Chapter 22
Stability Analysis of Explicit and Semi-implicit Euler Methods for Solving Stochastic Delay Differential Equations

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Abstract This paper dealt with the stability analysis of explicit and semi implicit Euler methods in approximating the solutions of linear stochastic delay differential equations (SDDEs). It has been proved that the methods are convergent with strong order 0.5 and are numerically stable in general mean square (GMS) and mean square (MS) sense for certain conditions. A comparative study of the stability explicit and semi implicit Euler methods in approximating the solutions of SDDEs are performed to visualize the theoretical results. Numerical experiments are conducted by applying both methods to linear SDDEs.

Keywords Stochastic delay differential equations · Mean square stable Explicit method · Implicit method · General mean square stable

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1 Introduction

Most of the natural systems around us are subjected to the presence of delayed feedback and are influenced by the uncontrolled environmental noise. For instance, the growth of the cancer cells is non-instantaneous but responds only after some time lag, $r > 0$. Cancer cells also subject to the uncontrolled factors such as blood pressures variations and individual characteristics like genes and stress impacts [1]. The most suitable mathematical equations describing such system is stochastic delay differential equations (SDDEs). In SDDEs, time delay and noisy behaviour are incorporating to its deterministic counterpart. Due to the presence of both effects, solving SDDEs is not an easy task. Analytical solutions of SDDEs are often unavailable. Thus, numerical methods provide a way to solve problem. The development of numerical methods for SDDEs is now among of the research interest. Amongst of the recent works are [2–4]. They proposed explicit numerical methods for solving SDDEs. The convergence and stability analysis of the semi-implicit method for linear SDDEs has been presented in [5]. It is the aimed of this paper to investigate the performance of explicit and semi-implicit Euler methods in approximating the solution of SDDEs. This paper is arranged as follows; Sect. 2 presents the mean-square stability properties of explicit and semi implicit Euler methods. Numerical experiment is conducted in Sect. 3. Concluding remarks are given in Sect. 4.

2 Stochastic Delay Differential Equations

Consider SDDEs of Ito form

$$\begin{aligned} dX(t) &= f(X(t), X(t-r))dt + g(X(t), X(t-r))dW(t), \quad t \in [-r, T] \\ X(t) &= \Phi(t), \quad t \in [-r, 0] \end{aligned} \tag{1}$$

where $\{W_t : t \in \mathbb{R}\}$ is a standard Wiener process with $W_0 = 0$ and the increments $W(t) - W(s) \sim N(0, t-s)$, $0 \leq s \leq t$. $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are drift and diffusion functions, respectively and $\Phi(t)$ is an initial function defined on interval $[-r, 0]$ which is $F_0 -$ measurable and right continuous, $E\|\Phi\|^2 < \infty$, where $\|\Phi\| \sup_{-r \leq s \leq 0} |\Phi(s)|$ $\Phi(t)$ does not depends on $W(t)$ and $r > 0$ is a positive fixed delay. Linear SDDEs is written by

$$\begin{aligned} dX(t) &= [aX(t) + bX(t-1)]dt + [cX(t) + dX(t-1)]dW(t), \quad t \in [-r, T] \\ X(t) &= 1 + t, \quad t \in [-r, 0] \end{aligned} \tag{2}$$

2.1 Euler Scheme for SDDEs

Let consider

$$X_{n+1} = X_n + [\alpha(aX_{n+1} + bX_{n+1-m}) + (1-\alpha)(aX_n + bX_{n-m})]h + (cX_n + dX_{n-m})\Delta W_n \tag{3}$$

where α is a parameter with $0 \le \alpha \le 1$ and $h > 0$ satisfies $r = mh$, for positive integer, m and $t_n = nh$. X_n is an approximate solution to $X(t_n)$. If $t_n \le 0$, $X_n = \Phi(t_n)$. The increment $\Delta W_n = W(t_{n+1}) - W(t_n)$ are independent and normally distributed with mean zero and variance, Δt.

A method is said to be explicit if $\alpha = 0$, hence we have

$$X_{n+1} = X_n + (aX_n + bX_{n-m})h + (cX_n + dX_{n-m})\Delta W_n \tag{4}$$

A semi-implicit Euler scheme is given by (2) for $0 < \alpha \le 1$.

2.2 Convergence and Mean Square Stability of Euler Scheme

The following results are cited from [5].

