times was motivated by the need of fixing time for the various religious sacrifices which were performed at different times in different seasons. For this the knowledge of the Sun's yearly motion was necessary. The ancient Hindus determined the solstices and the equinoxes and defined the seasons with reference to them. The *Kauṣītaki-brāhmaṇa* records

* K. S. Shukla, *Indian Journal of History of Science*, Vol. 4, Nos. 1–2 (1969), pp. 99–106.

[1] Winternitz, M., *A History of Indian Literature*, Vol. 1, Calcutta (1959), p. 271.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_23

the occurrence of the winter solstice on the new moon day of *Māgha*.[2] It is also stated there that the year ended with the full moon at the *Pūrva-Phālgunī*,[3] and that the spring commenced one day after the new moon of *Caitra*.[4] This shows that the beginning of the year under the *amānta* reckoning synchronised with the beginning of the seasons.

The year of vedic astronomy seems to have been a tropical one. The months were lunar and measured from full moon to full moon and also from new moon to new moon. There is evidence to show that to make the lunar year correspond to the solar year 12 days were intercalated after every lunar year and one month was dropped after every 40 years.[5] At a later stage this correspondence was established by evolving a cycle of five solar years with 62 lunar months. This cycle was called a *yuga*.

The *Vedāṅgajyotiṣa* is the earliest Hindu work dealing exclusively with astronomy. It represents the rudimentary vedic astronomy developed by the Hindus about 500 years before the beginning of the Christian era and shows that at that remote past they considered astronomy as a separate subject of study and realised its importance. The *Vedāṅgajyotiṣa* has come down to us in two recensions, viz. the *Ṛgvedic* recension (called *Ārca-jyotiṣa*) and the *Yajurvedic* recension (called *Yājuṣa-jyotiṣa*). The former contains 36 verses and the latter 43 verses, of which 31 verses are common. Both the recensions are thus practically the same and give an account of months, years, days and day-divisions, *nakṣatras*, new moons and full moons, solstices, and seasons occurring in the cycle of five solar years, which is taken to begin at the winter solstice in the beginning of the month *Māgha* when the Sun and the Moon simultaneously crossed into the *nakṣatra Śraviṣṭhā*. Both state the *tithi* (lunar date), *nakṣatra*, and month in which the Sun commenced its northward and southward journeys in the five-year cycle, give the amount by which the day increased or decreased during the two journeys of the Sun (in terms of water of the water-clock) and lay down rules for determining (i) the beginning of a season, (ii) the positions of the Sun and the Moon, (iii) the *nakṣatra* corresponding to a given *tithi*, (iv) the position of the Sun on its diurnal circle at the end of a *tithi*, (v) the time when the Sun crossed a *nakṣatra*, and (vi) the length of a day. The five-year cycle contains, 1830 civil days, 1835 sidereal days, 1800 *saura* days, 62 lunar months, 5 revolutions of the Sun, and 67 revolutions of the Moon. It is noteworthy that the phenomenon of the winter solstice at the beginning of the *nakṣatra Śraviṣṭhā* with which the five-year cycle is taken to begin occurred about 1200 BC.

There is also a third recension called the *Atharva-jyotiṣa* which belongs to

[2] *Kauṣītaki-brāhmaṇa*, xi. 3.
[3] *Ibid*, v. 1.
[4] *Ibid*, xix. 3.
[5] Law, N. N., *Age of the Ṛgveda*, pp. 20, 28–29.

a later date. It mentions the names of the seven planets[6] and the weekdays and in addition to the *tithi*, *nakṣatra*, and *yoga* (which were already known) gives the names of the seven *karaṇas* of the Hindu calendar. The *Atharva-jyotiṣa* consists of 162 verses and deals with both astronomy and astrology. The teachings of the *Ārca-jyotiṣa* are ascribed to sage Lagadha, and those of the *Atharva-jyotiṣa* to Svayambhū and Bhṛgu.

The Vedas and likewise the *Vedāṅgajyotiṣa* have survived the ravages of time because they were religious works and were studied in their original form. This was not the case with the works dealing with the sciences and the arts. With the emergence of new discoveries, new techniques or tools, or new style of writing, older works had either to be revised or recast or had to be discarded and replaced by new ones. This accounts for a big gap in the existing Sanskrit literature. On the one hand, we have the religious works comprising the vedic literature and, on the other, works written in an entirely new and different style belonging to the early centuries of the Christian era. Practically no scientific work of the intervening period ranging from c. 500 BC to c. 500 AD is available.

From the writings of Varāhamihira (died 587 AD) we gather that works on astronomy written during this intervening period were known as *siddhāntas*. Varāhamihira in his *Pañcasiddhāntikā* summarised the teachings of five of these *siddhāntas*, viz. (1) the *Paitāmahasiddhānta*, (2) the *Saurasiddhānta*, (3) the *Vasiṣṭhasiddhānta*, (4) the *Romakasiddhānta*, and (5) the *Pauliśasiddhānta*. Of these *siddhāntas*, none in its original form is now available.

The five *siddhāntas* summarised by Varāhamihira were written during the early centuries of the Christian era. It is probable that the *Paitāmahasiddhānta* (which is the earliest of the five) was written in 80 AD, this being the epoch mentioned in Varāhamihira's version of that work. The *Vasiṣṭhasiddhānta* was written prior to 269 AD as is shown by the fact that Sphujidhvaja Yavaneśvara, who wrote his *Yavanajātaka* in that year, refers to this work. The other *siddhāntas* were written later. During the early centuries of the Christian era the Indians were in touch with the Greeks and the Romans, and the Babylonian and Greek astronomical texts may have been accessible to them. This accounts for the traces of the Babylonian and Greek influences which are noticeable in the works summarised by Varāhamihira. Neugebauer[7] has shown that some of the astronomical constants in the *Vasiṣṭha* and the *Pauliśasiddhāntas* are inspired by the Babylonian linear astronomy. The *Romaka* and the *Saurasiddhāntas* likewise bear traces of the Greek influence. It is, however, intriguing to find that the refinements introduced by Ptolemy in the Greek astronomy remained unknown to the Hindus.

[6] The seven planets are mentioned in the *Taittirīya-āraṇyaka* and the *Maitrāyaṇī-upaniṣad* also.

[7] Neugebauer, O., *The Exact Sciences in Antiquity*, second edition, 1957.

The progress recorded by the publication of the five *siddhāntas* was indeed the result of the great Renaissance in Hindu mathematics which began some time before the Christian era.[8] The invention of the zero and the decimal place value system of notation and the development of the decimal arithmetic in India in the early centuries of the Christian era led to the development of mathematics in general including algebra and trigonometry. The availability of the decimal arithmetic along with the refined algebraic and trigonometrical tools revolutionised calculations and methods in astronomy. The Renaissance in Hindu astronomy which appears to have begun in the third or fourth century AD continued right up to the twelfth century when due to the advent of the Muslims in India and consequent unsettled political conditions further progress was stopped at least in north India. The writings of Āryabhaṭa I (born 476 AD), Brahmagupta (628 AD), Lalla (c. 749 AD), Vaṭeśvara (904 AD), Āryabhaṭa II (c. 950 AD), Śrīpati (c. 1039 AD), and Bhāskara II (1150 AD) represent landmarks in this era of progress of Hindu astronomy. One very unfortunate consequence of the writings of these eminent scholars has been that the study of the earlier texts was given up, so that they have been lost and we have little authentic material to reconstruct the history of astronomy from the time of the *Vedāṅgajyotiṣa* to the end of the fifth century AD when the *Āryabhaṭīya* was written by Āryabhaṭa I.

The *Āryabhaṭīya* of Āryabhaṭa I written about the end of the fifth century AD is the earliest work on astronomy of the Renaissance period that has been preserved. It is a small work consisting of 121 verses distributed over four chapters, of which Chapter I gives the astronomical constants and the sine table, Chapter II deals with mathematics, Chapter III defines the divisions of time and explains the motion of the planets with the help of eccentrics and epicycles, and Chapter IV describes the armillary sphere and gives rules relating to various problems of spherical astronomy including the calculation and graphical representation of the eclipses and the visibility of the planets. The *Āryabhaṭīya* proved to be a work of great merit, much superior to the earlier *siddhāntas* and won for its author a great name as an original mathematician and astronomer. It laid the foundation of a new school of astronomy, the Āryabhaṭa school, which flourished in India south of the river Narmada. The main centres of this school existed in Aśmaka and Kerala. The Aśmaka school was probably founded by Āryabhaṭa I himself and the main exponents of this school were Bhāskara I (629 AD) and Lalla (749 AD), etc. To the Kerala school belonged Haridatta (683 AD), Govinda Svāmī, Śaṅkaranārāyaṇa (869 AD), Udayadivākara (1073 AD), and others. The astronomers of the Kerala school utilised the constants of the *Āryabhaṭīya* to develop two new systems

[8] B. Datta, 'The Scope and Development of the Hindu Gaṇita', *Indian Historical Quarterly*, Vol. V, No. 3 (1929), p. 484.

of astronomy, the *Parahita* and the *Vākya*, which received wide popularity in that country.

The works of Āryabhaṭa I and later Hindu astronomers were either modifications of earlier works or based on them. From what we know about the earlier *siddhāntas* and the *Pañcasiddhāntikā* we are certain that the *Āryabhaṭīya* and later works followed the same general pattern of the earlier *siddhāntas* and enunciated rules and methods most of which were already well known. It may be that the arrangement of the subject-matter was changed and the style of expression and the language were improved.

In astronomy the epoch as well as the elements by which the mean motions were determined had to be changed from time to time, as we do even now as a result of observation. It may be that Āryabhaṭa I was the first to make considerable changes in the elements. In fact, it has been claimed by his followers that his elements continue to give correct results consistent with observation even after the lapse of long time and in far-off places.[9] It is this change in the elements that accounts in our opinion more for the popularity of the *Āryabhaṭīya* than his theory of rotation of the earth which did not in any way modify the methods of calculation based on a stationary earth. Āryabhaṭa I himself, however, clearly states that he has based his *Āryabhaṭīya* on the teachings of the *Svāyambhuvasiddhānta* (i.e. the *Paitāmaha* or *Brahmasiddhānta*), which was already held in high esteem at Kusumapura.[10] About 130 years later, Brahmagupta based his *siddhānta* on the same *Brahmasiddhānta.* During these 130 years a number of other works were written, based on the *Āryabhaṭīya* and the earlier *siddhāntas.* For example, Lāṭadeva, a direct pupil of Āryabhaṭa I, wrote two works in one of which he reckoned the day from midnight at Laṅkā and in the other from sunset at Yavanapura.[11] According to Varāhamīra, Lāṭadeva was the author of the commentaries on the *Romaka* and *Pauliśa-siddhāntas.*[12] During the same period Śrīṣeṇa brought out a redaction of the *Romakasiddhānta*, Vijayanandī and Viṣṇucandra independently brought out new editions of the *Vasiṣṭhasiddhānta.*[13]

In the early part of the seventh century, we find Bhāskara I writing works based on the *Āryabhaṭīya* wherein he does not change the elements of the *Āryabhaṭīya* but has interpreted the teachings of Āryabhaṭa I according to the traditions of the Aśmaka school and has made additions to Āryabhaṭa I's system to simplify astronomical calculations. At the same time, Brahmagupta brought out another recast, a very comprehensive one, containing 1008 stanzas, of the *Brahmasiddhānta.* His elements differ from those contained in

[9] *Laghubhāskarīya*, i. 2.
[10] *Āryabhaṭīya*, ii. 1.
[11] *Pañcasiddhāntikā*, xv. 18; *Siddhāntaśekhara* ii. 10.
[12] *Pañcasiddhāntikā*, xv. 18.
[13] *Brāhmasphuṭasiddhānta*, xi. 48–51.

the *Āryabhaṭīya*. Brahmagupta and his followers claimed greater accuracy for the *Brāhmasphuṭasiddhānta*, which was adopted as a standard textbook on astronomy in North India. The superiority of Brahmagupta's constants was recognised even by the followers of Āryabhaṭa I who introduced *bīja* correction to modify the constants given by Āryabhaṭa I. Thus, the astronomers of the Aśmaka school, headed by Lalla, introduced a correction taking 499 AD as the origin, whereas the astronomers of the Kerala school introduced a similar correction with 522 AD as the origin. The process of modification and improvement of the existing works continued till the middle of the twelfth century AD when Bhāksara II wrote his *Siddhāntaśiromaṇi*. After the time of Bhāksara II no significant progress was made in the field of astronomy.

Works on Hindu astronomy differ from one another either in the astronomical constants or in the details of calculation. The astronomical constants were subject to correction from time to time as a result of observation, and the methods of calculation were improved with the advance of mathematical knowledge. On basic principles and theories, there is complete unanimity. The following hypotheses are inherent in all of them:

Hypothesis 1: The mean planets revolve in geocentric circular orbits.

Hypothesis 2: The true planets move in epicycles or in eccentrics.

Hypothesis 3: All planets have equal linear motion in their respective orbits.

The Hindu astronomers, unlike their Greek counterparts, have established an epoch when all the planets were in zero longitude. According to Āryabhaṭa I the epoch, when the Sun, the Moon, Mars, Mercury, Jupiter, Venus and Saturn were last in zero longitude, was sunrise at Laṅkā (a hypothetical place at the intersection of the equator and the meridian of Ujjain) on Friday, 18 February, 3102 BC. The period from one such epoch to the next, according to Āryabhaṭa I, is 1,080,000 years. When the Moon's apogee and the Moon's ascending node are included in the list of the planets, the above mentioned period becomes 4,320,000 years, which is defined as the duration of a *yuga*. Thus, a *yuga* is a period of time which begins and ends when the Sun, the Moon, Mars, Mercury, Jupiter, Venus, Saturn, the Moon's apogee, and the Moon's ascending node are in zero longitude. It consists of four periods of 1,080,000 years, which are called quarter *yugas* and bear the names *Kṛtayuga*, *Tretā*, *Dvāpara*, and *Kaliyuga*. The current quarter *yuga* is the current *Kaliyuga* which is assumed to have begun at sunrise at Laṅkā on Friday, 18 February, 3102 BC. A bigger period than the *yuga* is called *kalpa*. According to Āryabhaṭa I, a *kalpa* consists of 1,008 *yugas*, and $459\frac{3}{4}$ *yugas* had elapsed at the beginning of the current *Kaliyuga* since the beginning of the current *kalpa*. The Hindu astronomical works called *Siddhānta* adopt the time of creation as the epoch of calculation whereas those called *Tantra* adopt the beginning of *Kaliyuga* as the epoch of calculation. Both these epochs are epochs of zero longitude, i.e., at these epochs the longitudes of the planets are zero.

The epoch of zero longitude is useful in the computation of the mean longitude of a planet. The Hindu astronomers calculate the *ahargaṇa*, i.e., the number of days elapsed since the epoch chosen and then by the application of the following formula (based on Hypothesis I) to determine the mean longitude of a planet:

$$\text{mean longitude} = \frac{R \times A}{C};$$

where

$R =$ number of revolutions of the planets in a *yuga* (or *kalpa*),

$C =$ number of days in a *yuga* (or *kalpa*), and

$A =$ *ahargaṇa*.

The true geocentric longitude of a planet is derived from its mean longitude by applying the following corrections:

1. Correction for local longitude (*deśāntara*)

2. Equation of the centre (*bāhuphala*)

3. Correction for the equation of time due to the eccentricity of the ecliptic (*bhujāvivara*)

4. Correction for local latitude (*cara*), in the case of the Sun and the Moon, and an additional correction called *śīghraphala* in the case of the other planets. The method of applying these corrections in the case of the planets other than the Sun, however, is not the same with all the astronomers.

Besides the above-mentioned corrections, a few more corrections were devised by later astronomers on the basis of continued observations. Vaṭeśvara (904 AD), for example, gave a lunar correction, which consists of the deficit of the Moon's equation of the centre and the 'evection', and Bhāskara II (1150 AD) gave another lunar correction which corresponds to the 'variation'. Śrīpati (1039 AD) prescribed a general correction which was meant to account for the *equation of time* due to the obliquity of the ecliptic.

The corrections applied to the mean longitude to get the true geocentric longitude were based on the epicyclic theory. Comparison of the Hindu epicyclic theory, as given by Āryabhaṭa I and his followers with that of the Greeks, reveals striking differences between the two theories. Whereas the epicycles of the Greek astronomers do not undergo any variation in size and remain the same at all places, the epicycles of Āryabhaṭa I and other Hindu astronomers are different in size in the beginnings of the odd and even quadrants and vary in size from place to place.

The longitudes of the Sun and the Moon were used in computing the elements of the Hindu Calendar (*Pañcāṅga*), viz. *tithi*, *nakṣatra*, *karaṇa* and *yoga*, and the times of the eclipses. Hindu astronomers were specially interested in the calculation and the projection of the eclipses as they had an important bearing on their religious observances. The Moon and its motion with respect to the *nakṣatras* has been a subject of study from the vedic times. The *Āryabhaṭīya* and other later works treat of the rising and setting, the phases, and the elevation of the horns of the Moon, as also deal with the conjunction of the Moon with the prominent stars or the *nakṣatras*. Among other topics dealt with in the *Āryabhaṭīya* and other works may be mentioned the heliacal risings and settings of the planets and the conjunction of the planets with the prominent stars of the *nakṣatras*.

The Hindu astronomers did not possess the telescope. They made their observations with the naked eye using suitable devices for measuring the angles. Their astronomy therefore remained confined to the study of the Sun, Moon, and the planets.

Main characteristics and achievements of ancient Indian astronomy in historical perspective *

1 Introduction

Ancient Indian astronomy may be classified into two main categories: (1) the vedic astronomy and (2) the post vedic astronomy. The vedic astronomy is the astronomy of the vedic period, i.e., the astronomy found in the vedic *saṃhitās* and *brāhmaṇas* and allied literature. The principal avocation of the people in the vedic times being the performance of the vedic sacrifices at the times prescribed by the *śāstras*, it was necessary to have accurate knowledge of the science of time so that the times prescribed for performing the various vedic sacrifices could be correctly predicted well in advance. Astronomy in those times, therefore, was essentially the science of time-determination. It centred round the Sun and Moon and its aim was to study the natural divisions of time caused by the motion of the Sun and Moon, such as days, months, seasons, and years, special attention being paid to the study of the times of occurrence of new moons, full moons, equinoxes, and solstices.

2 Vedic astronomy

The *Ṛgveda* (1. 52. 11; 10. 90. 14), which is believed to be the earliest of the *Vedas*, describes the universe as infinite and made up of the Earth, the atmosphere, and the sky. According to the *Taittirīya-saṃhitā* (7. 5. 23), fire rests in the Earth, the air in the atmosphere, the Sun in the sky, and the Moon in the company of the *nakṣatras* (zodiacal star-groups). The *Ṛgveda* (1. 105. 10; 4. 50. 4; 10. 123. 1; also see *Śatapatha-brāhmaṇa*, 4. 2. 1) refers to the five planets as gods and mentions *Bṛhaspati*[1] (Jupiter) and *Vena* (Venus)

* K. S. Shukla, in *History of Oriental Astronomy* (Proceedings of an International Astronomical Union Colloquium No. 91, New Delhi, India, 13–16 November, 1985, Edited by G. Swarup, A. K. Bag and K. S. Shukla), Cambridge University Press, Cambridge, 1987, pp. 9–22.

[1] According to the *Taittirīya-brāhmaṇa* 3. 1. 1, "Jupiter when born was first visible in the *nakṣatra Tiṣya* (*Puṣya*)".

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_24

by name.[2] It also mentions the thirty-four lights which, in all probability, are the Sun, the Moon, the five planets and the twenty-seven *nakṣatras* (*Ṛgveda*, 10. 55. 3).

The *Ṛgveda* (8. 58. 2; 1. 95. 3; 1. 164. 14) describes the Sun as the sole light-giver of the universe, the cause of the seasons, the controller and lord of the world (*Aitareya-brāhmaṇa* 2. 7 describes Sun as the cause of the wind). The Moon is called *Sūrya-raśmi*, i.e., one which shines by sunlight (*Taittirīya-saṃhitā* 3. 4. 7. 1). The Moon's path was divided into 27 equal parts, because the Moon took about $27\frac{1}{3}$ days in traversing it. These parts as well as the stars lying in their neighbourhood were called *nakṣatras* and given the names *Kṛttikā* etc. When the constellation called *Abhijit* (Lyra) was included in the list of *nakṣatras*, their number was stated as 28. Of these *nakṣatras*, *Tiṣya* (or *Puṣya*), *Aghā* (or *Maghā*), *Arjunī* (or *Phalgunī*), *Citrā*, and *Revatī* are mentioned in the *Ṛgveda* (5. 54. 13; 10. 64. 8.; 10. 85. 13; 4. 51. 2; 4. 51. 4). The *Taittirīya-saṃhitā* (4. 4. 10. 1-3; see also *Atharva-saṃhitā*, 19. 7. 2–5; *Kāṭhaka-saṃhitā*, 39. 13; *Maitrāyaṇī-saṃhitā*, 2. 13. 20) and the *Taittirīya-brāhmaṇa* (1. 5. 1; 3. 1. 1–2; 3. 1. 4–5) give the names of the 28 *nakṣatras* along with those of the deities supposed to preside over them. The *Śatapatha-brāhmaṇa* (10. 5. 4. 5) gives the names of the 27 *nakṣatras* as well as those of the 27 *upa-nakṣatras*. The *nakṣatras* were categorised into male, female, and neuter as well as into singular, dual, and plural. It seems that the prominent stars of each *nakṣatra* were counted and classified in order of their brilliance.

Some constellations other than the *nakṣatras* were also known. The *Ṛgveda* (1. 24. 10; 10. 14. 11; 10. 63. 10) mentions the *Ṛkṣas* or Bears (the Great Bear and the Little Bear), the two divine Dogs (Canis Major and Canis Minor), and the heavenly Boat (Argo Navis). The Great Bear was also known as *Saptarṣi* (the constellation of the seven sages) and was mentioned by this name in *Śatapatha-brāhmaṇa*[3] (2. 1. 2. 4) and the *Tāṇḍya-brāhmaṇa* (1. 5. 5). The golden Boat (Argo Navis) is mentioned in the *Atharvaveda* (5. 4. 4; 6. 95. 2) also. The *Aitareya-brāhmaṇa* (13. 9) mentions the constellation of *Mṛga* or Deer (Orion) and the star *Mṛgavyādha* (Sirius), and narrates an interesting story regarding them.

Besides the Sun, the Moon, and the *nakṣatras*, mention is also made of some of the other heavenly bodies and heavenly phenomena. For example, *ulkā* (meteors) and *dhūmaketu* (comets) have been mentioned in the *Atharvaveda* (19. 9. 8–9, 19. 9. 10). Eclipses have been mentioned and described as caused by *Svarbhānu* or *Rāhu*. The *Ṛgveda* (5. 40. 5–9) describes an eclipse of the

[2] Other planets are not mentioned by name in the early vedic literature. But *Śani* (Saturn), *Rāhu* (Moon's ascending node), and *Ketu* (Moon's descending node) are mentioned in the *Maitrāyaṇī-upaniṣad*, 7.6.

[3] According to *Śatapatha-brāhmaṇa* (2. 1. 2. 4) the Great Bear was originally called *Ṛkṣa* but later the name *Saptarṣi* was given to it.

Sun as brought about by *Svarbhānu*. The *Tāṇḍya-brāhmaṇa* (4. 5. 2; 4. 6. 13; 6. 6. 8; 14. 11. 14–15; 23. 16. 2) mentions eclipses as many as five times. Eclipses have been mentioned in the *Atharvaveda* (19. 9. 10), the *Gopatha-brāhmaṇa* (8. 19) and the *Śatapatha-brāhmaṇa* (5. 3. 2. 2) also.

The day, called *vāsara* or *ahan* in the vedic literature, was reckoned from sunrise to sunrise. The variability of its length was known. The *Ṛgveda* (8. 48. 7) invoking Somarāja says: "O Somarāja, prolong thou our lives just as the Sun increases the length of the days." Six days were taken to form a *ṣaḍaha* (six-day week); 5 *ṣaḍahas*, a month; and 12 months, a year. As to the names of the six days of a *ṣaḍaha*, there is no reference in the vedic literature. However, the six-day week was later replaced by the present seven day week (*saptāha*) which had attained popularity and was in general use at the time of composition of the *Atharva-jyautiṣa*.

The duration of daylight, reckoned from sunrise to sunset, was divided into two parts called *pūrvāhṇa* (forenoon) and *aparāhṇa* (afternoon), three parts called, *pūrvāhṇa*, *madhyāhna*, and *aparāhṇa*, four parts called *pūrvāhṇa madhyāhna*, *aparāhṇa* and *sāyāhna*,[4] and five parts called *prātaḥ*, *saṅgava*, *madhyāhna*, *aparāhṇa*, and *sāyāhna* (*Śatapatha-brāhmaṇa*, 2. 2. 3. 9). The days and nights were also divided into 15 parts each, and these parts were called *muhūrta*. The *muhūrtas* falling during the days of the light and dark fortnights as well as those falling during the nights of the light and dark fortnights were given specific names (*Taittirīya-brāhmaṇa*, 3. 10. 1. 1–3). The fifteen days and nights of the light fortnight as well as the fifteen days and nights of the dark fortnight were also assigned special names (*Taittirīya-brāhmaṇa*, 3. 10. 1. 1–3; 3. 10. 10. 2).

On the analogy of a civil day, a lunar day was also sometimes reckoned from one moonrise to the next and the name *tithi* was given to it (*Aitareya-brāhmaṇa*, 32. 10). The use of the term *tithi* in the sense in which it is used now occurs in the *Vedāṅga-jyautiṣa* (*Ārca-jyautiṣa*, 20, 21, 31; *Yājuṣa-jyautiṣa*, 20–23, 25, 26). It does not occur in the vedic *saṃhitās* and *brāhmaṇas*, but there are reasons to believe that *tithis* were used even in those times.

The year, generally called by the terms *samā*, *vatsara*, and *hāyana* in the vedic literature, was seasonal or tropical and was measured from one winter solstice to the next, but in due course it was used in the sense of a sidereal year. In the early stages, therefore, the names of the seasons were used as synonyms of a year. The *Kauṣītaki-brāhmaṇa* (19. 3) gives an interesting account of how the year-long sacrifice was commenced at one winter solstice and continued until the next winter solstice: "On the new moon of *Māgha* he (the Sun) rests, being about to turn northwards. They (the priests) also rest,

[4]Of these names the first three occur in *Ṛgveda*, 5. 76. 3; and *sāyam* (evening) occurs in *Ṛgveda*, 8. 2. 20; 10. 146. 3, 40. Kauṭilya (*Arthaśāstra*, 1.19), Dakṣa, and Kātyāyana divided the day and night each into eight parts.

Table 1: Vedic seasons (*Taittirīya-saṃhitā*, 4. 3. 2; 5. 6. 23; 7. 5. 14) and months (*Taittirīya-saṃhitā*, 1. 4. 14; 4. 4. 11)

	Seasons		Months
1.	*Vasanta* (Spring)	1.	*Madhu*
		2.	*Mādhava*
2.	*Grīṣma* (Summer)	3.	*Śukra*
		4.	*Śuci*
3.	*Varṣā* (Rainy)	5.	*Nabhas*
		6.	*Nabhasya*
4.	*Śarada* (Autumn)	7.	*Iṣa*
		8.	*Ūrja*
5.	*Hemanta* (Winter)	9.	*Sahas*
		10.	*Sahasya*
6.	*Śiśira* (Chilly Winter)	11.	*Tapas*
		12.	*Tapasya*

being about to sacrifice with the introductory *Atirātra*. Thus, for the first time, they (the priests) obtain him (the Sun). On him they lay hold with the *Caturviṃśa* rite; that is why the laying hold rite has that name. He (the Sun) goes north for six months; him they (the priests) follow with six day rites in continuation. Having gone north for six months, he (the Sun) stands still, being about to turn southwards. They (the priests) also rest, being about to sacrifice with the *Viṣuvanta* (summer solstice) day. Thus, for the second time, they obtain him (the Sun). He (the Sun) goes south for six months; they (the priests) follow him with six day rites in reverse order. Having gone south for six months, he (the Sun) stands still, being about to turn north; and they (the priests) also rest, being about to sacrifice with the *Mahāvrata* day. Thus, they (the priests) obtain him (the Sun) for the third time".

The *Taittirīya-brāhmaṇa* (3. 9. 22) calls the year "the day of the gods", the gods being supposed to reside at the north pole.

The year was supposed to consist of six seasons and each season of two (solar) months. The relation between the seasons and months was as shown in Table 1.

Two (solar) months commencing with the winter solstice were called *Śiśira*; the next two months, *Vasanta*; and so on. Sometimes *Śiśira* and *Hemanta* were treated as one season and the number of seasons was taken as five (*Aitareya-brāhmaṇa*, 1. 1; *Taittirīya-brāhmaṇa*, 2. 7. 10)

The lunar or synodic month was measured from full moon to full moon or from new moon to new moon (*Taittirīya-saṃhitā*, 7. 5. 6. 1) as is the case even now. The names *Caitra* etc. based on the *nakṣatras* in which the Moon becomes full do not occur in the early *saṃhitās* and *brāhmaṇas* but such terms as *phalgunī-pūrnamāsī*, *citrā-pūrnamāsī*, etc. are found to occur in the *Taittirīya-saṃhitā* (7. 4. 8). They occur in the *Saṅkhāyana* and *Tāṇḍya-brāhmaṇas*, the *Vedāṅga-jyautiṣa*, and the *Kalpa-sūtras*.[5] Twelve lunar months constituted a lunar year. In order to preserve correspondence between lunar and solar years, intercalary months were inserted at regular intervals. Mention of the intercalary month is made in the *Ṛgveda* (1. 25. 8), but how it was arrived at and where in the scheme of months it was introduced in that time is not known. The *Vedāṅga-jyautiṣa* prescribes insertion of an intercalary month after every 30 lunar months (*Yājuṣa-jyautiṣa*, 37). Thus, a year sometimes contained 12 lunar months and sometimes 13 lunar months. The *Taittirīya-saṃhitā* (5. 6. 7) refers to 12 as well as 13 months of a year and calls the thirteenth (intercalary) month by the names *saṃsarpa* and *aṃhaspati* (1. 4. 14). The *Vājasaneyī-saṃhitā* (7. 30; 22. 31) calls the intercalary month on one occasion by the name *aṃhasaspati* and on another by the name *malimluca* (22. 30). In later works the synodic month with two *saṃkrantis* is called *aṃhaspati*, the synodic month without any *saṃkrānti*, occurring before it, is called *saṃsarpa*, and the synodic month without any *saṃkrānti* occurring after it is called *adhimāsa* (intercalary month, *Tantrasaṃgraha* i. 8)

Originally the lunar (or synodic) months *Caitra* etc. were named after the *nakṣatras* occupied by the Moon at the time of full moon. But in due course they were linked with the solar months. Thus, the lunar month (reckoned from one new moon to the next) in which the Sun entered the sign Aries was called *Caitra* or *Madhu*; that in which the Sun entered the sign Taurus was called *Vaiśākha* or *Mādhava*; and so on. The lunar month in which the Sun did not enter a new sign was treated as an intercalary month.

Periods bigger than a year are also met with in the vedic literature. They were called *yuga*. One such *yuga* consisted of 5 solar years. The five constituent years of this *yuga* were called *saṃvatsara, parivatsara, idāvatsara, anuvatsara* and *idvatsara*. The *Ṛgveda* (7. 103. 7–8) mentions two of these, viz. *saṃvatsara* and *parivatsara*. The *Taittirīya-saṃhitā* (5. 5. 7. 1–3), the *Vājasaneyi-saṃhitā* (27. 45; 30. 16), and the *Taittirīya-brāhmaṇa* (3. 4. 11; 3. 10. 4), mention all the five names, with some alteration. The *Taittirīya-saṃhitā* calls them *saṃvatsara*, *parivatsara*, *idāvatsara*, *iduvatsara*, and *vatsara*; the *Vājasaneyi-saṃhitā*, *saṃvatsara*, *parivatsara*, *idāvatsara*, *idvatsara*, and *vatsara*, and the *Taittirīya-brāhmaṇa saṃvatsara*, *parivatsara*, *idāvatsara*, *idvatsara*, and *vatsara* respectively. The names *Kṛta*, *Tretā*, *Dvāpara*, and *Kali*

[5] *Māgha* is mentioned in *Sāṅkhāyana-brāhmaṇa* (= *Kauṣītaki-brāhmaṇa*) 19. 3; *Phālguna* in *Tāṇḍya-brāhmaṇa*, 5. 9. 7–12; and *Srāvana*, *Māgha* and *Pauṣa* in *Ārca-jyautiṣa*, 5, 6, 32, and 34 and *Yājuṣa-jyautiṣa* 5, 6, and 7; and *Mārgaśīrṣa* and *Śrāvana* in *Āśvalāyana-gṛhyasūtra*, 2. 3. 1 and 3. 5. 2 respectively.

which are used in later astronomy as the names of longer *yugas* are also used in the vedic literature to indicate different grades, each inferior to the preceding. But *Dvāpara*, as a unit of time, is found to be used in the *Gopatha-brāhmaṇa* (1. 1. 28).

The earliest work which exclusively deals with vedic astronomy is the *Vedāṅga-jyautiṣa*. It is available in two recensions, *Ārca-jyautisa* and *Yājuṣa-jyautiṣa*. Both the recensions are essentially the same; a majority of the verses occurring in them being identical. The date of this work is controversial, but the situation of the Sun and Moon at the beginning of the *yuga* of five years mentioned in this work, according to T. S. Kuppanna Sastry, existed about 1150 BC or about 1370 BC, according as the first point of *nakṣatra Śraviṣṭhā* stated there means the first point of the *nakṣatra*-segment *Śraviṣṭhā* or the *nakṣatra*-group *Śraviṣṭhā* (Sastry 1984, 3, p. 13). This work defines *jyotiṣa* (astronomy) as the science of time-determination and deals with months, years, *muhūrtas*, rising *nakṣatras*, new moons, full moons, days, seasons, and solstices. It states rules to determine the *nakṣatra* occupied by the Sun or Moon, the time of the Sun's or Moon's entry into a *nakṣatra*, the duration of the Sun's or Moon's stay in a *nakṣatra*, the number of new moons or full moons that occurred since the beginning of the *yuga*, the position of the Sun or Moon at the end of a new moon or full moon day or *tithi*, and similar other things. It gives also the measure of the water-clock, which was used to measure time, and tells when an intercalary month was to be added or a *tithi* was to be omitted. In short, it gives all necessary information needed by the vedic priest to predict times for the vedic sacrifices and other religious observances.

The five-year *yuga* of the *Vedāṅga-jyautiṣa* contained 61 civil, 62 lunar, and 67 sidereal months. The year consisted of 366 civil days which were reckoned from sunrise to sunrise. After every thirty lunar months one intercalary month was inserted to bring about concordance between solar and lunar years. Similarly, to equate the number of *tithis* and civil days in the *yuga* of five solar years, the thirty full moon *tithis* which ended between sunrise and midday were omitted. There were six seasons of equal duration in every year, each new season beginning after every 61 days. Besides *tithis* and *nakṣatras*, the *yoga* called *Vyatīpāta* was also in use.

The five-year *yuga* was taken to commence at the winter solstice occurring at the beginning of the first *tithi* of the light half of the month *Māgha*. Since the Sun and Moon were supposed to occupy the same position at the beginning of each subsequent *yuga* and all happenings in one *yuga* were supposed to be repeated in the subsequent *yugas* in the same way, the calendar constructed on the basis of the *Vedāṅga-jyautiṣa* was meant to serve for a long time.

The *Vedāṅga-jyautiṣa* astronomy suffered from two main defects. Since there are actually 1826.2819 days in a *yuga* of five solar (sidereal) years and not 1830 as stated in the *Vedāṅga-jyautiṣa*, therefore if one *yuga* was taken to commence at a winter solstice the next one commenced about four days later than the next winter solstice and not at the next winter solstice. Similarly, since there are actually 1830.8961 days in a period of 62 lunar months and not 1830 as stated in the *Vedāṅga-jyautiṣa*, therefore there was a deficit of about one *tithi* in the *yuga* of five solar years. These discrepancies must have been rectified but we do not know when and how this was done.

There is one more work on *jyotiṣa* belonging to the later vedic period. It is known as *Atharva-jyautiṣa*. This work describes the *muhūrtas*, *tithis*, *karaṇas*, *nakṣatras*, and week days, and prescribes the deeds that should be performed in them. The names of the lords of the week days stated in this work viz. *Āditya* (Sun), *Soma* (Moon), *Bhauma* (the son of Earth), *Bṛhaspati*, *Bhārgava* (the son of *Bhṛgu*), and *Śanaiścara* (the slow-moving planet), are undoubtedly of Indian origin and must have been in use in India from very early times.[6]

3 Post-Vedic astronomy

In the post-vedic period the scope of astronomy was widened. Astronomy outgrew its original purpose of providing a calendar to serve the needs of the vedic priests and was no longer confined to the study of the Sun and Moon. The study of the five planets was also included within its scope and it began to be studied as a science for its own sake. While further improvement of luni-solar astronomy continued, astronomers now devoted their attention towards the study of the planets which were known in the vedic period and were now well known. In the initial stages their synodic motion was studied. Astronomers noted the times of their first and last visibility, the duration of their appearance and disappearance, the distance from the Sun at the time of their first and last visibility, the times of their retrograde motion, the distances from the Sun at the times of their becoming retrograde and re-retrograde, and so on. Study was also made of their motion in the various zodiacal signs under different velocities called *gatis* (viz. very fast, fast, mean, slow, very slow, retrograde, very retrograde, and re-retrograde) and along their varying paths called *vīthīs*. The synodic motion of a planet was called *grahacāra* and it was elaborately recorded in the astrological works particularly the *saṃhitās*, the earlier works of the Jainas, the earlier *purāṇas*, and the earlier *siddhāntas* such as the *Vasiṣṭhasiddhānta* and the *Pauliśasiddhānta*. These records were analysed and in the beginning crude methods or empirical formulae were evolved to get the longitudes of the planets. Later on a systematic theory was established which gave rise to the astronomy of the later *siddhāntas*.

Of the astronomical works written in this period, the *Vasiṣṭhasiddhānta* is

[6]As regards the origin of the week-days, see Kāne, P. V. (1974) under references.

the earliest. Vasiṣṭha and his teachings have been mentioned in the *Yavanajātaka* of Sphujidhvaja Yavaneśvara which was written about 269 AD. From the summary of the *Vasiṣṭhasiddhānta* in the *Pañcasiddhāntikā* of Varāhamihira we learn that this work made improvement in the luni-solar astronomy and besides describing the synodic motion of the planets gave empirical formulae for knowing the positions of the planets Jupiter and Saturn. The *Vedāṅgajyautiṣa* sidereal year of 366 days was replaced by Vasiṣṭha by the sidereal year of 365. 25 days (Neugebauer & Pingree 1971, ii. 1). To obtain the Sun's longitude use was made of a table giving the Sun's motion in the various zodiacal signs (Neugebauer & Pingree 1971, ii. 1). The Moon's longitude was obtained in a special way. One anomalistic revolution of the Moon was divided into 248 equal parts called *pada*, each *pada* corresponding to 1/9 of a day. The period of the Moon's one anomalistic revolution was called *gati*, and that of 110 anomalistic revolutions *ghana*. It was assumed that the Moon moved through 111 revolutions $-\frac{3}{4}$ signs $+\,2$ mins. in one *ghana* and 1 rev. $(185-\frac{1}{10})$ mins. in one *gati*. First the Moon's anomalistic motion since the epoch was obtained in terms of *ghanas*, *gatis*, and *padas*, and then the Moon's motion corresponding to this was obtained and added to the Moon's position at the epoch (Neugebauer and Pingree 1971, ii. 2–6). To obtain the Moon's motion for *p padas* in the first half of its anomalistic revolution, the formula used was:

$$\text{Moon's motion for } p \text{ } padas \text{ in the first half of its anomalistic revolution} = p \text{ degrees} + \frac{[1094+5(p-1)]p}{63} \text{ mins.}$$

And to obtain the Moon's motion for *p padas* in the second half of its anomalistic revolution, the formula used was:

$$\text{Moon's motion for } p \text{ } padas \text{ in the second half of its anomalistic revolution} = p \text{ degrees} + \frac{[2414-5(p-1)]p}{63} \text{ mins.}$$

(Neugebauer & Pingree 1971, ii. 6)

In the case of Jupiter, starting from the point of zero longitude, its sidereal revolution was divided into 391 equal parts, called *padas*, divided into three unequal segments, the first segment containing 180 *padas*, the second containing the next 195 *padas*, and the third containing the remaining 16 *padas*. When Jupiter was at the end of *p padas* of the first segment, its longitude $\lambda_1(p)$ was given by the formula:

$$\lambda_1(p) = \frac{p(1456-p)}{24} \text{ mins.};$$

when at the end of *q padas* of the second segment, its longitude $\lambda_2(q)$ was given by the formula:

$$\lambda_2(q) = \lambda_1(180) + \frac{q(1165+q)}{24} \text{ mins.};$$

and when at the end of r *padas* of the third segment, its longitude $\lambda_3(r)$ was given by the formula:

$$\lambda_3(r) = \lambda_2(195) + \frac{r(1486-r)}{24} \text{ mins.}$$

(Neugebauer & Pingree 1971, xvii. 9–10)

Similarly, in the case of Saturn, starting with the point of zero longitude, its sidereal revolution was divided into 256 equal parts, called *padas*, divided into three segments, the first segment consisting of 30 *padas*, the second consisting of the next 127 *padas*, and the third consisting of the remaining 99 *padas*. When Saturn was the end of p *padas* of the first segment, its longitude $\lambda_1(p)$ was given by the formula:

$$\lambda_1(p) = \frac{p(2416+2p)}{27} \text{ mins.};$$

when at the end of q *padas* of the second segment, its longitude $\lambda_2(q)$ was given by the formula:

$$\lambda_2(q) = \lambda_1(30) + \frac{q(2519-2q)}{27} \text{ mins.};$$

and when at the end of r *padas* of the third segment, its longitude $\lambda_3(r)$ was given by the formula:

$$\lambda_3(r) = \lambda_2(127) + \frac{r(2037+2r)}{27} \text{ mins.}$$

(Neugebauer & Pingree 1971, xvii. 16–17)

The above formulae show that at the time of their formulation the longitude of Jupiter's apogee was 165.7 degrees and that of Saturn's apogee 220.8 degrees approximately. In the case of the other three planets no such empirical formulae could be devised and recourse was taken to their motion from one heliacal rising to the next.

A notable feature of the *Vasiṣṭhasiddhānta* is that it makes use of signs which were not used up to the *Vedāṅga* period, and reckons the longitudes of the planets from the first point of Aries.

Further progress in astronomy is recorded in the *Pauliśasiddhānta*. Varāhamihira has described the *Vasiṣṭhasiddhānta* as inaccurate but the *Pauliśasiddhānta* as accurate (Neugebauer & Pingree 1971, i. 4).

The length of the sidereal year, according to the *Pauliśasiddhānta*, is 365 days 6 hours 12 seconds (Neugebauer & Pingree 1971, iii. 1). This value is better than 365 days 6 hours given by Vasiṣṭha. Vasiṣṭha used approximate rules to get the longitudes of the Sun and Moon, Pauliśa, in the case of the Sun, first obtains the mean longitude and then applies correction for the equation of the centre to get the true longitude. He states a table giving the equation of the centre for the Sun for the intervals of 30 degrees starting from the point lying 20 degrees behind the point of zero longitude (Neugebauer & Pingree 1971, iii. 1–3).

According to Vasiṣṭha, the Moon's motion on the first day of its anomalistic revolution when it is least is 702′; thereafter it increases and reaches the maximum value of 879′ (Neugebauer & Pingree 1971, iii, 4). According to the tables prepared by the followers of Āryabhaṭa the minimum value of the Moon's motion on the first day of its anomalistic revolution is 722′ and the maximum daily motion near its perigee[7] is 859′, the former being 20′ greater and latter 20′ less than the values given by Vasiṣṭha. The values given by Vasiṣṭha are evidently gross. Pauliśa applied two corrections one after the other to the Moon's motion given by Vasiṣṭha but the rules summarised by Varāhamihira have not been understood so far (Neugebauer & Pingree 1971, iii. 5–8).

Pauliśa calls the Moon's ascending node by the name "*Rāhu*'s head", and takes 6795 days as the period of its sidereal revolution. The corresponding periods, according to Āryabhaṭa, Ptolemy, and modern astronomers are 6794.7 days, 6796.5 days, 6793 days respectively. The value given by Pauliśa is evidently closer to that of Āryabhaṭa.

The *Pauliśasiddhānta* deals also with the motion of the planets, the visibility of the Moon, and the eclipses. In the treatment of the planetary motion, it gives the distances from the Sun at which the planets rise or set heliacally and become retrograde and re-retrograde. Table 2 gives the synodic periods of the planets according to Vasiṣṭha, Pauliśa, Āryabhaṭa, Ptolemy, and modern astronomers.

Pauliśa's treatment of the visibility of the planets and the eclipses is very approximate.

A notable feature of the *Pauliśasiddhānta* is the mention of the *viṣuva* and *ṣaḍaśītimukha saṃkrāntis*.

The *Pauliśasiddhānta* was followed by the *Romakasiddhānta*. This *siddhānta* bears the impact of the teachings of the Greek astronomers. The day is reckoned from sunset at Yavanapura (Alexandria in Egypt). To obtain the mean positions of the Sun and Moon a luni-solar *yuga* of 2850 years was defined and astronomical parameters were stated for this period (Neugebauer & Pingree 1971, i. 15–16). The length of the year used in this work was

[7] Vide *Candravākyāni*. See Kuppanna Sastri & Sarma (1962) under references. For *Candrasāraṇī*, see *Sūrya-candra-sāraṇī*. Ms. No. 1657 of the Akhila Bharatiya Sanskrit Parishad, Lucknow.

Table 2: Synodic periods in days.

Planet	Vaśiṣṭha	Pauliśa	Āryabhaṭa	Ptolemy	Modern
Mars	779.955	799.978	779.92	779.943	779.936
Mercury	115.879	115.875	115.87	115.879	115.877
Jupiter	398.889	398.885	398.889	398.886	398.884
Venus	583.909	583.906	583.89	584.000	583.921
Saturn	378.1	378.110	378.08	378.093	378.092

365.246 days (Neugebauer & Pingree 1971, viii. 1.) which is exactly the same as given by the Greek astronomers Hipparchus and Ptolemy. It was really the value of the tropical year but it was used in the *Romakasiddhānta* as the value of the sidereal year. As the value of the sidereal year it was worse than that given by Vasiṣṭha. The longitude of the Sun's apogee stated in the *Romakasiddhānta* (Neugebauer & Pingree 1971, viii. 2) was 75°. This was the same as given by Hipparchus when reckoned from the point of zero longitude of Indian astronomy. The period of a sidereal revolution of the Moon's ascending node according to the *Romakasiddhānta* was 6796.29 days. This is also almost the same as the value 6796.5 days given by Ptolemy. The maximum equation of the centre for the Sun adopted in the *Romakasiddhānta* (Neugebauer & Pingree 1971, viii. 3, 6). was $2°23'23''$ and that for the Moon $4°56'$. The corresponding values given by Ptolemy are $2°23'$ and $5°1'$ respectively. Romaka's treatment of the solar eclipse was similar to that found in the later works on Indian astronomy but the rules given are very approximate (Neugebauer & Pingree 1971, viii). It may be that Varāhamihira himself has condensed them. The *Romakasiddhānta* did not deal with the planets.

Perfection in astronomy was brought about by Āryabhaṭa who carried out his observations at Kusumapura (modern Patna). He was successful in giving quite accurate astronomical parameters and better methods of calculation. Roger Billard (Billard 1971, pp. 81–83) has analysed these parameters and has shown that they were based on observations made around 512 AD.

Āryabhaṭa wrote two works on astronomy, in one reckoning the day from midnight to midnight and in the other from sunrise to sunrise, in the former dealing with the subject in detail and in the latter briefly and concisely. Both the works proved to be epoch-making and earned a great name for the author. The larger work was popular in northern India and was summarised by Brahmagupta in his *Khaṇḍakhādyaka* which was carried to Arabia and translated into Arabic. This work has been in use by the *pañcāṅga* makers in Kashmir till recently. The *Sūryasiddhānta* which was summarised by

Varāhamihira and declared by him as the most accurate work was simply a redaction of the larger work of Āryabhaṭa. The smaller work of Āryabhaṭa called the *Āryabhaṭīya* was studied in south India from the seventh century to the end of the nineteenth century. This work was also translated into Arabic, by Abu'l Hasan al-Ahwāzī.

Āryabhaṭa 's astronomy is based on three fundamental hypotheses viz.

1. That the mean planets revolve in geocentric circular orbits
2. That the true planets move in epicycles or in eccentrics
3. That all planets have equal linear motion in their respective orbits.

Āryabhaṭa's epicyclic theory differs in some respects from that of the Greeks. In Āryabhaṭa's theory there is no use of the hypotenuse-proportion in finding the equation of the centre. Moreover, unlike the epicycles of the Greek astronomers which remain the same in size at all places, Āryabhaṭa's epicycles vary in size from place to place.

The main achievements of Āryabhaṭa are:

1. His astronomical parameters which were well known for yielding accurate results
2. His theory of the rotation of the Earth which was described by him as spherical like the bulb of the *kadamba* flower
3. The introduction of sines by him
4. His value of $\pi = 3.1416$
5. Fixation of the Sun's greatest declination at $24°$ and the Moon's greatest celestial latitude at $4°30'$. These values were adopted by all later Indian astronomers.
6. Integral solution of the indeterminate equation of the first degree viz. $ax + c = by$, a, b, and c being constants.

The pattern set by the works of Āryabhaṭa was followed by all later astronomers. The works written by later astronomers differ either in the presentation of the subject matter, or in the astronomical constants which were revised from time to time on the basis of observation, or in the methods of calculation which were improved from time to time. A few new corrections which were not known in the time of Āryabhaṭa were discovered and used by later astronomers. Thus Mañjula (also called Muñjāla) discovered the lunar correction called "evection" and Bhāskara II another lunar correction called "variation".

According to Āryabhaṭa the Sun, the Moon, and the planets were last conjunction in zero longitude at sunrise at Laṇkā[8] on Friday, February 18, 3102 BC. This was chosen by him as the epoch of zero longitude to calculate the longitudes of the planets. The period from one such epoch to the next, according to him, is 10,80,000 years. This he has defined as the duration of a quarter *yuga*. Likewise the period of 43,20,000 years is called a *yuga*. At the beginning and end of this *yuga* the Moon's apogee and ascending node too are supposed to be in conjunction with the Sun, Moon, and the planets at the point of zero longitude. The revolution-numbers of the planets stated by Āryabhaṭa are for this *yuga*. The astronomical parameters and the rules stated by Āryabhaṭa are sufficient to solve all problems of Indian astronomy. The main problems dealt with by Āryabhaṭa and other later astronomers are the determination of the elements of the Indian *pañcāṅga*, calculation and graphical representation of the eclipses of the Sun and Moon, rising and setting of the Moon and the planets, the Moon's phases and the elevation of the Moon's horns and their graphical representation, and the conjunction of the planets and stars.

The ancient Indian astronomers did not posses the telescope. They made their observations with the naked eye using suitable devices for measuring angles. Their astronomy therefore remained confined to the study of the Sun, Moon, and the planets.

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[8] Laṅkā is the hypothetical place on the equator where the meridian of Ujjain intersects it.

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Discussion

L. C. Jain: Could you comment on whether the Jaina Astronomy (*Sūrya Prajñapati*, *Candra Prajñapati* or *Tiloyapannatti*) was motivated by *Vedāṅga Jyotiṣa* or originated independently?

K. S. Shukla: It was motivated by *Vedāṅga Jyotiṣa.*

On three stanzas from the *Pañcasiddhāntikā* *

1. We will here consider three stanzas from the *Pañcasiddhāntikā* of Varāhamihira (c. 550 AD), edited by G. Thibaut and S. Dvivedī (1889 AD). These stanzas were examined by us while comparing the astronomical constants of the midnight day-reckoning of Āryabhaṭa I (499 AD), as given by his follower Bhāskara I (629 AD), with those of the old *Sūryasiddhānta*, as summarised by Varāhamihira. This comparison revealed to us that the astronomical constants ascribed to Āryabhaṭa I's midnight day-reckoning were in general agreement with those found in Varāhamihira's version of the *Sūryasiddhānta*. The differences were, however, found to exist as regards the distances from the Sun at which the planets become visible and as regards the distances and diameters of the Sun and the Moon. It was soon discovered that the differences were not real but were due to the emendations made in the traditional text of the *Pañcasiddhāntikā* by the editors.

2. Of the above-mentioned three stanzas, one is stanza 12 of the seventeenth chapter. It states the distances of the planets from the Sun at which they rise heliacally, and runs as follows:

Traditional text

स्फुटदिनकरांतरांशा-
श्चन्द्रादीनां च दर्शनीज्ञेयाः ।
विशतिरूनावसुशशि
शिखिमुनिनवरुद्रेदियैः क्रमशः ॥

Text as emended by Thibaut and Dvivedī

स्फुटदिनकरान्तरांशा-
श्चन्द्रादीनां च दर्शने ज्ञेयाः ।
विंशतिरूना वसुशशि-
शिखिमुनिनवकेन्द्रियैः क्रमशः ॥

The emended version, translated by Thibaut, is as follows:

> The degrees of the distances from the sun at which the true planets become visible are 12 for the moon, 19 for Mars, 17 for Mercury, 13 for Jupiter, 11 for Venus, 15 for Saturn.

* K. S. Shukla, *Ganita*, Vol. 5, No. 2 (1954), pp. 129–136.

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Table 1: Distances from the Sun at which the planets become visible.

Planet	Distance according to			
	Modern *Sūryasiddhānta*	Āryabhaṭa I and Bhāskara I	Brahmagupta	The above emended text
Moon	12°	12°	12°	12°
Mars	17°	17°	17°	19°
Mercury	12° to 14°	13°	13°	17°
Jupiter	11°	11°	11°	13°
Venus (when direct)	10°	9°	10°	11°
Saturn	15°	15°	15°	15°

The constants given in this stanza and those given in the modern *Sūrya-siddhānta* and by Āryabhaṭa I and Brahmagupta (628 AD) are exhibited in Table 1.

The table shows that the constants given in the emended text differ from those given by the other Hindu authorities in the case of Mars, Mercury, Jupiter, and Venus and that the differences are such as to throw doubt in the correctness of the emended text. It appears from the comparison of the last three columns that the error, if any, in the emended text lies between the words giving the constants for the Moon and Mars and between the words giving the constants for Venus and Saturn. Note that the constants for Mars, Mercury, and Jupiter in the second and third columns have shifted bodily by one space downwards in the last column.

Let us now examine the traditional text to see whether it gives any clue to the above discrepancy. We observe that

(i) it is inconsistent with the subject matter, as the number of constants mentioned there is seven, whereas the number of planets to which those constants correspond is only six; and

(ii) it is metrically defective, as there are 14 syllables in place of 12 in the third quarter.

Turning to the emended text, we find that Thibaut and Dvivedī have got rid of the above defects of the traditional text by replacing the word *rudra* (meaning 11) by the suffix *ka*. And this drastic change, made in the traditional text, is indeed the cause of the whole trouble.

The most plauseworthy emendation of the text at this place would be the deletion of the superfluous word *śaśi* (meaning Moon or 1). With this emendation the stanza would run

स्फुटदिनकरान्तरांशा-
श्चन्द्रादीनां च दर्शने ज्ञेयाः ।
विंशतिरूना वसुशिखि-
मुनिनवरुद्रेन्द्रियैः क्रमशः ॥

and mean

> The degrees of the true distances from the Sun at which the Moon and others become visible are 12, l7, 13, 11, 9, and 15 respectively.

One may easily see that these constants are exactly the same as prescribed by Āryabhaṭa I, and also not much different from those occurring in the modern *Sūryasiddhānta*.

3. The other two stanzas are stanzas 15 and 16 of the ninth chapter. They deal with the distances and diameters of the Sun and the Moon and run as follows:

Traditional Text

मुनिकृतगुणेन्द्रियघ्नः
स्फुटकर्णः खकृतभाजितोऽर्कस्य ।
कक्षेति चन्द्रकरणों
दृघ्नः कक्षा शशांकस्य ॥
खखवसुखमुनींद्रविषया
भानोः खकृतर्तुसुगुणाः शशिनः ।
तात्कालिकमानार्थं
स्फुटकक्षाभ्यां पृथग्विभजेत् ॥

Text as emended by Thibaut and Dvivedī

मुनिकृतगुणेन्द्रियघ्नः
स्फुटकर्णः खार्कभाजितोऽर्कस्य ।
कक्षेति चन्द्रकर्णो-
ऽग्निघ्नः कक्षा शशाङ्कस्य ॥
खवसुखमुनीन्द्रविषया
भानोः खकृतर्तुसुरगुणाः शशिनः ।
तात्कालिकमानार्थं
स्फुटकक्षाभ्यां पृथग्विभजेत् ॥

The emended text, translated by Thibaut, runs:

> The true hypotenuse multiplied by 5347 and divided by 120 gives the *kakshā* of the sun; the true hypotenuse of the moon multiplied by 3 gives the *kakshā* of the moon. Take 5147080 for the sun and 333640 for the moon and, in order to find their (apparent) dimensions for a given time, divide those two quantities separately by the true distances in *yojanas*.

In the notes that follow this translation, Thibaut interprets these stanzas as containing the following formulae:

$$\text{Sun's true distance in } \textit{yojanas} = \frac{5347 \times (\text{Sun's mean distance in mins.})}{120};$$

$$\text{Moon's true distance in } \textit{yojanas} = 3 \times (\text{Moon's mean distance in mins.});$$

$$\text{Sun's true diameter in minutes} = \frac{5147080}{\text{Sun's true distance in } \textit{yojanas}};$$

$$\text{Moon's true diameter in minutes} = \frac{333640}{\text{Moon's true distance in } \textit{yojanas}}.$$

Both Thibaut and Dvivedī derive the first two formulae by assuming 5347 and 360 to be the mean distances in *yojanas* of the Sun and the Moon respectively. But this assumption does not agree with the numbers used in the last two formulae, as they yield 962.6 and 926.8 minutes for the mean diameters of the Sun and the Moon respectively, which is wrong. These numbers are about 30 times greater than the real diameters. Thibaut and Dvivedī, therefore, prescribe the division by 30 of the diameters obtained by the application of the third and fourth formulae. Dvivedī thinks that this division by 30 has been omitted in the text probably because, in the time of Varāhamihira, this operation was obligatory by convention. Thibaut is, however, doubtful of the correctness of the text, and writes:

> But for some reason or other their text—provided it be correct—does not mention the division by 30.

Thibaut and Dvivedī's assumption that the numbers 5347 and 360 denote the distances (in *yojanas*) of the Sun and Moon respectively is incompatible with their assumption in the next chapter[1] of the number 146 for the diameter (in *yojanas*) of the Sun. The last mentioned number should have been

[1] See Thibaut's and Dvivedī's notes on *Pañcasiddhāntikā*, x. 1.

Table 2: Mean distances and diameters (in *yojanas*) of the Sun and Moon according to the modern *Sūryasiddhānta* and the midnight day-reckoning of Āryabhaṭa I

(a) Actual

	Modern *Sūryasiddhānta*		Midnight day-reckoning of Āryabhaṭa I	
	Distance	Diameter	Distance	Diameter
Sun	689378	6500	689358	6480
Moon	51566	480	51566	480

(b) As abraded by 42.97

	Modern *Sūryasiddhānta*		Midnight day-reckoning of Āryabhaṭa I	
	Distance	Diameter	Distance	Diameter
Sun	16043	151	16040	150.8
Moon	1200	11.4	1200	11.4

more appropriately taken to be 151 *yojanas.* Table 2 shows that the correct distances of the Sun and the Moon conforming to the diameter 151 *yojanas* of the Sun are 16040 and 1200 *yojanas* and not 5347 and 360 *yojanas* as assumed by Thibaut and Dvivedī.

The inconsistencies in the interpretation of Thibaut and Dvivedī are due, as in the previous case, to the changes made by them in the traditional text. For example, *khakṛta* (meaning 40) has been changed into *khārka* (meaning 120), and *dṛ* has been changed into *agni.*

Such drastic changes are not necessary; emendation of the obvious clerical errors is enough to secure mathematically correct meaning. With these minor corrections, the text reads:

मुनिकृतगुणेन्द्रियघ्नः
स्फुटकर्णः खकृतभाजितोऽर्कस्य ।
कक्षेति चन्द्रकर्णो
दिग्घ्नः कक्षा शशाङ्कस्य ॥
खवसुखमुनीन्दुविषया

भानोः खकृततर्तु(व)सुगुणाः शशिनः ।
तात्कालिकमानार्थं
स्फुटकक्षाभ्यां पृथग्विभजेत् ॥

This gives the following four formulae:

$$\text{Sun's true distance in } \textit{yojanas} = \frac{5347 \times (\text{Sun's true distance in minutes})}{40};$$

$$\text{Moon's true distance in } \textit{yojanas} = 10 \times (\text{Moon's true distance in minutes});$$

$$\text{Sun's true diameter in minutes} = \frac{517080}{\text{Sun's true distance in } \textit{yojanas}};$$

$$\text{Moon's true diameter in minutes} = \frac{38640}{\text{Moon's true distance in } \textit{yojanas}}.$$

These may be derived as follows:

Assuming 16040 and 1200 *yojanas* to be the distances of the Sun and Moon respectively,[2] we have

$$\text{Sun's true distance in } \textit{yojanas} = \frac{16040 \times (\text{Sun's true distance in minutes})}{120},$$

120 being the value of the radius (in minutes) used in the *Pañcasiddhāntikā*. Thus,

$$\text{Sun's true distance in } \textit{yojanas} = \frac{5347 \times (\text{Sun's true distance in minutes})}{40}.$$

Similarly,

$$\text{Moon's true distance in } \textit{yojanas} = \frac{1200 \times (\text{Moon's true distance in mins.})}{120} = 10 \times (\text{Moon's true distance in mins.}).$$

Now assuming that the Sun's mean diameter is $32\frac{95}{401}$ minutes and the Moon's mean diameter 32.2 minutes, we have

$$\text{Sun's true diameter in minutes} = \frac{(\text{Sun's mean diameter in minutes}) \times (\text{Sun's mean distance in } \textit{yojanas})}{\text{Sun's true distance in } \textit{yojanas}} = \frac{517080}{\text{Sun's true distance in } \textit{yojanas}}.$$

[2] See Table 2b.

Similarly,

$$\text{Moon's true diameter in minutes} = \frac{38640}{\text{Moon's true distance in } \textit{yojanas}}.$$

(**ed.** The following note is given as a footnote to the above equation in the original:) The word '*yojana*' has been used above in the general sense of a 'linear unit'. It should not be confused with the terrestrial *yojana* of the *Pañcasiddhāntikā* which is equal to 7.8 miles approximately.

The *Pañcasiddhāntikā* of Varāhamihira (1) *

The *Pañcasiddhāntikā* of Varāhamihira is one of the most important sources for the history of Hindu astronomy before the time of Āryabhaṭa I (b. 476 AD). Two editions of this work (both furnished with English translation and commentary) have appeared, one in 1889 under the editorship of G. Thibaut and S. Dvivedi, and the other in two parts in 1970 and 1971 under the editorship of O. Neugebauer and D. Pingree. But even now the contents of the work are at places not correctly understood. The object of the proposed series of papers is to deal with certain passages of the work which have not been properly understood so far. In the present paper, which is the first of the series, I propose to deal with four topics, viz. (i) criticism of Viṣṇucandra and Romaka by Pauliśa, (ii) the declination table of Varāhamihira, (iii) the fifth correction for Mercury and Venus in the old *Sūryasiddhānta*, and (iv) a traditional correction of the Pauliśa school for the longitude of the Moon's ascending node.

1 Viṣṇucandra and Romaka criticised by Pauliśa

The following seven verses (**ed.** see Table 1) occurring in the end of the third chapter of the *Pañcasiddhāntikā*, which contains the teachings of the *Pauliśa-siddhānta*, were not clear to G. Thibaut and S. Dvivedi and so these verses were left uninterpreted by them in their edition of the *Pañcasiddhāntikā*.

D. Pingree, whose edition of the *Pañcasiddhāntikā* appeared in 1970, has translated the above verses as follows:

32. If the beginning (*pratipatti*) occurs when there is separation of *tithi* and *nakṣatra*, then it is good. But it is not so in a *bhadrā tithi* and Viṣṇu's *nakṣatra* (Śravaṇa): for thus does the world disappear.

33. There is not simultaneously everywhere a rising of the Sun or its setting. In what place is its setting? From that basis they know what has passed of the day.

* K. S. Shukla, *Indian Journal of History of Science*, Vol. 9, No. 1 (1974), pp. 62–76. (Updated version of the paper originally published in *Gaṇita*, Vol. 24, No. 1 (June 1973), pp. 59–73. This paper was read at the seminar organised by the Indian National Science Academy, New Delhi, on the occasion of the 500th Birth Anniversary of Nicolaus Copernicus on February 19–20, 1973.)

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Table 1

Manuscript Text	Emended Text
तिथिनक्षत्रच्छेदा- प्रतिपत्तिर्यदि तथा ततः साधुः । न तथा च भद्रविष्णो- स्तथा विनिवर्तते लोकः ॥३२॥	तिथिनक्षत्रच्छेद- प्रतिपत्तिर्यदि तथा ततः साधुः । न तथा च भद्रविष्णो- स्तथापि विनिवर्तते लोकः ॥३२॥
न युगपदुदयो भानु- रस्तमयो वापि भवति सर्वत्र । कस्मिन् देशेस्तमये पादादित्ये न भक्तिमिंदुः ॥३३॥	न युगपदुदयो भानो- रस्तमयो वापि भवति सर्वत्र । कस्मिन् देशेऽस्तमयः पादाद्दिनेन भुक्तं विदुः ॥३३॥[1]
मार्गादुपेतमेतत् काले लघुता न तावदतिदूरे । षविषयभूताष्टरसै- रब्दैः पश्यास्य विनिपातम् ॥३४॥	मार्गादुपेतमेतत् काले लघुता न तावदतिदूरे । खविषयभूताष्टरसै- रब्दैः पश्यास्य विनिपातम् ॥३४॥
रोमकमहर्गणं पा- दमर्कमिंदुं च गणयतां तां ग्राह्य । चैत्रस्य पौर्णमास्यां नवमी नक्षत्रमादित्यम् ॥३५॥	रोमकहर्गणं पा- दमर्कमिंदुं च गणयतां ग्राह्य । चैत्रस्य पौर्णमास्यां नवमी नक्षत्रमादित्यम् ॥३५॥
कालापेक्षा विधय- श्रौताः स्मार्ताश्च तदपचारेण । प्रायश्चित्ती भवति द्विजो यतोतोधिगम्येदम् ॥३६॥	कालापेक्षा विधयः श्रौताः स्मार्ताश्च तदपचारेण । प्रायश्चित्ती भवति द्विजो यतोऽतोऽधिगम्येदम् ॥३६॥
कुकरणविदो द्विजो ये कथयंत्यस्फुट सत्यं ... । कुकरणकारसहि- ते क्षणं नरके कृतवासाः ॥३७॥	कुकरणविदो द्विजो ये कथयन्त्यस्फुट(म)सत्यं (च गणितम्) । कुकरणकारसहि(ताश्च) ते क्षणं नरके कृतवासाः ॥३७॥[1]
स्फुटगणितविदिह लब्ध्वा धर्मार्थयशांसि दिनकरादीनां ॥३८॥	स्फुटगणितविदिह लब्ध्वा धर्मार्थयशांसि दिनकरादीनाम् ॥३८॥

[1] Emended by D. Pingree

34. This is arrived at from a method; there is no quickness in so very long a time. Look at its (the world's) destruction in 68550 years.
35. Taking the Romaka *ahargaṇa* as the basis, let one calculate (the longitudes of) the Sun and the Moon on the full moon (*tithi*) of Caitra; on the ninth (*tithi*) the *nakṣatra* is Āditya (Punarvasu).
36. The *śrauta* and *smārta* regulations depend on time; because a twice-born through offending them is a *prāyaścittī* (i.e., he has to perform propitiatory rites), therefore he studies this (i.e., time).
37. Whatever twice-born men, knowing a bad *karaṇa*, say that (astronomical) calculations are inaccurate and false, they, together with the makers of bad *karaṇas*, instantly make their homes in hell.
38. (But) one who knows accurate calculations of the Sun, and so on, obtains *dharma*, wealth, and praise in this world.

