Trends in Mathematics

Ayman Badawi
Jim Coykendall
Editors

Advances in Commutative Algebra

Dedicated to David F. Anderson

Birkhäuser

Birkhäuser

Trends in Mathematics

Trends in Mathematics is a series devoted to the publication of volumes arising from conferences and lecture series focusing on a particular topic from any area of mathematics. Its aim is to make current developments available to the community as rapidly as possible without compromise to quality and to archive these for reference.

More information about this series at http://www.springer.com/series/4961

Ayman Badawi · Jim Coykendall
Editors

Advances in Commutative Algebra

Dedicated to David F. Anderson

Birkhäuser

Editors
Ayman Badawi
Department of Mathematics and Statistics
American University of Sharjah
Sharjah, United Arab Emirates

Jim Coykendall
Department of Mathematical Sciences
Clemson University
Clemson, SC, USA

ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-981-13-7027-4 ISBN 978-981-13-7028-1 (eBook)
https://doi.org/10.1007/978-981-13-7028-1

Library of Congress Control Number: 2019933704

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

When David F. Anderson retired in September 2017, his many friends and colleagues from around the world felt it appropriate to acknowledge his contributions to the wide-ranging area of commutative algebra by writing an edited book in his honor. The enthusiastic response to this project led directly to this edited book; we are most thankful to all the contributors who helped to make this work possible. However, the material appearing in this book is not of the usual conference proceedings type: The editors have tried to present a balanced mix of survey papers, which will enable experts and non-experts alike to get a good overview of developments across a range of areas of commutative algebra outlining the work of David F. Anderson, along with research papers presenting some of the most recent developments in commutative algebra. Every effort has been made to make these research papers easily accessible in their introductory sections. We hope that the material will be of interest to both beginning graduate students and experienced researchers alike. The topics covered are, inevitably, just a cross section of the vast areas of commutative algebra, but they reflect in a strong way the areas in which David F. Anderson contributed so much.

The book contains surveys and recent research developments. Finally, we would like to express our sincere thanks to the colleagues who contributed papers so enthusiastically, to the many experts who acted as referees for all the papers, to the professional staff at Springer, and in particular to Shamim Ahmad and Shubham Dixit, for their help in producing a volume which we hope is an appropriate recognition of our friend David F. Anderson.

Sharjah, United Arab Emirates — Ayman Badawi
Clemenson, USA — Jim Coykendall

Contents

About the Editors

Ayman Badawi is Professor at the Department of Mathematics and Statistics, the American University of Sharjah, the United Arab Emirates. He earned his Ph.D. in Algebra from the University of North Texas, USA, in 1993. He is an active member of the American Mathematical Society and honorary member of the Middle East Center of Algebra and its Applications. His research interests include commutative algebra, pi-regular rings, and graphs associated to rings.

Jim Coykendall is Professor of Mathematical Sciences at Clemson University, South Carolina, USA. He earned his Ph.D. from Cornell University in 1995, and has held various academic positions at the California Institute of Technology, the University of Tennessee, Cornell University, Lehigh University, and North Dakota State University. He has successfully guided 12 Ph.D. students. His research interests include commutative algebra and number theory.

Introduction

David F. Anderson is a great teacher, a friend, and a supportive colleague. I have admired his huge knowledge and his passion for commutative ring theory. It is a privilege that I had the opportunity to write 17 papers with him so far. In spite of his busy schedule, he was always willing to find the time for me to discuss mathematics and research projects.

—Ayman Badawi

I first met David in the summer of 1987 when I was fortunate enough to have been assigned to him as his research mentee at the University of Tennessee's REU program, and I was lucky enough to be able to repeat the experience in the summer of 1988. These two summers were incredibly formative in my personal mathematical experience. During those summers, I discovered David to be incredibly gifted at sharing his understanding of mathematics (he is probably the best expositor of mathematics that I know). It is interesting to note that although my PhD was technically in algebraic number theory at Cornell University, my work now more closely approximates the mathematics that David turned me on to in the late 80s. The "Anderson Effect" has truly shaped my career both before and after my graduate school education. I have greatly admired David's work from both near and afar. I have been privileged to have benefitted from his mathematics, and I continue to aspire to his accomplishments with mathematics and its exposition and teaching.

—Jim Coykendall

David F. Anderson

David F. Anderson who retired in September 2017 is one of the leading algebraists of his generation. He was born in Fort Dodge, Iowa, and grew up in Gowrie, Iowa, a town of about 1000 people located 80 miles northwest of Des Moines. His father was a rural mail carrier, and his mother was an elementary school teacher. His father's cousin was E. F. Lindquist, a professor at the University of Iowa, who helped develop the ACT test and the GED examination and has the patent for the first optical-mark scanner. Much of his childhood was spent riding bikes, playing football, delivering newspapers, launching rockets, and being a Boy Scout. His interest in Abstract Algebra started from a summer program that he attended at the University of Iowa in 1966. He graduated from Prairie Community High School in 1967 along with 58 others, including another future mathematician (his twin brother Dan Anderson, now at the University of Iowa).