Theorem 1 ([5]) *Assume that* $ah\alpha < 1$. *The numerical solution produced by* (3) *is convergent to the exact solution of* (1) *in the mean square sense with order of 0.5, i.e. there exists a positive constant* C *such that*

$$\max_{1 \le n \le N} \left(E(\varepsilon_n)^2\right)^{\frac{1}{2}} \le Ch^{\frac{1}{2}} \quad \text{as } h \to 0 \tag{5}$$

where $\varepsilon_n = X(t_n) - X_n$ *is defined as a global error.*

Lemma 1 ([5]) *If*

$$a < -|b| - \frac{1}{2}c^2 \tag{6}$$

then the solution of (2) *is mean square stable, that is*

$$\lim_{t \to \infty} E|X(t)|^2 = 0 \tag{7}$$

Definition 1 ([5]) Under condition (6) a numerical method is said to be general mean square stable (GMS), if there exists $h_0(a,b,c,d) > 0$, such that any application of the method to problem (1) generates numerical approximations, X_n which satisfy

$$\lim_{n\to\infty} E|X_n|^2 = 0 \tag{8}$$

for every stepsize $h = \frac{r}{m}$.

Definition 2 ([5]) Under condition (6) a numerical method is said to be mean square stable (MS), if there exists $h_0(a,b,c,d) > 0$, such that any application of the method to problem (1) generates numerical approximations, X_n which satisfy

$$\lim_{n\to\infty} E|X_n|^2 = 0 \tag{9}$$

for all $h \in (0,h_0(a,b,c,d))$ with $h = \frac{r}{m}$.

Theorem 2 ([5]) *Under condition* (6) *and let*

$$K = \frac{|a|+|b|}{2|a|} + \frac{2a+2|b|+(|c|+|d|)^2}{2|a|(|a|+|b|)} \tag{10}$$

If $K < 0$, *then for all* $\alpha \in [0, 1]$, *the Euler method* (3) *is GMS stable. If* $K \geq 0$, *then for* $\alpha \in (K, 1]$, *the Euler method* (3) *is GMS stable and for* $\alpha \in [0, K]$, *it is MS-stable and* $h_0(a,b,c,d) = \min\{h', h''\}$, *where* $h' = \max\{h_1, h_2\}$, $h'' = \max\left\{\frac{1}{|a|}, h_2\right\}$ *and* $h_1 = \min\left\{\frac{1}{|a|}, \frac{-(2a+2|b|+(|c|+|d|)^2}{(a+|b|)^2}\right\}$, $h_2 = \frac{-(2a+2|b|+(|c|+|d|)^2}{(a+|b|)^2}$.

3 Numerical Experiments

Let consider linear SDDE (2) with sets of coefficients are given in Table 1.

By Theorem 2, SDDE (2) is GMS-stable for set of coefficients C1 if $0.1693 < \alpha \leq 1$ and it is MS-stable for $0 \leq \alpha \leq 0.1693$. For C2, a linear SDDE (2) is GMS-stable for $0 \leq \alpha \leq 1$ when $h \in (0,1.250)$. For C3, the solution obtained is GMS-stable for $\alpha \in (0,1]$ and it is MS-stable if $\alpha = 0$ when $h \in (0,1.0)$.

Table 1 Coefficients of linear SDDEs and the corresponding values of K and h_0

Coefficients	a	b	c	d	K	$h_0(a,b,c,d)$
C1	−2	0.2	0.5	0.0	0.1693	0.6921
C2	−0.8	0.2	0.2	0.2	−0.0250	1.2500
C3	−1.0	0.2	0.2	0.2	0.0000	1.0000

Theoretically, it shows that a set of coefficients C1 produce unstable solution if the explicit Euler method is applied, but it is GMS-stable and MS-stable for semi-implicit method under certain values of α. Moreover, the solution is GMS-unstable for $\alpha < 0.1693$. Linear SDDE generates from C2 is GMS-stable for both methods. Linear SDDE generates from C3 is MS-stable for both methods and GMS-unstable for explicit method. The theoretical results are confirmed by applying explicit Euler method (4) and semi implicit (3) with fixed parameter $\alpha = 0$, 0.1 and 1.0. Table 2 shows the corresponding methods for each α.

In Figs. 1, 2 and 3, the stepsize is fixed to $h = \frac{1}{10}$ and the parameter α is changed according to Table 2.