O. Neugebauer and D. Pingree have supplemented the above translation by the following commentary:

> These verses are evidently based on some obscure speculation in *Romakasiddhānta* about the duration of creation.
>
> The separation of *tithi* and *nakṣatra* presumably means that at the first *tithi* of the month the Moon is not in the first *nakṣatra*, Aśvinī; this separation is supposed to be an auspicious *muhūrta* for the *pratipatti*, i.e. the beginning of any action (or the beginning of creation?). However, if on a *bhadrā tithi* (the 2nd, 7th, or 12th in any *pakṣa*) the Moon is in Śravaṇa (Sagittarius 10° to 23° 20′), the *muhūrta* is inauspicious. The inauspiciousness arises from the fact that the creation ceases at such a *yuga*, i.e. when the conjunction of the Sun and Moon (the first *tithi*) occurs in Uttarāṣāḍha, i.e. at the winter solstice. This is reminiscent of Hellenistic speculations regarding a "world-year".
>
> The 68550 years in verse 34 is derived from the *Romakasiddhānta*; it is equal to $24 \times 19 \times 150 + 150$, where $19 \times 150 = 2850$ years is the Romaka's *yuga* (*cf.* ch. 1, vs. 15). The significance of this computation is obscure.
>
> The meaning of verse 35 also defies comprehension. Dikshit has indeed demonstrated that, by the elements of Varāhamihira's *Sūryasiddhānta*, the Caitra whose *pratipad* is used as epoch in this

karaṇa is *pūrṇimānta*; but there is no reason to compute the longitudes of the Sun and Moon for the *pūrṇima* of that month. Moreover, at *Caitrapūrṇimā* the Moon must be close to Libra 0° so that the Moon on the ninth *tithi* is far from Punarvasu (Gemini 20° to Cancer 3°20′). The reference to Punarvasu rather suggests an ecpyrosis at the summer solstice as we had a cataclysm at the winter solstice (vs. 32), but the text as it stands does not allow us to arrive at this interpretation.

The above translation and commentary clearly shows that Neugebauer and Pingree have not understood the real import of the text and are guided by conjectures only. They are indeed off the track. The verses in question, in fact, constitute a criticism of Viṣṇucandra and Romaka whose *tithis* and *nakṣatra*s were showing a wide divergence from the actual ones. The following modified translation would make the contents quite clear:

32. If the end (*cheda*) or commencement (*pratipatti*) of *tithi* and *nakṣatra* is as it should be, then it is good. But that of Śrī Viṣṇu(candra)[1] is not so; even then people (instead of discarding him) revert to him.

33. There is not simultaneously everywhere (on the same meridian) a rising of the Sun or its setting. In what meridian (lit. place) is its setting? From that basis they say what has passed of the day.[2]

34. From the tradition (of the *śāstras*) it is learnt that there is no decrease in time even after a lapse of enormous time. (But) look at its (the world's) destruction in 68550 years (advocated by Romaka).

35. For those who calculate (the longitudes of) the Sun and Moon on the full moon day of Caitra, taking the Romaka *ahargaṇa* as the basis, it is the ninth (*tithi*) and the Punarvasu *nakṣatra* (and not the full moon *tithi* and the Citrā *nakṣatra* as it should be).

36. The *śrauta* and *smārta* regulations depend on time; because a twice-born through offending them is a *prāyaścittī* (i.e. he has to perform propitiatory rites), therefore he studies this (time-ascertaining science of astronomy).

37. Those twice-born who, having studied bad *karaṇas*, declare inaccurate and false calculations, they, together with the authors of bad *karaṇas*, instantly make their homes in hell.

[1] Bhadraviṣṇu = Bhadra (=Śrī) + Viṣṇu (=Viṣṇucandra).

[2] This is a criticism of the rule which seeks to tell the time of a place on one meridian from the time of a place on another meridian by using the difference of longitudes of the two places only. In fact, correction due to difference in latitudes of the two places has also to be made.

38. (But) one who knows accurate calculations of the Sun, etc., obtains *dharma*, wealth, and praise in the world.

This translation is self-explanatory and on the basis of it one can easily draw the following conclusions:

1. In the time of Pauliśa, Viṣṇucandra's edition of the *Vasiṣṭhasiddhānta* was not yielding correct *tithis* and *nakṣatras*. But Viṣṇucandra was a popular astronomer and had a great following.

2. Calculations based on the *Romakasiddhānta* were showing an error of six *tithis* and seven *nakṣatras*.

3. Pauliśa, like Āryabhaṭa I, believed that time had no beginning or end, but Romaka held the contrary view.

Criticism of Viṣṇucandra and Romaka in the *Pauliśasiddhānta* further shows that *Pauliśasiddhānta* was written subsequent to the *siddhāntas* of Viṣṇucandra and Romaka. The statement of Varāhamihira, viz.

रोमकसिद्धान्तेऽयं नातिचिरे पौलिशेऽप्येवम्।

in ch. 1, vs. 10 is thus significant and should be understood to mean:

> This is according to the *Romakasiddhānta*; so it is also according to the *Pauliśasiddhānta* which is not much old.

This is the natural and straightforward meaning of the above hemistich.

Occurrence of criticism of Viṣṇucandra, Romaka, Vijayanandī and Pradyumna in the writing of a person like Varāhamihira shows that Brahmagupta's critical remarks against them are not totally baseless and unjustified. Sarcastic remarks against the Romakas are also found in the writings of Bhāskara I who was a contemporary of Brahmagupta. It is significant that Pauliśa has not been criticised by Brahmagupta or others.

2 The declination table of Varāhamihira

We now turn to verses 16–18(i) of ch. IV of the *Pañcasiddhāntikā*. Thibaut and Dvivedi were unable to interpret these verses and the credit of interpreting them for the first time is again due to D. Pingree. Pingree supposed that these verses contained the declination-differences for every $7°30'$ of the ecliptic (beginning with the first point of Aries) corresponding to the obliquity of the ecliptic equal to $23°40'$. So he emended the text as follows:

Manuscript Text	As emended by D. Pingree
जीवाध्यार्द्धशतांशाः	जीवा व्यश्व्यर्धशतांशााः
सैकाः षष्टिदिनेशकाष्ठांतः ।	साङ्कलिप्ता दिनेशकाष्ठातः ।
चंद्रस्य सविक्षेप-	चंद्रस्य स विक्षेप-
स्तदपक्रमराशिपादेन्यः ॥१६॥	स्तदपक्रमो राशिपादेभ्यः ॥१६॥
लिप्ताशतमासीत-	लिप्ताशतमशीतिं
दशास्त्रिषयुक्तमिंद्रियमनूनां ।	दशात्रिसंयुक्तामिन्द्रियमनूनाम् ।
गविसेमनुभवमुनि-	गवि मनुभवमुनिरूपै-
रूपैश्चगुणैः संयुतं च शतं ॥१७॥	श्च (त्रि)गुणैः संयुतं च शतम् ॥१७॥
नवतिस्त्रियुता षष्टि-	नवतिस्त्रियुता षष्टि-
श्चत्वारिंशछिवाश्च मिथुनान्तरे ।	श्चत्वारिंशच्छिवाश्च मिथुनान्ते ।

And his translation runs as follows:

16. The Sine of the maximum declination (*kāṣṭhā*) of the Sun is 50 minus 2 (= 48) parts and 9 minutes. (As) there is a latitude of the Moon, (so) is there a declination (of the Sun; it is) for fourths of a sign:

17. 180 minutes, plus 10 (= 190), plus 3 (= 183), minus 5 (= 175), and minus 14 (= 166); in Taurus 100 plus 14 times 3 (= 142), plus 11 times three (= 133), plus 7 times 3 (= 121), and plus 1 times 3 (= 103);

18. 90, 60 plus 3 (= 63), 40 plus 3 (= 43), and 11 at the end of Gemini.

The declination-differences given above are exhibited in Table 2 which also gives the corresponding modern values when the obliquity of the ecliptic $\epsilon = 23°40'$. The value $48'9''$ of the Sine of the Sun's maximum declination given above corresponds to the obliquity of the ecliptic equal to $23°40'$.

Comparison of the textual values with the modern ones in Table 2 clearly shows that there is a significant difference between the two. We cannot expect such a wrong table from Varāhamihira. Evidently Pingree has missed the target and has not been able to interpret the text correctly. Had he checked the accuracy of his values by comparing them with the modern ones he must have saved himself from committing the error. He has also missed to see that according to Varāhamihira, $\text{Sin}(23°40') = 48'9''$, and not 48 parts and 9 minutes as stated by him.

In fact, there is no need of changing the text to that extent. The following minor emendation of the text would be sufficient to rectify it:

Manuscript Text	Emended Text
जीवाध्यर्द्धशतांशाः	जीवाऽध्यर्धशतांशाः
सैकाः षष्टिदिनेशकाष्ठांतः ।	सैका षष्टिर्दिनेशकाष्ठाऽतः ।
चंद्रस्य सविक्षेप-	चंद्रस्य सविक्षेप-
स्तदपक्रमराशिपादेन्यः ॥१६॥	स्तदपक्रमो राशिपादेभ्यः ॥१६॥
लिप्ताशतमासीत-	लिप्ता साशीतिशतं
दशस्त्रिषयुक्तमिंद्रियमनूनां ।	मेषे त्रिखयुक्तमिंद्रियमनूनम् ।
गविसेमनुभवमुनि-	गवि मनुभवमुनिरूपै-
रूपैश्चगुणैः संयुतं च शतं ॥१७॥	श्च(तु)र्गुणैः संयुतं च शतम् ॥१७॥
नवतिस्त्रियुता षष्टि-	नवतिस्त्रियुता षष्टि-
श्चत्वारिंशछिवाश्च मिथुनांतरे ।	श्चत्वारिंशच्छिवाश्च मिथुनान्ते ।

This emendation does not interfere with the numerical parameters given in the text and is intended simply to rectify the grammatical error in the first half of verse 17 (Pingree has overlooked it) and to supply the missing word *meṣe* (meaning "in Aries") in view of the presence of the words *gavi* (meaning "in Taurus") and *mithunānte* (meaning "at the end of Gemini"). Thus we have interchanged the words *māsīta* (corrected as *sāśīti*) and *śata* (corrected as *śataṃ*) and replaced the unnecessary word *daśa* by *meṣe*. We have also inserted the missing letter *tu* in the last quarter of verse 17; Pingree had inserted *tri*. The unnecessary letter *se* has been removed from the third quarter of verse 17, as was also done by Pingree.

With the above emendation the text may be translated as follows:

16. The Sine (= $120' \times$ sine) of the Sun's maximum declination is $\frac{61}{75}$ of a degree or $48'48''$ (*saikā ṣaṣtiḥ* = 60 + 1; *adhyardhaśatāṃśāḥ* = *adhi* + *ardhaśatāṃśāḥ* = *adhyardha*+*ardhaśatāṃśāḥ* = one and a half times 50). With the help of it one may calculate the Sun's declination (for the desired time). That (declination) plus the Moon's latitude is the Moon's declination. The declinations arising from the successive quarters of the zodiacal signs are the following:

17. In Aries, 180 plus 3 (= 183), plus 0 (= 180), minus 5 (= 175), and minus 14 (= 166) minutes; in Taurus, 100 plus 4 times 14 (= 156), plus 4 times 11 (= 144), plus 4 times 7 (= 128), and plus 4 times 1 (= 104) minutes;

18. (then) 90, 60 plus 3 (= 63), 40, and 11 (minutes) at the end of Gemini.

Since $\frac{61}{75}$ of a degree is equal to $48'48''$ which is the Sine of 24° according to Varāhamihira (vide ch. IV, vs. 24), it follows that the declination-differences given in the above verses correspond to the obliquity of the ecliptic equal to

Table 2: Declination-differences for every $7°30'$ of the Sun's longitude (λ) when $\epsilon = 23°40'$.

λ	$\Delta\delta$ (modern) (correct to half a minute)	$\Delta\delta$ (textual)	Difference
$7°30'$	$3°0'$	$180' + 10' = 3°10'$	$+ 10'$
$15°$	$2°57'30''$	$180' + 3' = 3°\ 3'$	$+ 5'30''$
$22°30'$	$2°52'30''$	$180' - 5' = 2°55'$	$+ 2'30''$
$30°$	$2°44'30''$	$180' - 14' = 2°46'$	$+ 1'30''$
$37°30'$	$2°34'$	$100' + 42' = 2°22'$	$- 12'$
$45°$	$2°20'30''$	$100' + 33' = 2°13'$	$- 6'30''$
$52°30'$	$2°5'$	$100' + 21' = 2°\ 1'$	$- 4'$
$60°$	$1°46'30''$	$100' + 3' = 1°43'$	$- 2'30''$
$67°30'$	$1°25'30''$	$90' = 1°30'$	$+ 4'30''$
$75°$	$1°2'30''$	$63' = 1°\ 3'$	$+ 0'30''$
$82°30'$	$0°38'30''$	$43' = 0°43'$	$+ 4'30''$
$90°$	$0°13'$	$11' = 0°11'$	$-2'$
Total	$23°40'$	$23°40'$	0

24°. We give below in Table 3 the declination-differences stated in the above verses along with the corresponding modern values, taking the obliquity of the ecliptic (ϵ) to be equal to 24°. The differences between the two are also noted.

Table 3 shows that the values given in the text are generally in agreement with the modern ones. This proves that our interpretation of the text is correct. The value of the Sine of the Sun's maximum declination according to our interpretation is exactly the same as that given by Varāhamihira in the same chapter (in vs. 24).

3 The fifth correction for Mercury and Venus in the old *Sūryasiddhānta*

In the old *Sūryasiddhānta* school, the true longitudes of the superior planets (Mars, Jupiter and Saturn) were obtained by applying the following four corrections:

Table 3: Declination-differences for every $7°30'$ of the Sun's longitude (λ) when $\epsilon = 24°$.

λ	$\Delta\delta$ (modern) (correct to half a minute)	$\Delta\delta$ (textual)	Difference
$7°30'$	$3°2'30''$	$180' + 3' = 3°3'$	$+0'30''$
$15°$	$3°$	$180' + 0' = 3°$	
$22°30'$	$2°54'30''$	$180' - 5' = 2°55'$	$+0'30''$
$30°$	$2°47'$	$180' - 14' = 2°46'$	$-1'$
$37°30'$	$2°36'$	$100' + 56' = 2°36'$	
$45°$	$2°23'$	$100' + 44' = 2°24'$	$+1'$
$52°30'$	$2°6'30''$	$100' + 28' = 2°8'$	$+1'30''$
$60°$	$1°48'$	$100' + 4' = 1°44'$	$-4'$
$67°30'$	$1°27'$	$90' = 1°30'$	$+3'$
$75°$	$1°3'30''$	$60' + 3' = 1°3'$	$-0'30''$
$82°30'$	$0°39'$	$40' = 0°40'$	$+1'$
$90°$	$0°13'$	$11' = 0°11'$	$-2'$
Total	$24°00'$	$24°00'$	0

For obtaining the true longitude of the planet's apogee:

1. Half *śīghraphala* to the longitude of the planet's apogee (reversely).
2. Half *mandaphala* to the corrected longitude of the planet's apogee (reversely).

For obtaining the true longitude of the planet:

3. Entire *mandaphala* (calculated with the help of the true longitude of the planet's apogee) to the mean longitude of the planet.
4. Entire *śīghraphala* to the corrected mean longitude (called true-mean longitude) of the planet.

In the case of the inferior planets (Mercury and Venus) a fifth correction (called *pañcama saṃskāra*) was applied in addition to the above mentioned four corrections. In the case of Mercury this correction was calculated and applied in accordance with the following rule:

Subtract the longitude of the Sun's apogee from the longitude of Mercury's *śīghrocca*; multiply the Rsine of the resulting difference by the Sun's epicycle and divide by 360; the quotient gives the fifth correction for Mercury. Apply it to the longitude of Mercury (as corrected for the above mentioned four corrections) like the *mandaphala* of the Sun, i.e., subtract it when Mercury's *śīghrocca* minus Sun's apogee is less than 180° and add it when otherwise.

This correction has been stated in verse 21, chap. XVI (Pingree's edition) of the *Pañcasiddhāntikā*, the correct text of which runs as follows:

सर्वे स्फुटाः स्युरेवं ज्ञस्य तु शीघ्राद्विहाय रविमन्दम् ।
रविपरिधिनतं बाहुं बुधेऽर्कवत् क्षयधनं कुर्यात् ॥२१॥

In Thibaut and Dvivedi's edition of the *Pañcasiddhāntikā* the reading is *budhaphalavat* in place of *budhe'rkavat*, so their interpretation of the text has become erroneous. This rule, however, has been mentioned by Lalla in his *Śiṣyadhīvṛddhida* (I, ii. 37 (ii)) and is stated correctly there.

Pingree supposed that the above correction was applicable not only to Mercury but to Venus as well, so he has emended the text as follows:

सर्वे स्फुटाः स्युरेवं ज्ञेड्येषु शीघ्राद्विहाय रविमन्दम् ।
रविपरिधिनतं बाहुं बुधे कवौ क्षयधनं कुर्यात् ॥२१॥

In doing so Pingree was probably guided by the consideration that in the school of Āryabhaṭa I in the matter of planetary correction Mercury and Venus go together. But from the writings of astronomer Sumati, who belongs to the school of the old *Sūryasiddhānta*, we now know definitely that the above correction was meant for Mercury and Mercury alone. Sumati writes:[3]

> अर्कोच्चं बुधशीघ्रोच्चे शोध्य ज्याघ्नं शराश्विभिः ।
> भक्तं रूपाब्धिकोषैस्तु क्षयक्षेपबुधस्फुटम् ॥
>
> बुधस्य पंचमं कर्म सूर्यवत्संस्फुटीकृतम् ॥
>
> Having subtracted the longitude of the Sun's apogee from the longitude of Mercury's *śīghrocca*, multiply the Rsine thereof by 25 and divide by 641;[4] application of this (quotient) as a negative or positive correction (to the longitude of Mercury as corrected for the four corrections) gives the true longitude of Mercury.
>
> The fifth correction for Mercury should be applied like the correction for the Sun.

In the case of Venus, the fifth correction is always subtractive. Its value is found to be stated in three different forms:

[3] *Sumati-mahātantra* (MS., British Museum).
[4] $\frac{\text{Sun's epicycle}}{360} = \frac{14}{360} = \frac{25}{641}$.

1. Half the Sun's *mandaphala*.

2. $10 \times \frac{\text{radius}}{514}$ minutes, where radius $= 3438'$.

3. 67 minutes.

It can be easily verified that all the three forms yield the same value, viz. 67 minutes of arc. Form (3) is found in the *Pañcasiddhāntikā*; form (1) is mentioned in the *Śiṣyadhīvṛddhida* of Lalla. Sumati gives all the three forms. Writes he:

> व्यासार्धं दशभिर्निघ्नं शक्रबाणैर्विभाजयेत् ।
> भानोर्भूप्रतिचक्रार्धं स्फुटशुक्रे विशोधयेत् ॥
>
> शुक्रस्य पञ्चमं कर्म सप्तषष्टिकलैः क्षयम् ।
> The radius multiplied by 10 and divided by 514, or half the distance between (the centres of) the Earth and the Sun's eccentric should be subtracted from the true longitude of Venus (i.e., from the longitude of Venus as corrected for the four corrections).
>
> The fifth correction for Venus is the subtraction of 67 minutes of arc.

When Āryabhaṭa I wrote his *Āryabhaṭa-siddhānta* based on the old *Sūrya-siddhānta*, he dropped the fifth correction. And later on when Brahmagupta wrote his *Khaṇḍakhādyaka* based on the *Āryabhaṭa-siddhānta*, he followed Āryabhaṭa I and did not use the fifth correction. From Lalla's statement in his *Śiṣyadhīvṛddhida* we learn that it was in regular use in his time. Mallikārjuna Sūri (1178 AD), who has written a commentary on the *Śiṣyadhīvṛddhida*, does not seem to be aware of the school to which the correction belonged. He has ascribed it to the followers of Āryabhaṭa I.

When the old *Sūryasiddhānta* was revised and given the present form, the fifth correction was considered superfluous and was discarded.

4 A traditional correction of the Pauliśa school for the longitude of the Moon's ascending node

In Chapter VI of the *Pañcasiddhāntikā* where Varāhamihira deals with the calculation of a lunar eclipse according to the *Pauliśasiddhānta*, there occurs the following verse having reference to a correction to be applied to the longitude of the Moon's ascending node:

> राहोः सषड्कृतिकलं हित्वांशं तच्छशांकविवरांशैः ।
> ग्रहणं त्रयोदशान्तः पञ्चदशान्तस्तमस्तस्य ॥२॥

The same verse with some alteration reappears in Chapter VII, which deals with the calculation of a solar eclipse according to the same *Pauliśasiddhānta*:

राहोः सषड्कृतिकलं हित्वांशं तच्छशांकविवरांशैः ।
ग्रहणं त्रयोदशान्तः शशिनो भानोस्तथाष्टान्तः ॥५॥

These verses have been translated by Thibaut and Pingree as follows.

Thibaut's translation:

2. Deduct from the longitude of Rāhu twenty-six minutes, and thereupon take the degrees intervening between Rāhu and the Moon. If these degrees are within thirteen, there is an eclipse; if within fifteen, there is the shadow of an eclipse.
5. Deduct twenty-six minutes from the longitude of Rāhu, and take the degrees intervening between Rāhu and the Moon. If they are within thirteen, there takes place an eclipse of the Moon; and an eclipse of the Sun, if they are within eight.

Pingree's translation:

2. Put down the degrees of the ascending node increased by 36 (or 26?) minutes. (Operate) with the degrees of the difference between this and (the longitude of) the Moon; if they are within 13°, there is an eclipse, and if within 15°, a darkening of it (the Moon).
5. Put down the degrees of the ascending node increased by 36 (or 26?) minutes. (Operate) with the degrees of the difference between this and (the longitude of) the Moon; if they are within 13°, there is an eclipse of the Moon, and if within 8°, an eclipse of the Sun.

A close scrutiny reveals that the translation of the first line of each of the above two verses as given by both Thibaut and Pingree is not correct, because

राहोः सषड्कृतिकलं अंशं हित्वा

actually means "having subtracted one degree together with thirty six minutes". The above two verses should therefore be translated as follows:

2. One degree and thirty-six minutes having been subtracted from (the longitude of) the Moon's ascending node, if the degrees arising from the difference of that (corrected longitude of Moon's ascending node) and (the longitude of) the Moon are within thirteen, there is an eclipse (of the Moon), and if within fifteen, there is a darkening of that (Moon).

5. One degree and thirty-six minutes having been subtracted from (the longitude of) the Moon's ascending node, if the degrees arising from the difference of that (corrected longitude of the Moon's ascending node) and (the longitude of) the Moon are within thirteen, there is an eclipse of the Moon, and if within eight, there is an eclipse of the Sun.

The correctness of this translation is confirmed by the fact that the correction of $1°36'$ to the longitude of the Moon's ascending node was in regular use amongst the followers of the *Khaṇḍakhādyaka* of Brahmagupta (b. 598 AD). Although this correction was not mentioned in the *Khaṇḍakhādyaka*, the followers of the *Khaṇḍakhādyaka* made use of it as a traditional correction. The following verse occurring in a manuscript[5] of the *Khaṇḍakhādyaka* in the collection of the Akhila Bharatiya Sanskrit Parishad, Lucknow, throws light on this tradition:

> **पातस्य सम्प्रदायाद् विशोधयेदेकमंशकं लिप्ताः ।**
> **षट्त्रिंशत्स्फुटपातस्स भवति सर्वत्र साधने योग्यः ॥**[6]
>
> From (the longitude of) the Moon's ascending node one should, following the tradition, subtract one degree and thirty six minutes. Then is obtained the true (longitude of the) Moon's ascending node, which is fit for use in all calculations.

This verse is also mentioned in Bina Chatterjee's edition of the *Khaṇḍakhādyaka* (Vol. II, p. 8, footnote, lines 10–11), where it runs as:

> **पातस्य सम्प्रदायाद्विशोधयेदेकमंशकं लिप्ताः ।**
> **षड्विंशतिः स्फुटपातः स भवति सर्वत्र साधने योग्यः ॥**

The reading **षड्विंशतिः** given here is undoubtedly wrong, firstly because in the same edition elsewhere[7] the correction in question has been expressly stated as "ninety six minutes" (**षण्णवतिः कलाः**) and secondly because the reading **षड्विंशतिः** does not fit in in the metre of the verse. With this reading the third quarter of the verse contains 13 syllabic instants (*mātrās*), whereas in fact there should be 12 syllabic instants only.

It is noteworthy that the commentators of the *Khaṇḍakhādyaka* have prescribed the use of the above correction if the longitude of the Moon's ascending node was calculated according to the rule given in the *Pūrva Khaṇḍakhādyaka* and have forbidden its use if the longitude of the Moon's ascending node was calculated according to the rule given in the *Uttara Khaṇḍakhādyaka.* Thus writes the commentator Pṛthūdaka (864 AD):

[5] Accession No. 1662; script: Śāradā.

[6] This verse occurs in the manuscript after verse 14 of chapter I of *PKK* (= *Pūrva Khaṇḍakhādyaka*).

[7] See comm. on *PKK*, p. 104, line 23 and p. 120, line 4. Also see comm. on *UKK* (= *Uttara Khaṇḍakhādyaka*), ch. 1, vs. 3, p. 177, line 14.

तस्मात् षण्णवतिः कलाः संशोध्याः सम्प्रदायावच्छेदाः। पारम्पर्येणैवं कृते कर्मयोग्यश्चन्द्रपातो भवति।[8]
उत्तरकृताच्चन्द्रपातात् षण्णवतिः कला न शोध्या इति।[9]

From that (i.e. the longitude of the Moon's ascending node calculated according to *Pūrva Khaṇḍakhādyaka*) one should subtract the traditional correction of 96 minutes. This correction having been applied in accordance with the tradition, the longitude of the Moon's ascending node becomes fit for use in calculations.

From the longitude of the Moon's ascending node calculated from (the rule given in) the *Uttara Khaṇḍakhādyaka*, 96 minutes should not be subtracted.

So also writes the commentator Bhaṭṭotpala (968 AD):

अंशः सषड्कृतिकलः शोध्यः पातस्य पूर्वस्य।[10]

अनेन प्रकारेण कृतस्य चन्द्रपातस्य षण्णवतिः कला न शोध्याः।[11]

One degree together with thirty-six minutes should be subtracted from (the longitude of) the Moon's ascending node calculated according to *Pūrva* (*Khaṇḍakhādyaka*).

Ninety-six minutes should not be subtracted from the longitude of the Moon's ascending node if it is calculated by this method (of the *Uttara Khaṇḍakhādyaka*).

Note that the language used by Bhaṭṭotpala in his first statement is exactly similar to that used by Varāhamihira.

One may ask the question: How is it that the correction prescribed for application to the longitude of the Moon's ascending node by the *Pauliśa-siddhānta* of Varāhamihira was regarded as traditional by the followers of the *Pūrva Khaṇḍakhādyaka*? The reason seems to be that at a certain stage the followers of the *Pauliśasiddhānta* fell in line with the followers of the *Āryabhaṭa-siddhānta*. They revised the old *Pauliśasiddhānta* in the light of the teachings of the *Āryabhaṭa-siddhānta* and adopted the *Pūrva Khaṇḍakhādyaka* (which was based on the *Āryabhaṭa-siddhānta*) as a work of their own school. Quotations from the *Pauliśasiddhānta* which are found to occur in the writings of Pṛthūdaka (864 AD), Bhaṭṭotpala (968 AD), Āmarāja (c. 1200 AD) and the Persian scholar Al-Bīrūnī (b. 973 AD) leave no room to doubt that the revised

[8] See *Khaṇḍakhādyaka* (P. C. Sengupta's edition), ch. 1, vs. 14 (comm.), p. 13, lines 16–18. Also see p. 13, lines 26–27, and ch. IV, vs. I (i) (comm.), p. 91, lines 13–14.
[9] *Ibid*, *Khaṇḍakhādyakottaram*, vs. 2 (comm.), p. 150, lines 25–26.
[10] See *Khaṇḍakhādyaka* (Bina Chatterjee's edition), Vol. I, p. 163, line 6. Also see Vol. II, p. 104, lines 23–24 and p. 120, line 4.
[11] *Ibid*, Vol. II, *tithinakṣatrottarādhyāyaḥ*, vs. 3 (comm.), p. 177, lines 13–14.

Pauliśasiddhānta was in conformity with the teachings of Āryabhaṭa I under the midnight day-reckoning. It is noteworthy that the commentators of the *Khaṇḍakhādyaka* have shown special preference to *Pauliśasiddhānta* in their citations from the ancient *siddhāntas*.

The followers of the *Uttara Khaṇḍakhādyaka* did not apply the above correction because the *Uttara Khaṇḍakhādyaka* conformed to the teachings of the *Brāhmasphuṭasiddhānta* of Brahmagupta and such a correction was not prescribed there.

Note

The correction of $-96'$ for the Moon's ascending node shows its appearance in the school of Āryabhaṭa I under the sunrise day-reckoning also. For example, the *bīja* correction prescribed for the Moon's ascending node in the verses

> शाके नखाब्धिरहिते शशिनोऽक्षदस्रैः
> तत्तुङ्गतः कृतशिवैस्तमसः षडङ्कैः ।
> शैलाब्धिभिः सुरगुणोर्गुणिते सितोच्चा-
> च्छोध्यं त्रिपञ्चकुहतेऽभ्रशराक्षिभक्ते ॥
> स्तम्बेरमाम्बुधिहते क्षितिनन्दनस्य
> सूर्यात्मजस्य गुणितेऽम्बरलोचनैश्च ।
> व्योमाग्निवेदनिहते विदधीत लब्धं
> शीतांशुसूनुचलतुङ्गकलासु वृद्धिम् ॥

ascribed to astronomer Lalla is based in the assumption that in the year 420 *Śaka* (= 498 AD) the *bīja* correction for the Moon's ascending node was zero and that in the year 670 *Śaka* (= 748 AD) it decreased to $-96'$. Similarly, the *bīja* correction prescribed for the Moon's ascending node in the verses

> चन्द्रे बाणकरा बीजाश्चन्द्रोच्चे मनुभूमयः ।
> कुजे शून्यशरा ज्ञेयाः खाग्निवेदा बुधस्य तु ॥
> गुरोः खपञ्च विज्ञेयाः शुक्रे खाक्षनिशाकराः ।
> शनेः शशिकराः प्रोक्ता राहोः षण्णवतिः स्मृताः ।
> भवभानूनिते शाके बीजघ्ने शबरोद्धृते ।
> फलं लिप्ता विलिप्ताश्च ज्ञारार्कीणां धनं भवेत् ।
> राहुचन्द्रोच्चजीवानामृणं कार्यं भृगोरपि ॥

mentioned in Haridatta's *Grahacāranibandhanasaṃgraha* (vv. 19–22(i)) and quoted by Sūryadeva in his commentary on the *Laghumānasa* (*dhruvaka-nibandha*, 1–2) and by Nīlakaṇṭha in his commentary on the *Āryabhaṭīya* (iv. 48) is based on the assumption that in the year 444 *Śaka* (= 522 AD) the *bīja* correction for the Moon's ascending node was zero and that in the year 679 *Śaka* (= 757 AD) it decreased to $-96'$. Assumption of $-96'$ as the *bīja* correction for the Moon's ascending node in the years 748 and 757 AD seems to have been due to the influence of the followers of the *Pūrva Khaṇḍakhādyaka*. It must however be noted that whereas the followers of the *Pūrva Khaṇḍakhādyaka* used it as a fixed *bīja*, the followers of the *Āryabhaṭīya* used it as a variable *bīja* taking its value to be $\frac{-96}{250}$ or $\frac{-96}{235}$ minutes of arc per annum.

The *Pañcasiddhāntikā* of Varāhamihira (2) *

This is the second paper of this series.[1] In this paper I propose to deal with three topics, viz. (1) the epoch of Varāhamihira's *Romakasiddhānta*, (2) Vasiṣṭha's theory for the Moon's motion, and (3) the 30 days of the Parsi calendar.

1 The epoch of Varāhamihira's *Romakasiddhānta*

Verses 8–10 of Chapter 1 of the *Pañcasiddhāntikā* in G. Thibaut and S. Dvivedi's edition, run as follows:

> सप्ताश्विवेदसंख्यं शककालमपास्य चैत्रशुक्लादौ ।
> अर्धास्तमिते भानौ यवनपुरे सोमदिवसाधे ॥८॥
> ....
> रोमकसिद्धान्तेऽयं .... ॥१०॥
>
> (To calculate the *ahargaṇa* first) subtract 427 *Śaka* years, which elapsed at sunset at Yavanapura, the beginning of the light half of Caitra synchronising with the commencement of Monday, (from the given *Śaka* year) .... This is according to the *Romakasiddhānta*.

This shows that the epoch of calculation adopted in Varāhamihira's *Romaka-siddhānta* is sunset at Yavanapura (Alexandria), beginning of Monday and the first *tithi* of the light half of Caitra, *Śaka* 427 (505 AD).

David Pingree, following S. B. Dikshit, has replaced सोमदिवसाद्ये in the above verse by भौमदिवसाद्ये and thus he takes the above epoch at the beginning of Tuesday instead of at the beginning of Monday.

There are reasons to believe that the above epoch occurred at the beginning of Monday, as stated by G. Thibaut and S. Dvivedi, and not at the beginning of Tuesday, as stated by Dikshit and Pingree. A very simple and straightforward proof may be given in support of this assertion. This is as follows:

In verse 17 of the same chapter, Varāhamihira takes 2227 as the *ahargaṇa* for the above mentioned epoch reckoned since the commencement of a seven-civil year cycle starting with a Sunday. Since

$$2227 \equiv 1 \pmod{7}$$

* K. S. Shukla, *Gaṇita*, Vol. 28, 1977, pp. 99–116.

[1] The first paper of this series appeared in *Gaṇita*, Vol. 24, No. 1 (1973), pp. 59–73.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_27

it is clear, without any shadow of doubt, that the above epoch of Varāhamihira's *Romakasiddhānta* is the beginning of Monday and not the beginning of Tuesday.

Verse 17, referred to above, and verse 18 following it, contain a rule for finding the lord of the current civil year. These are:

> मुनियमयमद्वियुक्ते द्युगणे शून्यद्विपञ्चयमभक्ते ।
> प्रतिराश्य खर्तुदहनैर्लब्धं वर्षाणि यातानि ॥१७॥
> तानि प्रपन्नसहितान्यग्निगुणान्यश्विवर्जितानि हरेत् ।
> सप्तभिरेवं शेषो वर्षाधिपतिः क्रमात् सूर्यात् ॥१८॥
>
> Add 2227 to the *ahargaṇa* (reckoned from the epoch of the *Pañcasiddhāntikā*) and divide (the resulting sum) by 2520 (= 7 × 360, the number of days in a 7-civil year cycle): (the quotient gives the number of the 7-civil year cycles elapsed since the epoch occurring 2227 days before the above mentioned epoch of Varāhamihira's *Romakasiddhānta*).
>
> In another place divide the remainder of the division by 360: the quotient (of this division) gives the number of civil years elapsed since the beginning of the current 7–civil year cycle.
>
> To these civil years add 1, for the current civil year, then multiply by 3, then subtract 2, and then divide by 7: the remainder counted with the Sun, in the order of the lords of the week days, gives the lord of the current civil year.

It means that a 7–civil year cycle commenced 2227 days prior to the epoch of Varāhamihira's *Romakasiddhānta*, when it was the beginning of a Sunday.

That a 7–civil year cycle actually commenced 2227 days before the epoch of Varāhamihira's *Romakasiddhānta* and the next one 293 days after that epoch is confirmed from the fact that astronomer Sūryadeva Yajvā (b. 1191 AD), in his commentary on the *Āryabhaṭīya*, iii. 5, takes the beginning of a 7-civil year cycle on Sunday *Kali-Ahargaṇa* 1317416, which occurred 293 days after the epoch of Varāhamihira's *Romakasiddhānta*. For, the *Kali-Ahargaṇa* for the epoch of Varāhamihira's *Romakasiddhānta* is 1317123, and 1317416 – 1317123 = 293.

We may point out here that the next verse (i.e., vs. 19) of the same chapter, as stated by Thibaut and Dvivedi and Pingree, is incorrect. The correct reading is that stated by Bhaṭṭotpala (in his commentary on *Bṛhatsaṃhitā*, ch. 2, Sudhākara Dvivedi's ed., p. 32), viz.

> त्रिंशद्भक्ते मासाः प्रपन्नसहिता द्विसङ्गुणा व्येकाः ।
> सप्तोद्धृतावशेषे मासाधिपतिस्तथैवार्कात् ॥१९॥

(The same remainder) divided by 30 (instead of 360) gives the number of months elapsed since the beginning of the current 7-civil year cycle. These increased by 1, for the current month, then multiplied by 2, then diminished by 1, and then divided by 7, and the remainder counted from Sunday, as before, gives the lord of the current month.

The reading कार्याः, given by Thibaut etc., in place of व्येकाः is not correct as it does not lead to a correct rule. For the correct rule, one might also refer to *Brāhmasphuṭasiddhānta*, xiii. 43 and *Siddhāntaśekhara*, ii. 88.

2 Vasiṣṭha's theory for the Moon's motion

The *Vasiṣṭha-samāsa-siddhānta*, summarised by Varāhamihira in Chapter 2 of the *Pañcasiddhāntikā*, ascribes to Moon a regular zigzag motion which increases step by step during the first half of its anomalistic revolution and decreases in the same way during the second half of its anomalistic revolution, the increase and decrease being in an arithmetic progression. This motion has not been correctly interpreted so far and is being explained here for the first time. (**ed.** See Figure 1. Figure caption added.)

Let $ABPC$ be the Moon's orbit, A its apogee and P its perigee. Starting from A and proceeding anti-clockwise, let the orbit be divided into 248 equal divisions, called *padas* or steps (the Moon's steps). Thus, there are 124 *padas* or steps in the first half of its anomalistic revolution (*gati* or motion). One *pada* or step is supposed to be comprised of $\frac{1}{9}$ of a day, and the whole of the anomalistic motion or *gati*, of $\frac{248}{9}$ days (= 27 days 13 hours 20 mins.). A period of 3031 days is supposed to form a solid block (*ghana*), in which the Moon is supposed to make 110 anomalistic revolutions, approximately. In a period of 16 *ghanas*, it is supposed that the Moon makes complete sidereal revolutions.

The Moon is supposed to be slowest in the 1st *pada*, its motion in this *pada* being $\frac{4874}{63}$ or $77+\frac{23}{63}$ minutes. Thereafter, the motion increases at the rate of $\frac{10}{63}$ mins. per *pada*. The motion in the 124th *pada* is thus $\frac{4874}{63}+\frac{1230}{63}$ or $\frac{6104}{63}$ mins.

As the Moon passes from the 124th *pada* to the 125th *pada*, its motion is supposed to increase by $\frac{10}{7}$ mins., this being the average rate of increase per *pada* (the total increase in motion during 9 *padas* comprising a day in the first half-*gati* being $\frac{90}{7}$ mins.).

Thus in the 125th *pada*, or the 1st *pada* of the second half revolution (or second half-*gati*), the Moon's motion amounts to

$$\frac{6104}{63}+\frac{10}{7} \text{ or } \frac{6194}{63} \text{ mins.}$$

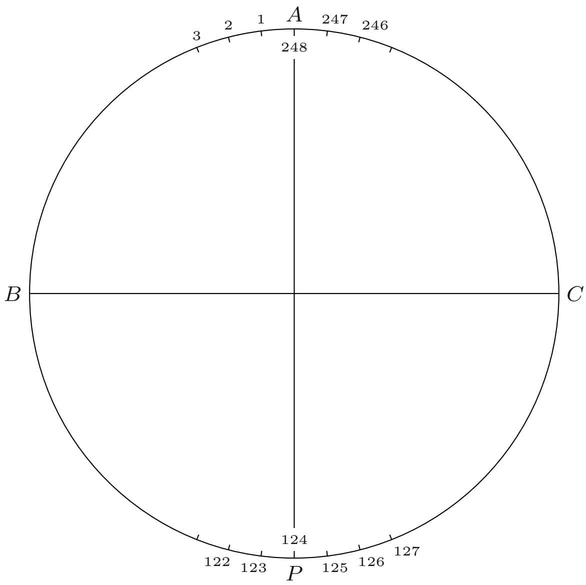

Figure 1: Vasiṣṭha's theory for the moon's motion.

This is supposed to be the maximum velocity of the Moon. Thereafter, the motion decreases at the rate of $\frac{10}{63}$ min. per *pada*, the motion in the 248th *pada* being $\frac{6194}{63} - \frac{1230}{63}$ or $\frac{4964}{63}$ mins.

As the Moon passes from the 248th *pada* to the 1st *pada*, its motion is supposed to get diminished by $\frac{10}{7}$mins., this being the average rate of decrease per *pada* (the total decrease in motion during 9 *padas* comprising a day in the second half-*gati* being $\frac{90}{7}$ mins.).

The above pattern of the Moon's motion is exhibited in tabular form in Table 1.

According to the above theory, the Moon's motion for P *padas* (steps) in the first half of the anomalistic revolution (or the first half-*gati*)

$$= \frac{P}{2}\left[2 \times \frac{4874}{63} + \frac{10}{63} \times (P-1)\right] \text{mins.}$$
$$= P \text{ degrees} + \frac{P}{63} \times [1094 + 5(P-1)] \text{mins.}, \qquad (1)$$

as stated in *PSi* (= *Pañcasiddhāntikā*), ii. 5-6.

Similarly, the Moon's motion for P *padas* (steps) in the second half of the anomalistic revolution (or the second half-*gati*)

$$= \frac{P}{2}\left[2 \times \frac{6194}{63} - \frac{10}{63} \times (P-1)\right] \text{mins.}$$

Table 1: Moon's motion *pada*-wise.

1st half-*gati*		2nd half-*gati*	
Pada or Step	Motion in minutes	*Pada* or Step	Motion in minutes
1st	$\frac{4874}{63}$ or $77+\frac{23}{63}$	1st	$\frac{6194}{63}$ or $98+\frac{20}{63}$
2nd	$\frac{4874}{63}+\frac{10}{63}$	2nd	$\frac{6194}{63}-\frac{10}{63}$
3rd	$\frac{4874}{63}+\frac{20}{63}$	3rd	$\frac{6194}{63}-\frac{20}{63}$
...	...	...	...
...	...	...	...
122nd	$\frac{4874}{63}+\frac{1210}{63}$	122nd	$\frac{6194}{63}-\frac{1210}{63}$
123rd	$\frac{4874}{63}+\frac{1220}{63}$	123rd	$\frac{6194}{63}-\frac{1220}{63}$
124th	$\frac{4874}{63}+\frac{1230}{63}$ or $\frac{6104}{63}$	124th	$\frac{6194}{63}-\frac{1220}{63}$ or $\frac{4964}{63}$
125th	$\frac{6104}{63}+\frac{10}{7}$ or $\frac{6194}{63}$	125th	$\frac{4964}{63}-\frac{10}{7}$ or $\frac{4874}{63}$

$$= P \text{ degrees} + \frac{P}{63} \times [2414 - 5(P-1)] \text{ mins.}, \tag{2}$$

as stated in *PSi*, ii. 5–6.

Substituting $P = 124$ in (1), the total motion of the Moon during the first half-*gati* comes out to be

$$\begin{aligned} &= 124^\circ + \frac{124}{63} \times (1094 + 5 \times 123) \text{ mins.} \\ &= 180^\circ + 3\frac{47}{63} \text{ mins., or } 180^\circ 4' \text{ approx.,} \end{aligned} \tag{3}$$

as stated in *PSi*, ii. 5.

Similarly, substituting $P = 124$ in (2), the total motion of the Moon during the second half-*gati*

$$\begin{aligned} &= 124^\circ + \frac{124}{63} \times (2414 - 5 \times 123) \text{ mins.} \\ &= 183^\circ \left(\frac{56}{63}\right)'. \end{aligned} \tag{4}$$

Adding (3) and (4), the Moon's motion for a *gati*

$$\begin{aligned} &= 363^\circ + 4\frac{40}{63} \text{ mins.} \\ &= 1 \text{ rev. } \left(185 - \frac{23}{63}\right)' \end{aligned}$$

$$= 1 \text{ rev. } \left(185 - \frac{1}{3}\right)' \text{ approx.}$$

According to Vasiṣṭha (see *PSi*, ii. 3–4):

$$\text{Moon's motion for a } gati = 1 \text{ rev. } \left(185 - \frac{1}{6}\right)' \text{ approx.}$$

$$\text{Moon's motion for a } ghana = 110 \text{ revs. } 11 \text{ signs } 7^\circ 32'$$
$$= -\frac{3}{4} \text{ signs} + 2', \text{ neglecting complete revs.}$$

Pauliśa-siddhānta

Let us now revert to Table 1. The motion of the Moon for the first 9 *padas* of the first half-*gati*

$$= \frac{9}{2}\left[2 \times \frac{4874}{63} + \frac{10}{63} \times 8\right] \text{ mins.}$$
$$= 702'.$$

The motion for the next 9 *padas* is $702 + \frac{90}{7}$ mins.; for the next 9 *padas*, $702 + \frac{180}{7}$ mins.; and so on. The motion for the first 9 *padas* of the second half-*gati*.

$$= \frac{9}{2}\left[2 \times \frac{6194}{63} - \frac{10}{63} \times 8\right] \text{ mins.}$$
$$= 879\frac{1}{7} \text{ or } 879 \text{ mins. approx.}$$

Hence, we have the following table (**ed.** see Table 2) for the Moon's motion for the successive days of the first and second half-*gatis*, according to Pauliśa (see *PSi*, iii. 4).

Since the minimum daily motion of the Moon is taken as 702 mins. and the maximum daily motion as $879\frac{1}{7}$ mins., it follows that the mean daily motion of the Moon, according to Vasiṣṭha, is

$$\frac{1}{2}\left(702 + 879\frac{1}{7}\right) \text{ mins. or } 790'\ 34''.$$

Lunar Tables

Lunar tables giving Moon's motion for each day of the *ghana* period of 3031 days were in vogue in Kaśmīr until about the 14th century AD. One such lunar table, known as *Candra-sāraṇī*, based on the parameters of the *Uttara-Khaṇḍakhādyaka*, occurs in a manuscript, in Śāradā script (Accession No.

Table 2: Moon's motion day-wise.

1st half-*gati*		2nd half-*gati*	
Day	Motion in minutes	Day	Motion in minutes
1st	702	1st	879
2nd	$702 + \frac{90}{7}$	2nd	$879 - \frac{90}{7}$
3rd	$702 + \frac{180}{7}$	3rd	$879 - \frac{180}{7}$
4th	$702 + \frac{270}{7}$	4th	$879 - \frac{270}{7}$
...	...	...	...
...	...	...	...

1657), in the collection of the Akhila Bharatiya Sanskrit Parishad, Lucknow. This table is in 379 columns and 8 rows.

Lunar tables, called *Candra-vākyas*, giving the Moon's motion for 1–248 anomalistic days are in use in certain parts of South India even today. One such table, ascribed to Vararuci, occurs as Appendix II in T. S. Kuppanna Sastri and K. V. Sarma's edition of the *Vākya-karaṇa*.

3 The 30 days of the Parsi calendar

> The Parsi Calendar is that of the old Persians when they were Zoroastrians. The year of this calendar consists of 12 months (Mah) each of 30 days (Roz), and 5 additional days at the end called Gathas. Each of the months and days bears a name, that of God or one of his angels, and the five additional days bear the names of the five Gathas, the oldest and most sacred hymns of the Avesta. The names commonly used by the Parsis of India for the months, days, and Gathas are given in Tables 3, 4, and 5 respectively.

Table 3: Names of the twelve months.

1.	Farwardin	4.	Tir	7.	Meher	10.	De
2.	Ardibahesht	5.	Amordad	8.	Avan	11.	Bahman
3.	Khurdad	6.	Shahrivar	9.	Adar	12.	Spandarmad

In *PSi*, i, 23–25, Varāhamihira writes:

Table 4: Names of the 30 days.

1.	Ahurmazd	11.	Khurshed	21.	Ram
2.	Bahman	12.	Mah	22.	Govad
3.	Aradibahesht	13.	Tir	23.	Depdin
4.	Shahrivar	14.	Gosh	24.	Din
5.	Spandarmad	15.	Dep-Meher	25.	Ashisvang
6.	Khurdad	16.	Meher	26.	Ashtad
7.	Amordad	17.	Sarosh	27.	Asman
8.	Depadar	18.	Rashna	28.	Zamyad
9.	Adar	19.	Farwardin	29.	Marespand
10.	Avan	20.	Behram	30.	Aneran

Table 5: Names of the 5 Gathas

1.	Ahunavad	3.	Spentomad	5.	Vahishtoyasht
2.	Ustavad	4.	Vohukhshathra[1]		

[1] See M. P. Khareghat's introduction (p. 27) to "100 Years' Indian Calendar containing Christian, Samvat, *Śaka*, Bengali, Mulki, Mugee, Burmese, Yazdejardi, Fasli, Nauroz and Hizri Eras with their corresponding dates from 1845 to 1944 AD by Jagjivan Ganeshji Jethabhai, Limbdi-Kathiawar (3rd Edition 1932)".

द्युगणे रूपाभ्यधिके पञ्चर्तुगुणोद्धृते मगाब्दाः स्युः ।
त्रिंशद्भक्ते शेषं ज्ञेयं राश्यंशकेन्द्राणाम् ॥२३॥
कमलोद्भवः प्रजेशः स्वर्गः शस्त्रं द्रुमान्नवासांसि ।
कालानलाभ्ररवयः शशीन्द्रगोनियतयः क्रमशः ॥२४॥
हरभवगुहपितृवरुणा बलदेवसमीरणौ यमश्चैव ।
वाक् श्रीधनदौ निरयो धात्री वेदाः परः पुरुषः ॥२५॥[2]

Add 1 to the *ahargaṇa* and then divide by 365: the quotient de-

[2] Verses 24 and 25 have been incorrectly deciphered by the other scholars. Thibaut and Dvivedi read them as:

कमलोद्भवः प्रजेशः स्वर्गीशश्चन्द्रमान्यवासांसि ।
कमलानलान्तरवयः शशीन्द्रगोनिर्ऋतयः क्रमशः ॥२४॥
हरभवगुहपितृवरुणा बलदेवसमीरणौ यमश्चैव ।
वाक् श्रीधनदौ गिरयो धात्री वेधाः परः पुरुषः ॥२५॥

David Pingree's reading is:

कमलोद्भवः प्रजेशः स्वर्गे(श)शास्तृरुद्रमन्युवसवः ।
कमलानलान्तरवयः शशीन्द्रगोनिर्ऋतयः क्रमशः ॥२४॥
हरभवगुरुपितृवरुणा बलदेवसमीरणौ यमश्चैव ।
वाक् श्रीधनदौ गिरयो धात्री वेधाः परः पुरुषः ॥२५॥

notes the number of elapsed years of the Zoroastrians (Magas). Divide the remainder by 30: the remainder is to be known as (the days) belonging to the lords of the 30 degrees of the signs, which are: Kamalodbhava, Prajeśa, Svarga, Śastra, Druma, Anna, Vāsa, Kāla, Anala, Abhra, Ravi, Śaśi, Indra, Go, Niyati, Hara, Bhava, Guha, Pitṛ, Varuṇa, Baladeva, Samīraṇa, Yama, Vāk, Śrī, Dhanada, Niraya, Dhātrī, Veda and Para Puruṣa.

One can easily see that in the above passage, Varāhamihira is referring to the years and the days of the the Parsi Calendar. which is used by the Parsi community in India even today. The names Kamalodbhava, etc., are the Sanskrit renderings of the names of the Parsi days, Ahurmazd, etc., as Table 6 will show. In order to facilitate comparison, English translation of the names is given where necessary. The correctness of the names given by Varāhamihira can be easily verified from comparison with the corresponding names given by astronomer Vaṭeśvara (904 AD), which are mentioned in the third column of the table.[3]

Table 6: Names of the 30 days of the Parsi months.

	Parsi name	Name given by Varāhamihira	Name given by Vaṭeśvara
1.	Ahurmazd (Lord God)	Kamalodbhava (Lotus-born)	Brahmā
2.	Bahman (Protector of creatures, Brahman)	Prajeśa (Protector of creatures)	Prajāpati (Protector of creatures)
3.	Ardibahesht (Holder of the keys of heaven)[4]	Svarga (Heaven)	Dyauḥ (Heaven)
4.	Shahrivar (Lord of pure metal)	Śastra (Weapon)	Śastra (Weapon)
5.	Spandarmad (Charitable)	Druma (Tree)	Taru (Tree)
6.	Khurdad (Lord of festivals)	Anna (Food)	Anna (Food)
7.	Amordad	Vāsa (Residence)	Vāsa (Residence)

[3]Vaṭeśvara's text runs thus (*Vaṭeśvarasiddhānta*, ch. 1, sec. 5, vss. 117(c-d)–118):

ब्रह्मा प्रजपतिर्द्यौः शस्त्रं तर्वन्नवासांसि ॥११७॥
कालाग्निखरविशशीन्द्रगोनियतिसवितृगुहाजपितृवरुणाः ।
हलिवायुयमा वाक्छ्रीधनदनिरयभूमिवेदपरपुरुषाः ॥११८॥

[4]See M. P. Kharegat, "On the interpretation of certain passages in the *Pañcasiddhāntikā* of Varāhamihira, an old Hindu Astronomical Work", JBBRAS, Vol. XIX, 1895–97.

Table 6: Names of the 30 days of the Parsi months (continued).

	Parsi name	Name given by Varāhamihira	Name given by Vaṭeśvara
8.	Depadar (Associate of Ahurmazd)	Kāla (Yama)	Kāla (Yama)
9.	Adar (Fire)	Anala (Fire)	Agni (Fire)
10.	Avan (Waters)	Abhra (Filled with water, cloud)	Kha (Same as Abhra)
11.	Khurshed (Sun)	Ravi (Sun)	Ravi (Sun)
12.	Mah (Moon)	Śaśī (Moon)	Śaśī (Moon)
13.	Tir (Distributor of water)	Indra (God of rain)	Indra (God of rain)
14.	Gosh (Cow)	Go (Cow)	Go (Cow)
15.	Depmeher (Ahurmazd's associate)	Niyati (Destiny)	Niyati (Destiny)
16.	Meher (Sun)	Hara (=Meher)	Savitṛ (Sun)
17.	Sarosh (Protector of the living and dead)	Bhava (Śiva)	Guha
18.	Rashna	Guha	Aja (Unborn God)
19.	Farwardin (Farohars of the dead)	Pitr (Manes)	Pitṛ (Manes)
20.	Behram (or Varenes)	Varuṇa	Varuṇa
21.	Ram	Baladeva (=Balarāma)	Hali (=Balarāma)
22.	Govad (Wind)	Samīraṇa (Wind)	Vāyu (Wind)
23.	Depdin (Ahurmuzd's associate)	Yama	Yama
24.	Din	Vāk (Speech)	Vāk (Speech)
25.	Ashisvang (Righteous)	Śrī (Righteousness)	Śrī (Righteousness)
26.	Ashtad (Angel created by Mazda)	Dhanada (Bestower of wealth, Kubera)	Dhanada (Bestower of wealth, Kubera)
27.	Asman (Sky)	Niraya (Hell)	Niraya (Hell)
28.	Zamyad (Earth)	Dhātrī (Earth)	Bhūmi (Earth)
29.	Marespand (Zarathushtrian and law religion)	Veda	Veda
30.	Aneran (Endless lights of shining heaven)	Paraḥ Puruṣaḥ (Supreme Being)	Parapuruṣa (Supreme Being)

I am grateful to Shri H. H. Vach, Joint Secretary, Parsi Punchayat, Bombay, for supplying me the following information regarding the meanings of the 30 Parsi days and the attributes of the corresponding 30 deities:

1. Ahurmazd—All-knowing Existing Lord. He is Lord God, whose 72 Avestan names are given in Hormazd Yasht, and 101 names are in Pazend. In Yasna Ha 1, Ahurmazda is called shining, glorious, greatest, best, wisest, most well-shaped, most attainable through holiness, who created and nourished us, who is the most bountiful spirit.

2. Bahman—Good mind. Attributes: Victorious friend, best amongst all creations, lord of inborn and acquired wisdom, created by Mazda (God), Protector of Cattle.

3. Ardibahesht—Best Righteousness. Attributes: most handsome; presiding on fire.

4. Shahrivar—Lord of pure metal; kind, nourisher of the poor; 'free-willed'

5. Spandarmad—Bountiful obedience; deity of earth; good, broad visioned, charitable, holy, Mazda-created.

6. Khurdad—Deity of waters—fullness; lord of seasons and of holy festivals.

7. Amordad—(Mordad is a misnomer)—deity of trees—immortality; lord of increase of herd of cattle, of corn, and of white Haoma.

8. Depadar—Associate of God Mazda, so called as the day precedes Adar' day. Attribute—'creator'.

9. Adar—Fire; son of Ahur Mazda, purifier, possessing lustre, and glory of Hormazd, health-giver, beneficent, fighter against demons.

10. Avan—Waters; good, undefiled, holy.

11. Khurshed—Shining Sun—immortal, brilliant, having swift horses (i.e., rays).

12. Mah—Moon, having seed of earth, only created, and of many varieties.

13. Tir—Star Sirius—shining, having glory, who works with star Satves, brave and distributor of waters.

14. Gosh—Also knows as Dravasp; a female deity—'giving health to horses', i.e., to all animals; brave, Mazda-created, holy.

15. Depmeher—Associate of God Mazda; so called because the day precedes Meher' day.

16. Meher—Light, ether; 'having broad postures'—1000 ears and 1000 eyes, well-known Yazata or angel; Avesta word Mithra = love, contract, etc.

17. Sarosh—Humility. The angel protects the living as well as the souls of the dead. 'Holy, strong, protector of body, with strong weapons, handsome, victorious'.

18. Rashna—Most righteous rectitude, making the world prosperous.

19. Farwardin—Farohar or Guardian Spirit—angel presiding over strong, valiant Farohars of the dead.

20. Behram—Smiter of evil enemy—well-shaped angel giving greatest victory.

21. Ram—Spiritual joy. The angel presiding over pure, health-giving air.

22. Govad—Good wind blowing above, below, forward, backward, with brave defence.

23. Depdin—Associate angel of God Mazda, so called because the day precedes the day 'Din'.

24. Din—Religion. Angel presiding over 'most righteous holy knowledge and of the good Mazda-worshipping religion'.

25. Ashisvang—Good, righteous. The angel presiding over good rectitude, wisdom, truthfulness and justice. Wealth of good life.

26. Ashtad—Universal law. Presiding angel, created by Mazda, furthering the world, brilliant through order and intelligence-holding.

27. Asman—The sky, containing the abode of paradise (Best Existence), full of happiness for the Holy.

28. Zamyad—Good Creation. The angel presides over lands, countries, mountains.

29. Marespand—Beneficent, holy spells that are effective, happiness-giving, anti-demoniac, expounding Zarathushtrian law and religion.

30. Aneran—Endless natural lights of shining Heaven, beyond the Bridge of Selection.

Each of the 30 days are supposed to be presided over by the deity of the same name. Deities of days 1 to 7 are known as 'Amshaspands' or arch-angels, Ahurmazd being considered one of them. Each Amshaspand has corresponding three or four associates or angels out of the thirty deities. They are thus classed:

1. Ahurmazd's associates are Nos: 8, 15, 23.
2. Bahman's associates are Nos: 12, 14, 21.
3. Ardibehesht's associates are Nos: 9, 17, 20.
4. Shahrivar's associates are Nos: 11, 16, 27, 30.
5. Spandarmad's associates are Nos: 10, 24, 25, 29.
6. Khurdad's associates are Nos: 13, 19, 22.
7. Amordad's associates are Nos: 18, 26, 28.

Nos. 5, 10, 14, 24 and 25 are female deities.

Lucky and unlucky days

On this point Shri Vach gives the following information.

> Originally all the thirty days of the month were auspicious and good, the 30 deities presiding over them were good, as can be seen above. In course of time contact with other nations of the world made the Iranians adopt some of their customs; and hence in some Pahlavi and Persian books of a much later period the idea of lucky and unlucky days crept in.

Referring to one such Pahlavi book, Dossabhai Framji Karaka, in his "*History of the Parsis*", London, 1884, Volume I, page 33, says:

> Madigan i Si Ruz describes in detail peculiar virtues of each day of Zoroastrian month. Great stress is laid in it upon the importance of each day in its bearing upon certain relations and transactions of life. Every single day is set apart as the fittest and most auspicious for certain special works of either devotion or worldly business. Some are best for beginning a journey or voyage; others for regulation of matters of domestic economy; some again for social gatherings and festivities; others again for pursuit of learning; while not a few reserved for rest and contemplation. Thus the Zoroastrian in his spiritual and temporal life is to be guided in the selection of a proper time for every new work by a knowledge of the auspiciousness or otherwise of the several days of the month.

A translation of this Pahlavi treatise is given at pages 133–143 of Karaka's book.

The attributes of the 30 angels are described in Persian verse by an unknown writer from sayings of Dastur Nosherwan Murzban of Kerman. [See (1) Rivayet of Dastur Darab Hormazdiar; Text, by Maneckji Rustomji Unvala; Vol. II pages 164–192. (2) Persian Rivayets of Hormazyar Framroze and others; by B. N. Dhanbhar; page 579. (3) Gujrati version of the Rivayet; Edited by Rustomji J. Dastur Meherji Rana; pages 565–68.]

Dr. Louis H. Gray of Columbia University, in his paper on "Alleged Zoroastrian Ophiomancy and its possible origin", pages 454–464, published in the "Dastur Hoshang Memorial Volume", 1918, says:

> One of the most curious superficial phenomena of the Zoroastrian faith is its intense horror of the serpent. Yet there is an elaborate system of divination from snakes.

Gray thinks this due to Babylonian influence.

The Rivayet contains a short tract in Persian verse, entitled "Mar Nameh", i.e., Book about Snakes, mentioning the effect, good or bad, at the sight of a snake on any one of the 30 days of the month. [See; (1) Unvala, II, pages 193–194. (2) Dhabhar, page 579. (3) Meherjirana, page 569.]

According to Gray, a Persian manuscript contains omens from seeing a snake on week days and at time of the moon entering the 12 signs of the Zodiac (pages 456–457, Hoshang Mem. Volume). [See *Burj Nameh* Rivayet; (1) Unvala II, 194. (2) Dhabhar, page 579. (3) Meherjirana, page 569.]

Al-Bīrūnī the great mathematician and astronomer of Persia who lived a thousand years ago, has, in his Arabic-Persian work on "*Vestiges of the Past*" which is translated in English by Dr. C. Edward Sachau of the Royal University of Berlin and published at London in 1879 under the title, "*The Chronology of Ancient Nations*", thus stated:

> The Persians divide all days of the year into preferable and lucky days and unlucky and detested ones. Further they have certain rule regarding appearance of snakes on different days of the month.
>
> Day 'Mah' (12th) is considered a preferable day as God created moon for distributing good till moon begins to wane.
>
> Two days of conjunction and opposition are unlucky. On conjunction madness and epilepsy occur. Child born is of imperfect health. Everything planted grows scanty.
>
> Al-Kindi says conjunction is detested as moon is being burned. At opposition full moon is detested since it requires light of sun. (*Chronology*, page 219).

Table 7: Unlucky days of the month.

No. and name of day		Omen as per *Mar Nameh*	Omen according to Al-Bīrūnī
2.	Bahman	Great sorrow	Illness and disease
3.	Ardibehesht	Death of relative	Death in family
7.	Amordad	Regret	Illness, disease
12.	Mah	Ruin of affairs	Evil after noon
15.	Depmeher	...	Illness, convalescence
18.	Rashna	Increase of defects, failures	Illness during journey
20.	Behram	Evil omen	Death in family
21.	Ram	Warfare and quarrel	...
22.	Govad	Destruction of property	Suspicion of theft
23.	Depdin	Trouble and loss	Illiness and disease
25.	Ashisvang	Cause for sorrow	Bad and blameable
27.	Asman	Grave accusations	Accusation of lying
28.	Zamyad	...	Calamity in family
30.	Aneran	Grief and anxiety	Punishment for fornication

The following days of the year are unlucky, according to AI-Bīrūnī when snake is sighted: 9th and 22nd of Farwardin month; 15th of 2nd month Ardibehesht; 11th and 30th of Khordad; 13th of Tir; 20th of Amordad; 3rd of Shahrivar; 20th of Adar; 20th of De; 2nd of Bahman; and 5th of last month Spandarmad. (*Chronology*, page 218).

The following (**ed.** see Table 7) are unlucky days of the month, when a snake is sighted according to *Mar Nameh* and according to AI-Bīrūnī.

According to the Hindu astronomer Vaṭeśvara (*VSi*, ch. 1, sec. 5, vs. 120), the unlucky days are those assigned to Śastra, Kāla, Agni, Niyati and Yama (i.e., Shahrivar, Depadar, Adar, Depmeher, and Depdin).

Āryabhaṭa I's astronomy with midnight day-reckoning *

1 Introduction

Āryabhaṭa I (born 476 AD) is generally known as the author of the *Āryabhaṭīya* and his fame as a mathematician and astronomer of the first rank is assessed on the basis of that work alone. No other work written by him is available to us. But there are reasons to believe that he was the author of at least one more work on astronomy. References to two works of Āryabhaṭa I, one in which the day was reckoned from sunrise (and is known as *Āryabhaṭīya*) and the other in which the day was reckoned from midnight, have been made by posterior Hindu astronomers. For instance Varāhamihira (died 583 AD) has written:

> Āryabhaṭa maintains that the beginning of the day is to be reckoned from midnight at Laṅkā; and the *same teacher* again says that the day begins from sunrise at Laṅkā.[1]

Brahmagupta (628 AD) has criticised Āryabhaṭa I for making contradictory statements in his two works:

> When it is (once) stated that the number of revolutions performed in a *yuga* by the sun is 4,32,000, the duration of the *yuga* is fixed. Why, then, is there a difference of 300 civil days (in the two works of Āryabhaṭa)?[2]
>
> In a period of 14,400 years, there would be a difference of one civil day, according to the works (of Āryabhaṭa) following the sunrise and the midnight day-reckonings.[3]

* K. S. Shukla, *Gaṇita*, Vol. 18, No. 1 (June 1967), pp. 83–105.

[1] लंकार्धरात्रसमये दिनप्रवृत्तिं जगाद चार्यभटः ।
स एव भूयः चार्कोदयात् प्रभृत्याह लंकायाम् ॥ (*PSi*, xv. 20)

[2] युगरविभगणाः ख्युघ्रीति यत् प्रोक्तं तत् तयोर्युगं स्पष्टम् ।
त्रिशती रव्युदयानां तदन्तरं हेतुना केन ॥ (*BrSpSi*, xi. 5)

[3] अधिकैः शतैश्चतुर्भिर्वर्षसहस्रैश्चतुर्दशभिरेकः ।
युगयातैर्दिनवारान्तरमौदयिकार्धरात्रिकयोः ॥(*BrSpSi*, xi. 13)

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_28

> (On account of the difference in daily motions of the planets prevailing in the two works of Āryabhaṭa) the mean longitude calculated according to the work with midnight day-reckoning may be smaller than that calculated according to the (other) work with sunrise day-reckoning by one-fourth of the planet's daily motion. Which of the two is correct is not certain. Therefore, not one of them is correct.[4]

References to Āryabhaṭa I's work with midnight day-reckoning are found to occur also in Al-bīrūnī's *India* and Nīlakaṇṭha's (c. 1500 AD) commentary on the *Āryabhaṭīya*.

It seems that, unlike the *Āryabhaṭīya*, the midnight astronomy taught by Āryabhaṭa I was more popular in the northern parts of India than in the southern. In the latter half of the seventh century AD when Brahmagupta wrote his *Khaṇḍakhādyaka*, scholars and Pandits in north India had a tendency to regard Āryabhaṭa I's midnight astronomy as the most accurate and generally based their calculations on it. This is probably the reason why the celebrated Brahmagupta who once bitterly criticised and ridiculed Āryabhaṭa I turned out to be his ardent follower in old age and based his *Khaṇḍakhādyaka* on his teachings. In the opening stanzas of the *Khaṇḍakhādyaka*, Brahmagupta writes:

> Having bowed in reverence to God Mahadeva, the cause of creation, maintenance and destruction of the world, I set forth the *Khaṇḍakhādyaka* which will yield the same results as the work of Āryabhaṭa.
>
> As it is generally not possible to perform calculations pertaining to marriage, nativity, etc., every day by the work of Āryabhaṭa, hence this smaller work giving the same results.[5]

The *Khaṇḍakhādyaka* being handy and more convenient for every day use than the work of Āryabhaṭa I on which it was based, was preferred to the latter and soon became a popular work in north India. A number of commentaries (*bhāṣya, vyākhyā, udāharaṇa*, etc.) were also written on it by scholars who hailed from Kashmir, Nepal, the Punjab, Uttar Pradesh, Saurāṣṭra, Orissa, etc. With the growing popularity of the *Khaṇḍakhādyaka*, Āryabhaṭa I's midnight astronomy fell into the background and was ultimately lost and

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[4] औदयिकाद्दिनभुक्तेस्तुर्यांशेनार्धरात्रिको भवत्यूनः ।
कतरः स्फुटं न निश्चितमनयोः स्फुटमेकमपि नातः ॥ (*BrSpSi*, xi. 14)

[5] प्रणिपत्य महादेवं जगदुत्पत्तिस्थितिप्रलयहेतुम् ।
वक्ष्यामि खण्डखाद्यकमाचार्यार्यभटतुल्यफलम् ॥१॥
प्रायेणार्यभटेन व्यवहारः प्रतिदिनं यतोऽशक्यः ।
उद्वाहजातकादिषु तत्समफललघुतरोक्तिरतः ॥२॥ (*KK*, i. 1–2)

forgotten. The result is that today we know of Āryabhaṭa I as the author of the *Āryabhaṭīya* alone.

The discovery of the works of Bhāskara I (629 AD) has provided a new source of knowledge regarding ancient Hindu astronomy. To our great fortune, in the *Mahābhāskarīya* there is material which contributes to our information regarding the lost work of Āryabhaṭa I. Bhāskara I has distinguished the two works of Āryabhaṭa I by calling the *Āryabhaṭīya* by the name *Āryabhaṭa-tantra* or *Bhaṭa-tantra* and referring to the other simply as *Tantrāntara* ("another *tantra*") without giving to it any particular name. In chapter VII of the *Mahābhāskarīya*, which deals with the astronomical elements, Bhāskara I has pointed out the differences between the two works of Āryabhaṭa I.

Further light in this direction is thrown by the commentators of the *Sūrya-siddhānta*, such as Rāmakṛṣṇa Ārādhya (1472 AD), Tamma Yajvā (1599 AD) and Bhūdhara (1572 AD), who have called Āryabhaṭa I's work with midnight day-reckoning by the name *Āryabhaṭasiddhānta* and has given the details of certain astronomical instruments described in that work. Rāmakṛṣṇa Ārādhya has even quoted 34 verses of the chapter on "Astronomical Instruments" from that work. Tamma Yajvā informs us that he himself wrote a work entitled *Siddhānta-sārvabhauma* which he based on the *Āryabhaṭasiddhānta*. Hence, there remains little doubt regarding the existence of a work with midnight day-reckoning coming from the pen of Āryabhaṭa I.

In the present paper we propose to set forth our present information regarding the *Āryabhaṭasiddhānta*. We shall first state the distinguishing features of the two works of Āryabhaṭa I, the *Āryabhaṭīya* and the *Āryabhaṭasiddhānta*, and then shall discuss the astronomical instruments stated in the 34 verses as quoted by Rāmakṛṣṇa Ārādhya from the *Āryabhaṭasiddhānta*. We shall also invite the attention of our readers to certain other verses which are found ascribed to Āryabhaṭa I in later works.

2 Distinguishing features of the two works of Āryabhaṭa I

According to Bhāskara I, the two works of Āryabhaṭa I differed mainly in three things: (i) Astronomical constants, (ii) Calculation of planetary longitudes, (iii) Calculation of planetary latitudes.

2.1 Astronomical constants

The differences in the astronomical constants in the two works of Āryabhaṭa I are exhibited in tabular form in Tables 1, 2, 3, 4, and 5.

The numbers in the second and the third columns of Table 1 are in the ratio of 2 : 3 approximately. This is due to the fact that the measures of the *yojanas*

Table 1: Diameters and distances of planets.

	Āryabhaṭīya (sunrise day-reckoning) in *yojanas*	*Āryabhaṭasiddhānta* (midnight day-reckoning) in *yojanas*
Earth's diameter	1050	1600
Sun's diameter	4410	6480
Moon's diameter	315	480
Sun's distance	4,59,585	6,89,358
Moon's distance	34,377	51,566
$\frac{\text{circumference of the sky}}{\text{revolutions of the Moon}}$	21,600	32,400

Table 2: Civil days, omitted lunar days, and revolutions of Mercury and Jupiter in a period of 43,20,000 years.

	Āryabhaṭīya	*Āryabhaṭasiddhānta*
Civil days	1,57,79,17,500	1,57,79,17,800
Omitted lunar days	2,50,82,580	2,50,82,280
Revolutions of Mercury	1,79,37,020	1,79,37,000
Revolutions of Jupiter	3,64,224	3,64,220

employed in the two works were in the ratio of 3 : 2. The circumference of a planet's concentric or deferent (or mean orbit) is supposed to be of 180 units (called degrees) in length and the dimensions in Table 4 are on the same scale.

2.2 Calculation of planetary longitudes

2.2.1 Sunrise day-reckoning

For obtaining the true longitudes of the superior planets (Mars, Jupiter and Saturn), Āryabhaṭa I's astronomy with sunrise day-reckoning prescribes the following corrections.

For obtaining corrected mean anomaly:

(i) Half *bāhu-phala* ("equation of the centre") to mean longitude.

(ii) Half *śīghra-phala* to the resulting longitude.

Table 3: Longitudes of planets' apogees (*mandocca*).

	Āryabhaṭīya	*Āryabhaṭasiddhānta*
Sun	78°	80°
Moon	118°	110°
Mercury	210°	220°
Jupiter	180°	160°
Venus	90°	80°
Saturn	236°	240°

Table 4: Dimensions of planets' *manda* epicycles.

	Āryabhaṭīya Beginning of		*Āryabhaṭasiddhānta*
	odd quadrant	even quadrant	
Sun	13°30′	13°30′	14°
Moon	31°30′	31°30′	31°
Mars	63°	81°	70°
Mercury	31°30′	22°30′	28°
Jupiter	31°30′	36°	32°
Venus	18°	9°	14°
Saturn	40°30′	58°30′	60°

Table 5: Dimensions of planets' *śīghra* epicycles.

	Āryabhaṭīya Beginning of		*Āryabhaṭasiddhānta*
	odd quadrant	even quadrant	
Mars	239°30′	229°30′	234°
Mercury	139°30′	130°30′	132°
Jupiter	72°	67°30′	72°
Venus	265°30′	256°30′	260°
Saturn	40°30′	36°	40°

For obtaining true longitude:

(iii) Entire *bāhu-phala* (as calculated from corrected mean anomaly) to mean longitude.

(iv) Entire *śīghra-phala* (as calculated from the resulting longitude) to the resulting longitude.

In the case of the inferior planets (Mercury and Venus), the corrections applied are as follows:

(i) Half *śīghra-phala* to mean longitude.

(ii) Entire *bāhu-phala* (as calculated from the resulting mean anomaly) to mean longitude.

(iii) Entire *śīghra-phala* to the resulting longitude.

2.2.2 Midnight day-reckoning

According to Āryabhaṭa I's astronomy with midnight day-reckoning, the true longitudes of all planets are obtained by applying the corrections as follows:

For obtaining corrected mean anomaly:

(i) Half *śīghra-phala* to mean longitude.

(ii) Half *bāhu-phala* to the resulting longitude.

For obtaining true longitude:

(iii) Entire *bāhu-phala* (as calculated from corrected mean anomaly) to mean longitude.

(iv) Entire *śīghra-phala* to the resulting longitude.

2.3 Calculation of planetary latitudes

2.3.1 Sunrise day-reckoning

In the case of the superior planets, the formula stated is

$$R \sin \beta = \frac{R \sin(\lambda - \Omega) \times R \sin i}{D},$$

where β is the latitude of the planet, λ the longitude of the planet, Ω the longitude of the planet's ascending node, i the inclination of the planet's orbit, and D is the distance of the planet in minutes.

In the case of the inferior planets, the formula is

$$R \sin \beta = \frac{R \sin(S - \Omega) \times R \sin i}{D},$$

where S is the longitude of the planet's *śīghrocca*, the other symbols being as before.

2.3.2 Midnight day-reckoning

Under midnight day-reckoning, two kinds of nodes (*pāta*) are defined: (i) *manda-pāta* and (ii) *śīghra-pāta*. Their longitudes are determined by the following formulae:

$$\begin{aligned}
\textit{manda-pāta} \text{ of Mars} &= (\text{apogee of Mars} - 3 \text{ signs}) + 1^\circ 30' \\
\textit{manda-pāta} \text{ of Jupiter} &= (\text{apogee of Jupiter} - 3 \text{ signs}) + 2^\circ \\
\textit{manda-pāta} \text{ of Saturn} &= (\text{apogee of Saturn} - 3 \text{ signs}) + 2^\circ \\
\textit{manda-pāta} \text{ of Mercury} &= (\text{apogee of Mercury} + 6 \text{ signs}) + 1^\circ 30' \\
\textit{manda-pāta} \text{ of Venus} &= (\text{apogee of Venus} + 6 \text{ signs}) + 2^\circ \\
\textit{śīghra-pāta} \text{ of Mars} &= (\textit{śīghrocca} \text{ of Mars} - 3 \text{ signs}) + 1^\circ 30' \\
\textit{śīghra-pāta} \text{ of Jupiter} &= (\textit{śīghrocca} \text{ of Jupiter} - 3 \text{ signs}) + 2^\circ \\
\textit{śīghra-pāta} \text{ of Mercury} &= \text{does not exist} \\
\textit{śīghra-pāta} \text{ of Venus} &= (\textit{śīghrocca} \text{ of Venus} + 6 \text{ signs}) + 2^\circ
\end{aligned}$$

For the celestial latitude of a planet the following formula is prescribed:

$$\begin{aligned}
\text{Celestial latitude} = &\ (\text{celestial latitude derived from } \textit{manda-pāta}) \\
&\pm (\text{celestial latitude derived from } \textit{śīghra-pāta}),
\end{aligned}$$

where + or − sign is taken according as the two latitudes are of like or unlike directions, and the celestial latitude from the *manda* or *śīghra-pāta* is determined by the formula:

$$R \sin(\text{celestial latitude}) = \frac{R \sin(\text{planet} - \textit{pāta}) \times R \sin i}{D},$$

where the symbols D and i have the same meanings as before.[6]

[6] See *MBh*, vii. 28(ii)–32. The concept of the *manda-pāta* and the *śīghra-pāta* and the method of deriving the celestial latitude from them are erroneous and represent a very old phase of Hindu astronomy. Such a concept is not found to occur in any other Hindu work on astronomy known to us.

3 Relationship of Āryabhaṭa I's midnight astronomy with other works

3.1 Āryabhaṭa I's midnight astronomy and the *Khaṇḍakhādyaka*

A comparison of the *Khaṇḍakhādyaka* with Āryabhaṭa I's midnight astronomy (as described by Bhāskara I) confirms Brahmagupta's claim that the former is based on the latter. The former work agrees with the latter in all respects excepting one which relates to the calculation of the celestial latitude of a planet. In that particular item Brahmagupta instead of following the incorrect method of Āryabhaṭa I's midnight astronomy makes use of the method of the sunrise system which has been described in detail by Bhāskara I in his works.

3.2 Āryabhaṭa I's midnight astronomy and Varāhamihira's *Sūryasiddhānta*

The original *Sūryasiddhānta* does not exist, but a rough idea of the contents of that work may be had from chapters i, ix, x, xi, xvi and xvii of the *Pañcasiddhāntikā* of Varāhamihira in which that work has been summarised.

A comparison of Āryabhaṭa I's midnight astronomy with the above mentioned chapters of the *Pañcasiddhāntikā* shows that:

(i) The astronomical constants in the two works agree. For example, the revolutions of the planets, the longitudes of the apogees of the planets, the dimensions of the epicycles and the limits for the visibility of the planets, etc., stated in those works are the same. The method of finding the true longitudes of the planets ascribed to Āryabhaṭa I's midnight astronomy is also the same as found in the *Pañcasiddhāntikā* (xvii. 4–9).

(ii) The actual diameters and distances of the Sun and the Moon are not stated in Varāhamihira's version of the *Sūryasiddhānta*, but their abraded values used in the *Pañcasiddhāntikā* (ix. 15–16) very nearly correspond to those of Āryabhaṭa I's midnight astronomy.[7]

The astronomical constants given in the *Sumati-mahātantra*, which too was based on the old *Sūryasiddhānta*, are also exactly the same as those of Āryabhaṭa I's midnight astronomy.

It appears therefore that Āryabhaṭa I's midnight astronomy was based on the old *Sūryasiddhānta*. This is the reason that certain later writers have regarded Āryabhaṭa I as an incarnation of God Sun, the promulgator of the orig-

[7] It may be mentioned that verse 12 of chapter xii of the *Pañcasiddhāntikā* as well verses 15 and 16 of chapter ix of the same work, as edited by G. Thibaut and S. Dvivedi, are incorrect. See my paper entitled "On three stanzas from the *Pañcasiddhāntikā*" in *Gaṇita*, **5** (2). pp. 129–136.

inal *Sūryasiddhānta*, as is evident from the following verse quoted by H. Kern in his introduction to the *Āryabhaṭīya*:

> सिद्धान्तपञ्चकविधावपि दृग्विरुद्ध-
> मौढ्योपरागमुखखेचरचारकॢप्तौ ।
> सूर्यः स्वयं कुसुमपुर्यभवत् कलौ तु
> भूगोलवित् कुलप आर्यभटाभिधानः ॥

References to the *Āryabhaṭasiddhānta* in the commentaries of the *Sūryasiddhānta* too point to the same conclusion.

3.3 Āryabhaṭa I's midnight astronomy and the *Puliśa-siddhānta*

The *Puliśa* (or *Pauliśa*)-*siddhānta* quoted by Al-bīrūnī in his *India* and by Bhaṭṭotpala in his commentary on the *Bṛhat-saṃhitā* of Varāhamihira also agrees with Āryabhaṭa I's midnight astronomy. The *Pauliśa-siddhānta* quoted by Āmarāja (c. 1200 AD) in his commentary on the *Khaṇḍakhādyaka*, however, differs from Āryabhaṭa I's midnight astronomy in the case of one or two astronomical constants. But it is interesting to note that Āmarāja on more than one occasion in that commentary cites the *Pauliśa-siddhānta* as an authority, while interpreting the rules of the *Khaṇḍakhādyaka*.

The above discussion seems to suggest that

(i) Āryabhaṭa I's midnight astronomy was based on the old *Sūryasiddhānta*.

(ii) The *Puliśa-siddhānta* quoted by Bhaṭṭotpala and Al-bīrūnī was a recast of that work, made up-to-date on the basis of Āryabhaṭa I's midnight astronomy.

(iii) The *Puliśa-siddhānta* quoted by Āmarāja was a subsequent redaction of the same work in which certain new additions and alterations were made.

It may be added that the *Puliśa-siddhānta* too reckoned the day from midnight.

4 The astronomical instruments of the *Āryabhaṭasiddhānta*

We have already referred to the 34 verses of the *Āryabhaṭasiddhānta* which have been quoted by Rāmakṛṣṇa Ārādhya in his commentary on the *Sūrya-siddhānta*. These verses have been taken from the *Yantrādhyāya* ("Chapter on Astronomical Instruments") of that work, and give a detailed account of (i) the *Chāyā-yantra*, (ii) the *Dhanu-yantra*, (iii) the *Yaṣṭi-yantra*, (iv) the

Cakra-yantra, (v) the *Chatra-yantra*, (vi) the water instruments, (vii) the *Ghaṭikā-yantra*, (viii) the *Kapāla-yantra*, and (ix) the *Śaṅku-yantra*.

Rāmakṛṣṇa quotes the above mentioned 34 verses with an introductory sentences as follows:

> ... आर्यभटसिद्धान्तोक्तयन्त्रानुसारेण। तत्कृतयन्त्राध्यायश्लोका विलिख्यन्ते–
>
> ... according to the astronomical instruments mentioned in the *siddhānta* of Āryabhaṭa. (Below) are indited the verses from the chapter on the astronomical instruments, written by him—

4.1 छायायन्त्राणि (The shadow instrument)

> दिङ्मध्यात्सप्तपञ्चाशदङ्गुलैस्त्रिज्यकांशकैः ।
> लिखेद्वृत्तं च चक्रांशचिह्नितं सममण्डलम् ॥१॥
> चराग्रज्याद्युनाडीभिः छायायन्त्राणि साधयेत् ।
> समवृत्तविदिक्छायाकर्णाभ्यां क्रान्तिदोर्गुणाः ॥२॥
> समवृत्ते स्वदिश्यग्रां दद्यात् प्राच्यपराशयोः ।
> चरज्यानामथाग्राङ्कान् दिङ्मध्यात् स्वदिशि न्यसेत् ॥३॥
> तदग्रबिन्दुतो वृतं वृत्तान्ताग्रं लिखेत्तु तत् ।
> स्वाहोरात्रदलं तत्र स्पष्टा नाड्यः स्वशङ्कुभिः ॥४॥
> स्वाहोरात्रदलेंऽशाः स्युः षड्गुणा दिननाडिकाः ।
> अग्रान्तेऽस्तोदयार्कौ च याम्यार्धे पूर्वपश्चिमे ॥५॥
> तत्पूर्वापररेखातो दक्षिणार्धं च तत्स्मृतम् ।
>
> 1. Construct a perfect circle (*samamaṇḍalaṃ vṛttaṃ*) with radius equal to 57 *aṅgulas*, the number of degrees in a radian, and on (the circumference of) it mark the 360 divisions of degrees.
>
> 2. Then construct shadow instruments (for every day of the year) with the help of the Rsine of the Sun's ascensional difference, the Rsine of the Sun's *agrā*, and the *nāḍīs* of the duration of the day (in the following manner):
>
> Determine the Rsines of the Sun's declination and of the Sun's longitude from the *samavṛttacchāyākarṇa* (i.e. the hypotenuse of the shadow of the gnomon when the Sun is on the prime vertical) or from the *vidik-chāyākarṇa* (i.e. the hypotenuse of the shadow when the Sun is in a mid-direction).
>
> 3. On the perfect circle (drawn above), lay off the (Sun's) *agrā* in its own direction (north or south) in the east as well as in the west (and at each place put down a point). Again lay off the (Sun's) *agrā* corresponding to the Sun's ascensional difference in its own direction from the centre of the circle.

> 4. With that end of the (Sun's) *agrā* (as centre) draw a circle passing through the (two) points marked on the circle: this circle denotes the Sun's diurnal circle. On (the southern half of) that circle, put down marks indicating true *ghaṭīs* with the help of the corresponding positions of the gnomon.[8]
>
> 5. These *ghaṭīs* of the day, multiplied by six, are the degrees on the diurnal circle. At the two points marked at the ends of the Sun's *agrā* in the east and west, are the positions of the Sun at rising and setting.
>
> 6(i). Half of the diurnal circle lying towards the south of the rising-setting line (of the Sun) is called the southern half of the diurnal circle.

In Figure 1 (**ed.** figure caption added), let $ENWS$ be the perfect circle (referred to above), in which E, W, N and S are the east, west, north, and south cardinal points respectively. Let A and B be the points at the ends of the Sun's *agrā*, laid off in the east and west in their proper direction. Also let C be the end of the Sun's *agrā* laid off from the centre O of the circle.

Let BDA be the circle drawn with C as centre and passing through A and B. This has been called "the diurnal circle". AB is the rising-setting line; A is the point where the Sun rises, and B the point where the Sun sets.

The southern half of the diurnal circle BDA was marked by the divisions of *ghaṭīs* and degrees as follows:

A gnomon was held on the plane of the circle at right angles to it in such a way that the end of its shadow at that moment was at the centre of the circle. By holding the gnomon in this way, marks were made on the diurnal circle where the shadow of the gnomon intersected it at the end of each *ghaṭī* after sunrise. These marks indicated the ending moments of the *ghaṭīs* of the day. Intervals between these graduations were then divided into six equal parts, called degrees.

The *chāyā-yantra* (shadow-instrument), thus constructed, was used for determining the *ghaṭīs* and degrees elapsed (since sunrise) at any time of the day. For this purpose one had to find the point where the instantaneous shadow of the gnomon crossed the diurnal circle, and to read the graduations between that point and the point A.

One *chāyā-yantra* served the purpose of giving time for one day only. So 365 such instruments were constructed, one for each day of the year.

Shadow-instruments (*chāyā-yantra*) of the type described above are unusual, as they are not found to occur in any other work on Hindu astronomy known to us.

[8] In each position the gnomon is to be held in such a way that the end of the shadow may lie at the centre of the circle.

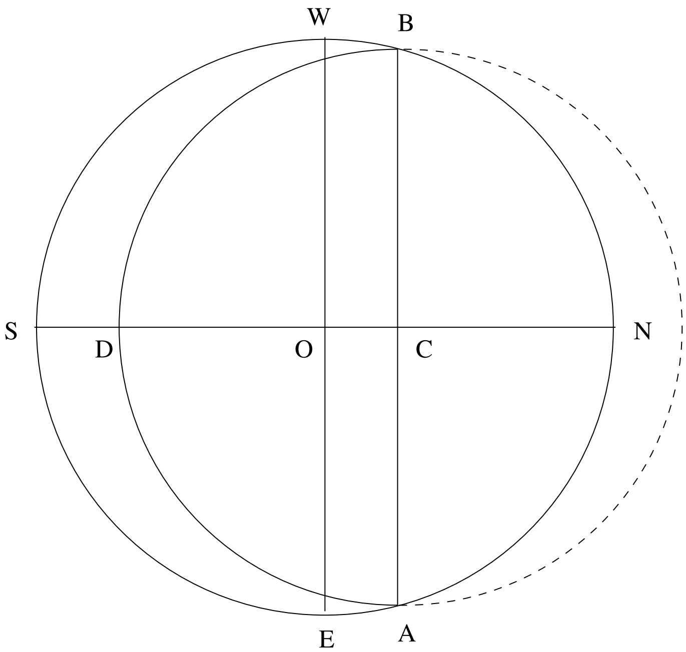

Figure 1: The *chāyā-yantra*.

4.2 धनुर्यन्त्रम् (The semi-circle)

वृत्तव्यासो धनुर्ज्या स्याद् व्यासार्धं धनुषः शरः ॥६॥
शङ्कुच्छाया धनुर्ज्यायां दिङ्मध्यात्त्विष्टभा सदा ।
प्रागग्रं धनुषो वृत्ते भ्रामयेदर्कदिङ्मुखम् ॥७॥
चापाग्रोदयमध्यांशाः षड्भिर्भाज्या दिने गताः ।

6(ii). The chord of the *dhanuryantra* is equal to the diameter of the circle (i.e. the perfect circle), and its arrow is equal to the radius. It is mounted on the circle vertically with the two ends of its arc coinciding with the east and west points.

7. The eastern end of the *dhanuryantra* should be moved along (the circumference of) the circle until the *dhanuryantra* is towards the Sun.[9] The shadow of the gnomon will then fall along the chord of the *dhanuryantra*, and (the shadow-end being at the centre of the circle) the distance of the gnomon as measured from the centre of the circle, will always be equal to the shadow for the desired

[9]The *dhanuryantra* resembles a semi-circular arc. It is held vertically with the two ends of its arc coinciding with the east and west points of the circle drawn on the ground.

time.[10]

8(i). The degrees intervening between the (eastern) end of the *dhanuryantra* and the rising point of the Sun divided by six, give the *ghaṭīs* elapsed in the day.

4.3 यष्टियन्त्रम्

वृत्तव्यासदलं यष्टिस्त्रिज्यांशाङ्गुलसम्मिता ॥८॥
दिङ्मध्येऽर्कोन्मुखी धार्या यष्टिः कर्णस्तदुन्नतिः ।
शङ्कुस्तस्यैव मूलात्तु छाया दिङ्मध्यगा सदा ॥९॥
यष्ट्यग्रोदयमध्यांशाः षड्भिर्भाज्या दिने गताः ।

8(ii)–10(i). The *yaṣṭi-yantra*[11] which is equal in length to the semi-diameter of the (perfect) circle with as many graduations of *aṅgulas* as there are degrees in a radian (i.e. 57) should be held at the centre of the circle towards the Sun. The *yaṣṭi* then denotes the hypotenuse, its elevation denotes the gnomon, and the distance from the foot of the gnomon up to the centre of the circle always denotes the shadow (of that gnomon). The degrees intervening between the end of the *yaṣṭi* and the rising point of the Sun, divided by six, give the *ghaṭīs* elapsed in the day.

4.4 चक्रयन्त्रम् (The circle)

भगणांशाङ्कितं चक्रं सरन्ध्रं विषुवत्यथ ॥१०॥
धनू रव्युन्मुखं कृत्वा चापवच्चक्रयन्त्रकम् ।
कल्पयेल्लम्बशङ्कोर्वा छायानाड्यश्च यष्टिवत् ॥११॥

10(ii)–11. The *cakra-yantra* ("an instrument resembling a circular hoop") bears (on its circumference) 360 marks of degrees and has (two) holes at the equinoctial points. Pointing the arc of the *cakra-yantra* towards the Sun, like the (arc of the) *dhanuryantra*, the shadow of the gnomon as also the *nāḍīs* elapsed in the day should be ascertained as in the case of the *yaṣṭi-yantra*.

The instruments *dhanu*, *cakra*, and *yaṣṭi*, have been mentioned in the *Sūrya-siddhānta*,[12] but details are not given there. The *cakra* instrument is mentioned in the *Pañcasiddhāntikā*,[13] where it is described as follows:

[10]The gnomon is represented in this case by means of a cord suspended from the Sun's position on the arc of the *dhanuryantra*.
[11]The *yaṣṭi-yantra* resembles a cylindrical stick.
[12]xiii. 20.
[13]xiv. 21–22.

> Take a circular hoop, on whose circumference the 360 degrees are evenly marked, whose diameter is equal to one *hasta*, and which is half an *aṅgula* broad. In the middle of the breadth of that hoop make a hole. Through this small hole made in the circumference allow a ray of the Sun at noon to enter in an oblique direction. The degrees, intervening on the lower half of the circle between (the spot illumined by the ray and) the spot reached by a string hanging perpendicularly from the centre of the circle, represent the degrees of the zenith distance of the mid-day Sun.[14]

How the time was determined by this instrument is not explained there.

The methods of determining time with the help of the *cakra* and *dhanu* instruments are given by Brahmagupta,[15] Lalla,[16] Śrīpati,[17] and Bhāskara II.[18] They differ from that given above.

As regards the *yaṣṭi* instrument, the details given by later astronomers are the same as those mentioned above. Lalla's procedure for finding time is, however, different.[19]

शङ्कुभ्रमप्रकारेण प्रोक्ता नाड्यश्च तत्प्रभा ।
अधुना भाभ्रमान्नाड्यः तच्छाया च कथ्यते ॥१२॥

12. Above have been stated the methods of obtaining the shadow of the gnomon and the *nāḍīs* (elapsed in the day) by the movement of the gnomon. Now will be stated the method of finding the *nāḍīs* (elapsed) and also the shadow (of the gnomon) by the movement of the shadow of the gnomon (set up at the centre of the circle).

4.5 छत्रयन्त्रम् (The umbrella)

छत्रं वेणुशलाकाभिः कृत्वा चक्रांशसंख्यया ।
दिङ्मध्ये समवृत्तञ्च कल्पयेच्छत्रयन्त्रकम् ॥१३॥
छत्रदण्डञ्च तन्मध्ये व्यासार्धं शङ्कुरेव सः ।
स्वाहोरात्रदलं सौम्यं व्यस्ताग्रं भाभ्रमाह्वयम् ॥१४॥
षड्गुणा दिननाड्योंऽशाः सौम्यार्द्धे छत्रयन्त्रतः ।
अग्रान्तेऽर्कोदयास्ते च प्रत्यक्प्राक् प्रभा स्थिता ॥१५॥
तत्प्रत्यगन्तमस्ताख्यं प्रागन्तमुदयाह्वयम् ।
अस्ताख्यादुदयस्यान्तं छायाकालांशकाः स्थिताः ॥१६॥

[14] *Psi*, xvi. 21–22 (Thibaut's translation).
[15] *BrSpSi*, xxii. 8–16, 18.
[16] *ŚiDVṛ*, II, xi. 20–21, 22.
[17] *SiŚe*, xix. 12–13.
[18] *SiŚi*, II, xi. 10–12, 15(ii).
[19] See *ŚiDVṛ*, II, xi. 46–47.

छत्रमध्यस्थशङ्कोस्तु छायैवेष्टप्रभा सदा ।
छायाग्रास्ताख्यमध्यांशा षड्भिर्नाड्यो दिवा गताः ॥१७॥

13. Construct a *chatra-yantra* ("an instrument resembling an umbrella") by bamboo-needles, mark (the circumference of) it with the 360 divisions of degrees, and set it at the centre of the (perfect) circle. Or, treat the perfect circle itself as a *chatra-yantra*.

14. The rod of the *chatra-yantra*, in the middle of it, equal to the radius, is the gnomon; the northern half of the diurnal circle drawn through the end-points of the (Sun's) *agrā*, laid off in the contrary direction (in the west and the east), is the so called "path of shadow".[20]

15. The *nāḍīs* of the day, multiplied by six, are the degrees in (the diurnal circle lying in) the north half of the *chatra-yantra*. Towards the end-points of the (Sun's) *agrā*, in the west and the east, falls the shadow at sunrise and sunset respectively.

16. The end (of the Sun's *agrā*) in the west is (therefore) called the "setting point (*asta*)"; and the end (of the Sun's *agrā*) in the east the "rising point (*udaya*)". From the "setting point" to the "rising point" (on the northern half of the diurnal circle) lie (the graduations of) the degrees of time in a *chatra-yantra*.

17. The shadow cast by the gnomon, situated in the middle of the *chatra*, is always the shadow for the desired time. The degrees (on the diurnal circle) intervening between the end of the shadow and the "setting point", divided by six, give the *nāḍīs* elapsed in the day.

The *chatra-yantra* is not found to have been mentioned in any other work known to us. But a similar instrument called *pīṭha-yantra* occurs in the works of Lalla and Śrīpati.[21]

4.6 तोययन्त्राणि (Water instruments)

स्तम्भं सद्विलसम्पूर्णं तोयं रन्ध्रे तु योजयेत् ।
तन्मुक्तकालसम्भाज्यः स्तम्भायामोऽङ्गुलात्मकम् ॥१८॥
अङ्गुलानां मितिः स्तम्भे प्रतिनाडीं तु यन्त्रके ।
नाड्याख्याद् भूतलच्छिद्रात् पूर्यादम्बुघटीतलम् ॥१९॥
बीजमेतत् घटीमानं यन्त्रेषु स्तम्भसूत्रयोः ।
यन्त्रे बद्धनरे शिल्पे युद्धे मेषादिकेऽपि च ॥२०॥

[20] In fact, this is not the path of shadow.

[21] See *ŚiDVṛ*, II, xi. 24–25 and *SiŚe*, xix. 15–16.

अन्तःसुषिरमेवं तन्मयूरं वानरं तथा ।
स्थाप्य स्तम्भे तु सम्पूर्णे यन्त्रे षष्ट्यङ्गुलोच्छ्रिते ॥२१॥
सुश्लक्ष्णकीलकं सूक्ष्मं यन्त्राङ्कपरिकल्पितम् ।
षट्यङ्गुलेन सूत्रेण वेष्टयेत् षष्टिवेष्टनैः ॥२२॥
तं प्रक्षिपेन्नरे मूर्ध्नि निर्गच्छन् कर्णरन्ध्रयोः ।
पार्श्वयोर्निक्षिपेत् सूत्रं मयूरे वानरेऽपि वा ॥२३॥
मध्यवेष्टितसूत्राग्रे बध्वाऽलाबुं सपारदाम् ।
नरोपरि जले क्षिप्त्वा गुदच्छिद्रेऽम्बु मोचयेत् ॥२४॥
मयूरे वानरे वेत्थं बध्वाऽलाबुं सपारदाम् ।
नाभिरन्ध्रादधः स्तम्भजले क्षिप्त्वाऽम्बु मोचयेत् ॥२५॥
प्रतिनाडीं जलं छिद्रान्निर्गच्छत्येकमङ्गुलम् ।
स्तम्भेऽलाबुर्बिलान्तस्थाऽधो याति तथाऽङ्गुलम् ॥२६॥
तद्यन्त्रमध्यकीलस्थवेष्टनं चैकमङ्गुलम् ।
अलाबुकर्षणे सूत्रं अधो याति बिलोन्मुखम् ॥२७॥
तत्कीलाग्रेऽपरं सूत्रं नाडीज्ञानाय लम्बयेत् ।
तद्वेष्टनानि यावन्ति तावत्यो घटिका गताः ॥२८॥

18. Construct a pillar with an excellent (cylindrical) cavity inside itself. Fill up the cavity with water (and then open the hole at the bottom of the pillar so that the water may flow out). By the time (in *ghaṭīs*) taken by the water to flow out completely, divide the whole length of the pillar. This gives the measure of an *aṅgula* (which corresponds to a *ghaṭī*).

19. On the pillar mark the *aṅgulas* corresponding to each *ghaṭī*. The water corresponding to one *ghaṭī* flowing out from the hole (at the bottom of the pillar) in the level of the ground, completely fills a *ghaṭikā* vessel (in one *ghaṭī*).

20(i)–22. This measure of a *ghaṭī* is the basis (*bīja*) (for the determination) of (the height of) the pillar and of (the length of) the cord to be used in connection with the (time) instruments. Having tied around the pillar a man or a pair of fighting rams of craftsmanship (keeping the head of the figure just above the top of the pillar), or having surmounted the pillar by the figures of a peacock or monkey, bearing a cavity inside it, thus making the whole instrument sixty *aṅgulas* in height, take a smooth fine (cylindrical) needle with its periphery equal to a unit of graduation (i.e. one *aṅgula*) on the instrument, and on it wrap a cord of sixty *aṅgulas* in sixty coils.

23. Place this needle within the head of the man passing through the holes of the ears, or, in the case of peacock or monkey, support the needle (over the holes) on the two sides (of the body).

24. Having tied a gourd containing (an appropriate quantity of) mercury to the end of the cord wrapped round the needle, place it on the water (inside the pillar, through the hole) at the top of the man, and then let the water flow out through the hole at the anus.

25. Similarly, in the case of peacock or monkey, tying a gourd containing mercury (as before), throw it on the water of the pillar through the navel hole and release the flow of water.

26–27. An *aṅgula* of water now flows out in a *nāḍī*, as also the gourd within the pillar goes down by *aṅgula*. The cord wrapped around the needle, within the instrument, also goes down towards the hole underneath due to the pull of the gourd.

28. At one extremity of the needle, protruding outside the instrument, suspend another cord to know the *nāḍīs* elapsed. The number of coils made by this cord on the needle will indicate the *nāḍīs* elapsed.

Water instruments, such as *nara*, *mayūra*, and *vānara*, are mentioned in the *Sūryasiddhānta*,[22] but details of construction of those instruments are not given there. Similar instruments are also mentioned by Brahmagupta,[23] Lalla,[24] and Śrīpati,[25] but the details given by them differ from those given above.

4.7 घटिकायन्त्रम्

वृत्तं ताम्रमयं पात्रं कारयेद्दशभिः पलैः ।
षडङ्गुलं तदुत्सेधो विस्तारो द्वादशानने ॥२९॥
तस्याधः कारयेच्छिद्रं पलेनाष्टाङ्गुलेन तु ।
इत्येतद्घटिकासंज्ञं पलषष्ट्यम्बुपूरणात् ॥३०॥

29–30. One should get a hemispherical bowl manufactured of copper, ten *palas* in weight, six *aṅgulas* in height, and twelve *aṅgulas* in diameter at the top. At the bottom thereof, let a hole be made by a needle eight *aṅgulas* in length and one *pala* in weight.

This is the *ghaṭikā*-(*yantra*), (so named) because it is filled up by water in a period of 60 *palas* (i.e. one *ghaṭī*).[26]

[22] xiii. 21.
[23] See *BrSpSi*, xxii. 50.
[24] See *ŚiDVṛ*, II, xi. 12–17.
[25] See *SiŚe*, xix. 9–11.
[26] *Cf.* *ŚiDVṛ*, II, xi. 34–35 and *SiŚe*, xix. 19–20.

4.8 कपालयन्त्रम् (The bowl)

स्वेष्टं वाऽन्यदहोरात्रे षष्ट्याऽम्भसि निमज्ज्ञति ।
ताम्रपात्रमधश्छिद्रमम्बुयन्त्रं कपालकम् ॥३१॥

31. Any other copper vessel made according to one's liking, with a hole in the bottom, which sinks into water 60 times in a day and night, is the water instrument called *kapāla*.

This description of the *kapāla-yantra* agrees with that given in the *Sūrya-siddhānta*.[27]

4.9 शङ्कुविभेदाः (*Śaṅku-yantras* or gnomons)

तले द्व्यङ्गुलविस्तारः समवृत्तो द्वादशोच्छ्रयः ।
सारदारुमयः शङ्कुर्द्वितीयो द्वादशाङ्गुलः ॥३२॥
सूच्यग्रस्थूलमूलोऽन्यस्तदुत्सेधस्तलाग्रयोः ।
सतिर्यग्वेधसूच्योस्तु लम्बसूची स्फुटो नरः ॥३३॥
तुल्याग्रस्तलवृत्तोऽन्यः शङ्कुः स्याद् द्वादशाङ्गुलः ।
या व्यक्ता शङ्कुभा यन्त्रात् सा व्यक्तैव नतप्रभा ॥३४॥

32–33. (The first kind of gnomon is) two *aṅgulas* in diameter at the bottom, uniformly circular (i.e. cylindrical), twelve *aṅgulas* in height, and made of strong timber.

The second kind of gnomon is twelve *aṅgulas* (in height), pointed at the top, and massive at the bottom (i.e., conical in shape). (Associated with it is) another true gnomon of the same height, mounted vertically on two (horizontal) nails fixed (to the previous gnomon) at the top and bottom thereof.

34. Another (third) kind of gnomon (which is more handy) is that having equal circles at the top and bottom (i.e. cylindrical) and of twelve *aṅgulas*.

Whatever shadow of the gnomon is seen to be cast by this instrument is indeed the projection of the (Sun's) zenith distance (i.e. the Rsine of the Sun's zenith distance).

Of the above three kinds of gnomons, the first is mentioned by Lalla[28] and Śrīpati,[29] and the third by Bhāskara II.[30] That of the second kind is unusual.

[27] xiii. 23.
[28] See *ŚiDVṛ*, II, xi. 31–32.
[29] See *SiŚe*, xix. 18.
[30] See *SiŚi*, II, xi. 9.

4.10 Observations

It is noteworthy that Āryabhaṭa I has devoted as many as 34 verses to the treatment of the shadow and water instruments only. This shows that the chapter on the astronomical instruments of the *Āryabhaṭasiddhānta* must have been fairly large, for it must have included the discussion of the armillary sphere (*gola*) and the self-rotating device (*svayaṃvaha*) etc., which have received special treatment in the *Āryabhaṭīya* and the *Sūryasiddhānta* etc. Judging from the size of the chapter on the astronomical instruments, it may be easily inferred that the *Āryabhaṭasiddhānta* as a whole must have been a voluminous work on astronomy. The opening stanzas of the *Khaṇḍakhādyaka* too point to the same conclusion.

The composition of the above verses in the *anuṣṭubh* metre is another notable point. This shows that the *Āryabhaṭasiddhānta*, instead of being in the *āryā* metre like *Āryabhaṭīya*, was written in the *anuṣṭubh* metre following the usual style of the old *siddhānta* works.

The subject matter treated in the above verses too is no less important and interesting, for it throws light on the astronomical instruments which were in vogue in India in the fifth century AD.

5 Other verses ascribed to Āryabhaṭa

(a) Śaṅkaranārāyaṇa (869 AD) in his commentary on the *Laghubhāskarīya* of Bhāskara I informs that some astronomers in his time attributed the authorship of the following two verses to Āryabhaṭa I:

> वस्वेकेषुयुगघ्नं मनुयुगमर्कादिमध्यमचतुर्णाम् ।
> धनमृणमृणमृणमथ कृतिगुणितं चक्रेशभैर्लब्धम् ॥
> भौमाङ्गिरश्शनीनां देयमृणं देयमब्धिनन्दहृते ।
> सितबुधयोर्हेयं देयं सप्तहतं बुधस्योच्चे ॥

Severally multiply the elapsed *yugas* of the current Manu (i.e. $27\frac{3}{4}$) by 8, 1, 5, and 4, and treating the results as minutes, apply them positively, negatively, negatively, and negatively to the mean longitudes of the Sun, Moon, Moon's apogee, and Moon's ascending node respectively. Next, multiply the elapsed *yugas* of the current Manu (i.e. $27\frac{3}{4}$) by 20 and severally divide the product by 12, 11, and 27, and treating the quotients as minutes, apply them positively, negatively, and positively to the mean longitudes of Mars, Jupiter, and Saturn respectively. Next, divide that (product of the elapsed *yugas* of the current Manu and 20) severally by 4 and 9, and treating the quotients as minutes, apply the former negatively

> to the mean longitude of the *śīghrocca* of Venus, and the latter, as multiplied by 7, positively to the mean longitude of the *śīghrocca* of Mercury.

The above correction has been called "the *Manuyuga* correction". Parameśvara (1380–1450 AD) in his commentary on the *Laghubhāskarīya* (i. 37) of Bhāskara I has included it among the five *bīja* corrections listed by him, and has remarked that one should apply that one of them which makes computation correspond with observation.

Both the above verses are in the *āryā* metre, but they are not found to occur in the *Āryabhaṭīya* which is also in the same metre. They could not possibly have belonged to the *Āryabhaṭasiddhānta*, for as we have seen, it was composed in the *anuṣṭubh* metre. So if the above verses are from the pen of Āryabhaṭa I, they must be regarded as being detached compositions having no special connection with any of the works composed by him. Bhāskara I in his commentary on the *Āryabhaṭīya* has indeed referred to certain detached verses composed by Āryabhaṭa I.

(b) The following two verses in the *anuṣṭubh* metre are also found to be ascribed to Āryabhaṭa I by Mallikārjuna Sūri (1178 AD) in his commentary on the *Sūryasiddhānta*:

> यो यो भागः परः सूक्ष्मः राशेरुदयमागतः ।
> पुनस्तस्योदयो ज्ञेयो दिवसो भोदयात्मकः ॥
> नाक्षत्रसावनादीनां स्वस्वसावनदिनानि ।
> यस्मात्तस्मादर्धं ढुण्डिगमनमन्दबुद्धीनाम् ॥

6 Concluding remarks

We thus see that Āryabhaṭa I was the author of at least two works on astronomy which were written in the following chronological order:

(i) *Āryabhaṭasiddhānta*

(ii) *Āryabhaṭīya.*

The former work was a voluminous work based on the old *Sūryasiddhānta.* The latter work, now well known, is a small work comprising 121 verses in *āryā* metre. It deals with both mathematics and astronomy and claims to be based on the *Svāyambhuva-siddhānta.*

The *Āryabhaṭīya*, though based on the *Svāyambhuva-siddhānta*, was the result of the author's deep study and continued observations and incorporated the discoveries made by the author. This work gave birth to a new school of astronomy (the Āryabhaṭa school) whose exponents called themselves "followers of Āryabhaṭa" and were scattered throughout the length and breadth of South India. These devoted followers of Āryabhaṭa set up schools of astronomy at various places in that region and zealously undertook the task of teaching and propagating the system of astronomy propounded by Āryabhaṭa in his *Āryabhaṭīya*. One of the most important of these schools of astronomy was located at Aśmaka, a territory lying between the rivers Naramadā and Godāvarī. This school was perhaps the oldest and established by Āryabhaṭa I himself. It produced great men like Lāṭadeva and Bhāskara I who earned a great name and fame as scholars and teachers of astronomy. Bhāskara I explained and elucidated the teachings of Āryabhaṭa I by writing a comprehensive commentary on the *Āryabhaṭīya* and two text-books on astronomy viz. the *Mahābhāskarīya* and the *Laghubhāskarīya*. These works added freshness, vigour and life to the teachings of Āryabhaṭa and gave impetus to the study of the *Āryabhaṭīya*.

Astronomers residing to the north of the river Narmadā were followers of the *Sūryasiddhānta*. They adopted the *Āryabhaṭasiddhānta* for obvious reasons, but did not give much credit to the teachings of the *Āryabhaṭīya*. Brahmagupta even went to the extent of criticising that work as harshly as he could possibly do. But he too hailed the teachings of the *Āryabhaṭasiddhānta* and wrote his calendrical work the *Khaṇḍakhādyaka* on the basis of that work.

With the growing popularity of the *Khaṇḍakhādyaka* in the northern India and that of the *Āryabhaṭīya* in the southern, the study of the *Āryabhaṭasiddhānta* was ultimately given up; still it continued to be used as a reference book by the scholars and teachers who wrote commentaries on astronomical works. Our present knowledge of the *Āryabhaṭasiddhānta*, as we have seen above, is indeed due to the references made by the commentators Mallikārjuna Sūri, Rāmakrṣṇa Ārādhya, Tamma Yajvā, and Bhūdhara in their respective commentaries on the *Sūryasiddhānta*. The commentator Tamma Yajvā besides making references to the *Āryabhaṭasiddhānta* wrote a book entitled "*Siddhānta-sārvabhauma*" which he based on the *Āryabhaṭasiddhānta* and has quoted from that work. It is interesting to note that the quotations dealing with the astronomical instruments contain practically the same matter as given in the foregoing 34 verses from the *Āryabhaṭasiddhānta*. Tamma Yajvā's *Siddhānta-sārvabhauma*, when discovered, will be of great value to historians of mathematics and astronomy as it is expected to throw further light on the work of Āryabhaṭa I now lost to us.

Glimpses from the *Āryabhaṭasiddhānta* *

1 Introduction

In a paper entitled "Āryabhaṭa I's astronomy with midnight day-reckoning" published by me nine years ago in the *Gaṇita* (Vol. 18, No. I, 1967), I had adduced concrete and conclusive evidence to show that Āryabhaṭa I, the celebrated author of the *Āryabhaṭīya*, wrote one more work on astronomy which was known as *Āryabhaṭasiddhānta.* Whereas in the *Āryabhaṭīya* the day was reckoned from one sunrise to the next, in the *Āryabhaṭasiddhānta* the day was reckoned from one midnight to the next. This latter work of Āryabhaṭa which adopted midnight day-reckoning was first mentioned by Brahmagupta (628 AD) of Bhinmal in Rajasthan, who was so much impressed by its wide popularity that he epitomised the teachings of this work in his calendrical work bearing the title "Food prepared with sugar candy" (*Khaṇḍakhādyaka*). The notable points of difference of this work of Āryabhaṭa I from his other work (viz. the *Āryabhaṭīya*) were recorded by his scholiast Bhāskara I (629 AD) hailing from Valabhī in Gujarat, in Ch. vii of his *Mahābhāskarīya.* The above work of Āryabhaṭa was also mentioned by Varāhamihira (died 587 AD) of Kāpitthaka near Ujjain, Govindasvāmī (ninth century) of Kerala, Mallikārjuna Sūri (1178 AD) of Veṅgī in Āndhra, Maithila Caṇḍeśvara of Benaras, Rāmakṛṣṇa Ārādhya (1472 AD) of Āndhra, Bhūdhara (1572 AD) of Kampil in Uttar Pradesh, and Tamma Yajvā (1599 AD) of Ahobila in Āndhra. This work of Āryabhaṭa was famous for its description of the astronomical instruments particularly the water clocks, and has been remembered by the commentators of the *Sūryasiddhānta* while commenting on the *Yantrādhyāya* of that work. The commentator Rāmakṛṣṇa has even quoted as many as 34 verses from that work. These verses were discussed by me in the said paper. But this is not all that is known regarding that work. The above mentioned commentators of the *Sūryasiddhānta* have given some more information regarding the contents of that work which is otherwise unknown to us. The object of the present paper is to throw light on this information.

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* K. S. Shukla, *Indian Journal of History of Science*, Vol. 12, No. 2 (1977), pp. 181–186.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_29

2 Time from shadow

Mallikārjuna Sūri (1178 AD) states the following method for finding time from the gnomonic shadow and ascribes it to the *Āryabhaṭasiddhānta*:

> When the Sun is in the signs of Scorpion, (Sagittarius), Capricorn or Aquarius, then, as a rule, and elsewhere too, if it is within two *ghaṭīs* before or after noon, the measure of the gnomonic shadow in terms of digits (*aṅgulas*) is equīvalent to time in terms of *ghaṭīs*. At that time one might get an approximate estimate of time in the manner stated in the *Āryabhaṭasiddhānta*. If you ask how, then proceed like this: If it is $\frac{1}{2}$ of a *ghaṭī*, or 1 *ghaṭī*, or $1\frac{1}{2}$ *ghaṭīs*, or any number of *ghaṭīs* not exceeding two before noon, then (having constructed a circle on level ground and having drawn the east-west and north-south lines through its centre) set up a gnomon of 9 digits on the line directed towards the east from the centre of the circle in such a way that the tip of the shadow may fall on the north-south line passing through the centre of the circle. Then if the distance between the centre of the circle and the foot of the gnomon is $\frac{1}{2}$ of a digit, it would indicate that $\frac{1}{2}$ of a *ghaṭī* is to elapse before noon. If the distance is one digit, it is 1 *ghaṭī* before noon, and if $1\frac{1}{2}$ digits, then it is $1\frac{1}{2}$ *ghaṭīs* before noon.
>
> If the desired time is (within 2 *ghaṭīs*) after noon, then one should set up a gnomon of 9 digits on the line going towards the west from the centre. If the distance between the centre and the gnomon is $\frac{1}{2}$ of a digit, it is $\frac{1}{2}$ of a *ghaṭī* past noon; if 1 digit, it is 1 *ghaṭī* past noon; and if $1\frac{1}{2}$ digits, it is $1\frac{1}{2}$ *ghaṭīs* past noon. But this method would work only when the gnomon is set up in such a way that the tip of its shadow falls on the north-south line (passing through the centre).[1]

Tamma Yajvā,[2] too, has ascribed this method to the *Āryabhaṭasiddhānta*.

3 Possibility of an eclipse

Mallikārjuna Sūri informs us that the possibility of an eclipse was discussed in the *Āryabhaṭasiddhānta* in the following way:

> Adding 6 signs to the Sun at a *parva* (full moon or new moon), one gets the Earth's shadow. This is the eclipser of the Moon. When it is equal to the Moon's node, we have a total eclipse of

[1]Mallikārjuna's commentary on *Sūryasiddhānta*, iii. 35.
[2]In his commentary on *Sūryasiddhānta*, iii. 35.

the Moon. A solar eclipse will also be total provided the Moon's latitude corrected for parallax happens to be zero at that time. Even when the Moon's latitude exists, a partial eclipse of the Sun will be possible provided the parallax in latitude is less than half the sum of the diameters of the eclipsed and eclipsing bodies. Thus, at places where the equinoctial midday shadow is 1 digit, parallax in latitude is always less than half the sum of the eclipsed and eclipsing bodies. Where the equinoctial midday shadow is 5 digits, there the parallax in latitude is sometimes less and sometimes equal to half the sum of the diameters of the eclipsed and eclipsing bodies. When the parallax in latitude amounts to half the sum of the eclipsed and eclipsing bodies, then, if the Moon's latitude is zero, a solar eclipse does not occur. Where the equinoctial midday shadow is 9 digits, there the parallax in latitude is sometimes less than, sometimes equal to, and sometimes greater than half the sum of the diameters of the eclipsed and eclipsing bodies. When the former is equal to or greater than the latter, a solar eclipse is not possible provided the Moon's latitude (at new Moon) is zero. But all this happens only when the longitude of the eclipsed body is equal to that of the Moon's node.

When the distance of the Shadow or the Sun from the Moon's node is 12° and also if the Moon's velocity per day amounts to 12°20′, a lunar eclipse certainly does not occur anywhere. When this distance is less than 12° and the Moon's velocity greater than 12°20′, then there is a possibility of a lunar eclipse. But when the distance is 13° and the Moon's velocity 13°20′, even then a lunar eclipse is impossible. When the distance is less than that (and the Moon's velocity greater than 13°20′), there is a possibility of a lunar eclipse. Again, when the distance of the Shadow or the Sun from the Moon's ascending node exceeds 14° and also if the Moon's velocity is 14°20′, a lunar eclipse is impossible. (In fact) when the distance is 14°, a lunar eclipse is always impossible. In that case, the Moon's velocity does not play any role. Hence, one should proceed to calculate a lunar eclipse only when the said distance is less than 14°.

In the case of a solar eclipse too, at places where the equinoctial midday shadow is 1 digit and the distance of the Earth's shadow or the Sun from the Moon's ascending node amounts to 14°, a solar eclipse is impossible. At those very places, if the said distance is less than 14°, there is a possibility of a solar eclipse. Where the equinoctial midday shadow is 5 digits and the said distance

> is 16°, a solar eclipse is impossible. But if the said distance is less than 16°, there is a possibility of a solar eclipse. In a place where the equinoctial midday shadow is 9 digits and the Sun is at the last point of the sign Gemini, the length of the day amounts to 36 *ghaṭīs*; and when at the end of the sign Sagittarius, the day amounts to 24 *ghaṭīs*. Where the equinoctial midday shadow exceeds 9 digits, there is no habitation. Hence, knowledge of occurrence of eclipses for those places is of no use. People do not live beyond 600 *yojanas* from the equator. The region lying north of that is inaccessible to man.
>
> All this has been explained in detail in the *Āryabhaṭasiddhānta*.[3]

Tamma Yajvā and Rāmakṛṣṇa Ārādhya too have included the above discussion of the possibility of an eclipse in their commentaries on the *Sūryasiddhānta*.[4]

4 Lord of the *parva* (New Moon or Full Moon)

Mallikārjuna Sūri says:

> Although the method of finding the lord of the *parva* is not discussed here, but this topic having been discussed in the *Āryabhaṭasiddhānta* and also because it is necessary at this place it is being stated here. The sum of the revolutions performed since the beginning of creation by the Sun and the Moon's ascending node at the desired *parva*, multiplied by 2, gives the number of *parvas* elapsed since the time of creation. (These are presided over by the seven lords Brahmā, Indra, Śakra, Kubera, Varuṇa, Agni, and Yama in the serial order). Dividing the number of *parvas* elapsed by 7, we get the number of complete cycles of the *parvas*. The number that is obtained as the remainder, counted with Brahmā, gives the lord of the current *parva*.[5]

Tamma Yajvā has also quoted this rule.[6]

It is noteworthy that Mallikārjuna quotes the above rule for the reason that it was stated in the *Āryabhaṭasiddhānta*. This probably suggests that the *Āryabhaṭasiddhānta* was an important work of the *Sūryasiddhānta* school.

[3]Mallikārjuna's commentary on *Sūryasiddhānta*, iv. 6.
[4]iv. 6.
[5]Mallikārjuna's commentary on *Sūryasiddhānta*, iv. 7–8.
[6]See Tamma Yajvā's commentary on *Sūryasiddhānta*, i. 67.

5 Observation of the planets

The *Sūryasiddhānta*, the *Brāhmasphuṭasiddhānta*, the *Śisyadhīvṛddhida* and other works on Hindu astronomy describe the method of observing the planets. According to Mallikārjuna Sūri, the same method occurred in the *Āryabhaṭasiddhānta* also. It might be explained briefly as follows:[7]

> First of all calculation was made of the length of the gnomonic shadow cast by the planet. The length of this gnomon was taken to be equal to the height of the observer's eye. Then two bamboos were set up vertically near the gnomon in the direction contrary to the shadow and a pipe (*yaṣṭi*) of 5 cubits in length was tied to these bamboos in the direction of the hypotenuse of shadow, in such a way that one end of the pipe was just at the top of the gnomon. This done, the planet was seen by the observer through the pipe by placing his eye at the top of the gnomon.
>
> Sometimes the planet was seen indirectly in water, oil, or mirror placed at the tip of the shadow. For this purpose another gnomon of equal length was set up in the direction of the shadow at a distance equal to double the shadow. The same method was used for the observation of the conjunction of two planets. When the two planets were in close conjunction, only one pipe was used. But when they were separated by a distance, two pipes were used, one directed towards one planet and the other towards the other. One end of each pipe was at the top of the gnomon, so that the observer could see both the planets with his eye at the top of the gnomon.
>
> This method was used also to see the Moon at its first visibility, the elevation of the lunar horns and the eclipses of the Sun and the Moon.

6 Distances of Mercury and Venus at rising or setting

Mallikārjuna Sūri writes:

> According to the *Āryabhaṭasiddhānta*, the time-degrees of the heliocentric distance of Venus at the time of its rising or setting are 9 when Venus is in swift motion, and 7 when in retrograde motion. The corresponding time-degrees of Mercury are 13 for swift motion and 12 for retrograde motion.[8]

[7] See Mallikārjuna's commentary on *Sūryasiddhānta*, vii. 12.
[8] Mallikārjuna's commentary on *Sūryasiddhānta*, ix. 9–11.

It might be mentioned here that the *Khaṇḍakhādyaka* gives 9 time-degrees for Venus and 13 time-degrees for Mercury which correspond to their swift motion. Those for retrograde motion are not given. From the above statement of Mallikārjuna Sūri we find that the *Āryabhaṭasiddhānta* contained time-degrees for swift as well as retrograde motions of Mercury and Venus.

7 Rising and setting of Canopus (*Agastya*)

Mallikārjuna Sūri writes:

> Canopus sets when the Sun's longitude amounts to 2 signs minus the local latitude. It rises when the Sun's longitude is 6 signs minus that. Thus we have stated here the view of the *Āryabhaṭasiddhānta* as an alternative method.[9]

That this method really belonged to the *Āryabhaṭasiddhānta* is confirmed by its occurrence in the *Pūrva-Khaṇḍakhādyaka*. See Bina Chatterjee's edition, Vol. 2, p. 147, lines 9–10.

8 The shadow instruments

Tamma Yajvā describes the shadow instruments of the *Āryabhaṭasiddhānta* as follows:

> We now describe the (shadow) instruments described there (i.e. in the *Āryabhaṭasiddhānta*). The gnomon (*śaṅku*) is of three kinds. The first is cylindrical in shape, 12 digits high, and 2 digits in diameter. The second is conical in shape, 2 digits in diameter at the base, 12 digits in height, pointed at the top, and having two horizontal holes at the bottom and top (separated by a distance of 12 digits) with two nails fixed to them. This gnomon having been set up vertically, a thread of 12 digits in length should be suspended between the two nails. The third gnomon is cylindrical in shape with small diameter, and its height is 12 digits. This (last) gnomon is fit for use by all people. The *yaṣṭi-yantra* (i.e. the pipe) is a smooth cylindrical pipe which is as many digits in length as there are degrees in a radian (i.e. 57). The *dhanur-yantra* (i.e. the semi-circle) is a semi-circle whose radius has as many digits as there are degrees in a radian, and whose circumference is graduated with the marks of degrees, and which is furnished with the chord and the arrow. The *cakra-yantra* (i.e. the circle)

[9]Mallikārjuna's commentary on *Sūryasiddhānta*, ix. 17.

> is a circle (or hoop) whose radius is as many digits in length as there are degrees in a radian, and whose circumference bears 360 marks of degrees as well as two holes one at each equinoctial point. The *chatra-yantra* (i.e. the umbrella) is constructed like the *cakra-yantra* with a vertical rod at its centre. The rod should be made as big as there are digits in a radian.[10]

9 The water-clocks

Describing the water-clocks of the *Āryabhaṭasiddhānta*, Tamma Yajvā says:

> Now I give the method of knowing time as taught by Āryabhaṭa and others. First of all one should construct a high cylindrical pillar with a (uniform cylindrical) cavity inside and a hole at the bottom, and fill it with water. Then, keeping an eye on the clock, allow the water to flow out through the hole at the bottom. Now divide the digits of the height of the pillar by the *ghaṭīs* in which the water has flown out (of the pillar): this would give the digits corresponding to one *ghaṭī*. The pillar should then be surmounted by the effigy of a peacock, man or monkey, or of a tortoise or some bird constructed out of bamboo pieces and leather. (A needle should then be inserted horizontally through the belly of the effigy; one end of a cord should be tied to it in the middle, some portion of the cord should be wrapped on it in coils, and the other end of the cord should be tied to a sling carrying a dry hollow gourd filled with some mercury; and the sling containing gourd should then be put on water in the cavity of the pillar underneath, so that, as the water flows out of the cavity through the hole at the bottom, the gourd goes down and the needle rotates.) A cord should also be hung at the end of the needle which is made to pass through the belly of the bird, for the knowledge of time. The number of coils that the thread makes on the needle indicates the number of *ghaṭīs* elapsed. Or, tie a cord to the sling carrying a dry gourd full of mercury and throw the sling into the cylindrical cavity full of water underneath the effigy of the man or peacock etc., and let the other end of the cord go out of the mouth of the man etc. and hang there. This outer cord should be made to pass through 60 beads, separated by equal intervals of one *ghaṭī*. This done, as the water flows out of the cavity, the beads enter the mouth of the man etc., one by one, at the end of every *ghaṭī*. In this way one

[10] Tamma Yajvā's commentary on *Sūryasiddhānta*, xiii, 20–21.

might have a proper knowledge of time.[11]

The above description of the shadow and water instruments closely agrees with that given in the verses cited by Rāmakṛṣṇa Ārādhya in his commentary on the *Sūryasiddhānta*. For details one might refer to *Gaṇita*, Vol. 18, No. 1, pp. 83–105.

[11] Tamma Yajvā's commentary on *Sūryasiddhānta*, xiii. 22–25.

Part V

Development of Siddhāntic Astronomy: Some Highlights

Early Hindu methods in spherical astronomy *

1 Introduction

The Hindu astronomers did not possess a general method for solving problems in spherical astronomy, unlike the Greeks who systematically followed the method of Ptolemy (c. 150 AD), based on the well known theorem[1] of Menelaus (90 AD). But, by means of suitable constructions within the armillary sphere, they were able to reduce many of their problems to comparison of similar right-angled plane triangles. In addition to this device, they sometimes also used the theory of quadratic equations, or applied the method of successive approximations. In spite of this, some of their problems could not be solved accurately till the twelfth century AD when complete and accurate solutions to all astronomical problems, based on the solution of right-angled or oblique spherical triangles, were obtained in India. This was done by the systematic use of the formula

$$R\sin b = R\sin c \times \frac{R\sin B}{R}$$

for the spherical triangle ABC, right-angled at C, and the credit of this achievement is due to the mathematician Nīlakaṇṭha (1500 AD) of Kerala in south India. Unfortunately, the discoveries made by him remained unknown in north India, and the astronomers there continued the work of perfecting methods in spherical astronomy up to the end of the seventeenth century.

It is worthy of note that all through the history of development of methods in spherical astronomy in India, we do not find any trace of the use of the theorem of Menelaus, or of the method of projection occurring in the Analemma of Ptolemy.[2]

* K. S. Shukla, *Gaṇita*, Vol. 19, No. 2 (1968), pp. 50–72.

[1] "If the sides BC, AC, and AB of a spherical triangle ABC be intersected by a transversal at the points L, M, and N respectively, then

$$\frac{\sin\widehat{BL}}{\sin\widehat{LC}}\frac{\sin\widehat{CM}}{\sin\widehat{MA}}\frac{\sin\widehat{AN}}{\sin\widehat{NB}} = 1."$$

[2] Braunmühl misunderstood the Hindu methods and supposed (*Geschichte der Trigonometrie*, pp. 38–42) that they were based on the method of projection given in the *Analemma*, but Sengupta has proved his assumption to be wrong. See P. C. Sengupta, *Greek*

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_30

Of the methods taught by Āryabhaṭa I (499 AD) and demonstrated by his scholiast Bhāskara I (629 AD), some are based on comparison of similar right-angled plane triangles, and others are derived from inference. Brahmagupta is probably the earliest astronomer to have employed the theory of quadratic equations and the method of successive approximations to solving problems in spherical astronomy. In the present paper we shall discuss the methods found in the works of Bhāskara I and shall also throw light on the innovations made by Brahmagupta.

2 Preliminary methods in plane trigonometry

2.1 Measurement of arcs

In Hindu trigonometry use of angles has not been made, and, instead of referring to the angle at a point, mention is made of an arc of a circle, centred at that point, subtending that angle. A circle is assumed to contain 3438 units in the radius and 21600 units in the circumference. These units are called *kalā* or *liptā* and correspond to minutes of arc in modern trigonometry. An arc is more commonly measured in *kalās* (minutes), but the other circular measures are also used. These are *rāśi* (sign), *aṁśa* or *bhāga* (degree), *vikalā* or *viliptā* (second), and *tatparā* (third). Besides these terms, Bhāskara I has also used the terms *bhavana* (sign), *maurika* (minute), *vimaurika* (second), and *sukṣmakā* (second).

It is customary to use the term *trijyā* (radius) for its length, i.e., $3438'$.[3] We have denoted the radius by the letter R, and have meant by it $3438'$ in accordance with the Hindu usage.

2.2 Trigonometrical functions

Only four trigonometrical functions have been used by the Hindus, viz. *jyā, koṭijyā, utkramajyā*, and *koṭyutkramajyā*. The corresponding functions in modern trigonometry are sine, cosine, versed-sine, and coversed-sine respectively. They are related by the following equalities:

$$\begin{aligned} \textit{jyā}\ \theta &= R \times \sin\theta = R\sin\theta \quad (\text{where } R = 3438'), \\ \textit{koṭijyā}\ \theta &= R\cos\theta, \\ \textit{utkramajyā}\ \theta &= R\,\mathrm{versin}\,\theta, \\ \textit{koṭyutkramajyā}\ \theta &= R\,\mathrm{covers}\,\theta. \end{aligned}$$

and Hindu Methods in Spherical Astronomy, Appendix II to his translation of the *Khaṇḍakhādyaka*, Calcutta (1934), pp. 172 ff. It may be mentioned that Sengupta is at places not very accurate in demonstrating the Hindu methods.