Table 2 Explicit and semi-implicit Euler methods

α	Method	Formula
0	Explicit	$X_{n+1} = X_n + (aX_n + bX_{n-m})h + (cX_n + dX_{n-m})\Delta W_n$
0.1	Semi-implicit	$X_{n+1} = X_n + [0.1(aX_{n+1} + bX_{n+1-m}) + 0.9(aX_n + bX_{n-m})]h + (cX_n + dX_{n-m})\Delta W_n$
1.0	Semi-implicit	$X_{n+1} = X_n + (aX_{n+1} + bX_{n+1-m})h + (cX_n + dX_{n-m})\Delta W_n$

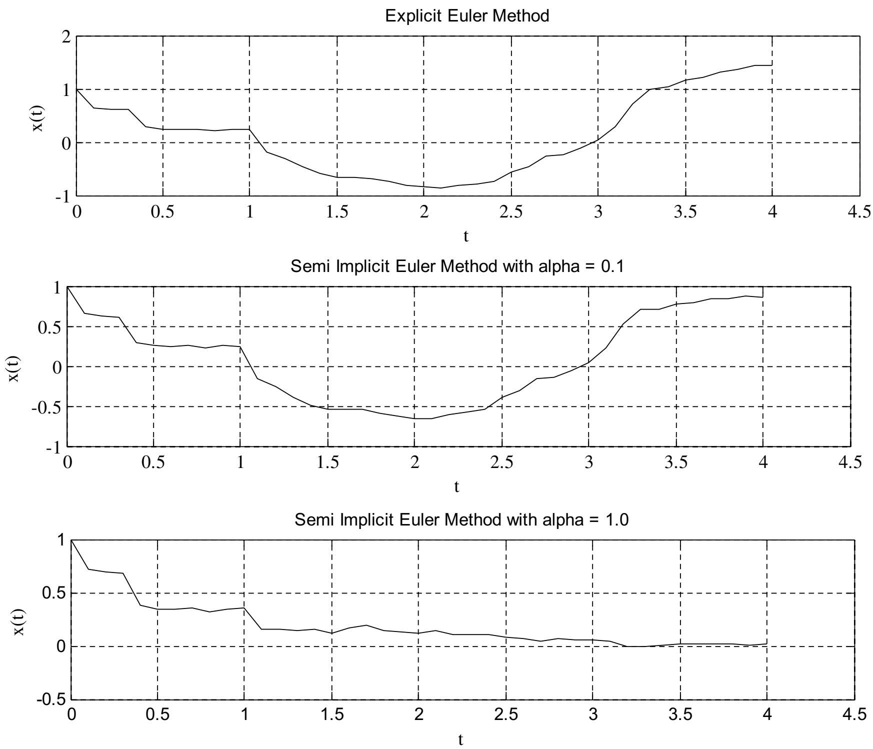

Fig. 1 Simulation result for C1 with fixed stepsize $h = \frac{1}{10}$ for $\alpha = 0$, 0.1 and 1.0