[3]It may be noted that 3438 denotes the number of minutes of arc in a radian.

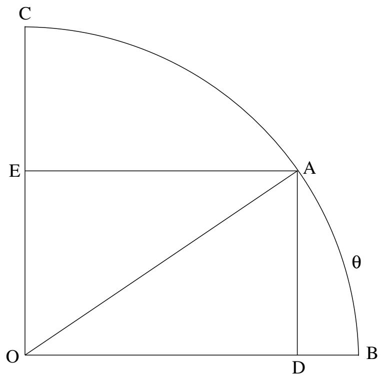

Figure 1

For convenience, we have omitted the multiplication sign, and have translated *jyā* by Rsine, *koṭijyā* by Rcosine, *utkramajyā* by Rversed-sine, and *koṭyutkramajyā* by Rcoversed-sine. Bhāskara I has used only three trigonometrical functions, *jyā*, *koṭijyā*, and *utkramajyā*. The function *koṭyutkramajyā* has been used by his commentator Śaṅkaranārāyāṇa (869 AD).[4]

2.3 The geometrical significance of *jyā* θ etc.

Let θ denote the number of minutes in the arc AB of a circle centred at O (see Figure 1), and suppose that AD is perpendicular to the radius OB. Also let OC be perpendicular to OB, and AE perpendicular to OC. Then *jyā* θ is defined by the length AD, *koṭijyā* θ by the length AE, *utkramajyā* θ by the length DB, and *koṭyutkramajyā* θ by the length EC.

It is easy to see that

$$AD = OA \sin\theta, \quad \text{or} \quad R\sin\theta,$$
$$\text{and} \quad OD = R\cos\theta.$$

2.4 Calculation of $R\sin\theta$, $\theta < 90°$

The value of the Rsine of an acute angle (or arc)[5] θ is calculated by means of a table of Rsine-differences.[6] The table mentioned by Bhāskara I is the same as given in the *Āryabhaṭīya*.[7]

[4] See his commentary on *LBh* (=*Laghubhāskarīya*), ii. 31–32; iv. 17; v. 13–14.

[5] As the radius of the circle is assumed to be one radian, it is immaterial whether we say "angle θ" or "arc θ".

[6] See our notes on *MBh* (=*Mahābhāskarīya*), iv. 3–4 (i).

[7] See *Ā* (=*Āryabhaṭīya*), i. 12.

Bhāskara I[8] has also stated the following approximate formula for calculating the Rsine of an acute angle without the use of a table:

$$R\sin\theta = \frac{R(180^\circ-\theta)\theta}{\dfrac{40500-(180^\circ-\theta)\theta}{4}},$$

where θ is in degrees.

This formula, in modern notation, may be written as

$$\sin\lambda = \frac{16\lambda(\pi-\lambda)}{5\pi^2-4\lambda(\pi-\lambda)},$$

where λ radians correspond to θ degrees.

Putting $\lambda = \frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{7}$, we get

$$\sin\left(\frac{\pi}{3}\right) = .8648\ldots,\quad \sin\left(\frac{\pi}{4}\right) = .70058\ldots,\ \text{and}\quad \sin\left(\frac{\pi}{7}\right) = .4313\ldots,$$

which are correct up to two places of decimals. The values of $\sin\pi$, $\sin\left(\frac{\pi}{2}\right)$, and $\sin\left(\frac{\pi}{6}\right)$, however, come out to be accurate.[9]

Bhāskara I ascribes the above approximate formula to Āryabhaṭa I.[10] It occurs in the *Brāhmasphuṭasiddhānta*[11] and in several later works also, such as the *Siddhāntaśekhara*,[12] and the *Līlāvatī*.[13] Occurrence of this formula in the *Mahābhāskarīya* and the *Brāhmasphuṭasiddhānta*, written about the same time, shows that it was well known in the first quarter of the seventh century AD.

2.5 Calculation of $R\sin\theta,\ \theta > 90^\circ$

Bhāskara I makes use of two methods for the purpose.

Method I

The first method is equivalent to the use of the following formulae:

$$\begin{aligned} R\sin(90^\circ+\phi) &= R\sin 90^\circ - R\ \operatorname{versin}\phi \\ R\sin(180^\circ+\phi) &= 90^\circ - R\ \operatorname{versin} 90^\circ - R\sin\phi \\ R\sin(270^\circ+\phi) &= R\sin 90^\circ - R\ \operatorname{versin} 90^\circ - R\sin 90^\circ + R\ \operatorname{versin}\phi, \end{aligned}$$

where $\phi < 90^\circ$.

[8] In his *Mahābhāskarīya* (vii. 17–19) and also in his commentary on *Ā*, i. 11.
[9] For details regarding the above formula, see our notes on *MBh*, vii. 17–19.
[10] See his commentary on *Ā*, i. 11.
[11] xiv. 23–24.
[12] iii. 17.
[13] See *Līlāvatī*, Ānandāśrama Sanskrit Series (No. 107), stanza 210, p. 212.

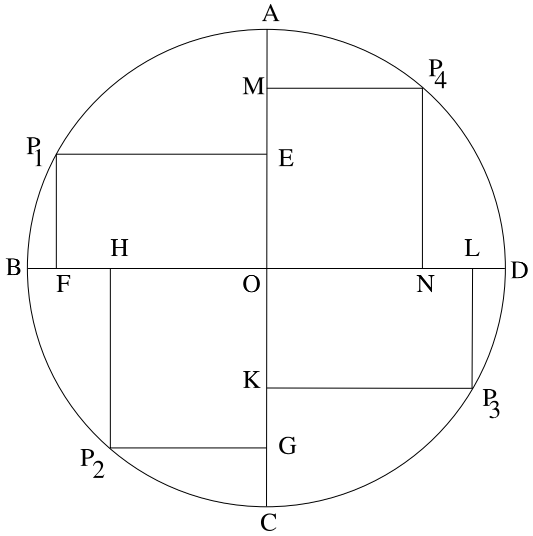

Figure 2

These formulae were used earlier in the *Āryabhaṭīya*,[14] but they were not explicitly stated there. W. E. Clark has, therefore, missed the exact significance of Āryabhaṭa I's rule in *Ā*, iii. 22 (i), although his interpretation is mathematically correct. Use of the above formulae is made by Brahmagupta also.[15] In later works they have been discarded in favour of Method II which is simpler.

Method II

This method may be explained as follows:

Let the circle $ABCD$ (See Figure 2) denote the mean orbit of a planet, and A the planet's apogee from which the mean anomaly is measured anticlockwise. AOB, BOC, COD, and DOA are the first, second, third, and fourth anomalistic quadrants.

When the planet is at P_1 in the first quadrant, the arc traversed is AP_1 and the arc to be traversed is P_1B; when the planet is at P_2 in the second quadrant, the arc traversed is BP_2 and the arc to be traversed is P_2C; when the planet is at P_3 in the third quadrant, the arc traversed is CP_3 and the arc to be traversed is P_3D; and when the planet is at P_4 in the fourth quadrant, the arc traversed is DP_4 and the arc to be traversed is P_4A.

[14] iii. 22 (i). Also see commentaries on it.

[15] See *BrSpSi* (*Brāhmasphuṭasiddhānta*), ii. 15–16.

In odd quadrants, the arc traversed is called *bhuja* and the arc to be traversed is called *koṭi*; and in even quadrants, the arc traversed is called *koṭi* and the arc to be traversed is called *bhuja*. The Rsine of the *bhuja* is defined to be the Rsine of the corresponding anomaly, and the Rsine of the *koṭi* is defined to be the Rcosine of the corresponding anomaly. The Rsine is positive in the first and the second quadrants, and negative in the third and fourth quadrants; the Rcosine is positive in the first and fourth quadrants, and negative in the second and third quadrants.

Thus we have

$$R\sin\widehat{AP_1} = P_1E, \qquad R\cos\widehat{AP_1} = P_1F;$$
$$R\sin\widehat{AP_2} = P_2G, \qquad R\cos\widehat{AP_2} = -P_2H;$$
$$R\sin\widehat{AP_3} = -P_3K, \qquad R\cos\widehat{AP_3} = -P_3L;$$
$$R\sin\widehat{AP_4} = -P_4M, \qquad R\cos\widehat{AP_4} = -P_4N.$$

It can be easily seen that this method is equivalent to the application of the following formulae:

$$\begin{aligned} R\sin\phi &= R\sin\phi, \\ R\sin(90^\circ+\phi) &= R\sin(90^\circ-\phi), \\ R\sin(180^\circ+\phi) &= -R\sin\phi, \\ \text{and}\quad R\sin(270^\circ+\phi) &= -R\sin(90^\circ-\phi), \end{aligned}$$

where $\phi < 90^\circ$.

3 Solution of right-angled spherical triangles

Given two elements (not both angles) of a right-angled spherical triangle, Bhāskara I could solve it completely.[16] The results enunciated by him correspond to the following relations between the sides and angles of a spherical triangle ABC, right-angled at C: (see Figure 3)

$$\begin{aligned} \sin b &= \sin c \sin B \\ \sin c \cos B &= \cos b \sin a \\ \cos c &= \cos a \cos b \\ \tan c \cos B &= \tan a \\ \tan B \sin a &= \tan b. \end{aligned}$$

As already pointed out, Bhāskara I did not adopt any systematic method for deriving the above results. The methods used by him can be best illustrated by means of his own problems.

[16] The methods used by him are the same as taught by Āryabhaṭa I.

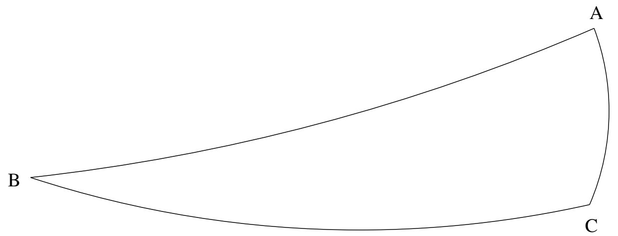

Figure 3

3.1 Problem 1

Given the longitude (λ) of the Sun, and the obliquity (ϵ) of the ecliptic, to find the declination (δ) of the Sun.

In Figure 4, let O be the centre of the armillary sphere; BD a quadrant of the equator, P its pole; BE a quadrant of the ecliptic, A the Sun. PGB, PAC, and PED are the secondaries to the equator. G is the point where the Sun's diurnal circle (represented by the dotted circle) intersects PGB. B is evidently the first point of Aries.

Let CH be the perpendicular from C to OB, AK the perpendicular from A to IG, KL the perpendicular from K to OB, and EF the perpendicular from E to OP. Join OC, OE, and LA. OE and LA are the evidently perpendicular to OB.[17]

In the triangles AKL and EFO, right-angled at K and F respectively, we have

$$KL = R\sin\widehat{AC}, \quad \text{i.e. } R\sin\delta,$$
$$LA = R\sin\widehat{BA}, \quad \text{i.e. } R\sin\lambda,$$

and

$$FO = R\sin\widehat{ED}, \quad \text{i.e. } R\sin\epsilon,$$
$$OE = R, \quad \text{the radius of the armillary sphere.}$$

These triangles are evidently similar, and by comparing them we have

$$\frac{KL}{FO} = \frac{LA}{OE},$$

giving[18]

$$R\sin\delta = \frac{R\sin\lambda \times R\sin\epsilon}{R}.$$

[17]All lines within the armillary sphere were shown by means of threads.
[18]See *MBh*, iii. 6(i).

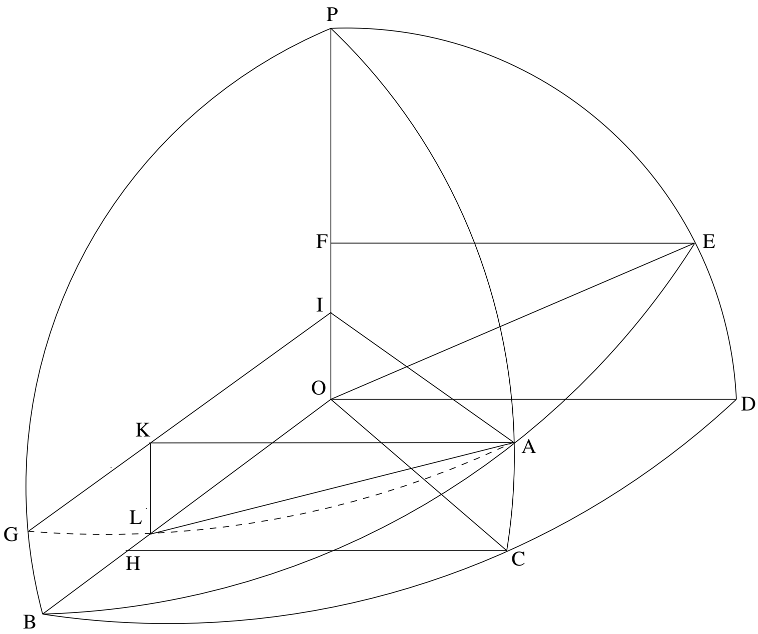

Figure 4

This formula corresponds to

$$\sin\delta = \sin\lambda \sin\epsilon$$

of modern astronomy. If the radius of the armillary sphere be taken to be unity, and ABC be any spherical triangle right-angled at C drawn on it, the above procedure will give

$$\sin b = \sin c \sin B,$$

which is the relation between b, c, and B in modern spherical trigonometry.

Note: The triangle AKL is known in Hindu astronomy by the name *krānti-kṣetra* ("declination triangle"). It may be noted that the sides of the triangle FOE are equal to the Rsines of the angles of the triangle AKL, and the comparison of the triangles AKL and FOE is equivalent to the use of the Rsine-formula

$$\frac{R\sin A}{a} = \frac{R\sin B}{b} = \frac{R\sin C}{c}$$

corresponding to the sine-formula

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

for the plane triangle ABC. Whenever we use the above Rsine-formula, it should be understood that we are comparing two similar triangles.

Similar other problems solved by Bhāskara I

1. Given the longitude of the Sun, the longitude of the rising point of the ecliptic, and latitude of the place, to determine the altitude of the meridian-ecliptic point.[19]
2. Given the longitude of the Moon, the longitude of the Moon's ascending node, and the inclination of the Moon's orbit to the ecliptic, to find the latitude of the Moon.[20]
3. Given the longitude of the rising point of the ecliptic, the longitude of the Moon's ascending node, and the inclination of the Moon's orbit, to determine the latitude of the rising point of the ecliptic.[21]
4. Given the longitude of the meridian-ecliptic point and of the Moon's ascending node, to determine the latitude of the meridian-ecliptic point.[22]
5. Given the zenith distance of the meridian-ecliptic point and the amplitude (*agrā*) of the rising point of the ecliptic, to determine the arcual distance between the central-ecliptic and meridian-ecliptic points.[23]
6. Given the Sun's declination, and the obliquity of the ecliptic, to obtain the Sun's longitude.[24]
7. Given the zenith distance and declination of the Sun when it is on the prime vertical, to determine the hour angle of the Sun.[25]

3.2 Problem 2

Given the longitude (λ) and declination (δ) of the Sun, and the obliquity (ϵ) of the ecliptic, to obtain the right-ascension (α) of the Sun.

See Figure 4. Comparing the similar triangles AKL and EFO, we have

$$\frac{AK}{LA} = \frac{EF}{OE},$$
$$\text{or} \quad AK = \frac{EF \times LA}{OE}. \tag{1}$$

[19] *MBh*, iii. 21.
[20] *LBh*, iv. 8.
[21] *MBh*, v. 14.
[22] *MBh*, v. 16 (ii).
[23] *MBh*, v. 19.
[24] *MBh*, iii. 16.
[25] *MBh*, iii. 40.

Again, comparing the similar triangles AKI and CHO, we have

$$\frac{CH}{OC} = \frac{AK}{IA},$$
$$\text{or} \quad CH = \frac{AK \times OC}{IA}.$$

Using (1), we get

$$\begin{aligned} CH &= \frac{EF \times LA}{OE} \times \frac{OC}{IA} \\ &= \frac{LA \times EF}{IA}, \quad \text{because} \quad OC = OE. \end{aligned}$$

Now, in Figure 4, $CH = R\sin\alpha$, $LA = R\sin\lambda$, $EF = R\cos\epsilon$, and $IA = R\cos\delta$, therefore

$$R\sin\alpha = \frac{R\sin\lambda \times R\cos\epsilon}{R\cos\delta},$$

which is the formula stated by Bhāskara I.[26]

If the radius of the armillary sphere be taken to be unity and the triangle ABC be assumed to be any spherical triangle right-angled at C, the above procedure will give

$$\sin c \times \cos B = \cos b \times \sin a,$$

which is the relation between a, b, c, and B in modern spherical trigonometry.

3.3 Problem 3

Given the declination (δ) of the Sun when it is on the prime vertical, and the latitude (ϕ) of the place, to determine the Sun's altitude (a).

Figure 5 is a triangle drawn within the armillary sphere. S is the Sun (on the armillary sphere), SA is the perpendicular drawn from S to the plane of the horizon. Since S is on the prime vertical, SA is evidently perpendicular to the east-west line. SB is perpendicular to the Sun's rising-setting line. AB evidently denotes the distance between the east-west and the Sun's rising-setting lines. AC is perpendicular to SB.

In the triangle SAB right-angled at A, we have

$$\begin{aligned} SA &= R\sin a, \\ AB &= R\sin(\text{Sun's } \textit{agrā}), \\ \angle SBA &= 90^\circ - \phi, \\ \text{and} \quad \angle ASB &= \phi. \end{aligned}$$

[26] *MBh*, iii. 9.

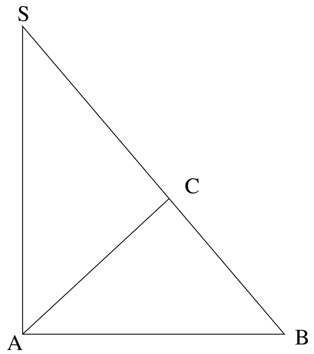

Figure 5

And in the triangle SCA, right-angled at C, we have

$$\begin{aligned} SA &= R\sin a, \\ AC &= R\sin\delta, \\ \angle ASC &= \phi, \\ \text{and}\quad \angle SAC &= 90^\circ - \phi. \end{aligned}$$

Applying the Rsine-formula to the triangle SCA, we obtain

$$\frac{SA}{R\sin\angle SCA} = \frac{AC}{R\sin\angle ASC},$$

$$\text{or}\quad SA = \frac{AC \times R\sin\angle SCA}{R\sin\angle ASC},$$

$$\text{i.e.,}\quad R\sin a = \frac{R\sin\delta \times R}{R\sin\phi}, \tag{2}$$

which is the result stated by Bhāskara I.[27]

In modern astronomy, the above problem requires the solution of the triangle SZP (see Figure 6), in which Z is the zenith, P is the celestial north pole, and S the Sun on the prime vertical. $ZP = 90^\circ - \phi$, $ZS = 90^\circ - a$, $SP = 90^\circ - \delta$, and $\angle SZP = 90^\circ$. Applying the cosine formula of modern spherical trigonometry, we get $\sin a = \frac{\sin\delta}{\sin\phi}$ which is the same as (2) in modern form.

The above method illustrates how the Hindus solved a right-angled spherical triangle when the three sides were given. Their result corresponds to the relation

$$\cos c = \cos a \cos b.$$

for a triangle ABC, right-angled at C, in modern spherical trigonometry.

[27] See *LBh*, iii. 22.

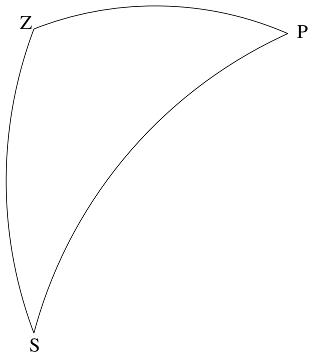

Figure 6

Other similar problems solved by Bhāskara I

1. Given the longitude (λ) of the Sun when it is on the prime vertical, the obliquity of the ecliptic (ϵ), and the latitude (ϕ) of the place, to find the Sun's altitude (a).[28]

   Bhāskara I gives the formula:

$$R\sin a = \frac{R\sin(\text{Sun's } \textit{agrā}) \times R\cos\phi}{R\sin\phi},$$

$$\text{where} \quad R\sin(\text{Sun's } \textit{agrā}) = \frac{R\sin\lambda \times R\sin\epsilon}{R\cos\phi}.$$

   In Figure 4, if we assume BC to be the equator, BA to be the local horizon, and the dotted circle GA to be the Sun's diurnal circle, then in the triangles AKL and EFO, we have

$$AL = R\sin(\text{Sun's } \textit{agrā}),$$
$$KL = R\sin\delta,$$
$$\text{and} \qquad OF = R\cos\phi.$$

   Therefore, comparing those triangles, we get

$$\frac{AK}{OE} = \frac{KL}{OF},$$

   giving

$$R\sin(\text{Sun's } \textit{agrā}) = \frac{R\sin\delta \times R}{R\cos\phi} = \frac{R\sin\lambda \times R\sin\epsilon}{R\cos\phi},$$

[28] *MBh*, iii. 37–38.

because

$$R\sin\delta = \frac{R\sin\lambda \times R\sin\epsilon}{R}.$$

Now from the triangle SAB, right-angled at A, in Figure 5, we have

$$SA = \frac{AB \times R\sin(\angle SBA)}{R\sin(\angle ASB)},$$

$$\text{i.e.,}\quad R\sin a = \frac{R\sin(\text{Sun's } agr\bar{a}) \times R\cos\phi}{R\sin\phi}.$$

2. Given the altitude of the Sun when it is on the prime vertical, and the latitude of the place, to obtain the Sun's longitude.[29]

3. Given the longitude of the rising point of the ecliptic, the obliquity of the ecliptic, and the latitude of the place, to determine the amplitude of the rising point of the ecliptic.[30]

3.4 Problem 4

Given the longitude (λ) of the Sun when it is on the prime vertical, the obliquity (ϵ) of the ecliptic, and the latitude (ϕ) of the place, to find the hour angle (H) of the Sun.

Bhāskara I gives the formula[31]

$$R\cos H = \frac{R\sin\lambda \times R\sin\epsilon \times R\cos\phi}{R\cos\delta \times R\sin\phi}. \tag{3}$$

The problem is really to obtain the hour angle H, when the declination δ of the prime vertical Sun, and the local latitude ϕ are known.

From the triangle SCA, right-angled at C, in Figure 5, we have

$$SC = \frac{R\sin\delta \times R\cos\phi}{R\sin\phi}.$$

But SC bears the same ratio to the radius of the Sun's diurnal circle as $R\cos H$ bears to the radius of the equator, therefore

$$SC = \frac{R\cos H \times R\cos\delta}{R}.$$

Hence we get

$$R\cos H = \frac{R\sin\delta \times R\cos\phi \times R}{R\cos\delta \times R\sin\phi}. \tag{4}$$

[29] *MBh*, iii. 41.
[30] *MBh*, v. 13.
[31] See, *MBh*, iii. 39.

Formula (3) is obtained by substituting $\frac{R\sin\lambda\times R\sin\epsilon}{R}$ for $R\sin\delta$, (vide Problem 1).

Formula (4) corresponds to the relation

$$\cos B = \frac{\tan a}{\tan c}$$

between a, c, and B of a spherical triangle ABC, right-angled at C, in modern spherical trigonometry.

3.5 Problem 5

Given the declination (δ) of the Sun, and the latitude (ϕ) of the place, to determine the ascensional difference (c) of the Sun.

In Figure 4, let us suppose that BA is the local horizon, BC the equator and P its pole, and the dotted circle the Sun's diurnal circle. Then in the triangle ABC, $BC = c$, $AC = \delta$, and $< ABC = 90° - \phi$. Also $CH = R\sin c$, $KL = R\sin\delta$, $EF = R\sin\phi$, and $FO = R\cos\phi$.

Comparing the similar triangles AKL and EFO, we get

$$\frac{AK}{EF} = \frac{KL}{FO},$$

$$\text{or} \quad AK = \frac{KL \times EF}{FO},$$

$$\text{i.e.,} \quad AK = \frac{R\sin\delta \times R\sin\phi}{R\cos\phi}. \tag{5}$$

Again, comparing the similar triangles CHO and AKI, we have

$$\frac{CH}{AK} = \frac{OC}{IA},$$

$$\text{or} \quad CH = \frac{OC \times AK}{IA},$$

$$\text{i.e.,} \quad R\sin c = \frac{R \times AK}{R\cos\delta}$$

$$= \frac{R\sin\delta \times R\sin\phi \times R}{R\cos\delta \times R\cos\phi}, \quad \text{using (5)},$$

which is the formula stated by Bhāskara I.[32]

This formula corresponds to

$$\sin a = \frac{\tan b}{\tan B}$$

for a spherical triangle ABC, right-angled at C, in modern trigonometry.

[32] See *MBh*, iii. 7.

Approximate solution

The following problem shows that Bhāskara I was unable to write down a relation involving two angles of a spherical triangle.

3.6 Problem 6

Given the Sun's longitude (λ), and the obliquity (ϵ) of the ecliptic, to obtain the *ayanavalana* (av).

In Figure 4, as already explained, BC is the equator, BA is the ecliptic, and A is the Sun. Thus, in spherical triangle ABC

$$\begin{aligned} AB &= \lambda, \\ \angle ABC &= \epsilon, \\ \angle BAC &= 90^\circ - av, \\ \text{and} < ACB &= 90^\circ. \end{aligned}$$

Both Bhāskara I and Brahmagupta were unable to get the desired relation between λ, ϵ and av, and have given only approximate solutions to the above problem. Thus, Brahmagupta gives

$$R\sin(av) = \frac{R\sin\epsilon \times R\sin(90^\circ + \lambda)}{R},$$

and Bhāskara I states

$$R\sin(av) = \frac{R\sin\epsilon \times R\,\text{versin}(90^\circ + \lambda)}{R},$$

Both these approximations are divided from inference.[33] It may be mentioned that Brahmagupta's approximation is better than that of Bhāskara I. The accurate solution to the above problem was given by Bhāskara II (1150 AD).[34]

4 Solution of oblique spherical triangles

As regards an oblique spherical triangle, Bhāskara I could write down the relation between the three sides and any one of the angles in the form

$$R\cos a = \frac{R \times R\cos b \times R\cos c + R\sin b \times R\sin c \times R\cos A}{R^2},$$

corresponding to the cosine formula of modern spherical trigonometry. He has solved the following problems accurately.

[33] For details see our notes on *MBh*, v. 45.
[34] See *SiŚi*, I, v. 21(ii)–22(i). Also see *SiŚi*. II, viii. 32 (ii) ff.

4.1 Problem 7

Given the Sun's declination (δ), the time elapsed since sunrise (say t), and the latitude (ϕ) of the place, to obtain the Sun's altitude (a).

Bhāskara I's solution to this problem is[35]

$$R\sin a = \frac{C \times R\cos\phi}{R},$$

where,

$$C = \frac{R\sin(t - \text{Sun's ascensional difference}) \times R\cos\delta}{R} + \frac{R\sin\phi \times R\sin\delta}{R\cos\phi}.$$

Since

$$t - \text{Sun's ascensional difference} = 90^\circ - H,$$

where H is the Sun's hour angle (east), therefore the above result reduces to

$$\begin{aligned} R\sin a &= \left[\frac{R\cos H \times R\cos\delta}{R} + \frac{R\sin\phi \times R\sin\delta}{R\cos\phi}\right] \times \frac{R\cos\phi}{R} \\ &= \frac{R \times R\sin\phi \times R\sin\delta + R\cos\phi \times R\cos\delta \times R\cos H}{R^2}, \end{aligned}$$

which corresponds to the following formula in modern astronomy:

$$\sin a = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H.$$

Other similar problems solved by Bhāskara I are:

1. Given the declination and altitude of the Sun, and the latitude of the place, to obtain the time elapsed since sunrise.[36]

   This is the converse of Problem 7.

2. Given the declination (δ) and altitude (a) of the Sun, and the latitude (ϕ) of the place, to determine the distance of the Sun's projection on the plane of the celestial horizon from the east-west line (i.e. the so called *bāhu*.)

   Assuming the Sun's azimuth A to be less than 90°, Bhāskara I's formula for the *bāhu* is:

$$b\bar{a}hu = \frac{R\sin a \times R\sin\phi}{R\cos\phi} + \frac{R \times R\sin\delta}{R\cos\phi}.$$

   But in terms of A and a,

$$b\bar{a}hu = \frac{R\cos A \times R\cos a}{R}.$$

[35] See *MBh*, iii. 25–26, 29–30(i), 30(ii)–31(i).
[36] *MBh*, iii. 27–29.

It follows that

$$\frac{R\cos A \times R\cos a}{R} = \frac{R\sin a \times R\sin\phi}{R\cos\phi} + \frac{R \times R\sin\delta}{R\cos\phi},$$

which corresponds to the following relation in modern astronomy:

$$\sin\delta = \sin a \sin\phi + \cos a \cos\phi \cos A.$$

Approximate solutions

Both Bhāskara I and Brahmagupta could not derive the correct relation between the four contiguous elements of an oblique spherical triangle, and have given only approximate solutions in such a case (see Problem 8). They were also probably unaware of the relation between the opposite sides and angles of an oblique triangle.

4.2 Problem 8

Given the hour angle (H) and the declination (δ) of the Sun, and the latitude (ϕ) of the place, to obtain the *akṣavalana*.

The *akṣavalana* is the angle subtended at the Sun by the arc of the celestial sphere joining the north pole of the equator and the north point of the horizon. Bhāskara I's solution to the above problem is[37]

$$R\sin(\textit{akṣavalana}) = \frac{R\,\text{versin}\,H \times R\sin\phi}{R}.$$

Brahmagupta's solution is[38]

$$R\sin(\textit{akṣavalana}) = \frac{R\sin H \times R\sin\phi}{R}.$$

Both the solutions are approximate and based on inference. The latter is the better than the former.[39]

The accurate solution to the above problem was given by Bhāskara II (1150 AD). Stated in modern notation, his solution is[40]

$$\sin(\textit{akṣavalana}) = \frac{\sin H \times \sin\phi}{\cos y},$$

where

$$\cos y = \sin\delta\cos\phi - \cos\delta\sin\phi\cos H.$$

[37]See *MBh*, v. 42.
[38]See *BrSpSi*, iv. 6.
[39]Brahmagupta has stated a still better formula in *BrSpSi*, x. 17.
[40]See *SiŚi*, II, viii. 66(ii)–67.

Note: Bhāskara I's formulae for the visibility corrections also are approximate, but they need not be mentioned here. The reader is referred to *MBh*, vi. 1–3, where they have been discussed in detail.

5 New methods introduced by Brahmagupta

5.1 Use of the theory of quadratic equations

Bhāskara I seems to have felt difficulty in cases where the required quantity is obtained in terms of its tangent (i.e. Rsine/Rcosine), and not in terms of its Rsine or Rcosine, for he has not considered any such problem. Brahmagupta has resolved the above difficulty by making use of the theory of quadratic equations, as the following problem solved by him will show.

5.1.1 Problem 9

Given the hour angle (H) of the Sun when it is on the prime vertical, and the latitude (ϕ) of the place, to obtain the declination (δ) of the Sun.

Brahmagupta states the result as follows:[41]

$$R\sin\delta = \frac{R\cos H \times s}{\sqrt{12^2 + \left(\frac{R\cos H \times s}{R}\right)^2}},$$

where s denotes the equinoctial midday shadow. Replacing $\frac{s}{12}$ by $\frac{R\sin\phi}{R\cos\phi}$, the above result may be written as

$$R\sin\delta = \frac{R\cos H \times R\sin\phi}{\sqrt{(R\cos\phi)^2 + \left(\frac{R\cos H \times R\sin\phi}{R}\right)^2}}$$

which is derived from the formula[42]

$$R\cos H = \frac{R\sin\delta \times R\cos\phi \times R}{R\cos\delta \times R\sin\phi}$$

by solving it as a quadratic in $R\sin\delta$.

5.1.2 Problem 10

Given the Sun's ascensional difference (c), and the latitude (ϕ) of the place, to obtain the Sun's declination (δ).

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[41] See *BrSpSi*, xv. 24–25.
[42] See Problem 4.

Brahmagupta gives the following formula:[43]

$$R\sin\delta = \frac{12R \times R\sin c}{\sqrt{12^2(R\sin c)^2 + (R \times s)^2}},$$

which is obtained from the result[44]

$$R\sin c = \frac{R\sin\delta \times R\sin\phi \times R}{R\cos\delta \times R\cos\phi}$$

by solving it as a quadratic in $R\sin\delta$, and then replacing $\frac{R\sin\phi}{R\cos\phi}$ by $\frac{s}{12}$.

5.1.3 Problem 11

Given the declination (δ) of the Sun when its azimuth is 135°, and the latitude of the place (ϕ), to find the altitude (a) of the Sun.

Brahmagupta states the result as follows:[45]

$$R\sin a = \sqrt{x + y^2} \pm y,$$

$$\text{where} \quad x = \left[\frac{1}{2}R^2 - \{R\sin(agr\bar{a})\}^2\right] \times \frac{144}{(72^2 + s^2)},$$

$$\text{and} \quad y = \frac{12s \times R\sin(agr\bar{a})}{72^2 + s^2},$$

s being the equinoctial midday shadow of a gnomon of 12 units (*aṅgulas*), + or − sign being taken according as the Sun is in the northern or southern hemisphere.

In Figure 7, let S be the Sun (on the armillary sphere) when its azimuth is 135°, SA the perpendicular from S to the plane of the horizon, and SB the perpendicular from S to the Sun's rising-setting line. Then

$$SA = R\sin a,$$

$$\text{and} \quad SBA = 90° - \phi,$$

and it can be easily shown that

$$AB = \frac{R\cos a}{\sqrt{2}} \pm R\sin(agr\bar{a}).$$

Therefore from the triangle SAB, we get

$$R\sin a = \left[\frac{R\cos a}{\sqrt{2}} \pm R\sin(agr\bar{a})\right] \times \frac{R\cos\phi}{R\sin\phi}$$

$$= \frac{12}{s}\left[\frac{R\cos a}{\sqrt{2}} \pm R\sin(agr\bar{a})\right].$$

[43] See *BrSpSi*, xv. 36–38.

[44] See Problem 5.

[45] See *BrSpSi*, iii. 54–56.

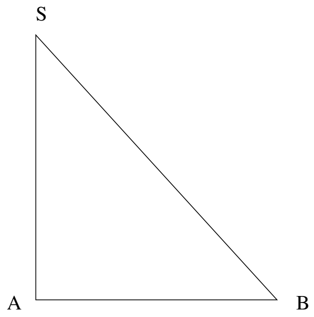

Figure 7

Solving this as a quadratic in $R\sin a$, we get the result stated by Brahmagupta.

5.2 Use of the method of successive approximations

The method of successive approximations is a common method in Hindu astronomy and has been extensively employed by both Bhāskara I[46] and Brahmagupta. But its use in solving problems in spherical astronomy is probably first made by Brahmagupta. He has used this method in solving the following problem.

5.2.1 Problem 12

Given the time (t) elapsed since sunrise when the Sun is on the prime vertical, and the latitude (ϕ) of the place, to obtain the Sun's longitude (λ).[47]

Brahmagupta's method of solving this problem may be explained as follows:[48]

From Figure 5, it is clear that SA $(= R\sin a)$ can be easily obtained if, in addition to latitude (ϕ), we also know the length SB. Since SB is not known, we assume[49]

$$SB = R\sin t$$

as the first approximation I_1 to SB. Then from the triangle SAB, the first approximation $R\sin a_1$ to $R\sin a$ is given by

$$R\sin a_1 = \frac{R\sin t \times R\cos\phi}{R}.$$

[46]See *MBh*, iv. 8–12, 19–20, 55(ii); v. 24–27(i), 34–37, 74(ii)–76; vi. 27, 35–36, 28–31, 32–34, 37–38, 39–40, 49–51, 56–60; viii. 5.

[47]This problem is essentially the same as Problem 9.

[48]See *BrSpSi*, xv. 21–23.

[49]In fact, $SB = \frac{R\sin t \times R\cos\delta}{R}$.

Now using the formula[50]

$$R\sin\lambda = \frac{R\sin a \times R\sin\phi}{R\sin\epsilon},$$

where ϵ is the obliquity of the ecliptic, the first approximation $R\sin\lambda_1$ to $R\sin\lambda$ is given by

$$R\sin\lambda_1 = \frac{R\sin a_1 \times R\sin\phi}{R\sin\epsilon} = \frac{R\sin t \times R\cos\phi \times R\sin\phi}{R\sin\epsilon \times R}.$$

From this value of λ_1, calculate the value of SB anew. This will give the second approximation I_2 for SB. The second approximation $R\sin\lambda_2$ to $R\sin\lambda$ is then obtained by the equation

$$R\sin\lambda_2 = \frac{R\sin\lambda_1 \times I_2}{I_1} = \frac{R\sin\lambda_1 \times I_2}{R\sin t}.$$

Proceeding as above till the successive approximations to λ are the same up to the desired unit, we obtain the required value of λ.

[50] See *MBh*, iii. 41.

Use of hypotenuse in the computation of the equation of the centre under the epicyclic theory in the school of Āryabhaṭa I ??? *

The present paper refutes the assertion of T. S. Kuppanna Shastri that the use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory is one of the principal characteristics of the school of Āryabhaṭa I. It has been shown that the followers of Āryabhaṭa I, like other Hindu astronomers, did not employ the hypotenuse in calculating the equation of the centre under the epicyclic theory. The reason for not using the hypotenuse is explained and the views of the prominent Hindu astronomers, such as Bhāskara I, Govinda Svāmi, Parameśvara, Nīlakaṇṭha, and others are cited in support.

1 Introduction

T. S. Kuppanna Shastri in a paper entitled "The school of Āryabhaṭa and the peculiarities thereof" published in an earlier issue of this Journal[1] has proclaimed that the use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory is an important characteristic of the school of Āryabhaṭa I. Writes he:

> Another important peculiarity of this school is the use of the true hypotenuse in the computation of the equation of the centre. The use of the hypotenuse in the equation of conjunction is common and accepted by all schools, as justified by the eccentric or epicyclic theory of the motion of the planets, which can be readily seen from a geometrical representation of the motion. By the same logic, the hypotenuse should be used for the equation of the centre also, the theory being essentially the same. That is why this school uses it, as a geometrical consequence of this theory set forth by Āryabhaṭa in *Kālakriyā*: 17–21, combined with the theory of uniform motion given in *Kāla*: 12–14. Thus, in the *Mahābhāsk.*, IV, 8–12, the manner of getting the true hypotenuse as based on the

* K. S. Shukla, *Indian Journal of History of Science*, Vol. 8, Nos. 1–2 (1973), pp. 43–57.
[1] *Indian Journal of History of Science*, Vol. 4, Nos. 1–2, pp. 126–134.

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theory of epicycles is given, and in 19–20 the same as based on the eccentric theory. In 21, the approximate sine equation of the centre is asked to be multiplied by the radius and divided by the true hypotenuse to get the correct sine equation of the centre. *Vaṭ. Sid. Spaṣṭādhikāra*, II, 3–4, gives the method of getting the true hypotenuse, and III, 11 instructs its use to divide the approximate equation of the centre to get the correct one.

The use of the hypotenuse is not only a logical result of the theory, but it will also give a better result. It supplies part of the second term of the modern correct equation of the centre. Neglecting powers of e (eccentricity) higher than the square, the first two terms are $2e\ \sin\ m - \frac{5}{4}\ e^2\ \sin\ 2m$, where m is the mean anomaly reckoned from the higher apsis, as in Hindu astronomy. The distance between the centres of the original and eccentric circles is equal to $2e$. It is also the radius of the epicycle. According to the theory, correct sine equation of the centre $= 2e\ \sin\ m \div h$ (=hypotenuse). But

$$h = \frac{\sin m}{\sin(m - \text{eq. cent.})},$$

if the radius of the eccentric circle is taken as unity. Therefore,

$$\begin{aligned}\sin(\text{eq. cent.}) &= \frac{2e \sin m \times \sin(m - \text{eq. cent.})}{\sin m} \\ &= 2e \sin(m - \text{eq. cent.}) \\ &= 2e \sin(m - 2e \sin m) \quad \text{(since eq. cent. is small)} \\ &= 2e \sin m - 4e^2 \sin m \cos m \\ &= 2e \sin m - 2e^2 \sin\ 2m.\end{aligned}$$

Though we get $2e^2$ as the coefficient of the second term, instead of the correct $\frac{5}{4}\ e^2$, it will not make much difference, being the second power of e. Also, the point is that we get the term instead of neglecting it. Using the Moon's epicycle of $31\frac{1}{2}$ degrees, which gives $\frac{7}{80}$ as the value of $2e$, we get for the second term—$13' \sin 2m$, the same as the modern correct one. (The apparent complete agreement is due to the Hindu coefficient of the first term being defective by about a fifth.).

Bhāskarācārya II discusses the point, why other schools do not use the hypotenuse for the equation of centre. He says that some do not use it thinking that the difference is small. This depends upon what we consider small and negligible and may be accepted. But the other argument he gives, quoting his master Brahmagupta,

> that the theory itself is that the epicycle, instead of being uniform, is proportionate to the true hypotenuse and has to be multiplied by it and divided by the radius, and therefore, the division by the true hypotenuse is cancelled out, is untenable, for this kind of argument helps only to shut out a tolerably good theory already existing and nothing more, and is just a way of escape, as pointed out by Caturvedācārya in his commentary on the *Brāhmasphuṭa-siddhānta* (cf. *Siddhāntaśiromaṇi*: *Gola*: *Chedyaka*; and commentary thereon).

The above statement does not reflect a correct understanding of the school of Āryabhaṭa I. In paragraph 1, Kuppanna Shastri tells us that in *Mahābhāskarīya*, iv. 21, the approximate sine equation of the centre is asked to be multiplied by the radius and divided by the true hypotenuse to get the correct sine equation of the centre. The same are stated to be the contents of *Vaṭeśvarasiddhānta*, II, iii. 11. But, contrary to what Kuppanna Shastri has said, both *Mahābhāskarīya*, iv. 21 and *Vaṭeśvarasiddhānta*, II, iii. 11 state the following formula and its application:

$$R\sin(\textit{spaṣṭabhuja}) = \frac{R\sin m \times R}{H},$$

where m is the *madhyamabhuja* (i.e. mean anomaly reduced to *bhuja*).

The formula

$$\sin(\text{equation of centre}) = 2e\sin m \div h,$$

on which Kuppanna Shastri bases his conclusions in paragraph 2 does not occur even in any nook or corner of the school of Āryabhaṭa I. There is not even a smell of it. The formula which has been actually used by the followers of Āryabhaṭa I is

$$R\sin(\text{equation of centre}) = \frac{\text{tabulated } \textit{manda} \text{ epicycle} \times R\sin m}{80}, \qquad (1)$$

the denominator being 80 instead of 360 because the tabulated *manda* epicycle is abraded by $4\frac{1}{2}$, or in the notation of Kuppanna Shastri,

$$\sin(\text{equation of centre}) = 2e\sin m.$$

Kuppanna Shastri seems to have been misled by the use of the true hypotenuse (*mandakarṇa* obtained by iteration) in the formula for the planet's *spaṣṭabhuja*, viz.

$$R\sin(\textit{spaṣṭabhuja}) = \frac{R\sin m \times R}{H}, \qquad (2)$$

where H is the true *mandakarṇa* (obtained by iteration), or, in the notation of Kuppanna Shastri,

$$\sin(\text{m} - \text{eq. centre}) = \sin m \div h.$$

He has missed to see that equation (1) is based on the tabulated *manda* epicycle which is false (*asphuṭa*) and on which the planet does not move, whereas equation (2) relates to the true eccentric on which the planet actually moves.

Kuppanna Shastri has also misquoted Bhāskara II to suit his purpose. In the passage under reference, Bhāskara II has said that Caturvedācārya Pṛthūdaka, who held views similar to those of Kuppanna Shastri, was not correct, and that Brahmagupta, whose views have been declared to be untenable by Kuppanna Shastri, was correct.

It would be interesting to note that whereas Kuppanna Shastri declares the use of hypotenuse in the computation of the equation of the centre to be an important peculiarity of the school of Āryabhaṭa I, the great scholiasts of Āryabhaṭa I, such as Bhāskara I, Govinda Svāmi, Parameśvara, and Nīlakaṇṭha, have taken pains to demonstrate why the hypotenuse has not been used in the computation of the equation of the centre.

The object of the present paper is to explain why the hypotenuse has not been used in the computation of the equation of the centre under the epicyclic theory and also to give the views of the prominent Hindu astronomers on this point.

2 Tabulated *manda* epicycles, true or actual *manda* epicycles, and computation of the equation of the centre

The *manda* epicycles whose dimensions are stated in the Hindu works on astronomy are not the actual epicycles on which the true planet (in the case of the Sun and Moon) or the true-mean planet (in the case of the star-planets, Mars, etc.) moves. Āryabhaṭa I has given two sets of the *manda* epicycles one for the beginning of the odd quadrant and the other for the beginning of the even quadrant. If one wants to find the *manda* epicycle for any other place in the odd or even quadrant, one should apply the proportion stated in *Mahābhāskarīya*, iv. 38–39(i) or *Laghubhāskarīya*, ii. 31–32. The local *manda* epicycle thus obtained is called the true *manda* epicycle (*sphuṭa-manda-vṛtta*), but this too is false (*asphuṭa*). Writes Parameśvara (1430) in his *Ṣiddhāntadīpikā*:[2]

> स्फुटितान्यपि मन्दवृत्तान्यस्फुटानि भवन्ति, तेषां कर्णसाध्यत्वात्। अतः कर्णसाधित-वृत्तसाध्या भुजाकोटिफलकर्णा इति।

[2] *Mahābhāskarīya*, edited by T. S. Kuppanna Shastri, p. 224, lines 15–17.

The *manda* epicycles, though made true, are false (*asphuṭa*), because the true (actual) *manda* epicycles are obtained by the use of the (*manda*) *karṇa*. Therefore, (the true values of) the *bhujāphala*, *koṭiphala*, and *karṇa* should be obtained by the use of the (*manda*) epicycles determined from the (*manda*) *karṇa*.

But how are the *manda* epicycles made true by the use of the *mandakarṇa*? Lalla (c. 748) has answered this question. Says he:[3]

सूर्येन्दुमन्दगुणकौ मृदुकर्णनिघ्नौ
त्रिज्योद्धृतौ भवत एवमिह स्फुटौ तौ ।
ताभ्यां पुनश्च भुजकोटिफले विधाय
साध्ये श्रुती मुहुरतः स्वगुणौ श्रुती च ॥

The *manda* multipliers (= tabulated *manda* epicycles) for the Sun and Moon become true when they are multiplied by the (corresponding) *mandakarṇa*s and divided by the radius. Calculating from them the *bhujāphala* and *koṭiphala* again, one should obtain the *mandakarṇa*s (for the Sum and Moon as before); proceeding from them one should calculate the *manda* multipliers and the *mandakarṇa*s again and again (until the nearest approximations for them are obtained).

The process of iteration is prescribed because the (true) *mandakarṇa* is unknown and is itself dependent on the true *manda* epicycle. If the (true) *mandakarṇa* were known, the true *manda* epicycle could be easily determined from the formula:

$$\text{true } \textit{manda} \text{ epicycle} = \frac{\text{tabulated } \textit{manda} \text{ epicycle} \times \text{ true } \textit{mandakarṇa}}{R}. \quad (3)$$

What is true for the *manda* epicycles of the Sun and Moon is also true for the *manda* epicycles of the planets, Mars, etc. Bhāskara II, commenting on the above passage of the *Śiṣyadhīvṛddhida* of Lalla, observes:[4]

तथा कुजादीनां मन्दकर्मणि उक्तवत् कर्णमुत्पादयित्वा तेन स्वमंदपरिधिं हत्वा व्यासार्धेन विभजेत्, फलं कर्णवृत्ते परिधिः। तेन पुनरुक्तवद् भुजकोटिफले कृत्वा ताभ्यां मन्दकर्णमानयेत्। एवं तावत्कर्म कर्तव्यं यावदविशेषः।

मन्दपरिधिस्फुटीकरणं त्रैराशिकात् – यदि व्यासार्धवृत्ते एतावान् परिधिस्तत्कर्णवृत्ते कियानिति फलं कर्णवृत्तपरिधिः, कर्णवृत्तपरिधेरसकृत्करण च कर्णस्यान्यथाभूतत्वात्।

Similarly, in the *manda* operation of the planets, Mars, etc., too, having obtained the (*manda*) *karṇa* in the manner stated above,

[3] *Śiṣyadhīvṛddhida*, I, iii. 17.
[4] Bhāskara II's comm. on *Śiṣyadhīvṛddhida*, 1, iii. 17.

> multiply the *manda* epicycle by that and divide (the product) by the radius: the result is the (*manda*) epicycle in the *karṇavṛtta* (i.e., at the distance of the *mandakarṇa*). Determining from that the *bhujāphala* and the *koṭiphala* again, in the manner stated before, obtain the *mandakarṇa*. Perform this process (again and again) until there is no difference in the result (i.e., until the nearest approximation for the true *manda* epicycle is obtained).
>
> Conversion of the false *manda* epicycle into the true *manda* epicycle is done by the (following) proportion: If at the distance of the radius we get the measure of the (false) epicycle, what shall we get at the distance of the (*manda*) *karṇa*? The result is the *manda* epicycle at the distance of the (*manda*) *karṇa*. Iteration of the true *manda* epicycle is done because the (*manda*) *karṇa* is of a different nature (i.e. because the *mandakarṇa* is obtained by iteration).

From what has been stated above it is evident that *manda* epicycles stated in the works on Hindu astronomy correspond to the radius of the deferent and are false, whereas the true *manda* epicycles which are derived therefrom by formula (3) above correspond to the distance (true *mandakarṇa*) of the planet and are the actual epicycles on which the planet (in the case of the Sun and the Moon) or the true-mean planet (in the case of the planets Mars, etc.) moves.

Therefore, if we use the tabulated *manda* epicycle, we shall get

$$\textit{bhujāphala} = \frac{\text{tabulated } \textit{manda} \text{ epicycle} \times R\sin m}{80}, \tag{4}$$

where m is the planet's mean *mandakarṇa* (reduced to *bhujā*), the tabulated *manda* epicycle being abraded by $4\frac{1}{2}$ as is usual in the school of Āryabhaṭa I.

Since the tabulated *manda* epicycle corresponds to the radius of the deferent, there is absence of the hypotenuse-proportion and we have

$$\begin{aligned} R\sin(\text{equation of centre}) &= \textit{bhujāphala} \\ &= \frac{\text{tabulated } \textit{manda} \text{ epicycle} \times R\sin m}{80}, \end{aligned}$$

which is the formula used in the school of Āryabhaṭa I.

If we choose to use the true *manda* epicycle, we shall get

$$\text{true } \textit{bhujāphala} = \frac{\text{true } \textit{manda} \text{ epicycle} \times R\sin m}{80},$$

and since this true *bhujāphala* corresponds to the true *mandakarṇa*, therefore, applying the hypotenuse-proportion, we have

$$R\sin(\text{equation of centre}) = \frac{\text{true } \textit{bhujāphala} \times R}{H}, \tag{5}$$

where H is the true *mandakarṇa* (obtained by iteration).

Substituting the value of true *bhujāphala* and making use of formula (3), equation (5) reduces to

$$R \sin(\text{equation of centre}) = \frac{\text{tabulated } \textit{manda} \text{ epicycle} \times R \sin m}{80}.$$

But this result is the same as (4) which was obtained without the use of the hypotenuse-proportion. This explains why in the school of Āryabhaṭa I, the *mandakarṇa* (true hypotenuse) is not used in the computation of the equation of the centre under the epicyclic theory.

3 Views of astronomers of the school of Āryabhaṭa I

3.1 Bhāskara I (629)

In his commentary on the *Āryabhaṭīya*, Bhāskara I, the greatest authority on Āryabhaṭa I, raises the question as to why the hypotenuse was used in finding the *śīghraphala* but was not used in finding the *mandaphala* (i.e. equation of centre) and answers it. Writes he:[5]

> **अत्र शीघ्रफलं व्यासार्धेन संगुणय्य तदुत्पन्नकर्णेन भागलब्धं फलं धनमृणं वा। ...अनेनाथ मन्दोच्चफलमेवं कस्मान्न क्रियते? उच्यते – क्रियमाणेऽपि तावदेव तत्फलं भवतीति न क्रियते। कुतः? मन्दोच्चकर्णोऽविशिष्यते। तत्र चाविशेषितेन फलेन व्यासार्धं संगुणय्य कर्णेन भागे हृते पूर्वमानीतमेव फलं भवतीति। अथ किमिति शीघ्रोच्चकर्णो नाविशिष्यते? अभावादविशेषकर्मणः।**
>
> Here the *śīghra* (*bhujā*)*phala* is got multiplied by the radius and divided by the *śīghrakarṇa* and the quotient (obtained) is added or subtracted (in the manner prescribed) ...
>
> [Question:] How is it that the *manda* (*bhujā*)*phala* is not operated upon in this way (i.e. why is the *mandabhujāphala* not multiplied by the radius and divided by the *mandakarṇa*)? [Answer:] Even if it is done, the same result is obtained as was obtained before; that is why it is not done. [Question:] How? [Answer:] The *mandakarṇa* is iterated. Therefore when we multiply the iterated (*mandabhujā*)*phala* (i.e. true *mandabhujāphala*) by the radius and divide by the (true) *mandakarṇa*, we obtain the same result as was obtained before. [Question:] Now, how is it that the *śīghrakarṇa* is not iterated? [Answer:] This is because the process of iteration does not exist there.

[5] Bhāskara I's comm. on *Āryabhaṭīya*, iii, 22.

3.2 Govinda Svāmi (c. 800–850)

Govinda Svāmi, who is another important exponent of the school of Āryabhaṭa I, raises the same question and answers it is the same way. Writes he:[6]

कथं पुनरिदं मन्दफलं प्रतिमण्डले न प्रमीयते? कृतेऽपि पुनस्तावदेवेति। कथम्? मन्दोच्चकर्णस्य तावदविशेष उक्तः। अविशिष्टात् फलाद् व्यासार्धहतात् कर्णेन (हृतात्) पूर्वानीतमेव फलं लभ्यत इति। किमिति शीघ्रकर्णो नाविशिष्यते? अविशेषाभावात्।

[Question:] How is it that the *manda* (*bhujā*)*phala* is not measured in the *manda* eccentric (i.e. How is it that the *mandabhujāphala* is not calculated at the distance of the planet's *mandakarṇa*)? [Answer:] Even if that is done, the same result is got. [Question:] How? [Answer:] Because iteration of the *mandakarṇa* is prescribed. So when the iterated (i.e. true) *bhujāphala* is multiplied by the radius and divided by the (true *manda*) *karṇa*, the same result is obtained as was obtained before. [Question:] How is it that the *śīghrakarṇa* is not iterated? [Answer:] Because there is absence of iteration.

3.3 Parameśvara (1430)

So also writes the celebrated Parameśvara:[7]

मन्दस्फुटे तु कर्णस्याविशेषितत्वान्मन्दफलमपि अविशेषितं भवति। अविशिष्टात् पुनर्मन्दफलात् व्यासार्धताडिताद् अविशिष्टेन कर्णेन लब्धं प्रथमानीतमेव भजाफलं भवति।

In the case of the *manda* correction, the (*manda*) *karṇa* being subjected to iteration the *manda* (*bhujā*)*phala* is also got iterated (in the process). So, the iterated *manda* (*bhujā*)*phala* being multiplied by the radius and divided by the iterated *mandakarṇa*, the result obtained is the same *bhujāphala* as was obtained in the beginning.

3.4 Nīlakaṇṭha (c. 1500)

Nīlakaṇṭha, author of the *Mahābhāṣya* on the *Āryabhaṭīya* and an eminent authority on Āryabhaṭa I, says the same thing in his *Mahābhāṣya*:[8]

पूर्वं तु केवलमन्त्यफलमविशिष्टेन कर्णेन हत्वा व्यासार्धहृतमेवाविशिष्टमन्त्यफलम्। तदेव पुनर्व्यासार्धेन हत्वा कर्णेन हृतं पूर्वतुल्यमेव स्याद्, यत उभयोस्त्रैराशिककर्मणोर्मिथो वैपरीत्यं स्यात्। एतदुक्तं महाभास्करीयभाष्ये – कृतेऽपि पुनस्तावदेतेति। तस्मान्मन्दकर्मणि भुजाफलं न कर्णसाध्यम्। केवलमेव मध्यमे संस्कार्यम्। शीघ्रे तु

[6] *Mahābhāskarīya*, edited by T. S. Kuppanna Shastri, p. 224, lines 1–4.
[7] *Mahābhāskarīya*, edited by T. S. Kuppanna Shastri, p. 223, line 22, p. 224, lines 12–13.
[8] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 17–21, p. 43, lines 4–10.

कर्णवशाद् उच्चनीचवृत्तस्य वृद्धिह्रासाभावात् सकृदेव कर्णः कार्यः। भुजाफलमपि व्यासार्धेन हत्वा कर्णेन हृतमेव चापीकार्यम्।

Earlier, the iterated *antyaphala* (= radius of epicycle) was obtained by multiplying the uniterated *antyaphala* by the iterated hypotenuse and dividing (the product) by the radius. The same (i.e. iterated *antyaphala*) having been multiplied by the radius and divided by the (iterated) hypotenuse yields the same result as the earlier one, because the two processes of "the rule of three" are mutually reverse. The same has been stated in the *Mahābhāskarīyabhāṣya* (i.e. in the commentary on the *Mahābhāskarīya* by Govinda Svāmi): 'Even if that is done, the same result is got.' So in the *manda* operation, the *bhujāphala* is not to be determined by the use of the (*manda*) *karṇa*; the (uniterated) *bhujāphala* itself should be applied to the mean (longitude of the) planet. In the *śīghra* operation, since the *śīghra* epicycle does not vary with the hypotenuse, the *karṇa* should be calculated only once (i.e., the process of iteration should not be used). The *bhujāphala*, too, should be multiplied by the radius, (the product obtained) divided by the hypotenuse, and (the resulting quotient) should be reduced to arc.

What is meant is that if we first find the true *antyaphala* (radius of the true *manda* epicycle) by the formula

$$\text{true } antyaphala = \frac{\text{radius of uniterated } manda \text{ epicycle} \times H}{R},$$

and then apply the hypotenuse proportion, we shall again get the radius of the uniterated *manda* epicycle with which we started. So the final result, viz.

$$R\sin(\text{equation of centre}) = \frac{\text{radius of uniterated } manda \text{ epicycle} \times R\sin m}{R},$$

may be obtained directly without finding the radius of the iterated *manda* epicycle and then applying the hypotenuse-proportion.

3.5 Sūryadeva Yajvā (b. 1191)

The same thing has been stated in a slightly different way by the commentator Sūryadeva, who writes:[9]

अत्राचार्येण कक्ष्यामण्डलकलाभिर्मन्दनीचोच्चवृत्तानि पठितानि। अतस्तद्गतैव ज्या काष्ठीकृता कक्ष्यामण्डलकलासाम्यात्तत्स्थे मध्यग्रहे संस्क्रियते। कर्णानयने तु तद्वृत्तपरिणामाय त्रैराशिकं कृत्वा अविशेषेण कर्णः कर्तव्यः। शीघ्रवृत्तानि तु प्रतिमण्डलस्थान्येवाचार्येण पठितानि। अतः फलज्यायाःकक्ष्यामण्डलपरिणामार्थं त्रैराशिकं – कर्णस्येय

[9]Sūryadeva's comm. on *Āryabhaṭīya*, iii. 24.

ज्या व्यासार्धस्य केति? लब्धा फलज्या चापीकृता कक्ष्यामण्डलसदृशी मन्द(स्पष्ट)ग्रहे संस्क्रियते। कर्णानयनं तु सकृत्कर्मणैव कार्यम्।

Here the *Ācārya* (viz. *Ācārya* Āryabhaṭa I) has stated the *manda* epicycles in terms of the minutes of the deferent. So the (*manda-bhujāphala*) *jyā* which pertains to that (deferent) when reduced to arc, its minutes being equivalent to the minutes of the deferent, is applied (positively or negatively as the case may be) to (the longitude of) the mean planet situated there (on the deferent). In finding the (*manda*) *karṇa*, however, one should, having applied the rule of three in order to reduce the *manda* epicycle to the circle of the (*mandakarṇa*), obtain the (true *manda*) *karṇa* by the process of iteration. The *śīghra* epicycles, on the other hand, have been stated by the *Ācārya* for the positions of the planets on the (true) eccentric. So, in order to reduce the (*śīghrabhuja*) *phalajyā* to the concentric, one has to apply the proportion: If this (*śīghrabhujaphala*) *jyā* corresponds to the (*śīghra*) *karṇa*, what *jyā* would correspond to the radius (of the concentric)? The resulting (*śīghra*) *phalajyā* reduced to arc, being identical with (the arc of) the concentric is applied to (the longitude of) the true-mean planet. The determination of the (*śīghra*) *karṇa*, however, is to be made by a single application of the rule (and not by the process of iteration).

3.6 *Putumana Somayājī* (1732)

A glaring example of the fact that the astronomers of the school of Āryabhaṭa I regarded the *manda* epicycles as corresponding to the mean distances of the planets and the *śīghra* epicycles as corresponding to the actual distances of the planets is provided by the following rule occurring in the *Karaṇa-paddhati* (vii. 27) of Putumana Somayājī, a notable exponent of the Āryabhaṭa school:

Let $4\frac{1}{2} \times e$ be the periphery of a planet's *manda* epicycle at the beginning of the odd anomalistic quadrant and $4\frac{1}{2} \times e'$ the periphery of a planet's *śīghra* epicycle at the beginning of the odd anomalistic quadrant. Then, the planet being at its *mandocca* (apogee),

$$mandakarṇa = \frac{80 \times R}{80 - e},$$

and, the planet being at its *mandanīca* (perigee),

$$mandakarṇa = \frac{80 \times R}{80 + e}.$$

On the other hand, the planet being at its *śīghrocca*,

$$śīghrakarṇa = \frac{(80 + e') \times R}{80},$$

and the planet being at its *śīghranīca*,

$$\acute{s}\bar{\imath}ghrakar\d{n}a = \frac{(80 - e') \times R}{80}.$$

4 Views of astronomers of other schools

4.1 Brahmagupta's view: Caturvedācārya Pṛthūdaka's disagreement: Bhāskara II's judgement

The astronomers of the Brahma school also use false *manda* epicycles and likewise they do not make use of the hypotenuse in the computation of the equation of the centre under the epicyclic theory. Brahmagupta (628), the author of the *Brāhmasphuṭasiddhānta* and the main exponent of this school, explains the reason for not using the hypotenuse in finding the *mandaphala* as follows:[10]

> **त्रिज्याभक्तः परिधिः कर्णगुणो बाहुकोटिगुणकारः ।**
> **असकृन्मान्दे तत्फलमाद्यसमं नात्र कर्णोऽस्मात् ॥**
>
> In the *manda* operation (i.e., in finding the *mandaphala*), the *manda* epicycle divided by the radius and multiplied by the hypotenuse is made the multiplier of the *bāhu*(*jyā*) and the *koṭi*(*jyā*) in every round of the process of iteration. Since the *mandaphala* obtained in this way is equivalent to the *bhujāphala* obtained in the beginning, therefore the hypotenuse-proportion is not used here (in finding the *mandaphala*).

This is the same explanation as was given by the astronomers of the school of Āryabhaṭa I.

Caturvedācārya Pṛthūdaka (864), on the other hand, was of the opinion that the hypotenuse-proportion was not applied in finding the equation of the centre because it did not produce any material difference in the result. He has therefore remarked:[11]

> **अतः स्वल्पान्तरत्वात् कर्णो मन्दकर्मणि न कार्यः इति ।**
>
> So, there being little difference in the result, the hypotenuse-proportion should not be used in finding the *mandaphala*.

The celebrated Bhāskara II (1150), the author of the *Siddhāntaśiromaṇi*, has examined the views of both Brahmagupta and Caturvedācārya Pṛthūdaka and has given his verdict in favour of Brahmagupta's view. Writes he:[12]

[10] *Brāhmasphuṭasiddhānta*, *golādhyāya*, 29.
[11] Pṛthūdaka's comm. on *Brāhmasphuṭasiddhānta*, *golādhyāya*, 29.
[12] *Siddhāntaśiromaṇi golādhyāya*, *Chedyakādhikāra*, 36–37, comm.

यो मन्दपरिधिः पाठपठितः स त्रिज्यापरिणतः। अतोऽसौ कर्णव्यासार्धे परिणाम्यते। ततोऽनुपातः। यदि त्रिज्यावृत्तेऽयं परिधिस्तदा कर्णवृत्ते क इति। अत्र परिधेः कर्णो गुणस्त्रिज्या हरः। एवं स्फुटपरिधिस्तेन दोर्ज्या गुण्या भांशैर्भाज्या। ततस्त्रिज्यया गुण्या कर्णेन भाज्या। एवंसति त्रिज्यातुल्ययोः कर्णतुल्ययोश्च गुणहरयोस्तुल्यत्वान्नाशे कृते पूर्वफलतुल्यमेव फलमागच्छतीति ब्रह्मगुप्तमतम्। अथ यद्येवं परिधेः कर्णेन स्फुटत्वं तर्हि किं शीघ्रकर्मणि न कृतमित्याशङ्क्य चतुर्वेद आह। ब्रह्मगुप्तेनान्येषां प्रतारणपरमिदमुक्तमिति। तदसत्। चले कर्मणीत्थं किं न कृतमिति नाशङ्कनीयम्। यतः फलवासना विचित्रा। शुक्रस्यान्यथा परिधेः स्फुटत्वं भौमस्यान्यथा तथा किं न बुधादीनामिति नाशङ्क्यम्। अतो ब्रह्मोक्तिरत्र सुन्दरी।

The *manda* epicycle which has been stated in the text is that reduced to the radius of the deferent. So it is transformed to correspond to the radius equal to the hypotenuse (of the planet). For that the proportion is: If in the radius-circle we have this epicycle, what shall we have in the hypotenuse circle? Here the epicycle has the hypotenuse for its multiplier and the radius for its divisor. Thus is obtained the true epicycle. The *bhujajyā* is multiplied by that and divided by 360. That is then multiplied by the radius and divided by the hypotenuse. This being the case, radius and hypotenuse both occur as multiplier and also as divisor and so they being cancelled the result obtained is the same as before: this is the opinion of Brahmagupta. If the epicycle is to be corrected in this way by the use of the hypotenuse, why has the same not been not done in the *śīghra* operation? With this doubt in mind, Caturveda has said: “Brahmagupta has said so in order to deceive and mislead others.” That is not true. Why has that not been done in the *śīghra* operation, is not to be questioned, because the rationales of the *manda* and *śīghra* corrections are different. Correction of Venus’ epicycle is different and that for Mars’ epicycle different; why is that for the epicycles of Mercury etc. not the same, is not to be questioned. Hence what Brahmagupta has said here is right.

4.2 Śrīpati (c. 1039)

Śrīpati, author of the *Siddhāntaśekhara*, has expressed the same opinion as Brahmagupta has done. He has written:[13]

त्रिज्याहृतः श्रुतिगुणः परिधिर्यतो दोः-
कोट्योर्गुणो मृदुफलानयनेऽसकृत्स्यात् ।
स्यान्मन्दमाद्यसममेव फलं ततश्च
कर्णः कृतो न मृदुकर्मणि तन्त्रकारैः ॥

[13] *Siddhāntaśekhara*, xvi, 24.

> Since in the determination of the *mandaphala* the epicycle multiplied by the hypotenuse and divided by the radius is repeatedly made the multiplier of the *bhuja*(*jyā*), and the *koṭi*(*jyā*), and since the *mandaphala* obtained in this way is equal to the *bhujāphala* obtained in the beginning, therefore the hypotenuse-proportion has not been applied in the *manda* operation by the authors of the astronomical *tantras*.

4.3 Āditya Pratāpa

The same view was held by the author of the *Ādityapratāpa-siddhānta*, whose words are:[14]

> भवेत्कक्षाभवो मन्दपरिधिः प्रतिमण्डले ।
> मृदुकर्णगुणः स्पष्टः कक्षाव्यासदलोद्धृतः ॥
> तद्बाहुकोटितः प्राग्वत्कर्णः साध्योऽसकृत् स्फुटः ।
> तेन बाहुफलं भक्तं कक्षाव्यासार्धसङ्गुणम् ॥
> भवेन्मन्दफलं मध्यपरिध्युत्पन्नसम्मितम् ।
> यत्तेन न कृतः कर्णः फलार्थं मन्दकर्मणि ॥

> The *manda* epicycle corresponding to (the radius of) the orbit (concentric), when multiplied by the *mandakarṇa* and divided by the semi-diameter of the orbit (concentric) becomes true and corresponds to (the distance of the planet on) the eccentric. With the help of that (true epicycle), the *bāhu*(*jyā*), and the *koṭi*(*jyā*), should be obtained the true *karṇa* by proceeding as before and by iterating the process. Since the (true) *bāhuphala* divided by that (true *karṇa*) and multiplied by the semi-diameter of the orbit yields the same *mandaphala* as is obtained from the mean epicycle (without the use of the hypotenuse-proportion), therefore use of the hypotenuse-(proportion) has not been made for finding the *mandaphala* in the *manda* operation.

4.4 The *Sūryasiddhānta* school

The method prescribed in the *Sūryasiddhānta* for finding the equation of the centre is exactly the same as given by the exponents of the schools of Āryabhaṭa I and Brahmagupta and there is no use of the hypotenuse-proportion. The author of the *Sūryasiddhānta* has not even taken the trouble of finding the *manda* hypotenuse. So it may be presumed that the views of the author of the *Sūryasiddhānta* on the omission of the use of the hypotenuse in finding

[14] Āmarāja's comm. on *Khaṇḍakhādyaka*, i. 16, p. 33.

the equation of the centre were similar to those obtaining (sic) in the schools of Āryabhaṭa I and Brahmagupta.

5 Conclusion

From what has been said above it is clear that the hypotenuse has not been used in Hindu astronomy in the computation of the equation of the centre under the epicyclic theory. It is also obvious that with the single exception of Caturvedācārya Pṛthūdaka all the Hindu astronomers are unanimous in their views regarding the cause of omission of the hypotenuse. According to all of them the *manda* epicycles stated in the works on Hindu astronomy correspond to the radius of the planet's mean orbit and are therefore false.

Since the *manda* epicycle stated in the Hindu works corresponded to the radius of the planet's mean orbit, the true *manda* epicycle corresponding to the planet's true distance (in the case of the Sun and Moon) or true-mean distance (in the case of the planets Mars, etc.) was obtained by the process of iteration. The planet's true or true-mean distance (*mandakarṇa*) was also likewise obtained by the process of iteration.

Direct methods for obtaining the true *mandakarṇa* or true *manda* epicycle were also known to later astronomers. Mādhava (c. 1340–1425) is said to have given the following formula for the true *mandakarṇa*:[15]

$$\text{true } \textit{mandakarṇa} \text{ (or iterated } \textit{mandakarṇa}) = \frac{R^2}{\sqrt{R^2 - (\textit{bhujāphala})^2} \mathbin{\tilde{+}} \textit{koṭiphala}},$$

$\sim$ or $+$ sign being taken according as the planet is in the half orbit beginning with the anomalistic sign Capricorn or in that beginning with the anomalistic sign Cancer.

The following alternative formula is attributed by Nīlakaṇṭha (c. 1500) to his teacher (Dāmodara):[16]

$$\text{true } \textit{mandakarṇa} \text{ (or iterated } \textit{mandakarṇa}) = \frac{R^2}{\sqrt{(\text{true } \textit{koṭijyā} \mathbin{\tilde{+}} \textit{antyaphalajyā})^2 + (\text{true } \textit{bhujajyā})^2}},$$

$\sim$ or $+$ sign being taken according as the planet is in the half orbit beginning with the anomalistic sign Capricorn or in that beginning with the anomalistic sign Cancer.

[15] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 17–21, p. 47. Also see *Tantrasaṅgraha*, ii, 44.
[16] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 17–21, p. 48. Also see *Tantrasaṅgraha*, i. 46–47.

The following alternative formula occurs in the *Karaṇa-paddhati* (vii. 17, 18, 20(ii)) of Putumana Somayājī:

true *mandakarṇa* (or iterated *mandakarṇa*)

$$= \frac{R^2}{\sqrt{(R \pm \textit{koṭiphala})^2 + (\textit{bhujāphala})^2}},$$

+ or − sign being taken according as the planet is in the half-orbit beginning with the anomalistic sign Cancer or in that beginning with the anomalistic sign Capricorn.[17]

One can easily see that each of these formulae gives an exact expression for the iterated *mandakarṇa*.

6 Use of hypotenuse under the eccentric theory indispensable

The problem of finding the *spaṣṭabhuja* (true *manda* anomaly reduced to *bhuja*) under the eccentric theory is quite different. Here one has to take the planet on its true *manda* eccentric and has to apply the proportion: "When corresponding to the radius vector equal to the iterated *mandakarṇa* one gets the *madhyama bhujajyā*, what shall one get corresponding to the radius R of the concentric?" The result is the Rsine of the *spaṣṭabhuja* equal to

$$\frac{(\textit{madhyama bhujajyā}) \times R}{H},$$

where H is the true (or iterated) *mandakarṇa*.

It must be noted that the planet moves on the true *manda* eccentric whose centre is displaced from the Earth's centre by an amount equal to the radius of the true *manda* epicycle. Bhāskara I writes:[18]

> **परिधिचालनाप्रयोगेण स्फुटीकृतपरिधिना व्यासार्धं संगुणय्याशीत्या भागलब्धं प्रतिमण्डलभूविवरम्।**
>
> Multiply the radius by the epicycle rectified by the process of iteration and divide by 80: the quotient obtained is the distance between the centres of the eccentric and the Earth.

This shows that the Hindu epicyclic theory in which the equation of the centre is obtained directly without the use of the hypotenuse-proportion is much simpler than the Hindu eccentric theory in which the use of the iterated

[17] The *bhujāphala* and *koṭiphala* used in this formula are those derived from true *bhujajyā* and true *koṭijyā*. This formula was known to Mādhava and Nīlakaṇṭha also. See Nīlakaṇṭha's commentary on *Āryabhaṭīya*, iii. 17–21, pp. 48–49 and *Tantrasaṅgraha*, i. 51.

[18] Nīlakaṇṭha's comm. on *Āryabhaṭīya*, iii. 21.

hypotenuse is indispensable. It is for this reason that the use of the epicyclic theory has been more popular in Hindu astronomy than the eccentric theory. The *Sūryasiddhānta* and other works, which have avoided finding the iterated hypotenuse, have dispensed with the eccentric theory altogether.

7 Exceptions: Use of true *manda* epicycle

Munīśvara (1646) and Kamalākara (1658), who claim to be the followers of the *Siddhāntaśiromaṇi* of Bhāskara II and the *Sūryasiddhānta* respectively, are perhaps the only two Hindu astronomers who, disregarding the general trend of Hindu astronomy, have stated the dimensions of the true *manda* epicycles in their works and have likewise used the hypotenuse-proportion in finding the equation of the centre under the epicyclic theory. The formula for the equation of the centre given by them is:[19]

$$R \sin(\text{equation of centre}) = \frac{bhuj\bar{a}phala \times R}{H}, \tag{6}$$

where H is the *mandakarṇa*. Since they have used the true *manda* epicycle, they have obtained the *mandakarṇa* directly without making use of iteration; this is as it should be.

It is noteworthy that although Kamalākara makes use of the true *manda* epicycle and used formula (6) above, he does not forget to record the fact that the *bhujāphala* obtained directly by the use of the *manda* epicycle corresponding to the radius of the planet's mean orbit yields the same result as formula (6) above. Writes he:[20]

> त्रिज्याहतः कर्णहृतः कृतश्चेद्
> यथोक्त आद्यः परिधिः स्फुटः स्यात् ।
> तत्साधितं दोःफलचापमेव
> फलं भवेद्वोक्तफलेन तुल्यम् ॥

> The true (*manda*) epicycle as stated earlier when multiplied by the radius and divided by the hypotenuse becomes corrected (i.e. corresponds to the radius of the planet's mean orbit). The arc corresponding to the *bhujāphala* computed therefrom yields the equation of centre which is equal to that stated before.

[19] *Siddhāntasārvabhauma*, ii. 124 (i); *Siddhāntatattvaviveka*, ii. 207 (i).
[20] *Siddhāntatattvaviveka*, ii. 208.

Hindu astronomer Vaṭeśvara and his works *

1 Early references

The earliest references to Vaṭeśvara are found to occur in *Rasā'ilul'Bīrūnī*[1] and *Tārikh al-Hind*[2] of the Persian scholar Al-Bīrūnī (b. 973 AD) and in the *Siddhāntaśekhara*[3] of the Hindu astronomer Śrīpati (1039 AD). Al-Bīrūnī has also quoted some passages from the *Karaṇasāra*, a calendrical work of Vaṭeśvara.[4] According to Al-Bīrūnī, Vitteśvara (Vaṭeśvara) was a son of Mihdatta (Mahadatta) and belonged to the city of Nāgarapura.[5] From the passages quoted by Al-Bīrūnī from the *Karaṇasāra*,[6] we find that this work adopted the year 821 of the Śaka era (corresponding to the year 899 of the Christian era) as the origin of calculation, which shows that the *Karaṇasāra* was written in 899 AD, i.e. exactly four hundred years after the composition of the *Āryabhaṭīya* of Āryabhaṭa I. Śrīpati has mentioned the name of Vaṭeśvara amongst the first rate astronomers of India—Āryabhaṭa I, Brahmagupta, Lalla, and Sūrya. He has also utilised the *Vaṭeśvarasiddhānta* in writing his own *siddhānta*.

References to the *Vaṭeśvarasiddhānta*, the astronomical *siddhānta* written by Vaṭeśvara have also been noticed in Hindu works on astronomy. One passage ascribed to the *Vaṭeśvarasiddhānta*, consisting of four verses in *āryā* metre and dealing with the lunar correction corresponding to the modern "evection", was discovered by D. V. Ketkar[7] in Yallaya's commentary (1480 AD) on the *Laghumānasa* of Mañjula (932 AD). But neither these verses nor the correction contained in them finds its occurrence in the manuscripts of the *Vaṭeśvarasiddhānta* available to us. Another set of three verses in *āryā* me-

* K. S. Shukla, *Gaṇita*, Vol. 23, No. 2 (December 1972), pp. 65–74.

[1] See Mohammad Saffouri and Adnan Ifram, "Al-Bīrūnī on Transits", pp. 32, 142. "Al-Bīrūnī on Transits" is an English translation of the third treatise included in *Rasā'ilul' Bīrūnī* published by the Osmania Oriental Publications Bureau, Hyderabad-Deccan, in 1948.

[2] See Al-Bīrūnī's *India*, translated into English by E. C. Sachau, Vol. I, pp. 156, 392.

[3] xviii, 18.

[4] See Al-Bīrūnī's *India*, Vol. I, pp. 317, 392; Vol. II, pp. 54, 60, 79; and "Al-Bīrūnī on Transits", p. 32. Also see Al-Bīrūnī's "Exhaustive Treatise on shadows", ch. xxiii.

[5] *Cf.* Al-Bīrūnī's *India*, Vol. I, p. 156.

[6] *Cf.* Al-Bīrūnī's *India*, Vol. I, p. 392; Vol. II, p. 54.

[7] See *Ketakī-graha-gaṇitam*, pp. 127–128.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_32

tre, ascribed to the *Paṭṭīśvara-siddhānta* (probably *Vaṭeśvarasiddhānta*) and dealing with the astronomical phenomenon *Pāta*, is found quoted in Mallikārjuna's commentary (1178 AD) on the *Sūryasiddhānta* from some commentary on the *Laghumānasa*, but these verses are not exactly the same as their counterparts found in the manuscripts of the *Vaṭeśvarasiddhānta* known to us. It is probable that the verses quoted by Yallaya and Mallikārjuna belonged to the *Karaṇasāra* and have been ascribed to the *Vaṭeśvarasiddhānta* by inadvertence; or they might have occurred in the *Golādhyāya* of the *Vaṭeśvarasiddhānta*, which is not available to us completely. The celebrated Bhāskara II, author of the *Siddhāntaśiromaṇi*, has made a reference to certain scholars who believed that eight and a half years of Brahmā's life had then elapsed.[8] He is probably referring to Lalla and Vaṭeśvara who held this view.[9]

The *Vaṭeśvarasiddhānta* seems to have been a popular work amongst the scholars of the *Dharmaśāstra*. References to this work have been found to occur in the *Kālanirṇaya*,[10] in the *Kālasāra* of Gadādhara,[11] and in the works of Kamalākara Bhaṭṭa.[12]

2 Works of Vaṭeśvara: Existing manuscripts

No other work besides the *Karaṇasāra* and the *Vaṭeśvarasiddhānta* mentioned above has been ascribed to Vaṭeśvara. These are the only works coming from the pen of Vaṭeśvara now known to us. The *Karaṇasāra*, written in 899 AD, was his earlier work. The *Vaṭeśvarasiddhānta* was written five years later in 904 AD.

The *Karaṇasāra* is now lost and is known only through quotations in the writings of Al-Bīrūnī. The *Vaṭeśvarasiddhānta* has, however, come down to us, but we are aware of only two manuscripts of this work, both incomplete. Of these manuscripts, one belongs to the West Panjab University Library, Lahore (Pakistan). This manuscript begins as follows:

> श्रीकृष्णाय नमः। ब्रह्मावनीन्दुबुधशुक्रदिवाकरारजीवार्कसूनुभगुरून् पितरौ च नत्वा।
> ब्राह्मं ग्रहर्क्षगणितं महदत्तसूनुर्वक्ष्येऽखिलं स्फुटमतीव वटेश्वरोऽहम्॥

It breaks off in the course of the seventh *adhikāra* (chapter) and ends thus:

> द्युचरास्फु समीरितं बुधैर्गणितस्कन्दविशेषभाजनैः दिनशेषविधूद्यासवस्सहितारात्रि-गृतासवोप्यलम् रविवन्नरकर्णदीप्तयोर्मावन्दधमेणभुजादि च। प्राग्लग्नदृक्चन्द्रमसौ वि-लग्न

[8] *Siddhāntaśiromaṇi*, I, i (a). 26.

[9] According to both these writers $8\frac{1}{2}$ years and $\frac{1}{2}$ of a month of Brahmā's life had elapsed up to the beginning of the current *kalpa*.

[10] See P. V. Kane, "History of Dharmaśāstra", Vol. I, p. 376.

[11] See P. V. Kane, *ibid*, Vol. I, p. 617.

[12] See T. S. Kuppana Shastri, "The system of the *Vaṭeśvarasiddhānta", Indian Journal of History of Science*, Vol. 4, Nos. 1 & 2, p. 135.

The colophons at the ends of the chapters runs as follows:

1. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते मध्यगतिः प्रथमोधिकारः॥
2. श्रीमदानंदपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्फुटसिद्धान्ते स्वनामसंज्ञिते ... टगत्यधिकारो (द्विती)यः॥
3. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविंदावतारिते त्रिप्रश्नाध्यायस्तृतीयः॥
4. श्रीमदानंदपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविंदावतारिते चंद्रग्रहणाधिकारश्चतुर्थः॥
5. श्रीमदानंदपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविंदावतारिते रविग्रहणाधिकारः पंचमः॥
6. श्रीमदानंदपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविंदावतारिते उदयास्तमयाधिकारः षष्ठः॥

Ram Swarup Sharma and Mukunda Mishra's edition of the *Vaṭeśvarasiddhānta* was based on this manuscript. A photostat copy of this manuscript was kindly procured for our use by the librarian of the Lucknow University Library.

The other manuscript of the *Vaṭeśvarasiddhānta* was discovered by me in the collection of the late Pandit Girish Chandra Awasthi of the Oriental Department, Lucknow University. It was later sold at my instance to the Lucknow University Library. This manuscript, though incomplete, is larger than the other one. It contains all the eight *adhikāras* of the *Kālakriyā* (or *Grahagaṇita*) Part. Besides that, there are four additional chapters written by Govinda, the copyist of the manuscript which are meant to serve as a supplement to the third *adhikāra* of the *Vaṭeśvarasiddhānta.* There are also a few opening chapters of the *Golādhyāya* which is incomplete in the manuscript available to me. It breaks off in the middle of the chapter entitled *Graha-gola-bandha.* This manuscript begins as follows:

> ॥ श्रीकृष्णाय नमः ॥ ब्रह्मावनीन्दुबुधशुक्रदिवाकरारजीवार्कसूनुभगुरून्पितरौ च नत्वा।
> ब्राह्मं ग्रहर्क्षगणितं महदत्तसूनुर्वक्ष्येऽखिलं स्फुटमतीव वटेश्वरोहम्॥१॥

and ends thus:

> प्रागपरकपालयोः कुफललम्छनं क्षयमतस्खमूभुदलपृष्टगनरयोर्दृक्साम्यालम्छनं न मध्याह्ने उपपत्ति या प्रोक्ता प्रागपरे लम्छनेन तिथिविरहेलम्छने विरहेवनतेयाम्यो

The colophons at the ends of the chapters run as follows:

1. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते मध्यगतिः प्रथमोधिकारः॥
2. श्रीमदानंदपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्फुटसिद्धान्ते स्वनामसंज्ञिते ... टगत्यधिकारो (द्विती)यः॥
3. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविन्दावतारिते त्रिप्रश्नाध्यायस्तृतीयः॥
4. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविंदावतारिते चंद्रग्रहणाधिकारश्चतुर्थः॥
5. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविन्दावतारिते रविग्रहणाधिकारः पंचमः॥
6. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविन्दावतारिते उदयास्तमयाधिकारष्षष्ठः॥
7. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविन्दावतारिते शृङ्गोन्नत्यधिकारस्सप्तमः॥
8. श्रीमदानन्दपुरीयभट्टमहदत्तसुतवटेश्वरविरचिते स्वनामसंज्ञिते स्फुटसिद्धान्ते भट्टगोविन्दावतारिते समागमाधिकारोष्टमः॥॥ समाप्तेयं कालक्रियापट्टम्॥

The supplementary chapters of the copyist begin with the following opening remarks:

> अनन्तरं त्रिप्रश्नाधिकारमध्ये दौरण्डीयभट्टवाह्लिकसुतगोविंदविरचिते स्थाननिर्देशशङ्काद्यानयनविधिः खसितमासीद्यः स लिख्यते।

The colophons at the ends of the chapters run as follows:

1. कालस्थाननिर्देशशंक्वाद्यानयनाध्यायः॥
2. कालनिर्देशशंक्वाद्यानयने सर्व्वकालशंक्वाद्यानयनाध्यायः॥
3. स्थाननिर्देशशंक्वाद्यानयने सर्वस्थानशंक्वाद्यानयनाध्यायः॥
4. स्थाननिर्देशशंक्वाद्यानयने कोणशंक्वाद्यानयनाध्यायः॥ श्रीदौरण्डीयभट्टवह्लिकसुतगोविन्दकृतो वटेश्वरसिद्धान्ते त्रिप्रश्नाधिकारापरमध्येऽष्टमोध्यायः समाप्तः॥

The chapters of the *Golādhyāya* bear the titles: (i) *Gola-praśaṃsā*, (ii) *Gola-bandha*, (iii) *Chedyaka*, (iv) *Khagola-bandha*, (v) *Bhagola-bandha*, (vi) *Grahagola-bandha*, (vii) *Gola-vāsanā*, and (viii) *Bhūgolādhāya.* Of these chapters, those entitled *Gola-praśaṃsā*, *Gola-bandha*, *Chedyaka*, *Khagola-bandha*, *Bhagola-bandha*, and *Grahagola-bandha* are appended to the manuscript of the *Vaṭeśvarasiddhānta* after the supplementary chapters of the copyist. The last chapter (*Grahagola-bandha*) is not complete and breaks off in the course of the seventeenth verse. This chapter being incomplete, its name does not occur in the manuscript. We have called this chapter *Grahagola-bandha* because its subject matter agrees with that of the chapter of the same name in the *Śiṣyadhīvṛddhida* of Lalla. The chapter entitled *Gola-vāsanā* and *Bhūgolādhāya* are found to occur in the midst of the third supplementary chapter of the copyist. The opening lines of the *Gola-vāsanā* are missing; only the concluding eleven verses are available; the chapter ends with the name of the chapter. In the case of the other chapter, the first 12 verses only are preserved. The name of the chapter is not given in the manuscript but the subject matter is similar to that of the *Bhūgolādhāya* of the *Śiṣyadhīvṛddhida* of Lalla.

Both the manuscripts of the *Vaṭeśvarasiddhānta* available to us seem to be copies of the same manuscript and, as far as they go, agree very closely with each other.

3 ***Vaṭeśvarasiddhānta*: An outline**

The *Vaṭeśvarasiddhānta* is perhaps the largest work on Hindu astronomy. Whereas the *Āryabhaṭīya* contains 121 verses, the *Brāhmasphuṭasiddhānta* 1008 verses, the *Sūrya-siddhānta* 500 verses, the *Śiṣyadhīvṛddhida* of Lalla 639 verses, the *Siddhāntaśekhara* of Śrīpati 890 verses, and the *Siddhāntaśiromaṇi* of Bhāskara II 962 verses, the available manuscript of the *Vaṭeśvarasiddhānta* contains 1779 verses (approx). The first eight chapters of the *Vaṭeśvarasiddhānta* which deal with mathematical astronomy (*Kālakriyā* or *Grahagaṇita*) consist of 1308 verses.

The *Vaṭeśvarasiddhānta*, as available to us, may be divided into three parts:

1. *Kālakriyā* Part (dealing with mathematical astronomy).
2. Four additional chapters written by Govinda to supplement the *Tripraśnādhikāra* of the *Kālakriyā* Part.
3. *Golādhyāya*.

The *Kālakriyā* Part is divided into eight chapters (called *adhikāra*) and each of them is divided into sections. The headings of these chapters and sections and the number of verses comprising them are given in Table 1.

The four chapters appended to the *Kālakriyā* Part by the copyist run as follows:

Chapter No.	Chapter Name	No. of verses
1	कालस्थाननिर्देशशङ्क्वाद्यानयनाध्यायः	111
2	कालनिर्देशशङ्क्वाद्यानयने सर्वकालशङ्क्वाद्यानयनाध्यायः	64
3	स्थाननिर्देशशङ्क्वाद्यानयने सर्वस्थानशङ्क्वाद्यानयनाध्यायः	75
4	स्थाननिर्देशशङ्क्वाद्यानयने कोणशङ्क्वाद्यानयनाध्यायः	125

Table 1: Contents of the *Kālakriyā* Part of the *Vaṭeśvarasiddhānta.*

Chapters	Verses	Sections		Verses
1. मध्यमाधिकारः	358 vss. + 8 half vss.	1.	भगणनिर्देशः	21
		2.	मानविवेकः	14
		3.	द्युगण-विधिः	25 + 2
		4.	सर्वतोभद्रः	62
		5.	प्रत्यब्दशुद्धि-विधिः	116 + 5
		6.	करण-विधिः	10
		7.	कक्ष्याविधान-ग्रहानयनविधिः	24
		8.	देशान्तरविधिः	18 + 1
		9.	प्रश्नविधिः	21
		10.	तन्त्रपरीक्षाध्यायः	47
2. स्फुटगत्याधिकारः	261 + 4	1.	स्फुटीकरणविधिः	102 + 3
		2.	स्वोच्चनीचग्रहस्फुटीकरणविधिः	29
		3.	प्रतिमण्डलग्रहस्फुटीकरणविधिः	21
		4.	ज्याविधिना स्फुटीकरणविधिः	15
		5.	फलज्यास्फुटीकरणविधिः	37
		6.	तिथ्यानयनविधिः	42
		7.	प्रश्नविधिः	15 + 1
3. त्रिप्रश्नाधिकारः	387 + 15	1.	विषुवच्छायासाधनविधिः	33 + 3
		2.	लम्बज्याक्षज्यानयनविधिः	27
		3.	क्रान्तिज्यानयनविधिः	13
		4.	द्युज्यानयनविधिः	8
		5.	कुज्यानयनविधिः	15
		6.	अग्रानयनविधिः	14
		7.	स्वचरार्धज्याप्राणसाधनविधिः	22 + 5
		8.	लग्नादिविधिः	25 + 1
		9.	द्युदलभादिविधिः	47 + 1
		10.	इष्टच्छायाविधिः	39 + 1
		11.	सममण्डलप्रवेशविधिः	25 + 2
		12.	कोणशङ्कुविधिः	29 + 1
		13.	छायातोऽर्कानयनविधिः	29
		14.	छायापरिलेखविधिः	26
		15.	प्रश्नाध्यायः	35 + 1
4. चन्द्रग्रहणाधिकारः	40 + 1			
5. रविग्रहणाधिकारः	117 + 8	1.	लम्बनविधिः	32 + 1
		2.	अवनतिविधिः	6 + 1
		3.	स्थित्यर्धविधिः	9 + 1
		4.	परिलेखविधिः	28 + 3
		5.	पर्वज्ञानविधिः	14
		6.	लघूकरणविधिः	21 + 2
		7.	छेद्यकविधिः	7
6. उदयास्तमयाधिकारः	29 + 2			
7. शृङ्गोन्नत्याधिकारः	53 + 1			
8. समागमाधिकारः	43 + 1	1.	ग्रहयुतिविधिः प्रथमः	14
		2.	ग्रहयुतिविधिः द्वितीयः	29 + 1

The *Golādhyāya* of the *Vaṭeśvarasiddhānta* is incomplete. Our manuscript contains the following chapters only:

Chapter No.	Chapter Name	No. of verses
1	गोलप्रशंसा	9
2	गोलबन्धः	18 + 1
3	छेद्यकः	18 + 1
4	खगोलबन्धः	4
5	भगोलबन्धः	6
6	ग्रहगोलबन्धः	17
7	गोलवासना	11
8	भूगोलाध्यायः	12

4 Sharma and Mishra's edition of *Vaṭeśvarasiddhānta*

Ram Swarup Sharma and Mukunda Mishra brought out Part I of the *Vaṭeśvara-siddhānta* containing the first three *adhikāras*. Their edition was based on the manuscript belonging to West Panjab University Library, Lahore. The following comparison will reveal that Sharma and Mishra's edition does not give the full text of the first three *adhikāras* of the *Vaṭeśvarasiddhānta*. A large number of verses which actually occur in the manuscript have been silently omitted by them. Some of the verses have received drastic emendation at their hands.

Chapter	No. of verses in the manuscript	No. of verses in Sharma-Mishra's edition
1	358 + 8	290
2	261 + 4	184
3	387 + 15	222
Total	1006 + 27	696

This shows that Sharma and Mishra's edition does not give the whole text but about two-third of it.

5 Vaṭeśvara's date and place

In the closing verse of Section 1, Chapter 1, of the *Vaṭeśvarasiddhānta*, Vaṭeśvara himself gives the time of his birth as well as the time of composition of the *Vaṭeśvarasiddhānta*. Writes he:

> When 802 years had elapsed since the commencement of the Śaka era, my birth took place; and when 24 years had passed since my birth this *Siddhānta* was written by me with the grace of the heavenly bodies.

This shows that Vaṭeśvara was born in 880 AD and *Vaṭeśvarasiddhānta* was written in 904 AD. His work, *Karaṇasāra*, was written, as reported by Al-Bīrūnī, five years earlier in 899 AD.

In the opening verse of the *Vaṭeśvarasiddhānta*, Vaṭeśvara has called himself "a son of Mahadatta". The colophons at the ends of the various chapters of the *Vaṭeśvarasiddhānta* go a step further and declare him as being "the son of Bhaṭṭa Mahadatta resident of Ānandapura." This shows that Vaṭeśvara was the son of Bhaṭṭa Mahadatta and belonged to the place called Ānandapura.

Ānandapura has been identified by Cunningham and Dey with the town of Vadnagar in northern Gujarat situated to the south-east of Sidhpur (lat. 23.45 N, long. 72.39 E).[13] "Ānandapura or Vaḍnagar," writes Dey, "is also called Nāgara which is the original home of the Nāgara Brāhmaṇas of Gujarat. Kumārapāla surrounded it with a rampart. Bhadrabāhu Svāmī, the author of the *Kalpasūtra*, composed in 411 AD, flourished at the court of Dhruvasena II, King of Gujarat, whose capital was at this place."[14] That Vaṭeśvara's Ānandapura was the same place as Vadnagar or Nāgara is confirmed by the testimony of Al-Bīrūnī who has written that Vaṭeśvara belonged to the city of Nāgarapura. Nāgara and Nāgarapura are obviously one and the same.

Ānandapura seems to have been a great seat of Sanskrit learning. Āmarāja (c. 1200 AD), a commentator of the *Khaṇḍakhādyaka* of Brahmagupta, and Mādhava (1263 AD), a commentator of the *Ratnamālā* of Śrīpati, also belonged to this place. According to both these writers the equinoctial midday shadow at this place was 5 *aṅgulas* and 20 *vyaṅgulas* and the hypotenuse of the equinoctial midday shadow, 13 *aṅgulas* and 8 *vyaṅgulas*,[15] which shows that the latitude of Ānandapura was 24° north approximately. The latitude of Vadnagar is also approximately the same.

In the closing verse of Section 9, Chapter 3, Vaṭeśvara has mentioned Daśapura which seems to suggest that he had some association with that place. This Daśapura was probably the same place as has been identified with Mandasor (lat. 24.03° N, long. 75.08° E), which is situated in Madhya Pradesh and is not far from Vadnagar. What kind of association Vaṭeśvara actually had with this place is difficult to say.

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[13] *Cf.* Sir Alexander Cunningham, "The Ancient Geography of India," p. 416; Nundo Lal Dey, "The Geographical Dictionary of Ancient and Medieval India," p. 6.

[14] *Cf.* Nundo Lal Dey, *ibid.*

[15] See Āmarāja's commentary on *Khaṇḍakhādyaka*, iii. 1, p. 87, and *Bhāratīya Jyotiṣa Śāstra* (Marathi) by S. B. Dikshit, Second Edition, p. 471.

6 Govinda, copyist of the manuscript

We conclude this paper with a note on Govinda, the copyist of the second manuscript of the *Vaṭeśvarasiddhānta*.[16] As already pointed out, he has added four chapters by way of supplementing the third chapter, entitled the *Tripraśnādhikāra*, of the *Vaṭeśvarasiddhānta*. This shows that Govinda was a good astronomer who had the capability of saying something which did not occur even to such a great astronomer as Vaṭeśvara. The contents of the chapters written by him definitely add to the value of the *Vaṭeśvarasiddhānta*. A number of rules given by him are quite new and do not occur in any other work on Hindu astronomy.

We know of more than one Hindu astronomer who bore the name Govinda but our Govinda is quite different from them. From the colophons occurring at the end of his four chapters it appears that he was the son of Bhaṭṭa Vāhnika or Vahnika who lived at the place called Dauraṇḍa.

[16]There are reasons to believe that the manuscript available to us was not actually written by Govinda. It seems to have been transcribed from the copy originally made by Govinda or from another copy thereof.

The evection and the deficit of the equation of the centre of the Moon in Hindu astronomy *

Section I

1. Dhirendranath Mukhopadhyaya (1930) published a paper entitled "The Evection and the Variation of the Moon in Hindu Astronomy" wherein he showed that the Hindu astronomer Mañjula knew of a lunar correction which is equivalent to the deficit of the equation of the centre and the evection. P. C. Sengupta (1932) published another paper entitled "Hindu Luni-solar Astronomy" in which, among other things, he considered various formulae regarding this dual correction as given by Mañjula (932), Śrīpati (1039), and Candra Śekhara Siṃha (latter half of the 19th century). None of these papers, however, contains a complete or systematic study of this correction and in consequence some errors have crept in. The object of the present paper is to exhibit the central idea underlying the corrections prescribed by various Hindu authors and to explain them more thoroughly in the light of further investigations in the field of Hindu astronomy.

2. The discovery of this correction is one of the greatest achievements of the Hindus in the field of practical astronomy. Early Hindu astronomers made observations and recorded the differences between the observed and computed positions of the heavenly bodies. As early as Vedic times, the Hindus performed sacrifices when the planets occupied specified positions in the heavens. This practice continued for thousands of years. The record of such observations served as the basis for the foundation of the Hindu theoretical astronomy and later on supplied material on the basis of which corrections were made and refinements introduced from time to time. These observations continued over long periods led to the discovery of the above lunar inequality as well as of all the other inequalities.

3. Due to the fact that a good deal of the early Hindu astronomical literature has not been preserved, it is impossible to locate the exact date when the

* K. S. Shukla, *Proceedings of the Benares Mathematical Society*, New Series, Vol. 7, No. 2 (1945), pp. 9–28.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_33

lunar inequality was first detected in India and to trace a regular theoretical history of this subject. The available formulae giving this correction exhibit an advanced state of the subject and it is our belief that they must have taken centuries to develop.

4. At present this correction can be traced back to the time of Vaṭeśvara,[1] the well-known critic of Brahmagupta[2] (628). The works of Vaṭeśvara are not available to us, but from Yallaya's commentary (1482) on the *Laghumānasa* (932) we learn that the *Vaṭeśvarasiddhānta* contained this correction. In his commentary on the *Laghumānasa*, i(c), 1–2, Yallaya has actually quoted Vaṭeśvara's version of this correction. This is as follows:

एकादशभिर्भागैर्विवर्जितैः शुद्धचन्द्रगतिभागैः ।
स्फुटसूर्यात् चन्द्रोच्चं त्यक्त्वा तत्कोटिजीवा या ॥
गुणिता स्याद्गुणकारैर्धनर्णसंज्ञां प्रयात्येषा ।
शुद्धेन्दौ स्फुटसूर्यं विशोध्य कोटिज्यकां भुजज्यां वा ॥
ज्ञात्वा तयोर्धनाख्यामृणसंज्ञां वा यथोचितां कृत्वा ।
भुजकोटिज्ये गुणिते तेन गुणेनैव ते भुजे क्रमशः ॥
रूपेण पञ्चभिर्ये लिप्त्याद्ये शीतगोश्च तद्भुक्तौ ।
भवति फलं शशिलिप्त्यां गुणकभुजातुल्यभिन्ननामयुतौ ॥
कुर्याद्रूपाप्तं यत् धनमृणमिन्दोः क्रमाद्भुक्त्याम् ।
भिन्नाशाख्यौ स्यातां कोटिगुणौ तद्धनं क्षयं कुर्यात् ॥[3] By the multiplier obtained by subtracting eleven degrees from the Moon's true daily motion, in degrees (*bhāga*), multiply the Rcosine of the Sun's true longitude *minus* the longitude of the Moon's apogee (*ucca*). This comes positive or negative. (Next) having subtracted the Sun's true longitude from the Moon's true longitude and having obtained the Rsine and Rcosine thereof, and (then) having properly ascertained their signs—positive or negative—, multiply the Rsine and Rcosine (thus obtained) by the (previous) product (or epicyclic multiplier). The results should be respectively divided by 1 and 5 and applied as correction, in minutes (*liptās*), to the Moon's true longitude and true daily motion (in the following manner): The

[1] According to Śaṅkara Bālakṛṣṇa Dīkṣita, Vaṭeśvara's time is 899 AD.

[2] In his *Gaṇakataraṅgiṇī*, Sudhākara Dvivedī writes:

यथा ब्रह्मगुप्तेनार्यभटादीनां खण्डनं कृतं तथैव वटेश्वरेण स्वसिद्धान्ते बहुत्र ब्रह्मगुप्तखण्डनं कृतमस्ति।

[3] Dattarāja gives the last $1\frac{1}{2}$ verses thus:

भवति फलैः शशिलिप्त्यां गुणकभुजातुल्यभिन्ननामयुतौ ॥
कुर्याद्रूपाप्तं यत् धनमृणमिन्दोः क्रमाल्लिप्ताम् ।
भिन्नाशाख्यौ स्यातां कोटिगुणा तद्धनं क्षयं कुर्यात् ॥

Cf. Ketakī-grahagaṇitam, p. 128.

> result which is obtained on dividing by one should be applied as a positive or negative correction to the minutes of the Moon according as the multiplier and the Rsine are of like or unlike signs; (and) the product of the Rcosines is to be applied as a positive or negative correction to the Moon's true daily motion provided their signs satisfy the contrary condition.

5. This correction is also found in the *Laghumānasa* of Mañjula in exactly the same form as stated above by Vaṭeśvara. According to the commentator Yallaya, Mañjula has borrowed this correction from the *Vaṭeśvarasiddhānta* itself. Yallaya gives the following introductory line to Mañjula's stanzas regarding this correction:

> **अथ चन्द्रस्य ग्रहसमागमच्छाया शृङ्गोन्नतिदृक्साधने वटेश्वरसिद्धान्तोक्तदृक्कर्मविशेषं श्लोकद्वयेनाह–**
>
> Now, in the (next) two verses, (the author) gives a special visibility correction, the same which has been stated in the *Vaṭeśvara-siddhānta* in connection with the calculation of the Moon's conjunction with the planets, the Moon's shadow, the Moon's *śṛṅgonnati* and the Moon's longitude agreeing with observation.

And actually we find that Mañjula has only summarised the above verses of Vaṭeśvara[4]. He says:

> **इन्दूच्चोनार्ककोटिघ्ना गत्यंशा विभवा विधोः ।**
> **गुणो व्यर्केन्दुदोःकोट्यो रूपपञ्चाप्तयोः क्रमात् ॥**
> **फले शशाङ्कतद्गत्योर्लिप्ताद्ये स्वर्णयोर्वधे ।**
> **ऋणं चन्द्रे धनं भुक्तौ स्वर्णसाम्यवधेऽन्यथा ॥**[5]
>
> Multiply the degrees of the Moon's true daily motion[6] as diminished by 11 by the Rcosine of the true longitude of the Sun *minus* the longitude of the Moon's apogee. This is the multiplier of the Rsine and the Rcosine of the true longitude of the Moon diminished by that of the Sun respectively divided by 1 and 5. The

[4] Note the brevity and conciseness of Mañjula's composition. He states the correction in two verses while Vaṭeśvara gives the same in five.

[5] *Cf. Laghumānasa*, i(c), 1–2.

[6] From the word *śuddhacandragatibhāga* used by Vaṭeśvara it is obvious that Mañjula's corresponding word *gatyaṁśa* should mean "the true daily motion, in degrees" and not "the mean daily motion" which has been given as a translation of *gatyaṁśa* by D. Mukhopadhyaya (1930), P. C. Sengupta (1932), and N. K. Majumdar (1944). Sūryadeva Yajvā also observes:

> विधोश्चन्द्रस्य स्फुटभुक्तिं षष्ट्या आरोप्य भागान् कुर्यात् ते गत्यंशा इति उच्यन्ते।
>
> *Cf. Laghumānasa*, i(c). 1–2 (comm).

> results (thus obtained) are respectively the corrections, in minutes (*liptās*), of the Moon and its true daily motion. If in the above product one (factor) is positive and the other negative, the correction for the Moon is subtractive and that for its true daily motion additive. If both are of like sign, both positive or both negative, the corrections are contrary.

If S, M, and U respectively denote the true longitudes of the Sun, the Moon, and the Moon's apogee (*mandocca*), then the correction for the Moon's longitude stated above by Vaṭeśvara and Mañjula is

$$\mp\left(8\frac{2}{15}\right)\cos(S-U)\,[\text{Moon's true daily motion, in degrees} - 11]$$
$$\times\left(8\frac{2}{15}\right)\sin(M-S)\text{ minutes,} \qquad (1)$$

according as

$$\left(8\frac{2}{15}\right)\cos(S-U)\quad\text{and}\quad\left(8\frac{2}{15}\right)\sin(M-S)$$

are of unlike or like signs; and the correction for the Moon's true daily motion is

$$\pm\left(8\frac{2}{15}\right)\cos(S-U)\,[\text{Moon's true daily motion, in degrees} - 11]$$
$$\times\frac{\left(8\frac{2}{15}\right)\cos(M-S)}{5}\text{ minutes,} \qquad (2)$$

according as

$$\left(8\frac{2}{15}\right)\cos(S-U)\quad\text{and}\quad\left(8\frac{2}{15}\right)\cos(M-S)$$

are of unlike or like signs.

Expression (2) is clearly an approximate value of the differential of (1). For, R being the radius,

$$d\left\{\left(8\frac{2}{15}\right)\sin(M-S)\right\} = \left(8\frac{2}{15}\right)\cos(M-S)\; d\left\{\frac{M-S}{R}\right\}$$
$$= \frac{\left(8\frac{2}{15}\right)\cos\,(M-S)}{5},^{7}$$

the term involving the differential of $\left(8\frac{2}{15}\right)\cos(S-U)$ being neglected.[8]

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[7]For let $dM = 790'35''$, $dS = 59'8''$, and $R = 3438'$; then

$$\frac{d(M-S)}{R} = \frac{731'27''}{3438'} = \frac{1}{5}\text{ approx.}$$

[8]The error committed is generally negligible.

6. The general form of this correction appears in the *Siddhāntaśekhara* of Śrīpati (1039), who gives it thus:

त्रिभविरहितचन्द्रोच्चोनभास्वद्भुजज्या
गगननृपविनिघ्नी भत्रयज्याविभक्ता ।
भवति परफलाख्यं तत् पृथक्स्थं शरघ्नं
हृतमुडुपतिकर्णत्रिज्ययोरन्तरेण ॥
यदिह फलमवाप्तं तद्धनर्णं पृथक्स्थे
तुहिनकिरणकर्णे त्रिज्यकोनाधिकेऽथ ।
स्फुटदिनकरहीनादिन्दुतो या भुजज्या
स्फुटपरमफलघ्नी भाजिता त्रिज्ययाऽऽप्तम् ॥
शशिनि चरफलाख्यं सूर्यहीनेन्दुगोलात्
तदृणमुतधनं स्यादुच्चहीनार्कगोलः ।
यदि भवति हि याम्यो व्यस्तमेतद्विधेयं
स्फुटगणितदृगैक्यं कर्तुमिच्छद्भिरत्र ॥[9]

Deduct 90° from the longitude of the Moon's apogee and by that diminish the true longitude of the Sun and obtain the Rsine of that. Multiply that by 160′ and divide by the radius. This is known as the *paraphala* (i.e., the maximum correction). Set it down in two places. Multiply one by 5 and divide by the Moon's true distance as divided by its difference with the radius.[10] Add whatever is obtained here to or subtract that from the other placed elsewhere according as the Moon's true distance is less or greater than the radius. (Thus is obtained the *sphuṭaparamaphala*). Now diminish the true longitude of the Moon by that of the Sun and take its Rsine. Multiply it by the *sphuṭaparamaphala* and divide by the radius. Then is obtained the so-called *cara* correction of the Moon. (When

$$\{\text{Sun's true longitude} - (\text{longitude of Moon's apogee} - 90°)\}$$

is less than 6 signs) this is subtractive or additive according as

$$(\text{Moon's true longitude} - \text{Sun's true longitude})$$

is less or greater than 6 signs. When

$$\{\text{Sun's true longitude} - (\text{longitude of Moon's apogee} - 90°)\}$$

[9] *Cf. Siddhāntaśekhara*, xi. 2–4. The text of the above as printed by Babua Misra in his edition of the *Siddhāntaśekhara* (Calcutta University Press) is defective. Emendations have been made by us by comparison with the text of the above as found in the *MS* of Suryadeva Yajvā's commentary on the *Laghumānasa* in the Lucknow University.

[10] त्रिज्याकर्णयोरन्तरेण गुणिते स्फुटकलाकर्णेन हृते। (Sūryadeva Yajvā).

> is greater than 6 signs, the correction is reversed. This is the process performed by those who wish to tally computation with observation.[11]

Expressed mathematically, Śrīpati's correction is

$$\mp R\sin\{S-(U-90^\circ)\}\times\frac{160}{R}\times$$
$$\left[1\pm\frac{5(\text{Moon's true distance, in minutes}\sim R)}{\text{Moon's true distance, in minutes}}\right]\frac{R\sin(M-S)}{R}\ \text{minutes},\quad(3)$$

where, within the square brackets, + or − sign is to be taken according as

$$\text{Moon's true distance in minutes} \lesseqgtr R$$

and the correction is to be applied positively or negatively according as

$$R\sin\{S-(U-90^\circ)\}\quad\text{and}\quad R\sin(M-S)$$

are of unlike or like signs.[12]

It would be easily seen that the correction of Śrīpati may also be stated as

$$\pm\frac{R\cos(S-U)}{R}\,[\text{Moon's true daily motion, in minutes} - 630'35'']$$
$$\times\frac{R\sin(M-S)}{R}\ \text{minutes approx.},$$

---

[11]P. C. Sengupta gives a different translation of the above passage which seems to us to be incorrect. For the sake of comparison, however, we quote it here.

> From the moon's apogee subtract 90°, diminish the sun by the remainder left; take the "sine" of the result; multiply it by 160′ and divide by the radius; the result is the *caraphala*. Put it down in another place, multiply it by *śara* (i.e., $R\text{vers}(M-U)$ or versed sine of the Moon's distance from the apogee) and divide by the difference between the moon's distance (hypotenuse) and the radius; the result is called *parama*(*cara*)*phala*, which is to be considered to be positive or negative according as the hypotenuse put down in another place is less or greater than the radius. Multiply the "sine" of the moon which has been diminished by the apparent sun, by the apparent *paramaphala* and divide by the radius; the final result is to be called *caraphala* to be applied to the moon negatively or positively, if the moon *minus* the sun and the sun *minus* the moon's apogee (diminished by 90°) be of opposite signs; if these latter quantities be of the same sign the new equation should be applied in the inverse order by those who want to make the calculation of the apparent moon agree with observation.

In consequence, P. C. Sengupta gives the correction in the following form:

$$\mp\frac{160\times R\cos(S-U)\times R\sin(M-S)}{R\times R}\times\frac{\text{Rvers}\,(M-U)}{H-R},$$

H being the Moon's true distance (*mandakarṇa*).

[12]The corresponding correction for the Moon's true daily motion does not occur in the *Siddhāntaśekhara*.

according as

$$R\cos(S-U) \quad \text{and} \quad R\sin(M-S)$$

are of unlike or like signs, which form is analogous to the forms of Vaṭeśvara and Mañjula.

It will be noted that all the formulae stated above are but approximate. The approximation has been preferred because it gives the formulae a particular form. The correct form of the formula of Śrīpati, say, would be

$$\pm R\sin\{S-(U-90°)\} \times \frac{160}{R} \times \left[1 \pm \frac{(\text{Moon's true distance, in minutes} \sim R)}{\text{Moon's true distance, in minutes}}\right] \times \frac{R\sin(M-S)}{R}$$

according as

$$R\sin\{S-(U-90°)\} \quad \text{and} \quad R\sin(M-S)$$

are of unlike or like signs; or

$$\pm\frac{R}{H} \times \frac{160}{R} \times \frac{R\sin(M-S) \times R\cos(S-U)}{R}$$ [13]

according as

$$R\sin(M-S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs, which is equivalent to the form of Nīlakaṇṭha; or,

$$\pm 160 \times \frac{R\sin(M \sim S)}{R} \times \frac{R\cos(S-U)}{R} \times \frac{\text{Moon's true daily motion}}{\text{Moon's mean daily motion}}$$

according as

$$\mathrm{R}\sin(M \sim S) \quad \text{and} \quad \mathrm{R}\cos(S-U)$$

are of unlike or like signs, which exactly conforms to the result of Candra Śekhara Siṃha.

Śrīpati has introduced the number 5 in his formula to make it agree with the forms of Vaṭeśvara and Mañjula.

7. The error committed in the above formulae of Vaṭeśvara, Mañjula, and Śrīpati was recognised by Nīlakaṇṭha (1500), who states his rule in the following manner:

व्यर्केन्दुबाहुकोटिज्ये हते वीन्दूच्चभास्वतः ।
कोट्यर्धेन त्रिजीवाप्ते दशघ्नेन्दुकलाश्रुतौ ॥

[13] H denotes the Moon's true distance, in minutes.

> अयनैक्ये च भेदे च स्वर्णं कोटिजमेतयोः ।
> तद्बाहुफलवर्गैक्यमूलमिन्दुधरान्तरम् ॥
> त्रिज्याघ्नं बाहुजं तेन भक्तं स्वर्णं विधोः स्फुटे ।
> कर्केणादौ विधूच्चोनरवौ शुक्लेऽन्यथाऽसिते ॥[14]
>
> Divide by the radius the Rsine and the Rcosine of the Moon's true longitude *minus* the Sun's true longitude severally multiplied by half the Rcosine of the sun's true longitude *minus* the longitude of the Moon's apogee: (the results are, in *yojanas*, the *bāhuphala* and the *koṭiphala*). Add the *koṭiphala* to or subtract that from ten times the true distance of the Moon (viz. the Moon's *mandakarṇa*), in minutes (*kalās*), according as the Rcosines are of like or unlike signs. The square root of the sum of the squares of that and the *bāhuphala* is the distance (in *yojanas*) between (the true positions of) the Moon and the Earth.[15] By that divide the *bāhuphala* as multiplied by the radius and apply it as a positive or negative correction to the Moon according as the Sun *minus* the Moon's apogee is in the six signs beginning with Cancer or in those beginning with Capricorn provided that it is the light half of the lunar month; in the dark half the correction is to be reversed.

Stated mathematically, Nīlakaṇṭha's correction takes the following form:

$$\pm \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{R} \text{ minutes} \tag{4}$$

according as

$$R\sin(M-S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs, H_1 being the Moon's second true distance, in minutes.[16]

---

[14] *Cf. Tantrasaṅgraha*, viii. 1–3.

[15] This is also known as Moon's second true distance (*dvitīya-sphuṭa-karṇa*).

[16] As regards the corresponding correction for the Moon's true daily motion, Nīlakaṇṭha does not give any formula analogous to that given by Vaṭeśvara and Mañjula. He has, however, prescribed the following rule (*cf. Tantrasaṅgraha*, viii. 4) for obtaining the second true daily motion of the Moon:

> मध्यभुक्तिर्दशघ्नेन्दोस्त्रिज्याघ्ना योजनैर्हृता ।
> भूचन्द्रान्तरगैर्भुक्तिर्विधोरस्य स्फुटा मता ।।
>
> Ten times the Moon's mean daily motion (in minutes) multiplied by the radius and divided by the distance between (the true positions of) the Earth and the Moon (*bhūcandrāntara*), in *yojanas*, has been stated to be its (second) true daily motion (in minutes).

This gives the following formula:

Moon's second true daily motion =

$$\frac{\text{Moon's mean daily motion, in minutes} \times 10 \times R}{\text{Moon's second true distance, in } \textit{yojanas}} \text{ minutes.}$$

8. This correction also occurs in the *Siddhāntadarpaṇa* of Candra Śekhara Siṃha where it has been called *tuṅgāntara* and stated as follows:

> अभीष्टकालोत्थितचन्द्रमन्दात् पक्षे सिते सत्रिभसूर्यहीनात् ।
> कृष्णे त्रिभोनार्यमवर्जिताद्यत् केन्द्रं तदीया भुजमौर्विका या ॥
> साभ्राङ्गभूघ्नी त्रिगुणेन भक्ता स्फुटार्कचन्द्रान्तरदोर्गुणघ्नी ।
> त्रिज्योद्धृता लब्धमतः कलाद्यं गत्या विनिघ्नं प्रथमस्फुटेन्दोः ॥
> तन्मध्यगत्या विहृतं फलं स्यात् तुङ्गान्तरं तेन विहीनयुक्तः ।
> पर्यायतः सत्रिभवित्रिभार्कहीनेन्दुमन्दोच्चभवोक्तकेन्द्रे ॥
> तुलाधराजादिभषड्कनिष्ठे प्राक्सिद्धचन्द्रो भवति द्वितीयः ।[17]
>
> From the longitude of the Moon's apogee for the desired instant subtract the Sun's true longitude as increased by 3 signs if it is the light half of the lunar month and subtract the Sun's true longitude as diminished by 3 signs if it is the dark half of the lunar month. Treat the remainder as *kendra* and determine the Rsine thereof. Multiply that by 160 and divide by the radius; (again) multiply by the Rsine of the difference between the true longitudes of the Sun and the Moon and divide by the radius. Multiply the quotient, in minutes, thus obtained, by the daily motion of the first true Moon[18] and divide by the Moon's mean daily motion: the result (thus obtained) is known as *tuṅgāntara.* The true longitude of the Moon obtained before, when diminished or increased, *or* increased or diminished by that according as the *kendra* obtained by subtracting the Sun's true longitude as increased by 3 signs *or* decreased by 3 signs from the longitude of the Moon's apogee is in the six signs commencing with Libra or Aries respectively, becomes the second true longitude of the Moon.

This is equivalent to the following correction:

$$\pm\, 160 \times \frac{R\cos(M \sim S)}{R} \times \frac{R\sin(S-U)}{R} \times \frac{\text{daily motion of the first true Moon}}{\text{Moon's mean daily motion}} \text{ minutes}, \qquad (5)$$

where $+$ and $-$ signs are chosen according as

$$R\sin(M \sim S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs.

[17] *Cf. Siddhāntadarpaṇa, grahagaṇita*, vi. 7–10 (i).

[18] The first true Moon is the same as the true Moon. Similarly the first true longitude and the first true daily motion are the same as the true longitude and the true daily motion.

The corresponding correction for the Moon's true daily motion given by Candra Śekhara Siṃha is contained in the following lines:

> तुङ्गान्तरं यत् फलमत्र सिद्धं त्रिज्याहतं तत् प्रथमेन्दुभान्वोः ।
> विश्लेषदोर्ज्याप्तफलं तदीयान्तरोत्थकोटीगुणसङ्गुणञ्च ॥
> त्रिज्योद्धृतं तत् पुनरर्कचन्द्रगत्यन्तरघ्नं त्रिगुणाप्तलब्धम् ।
> योज्यं तदेव प्रथमेन्दुभुक्तिफले भवेद्भुक्तिफलं द्वितीयम् ॥
> तत् संस्कृतं मध्यगतौ पुरोवद्भवेद्द्वितीया रजनीशभुक्तिः ।[19]
>
> The result known as *tuṅgāntara*, which has been just obtained, should be multiplied by the radius and divided by the Rsine of the difference of the first true longitudes of the Moon and the Sun. The result should be multiplied by the Rcosine of the same difference and divided by the radius. That should be again multiplied by the motion-difference of the Sun and the Moon, and divided by the radius and that should be added to (or subtracted from) the Moon's first *bhuktiphala* (correction for motion): result is the Moon's second *bhuktiphala*. That applied (as a correction—positive or negative—) to the Moon's mean daily motion as before gives the Moon's second (true) daily motion.

Accordingly, the corresponding correction for the Moon's true daily motion is

$$\pm 160 \times \frac{R\cos(M \sim S)}{R} \times \frac{R\cos(S-U)}{R} \times \frac{\text{daily motion of the first true Moon}}{\text{Moon's mean daily motion}} \times \frac{\text{motion-difference of the Moon and the Sun}}{R} \text{ minutes,} \qquad (6)$$

$+$ or $-$ sign is to be taken according as

$$R\cos(M \sim S) \quad \text{and} \quad R\cos(S-U)$$

are of unlike or like signs.

Expression (6) is clearly an approximate differential of (5), the term involving the differential of $R\cos(S-U)$ having been neglected.

Correspondence between formulae (2) and (6) may be noted.

9. The above discussion clearly shows that there is striking similarity among the rules stated above. The differences are due to different maximum values of the correction taken by different authors.

[19] *Cf. l.c.* vi. 17–19 (i).

10. Although there is a general unity among the rules above, yet it is surprising to note, at first sight, that Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha have deviated from Vaṭeśvara and Mañjula regarding the sign of the correction for the Moon's longitude. Vaṭeśvara and Mañjula apply the correction negatively where Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha apply it positively and vice versa. The reason for this deviation is that all the Hindu astronomers including, of course, Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha agree among themselves in taking $R\sin\theta$ positive or negative according as

$$0 < \theta < 6 \text{ signs}$$

or

$$6 \text{ signs} < \theta < 12 \text{ signs},$$

whereas Vaṭeśvara and Mañjula take $R\sin\theta$ positive or negative according as

$$6 \text{ signs} < \theta < 12 \text{ signs}$$

or

$$0 < \theta < 6 \text{ signs}.$$

Mañjula gives his rule of sign as follows:

> **ग्रहः स्वोच्चोनितः केन्द्रं तदूर्ध्वाधोऽर्धजो भुजः ।**
> **धनर्णं पदशः कोटी धनर्णर्णधनात्मिका ॥**[20]
>
> The (mean or true-mean) longitude of the planet diminished by the (mean) longitude of the (*manda* or *śīghra*) *ucca* is known as (*manda* or *śīghra*) *kendra*. There the *bhuja*[21] (and the Rsine thereof) is positive or negative according as the *kendra* is greater or less than half a circle; and the *koṭi* (i.e., the complementary arc of the *bhuja*) (and the Rsine thereof) is *plus*, *minus*, *minus*, and *plus* in the respective quadrants.[22]

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[20] *Cf. Laghumānasa*, i(b), 1. The text given here agrees with that given by Sūryadeva Yajvā (b. 1191), Parameśvara (1409), and Yallaya (1482). N. K. Majumdar, however, gives *ṣaḍūrdha* instead of *tadūrdhva*. P. C. Sengupta's version is *ṣaḍūrdhvādardhajo*.

[21] "If the (mean or true-mean) planet is in the odd quadrant (the portion of) the *kendra* (which lies in that quadrant) is known as *bhuja* (and the complementary arc as *koṭi*); if the (mean or true-mean) planet is in the even quadrant (the portion of) the *kendra* (which lies in that quadrant) is called *koti* (and the complementary arc as *bhuja*)." (Lalla).

[22] Parameśvara says: **केन्द्रे तुलादिषड्राशिगते धनात्मको भुजः मेषादिषड्राशिगते ऋणात्मक इत्यर्थः।** Sūryadeva Yajvā says: **केन्द्रस्य च उर्ध्वार्धात् अधोऽर्धात् जातश्च भुजः क्रमात् धनसंज्ञं च ऋणसंज्ञं च भवतः। यदा केन्द्रं राशिषट्कादधिकं तदा ऊर्ध्वार्धे वर्तत इति ज्ञेयम्। यदा राशिषट्कादूनं तदा अधोऽर्धे वर्तत इति ज्ञेयम्।**

Literally translated the latter part of the above verse would give: "There the *bhuja* arising from the upper half-circle (commencing from the sign Libra) and the lower half-circle (commencing from the sign Aries) is positive and negative and the *koṭi* in the respective quadrants is positive, negative, negative, and positive."

Consequently in the four quadrants the signs of $R\sin\theta$ and $R\cos\theta$ taken by Vaṭeśvara and Mañjula are *minus, minus, plus, plus* and *plus, minus, minus, plus* respectively. As regards the sign of $R\sin\theta$, this convention, as already pointed out, does not agree with the general Hindu convention.

The conception of Vaṭeśvara and Mañjula is based, however, on the following two considerations:

(i) the *bhujaphala* (the equation of the centre) is a function of the Rsine of the *bhuja* while the *koṭiphala* (i.e., the correction for the radius) is a function of the Rsine of the *koṭi*; and

(ii) the *bhujaphala* (i.e., equation of the centre) is subtractive, subtractive, additive, and additive in the four successive quadrants while in the same quadrants the *koṭiphala* (i.e., the correction for the radius) is additive, subtractive, subtractive, and additive.

Following this Vaṭeśvara and Mañjula take the Rsine and the corresponding arc known as *bhuja* negative where the *bhujaphala* (i.e., the equation of the centre) is negative and positive where it is positive. Similarly, where the *koṭiphala* (i.e., the correction for the radius) is negative the Rcosine and likewise the corresponding arc known as *koṭi* is taken as negative and where the *koṭiphala* is positive, the Rcosine and the *koṭi* are taken as positive.

This explains the difference in sign in the corrections for the Moon's longitude given by Vaṭeśvara and Mañjula and those given by Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha.

11. Thus there has been established complete unity among the rules of Vaṭeśvara, Mañjula, Śrīpati, Nīlakaṇṭha, and Candra Śekhara Siṃha.

Section II

12. Vaṭeśvara and Mañjula call the expression

$$\left(8\frac{2}{15}\right)\cos(S-U)[\text{Moon's true daily motion, in degrees} - 11]$$

by the term *guṇa*, which has also been used to denote the epicyclic multiplier. Śrīpati calls

$$\frac{160}{R}\times R\sin\{S-(U-90°)\}$$

paraphala, which corresponds with the *antyaphala* i.e., the radius of the epicycle, and

$$R\sin\{S-(U-90^\circ)\}\times\frac{160}{R}\times \left[1\pm\frac{5(\text{Moon's true distance, in minutes}\sim R)}{\text{Moon's true distance, in minutes}}\right]$$

sphuṭa-parama-phala, which may be translated by the expression "corrected epicyclic radius". Nīlakaṇṭha has actually used the terms *bāhuphala* and *koṭiphala*, and says

$$\textit{bāhuphala}=\frac{R\sin(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ \textit{yojanas},$$

$$\text{and}\qquad \textit{koṭiphala}=\frac{R\cos(M-S)\times\frac{1}{2}R\cos(S-U)}{R}\ \textit{yojanas}.$$

These facts clearly indicate that the Hindu astronomers had also an epicyclic representation of the above correction. In what follows we shall explain this point of view.

13. The Hindu astronomers believed that the Earth did not always occupy its natural position (which coincides with the centre of the so-called *bhagola*) and that the dual correction above was due to its displacement. Let E (**ed.** See Figure 1) denote the natural position of the Earth's centre (*bhagolaghanamadhya*), the bigger circle round E the Moon's concentric (*kakṣāvṛtta*), the point U the position of the Moon's apogee on the concentric, M the true position of the Moon's centre and ES the direction of the Sun from the Earth's centre. The small circle round E has λ for its radius where λ denotes the maximum value of the dual correction at the Moon's distance given by the following table:

Authority	λ, in minutes (*kalās*)
Vaṭeśvara and Mañjula	144 approx.
Śrīpati and Candra Śekhara Siṃha	160
Nīlakaṇṭha	171.9

K is the point on the small circle opposite to U. KE_1 is perpendicular to SES_1 and E_1P to MEM_1. The point E_1 denotes, according to Hindu astronomers, the displaced position of the Earth's centre and is known as *ghanabhūmadhya*.

The circle with centre E and radius EE_1 (not shown in the figure) is treated as an epicycle and its radius EE_1 is known as the epicyclic radius[23] (*paraphala*). In consequence E_2P is known as *bāhuphala* (or *bhujāphala*) and EP

[23]It will be noted that the size of this epicycle does not remain constant. It depends on the

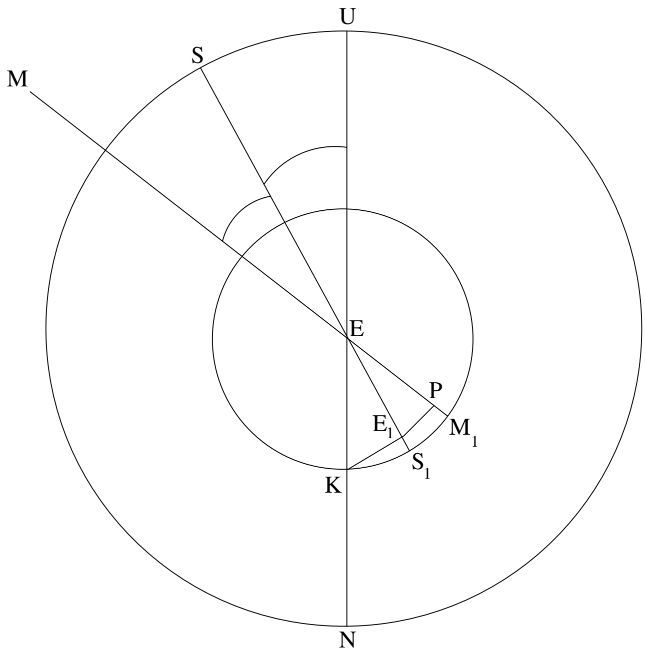

Figure 1

as *koṭiphala*.

$$\because \angle KEE_1 = S - U,$$
$$\text{and} \qquad \angle E_1EP = M - S,$$
$$\therefore EE_1 = \lambda \cos(S - U),$$
$$\text{whence} \qquad E_1P \text{ i.e., } \textit{bāhuphala} = \lambda \cos(S - U) \times \sin(M - S),$$
$$\text{and}^{24} \qquad EP \text{ i.e., } \textit{koṭiphala} = \lambda \cos(S - U) \times \cos(M - S).$$

positions of the Sun and the Moon's apogee. When $S - U$ is zero or 180°, it assumes its greatest size and coincides with the small circle drawn in the figure. When $S \sim U$ equals 90°, it reduces to the point-circle at E. It will be further noted that EE_1 denotes the epicyclic radius at the Moon's second true distance. When it is reduced to the distance R, it is known as corrected epicyclic radius (*sphuṭa-parama-phala*).

[24] If we put $\lambda = 171.9$ minutes or $\frac{R}{2}$ *yojanas*, we have

$$\textit{bāhuphala} = \frac{R \sin(M - S) \times \frac{1}{2} R \cos(S - U)}{R} \textit{ yojanas},$$
$$\text{and} \qquad \textit{koṭiphala} = \frac{R \cos(M - S) \times \frac{1}{2} R \cos(S - U)}{R} \textit{ yojanas},$$

which agree with the formulae stated by Nīlakaṇṭha.

Due to the displacement of the Earth's centre from E to E_1 the Moon's true distance, measured in minutes, (*mandakarṇa* or *sphuṭa-kalā-karṇa*) changes from EM to E_1M. The distance E_1M is known as the distance between (the true positions of) the Moon and the Earth (*bhūmyantara-karṇa*) or the Moon's second true distance (*dvitīya-sphuṭa-karṇa*) and is obtained by the following formula:

$$\text{Moon's second true distance, in minutes} = \Big[(\text{Moon's true distance, in } \textit{kalās} \pm \textit{koṭiphala}, \text{ in } \textit{kalās})^2 + (\textit{bāhuphala}, \text{ in } \textit{kalās})^2\Big]^{\frac{1}{2}}$$

according as the *koṭiphala* is positive or negative.[25]

Similarly, the dual correction due to the displacement of the Earth's centre from E to E_1 is obviously given by the following:

$$\begin{aligned}\text{the dual correction} &= \angle EME_1\\ &= R\sin^{-1}\left\{\frac{E_1P \times R}{E_1M}\right\}\\ &= \frac{E_1P \times R}{E_1M} \text{ approx.}\\ &= \frac{R}{H_1} \times \lambda \sin(M - S) \times \cos(S - U)^{26}\end{aligned}$$

where H_1 denotes the Moon's second true distance, in minutes.

Since the Moon's true distance is approximately equal to the Moon's second true distance, the Hindu astronomers have in general used the Moon's true distance, in minutes, in place of the Moon's second true distance, in minutes, in their formulae for the dual correction. The error is negligible. Nīlakaṇṭha, however, used the Moon's second true distance, in minutes.

14. The displacement of the Earth's centre conceived by Hindu astronomers not only changes the Moon's true distance but it also creates a change in the

[25] This corresponds with Nīlakaṇṭha's formula above.