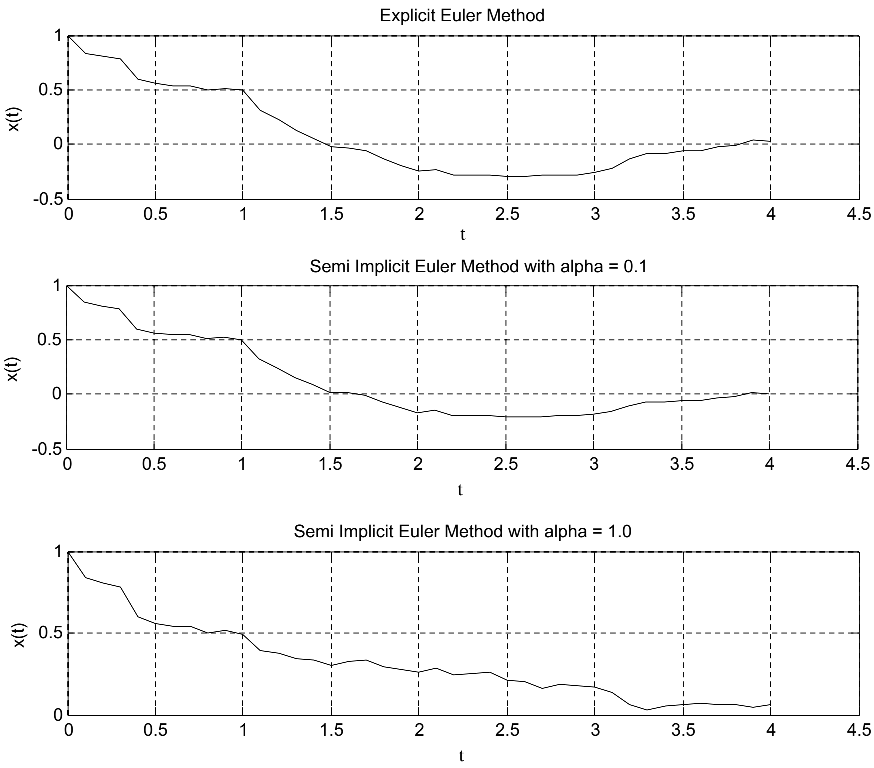

Fig. 2 Simulation result for C2 with fixed stepsize $h = \frac{1}{10}$ for α = 0, 0.1 and 1.0

For α = 0 and 0.1, the simulated results for C2 are mean-square unstable. This indicates that for small values of α (when the method is reduced to explicit scheme) the results become unstable for C2. However, for C1 and C3, the results tend to negative values for α = 0 and 0.1, hence indicates instability of the solution. However, the simulated result produced by semi-implicit method with α = 1.0 possess the stability in mean-square for C1, C2 and C3.

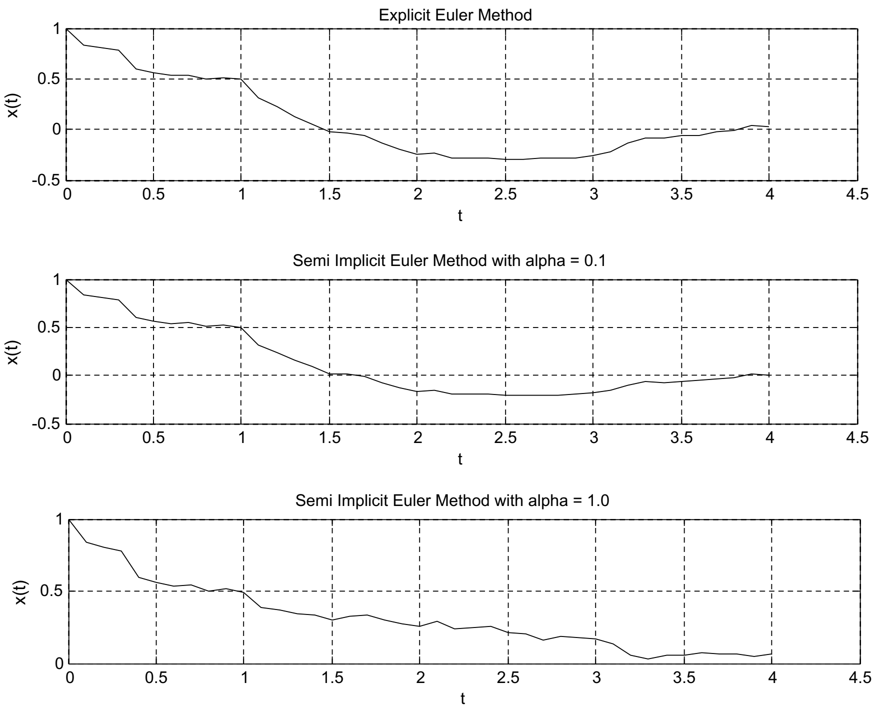

Fig. 3 Simulation result for C3 with fixed stepsize $h = \frac{1}{10}$ for α = 0, 0.1 and 1.0

4 Conclusions

It can be concluded that the stability of explicit and semi-implicit methods are influenced by the values of $h$and α. Small values of αproduce instable results compare than large value of α (for α = 1.0).

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References

1. Ditlevsen. S., Samson, A.: Stochastic Biomathematical Models, pp. 3–35. Springer, New York (2013)
2. Baker, C.T.H., Buckwar, E.: Numerical analysis of explicit one-step methods for stochastic delay differential equations. J. Comput. Math. **3**, 315–335 (2000)
3. Kuchler, U., Platen, E.: Strong discrete time approximation of stochastic differential equations with time delay. Math. Comput. Simul. **54**, 189–205 (2000)

4. Rosli, N., Bahar, A., Yeak, S.H., Mao, X.: A systematic derivation of stochastic taylor methods for stochastic delay differential equations. Bull. Malays. Math. Sci. Soc. **36**(3), 555–576 (2013)
5. Liu, M., Cao, W., Fan, Z.: Convergence and stability of the semi implicit Euler method for a linear stochastic differential delay equation. J. Comput. Appl. Maths. **170**, 255–268 (2004)