[26] Putting various values of λ in this formula, we obtain the formulae of Vaṭeśvara etc. in their modified form. For example, putting $\lambda = \frac{R}{2}$ *yojanas*, we obtain Nīlakaṇṭha's formula

$$\begin{aligned}\text{the dual correction} &= \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{R} \textit{ yojanas},\\ &= \frac{R}{H_1} \times \frac{R\sin(M-S) \times \frac{1}{2}R\cos(S-U)}{10 \times R} \text{ minutes.}\end{aligned}$$

distances of all other planets, howsoever small that change may be. Nīlakaṇṭha has considered the two particular cases relating to the lunar and solar distances:

(1) when the longitudes of the Sun and the Moon are the same; and

(2) when the longitudes of the Sun and the Moon differ by 180°.

He says:

> उच्चोनशशिकोटिज्यादलं पर्वान्तजं स्फुटम् ।
> स्फुटयोजनकर्णे स्वं जह्यात् कर्क्यादिजं ततः ॥
> स भूम्यन्तरकर्णः स्यात् तेन बिम्बकलां नयेत् ।
> स्फुटयोजनकर्णे स्वे मासान्ते शशिवद्रवेः ॥
> व्यस्तं पक्षान्तजं कार्यं रविभूम्यन्तराप्तये ।[27]
>
> The half of the true value of the Rcosine of the Moon's longitude *minus* the longitude of the Moon's apogee (*mandocca*) corresponding to the instant of geocentric conjunction or opposition of the Sun and the Moon should be added to the true distance of the Moon, in *yojanas*, (when the Moon is in the six anomalistic signs commencing with Capricorn) and subtracted from that when the Moon is in the six anomalistic signs beginning with Cancer. This gives the distance, in *yojanas*, between (the true positions of) the Moon and the Earth (*bhūmyantarakarṇa*). This is to be used in calculating the Moon's diameter, in minutes (*kalās*), (for the instant of geocentric conjunction or opposition). In order to obtain the distance (in *yojanas*) between (the true positions of) the Sun and the Earth, apply the same as a positive or negative correction to the Sun's true distance, in *yojanas*, as in the case of the Moon, provided that it is the end of the lunar month; if it is the end of the fifteenth lunar date, apply the same reversely.

From figures similar to that drawn above for the general case, it will be seen that, when the Sun and the Moon are in geocentric conjunction,

(i) the Sun and the Moon are in the same direction from the natural position of the Earth's centre (*bhagolaghanamadhya*) while the displaced position of the Earth's centre is in the contrary or the same direction according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer; and

(ii) $M - S = 0$, whence

$$bāhuphala = 0 \quad \text{and} \quad koṭiphala = \frac{1}{2} R\cos(M-U) \ \ yojanas.$$

[27] *Cf. Tantrasaṅgraha*, iv. 12–14 (i).

Consequently

$$\text{Sun's second true distance, in } \textit{yojanas} = \text{Sun's true distance, in } \textit{yojanas} \pm \frac{1}{2} R \cos(M - U) \ \textit{yojanas},$$

and

$$\text{Moon's second true distance, in } \textit{yojanas} = \text{Moon's true distance, in } \textit{yojanas} \pm \frac{1}{2} R \cos(M - U) \ \textit{yojanas},$$

according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer.

When the Sun and the Moon are in geocentric opposition, it will be, similarly, seen that

(i) the Moon and the Sun are on the opposite sides of the natural position of the Earth's centre (*bhagolaghanamadhya*) and the displaced position of the Earth is directed towards that of the Sun or the Moon according as the Moon is in the six anomalistic signs commencing with Capricorn or Cancer; and

(ii) $M - S = 180$, whence

$$\textit{bāhuphala} = 0 \quad \text{and} \quad \textit{koṭiphala} = \frac{1}{2} R \cos(M - U) \ \textit{yojanas}.$$

Consequently

$$\text{Sun's second true distance, in } \textit{yojanas} = \text{Sun's true distance, in } \textit{yojanas} \mp \frac{1}{2} R \cos(M - U) \ \textit{yojanas},$$

and

$$\text{Moon's second true distance, in } \textit{yojanas} = \text{Moon's true distance, in } \textit{yojanas} \pm \frac{1}{2} R \cos(M - U) \ \textit{yojanas},$$

according as the Moon is in the six anomalistic signs beginning with Capricorn or Cancer.

Hence Nīlakaṇṭha's rules above.

15. In the above graphical method of the Hindus the true position of the Moon remains unaffected whereas the position of the Earth goes on changing from time to time. Also the size of the epicycle ascribed to the Earth does not remain invariable. It depends upon the positions of the Sun and the Moon's apogee. It is maximum when the Sun crosses the Moon's line of apsides (*uccanīcarekhā*) and minimum when the Sun is at right angles to it. This variation in the size of the Earth's epicycle causes a variation in the eccentricity of the Moon's path which, it will be noted, always assumes its maximum value when the Sun crosses the Moon's line of apsides and its minimum value when the Sun is at right angles to it. This variation in the eccentricity of the Moon's orbit is obviously related to the dual correction. In fact the dual correction depends upon it. Young (1889) has actually said that the evection "depends upon the alternate increase and decrease of the eccentricity of the Moon's orbit, which is always a maximum when the sun is passing the moon's line of apsides, and a minimum when the sun is at right angles to it." According to Mañjula and Vaṭeśvara the maximum value of the eccentricity of the Moon's orbit comes out to be roughly about 0.0652 and according to Śrīpati and Candra Śekhara Siṃha about 0.0674; the minimum Hindu value of the Moon's eccentricity is about 0.0442. The corresponding maximum and minimum values given by Horrocks (1640) are 0.06686 and 0.04362 respectively. According to Young the eccentricity of the Moon's orbit varies from $\frac{1}{14}$ to $\frac{1}{22}$.

Section III

16. The Greek astronomer Ptolemy (140 AD) was also aware of this dual correction of the Moon.[28] He is said to have constructed an instrument by means of which he observed the Moon in all parts of its orbit and found

(i) that the computed positions of the Moon were generally different from the observed ones, the maximum amount of this difference noted by him being 159 minutes, and

(ii) that the difference between the observed and computed positions of the Moon attained its maximum value when $|M-S|$ equalled 90° and $S-U$

[28] In modern works we find that instead of this dual correction being attributed to Ptolemy, the evection is generally attributed to him. It should be noted that Ptolemy did not detect the evection alone but a mixture which contains the deficit of the equation of the centre and the evection. And this is what can be naturally expected from the ancient astronomers who took the maximum value of the Moon's equation of the centre smaller than its actual value. Some writers also make the erroneous statement that the ancient astronomers detected the evection because it can affect the time of an eclipse by about 6 hours at its maximum. The fact is that the ancient astronomers did not detect the evection separately, and if we call this dual correction by the name of evection, we find that it does not make any difference in the time of an eclipse. Even Ptolemy says that this correction is zero when $(M-S)$ is zero or 180°.

was either zero or 180°, and that it vanished altogether when $M - S$ equalled zero or 180°.

To represent this dual correction Ptolemy imagined an eccentric in the circumference of which the centre of the epicycle moved while the Moon moved on the circumference of the epicycle. Later on it was discovered by Copernicus (1543) that the lunar distances resulting from Ptolemy's hypothesis were totally at variance with the observations of the Moon's apparent diameter. Consequently he gave another method of representing the lunar inequality which is known as Copernicus's hypothesis.

17. Ptolemy had previously discovered that in quadrature when the equation of the centre assumed its maximum value viz. $5°1'$, the dual correction increased it to $7°40'$, which happened when the apse-line (*ucca-nīca-rekhā*) coincided with the direction of the Sun from the Earth's centre, but when the Sun's direction was perpendicular both to the apse–line and Moon's direction, the equation of the centre vanished with the dual correction. Consequently Ptolemy had fixed $6°20\frac{1}{2}'$ as the value of the mean of the two corrections. Copernicus took it as the corrected value of the maximum equation of the centre and treated it as the radius of the Moon's first epicycle. Thus the deficit of the equation of the centre was unconsciously added to it. The radius of the first epicycle conceived by Copernicus was likewise equal to M_1O (**ed.** See Figure 2). In order to account for the remaining correction viz. the evection, Copernicus took Om for the radius of the Moon's second epicycle and supposed the Moon M to move on it in the anti-clockwise direction from the point m in such a way that, at any instant,

$$\angle MOm = 2(M_1 - S_1),$$

where M_1 and S_1 are the mean positions of the Moon and the Sun respectively.

Copernicus's hypothesis also leads to the same form of expression for the Moon's dual correction as given by Hindu astronomers but it does not explain the variation of the eccentricity of the Moon's orbit which really causes the dual correction.[29]

19. According to modern lunar theory the relevant terms of the Moon's longitude are given by

$$\text{Moon's longitude} = M_1 - 377' \,\text{Sin}\,(M_1 - U) - 76' \,\text{Sin}\,\{2(M_1 - S_1) - (M_1 - U)\}. \qquad (7)$$

[29] **ed.** As per the sequence, the subsequent point of discussion should be numbered **18**. However, since in the original it jumps by one number, we have just retained it.

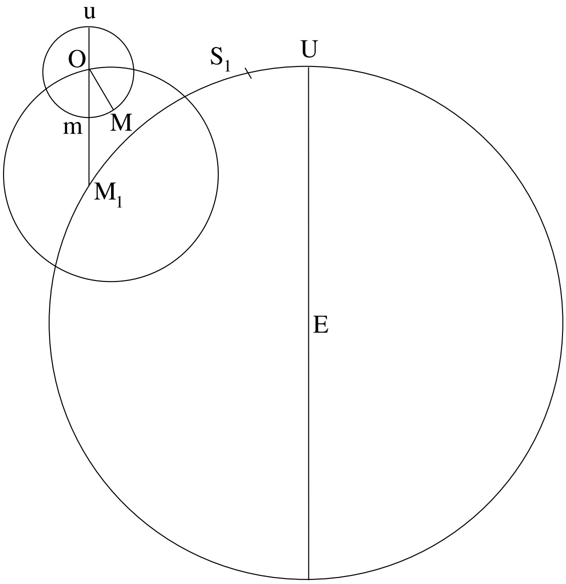

Figure 2

M_1 and S_1 denote the mean longitudes of the Moon and the Sun and U the longitude of the Moon's apogee.

The term $-377'$ Sin$(M_1 - U)$ denotes the equation of the centre and the term $-76'$ Sin$\{2(M_1 - S_1) - (M_1 - U)\}$ is known as evection.

If the equation of the centre viz. $-377'$ Sin$(M_1 - U)$ be broken up into two components $-301'$ Sin$(M_1 - U)$ and $-76'$ Sin$(M_1 - U)$ and the second component be combined with the evection-term, then formula (7) would become

$$\text{Moon's longitude} = M_1 - 301' \text{ Sin}\,(M_1 - U) \\ - 152' \text{ Cos}\,(S_1 - U) \times \text{Sin}\,(M_1 - S_1). \qquad (8)$$

Here, if the term $-301'$ Sin$(M_1 - U)$ be taken for the equation of the centre, then the term $-152'$ Cos $(S_1 - U) \times$ Sin$(M_1 - S_1)$ would give the deficit of the equation of the centre and the evection. The term $-301'$ Sin$(M_1 - U)$ corresponds to the Hindu equation of the centre and the term $-152'$ Cos $(S_1 - U) \times$ Sin$(M_1 - S_1)$ to the dual correction discussed above. Comparison of this term with the expressions for the dual correction given by Hindu astronomers proves the correspondence between the two as also the perfectness of the Hindu form. It would also be noted that the difference between the Moon's longitudes calculated according to Vaṭeśvara or Mañjula and by formula (7) would never exceed about 5 minutes.

20. The above discussion proves conclusively the soundness of the formulae for the Moon's dual correction given by various Hindu astronomers. We have also seen that this correction in its perfect form was known in India in the time of Vaṭeśvara (c. 899) or earlier. In view of the advanced state of the available Hindu formulae, we have every reason to believe that the correction was known much earlier in India, specially when we see that in Europe it was known in its proper from about 1400 years after it was actually detected. The graphic method of the Hindus which not only explains the dual correction but also the variation of the eccentricity of the Moon's orbit was known to the Hindus long before Copernicus gave his own. Hindus were, thus, the first to give the Moon's dual correction in its perfect form and the first to explain it properly.

21. I take this opportunity to express my thanks to Prof. A. N. Singh for help and guidance.

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Phases of the Moon, rising and setting of planets and stars and their conjunctions *

1 Introduction

It has been known from time immemorial that the Moon is intrinsically a dark body but looks bright as it is lighted by the Sun. There is an oft-quoted statement in the *Yajurveda*[1] which describes the Moon as sunlight. As the Moon revolves round the Earth its lighted portion that faces us is seen by us in successively increasing or diminishing amounts. These are called the phases of the Moon.

When the Sun and Moon are in the same direction, the face of the Moon which is turned towards us is completely dark. It is called new-moon and marks the beginning of the light fortnight. When the Moon is 12 degrees ahead of the Sun, it is seen after sunset in the shape of a thin crescent. As the Moon advances further this crescent becomes thicker and thicker night after night. When the Moon is 180 degrees away from the Sun, the Moon is seen fully bright. It is called full-moon. The light fortnight now ends and the dark fortnight begins. The phases are now repeated in reverse order until the Moon is completely dark at the end of the dark fortnight when the Sun and Moon are again in the same direction.

Vaṭeśvara says:

> The Sun's rays reflected by the Moon destroy the thick darkness of the night just as the Sun's rays reflected by a clean mirror destroy the darkness inside a house.[2]
>
> In the dark and light fortnights the dark and bright portions of the Moon (gradually) increase as the Moon respectively approaches and recedes from the Sun.[3]

* K. S. Shukla, in *History of Astronomy in India*, S. N. Sen and K. S. Shukla (eds.), Indian National Science Academy, New Delhi, 1985, pp. 212–251 (Originally published in *Indian Journal of History of Science*, Vol. 20, Nos. 1–4 (1985)).

[1] *Adhyāya* 18, *mantra* 40: सुषुम्णः सूर्यरश्मिश्चन्द्रमा गन्धर्वः।

[2] *VG*, iv. 23.

[3] *VG*, iv. 25.

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_34

> On the new-moon day the Moon is dark, in the middle of the light fortnight, it is seen moving in the sky half-bright; on the full moon day it is completely bright as if parodying the face of a beautiful woman.[4]

> The crescent of the Moon appears to the eye like the creeper of Cupid's bow, bearing the beauty of the tip of the Ketaka flower glorified by the association of the black bees, and giving the false impression of the beauty of the eyebrows of a fair-coloured lady with excellent eyebrows.[5]

> When the measure of the Moon's illuminated part happens to be equal to the Moon's semi-diameter, the Moon looks like the forehead of a lady belonging to the Lāṭa country (Southern Gujarat).[6]

Similar statements appear in the writings of Varāhamihira and other Indian astronomers.

1.1 Phase and *sita*

In modern astronomy the phase of the Moon is measured by the ratio of the central width of the illuminated part to the diameter. In Indian astronomy it is generally measured by the width of the illuminated part itself which is called *sita* or *śukla*. The width of the un-illuminated part, which is equal to 'the Moon's diameter minus the *sita*', is called *asita*.

The *Pūrva Khaṇḍakhādyaka* of Brahmagupta, which summarises the contents of Āryabhaṭa I's astronomy based on midnight day-reckoning, gives the following approximate rule to find the *sita* in the light half of the month:

> The difference in degrees between the longitudes of the Sun and the Moon, divided by 15, gives the *śukla* (in terms of *aṅgulas*).[7]

Stated mathematically, it is equivalent to the following formula:

$$sita = \frac{M - S}{15}\ aṅgulas,$$

where S and M denote the longitudes of the Sun and the Moon respectively, in terms of degrees. This formula may be obtained by substituting

$$\text{Moon's diameter} = 12\ aṅgulas$$

[4] *VG*, iv. 24.
[5] *VSi*, VII, i. 51.
[6] *VSi*, VII, i. 49 (c–d).
[7] *KK*, Part I, vii. 4 (a–b).

in the general formula:

$$sita = \frac{(M - S) \times \text{Moon's diameter}}{180}. \tag{1}$$

Bhāskara I (629), a follower of Āryabhaṭa I, who claims to have set out in his works the teachings of Āryabhaṭa I, however, gives the following rule:

> (In the light fortnight) multiply (the diameter of) the Moon's disc by the Rversed-sine of the difference between the longitudes of the Moon and the Sun (when less than 90°) and divide (the product) by the number 6876: the result is always taken by the astronomers to be the measure of the *sita*. When the difference between (the longitudes of) the Moon and the Sun exceeds a quadrant (i.e. 90°), the *sita* is calculated from the Rsine of that excess, increased by the radius.
>
> After full-moon (i.e. in the dark fortnight), the *asita* is determined from the Rversed-sine of (the excess over six or nine signs, respectively, of) the difference between the longitudes of the Moon and the Sun in the same way as the *sita* is determined (in the light fortnight).[8]

That is to say:

(i) In the light fortnight (*śukla-pakṣa*)

$$sita = \frac{R\,\text{versin}(M - S) \times \text{Moon's diameter}}{6876},$$

if $M - S \leq 3$ signs, i.e. if it is the first half of the fortnight; and

$$= \frac{[R + R\sin(M - S - 90°)] \times \text{Moon's diameter}}{6876},$$

if $M - S > 3$ signs, i.e. if it is the second half of the fortnight.

(ii) In the dark fortnight (*kṛṣṇa-pakṣa*)

$$asita = \frac{R\,\text{versin}(M - S - 180°) \times \text{Moon's diameter}}{6876},$$

if $M - S > 6$ signs, i.e. if it is the first half of the fortnight; and

$$= \frac{[R + R\sin(M - S - 270°)] \times \text{Moon's diameter}}{6876},$$

if $M - S > 9$ signs, i.e. if it is the second half of the fortnight.

[8] *MBh*, vi. 5(c–d)–7. Also see *LBh*, vi. 6–7.

Bhāskara I's contemporary Brahmagupta (628) gives the following rule, which is a via media between the above two rules:

> One half of the Moon's longitude minus Sun's longitude, multiplied by the Moon's diameter and divided by 90, gives the *sita*. This is the first result.
>
> When the Moon's longitude minus Sun's longitude, reduced to degrees, is less than or equal to 90°, take the *R*versed-sine of that; and when that exceeds 90°, take the *R*sine of the excess over 90° and add that to the radius. Multiply that by the measure of the Moon's diameter and divide by twice the radius (i.e. by 2 × 3438 or 6876). This is another result. The former result gives the *sita* in the night and the latter in the day. One half of their sum gives the same during the two twilights.[9]

That is:

(i) *sita* for night

$$= \frac{\left[\frac{(M-S)}{2}\right] \times \text{Moon's diameter}}{90},$$

$M - S$ being in degrees.

(ii) *sita* for day

$$= \frac{R\,\text{versin}(M-S) \times \text{Moon's diameter}}{2R},$$

if $M - S \leq 90°$; or

$$= \frac{[R + R\sin(M - S - 90°)] \times \text{Moon's diameter}}{2R},$$

if $M - S > 90°$. ($R = 3438$)

(iii) *sita* for twilights

$$= \frac{\textit{sita} \text{ for day} + \textit{sita} \text{ for night}}{2}.$$

These formulae obviously relate to the light half of the month.

Vaṭeśvara[10] (904) and Śrīpati[11] (1039) have followed Brahmagupta. Lalla[12] gives the two results stated by Brahmagupta, treating them as alternative.

[9] *BrSpSi*, vii. 11–13 (a–b).
[10] *VSi*, VII, i. 23–24.
[11] *SiŚe*, x. 16–19 (a–b).
[12] *ŚiDVṛ*, X, ix. 13, 14.

But whereas his commentator Mallikārjuna Sūri (1178) interprets them as alternative rules, his other commentator Bhāskara II (1150) makes no distinction between the rules of Brahmagupta and Lalla and interprets them in the light of Brahmagupta's rules. Bhāskara II has also attempted to explain why different formulae were prescribed for day, night, and twilights. He says:

> The first *sita*, being based on arc, is gross. This is to be used in the graphical representation of the Moon in the night, because then there is absence of the accompaniment of the Sun's rays. The second *sita*, being based on *R*sine, is accurate. This is to be used in the graphical representation of the Moon in the day, because then the Moon's rays being overpowered by the Sun's rays are not bright. During the twilights, the *sita* should be obtained by taking their mean value, because then the characteristic features of the day and night are medium.[13]

Āryabhaṭa II[14] (c. 950) and Bhāskara II[15] have prescribed the first result of Brahmagupta for all times. The method given in the *Sūryasiddhānta*[16] is also essentially the same. The difference exists in form only.

According to Bhāskara II[17] the *sita* amounts to half when the Moon's longitude minus Sun's longitude is $85°45'$, not when it is $90°$ as presumed by the earlier astronomers. This means that he understood that the *sita* varies as the elongation of the Earth from the Sun (as seen from the Moon) and not as the Moon's longitude minus Sun's longitude. However, he has not stated this fact expressly, nor has he attempted to obtain the Earth's elongation from the Sun. Instead, he has applied a correction to the Moon in order to get the correct value of the *sita*.[18]

The astronomers who succeeded Bhāskara II have calculated the *sita* from the actual elongation of the Moon from the Sun and not from the difference between the Moon's longitude and the Sun's longitude. Since the actual elongation of the Moon from the Sun was the same as the angular distance between the discs of the Sun and Moon, these astronomers have called it *bimbāntara* ("the disc interval") and have calculated the *sita* by using it in place of the Moon's longitude minus Sun's longitude.

The *sita* really varies as the versed sine of the elongation of the Moon from the Sun (or more correctly as the versed sine of the elongation of the Earth from the Sun as seen from the Moon), not as the elongation of the Moon

[13] Bhāskara II's commentary on *ŚiDVṛ*, ix. 14.
[14] *MSi*, vii. 7.
[15] See *SiŚi*, I, ix. 7, comm.
[16] x. 9.
[17] *SiŚi*, I, ix. 6; also his comm. on it.
[18] *Ibid.*

from the Sun (as measured on the ecliptic). So Brahmagupta's first result is gross and has been rightly criticised by the author of the *Valana-śṛṅgonnati-vāsanā*. "Brahmagupta and others (who have followed him)" says he, "have not considered the nature of the arc relation."[19] The rule given by Bhāskara I, however, is fairly good for practical purposes.

1.2 Special rules

Muñjala (932), the author of the *Laghumānasa*, gives the following ingenious rule:

> The number of *karaṇas* elapsed since the beginning of the (current) fortnight diminished by two and then (the difference obtained) increased by one-seventh of itself, gives the measure of the *sita* if the fortnight is white or the *asita* if the fortnight is dark.[20]

That is:

$$\mathit{sita} = (K-2)\left(1+\frac{1}{7}\right)\ \mathit{a\dot{n}gulas},$$

where K denotes the number of *karaṇas* elapsed in the light fortnight, the diameter of the Moon being assumed to be 32 *aṅgulas*.

As the Moon is visible only when it is at the distance of 12 degrees from the Sun, i.e. when 2 *karaṇas* have just elapsed, so the proportion is made here with $180-12=168$ degrees, instead of 180 degrees. If M and S denote the longitudes of the Moon and the Sun in terms of degrees, the proportion implied is: "When $M-S-12$ degrees amount to 168 degrees, the measure of the *sita* is 32 *aṅgulas*, what will be the measure of the *sita* when $M-S-12$ degrees have the given value?" The result is:

$$\begin{aligned}\mathit{sita} &= \frac{(M-S-12)\times 32}{168}\\ &= \left(\frac{M-S}{6}-2\right)\left(1+\frac{1}{7}\right)\\ &= (K-2)\left(1+\frac{1}{7}\right)\ \mathit{a\dot{n}gulas}.\end{aligned}$$

Similar is the rule stated by Gaṇeśa Daivajña (1520):

> The number of *tithis* elapsed in the light fortnight diminished by one-fifth of itself gives the measure of the *sita*.[21]

[19] *Gaṇitayuktayaḥ*, Tract no. 8, p. 49.
[20] *LMā*, viii. 3.
[21] *GLā*, xii. 3d.

That is:

$$sita = \left(1 - \frac{1}{5}\right) T \ \textit{aṅgulas},$$

T being the number of *tithis* elapsed in the light fortnight and the Moon's diameter being assumed to be equal to 12 *aṅgulas*.

Gaṇeśa Daivajña has applied proportion with the *tithis* elapsed in the light fortnight. His proportion is: "When on the expiry of 15 *tithis* the *sita* amounts to 12 *aṅgulas*, what will it amount to on the expiry of T *tithis*?" The result is:

$$sita = \frac{12T}{15} = \left(1 - \frac{1}{5}\right) T \ \textit{aṅgulas}.$$

Both the above rules are approximate.

It will be noticed that Gaṇeśa Daivajña's formula is the same as the first result of Brahmagupta. The difference is in form only.

1.3 Graphical representation of the *sita*

The Indian astronomers have also stated rules to exhibit the *sita* graphically. It enabled them to know which of the two lunar horns was higher than the other at the time of the Moon's first visibility, the knowledge of which is of importance in natural astrology.

Bhāskara I and other early astronomers have exhibited the *sita* by projecting the Sun and Moon in the plane of the observer's meridian. They have first constructed a right-angled triangle MAS, in which S denotes the projection of the centre of the Sun, M the projection of the centre of the Moon, and MA the projection of the altitude-difference of the Sun and Moon, all in the plane of the observer's meridian. AS, the horizontal side of this triangle, is called the base; MA, the vertical side, the perpendicular or upright; and MS, the hypotenuse.

Describing how the construction of the *sita* is to be done at sunset in the first quarter of the lunar month, Bhāskara I says:

> Lay off the base from the Sun in its own direction. (Then) draw a perpendicular-line passing through the head and tail of the fish-figure constructed at the end (of the base). (This) perpendicular should be taken equal to the *R*sine of the Moon's altitude and should be laid off towards the east. The hypotenuse-line should (then) be drawn by joining the ends of that (perpendicular) and the base.
>
> The Moon is (then) constructed by taking the meeting point of the hypotenuse and the perpendicular as centre (and the semi-diameter of the Moon as radius); and along the hypotenuse (from

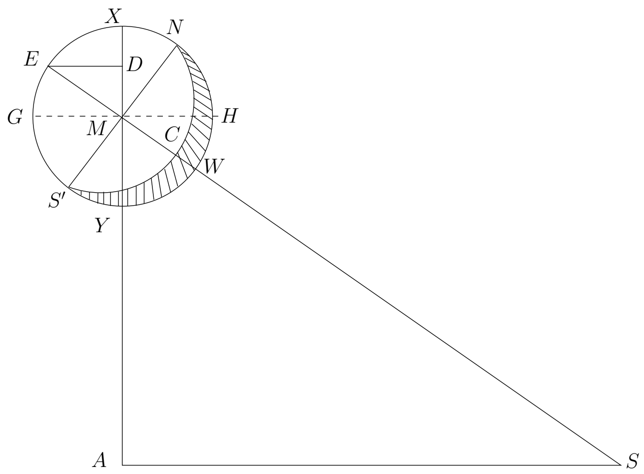

Figure 1

> the point where it intersects the periphery of the Moon) is laid off the *sita* towards the interior of the Moon.
>
> The hypotenuse (indicates) the east and west directions; the north and south directions should be determined by means of a fish-figure. (Thus are obtained the three points, viz.) the north point, the south point, and a third point obtained by laying off the *sita*. (Now) with the help of two fish-figures constructed by the method known as *triśarakarāvidhāna* draw the circle passing through the (above) three points. Thus is shown, by the elevation of the lunar horns which are illuminated by the light between two circles, the Moon which destroys the mound of darkness by her bundle of light.[22]

Figure 1 illustrates how the construction is made at sunset in the first quarter of the month. AS and M are the projections of the centres of the Sun and Moon in the plane of the observer's meridian, and MA the projection of the Moon's altitude in the same plane.

The triangle MAS is right-angled at A, SA is called the base, MA the perpendicular or upright, and MS the hypotenuse of this triangle. In the present case the base lies to the south of the Sun and the upright to the east of the base. The circle centred at M is the Moon's disc, i.e. the projection of the Moon's globe in the plane of the observer's meridian. The point W where MS intersects it is the west point of the Moon's disc. E, N, and S' are the

[22] *LBh*, vi. 12–17, also see *MBh*, vi. 13–17.

east, north, and south points. WC is the *sita* which has been laid off, in the present case, from the west point W towards the interior of the Moon's disc. NCS' is the circle drawn through N, C, and S'. The shaded portion of the Moon's disc between the circles NWS' and NCS' is the illuminated part of the Moon's disc; the remaining part of the Moon's disc does not receive light from the Sun and remains dark (*asita*) and invisible.

Let GH be drawn perpendicular to MA through M. Then the Moon's horn which is intersected by it lies to the north of the upright MA. This is the higher horn. The elevation of this higher horn is measured by the angle NMH. The other horn, viz. $CS'W$, which is not intersected by GH is the lower one. It lies to the south of the upright MA. So in the present case the northern horn is the higher one.

If the figure be held up with MA in the vertical position, the Moon in the sky will look like the shaded portion in the figure. This is what was intended.

The author of the *Sūryasiddhānta* has followed the method of Bhāskara I. Describing the construction of the *sita*, at sunrise in the last quarter of the month, he says:

> Set down a point and call it the Sun. From it lay off the base in its own direction. From the extremity of that lay off the upright towards the west. Next draw the hypotenuse by joining the extremity of the upright and the point assumed as the Sun. Taking the junction of the upright and the hypotenuse as centre and the semi-diameter of the Moon at that time as the radius, draw the Moon's disc. Now with the help of the hypotenuse (assumed as the east-west line), first determine the directions (relative to the Moon's centre). From the point where the hypotenuse intersects the Moon's disc lay off the *sita* towards the interior of the Moon's disc. Between the point at the extremity of the *sita* and the north and south points draw two fish-figures. From the point of intersection of the lines going through them (taken as centre) draw an arc of a circle passing through the three points. As the Moon looks (in the figure) between this arc and the eastern periphery of the Moon's disc, so it looks in the sky that day. If the directions are determined with the help of the upright, the horn which is intersected by the line drawn at right angles to the upright through the Moon's centre is the higher one. The shape of the Moon should be demonstrated by holding up the figure keeping the upright in a vertical position.[23]

The construction given by Lalla is more general. He says:

[23] *SūSi*, x. 10–14.

> Take a point on the level ground and assume it to be Sun. From this point lay off the base in its own direction (north or south). From the point reached lay off the upright. If the Moon is in the eastern hemisphere, the upright should be laid off towards the western direction; if the Moon is in the western hemisphere, it should be laid off towards the east. The hypotenuse should then be drawn by joining the extremity of the upright and the point assumed as the Sun. The Moon's disc should (then) be drawn by taking the junction of the hypotenuse and the upright as the centre. The hypotenuse-line here goes from west to east. The remaining (north and south) directions should be determined by means of a fish-figure. All this should be drawn very clearly with chalk. From the west point lay off the *sita* in the light fortnight or the *asita* in the dark fortnight (towards the interior of the Moon's disc). Taking the point thus reached, as also the north and south points (on the Moon's disc) as the centre draw two fish-figures. Where the mouth-tail lines of these fish-figures meet, taking that as the centre draw a neat circle passing through the *sita*-point inside the Moon's disc to exhibit the illuminated portion of the Moon. The direction in which the *aṅgulas* of the base have been laid off gives the direction of the depressed horn; the other horn is the elevated one.[24]

Āryabhaṭa II and Bhāskara II have omitted the construction of the triangle MAS. They have drawn the Moon directly with any point in the plane of the horizon as centre. Then they draw the direction-lines, i.e. the east-west and north-south lines. Assuming the north-south line as the same as the line XY of Figure 1, they lay off ED (drawn orthogonally to the upright) which they call *digvalana* ("direction-deflection"). Having thus obtained the point E they draw EW, the line joining the Sun and Moon. After this their procedure is the same as that of Bhāskara I. The *digvalana* ED is evidently equal to

$$\frac{SA \times \text{Moon's diameter}}{MS}$$

which follows from the comparison of the similar triangles MED and MAS. See Figure 1.

Brahmagupta does not project the Sun and Moon in the plane of the observer's meridian or any other plane. He keeps them where they are. So in the triangle MAS, which he constructs, M and S denote the actual positions of the centres of the Sun and Moon; AS is parallel to the north-south line of the horizon, and MA is perpendicular from M on this line. The Moon's disc

[24] *ŚiDVṛ*, ix. 15–19 (a–b).

is drawn in the plane of MAS with M as the centre. The laying off of the *sita* and the construction of the inner boundary of the *sita* is done as before.

Brahmagupta has been followed by Lalla and Śrīpati. Vaṭeśvara too follows Brahmagupta except in the case of sunset or sunrise where he follows Bhāskara I.

Bhāskara II has pointed out a fallacy in the method of Brahmagupta. He says:

> When the *sita* of the Moon is graphically shown in the way taught by him, using his base and hypotenuse, the lunar horns (shown in the figure) will not look like those seen in the sky. This is what I feel. Those proficient in astronomy should also observe it carefully. For, at a station in latitude 66°, the ecliptic coincides with the horizon and when the Sun is at the first point of Aries and the Moon at the first point of Capricorn the Moon appears vertically split into two halves by the observer's meridian and its eastern half looks bright. But this is not so in the opinion of Brahmagupta, for his base and upright are then equal to the radius. Actually, the tips of the lunar horns fall on a horizontal line when there is absence of the base, and on a vertical line when there is absence of the upright. Brahmagupta's base and upright then are both equal to the radius. Or, be as it may; I am not concerned. I bow to the great.[25]

Gaṇeśa Daivajña does not see any utility of the *parilekha* (graphical representation of the Moon). When from the direction of the (*dig*)*valana* itself, one can know which horn is high and which low, then, asks he, what is the use of the *parilekha*?[26]

1.4 The visible Moon

In the present problem, we are concerned with the actual Moon and not with its calculated position on the ecliptic. The Indian astronomers have found it convenient to use, in place of the actual Moon, that point of the ecliptic which rises or sets with the actual Moon. This point of the ecliptic is called "the visible Moon" (*dṛśya-candra*). This is derived from the calculated true Moon by applying to the latter a correction known as the visibility correction (*dṛkkarma* or *darśana-saṃskāra*). The early astronomers, from Āryabhaṭa I to Bhāskara II, have applied two visibility corrections, viz. the *ayana-dṛkkarma* and the *akṣa-dṛkkarma*. The former is the portion of the ecliptic that lies between the secondaries to the ecliptic and the equator going through the

[25] *SiŚi*, I, ix. 10–12.
[26] *GL*, xii. 4.

actual Moon, and the latter is the portion of the ecliptic that lies between the horizon and the secondary to the equator going through the actual Moon, the actual Moon being supposed to be on the horizon.

Āryabhaṭa I gives the following rule for deriving the above mentioned visibility corrections:

> Multiply the *R*versed-sine of the Moon's (tropical) longitude (as increased by three signs) by the Moon's latitude and also by the (*R*sine of the Sun's) greatest declination and divide (the resulting product) by the square of the radius: (the result is the *ayana-dṛkkarma* for the Moon). When the Moon's latitude is north, it should be subtracted from or added to the Moon's longitude, according as the Moon's *ayana* is north or south (i.e. according as the Moon is in the six signs beginning with tropical sign Capricorn or in the six signs beginning with the tropical sign Cancer); When the Moon's latitude is south, it should be added or subtracted (respectively).[27]

> Multiply the *R*sine of the latitude of the local place by the Moon's latitude and divide (the resulting product) by the *R*sine of the colatitude: (the result is the *akṣa-dṛkkarma* for the Moon). When the Moon is to the north (of the ecliptic), it should be subtracted from the Moon's longitude (as corrected for the *ayana-dṛkkarma*) in the case of the rising of the Moon and added to the Moon's longitude in the case of the setting of the Moon; when the Moon is to the south (of the ecliptic), it should be added to the Moon's longitude (in the case of the rising of the Moon) and subtracted from the Moon's longitude (in the case of the setting of the Moon).[28]

If β be the Moon's latitude and M the Moon's tropical longitude, then the above rules are equivalent to the following formulae:

$$\textit{ayana-dṛkkarma} = \frac{R\,\mathrm{versin}(M+90^\circ)\times\beta\times R\sin 24^\circ}{R\times R} \tag{2}$$

$$\text{and}\quad \textit{akṣa-dṛkkarma} = \frac{R\sin\phi\times\beta}{R\cos\phi}, \tag{3}$$

ϕ being the latitude of the place and 24° being the Indian value of the Sun's greatest declination.

The same formulae occur in the works of Lalla,[29] Vaṭeśvara,[30] and Śrīpati.[31]

[27] *Ā*, iv. 36.
[28] *Ā*, iv. 35.
[29] *ŚiDVṛ*, viii. 3 (a–b).
[30] *VSi*, VII, i. 9.
[31] *SiŚe*, ix. 4.

These formulae are approximate and were modified by the later astronomers. Brahmagupta[32] replaced (2) by the better formula:

$$\textit{ayana-dṛkkarma} = \frac{R\sin\,(M+90^\circ)\times\beta\times R\sin 24^\circ}{R\times R}.$$

This formula reappears in the *Mahāsiddhānta*[33] of Āryabhaṭa II in the form

$$\textit{ayana-dṛkkarma} = \frac{R\cos M\,\times\beta\times R\sin 24^\circ}{R\times R}.$$

Śrīpati, while retaining the use of the *R*versed-sine, has improved (2) by multiplying it by 1800 and dividing by the *asus* of the rising of the sign occupied by the Moon.[34] (The *asus* are the minutes of arc of the equator).

Bhāskara II has criticised the use of the *R*versed-sine and has applauded Brahmagupta for replacing the *R*versed-sine by the *R*sine. He has also given the following new formulae:[35]

(i) *ayana-dṛkkarma*

$$= \frac{R\sin(\textit{ayanavalana})\times\beta}{R\cos\delta}\times\frac{1800}{T}$$

where the *ayanavalana* is the angle between the secondaries to the equator and the ecliptic going through the Moon, δ the Moon's declination, and T the time of rising (in *asus*) of the sign occupied by the Moon.

(ii) *ayana-dṛkkarma*

$$= \frac{R\sin(\textit{ayanavalana})\times\beta}{R\cos(\textit{ayanavalana})}.$$

Formula (3) was modified by Bhāskara II. For his modified formulae the reader is referred to his *Siddhāntaśiromaṇi* (Part I, ch. vii, vss. 3, 6–8 and Part II, ch. ix, vs. 10).

1.5 Altitude of Sun and Moon

To determine the Sun's altitude for the given time one has to know the Sun's ascensional difference and the earth-sine. The Sun's ascensional difference is the difference between the times of rising of the Sun at the equator and at the local place. It is measured by the *asus* (minutes of equator) lying between the

[32] *BrSpSi*, vi. 3; also xxi. 66; x. 17.
[33] vii. 2, 3.
[34] *SiŚe*, ix. 6.
[35] *SiŚi*, I, vii. 4; vii. 5 (a–b).

hour circle through the east point (called the six o'clock circle) and the hour circle through the rising Sun. The formula used to obtain is:

$$R \sin c = \frac{R \sin \phi \times R \sin \delta \times R}{R \cos \phi \times R \cos \delta}$$

or, in modern notation,

$$\sin c = \tan \phi \tan \delta,$$

where c denotes the ascensional difference, δ the declination, and ϕ the latitude of the place.

The Sun's declination is obtained by the formula:

$$R \sin \delta = \frac{R \sin \lambda \times R \sin 24^\circ}{R}$$

where λ is the Sun's tropical longitude and 24° the Indian value of the obliquity of the ecliptic.

The earth-sine is the *R*sine of c reduced to the radius of the diurnal circle and is obtained by the formula:

$$\text{earth-sine} = \frac{R \sin c \, R \cos \delta}{R} = \frac{R \sin \phi \, R \sin \delta}{R \cos \phi}.$$

The Sun's ascensional difference and the earth-sine being thus known, the Sun's altitude can be determined. Bhāskara I gives the following rule to find the Sun's altitude when the time elapsed since sunrise in the forenoon or to elapse before sunset in the afternoon is known:

> The *ghaṭīs* elapsed (since sunrise) or those to elapse (before sunset), in the first half and the other half of the day (respectively), should be multiplied by 60 and again by 6: then they (i.e. those *ghaṭīs*) are reduced to *asus*. (When the Sun is) in the northern hemisphere, the *asus* of the Sun's ascensional difference should be subtracted from them and (when the Sun is) in the southern hemisphere, the *asus* of the Sun's ascensional difference should be added to them. (Then) calculate the *R*sine (of the resulting difference or sum) and multiply that by the day-radius (i.e. by $R \cos \delta$). Then dividing that (product) by the radius, operate (on the quotient) with the earth-sine contrarily to the above (i.e. add or subtract the earth-sine according as the Sun is in the northern or southern hemisphere). Multiply that (sum or difference) by the *R*sine of the co-latitude and divide by the radius: the result is the *R*sine of the Sun's altitude.[36]

[36] *LBh*, iii. 7–9.

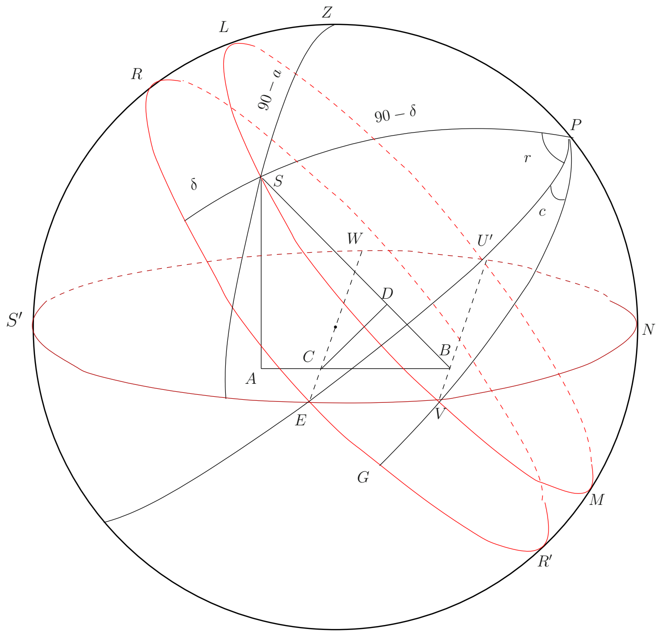

Figure 2

> When the Sun's ascensional difference cannot be subtracted from the given (time reduced to) *asus*, reverse the subtraction (i.e. subtract the latter from the former) and with the *R*sine of the remainder (proceed as above). In the night the *R*sine of the Sun's altitude should be obtained contrarily (i.e. by reversing the laws of addition and subtraction).[37]

That is, when the Sun is in the northern hemisphere,

$$R\sin a = \frac{\Big[\{R\sin(T \overset{\sim}{+} c) R\cos\delta\}/R \ \overset{+}{\sim}\ \text{earth-sine}\Big]\, R\cos\phi}{R},$$

where a denotes the Sun's altitude, δ the Sun's declination, T the time elapsed since sunrise in the forenoon or to elapse before sunset in the afternoon (reduced to *asus*), c the Sun's ascensional difference (in *asus*), and ϕ the local latitude, the sign $+$ or $\sim$ being chosen properly, depending on the Sun's position.

In Figure 2, which represents the celestial sphere for a place in latitude ϕ, $S'ENW$ is the horizon, E, W, N, and S' being the east, west, north, and

[37] *LBh*, ii. 11.

south points. Z is the zenith. RER' is the equator and P its north pole. S is the Sun and LSM its diurnal circle. VU' is the Sun's rising-setting line and EW the east-west line. SA is the perpendicular from the Sun on the plane of the horizon and SB on the rising-setting line; AB is the perpendicular to the rising-setting line. C is the point where AB intersects EW. SA is the Rsine of the Sun's altitude, AB is the Sun's *śaṅkutala*, CB the Sun's *agrā*, AC the Sun's *bhuja*, and SB the Sun's *iṣṭahṛti*.

It can be easily seen that

$$SB = \frac{R\sin(T-c)R\cos\delta}{R} + \text{earth-sine},$$

so that from the triangle SAB, right-angled at A, in which $\angle SBA = 90° - \phi$, we easily have

$$\begin{aligned} SA \text{ or } R\sin a &= \frac{SB \times R\cos\phi}{R} \\ &= \frac{[\{R\sin(T-c)R\cos\delta\}/R + \text{earth-sine}]R\cos\phi}{R}. \end{aligned}$$

Using modern spherical trigonometry, the rationale of this rule is as follows:

In Figure 2, $\angle VPS = T$ and $\angle VPE = c$, so that $\angle EPS = T - c$, and likewise $\angle ZPS = 90° - (T - c)$. Now in the spherical triangle ZPS, we have $ZS = 90° - a$, $ZP = 90° - \phi$, $SP = 90° - \delta$, and $ZPS = 90° - (T - c)$. Therefore, using cosine formula, we have

$$\begin{aligned} \cos ZS &= \cos ZP \cos SP + \sin ZP \sin SP \cos\angle ZPS \\ \text{or} \qquad \sin a &= \sin\phi\sin\delta + \cos\phi\cos\delta\sin(T-c), \end{aligned}$$

and multiplying by R and rearranging,

$$\begin{aligned} R\sin a &= \frac{[\{R\sin(T-c)R\cos\delta\}/R + R\sin\phi R\sin\delta/R\cos\phi]R\cos\phi}{R} \\ &= \frac{[\{R\sin(T-c)R\cos\delta\}/R + \text{earth-sine}]R\cos\phi}{R}. \end{aligned}$$

This is true when the Sun is in the northern hemisphere and above the horizon. Similar rationale may be given when the Sun is in the southern hemisphere or below the horizon.

The Moon's altitude is obtained in the same way. But in this case one has to use the Moon's true declination, i.e. the declination of the actual Moon. For this, in the present context, the early Indian astronomers, from Āryabhaṭa I to Bhāskara II, use the following approximate formula:

$$\text{Moon's true declination} = \delta \overset{+}{\sim} \beta,$$

where δ is the declination of the Moon's projection on the ecliptic and β the Moon's latitude.

In place of the time elapsed since sunrise or to elapse before sunset, one has to use the time elapsed since moon-rise or to elapse before moon-set. The methods used to find the time of moon-rise or moon-set will be described in the next chapter. (**ed.** See Section 2.)

1.6 Base and upright

The base SA of the triangle MAS (in Figure 1) is equal to the difference or sum of the Sun's *bhuja* and the Moon's *bhuja*. In case of the Sun and the Moon are both above the horizon, the difference is taken provided the Sun and Moon are on the same side of the east-west line; otherwise the sum is taken. The *bhuja* of a heavenly body is defined by the distance of its projection on the plane of the horizon from the east-west line, so that

$$\begin{aligned} \textit{bhuja} &= \text{distance of projection from the east-west line} \\ &= \text{distance of projection from the rising-setting line (called \textit{śaṅkutala}} \\ &\qquad \text{or \textit{śaṅkvagra})} \overset{+}{\sim} \text{distance between the east-west and} \\ &\qquad \text{rising-setting lines (called \textit{agrā})} \\ &= \textit{śaṅkutala} \overset{+}{\sim} \textit{agrā}. \end{aligned}$$

In Figure 2, S is the Sun. A is the Sun's projection on the plane of the horizon, AB is the Sun's *śaṅkutala*, CB is the Sun's *agrā*, and AC the Sun's *bhuja*. It is evident from the figure that in this case

$$\begin{aligned} AC &= AB - CB \\ \text{i.e.}\quad \text{Sun's } \textit{bhuja} &= \text{Sun's } \textit{śaṅkutala} - \text{Sun's } \textit{agrā}. \end{aligned}$$

The Sun's *śaṅkutala* AB is obtained from triangle SAB (of Figure 2) by using the sine relation:

$$\begin{aligned} AB \text{ or Sun's } \textit{śaṅkutala} &= \frac{SA \times R\sin\angle ASB}{R\sin\angle SBA} \\ &= \frac{R\sin a \times R\sin\phi}{R\cos\phi}. \end{aligned}$$

Bhāskara I says:

> The *R*sine of the Sun's altitude multiplied by the *R*sine of the latitude and divided by the *R* cosine of the latitude is the (Sun's) *śaṅkvagra*, which is always to the south of the rising-setting line.[38]

[38] *LBh*, iii. 16.

The Sun's *agrā* is obtained thus: In Figure 2, let CD be the perpendicular from C to SB. Then in the triangle CDB, right-angled at D, $CB =$ Sun's *agrā*, $CD = R\sin\delta$, $\angle CBD = 90° - \phi$, so that

$$CB \text{ or Sun's } agr\bar{a} = \frac{AD \times R\sin\angle CDB}{R\sin\angle CBD} = \frac{R\sin\delta \times R}{R\cos\phi}.$$

Brahmagupta says:

> The *R*sine of the declination multiplied by the radius and divided by the *R*sine of the co-altitude is the *agrā* which lies east-west in the plane of the horizon.[39]

The Moon's *bhuja* is obtained in the same way, using the Moon's true declination. The difference or sum of the Moon's *bhuja* and Sun's *bhuja* finally gives the base.

When the calculations are made for sunset, the Sun's *agrā* itself is the Sun's *bhuja*. In that case, the difference or sum of the Moon's *bhuja* and the Sun's *agrā* gives the base.

Bhāskara I says:

> From the *asus* intervening between the Sun and Moon (corrected for the visibility corrections) and from the Moon's earth-sine and ascensional difference, determine the *R*sine of the (Moon's) altitude; and from that find the (Moon's) *śaṅkvagra*, which is always south (of the rising-setting line of the Moon).
>
> The *R*sine of the difference or sum of the (Moon's) latitude and declination, according as they are of unlike or like directions is (the *R*sine of) the Moon's true declination. From that (*R*sine of the Moon's true declination) determine her day-radius, etc. Then multiply (the *R*sine of) the Moon's (true) declination by the radius and divide by (the *R*sine of) the co-latitude: then is obtained (the *R*sine of) the Moon's *agrā*.
>
> If that (*R*sine of the Moon's *agrā*) is of the same direction as the (Moon's) *śaṅkvagra*, take their sum; otherwise, take their difference. Thereafter take the difference of (the *R*sine of) the Sun's *agrā* and that (sum or difference), if their directions are the same; otherwise, take their sum: thus is obtained the base (*bāhu* or *bhuja*).[40]

[39] *BrSpSi*, iii. 64 (a–b).
[40] *LBh*, vi. 8–12 (a–b).

The difference of the *R*sines of the Moon's altitude and the Sun's altitude during the day or their sum during the night, obviously, gives the upright. When the calculations are made for sunset, the *R*sine of the Moon's altitude itself is the upright, as the Sun then is on the horizon and its altitude is zero.

The base and upright obtained in the above way are according to those astronomers who, like Bhāskara I, project the Sun and Moon in the plane of the meridian. Brahmagupta and his followers, who keep the Sun and Moon where they are, obtain their base and upright, which shall be called Brahmagupta's base and upright, thus:

$$\text{Brahmagupta's base} = b \overset{+}{\sim} b'$$

$$\text{Brahmagupta's upright} = \sqrt{(k \overset{+}{\sim} k')^2 + (R\sin a \overset{+}{\sim} R\sin a')^2},$$

where b, b'; k, k'; a, a' are the *bhujas*, uprights, and altitudes of the Sun and Moon respectively, derived in the manner described above.

It will be noted that whereas Brahmagupta's upright differs in length from that of Bhāskara I, his base is exactly equal to that of the latter.

2 Rising and setting of planets and stars

2.1 Heliacal rising and setting of the planets

When a planet gets near the Sun, it is lost in the dazzling light of the Sun and becomes invisible. The planet is then said to set heliacally. Sometimes later the planet comes out of the dazzling light and is seen again. It is then said to rise heliacally. In the case of the Moon, a special term *candra-darśana* ("Moon's first appearance") is used for its heliacal rising.

Brahmagupta says:

> A planet with lesser longitude than the Sun rises in the east, in case it is slower than the Sun; in the contrary case, it sets in the east. A planet with greater longitude than the Sun rises in the west, in case it is faster than the Sun; and sets in the west, in case it is slower than the Sun.[41]

The author of the *Sūryasiddhānta* says:

> Jupiter, Mars, and Saturn, when their longitude is greater than that of the Sun, go to their setting in the west; when it is lesser, to their rising in the east; so likewise Venus and Mercury, when retrograding. The Moon, Mercury, and Venus, having a swifter

[41] *BrSpSi*, vi. 2.

> motion, go to their setting in the east when of lesser longitude than the Sun; when of greater, go to their rising in the west.[42]

Vaṭeśvara's account is fuller and more explicit:

> A planet with lesser longitude (than the Sun) rises in the east if it is slower than the Sun; and sets in the east if it is faster than the Sun; whereas a planet with greater longitude (than the Sun) rises in the west if it is faster than the Sun, and sets in the west if it is slower than the Sun.
>
> The Moon, Venus, and Mercury rise in the west, whereas Saturn, Mars, and Jupiter and also retrograding Mercury and Venus rise in the east. These planets set in the opposite direction.[43]

Similar statements have been made by Lalla,[44] Āryabhaṭa II,[45] Śrīpati,[46] Bhāskara II[47] and others.

The distances from the Sun at which the heliacal rising or setting occurs is not the same for all the planets. It depends upon the size and luminosity of the planet. The larger or more luminous it is, the lesser will be its distance from the Sun at the time of its rising or setting.

The Indian astronomers state the distances of the planets from the Sun at the time of their first visibility ("rising") or last visibility ("setting") in terms of time-degrees i.e. in terms of time, between the time of rising or setting of the planet and that of the Sun, converted into degrees by the formula:

$$60 \textit{ ghaṭīs} \text{ or } 24 \text{ hours} = 360 \text{ degrees.}$$

Āryabhaṭa I says:

> When the Moon has no latitude it is visible when situated at a distance of 12 degrees (of time) from the Sun. Venus is visible when 9 degrees (of time) distant from the Sun. The other planets taken in the order of decreasing sizes (viz. Jupiter, Mercury, Saturn, and Mars) are visible when they are 9 degrees (of time) increased by two-s (i.e. when they are 11, 13, 15, and 17 degrees of time) distant from the Sun.[48]

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[42] *SūSi*, ix. 2–3.
[43] *VSi*, VI, 1–2.
[44] *ŚiDVṛ*, viii. I.
[45] *MSi*, ix. 1–2.
[46] *SiŚe*, ix. 2–3.
[47] *SiŚi*, I, viii. 4 (c–d)-5.
[48] *Ā*, iv. 4.

The same distances have been given in the *Āryabhaṭa-siddhānta* and the *Khaṇḍakhādyaka*,[49] and by Brahmagupta,[50] Lalla,[51] Vaṭeśvara,[52] and Śrīpati.[53] Those given by Āryabhaṭa II,[54] Bhāskara II,[55] and by the author of the *Sūryasiddhānta*[56] slightly differ in one or two cases.

Regarding Venus and Mercury, Brahmagupta says:

> Owing to its small disc, Venus (in direct motion) rises in the west and sets in the east at a distance of 10 time-degrees (from the Sun); and owing to its large disc, the same planet (in retrograde motion) sets in the west and rises in the east at a distance of (only) 8 times-degrees (from the Sun). Mercury rises and sets in a similar manner when its distance (from the Sun) is 14 time-degrees (in the case of direct motion) or 12 time-degrees (in the case of retrograde motion).[57]

So has been said by the author of the *Sūryasiddhānta*[58] and Śrīpati.[59]

To find the day on which a planet is to rise or set heliacally in the east or west, the Indian astronomers proceed as follows: In the case of rising or setting in the east, they first calculate for sunrise the longitudes of the Sun and the planet, the latter being corrected by the visibility corrections for rising. Then, using the table giving the times of rising of the signs for the local place, they calculate the time of rising of the portion of the ecliptic lying between the Sun and the corrected planet. This they convert into time-degrees, and then find the difference between these time-degrees and the time-degrees for rising or setting of the planet under consideration. If the planet is in direct motion they divide this difference by the degrees of difference between the daily motions of the Sun and the planet; and if the planet is in retrograde motion they divide that difference by the degrees of the sum of the daily motions of the Sun and the planet. The quotient obtained gives the days elapsed since or to elapse before the rising or setting of the planet in the east.

In the case of rising or setting in the west, they first calculate for sunset the longitudes of the Sun and the planet, the latter corrected by the visibility corrections for setting. Both these longitudes are increased by six signs. Then,

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[49] Part I, vi. 1.
[50] *BrSpSi*, vi. 6, 11, 12.
[51] *ŚiDvr*, viii. 5.
[52] *VSi*, VI, 3.
[53] *SiŚe*, ix. 8–3.
[54] *MSi*, ix. 3.
[55] *SiŚi*, I, viii. 6.
[56] x. 1; ix. 6–8.
[57] *BrSpSi*, vi. 11, 12; *KK*, II, v. 3–4.
[58] ix. 7.
[59] *SiŚe*, ix. 9.

Table 1: Time-degrees for the heliacal rising and setting.

Celestial body	Time-degrees according to Āryabhaṭa I	Brahmagupta	Lalla and Vaṭeśvara	Āryabhaṭa II	*Sūryasiddhānta* and Bhāskara II
Moon	12°	12°	12°	12°	12°
Mars	17°	17°	17°	17°	17°
Mercury	13°	13° (mean), 14°	13°	13°	14°
Mercury (retro)		12°	12°	12°30′	12°
Jupiter	11°	11°	11°	12°	11°
Venus	9°	9° (mean), 10°	9°	8°	10°
Venus (retro)		8°	8°	7°30′	8°
Saturn	15°	15°	15°	15°	15°

using the table giving the times of rising of the signs for the local place, they calculate the time of rising of the portion of the ecliptic lying between the Sun as increased by six signs and the corrected planet as increased by six signs. This they convert into time-degrees, and find the difference between these time-degrees and the time-degrees for rising or setting of the planet under consideration. If the planet is in direct motion they divide this difference by the degrees of difference between the daily motions of the Sun and the planet; and if the planet is in retrograde motion they divide that difference by the degrees of the sum of the daily motions of the Sun and the planet. The quotient obtained gives, as before, the days elapsed since or to elapse before the rising or setting of the planet in the west.

Bhāskara I says:

> (When the planet is to be seen) in the east, (its) visibility should be announced by calculating the time (of rising of the part of the ecliptic between the Sun and the planet corrected by the visibility corrections) by using the time of rising at the local place of that very sign (in which the Sun and the planet are situated); (when the planet is to be seen) in the west, (its) visibility should be announced by calculating the time (of setting of the part of the ecliptic between the Sun and the planet) by using the time of rising

of the seventh sign at the local place.[60]

Lalla describes the method as follows:

> (If the heliacal rising or setting of a planet) on the western horizon is considered, the true longitude of the Sun and the *dṛggraha* (i.e. the planet corrected by the visibility corrections for setting) should each be increased by six signs.
>
> Find the *asus* of rising of the untraversed part of the sign occupied by the planet with lesser longitude and the *asus* of rising of the traversed part of the sign occupied by the planet with greater longitude. To the sum of the two add the *asus* of rising of the intervening signs. The result divided by 60 gives the time-degrees of the planet's distance from the Sun. If these time-degrees are lesser than the time-degrees stated for the rising or setting of the planet, it must be understood that the planet is invisible.
>
> Find the difference expressed in minutes between the calculated time-degrees and the time-degrees for rising or setting of the planet. Divide it by (the minutes of) the difference of the daily motions of the Sun and the planet if they are moving in the same direction, or by (the minutes of) the sum if they are moving in opposite directions. The quotient gives the days elapsed since or to elapse before the rising or setting of the planet, which is to be understood by the following consideration.
>
> When the setting of a planet is considered, if the time-degrees for rising or setting of the planet are greater than the calculated time-degrees, know then that the planet has set heliacally before the number of days found above; if the former is lesser, the planet will set after so many days. When the rising is considered, then in the former case the planet will rise after the days calculated and in the latter case the planet has risen before the days calculated.[61]

Vaṭeśvara explains the method thus:

> In the case of rising in the east, find the *asus* of rising of the untraversed part of the sign occupied by the planet (computed for sunrise and corrected by the visibility corrections for rising), those of the traversed part of the sign occupied by the Sun (at sunrise); and in the case of setting in the west, in the reverse order.[62] Add them to the *asus* of rising of the intervening signs. (Then are

[60] *LBh.* vii. 3.
[61] *ŚiDVṛ*, viii. 5(c–d)–8.
[62] What is meant is that, in the case of rising or setting in the west, one should compute

obtained the *asus* of rising of the part of the ecliptic lying between the planet corrected by the visibility corrections and the Sun at sunrise. These divided by 60 give the time-degrees between the planet corrected by the visibility corrections and the Sun).

(To obtain the time-degrees corresponding to the traversed and untraversed parts), one should multiply the untraversed and traversed degrees by the *asus* of rising of the corresponding signs and divide (the products) by 30 and 60 (i.e. by 1800). The time-degrees divided by 6 give the corresponding *ghaṭīs*.

When the time-degrees between the planet corrected by the visibility corrections and the Sun are greater than the time-degrees for the planet's rising or setting, it should be understood that the planet has already risen (heliacally); if lesser, the rising has not yet taken place.

Divide their difference by the daily motion of the Sun diminished by the daily motion of the planet when the planet is in direct motion, and by the daily motion of the Sun increased by the daily motion of the planet when the planet is in retrograde motion: the result is the time (in days) which have to elapse before the planet will rise or set or elapsed since the rising or setting of the planet.[63]

Similar rules have been given by the other Indian astronomers.[64]

2.2 Heliacal rising and setting of the stars

The stars having no motion rise in the east and set in the west. The distance from the Sun at which they rise or set heliacally, according to the Indian astronomers, is 14 time-degrees or $6\frac{1}{3}$ *ghaṭīs*. In the case of Canopus (*Agastya*) this distance is 12 time-degrees or 2 *ghaṭīs* and in the case of Sirius (*Mṛgavyādha*) it is 13 time-degrees or $2\frac{1}{6}$ *ghaṭīs*. This is so because Canopus and Sirius are bright stars.

The point of the ecliptic which rises on the eastern horizon exactly when a star rises on the eastern horizon, is called the star's *udayalagna*; and the point of the ecliptic which rises on the eastern horizon exactly when a star sets on the western horizon, is called the star's *astalagna*. Similarly, the point

for sunset the longitude of the planet and apply to it the visibility correction of setting and also the longitude of the Sun. Both of these should be increased by six signs. One should then find the *asus* of rising of the traversed part of the sign occupied by the resulting planet as also the *asus* of rising of the untraversed part of the sign occupied by the resulting Sun.

[63] *VSi*, VI. 23(b–d)–25, 27.

[64] See e.g. *BrSpSi*, x. 32; vi. 7; x. 33; *SūSi*, ix. 16; *MSi*, ix. 4–5; *SiŚe*, ix. 12–13; 10; *SiŚi*, I, viii. 8(c–d)–10.

of the ecliptic occupied by the Sun when a star rises heliacally is called the star's *udayārka* or *udayasūrya*; and the point of the ecliptic occupied by the Sun when a star sets heliacally is called the star's *astārka* or *astasūrya*.

The positions of the stars are given in terms of their polar longitudes. So only one visibility correction, viz. the *akṣa-dṛkkarma*, has to be applied to them. When the *akṣa-dṛkkarma* for rising is applied to the polar longitude of a star, one gets the star's *udayalagna*; and when the *akṣa-dṛkkarma* for setting is applied to the polar longitude of a star and six signs are added to that, one gets the star's *astalagna*.

The *udayārka* for a star is obtained by calculating the *lagna* (the rising point of the ecliptic), by taking the Sun's longitude as equal to the star's *udayalagna* and the time elapsed since sunrise as equal to the *ghaṭīs* of the star's distance from the Sun at the time of its heliacal rising. The *astārka* for a star is obtained by calculating the *lagna*, by taking the Sun's longitude as equal to the star's *astalagna* and the time to elapse before sunrise as equal to the *ghaṭīs* of the star's distance from the Sun at the time of its heliacal visibility, and adding six signs to that.

Taking the case of Canopus and Sirius, Brahmagupta says:

> From the *udayalagna* of Canopus calculate the *lagna* at two *ghaṭīs* after sunrise by means of the times of rising of the signs (at the local place). The result is the *udayasūrya* of Canopus. Again from the *astalagna* (of Canopus) calculate the *lagna* at two *ghaṭīs* before sunrise, and add six signs to it. The result is the *astasūrya* of Canopus.
>
> In the same manner the *udayasūrya* and *astasūrya* of Sirius may be found. In this case $2\frac{1}{6}$ *ghaṭīs* should be used.
>
> Similarly, the *udayasūrya* and *astasūrya* of other stars should be calculated. In this case $2\frac{1}{3}$ *ghaṭīs* should be used.
>
> Canopus, Sirius, or any of the (other) stars rises or sets according as its *udayasūrya* or *astasūrya* is the same as the true Sun.[65]

Lalla says:

> On account of the motion of the provector wind, the rising of a star occurs with the rising of its *udayalagna*, and the setting of a star occurs with the rising of its *astalagna*.
>
> Two *ghaṭīs* plus one-third of a *ghaṭī* is the time-distance of a star from the Sun at the time of its heliacal rising or setting; that for Sirius, it is two *ghaṭīs* plus one-sixth of a *ghaṭī*; and that for Canopus, it is two *ghaṭīs*.

[65] *KK*, II, v. 8–10.

> The star whose *udayalagna* increased by the result due to that time-distance (i.e. by the arc of the ecliptic that rises in that time) happens to be equal to the Sun's longitude (at that time), rises heliacally; and the star whose *astalagna* diminished by the result due to that time-distance and also by six signs, happens to be equal to the Sun's longitude (at that time), sets heliacally.[66]

So also says Vaṭeśvara:

> When the longitude of the Sun is equal to the longitude of the star's *udayalagna* as increased by the result obtained on converting the 14 time-degrees for the star's heliacal rising or setting into the corresponding arc of the ecliptic (which rises at the local place in that time), the star rises heliacally; and when the longitude of the Sun is equal to the longitude of the *astalagna* as diminished by the result due to the time-degrees for the star's heliacal rising or setting and by half a circle, the star sets heliacally.
>
> When the star's *udayalagna* or *astalagna* is at a lesser distance from the Sun, the star is invisible; in the contrary case, the star is visible.[67]

A similar statement has been made by Bhāskara II.[68]

As regards the duration of a star's visibility or invisibility, Brahmagupta says:

> A star is visible as long as the Sun lies between its *udayasūrya* and *astasūrya*; otherwise, it is invisible.
>
> Find the difference between the star's *udayasūrya* and the Sun; or between the *astasūrya* and the Sun. Express the difference in minutes. Divide each difference by the daily motion of the Sun. The result gives respectively the number of days passed since the heliacal rising of the star and those which will pass before the star sets heliacally.[69]

Lalla says:

> As long as the Sun is between the star's *udayārka* and *astārka*, so long is the Sun heliacally visible, provided that the star's declination diminished or increased by the local latitude according as they are of like or unlike directions, is less than 90°.

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[66] *ŚiDVṛ*, xi. 16–17. Also see *SiŚe*, xii. 16–17.
[67] *VSi*, VIII, 19(c–d)–20.
[68] *SiŚi*, I, xi. 12–14.
[69] *BrSpSi*, x. 39; *KK*, II, v. 12.

> As long as the Sun is between the star's *astasūrya* and *udayasūrya*, so long is the star heliacally invisible.
>
> The difference between the two (i.e. the star's *udayasūrya* minus the star's *astasūrya*) expressed in minutes, when divided by the true daily motion of the Sun, gives the day (for which the star is invisible).[70]

Śrīpati says:

> As long as the Sun is between the star's *udayasūrya* and the star's *astasūrya*, so long is the star heliacally visible, and as long as the Sun is between the star's *astasūrya* and the star's *udayasūrya*, so long is the star invisible. The star, however, is seen as long as its zenith distance is less than 90°.[71]

Vaṭeśvara:

> Subtract the star's *astārka* from the star's *udayārka* and reduce the difference to minutes. Divide these minutes by the minutes of the Sun's daily motion.
>
> Then is obtained the number of days during which the star remains set heliacally.[72]

2.3 Stars always visible heliacally

The stars which are far away from the ecliptic do not fall prey to the dazzling light of the Sun. Such stars are always visible heliacally. The author of the *Sūryasiddhānta* says:

> Vega (*Abhijit*), Capella (*Brahmahṛdaya*), Arcturus (*Svātī*), α Aquilae (*Śravaṇa*), β Delphini (*Śraviṣṭhā*), and λ Pegasi (*Uttara-Bhādrapada*), owing to their (far) northern situation, are not extinguished by the Sun's rays.[73]

These stars have large latitudes and in their case the *astasūrya* exceeds the *udayasūrya*. The latter is the condition for a star's permanent heliacal visibility.

Brahmagupta says:

> The star whose *udayārka* is smaller than its *astārka* is always visible.[74]

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[70] *ŚiDVṛ*, xi. 18–19.
[71] *SiŚe*, xii. 18.
[72] *VSi*, VIII, ii. 21 (a–b).
[73] *SūSi*, ix. 18.
[74] *BrSpSi*, x. 38 (c–d); *KK*, II, v. 11 (c–d).

Lalla says:

> The star, whose *astārka* is greater than its *udayārka*, never sets heliacally.[75]

So also say the other Indian astronomers.[76]

2.4 Diurnal rising and setting

The rising of the heavenly bodies every day on the eastern horizon is called the diurnal rising of those heavenly bodies. Similarly the setting of the heavenly bodies on the western horizon is called their diurnal setting. It is this rising or setting that is meant when one talks of sunrise or sunset, moon-rise or moon-set.

The rising of the Sun does not present any difficulty, because it is taken as the starting point of time measurement. The rising and setting of the Moon are indeed of importance to the Indian astronomers. All astronomical works deal with them and give rules to find the time of moon-set or moon-rise in the light and dark fortnights of the month.

Bhāskara I gives the following rules to find the time of moon-set or moon-rise:

> In the light fortnight, find out the *asus* due to oblique ascension (of the part of the ecliptic) intervening between the Sun (at sunset) and the (visible) Moon (at sunset treated as moon-set) both increased by six signs and apply the method of successive approximations. This gives the duration of the visibility of the Moon (at night) (or, in other words, the time of moon-set).[77]
>
> Thereafter (i.e. in the dark fortnight) the Moon is seen (to rise) at night (at the time) determined by the *asus* (due to oblique ascension) derived by the method of successive approximations from the part of the ecliptic intervening between the Sun as increased by six signs and the (visible) Moon as obtained by computation, (the Sun and the Moon both being those calculated for sunset).[78]

Further he says:

> (In the light half of the month) when the measure of the day exceeds the *nāḍīs* (due to the oblique ascension of the part of the ecliptic) lying between the Sun and the (visible) Moon (computed

[75] *ŚiDVṛ*, xi. 20 (a–b).
[76] *VSi*, VIII, ii. 21 (c-d); *SiŚe*, xi. 21 (a-b); *SiŚi*, I, xi. 15.
[77] *MBh*, vi. 27.
[78] *MBh*, vi. 28.

> for sunset), the moon-rise is said to occur in the day when the residue of the day (i.e. the time to elapse before sunset) is equal to the *ghaṭīs* of their difference.[79]
>
> (In the dark half of the month) find out the *asus* due to the oblique ascension of the part of the ecliptic lying from the setting Sun up to the (visible) Moon; and therefrom subtract the length of the day. (This approximately gives the time of moon-rise as measured since sunset). Since the Moon is seen (to rise) at night when so much time, corrected by the method of successive approximations, is elapsed, therefore the *asus* obtained above should be operated upon by the method of successive approximations.[80]
>
> Or, determine the *asus* (due to the oblique ascension of the part of the ecliptic lying) from the (visible) Moon at sunrise up to the rising Sun; then subtract the corresponding displacements (of the Moon and the Sun) from them (i.e. from the longitudes of the visible Moon and the Sun computed for sunrise); and on them apply the method of successive approximations (to obtain the nearest approximation to the time between the visible Moon and the Sun computed for moon-rise, i.e. between the risings of the Moon and the Sun). The Moon, ..., rises as many *asus* before sunrise as correspond to the *nāḍīs* obtained by the method of successive approximations.[81]

Bhāskara I has given the details of the implied processes of successive approximations also.

Vaṭeśvara gives the methods of finding out the time of moon-rise and moonset thus:

> In the light half of the month the calculation of the time of rising of the Moon in the day is prescribed to be made from the positions of the Sun and the (visible) Moon (for sunrise) in the manner stated before; and that of the time of setting of the Moon at the end of the day (i.e. night) from the positions of the Sun and the (visible) Moon (for sunset), both increased by the six signs.
>
> In the dark half of the month, the (time of) rising of the Moon, when the night is yet to end, should be calculated by the process of iteration from the positions of the Sun and the (visible) Moon (for sunrise); and in the light half of the month, the (time of) rising of the Moon, when the day is yet to end, should be calculated by

[79] *MBh*, vi. 35.
[80] *MBh*, vi. 32–33.
[81] *MBh*, vi. 37–38.

> the process of iteration from the position of the Sun (for sunset) increased by six signs and the position of the (visible) Moon (for sunset).
>
> In the dark half of the month, the time of setting of the Moon, when the day is yet to elapse, should be obtained from the positions of the Sun and the (visible) Moon (for sunset), each increased by six signs.
>
> In the light half of the month, the same time (of setting of the Moon), when the night is yet to elapse, should be obtained from the positions of the (visible) Moon (for sunrise) increased by six signs and the position of the Sun (for sun rise).[82]

Similar methods have been prescribed by the other Indian astronomers also.

The phenomenon of moon-rise on the full moon day is of special importance and the Indian astronomers have dealt with this separately. Bhāskara I says:

> If (at sunset) on the full moon day the longitude of the Moon (corrected for the visibility corrections for rising) agrees to minutes with the longitude of the Sun (increased by six signs), then the Moon rises simultaneously with sunset. If (the longitude of the Moon is) less (than the other), the Moon rises earlier; if (the longitude of the Moon is) greater (than the other), the Moon rises later.
>
> (In the latter cases) multiply the minutes of the difference by the *asus* of the oblique ascension of the sign occupied by the Moon and divide by the number of minutes of arc in a sign, and on the resulting time apply the method of successive approximations (to get the nearest approximation to the time to elapse at moon-rise before sunset or elapsed at moon-rise since sunset).[83]

Lalla says:

> If the true longitude of the Moon, (corrected for the visibility corrections for rising), is the same as the true longitude of the Sun at sunset, increased by six signs, the Moon rises at the same time as the Sun sets; if greater, it rises later; and if less, it rises before sunset.
>
> If the true longitude of the Moon, (corrected for the two visibility corrections for setting) and increased by six signs, is the same as the true longitude of the Sun while rising, the Moon sets at that time; if greater, it sets after, and if less, before sunrise.[84]

[82] *VSi*, VII, 1, 2–5.
[83] *LBh*, vi. 20–21.
[84] *SiDVṛ*, viii. 9–10.

So also says Vaṭeśvara:

> When the true longitude of the Moon (for sunset), (corrected for the visibility corrections for rising), becomes equal to the longitude of the Sun (for sunset), increased by six signs, then the Moon, in its full phase, resembling the face of a beautiful lady, rises (simultaneously with the setting Sun), and goes high up in the sky, glorifying by its light the circular face of the earth freed from darkness, making the lotuses close themselves and the water lilies blossom forth.
>
> On the full-moon day, at evening, the Sun and the Moon, stationed in the zodiac at the distance of six signs, appear on the horizon like the two huge gold bells (hanging from the two sides) of Indra's elephant.[85]

2.5 Time-interval from rising to setting

In the case of the Sun, the time-interval from rising to setting is called the duration of sunlight or the duration of the day. Similarly, the time-interval from setting to rising is called the duration of the night. These are obtained by the formulae:

$$\text{duration of day} = 2\,(15 \textit{ ghaṭīs} \pm \textit{ ghaṭīs} \text{ of Sun's ascensional difference})$$
$$\text{duration of night} = 2\,(15 \textit{ ghaṭīs} \pm \textit{ ghaṭīs} \text{ of Sun's ascensional difference}),$$

the upper of lower sign being taken according as the Sun is to the north or south of the equator.

Brahmagupta says:

> 15 *ghaṭīs* respectively increased and diminished when the Sun is in the northern hemisphere, or respectively diminished and increased when the Sun is in the southern hemisphere, by the *ghaṭīs* of the Sun's ascensional difference, and the results doubled, give the *ghaṭīs* of the durations of the day and night, respectively.[86]

Lalla says:

> When the Sun's ascensional difference expressed in *ghaṭīs* is respectively added to and subtracted from 15 *ghaṭīs*, and the results are doubled, the lengths of night and day are obtained, provided the Sun is in the southern hemisphere beginning with Libra. The same give the lengths of day and night, if the Sun is in the northern hemisphere, beginning with Aries.[87]

[85] *VSi*, VII, 1, 7–8.
[86] *BrSpSi*, ii. 60; *KK*, I, iii. 3.
[87] *ŚiDVṛ*, ii. 20(c–d)–21.

So also has been stated by Śrīpati,[88] Bhāskara II,[89] and other Indian astronomers.

The duration from moon-rise to moon-set is called the length of the Moon's day and the duration from moon-set to moon-rise, the length of the Moon's night. The former is obtained by the formula:

length of Moon's day = (time of rising of the untraversed portion of the sign occupied by the Moon's *udayalagna*) + (time of rising of the traversed portion of the sign occupied by the Moon's *astalagna*) + (time of rising of the intermediate signs).

Vaṭeśvara says:

> The Moon's *udayalagna* increased by six signs gives the Moon's *astalagna*. Find the oblique ascension of that part of the ecliptic that lies between the two (i.e. between the Moon's *udayalagna* and *astalagna*) with the help of the oblique ascensions of the signs: (this gives the length of the Moon's day). The difference between half of it and 15 *ghaṭīs* is the Moon's ascensional difference.[90]

Vaṭeśvara's method of finding the Moon's *astalagna* is gross. For, here the motion of the Moon from moon-rise to moon-set has been neglected. The correct rule is: First find out the Moon's true longitude for the time of moon-rise; then increase it by half the Moon's daily motion; then apply to it the visibility corrections for setting; then add six signs to that: the result thus obtained will be the Moon's *astalagna*.

In the case of a planet or a star the length of the day is defined and obtained as in the case of the Moon.

Āryabhaṭa II gives the following rule to get a planet's *astalagna*:

> Calculate the true longitude of the planet for the time of its rising, apply to it one-half of the planet's daily motion, then correct it by the visibility corrections for the western horizon (i.e. for setting), and then add six signs to it. (The result is the planet's *astalagna*). Now find the time of rising of the traversed part of the decan occupied by it, to it add the time of rising of the untraversed part of the decan occupied by the planet's *udayalagna*, as also the times of rising of the intervening decans. The result is the length of the planet's day. Using this length of the planet's day,

[88] *SiŚe*, iii. 70.
[89] *SiŚi*, I, ii. 52.
[90] *VSi*, VII, i. 11.

> again calculate the true longitude of the planet for the time of its setting, and iterate the above process. Thus will be obtained the accurate longitude of the visible planet on the western horizon. That increased by six signs is the planet's *astalagna*.[91]

The planet's *udayalagna* and *astalagna* being known, the length of the planet's day is obtained by adding together the time of rising of the untraversed portion of the decan (or sign) occupied by the planet's *udayalagna*, the time of rising of the traversed portion of the decan (or sign) occupied by the planet's *astalagna*, and the times of rising of the intervening decans (or signs).

In the case of the stars too the method used to find the length of the day is the same. The stars being fixed, their *udayalagna* and *astalagna* remain the same for years. Āryabhaṭa II says:

> In the case of Canopus, the Seven Sages and the stars (in general) the *udayalagna* and the *astalagna* remain invariable for some years. Not so in the case with the ever moving planets, the Moon etc., because of their inconstancy.[92]

2.6 Stars that do not rise or set (circumpolar stars)

The stars whose direction is greater than or equal to the co-latitude of the place do not rise or set at that place. If the declination is north, these stars are always visible at the place; if south, they are always invisible there.

Bhāskara II says:

> The stars for which the true declination, of the northern direction, exceeds the co-latitude (of the local place), remain permanently visible (at that place). And the stars such as Sirius and Canopus etc. for which the true declination, of the southern direction, exceeds the co-latitude (of the local place), remain permanently invisible (at that place).[93]

3 Conjunction of planets and stars

3.1 Conjunction of two planets: *Samāgama* and *Yuddha* ("Union and encounter")

When two planets have equal longitudes they are said to be in conjunction. This conjunction of two planets is given different names depending on the participating planets. When the conjunction of a planet takes place with the

[91] *MSi*, x. 4–5.
[92] *MSi*, x. 8.
[93] *SiŚi*, I, xi. 16.

Sun, it is called *astamaya* (setting of the planet); when with the Moon, it is called *samāgama* (union), and when any two planets, excluding the Sun and Moon, are in conjunction, it is called *yuddha* (encounter).

Viṣṇucandra says:

> Conjunction (of a planet) with the Sun is called *astamaya* (setting); that with the Moon, *samāgama* (union); and that of Mars etc. with one another, *yuddha* (encounter).[94]

Brahmagupta says:

> Conjunction (of two planets), in which the Sun and Moon do not take part, is called *yuddha* (encounter); that of Mars etc. with the Moon, *samāgama* (union); and that with the Sun, *astamaya* (setting). (In the case of encounter) the planet that lies to the north of the other is the victor, but Venus is the victor (even) when it is to the south of the other.[95]

According to the *Sūryasiddhānta*:

> Of the star-planets (Mars etc.) there take place, with one another, *yuddha* (encounter) and *samāgama* (union); with the Moon, *samāgama* (union); with the Sun, *astamaya* (setting).[96]
>
> (In an encounter) Venus is generally the victor, whether it lies to the north or to the south (of its companion).[97]

The conjunction of two star-planets Mars etc. which has been defined above as *yuddha* (encounter), is further classified into five categories, depending on the distance between them at the time of their conjunction. Let d be the distance between their centres at the time of their conjunction, and s the sum of their semi-diameters. Then the conjunction is called:

1. *Ullekha* (external contact), when $d = s$;
2. *Bheda* (occultation), when $d < s$;
3. *Aṃśu-vimarda* (pounding or crushing of rays, friction of rays), when $d > s$;
4. *Apasavya* (dexter) when $d > s$ but $< 1°$ and one planet is tiny;
5. *Samāgama* (union), when $d > s$ and also $> 1°$ and the planets have large discs.

[94] *KK*, I, viii. 1 (Bhaṭṭotpala's comm.).
[95] *KK*, I, viii. 1.
[96] *SūSi*, vii. 1.
[97] *SūSi*, vii. 23 (a–b).

The *Sūryasiddhānta* says:

> The conjunction of two star-planets is called *ullekha* (external contact), when they touch each other (externally); *bheda* (occultation), when there is overlapping; *aṃśu-vimarda* (pounding or crushing of rays, or friction of rays), when there is mingling of rays of each other; *apasavya-yuddha* (dexter), when one planet has tiny disc and the distance between the two is less than one degree; *samāgama* (union), when the discs of the planets are large and the distance between them is greater than one degree.[98]

The *Sūryasiddhānta* further says:

> In the *apasavya-yuddha* (dexter encounter) the star-planet which is tiny, destitute of brilliancy, and covered (by the rays of the other), is the defeated one. (In general) the star-planet which is rough, colourless, struck down, and situated to the south, is the vanquished one. That situated to the north is the victor if it is large and luminous; that situated to the south too is the victor if it is powerful (i.e. large and luminous). When two star-planets are in proximity, there is *samāgama* (union) if both are luminous; *kūṭa* (confrontation) if both are small in size; and *vigraha* (conflict, or fight) if both are struck down. Venus is generally the victor whether it is to the north or to the south (of the other).[99]

The *Bhārgavīya* says:

> Hostility should be foretold when there is *apasavya* (dexter); war when there is *raśmi-saṃkula* (melee of rays); ministerial distress when there is *ullekha* (external contact); and loss of wealth when there is *bheda* (occultation).[100]

3.2 Conjunction in celestial longitude (*Kadambaprotīya-yuti*)

Āryabhaṭa I and his staunch follower Bhāskara I have dealt with the conjunction of the planets in celestial longitude (i.e. along the circle of celestial longitude or secondary to the ecliptic) and have given rules to find the time when such a conjunction occurs.

Bhāskara I says:

> If one planet is retrograde and the other direct, divide the difference of their longitudes by the sum of their daily motions; otherwise (i.e. if both of them are either retrograde or direct), divide

[98] *SūSi*, vii. 18–19.
[99] *SūSi*, vii. 20(c–d)–23(a–b).
[100] Quoted by Kapileśvara Chaudhary in his comm. on *SūSi*, vii. 18–19.

> the same by the difference of their daily motions; thus is obtained the time in terms of days etc. after or before which the two planets are in conjunction (in longitude). The velocity of the planets being different (from time to time), the time thus obtained is gross. One, proficient in the science of astronomy, should, therefore, apply some method to make the longitudes of the two planets agree to minutes. Such a method is possible from the teachings of the preceptor or by day to day practice.[101]
>
> In the case of Mercury and Venus, subtract the longitude of the ascending node from that of the *śīghrocca*: (thus is obtained the longitude of the planet as diminished by the longitude of the ascending node). The longitudes (in terms of degrees) of the ascending nodes of the planets beginning with Mars (i.e. Mars, Mercury, Jupiter, Venus, and Saturn) are respectively 4, 2, 8, 6, and 10, each multiplied by 10.
>
> The greatest latitudes, north or south, in minutes of arc, (of the planets beginning with Mars) are respectively 9, 12, 6, 12, and 12, each multiplied by 10. (To obtain the *R*sine of the latitude of a planet) multiply (the greatest latitude of the planet) by the *R*sine of the longitude of the planet minus the longitude of the ascending node (of the planet) (and divide by the "divisor" defined below).
>
> The product of the *mandakarṇa* and the *śīghrakarṇa* divided by the radius is the distance between the Earth and the planet: this is defined as the "divisor".
>
> Thus are obtained the minutes of arc of the latitudes (of the two planets which are in conjunction in longitude).
>
> From these latitudes obtain the distance between those two given planets (which are in conjunction in longitude) by taking their difference if they are of like directions or by taking their sum if they are of unlike directions. The true distance between the two planets, in minutes of arc, being divided by 4 is converted into *aṅgulas*.
>
> Other things should be inferred from the colour and brightness of the rays (of the two planets) or else by the exercise of one's own intellect.[102]

The method prescribed by Āryabhaṭa I in his work employing midnight day-reckoning was also practically the same. Brahmagupta has summarised it as follows:

[101] *MBh*, vii. 49–51.
[102] *LBh*, vii. 6–10.

> Divide the difference between the longitudes of the two planets (whose conjunction is under consideration) by the difference of their daily motions, if they are both direct or both retrograde, or by the sum of their daily motions, if one is direct and the other retrograde. The result is in days. If the slower planet is ahead of the other (and if both the planets are direct), the conjunction is to occur after the days obtained; if the quicker planet is ahead of the other, the conjunction has already occurred before the days obtained.
>
> Multiply the difference between the longitudes of the two planets by their own daily motions and divide (each product) by the difference or sum of their daily motions, as before. Subtract each result from the longitude of the corresponding planet, if the conjunction has already occurred, and add, if it is to occur, provided the planet is in direct motion. If it is retrograde, reverse the order of subtraction and addition. The planets will then have equal longitudes.
>
> From the longitudes of the two planets made equal up to minutes of arc, subtract the longitudes of their own ascending nodes (in the case of Mars, Jupiter, and Saturn). In the case of Mercury and Venus, the longitude of the ascending node should be subtracted from the *śīghrocca* of the planet. Multiply the *R*sine of that by the greatest latitude of the corresponding planet and divide by the last *karṇa* ("hypotenuse for the planet"): the result is the latitude of the planet.
>
> Take the difference or sum of the latitudes of the two planets (which are in conjunction in longitude) according as they are of like or unlike directions. Then is obtained the distance between the planets (at the time of their conjunction in longitude).[103]

3.3 Occultation (*Bheda-yuti*)

When the distance between the two planets in conjunction in longitude falls short of the sum of their semi-diameters the lower planet covers partly or wholly the disc of the higher planet. The situation is analogous to the solar eclipse where the Moon eclipses the Sun. In such a case the lower planet is treated as the Moon and the upper one as the Sun, and all processes prescribed in the case of a solar eclipse are gone through in order to obtain the time of contact and separation, immersion and emersion, etc.

Vaṭeśvara says:

[103] *KK*, I, viii. 3–6(a–b).

> When the distance between the two planets (which are in conjunction) is less than half the sum of the diameters of the two planets, there is occultation (*bheda*) of one planet by the other. The eclipser is the lower planet. All calculations (pertaining to this occultation), such as those for the semi-duration etc. are to be made as in the case of a solar eclipse.
>
> When the Moon occults a planet, the time of conjunction should be reckoned from moon-rise and for that time one should calculate the *lambana* (parallax-difference in longitude) and the *avanati* (parallax-difference in latitude). In case one planet occults another planet, the time of conjunction should be reckoned from the (occulted) planet's own rising and for that time one should calculate the *lambana* and the *avanati*.[104]

The whole process has been explained by Bhaṭṭotpala as follows:

> The planet which lies in the lower orbit is the occulting planet (or the occulter); it is to be assumed as the Moon. The planet which lies in the higher orbit is the occulted planet; it is to be assumed as the Sun. Then, assuming the time of conjunction (of the two planets) as reckoned from the rising of the occulted planet as the *tithyanta*, calculate the *lagna* for that *tithyanta*, with the help of (the longitude of) the occulted body, which has been assumed as the Sun, and the oblique ascensions of the signs. Subtracting three signs from that, calculate the corresponding declination (i.e. the declination of the *vitribha*).[105] Taking the sum of that (declination) and the local latitude when they are of like directions, or their difference when they are of unlike directions, calculate the *lambana* (for the time of conjunction) as in the case of a solar eclipse. When the longitude of the planets in conjunction is greater than (the longitude of) the *vitribha*, subtract this *lambana* from the time of conjunction; and when the longitude of the planets in conjunction is less than (the longitude of) the *vitribha*, add this *lambana* to the time of conjunction; and iterate this process: this is how the *lambana* is to be calculated. Then from the longitude of the *vitribhalagna* which has got iterated in the process of iteration of the *lambana*, severally subtract the ascending nodes of the two planets, and therefrom calculate the celestial latitudes of the two planets, as has been done in the case of the solar eclipse. Then taking the sum or difference of the declination of the *vitribhalagna*, the

[104] *VSi*, VIII, i, 7–8.
[105] The *vitribha* or *vitribhalagna* is the *lagna* (rising point of the ecliptic) minus *tribha* (three signs).

> latitude of the *vitribhalagna*, and the local latitude, each in terms of degrees, (according as they are of like or unlike directions), in the case of both the planets. Then applying the rule: "Multiply the *R*sine of those degrees of the sum and difference by 13 and divide by 40: the result is the *avanati*," calculate the *avanatis* for the two planets. Then calculate the latitudes of the occulted and the occulting planets in the manner stated in the chapter on the rising and setting of the heavenly bodies, and increase or decrease them by the corresponding *avanatis* according as the two are of like or unlike directions: the results are the true latitudes (of the occulted and occulting planets). Take the sum or difference of those true latitudes according as they are of unlike or like directions. The result of this is the *sphuṭa-vikṣepa*.
>
> Having thus obtained the *sphuṭa-vikṣepa*, one should see whether there exists eclipse-relation between this *sphuṭa-vikṣepa* and the diameters of the discs of the two planets. If the *sphuṭa-vikṣepa* is less than half the sum of the diameters of the two planets, this relation does exist; if greater, it does not. The totality of the occultation should also be investigated as before. Then, (severally) subtract the square of the *sphuṭa-vikṣepa* from the squares of the sum and the difference of the semi-diameters of the occulted and occulting planets, and take the square roots (of the results). Multiply them by 60 and divide by the difference or sum of the daily motions of the planets as before: then are obtained the *sthityardha* and the *vimardārdha*, (respectively). They are fixed (by the process of iteration) as in the case of a solar eclipse. The *sthityardha* and *vimardārdha* having been obtained in this way, they should be corrected by the *lambana* obtained by the process of iteration. Then the time of apparent conjunction should be declared as the time of the middle of the occultation; this diminished and increased by the (*spārśika* and *maukṣika*) *sthityardhas*, (respectively), the times of contact and separation (of the two planets); and the same diminished and increased by the (*spārśika* and *maukṣika*) *vimardārdhas* (respectively), the times of immersion and emersion.[106]

Bhāskara II explains the same as follows:

> When there is *bheda-yuti*, then one should compute the *lambana* etc. as in the case of a solar eclipse. There, the lower of the two planets is to assumed as the Moon and the upper one as the Sun. Why are they so assumed? To compute the *lambana* etc. But

[106] *KK*, I, viii. 5–6, comm.

the *lagna*, which is obtained in order to find the *vitribhalagna*, is to be computed from the actual Sun, not from the assumed Sun. For what time is the *lagna* calculated from the Sun? For the time of conjunction (in longitude of the two planets). What is meant is this: On the day the conjunction (of the two planets) takes place, find the *ghaṭīs* of the night elapsed at the time of conjunction. Therefrom calculate the Sun as increased by six signs, and therefrom the *lagna*. Then calculate the *vitribha* and then the corresponding *śaṅku* (i.e. *R*sine of the altitude of the *vitribha*). Then, applying the rule: "Multiply the *R*sine of the difference of that *vitribha* and the assumed Sun by 4 and divide by the radius, and so on," calculate the *lambana* and *nati*, as before. Then correct the time of conjunction by that *lambana*. But the *lambana* etc. should be applied only when the two planets are fit for observation. In this *bheda-yuti*, the north-south distance between the planets is the latitude; and the direction of the latitude is that in which the assumed Moon lies, as seen from the assumed Sun. Now is stated the peculiarity in the case of *parilekha* (graphical representation of the occultation). When the lower planet, which has been assumed as the Moon, is slower or retrograde, then one should understand that the contact (of the two planets) occurs towards the east and the separation towards the west. In the contrary case, one should understand that the contact occurs towards the west and the separation towards the east. We have stated here the (notable) points of difference in the case of *bheda-yoga*: there is no other difference in the procedure.[107]

3.4 Conjunction along the circle of position (*Samaprotīma-yuti*)

Conjunction in longitude, though theoretically sound and perfect, suffered from one practical setback viz. that there being no star at the pole of the ecliptic such a conjunction could not be observed with precision and so the calculated time of its occurrence could not be confirmed by observation. Brahmagupta noted that the stars *Citrā* (Spica) and *Svāti* (Arcturus), which, though of unequal longitudes, were seen daily to be in conjunction along the circle of position (*samaprota-vṛtta*). This conjunction was easily observable and agreement between computation and observation in this case could be established. Brahmagupta therefore gave preference to conjunction along the circle of position over conjunction in longitude.

To obtain the time when two planets are in conjunction along the circle of position, Brahmagupta first finds the time of their conjunction in longitude

[107] *SiŚi*, I, x. 7–9, comm.

and then he derives how much earlier or later conjunction along the circle of position takes place. He states two rules for the purpose, one gross and the other approximate.

3.5 Brahmagupta's gross rule

Brahmagupta's gross rule runs as follows:

> Find the *udayalagna* (the rising point of the ecliptic at the time of rising of the planet) and also the *astalagna* (the setting point of the ecliptic at the time of setting of the planet) of the two planets equalised up to minutes of arc (i.e. for the time of their conjunction in longitude). Then find the *ghaṭīs* of the day-lengths of the two planets by adding together the times of rising at the local place of (1) the untraversed part of the *udayalagna*, (2) the traversed part of the *astalagna* as increased by six signs, and (3) the intervening signs, (in each case). If out of the two planets (in conjunction in longitude), the planet with lesser *udayalagna* is such that its *astalagna* increased by six signs is smaller than the other planet's *astalagna*, increased by six signs, one should understand that the conjunction of the two planets along the circle of position is to occur;[108] if greater, one should understand that the conjunction of the two planets along the circle of position has already occurred.
>
> Now (in the case of both the planets) multiply the minutes of the difference between the planet's *astalagna* plus six signs and the *udayalagna* by the *ghaṭīs* of the planet's own day-length. The result (in each case) should be taken as negative or positive according as the *astalagna* plus six signs is smaller or greater than the *udayalagna*. In case these results are both negative or both positive, divide the minutes of the difference between the planets' own *udayalagnas* by the difference of the two results; in case one result is positive and the other negative, divide the same minutes by the sum of the two results. (This gives the time, in terms of *ghaṭīs*, to elapse before or elapsed since the conjunction of the two planets along the circle of position, at the time of their conjunction in longitude). By these *ghaṭīs* multiply the minutes of the difference between the planet's *udayalagna* and *astalagna*, the latter increased by six signs, and divide by the *ghaṭīs* of the planet's own day-length. By the resulting minutes increase or diminish the

[108] For, the planet whose rising point at rising and the rising point at setting are both smaller than of the other planet, has greater day-length than the other. So the latter is swifter than the other.

> planet's own *udayalagna* according as it is smaller or greater than the planet's (own) *astalagna* plus six signs: then is obtained the planets' common longitude at the time of their conjunction along the circle of position. In case it is less than the *udayalagna* for that time in the night or greater than the *udayalagna* plus six signs, the two planets will be seen (in the sky) in conjunction along the circle of position.[109]

Śrīpati, following Brahmagupta, has stated this rule in his *Siddhāntaśekhara*.[110] But Lalla and Vaṭeśvara have omitted it.

3.6 Brahmagupta's approximate rule

The second rule of Brahmagupta which is intended to give better time of conjunction (of two planets along the circle of position) than the first rule (stated above), runs as follows:

> Multiply the duration of day for the planet with greater day-length by the time (in *ghaṭīs*) elapsed (at the time of conjunction in longitude) since the rising of the planet with smaller day-length and divide by the duration of day for the planet with smaller day-length. When the resulting time is greater than the time elapsed (at the time of conjunction in longitude) since the rising of the planet with greater day-length, (it should be understood that) the conjunction of the two planets (along the circle of position) has already occurred; when less, (it should be understood that) the conjunction of the two planets (along the circle of position) is to occur.[111]
>
> The difference of the two times (in terms of *ghaṭīs*) is the "first". A similar result derived from the two planets, diminished or increased by their motion corresponding to "arbitrarily chosen *ghaṭīs*"[112] (as the case may be), is the "second". When the "first" and the "second" both correspond to conjunction past or to occur, divide the product (of the *ghaṭīs*) of the "first" and the "arbitrarily chosen *ghaṭīs*" by the *ghaṭīs* of the difference between the "first" and the "second"; in the contrary case (i.e. when out of the "first" and the "second", one corresponds to conjunction past and the other to conjunction to occur), divide the product by (the *ghaṭīs* of) the sum

[109] *BrSpSi*, ix. 13–18.
[110] *SiŚe*, xi. 21–27.
[111] (15).
[112] These are the *ghaṭīs* elapsed since or to elapse before conjunction along the circle of position, at the time of conjunction, chosen by conjecture.

> of the "first"and the "second". The resulting *ghaṭīs* give the *ghaṭīs* elapsed since or to elapse before the conjunction along the circle of position, at the time of conjunction in longitude, depending upon whether the "first" relates to conjunction past or to occur.
>
> The conjunction of two planets, along the circle of position, takes place when the result (in *ghaṭīs*) obtained on dividing by the *ghaṭīs* of the day-length of one planet, the product of the *ghaṭīs* elapsed since the rising of that planet and the *ghaṭīs* of the day-length of the other planet, is equal to the *ghaṭīs* elapsed since the rising of the other planet.[113]

This latter rule of Brahmagupta has been adopted by Lalla,[114] and Śrīpati.[115]

3.7 Alternative form of Brahmagupta's approximate rule

Brahmagupta has stated his approximate rule in the following alternative form also:

> Multiply the *nāḍīs* of the duration of day for the planet with smaller day-length by the *ghaṭīs* elapsed since the rising of the planet with greater day-length and divide by the *ghaṭīs* of the duration of day for the planet with greater day-length: the result is in terms of *nāḍīs*. When these *nāḍīs* are less than the *ghaṭīs* elapsed since the rising of the planet with smaller day-length, (it should be understood that) conjunction (along the circle of position) of the two planets has already occurred; when greater, (it should be understood that) conjunction is to occur. Assume the difference of the two, in terms of *ghaṭīs*, as the "first". Now multiply the daily motion of each planet by "the arbitrarily chosen *ghaṭīs*" and divide each product by 60: add the result to or subtract it from the longitude of the corresponding planet according as the conjunction has occurred or is to occur. Then obtain the difference similar to the "first" and call it "second". When both the differences, the "first" and the "second", correspond either to conjunction past or to conjunction to occur, divide the product of the "first" and the "arbitrarily chosen *ghaṭīs*" by the difference of the "first" and the "second"; in the contrary case (i.e. when out of the "first" and the "second", one corresponds to conjunction past and the other to conjunction to occur), divide that product by the sum of the "first" and the "second". The resulting *ghaṭīs* give the

[113] *BrSpSi*, ix. 22–25; *KK*, II, vi. 1–4.
[114] *ŚiDVṛ*, x. 17–20.
[115] *SiŚe*, xi. 28–31.

> *ghaṭīs* elapsed since or to elapse before the conjunction of the two planets (along the circle of position), depending upon whether the "first" relates to conjunction past or to conjunction to occur. If by applying the above rule once conjunction of the two planets is not arrived at, the rule should be iterated (until one does not get the conjunction of the two planets).[116]

This alternative form of Brahmagupta's approximate rule has been adopted by Vaṭeśvara who states it as follows:

> Multiply the duration of day for the planet with smaller day-length by the time (in *ghaṭīs*) elapsed since the rising of the planet with greater day-length, and divide by the duration of day for the planet with greater day-length. When the resulting time is greater than the time elapsed since the rising of the planet with smaller day-length, (it should be understood that) the conjunction (along the circle of position) of the two planets is to occur; in the contrary case, (it should be understood that) the conjunction has already occurred.
>
> The difference of the two times (in terms of *ghaṭīs*) is the "first". A similar difference derived from the "*ghaṭīs* arbitrarily chosen" (for *ghaṭīs* elapsed since or to elapse before conjunction) is the "second". When both the "first" and the "second" correspond either to conjunction past or to conjunction to occur, divide the product of the "first" and the "arbitrarily chosen *ghaṭīs*" by the *ghaṭīs* of the difference between the "first" and the "second"; in the contrary case (i.e. when out of the "first" and the "second", one corresponds to conjunction past and the other to conjunction to occur), divide that product by (the *ghaṭīs* of) the sum of the "first" and the "second". The resulting *ghaṭīs* give the *ghaṭīs* elapsed since or to elapse before the conjunction of the two planets (along the circle of position), depending on whether the "first" relates to conjunction past or to occur.[117]

Munīśvara has criticised conjunction along the circle of position advocated by Brahmagupta, for the reason that the time of such a conjunction will differ from place to place, and so it will create confusion in making astrological predictions. See *Siddhānta-sārvabhauma*, *Bhagrahayuti*, vs. 15, p. 543.

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[116] *BrSpSi*, x. 51–58.
[117] *VSi*, VIII, i. 12–14.

3.8 Āryabhaṭa II's rule

Āryabhaṭa II gives the following rule to find the time of conjunction in celestial longitude and that of conjunction along the circle of position:

> Divide the difference (in minutes) between the longitude of the two planets (whose conjunction is under consideration) by the difference between the daily motions (of the two planets), provided they are both direct or both retrograde; if one of the planets is retrograde (and the other direct), divide by the sum of the daily motions (of the two planets); the result gives the days elapsed since the conjunction of the two planets, in case the faster planet is greater than the other, and also if the planet with lesser longitude is retrograde (and the other direct). When both the planets are retrograde, the case is contrary to what happens when both the planets are direct. The two planets should then be calculated for the time of conjunction. Then the two planets become equal in longitude.
>
> When conjunction suitable for observation (i.e. along the circle of position) is required, then the two planets should be corrected for the *ayana-dṛkkarma* and *akṣa-dṛkkarma* also. The time when they become equal in longitude, is certainly the time of conjunction (along the circle of position).[118]

Indications of this rule occur in the *Sūryasiddhānta*[119] and the *Vaṭeśvara-siddhānta*[120] also. According to Kamalākara, a staunch follower of the *Sūrya-siddhānta*, however, the conjunction of the planets and stars taught in the *Sūryasiddhānta* is in celestial longitude.[121]

3.9 Conjunction in polar longitude (*Dhruvaprotīya-yuti*)

Bhāskara II has given rules for conjunction in celestial longitude as well as conjunction in polar longitude. But as there is no star at the pole of the ecliptic, conjunction in celestial longitude does not, says he, create confidence in the observer; while there being one at the pole of the equator, conjunction in polar longitude is better for observation. However, conjunction of two planets, in his opinion, really occurs when the two planets are nearest to each other and this happens when the two planets are in conjunction in celestial longitude only.[122] He has given no credit to conjunction along the circle of

[118] *MSi*, xi. 3(c–d)–6(a–b).
[119] vii. 7–12.
[120] VIII, i. 9.
[121] *SiTVi*, *Bhagrahayuti*, vss. 105–106.
[122] *SiŚi*, I, x. 4(c–d)–5, gloss.

position probably because it was not universal. He has not even mentioned this conjunction.

Bhāskara II's rule for the conjunction of two planets in polar longitude runs thus:

> Divide the minutes of the difference between the longitudes of the two planets by the difference of their daily motions (if both planets are direct or both retrograde); if one of them is retrograde (and the other direct), divide by the sum of the daily motions (of the planets): the result is the number of days elapsed since the conjunction of the two planets provided the slower planet has lesser longitude than the other, or if, one planet being retrograde, its longitude is lesser than that of the other. If otherwise, the conjunction occurs after the days obtained. If both the planets are retrograde, the result is contrary to that for direct planets. (This gives approximate time for conjunction. To get accurate time, proceed as follows:)
>
> (Calculate the longitudes of the planets for the time of conjunction and) apply the *ayana-dṛkkarma* (to them). Iterate the process until the time of conjunction is not fixed. When this is done, the two planets lie on the same great circle passing through the poles of the equator. The planets are then said to be in conjunction in the sky. If the *ayana-dṛkkarma* is not applied, the planets lie on the same secondary to the ecliptic.[123]

4 Conjunction of a planet and star

The conjunction of a planet and a star is treated in the same way as the conjunction of two planets and the rules in the two cases are similar. The only remarkable difference is that the stars, unlike the planets, are supposed to be points of light having no diameter and fixed in position having no eastward daily motion.

Bhāskara I says:

> All planets whose longitudes are equal to the longitude of the junction-star of a *nakṣatra*[124] are seen in conjunction with that star. (Of a planet and a star) whose longitudes are unequal, the time of conjunction is determined by proportion.[125]

[123] *SiŚi*, I, x. 3–5.
[124] The junction-stars are the prominent stars of the *nakṣatras*.
[125] *LBh*, viii. 5.

> The distance between a planet and a star (when they are in conjunction) is determined from (the sum or difference of) their latitudes.[126]

Brahmagupta says:

> If the longitude of a planet is less than the longitude (*dhruvaka*) of a star, their conjunction is to occur; if greater, their conjunction has already occurred. If the planet is retrograde, reverse is the case. The rest is similar to that stated in the case of the conjunction of two planets.[127]

Lalla says:

> If the longitude of a planet is greater than the longitude of the junction-star of a *nakṣatra*, their conjunction has already taken place; if less, it will take place. If the planet is retrograde, the contrary is the case. The rest is similar to that in the case of the conjunction of two planets.[128]

A similar statement has also been made by Vaṭeśvara,[129] Āryabhaṭa II,[130] Śrīpati,[131] the author of the *Sūryasiddhānta*,[132] and others.

In the case of occultation, Brahmagupta says:

> When a planet is on the same side of the ecliptic as the junction-star of a *nakṣatra*, the planet will occult the junction-star if its true latitude is greater than the latitude of the junction-star minus the semi-diameter of the planet or less than the latitude of the junction-star plus the semi-diameter of the planet.[133]

The occultation of a star by the Moon was considered important. So the occultation of certain prominent stars was specially noted and recorded by the Indian astronomers.

Bhāskara I says:

> The Moon, moving towards the south of the ecliptic, destroys (i.e. occults) the Cart of *Rohiṇī* (the constellation of Hyades), when its latitude amounts to 60 minutes; the junction-star of *Rohiṇī*

[126] *MBh*, iii. 71 (a–b).
[127] *KK*, I, ix. 7.
[128] *ŚiDvṛ*, xi. 4.
[129] *VSi*, VIII, ii. 4 (a–c).
[130] *MSi*, xii. 9.
[131] *SiŚe*, xii. 3.
[132] *SūSi*, viii. 15.
[133] *KK*, I, ix. 14. Also see *BrSpSi*, x. 4.

(i.e. Aldebaran), when its latitude amounts to 256 minutes; (the junction-star of) *Citrā* (i.e. Spica), when its latitude amounts to 95 minutes; (the junction-star of) *Jyeṣṭha* (i.e. Antares), when its latitude amounts to 200 minutes; (the junction-star of) *Anurādhā*,[134] when its latitude amounts to 150 minutes; (the junction-star of) *Śatabhiṣak* (i.e. λ Aquarii), when its latitude amounts to 24 minutes; (the junction-star of) *Viśākha*,[135] when its latitude amounts to 88 minutes; and (the junction-star of) *Revatī* (i.e. Zeta Piscium), when its latitude vanishes. When it moves towards the north (of the ecliptic), it occults the *nakṣatra Kṛttikā* (i.e. Pleiades), when its latitude amounts to 160 minutes; and the central star of the *nakṣatra Maghā*, when it assumes the greatest northern latitude. These minutes (of the Moon's latitude) are based on actual observation made by means of the *Yaṣṭi* instrument (i.e. the Indian telescope).[136]

Brahmagupta says:

The planet whose south latitude at 17° of Taurus exceeds 2°, occults the Cart of *Rohiṇī*.[137] The Moon, when it has the maximum north latitude, occults the third star of *Maghā*; when it has no latitude, it occults *Puṣya*, *Revatī*, and *Śatabhiṣak*.[138]

Lalla says:

The Moon, situated in the middle of the *nakṣatra Rohiṇī*, occults the Cart of *Rohiṇī*, when its southern latitude amounts to 2°40′; (the junction-star of) the *nakṣatra Rohiṇī*, when its southern latitude is 4°30′; the middle of the *nakṣatra Maghā*, when its north latitude amounts to 40°30′; and the *nakṣatras Revatī*, *Puṣya*, and *Śatabhiṣak*, when it is devoid of latitude.[139]

Vaṭeśvara says:

The planet, whose latitude at 17° of Taurus amounts to $1\frac{1}{2}$ degrees south, occults the Cart of *Rohiṇī*. The Moon with its (maximum) latitude south (i.e., 4°30′ S) covers the junction-star of *Rohiṇī*.[140]

Śrīpati similarly says:

[134] β or δ Scorpii.
[135] α or K Librae.
[136] *MBh*, iii. 71(c–d)–75(a–b). Also see *LBh*, viii. 11–16.
[137] *SūSi*, viii. 13 also.
[138] *BrSpSi*, x. 11–12; *KK*, ix. 15–16.
[139] *ŚiDVṛ*, xi. 11.
[140] *VSi*, VIII, ii. 10–11.

The planet whose southern latitude at 17° of Taurus exceeds 2° certainly occults the Cart of *Rohiṇī*. The Moon with its longitude equal to that of (the junction-star of) *Maghā* occults the third star of *Maghā*, when it has maximum (north) latitude; and the *nakṣatras Śatabhiṣak*, *Revatī*, and *Puṣya* when its longitude is equal to their longitudes.[141]

[141] *SiŚe*, vii. 8–9.

Part VI

Reviews and Responses

Vedic Mathematics: The deceptive title of Swamiji's book*

The title of the book, *Vedic Mathematics or Sixteen Simple Mathematical Formulae from the Vedas*, written by Jagadguru Svāmī Śrī Bhāratī Kṛṣṇa Tīrthajī Mahārāja, Śaṅkarācārya of Govardhana Matha, Puri, bears the impression that it deals with the mathematics contained in the Vedas—*Ṛgveda, Sāmaveda, Yajurveda* and *Atharvaveda.* This indeed is not the case, as the book deals not with Vedic Mathematics but with modern elementary mathematics up to the Intermediate standard. In his Foreword to Swamiji's book, V. S. Agrawala, the editor, writes:

> The question naturally arises as to whether the sūtras which form the basis of this treatise exist anywhere in the Vedic literature as known to us. But this criticism loses all its force if we inform ourselves of the definition of Veda given by Sri Sankaracharya himself as quoted below:
>
>> The very word "Veda" has this derivational meaning, i.e., the fountain-head and illimitable store-house of all knowledge. This derivation, in effect, means, connotes, and implies that the Vedas *should* contain within themselves all the knowledge needed by mankind relating not only to the so called "spiritual" (or otherworldly) matters but also to those usually described as purely "secular", "temporal", or "worldly" and also to the means required by humanity as such for the achievement of all-round, complete and perfect success in all conceivable directions and that there can be no adjectival or restrictive epithet calculated (or tending) to limit that knowledge down in any sphere, any direction or any respect whatsoever.
>>
>> In other words, it connotes and implies that our ancient Indian Vedic lore should be all-round, complete and per-

* K. S. Shukla, in *Issues in Vedic Mathematics*, New Delhi: Rashtriya Veda Vidya Pratishthan in association with Motilal Banarsidass, 1991, pp. 31–39 (This volume came out as Proceedings of the National Workshop on Vedic Mathematics held during 25–28 March, 1988, at the University of Rajasthan, Jaipur).

A. Kolachana et al. (eds.), *Studies in Indian Mathematics and Astronomy*,
Sources and Studies in the History of Mathematics and Physical Sciences,
https://doi.org/10.1007/978-981-13-7326-8_35

> fect and able to throw the fullest necessary light on all matters which any aspiring seeker after knowledge can possibly seek to be enlightened on.

> It is the whole essence of his assessment of Vedic tradition that it is not to be approached from a factual stand-point but from an ideal stand point, viz. as the Vedas as traditionally accepted in India as the repository of all knowledge *should be* and not what they are in human possession. That approach entirely turns the tables on all critics, for the authorship of Vedic mathematics then need not be laboriously searched in the texts as preserved from antiquity.

In his preface to his *Vedic Mathematics*, Swamiji has stated that the sixteen *sūtras* dealt with by him in that book were contained in the *Pariśiṣṭa* (the Appendix) of the *Atharvaveda*. But this is also not a fact;[1] for they are untraceable in the known *Pariśiṣṭas of the Atharvaveda* edited by G. M. Bolling and J. von Negelein (Leipzig, 1909–10). Some time in 1950 when Swamiji visited Lucknow to give a black-board demonstration of the sixteen *sūtras* of his 'Vedic Mathematics' at the Lucknow University, I personally went to him at his place of stay with Bolling and Negelein's edition of the *Pariśiṣṭas of the Atharvaveda* and requested him to point out the places where the sixteen *sūtras* demonstrated by him occurred in the *Pariśiṣṭas*. He replied off hand, without even touching the book, that the sixteen *sūtras* demonstrated by him were not in those *Pariśiṣṭas*, they occurred in his own *Pariśiṣṭa* and not in any other.

As regards the *Pariśiṣṭas* of the *Atharvaveda* referred to by Swamiji, V. S. Agrawala says:

> The Vedas are well-known as four in number, Ṛk, Yajus, Sāma and Atharva, but they have also the four Upavedas and the six Vedāṅgas all of which form an individual corpus of divine knowledge as it once was and as it may be revealed. The four Upavedas (associated with the four Vedas) are as follows:

Vedaṣ	*Upavedas*
Ṛgveda	Āyurveda
Sāmaveda	Gandharvaveda
Yajurveda	Dhanurveda
Atharvaveda	Sthāpatyaveda

[1] **ed.** The original contains the following editorial note here: "In order to put the matter in proper perspective the views of the Jagadguru Śaṅkarācārya contained in the chapter on Vedic mathematics in his book *Vedic Metaphysics* have been given at the end of this publication as Appendix II. (Editorial note)."

> In this list the Upaveda or Sthāpatya or engineering comprises all kinds of architectural and structural human endeavour and all visual arts. Swamiji naturally regarded mathematics or the science of calculations and computations to fall under this category
>
> In the light of the above definition and approach must be understood the author's statement that the sixteen *sūtras* on which the present volume is based form part of a Pariśiṣṭa of the Atharvaveda. We are aware that each Veda has its subsidiary apocryphal texts some of which remain in manuscripts and others have been printed but that formulation has not closed. For example, some Pariśiṣṭas of the Atharvaveda were edited by G. M. Bolling and J. von Negelein, Leipzig, 1909-10. But this work of Sri Sankaracharyaji deserves to be regarded as a new Pariśiṣṭa by itself and it is not surprising that the sūtras mentioned herein do not appear in the hitherto known Pariśiṣṭas.

V. S. Agrawala's verdict that the work of Śrī Śaṅkarācārya deserves to be regarded as a new *Pariśiṣṭa* by itself is fallacious. The question is whether any book written in modern times on a modern subject can be regarded as a *Pariśiṣṭa* of a Veda. The answer is definitely in the negative.

From what has been said above it is evident that the sixteen *sūtras* of Swamiji's *Vedic Mathematics* are his own compositions, and have nothing to do with the mathematics of the Vedic period. Although there is nothing Vedic in his book, Swamiji designates his Preface to the book as 'A Descriptive Prefactory Note on the Astounding Wonders of Ancient Indian Mathematics' and at places calls his mathematical processes as Vedic processes.

The deceptive title of Swamiji's book and the attribution of the sixteen *sūtras* to the *Pariśiṣṭas* of the *Atharvaveda*, etc., have confused and baffled the readers who have failed to recognise the real nature of the book, whether it is Vedic or non-Vedic. Some scholars, in their letters addressed to me, have sought to know whether the sixteen *sūtras* stated by Swamiji occurred anywhere in the Vedas or the Vedic literature.

Even the Rashtriya Veda Vidya Pratishthan, under whose auspices this Workshop on Vedic Mathematics has been organised, in their circular latter issued through the Ministry of Human Resource Development, are under the impression that the sixteen *sūtras* were actually reconstructed from materials in the various parts of the Vedas and the sixteen formulae contained in them were based on an Appendix of the *Atharvaveda*, which Appendix was not known to exist before the publication of Swamiji's book.

Let us now examine briefly the contents of that part of Swamiji's book which demonstrates the sixteen *sūtras*. These are divided into 40 chapters which run as follows:

Ch. 1 deals with the conversion of vulgar fractions into decimal or recurring decimal fractions. Here it may be remarked that nobody in the Vedic period could think of decimal or recurring decimal fractions. The decimal fractions were first introduced by the Belgian mathematician Simon Stevin in his book *La Disme* which was published in AD 1585. The decimal point (.) was used for the first time by Lemoch of Lemberg. The recurring decimal point ($.\dot{6}$ for .666 . . .) is the invention of Nicholas Pikes (AD 1788).

Chs. 2 and 3 deal with methods of multiplication and chs. 4 to 6 and 27 with methods of division. All these methods are quite different from the traditional Hindu methods.

Chs. 7 to 9 deal with factorisation of algebraic expressions, a topic which was never included in any work on Hindu algebra.

Ch. 10 deals with the H.C.F. of algebraic expressions. This topic also does not find place in Hindu works on algebra.

Chs. 11 to 14 and 16 deal with the various kinds of simple equations. These are similar to those occurring in modern works on algebra.

Chs. 1 and 20–21 deal with the various types of simultaneous algebraic equations. These are also similar to those taught to Intermediate students and do not occur in ancient Hindu works on algebra.

Ch. 17 deals with quadratic equations; ch. 18 with cubic equations; and ch. 19 with biquadratic equations.

Ch. 22 deals with successive differentiation, covering the theorems of Leibnitz, Maclaurin and Taylor, among others; ch. 23 with partial fractions; and ch. 24 with integration by partial fractions. These are all modern topics

Ch. 25 deals with the so called *Kaṭapayādi* system of expressing numbers by means of letters of the Sanskrit alphabet. It is called by Swamiji by the name 'the Vedic numerical code' although it has not been used anywhere in the Vedic literature.

Ch. 26 deals with the recurring decimals; ch. 28 with the so-called auxiliary fractions; and chs. 29 and 30 with divisibility and the so-called osculators. These topics too do not find place in the Hindu works on algebra.

Ch. 31 deals with the sum and difference of squares.

Chs. 32 to 36 deal with squaring and cubing, square-root and cube-root.

Ch. 37 deals with Pythagoras Theorem and ch. 38 with Appolonius Theorem.

Ch. 39 deals with analytical conics, and finally ch. 40 with miscellaneous methods.

From the contents it is evident that the mathematics dealt with in the book is far removed from that of the Vedic period. Instead, it is that mathematics which is taught at present to High School and Intermediate classes. It is indeed the result of Swamiji's own experience as a teacher of mathematics in his early life. Not a single method described is Vedic, but the Swamiji has declared all the methods and processes explained by him as Vedic and ancient.

Let us now say a few words regarding the mathematics which was known in the Vedic period, i.e. during the period ranging from c. 2500 BC to c. 500 BC.

Works of this period dealing exclusively with mathematics have not survived the ravages of time and our knowledge regarding mathematics of this period is based on the religious works of this period, viz. the Vedic Saṃhitās, the Brāhmaṇas and the Vedāṅgas. The religious works of the Buddhists and the Jainas and the *Āryabhaṭīya* of Āryabhaṭa (AD 499) give some idea of the development of mathematics from 500 BC to AD 500.

A study of the Vedic works reveals that by 500 BC the Hindus were well-versed in the use of numbers. They knew all the fundamental operations of arithmetic, viz. addition, subtraction, multiplication, division, squaring, cubing, square-root and cube-root. They were also well-versed in the use of fractional numbers and surds, mensuration and construction of simple geometrical figures, and could solve some algebraic problems also.

In arithmetic, they were masters of numbers and could use large numbers. They had developed an extremely scientific numeral terminology based on the scale of 10. In the *Yajurveda-saṃhitā* (*Vājasaneyi*, XVII.2) we have the following list of numeral denominations proceeding in the ratio of 10:

> *eka* (1), *daśa* (10), *śata* (100), *sahasra* (1000), *ayuta* (10000), *niyuta* (10^5), *prayuta* (10^6), *arbuda* (10^7), *nyarbuda* (10^8), *samudra* (10^9), *madhya* (10^{10}), *anta* (10^{11}), and *parārdha* (10^{12}).

The same list occurs in the *Taittirīya-saṃhitā* (IV.40.11.4 and VII.2.20.1), and with some alterations in the *Maitrāyaṇī* (II.8.14) and *Kāṭhaka* (XVII.10) *Saṃhitās* and other places.

The numbers were classified into even (*yugma*, literally meaning 'pair') and odd (*ayugma*, literally meaning 'not pair'). In two hymns of the *Atharvaveda* (XIX.22, 23), there seems to be a reference to the zero, as well as to the recognition of the negative number. The zero has been called *kṣudra* (trifling). The negative number is indicated by the term *anṛca*, while the positive number by *ṛca*.

The Vedic Hindus seem to have been interested in series or progressions of numbers as well. The following series are found to occur in the *Taittirīya-saṃhitā* (VII.2.12.17):

$$1, 3, 5, \ldots, 19, \ldots, 29, \ldots, 39, \ldots, 99;$$
$$2, 4, 6, \ldots, 20;$$
$$4, 8, 12, \ldots, 20;$$
$$5, 10, 15, \ldots, 100;$$
$$10, 20, 30, \ldots, 100.$$

The arithmetic series were classified into even (*yugma*) and odd (*ayugma*) series. The following examples of these two categories are given in the *Vājasaneyi-saṃhitā* (XVIII.24, 25):

$$4, 8, 12, \ldots 48;$$
$$1, 3, 5, \ldots 33.$$

Of these two series, the second one is found to occur also in the *Taittirīya-saṃhitā* (IV.3.10). In the *Pañcaviṃśa-brāhmaṇa* (XVIII.3) is given a list of sacrificial gifts which form the following series in geometrical progression:

$$24, 48, 96, 192, \ldots, 49152, 98304, 196608, 393216.$$

This series occurs also in the *Śrauta-sūtras.*

Some method for summing a series was also known. In the *Śatapatha-brāhmaṇa* (X.5.4.7), the sum of the series

$$3 \times (24 + 28 + 32 + \ldots \text{ to 7 terms})$$

is stated correctly as 756. And in the *Bṛhaddevatā* (III.13) the sum of the series

$$2 + 3 + 4 + \cdots + 1000$$

is stated correctly as 500499.

From the method indicated by Baudhāyana for the enlargement of a square by successive addition of gnomons, it seems that the following result was known to him:

$$1 + 3 + 5 + \cdots + (2n + 1) = n + 1.$$

From the following results occurring in the *Śulba-sūtras* we find that the Vedic Hindus knew how to perform fundamental operations with fractional numbers:

$$7\frac{1}{2} \div \frac{1}{25} = 187\frac{1}{2},$$

$$\left(2\frac{1}{7}\right)^2 + \left(\frac{1}{2} + \frac{1}{12}\right)\left(1 - \frac{1}{3}\right) = 7\frac{1}{2},$$

$$7\frac{1}{9} = 2\frac{2}{3},$$

$$7\frac{1}{2} \div \frac{1}{15} \text{ of } \frac{1}{2} = 225.$$

In geometry, the Vedic Hindus solved propositions about the construction of various rectilinear figures, combination, transformation and application of area, mensuration of areas and volumes, squaring of the circle and vice versa, and about similar figures. One theorem which was of the greatest importance to them on account of its various applications was the so-called Pythagoras Theorem. It has been enunciated by Baudhāyana (800 BC) thus: 'The diagonal of a rectangle produces both (area) which its length and breadth produce separately' (*Baudhāyana-Śulba*, 1.48). That is, the square described on the diagonal of a rectangle has an area equal to the sum of the areas of the squares described on its two sides.

The converse theorem, viz. 'If a triangle is such that the square on one side of it is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle', is not found to have been expressly stated by any Vedic geometrician. But its truth has been tacitly assumed by all of them and it has been most freely employed for the construction of a right angle.

In the course of construction of fire-altars, it was necessary to add together two or more figures such as squares, rectangles, triangles, etc., or subtract one of them from another. In the case of combinations of squares, mere application, repeated when necessary, of the Pythagoras Theorem was sufficient to get the desired result. But in the case of other figures, they had first to be transformed into squares before the theorem could be applied and the combined square was then used to be transformed into any desired shape.

The Vedic Hindus knew elementary treatment of surds. They were aware of the irrationality of $\sqrt{2}$ and attained a very remarkable degree of accuracy in calculating its approximate value, viz.

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}, \quad \text{nearly}$$

In terms of decimal fractions this works out to $\sqrt{2} = 1.4142156\ldots$. According to modern calculation $\sqrt{2} = 1.414213\ldots$, so that the Hindu approximation is correct up to the fifth place of decimals, the sixth place being too large.

There have been many speculations as to how the value of $\sqrt{2}$ was determined in that early time to such a high degree of approximation. The Kerala mathematician Nīlakaṇṭha (AD 1500) was of the opinion that Baudhāyana assumed each side of a square to consist of 12 units. Thus the square of its diagonal was equal to 2×12^2. Now

$$2 \times 12^2 = 288 = 289 - 1 = 17^2 - 1.$$

Therefore,

$$12\sqrt{2} = \sqrt{17^2 - 1} = 17 - \frac{2}{2 \times 17}, \quad \text{nearly.}$$

Hence

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}, \quad \text{nearly.}$$

The same hypothesis has been suggested by G. Thibaut (*Śulba-sūtras*, pp. 13ff). Several other methods have been given by B. Datta and others.

Another similar result is

$$\sqrt{3} = 1 + \frac{2}{3} + \frac{1}{3 \times 5} - \frac{1}{3 \times 5 \times 52}, \quad \text{nearly,}$$

whose derivation, as suggested by Nīlakaṇṭha, is as follows:

$$\begin{aligned}
\sqrt{3} &= \frac{1}{15}\sqrt{3 \times 15^2}, \\
&= \frac{1}{15}\sqrt{26^2 - 1}, \\
&= \frac{1}{15}\left(26 - \frac{1}{2 \times 26}\right), \quad \text{nearly,} \\
&= 1 + \frac{2}{3} + \frac{1}{3 \times 5} - \frac{1}{3 \times 5 \times 52}, \quad \text{nearly.}
\end{aligned}$$

A note on the *Rājamṛgāṅka* of Bhoja published by the Adyar Library *

K. Madhava Krishna Kumar Śarma of the Adyar Library published (1940) the "reconstructed" Sanskrit text of a calendrical work of Indian astronomy under the title: the *Rājamṛgāṅka* of Bhoja. The published text comprises two chapters: (1) *Madhyamādhikāra*, dealing with the mean motion of the planets in 34 stanzas, and (2) *Spaṣṭādhikāra*, dealing with the true motion of the planets in 52 stanzas. There are two additional stanzas bearing the name of the astronomer Rāma appended after the second chapter which for obvious reason do not form part of the text. The text is followed by three astronomical tables.

There is no chapter dealing with eclipses and the heliacal rising of the planets, etc., and the book ends rather abruptly. The last stanza of the *Rājamṛgāṅka*, viz.,

इत्युर्वीपतिवृन्दवन्दित-
पदद्वन्द्वेन सद्बुद्धिना ।
श्रीभोजेन कृतं मृगाङ्क-
करणं ज्योतिर्विदां प्रीतये ॥

which Dīkṣita quoted from a manuscript of that work consulted by him does not occur in Śarma's edition.[1] Three and a half other stanzas quoted by Dīkṣita from the same work do occur in his edition but there is remarkable difference of reading as can be seen by comparison:[2]

Dīkṣita's version	**Śarma's version**
नन्दाद्रीन्द्वग्निसंयुक्तान् भजेत्खाभ्राभ्रभानुभिः । शाकाब्दानविनष्टं तु भाजकाच्छेषमुत्सृजेत् ॥१७॥	नन्दात्यष्ट्यग्निसंयुक्तान् भजेत्खाभ्राभ्रभानुभिः । शाकाब्दानवशिष्टम् यद्- भाजकात्तत्समुत्सृजेत् ॥१६॥
तयोरल्पं द्विशत्याप्तं बीजं लिप्तादिकं पृथक् ।	तयोरल्पं त्रिगुणितं दशभक्तं विलिप्तिकाः ।

* K. S. Shukla, *Gaṇita*, Vol. 5, No. 2 (1954), pp. 149–151.

[1] See Dīkṣita, S. B., *Bhāratīya Jyotiṣaśāstra*, Second edition (1931), p. 238.

[2] The difference in the numbering of the verses is also notable.

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Dīkṣita's version	Śarma's version
त्रिभिः शरैर्भुवा द्व्यक्षैः बाणैस्तिथिभिरब्धिभिः ॥१८॥	त्रिभिः शरैर्भुवा द्व्यक्षैः बाणैस्तिथिभिरब्धिभिः ॥१७॥
द्विकेन यमलेनैवं गुण्यमर्कादिषु क्रमात् । स्वं ज्ञशीघ्रे धरासूनौ सूर्यपुत्रेऽपरेष्वृणम् ॥१९॥	द्वाभ्यां यमाभ्यां गुणितं बीजान्यर्कादिषु क्रमात् । स्वं ज्ञशीघ्रे धरासूनौ मन्दे पाते परेष्वृणम् ॥१८॥
शकः पञ्चाब्धिवेदोनः षष्टिभक्तोऽयनांशकाः ॥२५॥	शकात्कृताब्धिवेदोनात् षष्ट्याप्ता अयनांशकाः ॥२४॥

Āmarāja, son of Mahādeva, who lived at Ānandapura about 150 years after Bhoja, in his commentary on the *Khaṇḍakhādyaka*[3] of Brahmagupta, quotes the following four passages from the *Rājamṛgāṅka* but none of them occurs in Śarma's edition:

1. धनर्णगैः सहस्रांशोः फलाल्लङ्कोदयासुभिः ।
   हताद्राशिकलाप्तासुफलैः स्युः सूर्यवद्ग्रहाः ॥
2. त्रिभमन्त्यफलोपेतं पदमाद्यं द्वितीयकम् ।
   चक्रार्धान्तं व्यन्त्यफलं नवभान्तं तृतीयकम् ॥
   ततश्चतुर्थं चक्रान्तं पदानि प्रतिमण्डले ॥
3. स्पष्टमध्यमयोरैक्यं दलितं मध्यसंज्ञकम् ।
   स्पष्टमध्ययुतेरर्धात्कार्यं मन्दफलं ग्रहे ॥
   स्पष्टीकरणयोग्ये च तस्माच्छीघ्रफलं ततः ।
   तत्रैव त्वेवमसकृदवशेषात् स्फुटग्रहाः ॥
4. बिम्बार्धेन समं यस्य याम्यक्रान्तिभवं धनुः ।
   तस्मिन्मध्यदिनस्थेऽर्के या छाया विषुवत्यसौ ॥

The above-mentioned discrepancies in Śarma's edition create strong suspicion regarding its authenticity. There is reason to suspect that it is not the original and full text of the *Rājamṛgāṅka* but an abridged edition of that work by some later writer as is indicated by the second stanza of the first chapter wherein the author says:

> We give out *the essence of the Rājamṛgāṅka* for the computation of the planets.[4]

[3] This commentary was edited by Babua Misra and published by the Calcutta University in 1925.

[4] ब्रूमो राजमृगाङ्कस्य सारं सिद्ध्यै द्युसद्मनाम् ।

Review of *Rājamṛgāṅka* of Bhojarāja *

The *Rājamṛgāṅka* ascribed to the Paramāra king Bhoja of Dhar is the earliest *karaṇa* ("hand-book of astronomy") based on the teachings of the *Brāhmasphuṭasiddhānta* of Brahmagupta. It also incorporates at places teachings of the *Sūryasiddhānta*, the *Romakasiddhānta*, the *Khaṇḍakhādyaka* of Brahmagupta, the *Śiṣyadhīvṛddhida* of Lalla, the *Laghumānasa* of Mañjula, and other earlier works.

K. Madhava Krishna Sarma had earlier edited this work on the basis of the Adyar Library manuscript (shelf no. 8. D. 42). It contained 2 chapters only and was regarded as incomplete as 4 passages quoted from this work by Āmarāja in his commentary on the *Khaṇḍakhādyaka* were not found to occur in it.

David Pingree has now brought out a new edition giving the full text of this work on the basis of two manuscripts designated as F and G by him, the former acquired from the Bhandarkar Oriental Institute, Poona, and the latter from the Rajasthan Oriental Research Institute, Jodhpur. It contains 8 chapters dealing with mean motion, true motion, the three problems, rising and setting, elevation of Moon's horns, lunar eclipse, solar eclipse, and conjunction of planets.

Chapter 1 begins with the benediction **श्री गणेशाय नमः** which really does not come from the pen of Bhojarāja and should have been given in the apparatus. The editor has missed to see that vs. 6c is out of place (being a scribal repetition of vs. 9c). The correct version of vs. 6cd should have been

रविवारादिकः स स्यात् लङ्कामध्यार्यमोदयात् as in mss. A, C, D,

or

भवेदर्कादिः सप्ताप्तो लङ्कायां तपनोदये as in ms. D.

He has also missed to note that in framing the *Ahargaṇa* rule, 2 has been added to the *Caitrādi ahargaṇa*, so the epoch of the work is not sunrise of Tuesday, February 23, AD 1042 as stated by him in his introduction to the work, but 2 days earlier i.e., sunrise of Sunday, February 21, AD 1042 as stated by S. B. Dikshit in his *Bhāratīya Jyotiṣaśāstra*. Sunday sunrise has

* K. S. Shukla, *Gaṇita Bhāratī*, Vol. 14, Nos. 1–4 (1992), pp. 91–93 (The text *Rājamṛgāṅka* was edited by David Pingree; Aligarh Oriental Series, No. 7, Viveka Publications, Aligarh, 1987. Pages, 70).

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been clearly mentioned in the manuscripts. It is surprising that Prof. Pingree has missed to see it. There is also an error in vs. 15 where **शाकाब्दादवशिष्टं** is printed in place of **शाकाब्दानवशिष्टं**.

Vs. 7 of chap. 1 is based on a table but the table is missing. Vss. 7–8, 17–18, 20–21, 28–30, and 32–33 of chap. 2, as well as vss. 35 and 45–46 of chap. 3 too are based on tables but these tables are also missing. These tables did occur in mss. B, H, and I but have been omitted by the editor. This has rendered the text incomplete and obscure at those places.

In vs. 12 of chap. 3, **दिनपातोनं** should be read as **दिनयातोनं**. In vs. 5 of chap. 4, **सा यमज्येति** should be read as **साऽपमज्येति**. In vs. 61 of chap. 4, **द्विपद्द्रङ्गेति** should be read as **वियद्द्रङ्गेति** as in *Romakasiddhānta* (ix. 16); similarly **विशाखाग्रिम°** in vs. 63 should be read as **विशाखाश्विभ°** and **पश्चिमे** in vs. 64 as **पश्चिमा**.

Vs. 6 of chap. 3 has not been edited carefully. Two rules are stated there, not one as supposed by the editor. Thus, instead of

तज्ज्याप्ते घुदलार्कघ्ने नम्रत्रिज्ये प्रभाश्रुतिः ॥

there should be

तज्ज्याप्ते घुदलेऽर्कघ्ने नम्रत्रिज्ये प्रभाश्रुती ॥

In vs. 60ab of chap. 4, **ऽभीष्टधीवक्त्रसंज्ञिता** conveys no meaning. This hemistich should really be read as follows:

अपांवत्सस्य निकटे भेऽष्टधावक्रसंज्ञिता ।

as in ms. G. **अष्टधावक्रसंज्ञिता** means **अष्टावक्रसंज्ञिता**. It may be mentioned that Aṣṭāvakra is the name of a well known Indian sage (*ṛṣi*).

Variations in mss. designated as A, B, C, D, and E, each containing 2 chapters only, have been given in the Appendix, the manuscript used by Sarma being designated as D. Tables contained in mss. B, H, and I have been briefly described (not given in full) towards the end of the work.

The whole editing is based on collation and not much care has been taken to rectify the text and make it free from errors.

The four passages quoted by Āmarāja which did not occur in Sarma's edition do not occur in Prof. Pingree's edition also, although similarity is noted in one case. Manuscripts of this work existing in the libraries at Ahmedabad, Baroda (343), Jaipur, Jesalmere, Poona, and Udayapur have not been consulted. They might give some clue regarding the missing verses and reveal something new.

Prof. Pingree must be congratulated for bringing out the present edition for the benefit of scholars working in the field of Indian mathematics and astronomy.

Review of *Karaṇaratna* of Devācārya *

The present work is a very welcome addition to the series of works by which Prof. Shukla has sought to expand the available literature of Sanskrit astronomy. He has previously given scholars the work of Bhāskara, I, including his commentary on the *Āryabhaṭīya*, and also an edition of the *Sūryasiddhānta*. The present text is that of a short *karaṇa*, that is, handy set of astronomical formulae, in which the mean longitudes are referred to some epoch set around the time of composition, and expressed in terms of ordinary calendrical quantities, the Śaka year, *tithis* of current month, etc. The *Karaṇaratna* is known uniquely from a MS transcribed in AD 1097, from a palm leaf MS in Malayalam character. The date of composition would appear to be ca. AD 689, that at least being the epoch of the formulae. He also argues from various circumstances that the author, who is not known from any other source, was a southerner. Another work, which deals only with "*Mahāpāta* in the *Karaṇaratna*", is reproduced here also from two MSS in the Government Oriental Library, Mysore. This is an eclectic collection of verses, many of them decidedly later than AD 689, and which occasionally contradict the main work. Apart from the name given to it, it seems to have no connection at all with the *karaṇa*.

Prof. Shukla provides a full translation, together with an analysis of many of the technical details. Here and there I have had to disagree with some of the analysis, as I explain below. Shukla's explanation of basic technical notions is much less helpful than one would like, so that anyone unversed in the use of the calendrical terms, *avama*, *avamaśeṣa*, and the like, for example, will have to turn elsewhere.

The long introduction draws attention to all the unusual and important features, including: the calculation of solar and lunar longitudes in terms of the omitted lunar days (*avama*); the occurrence of three systems of corrections to the mean longitudes; the incorporation of a model of precession known hitherto from Āmarāja; a particularly detailed treatment of eclipses.

The opening verses provide the rules for determining the *ahargaṇa* and the mean longitudes of Sun, Moon, and planets. The epoch of the *karaṇa* is

* Raymond P. Mercier, *Gaṇita Bhāratī*, Vol. 4, Nos. 3–4 (1982), pp. 141–146 (The text *Karaṇaratna* was critically edited and translated into English with explanatory and critical notes and comments, etc., by Kripa Shankar Shukla, and got published as Hindu Astronomical and Mathematical Texts Series No. 5, Department of Mathematics and Astronomy, Lucknow University, 1979. Pages xii + 126).

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Table 1

	Āryabhaṭīya	*Karaṇaratna*	verse
Sun	335; 57, 7	336; 2, 18	i, 9–10, 15
		(335; 58, 36)	emended
Moon	347; 7, 34	347; 12, 4	i, 11
		(347; 8, 22)	emended
apogee	203; 6, 59	203; 1, 5	i, 12, 14–15
node	276; 34, 52	276; 34, 20	i, 13, 14–15
Mars	0; 47, 40	0; 47, 10	vii, 2
Mercury	48; 58, 2	48; 57, 14	vii, 3
Jupiter	192; 3, 9	192; 3, 0	vii, 4
Venus	265; 7, 7	265; 6, 38	vii, 5
Saturn	208; 58, 51	208; 58, 48	vii, 6

Sunrise 1st Caitra Śaka 511, separated by 1384306 days from the Kaliyuga; the Latin date is AD 689 Feb 26 (Julian date 1972772). This precise date is clearly established by Shukla in his commentary to i, 5–8. This epoch can hardly have been chosen for astronomical reasons. It may be contrasted, for example, with the epoch AD 638 March 21 of an earlier *karaṇa* known from calendrical usage in South East Asia. There was a total solar eclipse visible in central India on that date, very near to the time of the Spring Equinox.

This epoch is situated a fraction of a day later than the moment of the mean new Moon; the interval, called *avamaśeṣa*, is $\frac{644}{692}$ *tithis*, or $\frac{644}{703}$ days. Since the formulae of the *karaṇa* for the mean Sun, Moon, lunar apogee, lunar node, are given in terms of the lunar date, this value 644 of the *avamaśeṣa* must be used when determining those mean values at the epoch from these formulae. This point has been overlooked by Shukla, who is led consequently to propose a number of unnecessary emendations to the constant terms in the formulae.

The mean longitudes are intended to follow the *Āryabhaṭīya*, as indicated in i, 2, and one can see from Table 1 that this is achieved, at the time of the epoch, within one minute of arc in most cases.

In calculating from the *karaṇa* I have included none of the emendations which Shukla proposes, but have changed his reading in the case of the node, as noted below.

Notes on the formulae

Sun

There is certainly a need for some emendation of the parameter 699, which is given in i, 10a as *navanandarasa.* This parameter is used in the expression

$$(\textit{ahargaṇa} - \textit{avama})\left(1 + \frac{1}{699}\right)$$

which contributes to the degrees of the mean Sun. The value strictly implied by the *Āryabhaṭīya* is 656.56 . . . , a point quite overlooked by Shukla. If 699 were replaced by 657 (with no other change), the mean Sun and Moon at epoch would be closer to the calculated values, as seen in the above table.

Moon's apogee

The text of i, 12,

> 28 *yuta naṣṭadinaṃ 7-guṇamātma 968-aṃśayutam bhāgās taccandrocce* . . .

should be rendered

> to the *avama* days add 28, multiply by 7, increase the result by $\frac{1}{968}$, to obtain the degrees of the Moon's apogee, . . .

The complete formula is then

$$\text{apogee} = (\textit{avama} + 28)\left(7 + \frac{7}{968}\right) + 0;24 + \left[\frac{1}{99} - \frac{1}{(120 \times 60)}\right] \textit{avamaśeṣa.}$$

At epoch, when *avama* = 0, *avamaśeṣa* = 644, this gives 203; 1, 5.

Shukla's rendering, which may be expressed as

$$\left(7 + \frac{7}{968}\right) \textit{avama} + 28 + \ldots$$

is taken by him to agree with the *Āryabhaṭīya*, but he has made a large error in calculating from the latter: at the top of p. 11, 67580434614 should be 675844491014.

Moon's node

Shukla's calculation from the *Āryabhaṭīya* has two errors; for 13484806 read 1384306, for 193 read 205. He has, as in all the previous formulae, omitted to take the *avamaśeṣa* at epoch as 644, and so wrongly proposes to emend the 19 minutes of the formula (i, 13b) to 193.

When we come to the equations of Sun, Moon, and planets, we find that the *Āryabhaṭīya* has been followed quite strictly as regards the dimensions of the epicycles, but that two small differences are to be noted as regards the scheme of calculation. Firstly, the lunar equation is reduced by $\frac{1}{27}$ times the solar equation, a curious correction which is found also in the *Khaṇḍakhādyaka*. Secondly, the four–fold procedure in the calculation of the planetary equations is the same here for both the inner and outer planets, the same indeed as that used by the *Āryabhaṭīya* for the outer planets; in the case of the inner planets, the *Āryabhaṭīya* uses a shorter three–fold procedure.

It is interesting that the declination, ascension, and lunar latitude are given for 10° steps of the longitude. In the *Āryabhaṭīya* these functions are given for steps of 30°, but according to Bhāskara II (in his commentary to his *Grahagaṇita*, *Spaṣṭādhikāra*, v. 65), Āryabhaṭa calculated ascensions in steps of 10°. Bhāskara's remark has been wrongly understood to refer to the author of the *Mahāsiddhānta*, a work which we know now to be of the fifteenth century ([1], p. 161).† The attribution in this *karaṇa* therefore supports the view that Āryabhaṭa had, in some work now lost, calculated ascensions in steps of 10°.

The verses i, 16–21 describe three different systems of corrections to the mean longitudes of the *Āryabhaṭīya*. Two of these are already known, and have been studied by Billard ([1], p. 136 seq.) who has applied his delicate method of analysis to them. Their presence here may raise some problem of dating the *karaṇa*, or more reasonably, shows that it is composite in character, for Billard has shown that these systems could not be earlier than about AD 800. The first system (i, 16–8), which Shukla calls *śakābda*, gives a system known from the *Grahacāranibandhana-saṃgraha* (anon., 9th century), verses 17–22, and from a number of later sources. In fact these verses contain two systems, which Billard denotes **A** (v. 19–22) and **B** (v. 17–8), and in the *karaṇa* we have the lunar corrections from **A**, and the planetary corrections from **B**. The system was designed with great skill in such a way as to be accurate over the period AD 500–900, approximately; its construction is certainly to be dated towards the end of the ninth century. The second system (i, 19) which Shukla calls *kalpa* (following a reference to it by Parameśvara) is hitherto unknown. It involves, in practical terms, the addition of certain constants to the mean motions. For the mean Moon, for example, one subtracts $\frac{13}{8064}$ degrees per yuga elapsed from the beginning of the *kalpa*; this amounts to $27.75 \times \frac{13}{8064} \approx 0.74$ degrees at the Kaliyuga. I have examined the deviations, according to Billard's method, and do not find that they point cogently to any well defined date of observation. It may be that some of this text is corrupt. The third system (i, 20–1), which Shukla calls *manuyuga*, is known already

† Otherwise *Mahāsiddhānta* (of Āryabhaṭa II) is usually assigned the date circa 950 AD. —Editor (RCG)

from the commentary on the *Laghubhāskarīya* by Śaṅkaranārāyaṇa (AD 869). It is applied like the *kalpa* correction, amounting in practical terms to the addition of certain constants, Billard's analysis points clearly to a composite character, with an eventual completion ca. AD 820.

In i, 36 there is given a model of accession and recession which is of considerable interest. The argument of this motion is

$$A = \frac{360t}{7380}$$

where the time t is measured in years from Kaliyuga. The difficulty is that of deciding whether to take the *ayanāṃśa* as

$$-\arcsin(\sin 24 \sin A)$$

as Shukla wishes, or as

$$-\frac{24}{90}\,bhuja\,(A) = -\frac{24}{90}\arcsin(\sin A).$$

The difficult text is

> *tadoh (**ed.** sic) krāntijaliptikā ṛṇadhanaṃ syād*

which must be compared with two other closely similar accounts. Āmarāja gives, as Shukla notes, in his commentary to the *Khaṇḍakhādyaka* iii, 11, the very same model, but without any reference to the sine function,

> 3179 *yuk śākāt* 7380 *hṛtāt bhagaṇādeḥ krāntibhāgā ṛṇasvaṃ saumya-dakṣiṇāḥ.*

This is no longer, at least explicitly, a model of accession and recession, but simply a model of uniformly increasing precession, but at the same rate. Here *krāntibhāgā* can only mean 'divided by 24', with *krānti* being used as a word numeral.

In the *Mahāsiddhānta* (i, 11 and iii, 13) there is defined a model of accession and recession very nearly identical to this, but now expressed by a rather corrupt text,

> *ayanagrahadoḥ krāntijyācāpaṃ kendravaddhanarṇa syāt*

which might be thought to descend from the text of our *karaṇa.*

The rate of motion is 46.83 seconds per annum, or 46.25 according to the *Mahāsiddhānta.* This rate occurs in two other sources. In Bhāskara's commentary to the *Āryabhaṭīya*, which has been edited also by Shukla in 1976, we are told that according to the Romaka school, there was a model of accession and recession which may be expressed as

$$bhuja\,(A)$$

where A increased at the rate $180 \times \frac{137}{1894110}$ degrees per annum, that is 46.87 seconds p.a. Here unfortunately the zero point is not given. Further, as I have noted elsewhere [3], the comparison of the sidereal and tropical motions of the Sun in the Babylonian system A yields a rate of precession equal to 46.875 seconds p.a. It is certainly hard to avoid the conclusion that this well defined Indian usage, associated with the Romaka school, is historically related to much older material.

The relation to Hellenistic astronomy appears in another way, for this model given by Devācārya and Āmarāja, and indeed also the rather different one given in the later *Sūryasiddhānta* (iii, 9–12), both agree at the time of Hipparchus (ca. AD 126), when the *ayanāṃśa* $\doteqdot -9;20$. These models of precession therefore, as I have discussed at length elsewhere [2], are intended to relate tropical longitudes of the time of Hipparchus to the sidereal longitudes measured from the head of Revatī.

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A note on Raymond P. Mercier's review of "*Karaṇaratna* of Devācārya"*

Certain views expressed by Mercier have been discussed in this short note:

1 Curious lunar correction

Early Hindu astronomers have applied four corrections to mean longitude for mean sunrise at Laṅkā to get true longitude for true sunrise at the local place:

i *Deśāntara* ("Correction for local longitude")

ii *Bhujāphala* ("Equation of the centre")

iii *Bhujāntara* ("Correction for Sun's *Bhujāphala*")

iv *Cara* ("Correction for Sun's ascensional difference")

In the opinion of Mercier the *Bhujāntara* correction in the case of the Moon, stated in *Karaṇaratna*, i. 26, is a curious correction. As stated above this correction is a usual one and has been applied to the longitude of a planet (including the Sun and Moon) by all Hindu astronomers. *Karaṇaratna*, like *Khaṇḍakhādyaka*, states it in its abridged form but it is stated in general form in all Hindu *siddhāntas* including *Sūryasiddhānta*. Commenting on the rule of *Sūryasiddhānta* (ii, 46), Rev. E. Burgess remarks:

> By this rule, allowance is made for that part of the equation of time, or of the difference between mean and apparent solar time, which is due to the difference between the Sun's mean and true places.

It is not understandable what prompted Mercier to call it a "curious" correction.

2 Āryabhaṭa II's date

Bhāskara II has referred to Āryabhaṭa in connection with the ascensions of the "decans" (i.e., 10° steps of longitude). Since these ascensions of the decans exist in the *Mahāsiddhānta* (iv. 40–41) of Āryabhaṭa II, it was inferred

* K. S. Shukla, *Gaṇita-Bhāratī*, Vol. 6, Nos. 1–4 (1984), pp. 25–28.

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by S. B. Dikshit, S. Dvivedi and others that Āryabhaṭa II was anterior to Bhāskara II (1150 AD) and his date was tentatively fixed at 950 AD. David Pingree, too, on the same ground, has put him between 950 AD and 1150 AD. Mercier does not agree with all this. He asserts that Bhāskara II in that passage refers to Āryabhaṭa I and not to Āryabhaṭa II and in support of this view he says that Devācārya, the author of *Karaṇaratna*, who is a follower of Āryabhaṭa I, states the ascensions for 10° steps of longitude; but where, he does not say. Mercier's assertion that Devācārya states ascensions for 10° steps of longitude is wrong. Actually Devācārya, like Āryabhaṭa I, gives ascensions for 30° steps of longitude. See *Karaṇaratna*, iv. 6. Mercier's conclusion is therefore untenable.

3 *Śakābda*, *Manuyuga*, and *Kalpa* corrections

These three corrections were devised and used by the astronomers of Kerala Pradesh in South India. According to K. V. Sarma, an authority on Kerala astronomy, the *Śakābda* correction was devised by Haridatta who introduced the *Parahita* system of astronomy in Kerala in 683 AD. This is affirmed by the Malayalam and Sanskrit works on astronomy written in Kerala. This correction, says Mercier, was designed with great skill in such a way as to be accurate over the period AD 500–900. If it is so, how does it follow, as asserted by Mercier, that "its construction is certainly to be dated towards the end of the ninth century"? The *Manuyuga* correction occurs in *Karaṇaratna* and in Śaṅkaranārāyaṇa's commentary (869 AD) on the *Laghubhāskarīya* of Bhāskara I. Regarding the origin of this correction, Śaṅkaranārāyaṇa remarks:

> Another *madhyama-saṃskāra* (viz. *Manuyuga* correction) too was devised by Āryabhaṭa himself—this is what some (scholars) say.

It means that this correction was old and Śaṅkaranārāyaṇa was unaware of its origin. It it were devised in 820 AD as Mercier seems to think, this fact must have been known to Śaṅkaranārāyaṇa or at least to his teacher Govinda who lived about that time. This date is contradicted also by its occurrence in *Karaṇaratna*. The *Kalpa* correction has been stated in *Karaṇaratna* alone. It does not occur in any other work. The *Manuyuga* and *Kalpa* corrections are indeed old, certainly older than Devācārya. They do not seem to have gained any popularity, and were never used after the ninth century AD.

4 *Ayanacalana* or movement of the solstices or equinoxes

As regards the *ayanacalana* the Hindu astronomers are divided. Some follow the theory of oscillatory motion, while others the theory of progressive backward motion. Devācārya is a follower of the theory of oscillatory motion and takes 24° as the amplitude of oscillation. According to *Sūryasiddhānta* (iii. 9–10), which follows the same theory, the amplitude of oscillation is 27°. But, unlike *Sūryasiddhānta* which gives the amount of *ayanacalana* in the form

$$\frac{3}{10}\ (\textit{bhuja}\ A) \tag{1}$$

Karaṇaratna gives it in the form

$$\text{arc}\ (\sin 24^\circ \sin\ \textit{bhuja}\ A), \tag{2}$$

where A is the longitude of an imaginary body called the *Ayanagraha*.

There is no doubt that *Karaṇaratna* states it in the form (2). For the Sanskrit text *"taddoḥkrāntijaliptikāḥ"* undoubtedly means "The minutes corresponding to the declination derived from the *bhuja* of those (degrees of the longitude of the *Ayanagraha*)." Evidently the author of *Karaṇaratna* has used the proportion

$$R\sin 90^\circ : R\sin 24^\circ :: R\sin(\textit{bhuja}\ A) : R\sin(\textit{ayanacalana})$$

instead of the proportion

$$90^\circ : 24^\circ :: \textit{bhuja}\ A : \textit{ayanacalana}$$

used by *Sūryasiddhānta*.

The theory of oscillation is indeed incorrect but sometimes it gives a fairly good rate of *ayanacalana*. According to *Sūryasiddhānta*, for example, it is 54″ per annum.

Mercier's suggestion that the word "*krāntibhāgāḥ*" in the verse quoted by Āmarāja can only mean "divided by 24", besides being irrelevant to the context, is against the interpretation of Āmarāja who explains the verse as follows:

> Add 3179 to the (current) *Śaka* year and then divide by 7380: the result in involutions etc. is (the longitude of) the *Ayanagraha*. The degrees in the arc of the declination derived from that (longitude) are called the degrees of *ayanāṃśa*.

Exactly similar rule has been given by Āryabhaṭa II in his *Mahāsiddhānta* (iii. 13). Sudhākara Dvivedi and S. R. Sharma who have explained and translated *Mahāsiddhānta* did not see any flaw in the text giving that rule and have interpreted it as it should be. But to Mercier that text seems to be corrupt, because it does not state the *ayanacalana* in the form stated in *Sūryasiddhānta*. The fact is that both forms (1) and (2) are found to occur in Hindu works. It is, however, true that both forms are based on wrong hypotheses. Apart from that, the rule quoted by Āmarāja and that stated in *Mahāsiddhānta* give rather low rates of *ayanacalana*.

5 Sun's longitude

The formula for the Sun's longitude stated in *Karaṇaratna* as pointed out by Mercier is indeed gross. The Sanskrit text was however correct and there was no reason to alter it. So it was kept as found in the manuscript. Likewise keeping the divisor 699 intact and following the rationale intended by the author of *Karaṇaratna* the Sun's longitude was derived in the form

$$\begin{aligned}&(A - 24 - \textit{avama}\text{ days})\text{ degrees}\\ &\quad + \frac{(A - 33 - \textit{avama}\text{ days} - \textit{avamaśeṣa})}{699}\text{ degrees} + \frac{\textit{avamaśeṣa}}{2730\text{ mins.}}\end{aligned} \tag{3}$$

denoting the omitted days corresponding to *ahargaṇa* A by

$$\frac{\textit{avama}\text{ days} + \textit{avamaśeṣa}}{703}. \tag{4}$$

There was no need of the *avamaśeṣa* for the epoch, although it could be introduced by adding to (4) the expression

$$\frac{\textit{avamaśeṣa}\text{ for epoch} - 644}{703}$$

(equivalent to zero).

Mercier has suggested that the correct expression for the Sun's longitude should be

$$\begin{aligned}&(A - 23 - \textit{avama}\text{ days})\text{ degrees}\\ &\quad + \frac{(A - 32 - \textit{avama}\text{ days} - \textit{avamaśeṣa})}{657}\text{ degrees} + \frac{\textit{avamaśeṣa}}{1960\text{ mins.}}.\end{aligned} \tag{5}$$

There can be no objection to this because the divisor 657 is certainly better than 699, but it must be noted that the number 1960 does not agree with 657. Moreover, if form (5) is taken the Sun's longitude at epoch comes out to be equal to 11 signs $5°58'36''$ (as calculated by Mercier himself) which exceeds the value obtained from Āryabhaṭa I's constants by $1'29''$. If however, form (3) is taken, the Sun's longitude at epoch comes out to be $(-24\frac{33}{999})$ degrees or $(-24°2'50'')$ or 11 signs $5°57'10''$ which is only $3''$ in excess over that obtained from Āryabhaṭa I's constants.

Thanks are indeed due to Mercier for pointing out numerical errors in the case of Moon's apogee and ascending node.

The *yuga* of the *Yavanajātaka*: David Pingree's text and translation reviewed *

Introduction

The *Yavanajātaka* written by Sphujidhvaja Yavaneśvara in the third century AD was edited and translated into English by Prof. David Pingree in 1978. The last chapter (Ch. 79) of this work is called *Horāvidhi* and deals with luni-solar astronomy on the basis of a period of 165 years called *yuga* and the synodic motion of the planets. The text is marred by faulty editing, the incorrect readings being adopted and the correct ones given in the apparatus criticus, with the result that the translation is incorrect at places and the meaning really intended by the author is lost.

The object of the present paper is to study this chapter so as to bring out the meaning really intended by the author. The paper will be confined to the study of the *yuga* of the *Yavanajātaka* and its various constituents. In the process the relevant passages and their translation as given by Prof. Pingree will be reviewed and modified.

1 Time-measures

Verses 28–29 of Ch. 79 of the *Yavanajātaka* give a table of time-measures. Pingree's text and translation run thus:

त्रयः पलाः स्युः कुडवोऽष्टमश्च
तन्नाडिकाख्यं विदुरेकषष्टिम् ।
ताः षष्टिलिप्तापि च नाडिकाख्या
भवन्ति षष्टिर्द्युनिशा क्रमेण ॥२८॥
कला निमेषाष्टशता दशोना
विदुः कलास्त्रिंश [च] नाडिका तु ।
द्विनाडिकस्तु प्रथितो मुहूर्तो
मानप्रमाणादिविधिप्रसिद्धौ ॥२९॥

trayaḥ palāḥ syuḥ kuḍavo'ṣṭamaśca
tannāḍikākhyaṃ vidurekaṣaṣṭim |

* K. S. Shukla, *Indian Journal of History of Science*, Vol. 24, No. 4 (1989), pp. 211–223.

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tāḥ ṣaṣṭiliptāpi ca nāḍikākhyā
bhavanti ṣaṣṭirdyuniśā krameṇa ||28||
kalā nimeṣāṣṭaśatā daśonā
viduḥ kalāstriṃśa |*ca*| *nāḍikā tu* |
dvināḍikastu prathito muhūrto
mānapramāṇādividhiprasiddhau ||29||

A *kuḍava* is $3\frac{1}{8}$ *palas*, and 61 *kuḍavas* equal 1 *nāḍikā*. The *nāḍikās* are also each divided into 60 *liptās* ("minutes"); there are 60 *nāḍikās* in a nychthemeron. One *kalā* equals 790 (?) *nimeṣas*, one *nāḍikā* 30 *kalās*, and one *muhūrta* 2 *nāḍikās* in the accomplishment of the rules relating to measures and standards.

Remarks

(1) We find that according to this translation one *nāḍikā* is equal to 30 *kalās*, whereas in the formulation of the rules stated in vss. 11, 12, and 13 one *muhūrta* ("a period of 2 *nāḍikās*") is taken equal to 20 *kalās*, and in verse 31 also one *kalā* has been used in the sense of $\frac{1}{10}$ of a *nāḍikā* or $\frac{1}{20}$ of a *muhūrta*. This discrepancy is due to adoption of the incorrect reading "*kalāstriṃśa* |*ca*|" (in vs. 29b) in place of the correct reading "*kalāstā daśa*" which has been given in the apparatus criticus. Restoring the correct reading in place of the incorrect one, we find that the text gives the following table:

$3\frac{1}{8}$ *palas*	=	1 *kuḍava*
61 *kuḍavas*	=	1 *nāḍikā*
60 *liptās*	=	1 *nāḍikā*
60 *nāḍikā*s	=	1 nychthemeron
790 *nimeṣas*	=	1 *kalā*
10 *kalās*	=	1 *nāḍikā*
2 *nāḍikās*	=	1 *muhūrta* (or *kṣaṇa*).

Likewise

20 *kalās*	=	1 *muhūrta*
30 *muhūrtas*	=	1 nychthemeron.

It is these two relations that have been used in verses 11, 12, and 13. The same relations were given by Suśruta[1] and Parāśara.[2]

[1]See *Suśruta-saṃhitā*; *Sutrasthāna*, ch. vi. 4.
[2]See *Bṛhat-saṃhitā* with Bhattotpala's commentary, Sudhakara Dvivedi's edition, p. 24, lines 3–5.

(2) It is noteworthy that according to the *Vedāṅga-jyautiṣa*[3] too,

$$3\frac{1}{8}\ \textit{palas} = 1\ \textit{kuḍava},$$
$$\text{and}\quad 61\ \textit{kuḍavas} = 1\ \textit{nāḍikā}.$$

But there

$$1\ \textit{nāḍikā} = 10\frac{1}{20}\ \textit{kalās}.$$

It seems that Sphujidhvaja Yavaneśvara has taken

$$1\ \textit{nāḍikā} = 10\ \textit{kalās}$$

to avoid fractions, or he has followed Suśruta or Parāśara.

2 *Tithis* in the *yuga*

Verse 6 gives the number of *tithis* in the *yuga* ("a period of 165 years"). Pingree's text and translation run thus:

> क्रमेण चन्द्रः क्षयवृद्धिलक्ष्य-
> स्तिथिश्चतुर्मानविधानबीजः ।
> षट्पञ्चकाग्रे द्विशते सहस्रं
> तेषां युगे बिन्दुयुतानि षट् च ॥६॥
>
> *kramеṇa candraḥ kṣayavṛddhilakṣya-*
> *stithiścaturmānavidhānabījaḥ |*
> *ṣaṭpañcakāgre dviśate sahasraṃ*
> *teṣāṃ yuge binduyutāni ṣat ca ||6||*
>
> The Moon is to be characterised by waning and waxing in order. The *tithi* possesses the seed of the principles of the four (systems of time-) measurement. There are 60,265 (days) in a *yuga*.

Remarks

The last sentence of this translation is wrong. The number 60,265 as well as its designation as "days" both are incorrect. The word *ṣaṭpañcaka* means 6×5 i.e. 30, not 65; and the word "*teṣāṃ*" refers to *tithis*, not to civil days. Moreover, the number of civil days in a *yuga* is 60,272, not 60,265. See below.

The second half of the text really gives the number of *tithis* in the *yuga*, not the number of civil days in the *yuga* as supposed by Pingree. The error is due to faulty editing of the text. The adoption of the incorrect readings "°*kāgre dviśate*" and "*bindu*" in place of the correct readings "°*kāgrā dviśatī*"

[3] See *Yājuṣa-jyautiṣa*, vs. 24.

and "*viddhya*", respectively has spoiled the text. It is noteworthy that the correct readings are given in the apparatus.

The correct reading of the text is:

क्रमेण चन्द्रक्षयवृद्धिलक्ष्य-
स्तिथिश्चतुर्मानविधानबीजः ।
षट्पञ्चकाग्रा द्विशती सहस्रं
तेषां युगे विद्ध्ययुतानि षट् च ॥६॥

krameṇa candrakṣayavṛddhilaksya-
stithiścaturmānavidhānabījaḥ |
ṣaṭpañcakāgrā dviśatī sahasraṃ
teṣāṃ yuge viddhyayutāni ṣaṭ ca ||6||

The *tithi*, which is the indicator of the gradual waning or waxing of the Moon, is the seed of the principles of the four (systems of time-) measurement. Know that there are 60000 plus 1000 plus 200 and 6 × 5 (i.e. 61,230) of them (in a *yuga*).

That is,

$$\text{one } yuga = 61230 \; tithis,$$

or 2041 synodic months, as stated in vss. 9 and 20 c–d.

3 Civil days in the *yuga*

Verse 7 gives the number of civil days in the *yuga*. Pingree's text and translation run thus:

त्रिंशन्मुहूर्तं दिनरात्रमुक्तं
सूर्योदयात् कालबुधास्तदाहुः ।
तेषां शते द्वे त्रिशदेककाग्रे
षट् खायुतान्यर्कयुगं वदन्ति ॥७॥

triṃśanmuhūrtam dinarātramuktaṃ
sūryodayāt kālabudhāstadāhuḥ |
teṣām śate dve triśadekakāgre
ṣaṭ khāyutānyarkayugaṃ vadanti ||7||

A nychthemeron is said to consist of 30 *muhūrtas*; experts on time say that it beings with sunrise. They say that a *yuga* of the Sun consists of 61,230 (*tithis*).

Remarks

The second sentence of this translation, though mathematically correct, is not the correct translation of the second half of the text. The number 61330 and its designation as "*tithis*" both are wrong. The word "*triśat*" means 300, not 30; and it is difficult to interpret "*ekakāgre ṣaṭ khāyutani*" as meaning 61000. Also, the word "*teṣāṃ*" refers to nychthemera or civil days, not to *tithis*.

The second half of the verse really gives the number of civil days in a *yuga*, not the number of *tithis* in a *yuga* as supposed by Pingree. The error is due to the faulty editing of the text. The adoption of the incorrect readings "*triśadekakāgre*" and "*ṣaṭ khā°*" in place of the correct readings "*trikṛdaṣṭakāgre*" and "*ṣaṭkā°*" respectively has marred the text. It is noteworthy that the correct readings are given in the apparatus.

The correct reading of the text is:

> त्रिंशन्मुहूर्तं दिनरात्रमुक्तं
> सूर्योदयात् कालबुधास्तदाहुः ।
> तेषां शते द्वे त्रिकृदष्टकाग्रे
> षट्कायुतान्यर्कयुगं वदन्ति ॥७॥
>
> *triṃśanmuhūrtaṃ dinarātramuktaṃ*
> *sūryodayāt kālabudhāstadāhuḥ* |
> *teṣāṃ śate dve trikṛdaṣṭakāgre*
> *ṣaṭkāyutānyarkayugaṃ vadanti* ||7||
>
> A nychthemeron (civil day) is said to consist of 30 *muhūrtas*; experts on time say that it begins with sunrise. They say that a *yuga* of the Sun consists of 60000 plus 200 plus $3^2 \times 8$ (i.e. 60,272) of them (i.e. civil days).

That is,

$$\text{one } yuga = 60272 \text{ civil days.}$$

The word "*trikṛt*" means 3^2 i.e. 9, and the word "*trikṛdaṣṭaka*" $3^2 \times 8$ i.e. 72.

Further remarks on vss. 6 and 7

Pingree is aware of the fact that the second half of vs. 6 should contain the number of *tithis* in a *yuga* and the second half of vs. 7 the number of civil days in a *yuga*, but his text has landed him in trouble and he remarks: "A more logical order might be achieved by interchanging 6 c–d with 7 c–d." He also complains about Sphujidhvaja Yavaneśvara's way of expressing numbers in verse: "The extreme clumsiness with which Sphujidhvaja expresses numbers is a reflection of the fact that a satisfactory and consistent method of versifying them had not yet been devised in the late third century." But these remarks are uncalled for, as it is all due to the faulty edited text.

4 Civil days in a solar year

Verse 34 gives the number of civil days in a solar year. Pingree's text and translation run thus:

सपञ्चषष्टिं त्रिशतं दिनानां
द्यूनं द्विभिन्नं तु दिनांशकानाम् ।
त्र्यूनं शतार्धं दिनकृत्समा स्याद्
यया भवर्गं सविता भुनक्ति ॥३४॥

sapañcaṣaṣṭiṃ triśataṃ dinānāṃ
dyūnaṃ dvibhinnaṃ tu dināṃśakānām |
tryunaṃ śatārdhaṃ dinakṛtsamā syād
yayā bhavargaṃ savitā bhunakti ||34||

A year of the Sun consists of 365 days and 14; 47 sixtieths (*amśas*) of a day, in which the Sun traverses the signs.

Remarks

This translation is incorrect, because "14; 47 sixtieths" does not yield the value of the solar year according to Sphujidhvaja. For, according to this translation

$$\text{one solar year} = 6, 5; 14, 47 \text{ days},$$

whereas according to Sphujidhvaja

$$\text{one solar year} = 6, 5; 17, 5, 27, 16 \text{ days}.$$

The error is due to the adoption of the incorrect reading "*dyūnaṃ dvibhinnaṃ*" in place of the correct reading "*yugādvibhinnaṃ*" given in the apparatus.

The correct reading of the text is:

सपञ्चषष्टिं त्रिशतं दिनानां
युगाद्विभिन्नं तु दिनांशकानाम् ।
त्र्यूनं शतार्धं दिनकृत्समा स्याद्
यया भवर्गं सविता भुनक्ति ॥३४॥

sapañcaṣaṣṭiṃ triśataṃ dinānāṃ
yugādvibhinnaṃ tu dināṃśakānāṃ |
tryūnaṃ śatārdhaṃ dinakṛtsamā syād
yayā bhavargaṃ savitā bhunakti ||34||

A *yuga* of the Sun consists of 365 days and a fraction of a day equal to fifty minus three divided by (the number of years in) a *yuga*, in which the Sun traverses the signs.

That is,

$$\text{one solar year} = 365 + \frac{50-3}{165} \text{ civil days}$$
$$= \frac{60272}{165} \text{ civil days.}$$

This result confirms the statement of vs. 7 that there are 60,272 days in a *yuga* (consisting of 165 years).

5 Civil days in a solar month

Verse 11 defines a civil month and gives the number of civil days etc. in a solar month. Pingree's text and translation run thus:

> त्रिंशद्दिनाः सावनमास आर्क-
> स्त्र्यग्रैर्विशिष्टा दशभिर्मुहूर्तैः ।
> कलाचतुष्केण च पञ्चषट्कै-
> स्त्र्यग्र्यांशकैश्च द्विगुणैश्चतुर्भिः ॥११॥
>
> *triṃśaddināḥ sāvanamāsa ārka-*
> *stryagrairviśiṣṭā daśabhirmuhūrtaiḥ* |
> *kalācatuṣkeṇa ca pañcaṣaṭkai-*
> *stryagryāṃśakaiśca dviguṇaiscaturbhiḥ* ||11||
>
> A civil month equals 30 days, a solar month equals (a civil month) plus 13 *muhūrtas* and 4 *kalās* and 56 thirds and 2 fourths.

Remarks

Here the text is correct[4] but the translation incorrect. For, "*pañcaṣaṭka*" means 5 × 6 i.e. 30, not 56; also "*tryagryāṃśaka*" does not mean third, nor "*catur*" fourth. Moreover, according to this translation,

$$\text{one solar month} = 30; 26, 9, 52, 4 \text{ days}$$

whereas, according to Sphujidhvaja,

$$\text{one solar month} = 30; 26, 25, 27, 16 \text{ days.}$$

The correct translation is:

> A civil month equals 30 days, a solar month is greater (than that) by 10+3 *muhūrtas*, 4 *kalās*, and $\frac{2\times4}{5\times6+3}$ of a *kalā*.

[4]Read °*rviśisto* in place of °*rviśiṣṭā*.

Thus,

$$\text{one solar month} = 30 \text{ days} + 13 \ \textit{muhūrtas} + 4\frac{8}{33} \ \textit{kalās}$$
$$= \frac{60272}{1980} \text{civil days,}$$

because 20 *kalās* = 1 *muhūrta* and 30 *muhūrtas* = 1 civil day.

This result also confirms the statement of vs. 7 that there are 60272 civil days in a *yuga*.

6 Civil days in a synodic month

Verse 12 gives the number of civil days etc. in a synodic month. Pingree's text and translation run thus:

> अह्नस्तु षट्पञ्चकमेकहीनं
> क्षणाष्टकौ द्वौ द्विकलाविहीनौ ।
> कलालवाः सप्त शतं विदिष्टः
> समासभिन्नः शशिनः स मासः ॥१२॥
>
> *ahnastu ṣaṭpañcakamekahīnaṃ*
> *kṣaṇāṣṭakau dvau dvikalāvihīnau* |
> *kalālavāḥ sapta śataṃ vidiṣṭaḥ*
> *samāsabhinnaḥ śaśinaḥ sa māsaḥ* ||12||
>
> A (synodic) month of the Moon, which ends with a conjunction, consists of 29 days and 32 *kṣaṇas* minus 4 *kalās* and 107 sixtieths of a *kalā*.

Remarks

This translation is based on misinterpretation of the text and does not accord to the teaching of Sphujidhvaja. For, according to this translation,

$$\text{one synodic month} = 30; 3, 55, 34 \text{ days,}$$

whereas according to Sphujidhvaja.

$$\text{one synodic month} = 29; 31, 50, 14, 24 \text{ days.}$$

The error is really due to the adoption of the incorrect readings "*ahnastu*", "*śataṃ vidiṣṭaḥ*", and "*samāsabhinnaḥ*" in place of the correct readings "*ahnāṃ tu*", "*śatī dviṣaṣṭā*", and "*svamāsabhinnā*" respectively which are given in the apparatus.

Thus, the correct reading of the text is:

अह्नां तु षट्पञ्चकमेकहीनं
क्षणाष्टकौ द्वौ द्विकलाविहीनौ ।
कलालवाः सप्तशती द्विषष्टा
स्वमासभिन्ना शशिनः स मासः ॥१२॥

ahnāṃ tu ṣaṭpañcakamekahīnaṃ
kṣaṇāṣṭakau dvau dvikalāvihīnau |
kalālavāḥ saptaśatī dviṣaṣṭā
svamāsabhinnā śaśinaḥ sa māsaḥ ||12||

6 × 5 − 1 days, 2 × 8 *kṣaṇa*s (*muhūrtas*) minus 2 *kalās*, and a fraction of a *kalā* equal to 762 divided by (the number of) its own (i.e. synodic) months (in a *yuga*): this is (the length of) the (synodic) month of the Moon.

That is,

$$\text{one synodic month} = 29 \text{ days} + (16\ muh\bar{u}rtas - 2\ kal\bar{a}s) + \frac{762}{2041}\ kal\bar{a},$$

because there are 2041 synodic months in a *yuga*, $= \frac{60272}{2041}$ civil days, because 20 *kalās* = 1 *muhūrta* and 30 *muhūrtas* = 1 civil day.

This again confirms that there are 60272 civil days in a *yuga*.

7 Civil days in a sidereal month

Verse 13 gives the length of a sidereal month in terms of civil days, etc. Pingree's text and translation run thus:

आर्क्षस्तु कृत्त्रिर्द्विगुणस्तु कृच्च
क्षणाः क्षणार्धं च कलाश्च तिस्रः ।
कलांशकानां च त्रिसप्तकाग्रं
शतं विभक्तो दलितैः समासैः ॥१३॥

ārkṣastu kṛttrirdviguṇastu kṛcca
kṣaṇāḥ kṣaṇārdhaṃ ca kalāśca tisraḥ |
kalāṃśakānāṃ ca trisaptakāgraṃ
śataṃ vibhakto dalitaiḥ samāsaiḥ ||13||

A sidereal month consists of 27 days plus $8\frac{1}{2}$ *kṣaṇas* and 3 *kalās* and 137 sixtieths of a *kalā*: it is separated by half-conjunctions(?).

Remarks

The first line of the text is corrupt and the translation is arbitrary and wrong. "*Trisaptaka*" does not mean 37; it means 3×7 or 21. It is difficult to understand how the first line has been interpreted in that way.

According to the above translation,

$$\text{one sidereal month} = 27;17,10,34 \text{ days}$$

whereas, according to Sphujidhvaja,

$$\text{one sidereal month} = 27;19,18,39 \text{ days.}$$

The correct text is:

आर्क्षस्त्रिकृत्त्रिर्द्युगणस्त्रिकृच्च
क्षणाः क्षणार्धं च कलाश्च तिस्रः ।
कलांशकानां च त्रिसप्तकाग्रं
शतं विभक्तं दलितैः स्वमासैः ॥१३॥

ārkṣastrikṛttrirdyugaṇastrikṛcca
kṣaṇāḥ kṣaṇārdhaṃ ca kalāśca tisraḥ |
kalāṃśakānāṃ ca trisaptakāgram
śataṃ vibhaktaṃ dalitaiḥ svamāsaiḥ ||13||

A sidereal month consists of $3^2 \times 3$ days, 3^2 *kṣaṇas* (*muhūrtas*) plus half a *kṣaṇa*, 3 *kalās* plus a fraction of a *kalā* equal to 121 divided by half (the number) of its own (i.e. sidereal) months (in a *yuga*).

That is,

$$\text{one sidereal month} = 27 \text{ days} + 9\frac{1}{2}\ \textit{muhūrtas} + 3\frac{121}{1103}\ \textit{kalās},$$

because there are 2206 sidereal months (or Moon's revolutions) in a *yuga*,

$$= \frac{60272}{2206} \text{ civil days,}$$

because 20 *kalās* = 1 *muhūrta* and 30 *muhūrtas* = 1 civil day.

This is true because there are 60272 civil days and 2206 sidereal months in a *yuga*.

8 Intercalary days in a solar year

Verse 19(a–c) gives the number of intercalary days in a solar year and the number of intercalary months in a given number of solar years. Pingree's text and translation run thus:

एकादशैकाद[श] भागयुक्त्या
युगाद्गताब्दान् विहतान् विभज्य ।
षड्द्व्यङ्ककेनाधिकमासकास्ते
... ... ॥१९॥

ekādaśaikāda[*śa*] *bhāgayuktyā*
yugādgatābdān vihatān vibhajya |
ṣaṭpañcakenādhikamāsakāste
... ... ||19||

The number of years which have passed of the *yuga* is to be multiplied by 11; 11 and divided by 30: (the result is the number of lapsed) intercalary months.

Remarks

The text is correct with one exception that there should be "*yutya*" in place of "*yuktyā*" in the first line. But the translation is erroneous because the number 11; 11 (denoting $11\frac{11}{60}$) is wrong. There are $11\frac{1}{11}$ intercalary days in a solar year, not $11\frac{11}{60}$. The correct translation is:

> The number of years which have passed of the *yuga*, multiplied by $11\frac{1}{11}$ and divided by 30 gives the number of intercalary months (in that period).

This is true because there being 1980 solar months and 2041 synodic months in a *yuga*, there are 61 intercalary months in a *yuga*. Likewise there are $\frac{61\times30}{165}$ or $11\frac{1}{11}$ intercalary days in a year.

9 Omitted *tithis* in a *yuga*

Verse 5 given length of a *tithi* in terms of civil days, the length of a civil day in terms of *tithis*, and the number of omitted *tithis* in a *yuga*, Pingree's text and translation run thus:

दिनं चतुः षष्टिलवोनमाहु-
स्तिथिं प्रषष्ट्यन्त्यमहस्तु सर्वम् ।
द्विषष्टिभागं नवतिः सहस्रं
युगे त्वृतूनामपशुद्धशतम् ॥५॥

dinaṃ catuḥṣaṣṭilavonamāhu-
stithiṃ praṣaṣṭyantyamahastu sarvam |
dviṣaṣṭibhāgaṃ navatiḥ sahasraṃ
yuge tvṛtūnāmapaśuddhaśatam ||5||

They say that a *tithi* equals a day minus $\frac{1}{64}$th, but that every day equals a *tithi* plus $\frac{1}{60}$th. In a *yuga* there are 990 seasons (*ṛtu*), (each) consisting of 62 (*tithis*).

Remarks

1. This translation is incorrect, because

   (i) if one *tithi* consists of $1 - \frac{1}{64}$ civil day, a civil day cannot be equal to $1 + \frac{1}{60}$ *tithis*; and

   (ii) if there 990 seasons in a *yuga* and 62 *tithis* in a season, there must be 990×62 or 61380 *tithis* in a *yuga*. but according to vs. 6 there are only 61230 *tithis* in a *yuga*.

2. The text given by Pingree is faulty, because he has adopted the incorrect reading "*dviṣaṣṭibhāgaṃ navatiḥ*" in place of the correct reading "*triṣaṣṭibhāgena yutam*" and the incorrect reading "*tvṛtūnāmapaśuddhaśatam*" in place of the correct reading "*'vamānāmapasaptaṣaṭkam*". Partially correct readings occur in the apparatus.

3. The correct reading of the text is:

   दिनं चतुः षष्टिलवोनमाहु-
   स्तिथिं प्रषष्ट्यन्त्यमहस्तु सर्वम् ।
   त्रिषष्टिभागेन युतं सहस्रं
   युगेऽवमानामपसप्तषट्कम् ॥५॥

   dinaṃ catuḥṣaṣṭilavonamāhu-
   stithiṃ praṣaṣṭyantyamahastu sarvam |
   triṣaṣṭībhāgena yutaṃ sahasraṃ
   yuge'vamānāmapasaptaṣaṭkam ||5||

   They say that a *tithi* is equal to a day minus $\frac{1}{64}$ of a day, correct up to the sixtieth of a sixtieth (of a day, i.e. up to *vighaṭīs*), and a day equals a whole *tithi* plus $\frac{1}{63}$ of a *tithi*. The number of omitted *tithis* in a *yuga* is equal to 1000 minus 42 (i.e. 958).

This can be easily proved to be true. For, in a *yuga*

(i) no. of *tithis* = 61230, and no. of civil days = 60272. Therefore,

$$\text{one } tithi = \frac{60272}{61230} = 1 - \frac{1}{64} \text{ civil day,}$$

and

$$\text{one civil day} = \frac{61230}{60272} = 1 + \frac{1}{63} \ tithis.$$

Both the results are correct upto *vighaṭīs*.

(ii) no. of omitted *tithis* = *tithis* - civil days = $61230 - 60272 = 958$.

10 Conclusion

From the above discussion, we conclude that the *yuga* defined in the *Yavanajātaka* contains:

$$
\begin{aligned}
\text{Solar years} &= 165\\
\text{Solar months} &= 165 \times 12 = 1980\\
\text{Solar days} &= 165 \times 360 = 59400\\
\text{Civil days} &= 60272\\
\text{Synodic months} &= 2041\\
\text{Intercalary months} &= \text{synodic months} - \text{solar months}\\
&= 2041 - 1980 = 61 \text{ (vide vs. 10)}\\
\text{Synodic days or } \textit{tithis} &= 2041 \times 30 = 61230\\
\text{Omitted } \textit{tithis} &= \textit{tithis} - \text{civil days}\\
&= 61230 - 60272 = 958\\
\text{Sidereal months}&\\
\text{(or Moon's revolutions)} &= \text{synodic months} - \text{Sun's revolutions}\\
&= 2041 - 165 = 2206\\
\text{Risings of asterisms}&\\
\text{(or Earth's rotations)} &= \text{civil days} + \text{Sun's revolutions}\\
&= 60272 + 165 = 60437\\
\text{Risings of the Sun} &= \text{risings of asterisms} - \text{Sun's revolutions}\\
&= 60437 - 165 = 60272 \text{ (vide vs. 8)}\\
\text{Risings of the Moon} &= \text{risings of asterisms} - \text{Moon's revs.}\\
&= 60437 - 2206 = 58231 \text{ (vide vs. 8)}\\
\text{Solar year} &= 6, 5; 17, 5, 27, 16 \text{ days}\\
\text{Sun's mean daily motion} &= 0; 59, 7, 55, 28 \text{ degrees}\\
\text{Synodic month} &= 29; 31, 50, 14, 24 \text{ days}\\
\text{Sidereal month} &= 27; 19, 18, 39 \text{ days.}
\end{aligned}
$$

According to *Sūryasiddhānta*:

$$
\begin{aligned}
\text{Solar year} &= 6, 5; 15, 31, 3 \text{ days}\\
\text{Sun's mean daily motion} &= 0; 59, 8, 10, 10 \text{ degrees}\\
\text{Synodic month} &= 29; 31, 50 \text{ days}\\
\text{Sidereal month} &= 27; 19, 18 \text{ days.}
\end{aligned}
$$

Review of *Vaṭeśvarasiddhānta* and *Gola* of Vaṭeśvara *

It's a long and complicated text. I have read every line, and compared it with the manuscript readings conveniently given in the appendix. Prof. Shukla has demonstrated enormous acuity and ingenuity in editing the work; it puts to shame the old, partial edition by Ram Swarup Sharma and Mukund Mishra (who, however, anticipate many of Shukla's restorations, for which their work should have been acknowledged).

I would not, however, have gone as far as Shukla often does in re-writing the manuscripts or even inventing whole verses. This kind of editing strikes me as a distortion of the evidence, one based on the notion that the author could not have made the mistakes that the manuscript readings imply. I believe it would be better, however, to leave the manuscript readings which follow Vaṭeśvara's solid knowledge of grammar and prosody even when he may present a mathematically imprecise formula; the correct formula belongs in a commentary. I enclose some proposals I would make for returning the text to a state closer to that justified by the manuscript readings, including a number of passages where the meter or the grammar demands a reading different from Shukla's. The text is also marred by frequent occurrences of -स्स for -ः स, or of -ः for -स्, and other inconsistencies of external *sandhi* (it is the inconsistency which is disturbing, and not following the general practice of the manuscript). In many cases also alegores inserted into the text are *not* enclosed by [ ] as was intended. I have not noted these slips since they would occupy many more pages, and can be corrected by any careful reader.

It was a good idea to present the full text of the principal manuscript below the edited texts; more convenient, and useful, perhaps, would have been a facsimile in a separate volume of that manuscript. The reader cannot be sure what errors may have crept into the complicated presentation of the manuscript's readings. So far as I can see, manuscript B was used in only two places—V I, 27a–28b and VII 1, 9–11. In the latter case, the line numbers in the apparatus for Ms. A are incorrectly placed, and the statements about Ms. B are inconsistent. Since Ms. B appears to have been the manuscript used by

* David Pingree, *Indian Journal of History of Science*, Vol. 26, No, 1 (1991), pp. 115–122 (The text *Vaṭeśvarasiddhānta* and *Gola* of Vaṭeśvara, was critically edited with English translation and commentary by K. S. Shukla, Part I: Sanskrit text, Part II: English translation and comments, Indian National Science Academy, New Delhi, 1985–1986).

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Sharma and Miśra, one wonders whether or not some of their odder readings depend on that manuscript. Not enough is said of it to be able to answer this question.

A reproduction of Ms. A would also allow the reader to examine Govinda's contribution (it is surprising that Shukla seems nowhere to refer to his important article in *Gaṇita* 23 for 1972) and to reconstruct the original location in the manuscript and the order of the verses in the so-called *Gola*; this is not possible from the scanty information offered in pp. xxi and 302 of vol. I. I do not find Shukla's arguments for attributing this *Gola* to Vaṭeśvara (Vol. 2, p. liii) entirely convincing. Incidentally, there is a manuscript of a *Karaṇasāra* that may be Vaṭeśvara's at Kotah, it deserves to be investigated.

I have not worked through *all* of the English translation. What I have seen, of course, represents Shukla's heavily emended (or re-written) text, and therefore present Vaṭeśvara in a "better" light than the manuscript evidence really suggests. It would easily be possible to employ English terms for many of the terms that now appear in transliterated Sanskrit; the practice of using transliterations tends to keep this technical material inaccessible to historians of science who do not know Sanskrit and thereby defeats the purpose of a translation. Many of these English equivalents are given in the glossary appended to Vol. 1; they should be incorporated into the translation in Vol. 2.

If I have been critical of Shukla's work, it is only because I believe that some aspects of his editorial and translating policy do not conform to general practice. But I greatly admire his extraordinary ability to wrest sense from the frequently garbled text of the manuscripts at his disposal, and the brilliance of many of his emendations is overwhelming. I congratulate him on his fine achievement, and the Academy for having undertaken to publish his book.

Appendix

I 1

9. कल्पश्चतुं
13. all other planet names in genitive; so also should be *ravija*. The emended text has it in vocative; Though hyper metric, keep रविजस्य भुजङ्ग°
14. ख should mean only one number, i.e., 0. Read राशिकाश्वि°

I 2

1. Ms's प्राग्दिनं° correct
2. Read हि instead of द्वि°
3. Keep °मासका युगे
7. Read °स्वरा instead of °ग्रहा

I 3

5. Possibly दिनै रवेः प्रोक्तवद्भिभाजितां°
14. Need one add 14b, or simply understand 12c? 15b could also be omitted and the verses renumbered.

$$13 = 13\text{a–b},\ 14\text{a},\ 14\text{c}$$

$$14 = 14\text{d},\ 15\text{a} + 15\text{c–d}$$

16. Keep °वेदरसरामका (रामक = राम)
22. यातावमेन्दुदिनराशिरपि स्वशिष्ट्या युक्तोऽयुतोऽवमगणः... In युक्तोनितावमं°, tāv is both the dual ending and ta + av, the beginning of *avama*. Should one not keep एवं... रविद्युराशिरन्योन्यतो
24. °विहृता° instead of °भजिता°

I 4

1. पर्ययादिर्गगनेट,
3. °र्विहृतां°
17. चन्द्रश्चन्द्रः
24. अधिकाब्दं°
26. पर्ययादिस्तद्युक्तः

I 4

30. First line (30 a−b) rule for finding lapsed *avamas* according to Ms:

युगावमघ्नो द्युगणः क्वहोद्धृत - श्च वासरानापहरेद् दिनौघतः

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Should this not be retained? And the test corrupt *pāda* incorporated into the restoration at the rest of the verse?

37. भूदिनैश्च च°
41. °दया
42. ग [ति] श्चै [कै] कोट्या°
43. क्वहा [नि]?
44. युतिरन्तरं

I 5

3. वाराङ्का
13. प्रसाध्य चेह गदितवत् समाधिपम्
16. विधवर्षनाथः
21. रुद्रैर्हि°
25. I would not attempt to introduce a completely fabricated verse into the text; the information belongs in the commentary, but Vaṭeśvara's Sanskrit is irrecoverable.
42. °भिर्व°
46. The Ms. clearly implies the reading = द्विघ्नाच्च मासयोगात् स्याद्धानोर्मास°
47. The end is: मासाधिपाद्दिन [पः]
48. °गुणैर्नन्द°
54. द्युभूढं°?? What is this word?
57. स्वाब्देमासान्तरवि [र्युक्] अतोंऽशै …
after 59, two lines omitted, which seen to imply something like

> गतदिवसास्त्रिरूहोजैर्गुणिता भाजिता मध्य? गम्ययुग्गतविधुर्वा गत [द्यु] वियु-ग्गम्ययुगपरो द्युगणः

I can't explain the above; but it must be the basis of a reconstruction, not simply ignored.

I 5

60. Keep देयं
63. ग्रहणविविक्ते?
73. °[भागो]
74. °निहतचन्द्रदिवसेभ्यः
81d. लब्धोद्यु [ग] णो गुरोस्तु चैत्रशुक्लादेः
83. चतुघ्नेज्याब्दैर्युक्त [मी] ज्यवर्ष [प] तिः स्यात्
87. स [कलाः] शुद्धिविहीना वा दिवसाः
88. जीवाब्दान्तावमघटिका रुद्रहतास्त्रिखनगहृताः °ज्जीव [व] र्षतो
90. °इन्दुवेदै°
91. °हतात्, °महःपति°

93. °भिर्जी [वा]ब्दगणाद भार्गवचलोच्च [क]म्? °भिर्जैवगणाद भार्गवचलोच्चम्? °भिर्जीवगणाद्?
94. °भाजिता[ज्ञै]वात्?
95. °भिर्जैवाच्छ°?
96. °ताडिताज्ञैवात्
97. भुजगोयुगै°
98. नवभिर्भागः
100. र्नभकु°
105. स्युर्लब्ध°
114. अधिशेषात् खगुण°
115. गृहाद्यौ प्रोक्तवन्मध्यौ
116. °फलेनेष्ट°
120. हुताशशस्त्राणां

I 6

9. व्यगुपाताः? खगपाताः?

I 7

3. खाक्षिगिर्यब्धि°
18. बुधसितचलकक्ष्या योजन[ानि] युगाब्देषु?
20. °सुताङ्गिरः शनै°

I 8

3. °कृत शरदिग्भि°
7–8. जगुः प्रोक्तम् ॥ फलयोजनं ...
after II a half-verse omitted. It is corrupt, but should not simply be omitted.
13. पूर्वमु[द]याद्यरदलेन वासरादिः

I 9

11. इष्टग्त्रहमव°
20. ग्रहा[न्] समाधिपं सावनदिवसेशम्

I 10

1. शास्त्रलवमध्यये ऽहं
11. ओङ्कारादिन°
28. शास्त्रसम्मिते
29. °रस्तत्र तत् स्फटं
44. प्रकल्पितं
46. गणित गोलानाम्

II 1

8. नगकृतनखानि
16. तिथियुगरामाश्चन्द्र
25. रसेषवो° सागरा
28. °स्त्रिदश °चन्द्राः
36. रविपुत्राः = 2?
47. बाणास्त्रिभुजा नवाब्धयो
53. सन्त्यंशाख्याः?
58. ज्यान्तराहतां

69–70. apparatus भुक्तगण rest of line in 70 a–b वसुगुणो
text of 70: [लब्धः स्वार्धसङ्गुणो]

85. भुक्तज्यान्तक°
102. धनुर्दले धनुर्हृते[1]

---

[1] **ed.** Or धनुर्हते? Original unclear.

II 3

8. परफलं
11. °हता[2] तस्य मध्ये respecting the manuscript. Correction belongs in commentary.
14. [चरणे]
16. remove च

II 4

3. जीवाथैव°
5. remove च
10. °शोधितान्यृजूणि
13. °र्जिनैर्जगुर्भागैः

II 5

11. त्रिज्याम्
15. द्विघ्नान्त्यफलज्याकृतिनिघ्नामाप्तः कुटिलकोटिः
21. °च्छेवमूनः नाप्यस्ति

II 6

7. भाजिताः
12. किंतुघ्नम्
15. वैधृतिमेवं
20. विधेयाः

26. restoration denotes widely from Ms.
35. °र्युतोनितश्चन्द्रः °लिप्ताभिश्चैव

II 7

5. ग्रहगतिं क[र]णाधिकृत् स?
13. सम्यग्गणितस्फुटां
14. त्र्यहः स्पृग्

III 1

32. Restoration very far from ..

III 2

13. keep व्यस्ताक्षज्यावलम्बज्ये

[2] **ed.** Or हृता? Original unclear.

14. keep उत्क्रमलम्बाक्षज्ये क्रमात्पललम्बत्रिभगुणविवरे वा
15. चोत्क्रमपललम्ब[क]ज्ये स्तः
16. चोत्क्रमपललम्बकमौर्विके

III 5

9. पलभानृतलाभ्यासहृ[त्] क्षितिज्या
14. °भ[क्त वा क्षिति]ज्या

III 7

8. द्युज्याधृति
9. चरार्धज्या
11. क्रान्तिपलत्रिगुणव[धो लम्बद्युज्याव]धाप्तो वा
12. In reading closer to Ms., and so preferable
17. °धृति[कृत्या] वा भक्तः
18. पलभाक्षाग्रगुण भूगुनैर्गुणितात्
19. [अग्रका] धृतिहतिगुणैस्त्रि°? तैर्हारैर्भक्ते
23. द्युज्याहृदुन्नत°
25. खेर्दोर्ज्यातः
26. चोक्तवद्भवन्ति

III 8

3. स्वदि[नज्या]प्ता च

III 9

2. कोटिस्तज्ज्या चोज्ज्या
15. नरा [भवन्ति]
19. सौम्यै ऽप्यैक्य
22–23. restoration should be closer to Ms.
35. फलवियुगुदक् समेतो
36. very far from Ms.
37. It seems to have substituted the *agrā* for the *caradala* कुज्यान्त्याघ्नहृदक्षश्रवणाग्राग्रात्रिगुणघातो वा °घातो ऽग्रात्रिगुणयोश्च
46. remove [स्ति]

III 10

1. ऽर्कभूजान्त°
12. °द्रविविहृतेनेष्ट°
26. °पलभाहृत[ाः]
30. It has त्रिगुणस्य instead of अन्त्यस्य (which, in any case, is masculine against the feminine अन्त्या earlier in the verse). Why should be not be in error?
31. What of the Ms's two lines between 31 a–b and 31 c–d?
39. मूलं
40. यथा भवेज्ज्ञेयौ

III 11

4. सौम्यान्ययोर्गो° सूर्य उत्तरगोलगे
6. °दाद्यान्यावाद्यों
8. रविभुजजिनमौव्योर्व[व व]धात्
9. समनरः
15. पलभां
17. भक्ते तेन क[र्णेन]?
18. त्रिज्येष्ट[क्रा]न्तिगुण° remove वा
19. °नृतल[वध]हृत्
21. remove समना
25. °भुजदिनार्ध°
30. वार्ककृति° (वा at end of स)
31. संक्षेप[तो] नतोन्नतौ

III 12

4. चेष्टयुजाग्रया
7. वाक्षगुण°
12. समनरकुज्या°

III 13

7. लम्बगुणतद्धृतिवधान्
10. भाकर्णान्त्यक्रान्त्योर्वधेन द्युमौर्विका स्यादतः
26. गच्छतीनो मृगादेः °र्भषड्कम्
27. remove यत् °सङ्के ऽर्कस्य

III 14

5. रेखायां तु
11. छायाभ्रमावशेषै°
14. नवति[मि]तखाक्षे

III 15

1. °संख्या स्पष्टशब्दार्थमुक्तम्
3. वेत्ति दिशो ऽयमभा[ग्र] पलैर्यो वाद्युतिविभ्रमणा[द्]
6. °तमोरिजाभां
7. गणको[त्त]मः
14. विचित्रसुतन्त्रकृतश्रमः
24. तमोरौ
26. कथयति [तथा] ध्वां°
27. [तीक्ष्णगौ हुत]वहगे
35. प्रततसहशसेव

IV

1. °योर्ग्रहणे
2. स्वककर्ण आप्तः
13. सुरेषु तत्
21. °दलात् क्षेपकेन
23. तिमिरेन्दुमानयोः
28. °वियुता समेता स्फुटा
32. श्रतिश्चछाद्यच्छादकमान°
33. ग्राह्यग्राहकामान भेदकम्
35. °तिथिर्या प्रयाता Not necessary
41. प्रग्रहः

VI

1. विभ्रमो
7. तत्क्रमाद्?
9. another line lost, of which तयोर्वा is the end

11. युक्तो लघ्व्या दृग्गं
20. भवेल्लम्बनसंज्ञकं

V 1

26. Note that *pādas* b + c interchanged
27. °मूलविहृतो?
28. यतो ऽन्यथा
30. श्रवणयोस्तमसोस्तु?

V 2

2. विधोरितो ऽन्या?
6. [हि वा]?

V 3

7. ग्राहह्यग्राहकं — to avoid छाद्यच्छादक°

V 4

5. तेभ्यः
31. उष्णतेजसो न[दीप्ति] तैक्ष्ण्यात्

V 5

4. Should not print [तिश्रु] since it is not intended to be part of the text
5. स्थितिदलचन्द्रकलाः

V 6

3. स्पष्टवत् त्वृणक संयुते

V 7

4. छाद्यच्छादकं? (meter wrong)

VI

5. गुणः क्षेप°
10. समान्यदिकत्व ऋणं
11. तदृणं
16. ज्याविमौ
24. त्रिशद्भाजिताः खषड्भृता
25. [ल्पतरे]ष?

26. ऽल्प[तरे]षु?

VII 11

1. रविचन्द्रभुजांशकैर्विधोः
2. तेजसोर्वारान्ते?
4. ह्यसकृद्

VII 1

13. चन्द्रभ्रमणे[न]
34. श्रुतिरिष्टेनापवर्तनं येषाम्
40. युतेर्मूलम्
41. स्यादनुष्णगो

VIII 2

5. विश्वे ऽष्टौ
8. दिशि क्षिप्तां
13. साभिजितां
19. क्षयवृद्धी
23. नन्दक[ाश्व] सदा
28d. क्रतुरिति ये क्रमेण ते॥

Gola I

7. व्यलीकातु

II

8. First *pāda* in Ms. not used

III

14. °कुजान्तरज्या[क]।
15. र्यद्धरिजे

IV

2. दृश्य एण°
5. प्रोक्तो
6. स्फुटखेः
8. याम्योत्तरे [गत]
15. ये स्वचरार्धोनसंयुताः प्राक् कुजे समुदयन्ति

V

7. यात्यसौ