David F. Anderson attended Iowa State University, where he received his BS and MS in 1971. His mathematical skills were put to good use "counting cars" for the Iowa State Highway Commission during the summers of 1967 and 1968. During the summer of 1969, he attended an REU program in mathematics at St. Olaf College. Then, he went to the University of Chicago on an NSF Fellowship, where he received his Ph.D. in 1976. He joined the University of Tennessee in 1976 and had been Associate Department Head for Graduate Studies since 2001. He had been active in our REU program and was an AP Calculus Grader and Table Leader from 1987 to 1998. He is married with two grown children and five grandchildren. He enjoys reading about history and religion and in particular enjoys listening to college courses on CDs while commuting to work each day.

David F. Anderson published more than 160 papers in different branches of commutative algebra. His complete record can be seen on Math Science Net. Many of his publications appeared in very prestigious journals (e.g., *Proceedings of the American Mathematical Society*, *Journal of Algebra*, *Journal of Pure and Applied Algebra, Communications in Algebra*, *Journal of Algebra and Its Applications*). He was a keynote speaker for several American Mathematical Society meetings.

At the outset, let us stress that it is impossible in a few pages to give a detailed overview of the many research contributions made by David F. Anderson, and it will be for the later generations to assess his impact on the world of algebra. An obvious feature of David F. Anderson's research output is the number of coauthors, some 30 in total, but perhaps more surprising is the number of coauthors with whom he wrote multiple papers and the duration of these collaborations. Many coauthors had more than ten joint papers with him, and these collaborations endured for more than 20 years—Daniel. D. Anderson coauthored 31 papers; David D. Dobbs coauthored 25 papers; Ayman Badawi coauthored 17 papers; Muhammad Zafrullah coauthored 14 papers; Scot Chapman coauthored 13 papers; Marco Fontana coauthored 10 papers; John D. Lagrange coauthored 5 papers; and Salah-Edidne Kabbaj coauthored 4 papers. Many others had collaborations resulting in more than two joint papers. David F. Anderson always enjoyed this joint approach to working on a problem and often expressed the view that "it's fun working together"; he is a generous coauthor, quick to share ideas, but always demanding in terms of getting the best results possible. His ability to move from one topic to another is impressive.

Students of David F. Anderson

1. Kihne, Patricia, 1999
2. Kim, Hawankoo, 1998
3. LaGrange, John, 2008
4. Laska, Jason, 2010
5. Lewis, Elizabeth, 2015
6. Lynch, Ben, 2010
7. Rand, Ashley, 2013
8. Redmond, Shane, 2001
9. Smith, Jessee, 2014
10. Smith, Neal, 2004
11. Weber Darrin, 2017

Publication List of David F. Anderson

1. Anderson, David F.; Badawi, Ayman; Fahid Brahim. Weakly (m, n)-closed ideals and (m, n)-von-Neumann regular rings. *J. Korean Math. Soc.* 55(2018), No. 5, pp. 1031–1043.
2. Anderson, David F.; Chang, Gyu Whan; Zafrullah, Muhammad Graded Prüfer domains. *Comm. Algebra* 46 (2018), no. 2, 792–809.
3. Anderson, David F.; Weber, Darrin The zero-divisor graph of a commutative ring without identity. *Int. Electron. J. Algebra* 23 (2018), 176–202.
4. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Completely integrally closed Prüfer v-multiplication domains. *Comm. Algebra* 45 (2017), no. 12, 5264-5282.

5. Anderson, David F.; Badawi, Ayman The zero-divisor graph of a commutative semigroup: a survey. *Groups, modules, and model theory—surveys and recent developments,* 23–39, *Springer, Cham,* 2017.
6. Anderson, D. D.; Anderson, David F.; Chang, Gyu Whan Graded-valuation domains. *Comm. Algebra* 45 (2017), no. 9, 4018–4029.
7. Anderson, David F.; Chang, Gyu Whan; Zafrullah, Muhammad On locally AGCD domains. *J. Algebra Appl.* 16 (2017), no. 2, 1750028, 16 pp.
8. Anderson, David F.; Badawi, Ayman On (m,n)-closed ideals of commutative rings. *J. Algebra Appl.* 16 (2017), no. 1, 1750013, 21 pp.
9. Anderson, David F.; Chang, Gyu Whan Graded integral domains whose nonzero homogeneous ideals are invertible. *Internat. J. Algebra Comput.* 26 (2016), no. 7, 1361–1368.
10. Anderson, David F.; Lewis, Elizabeth F. A general theory of zero-divisor graphs over a commutative ring. *Int. Electron. J. Algebra* 20 (2016), 111–135.
11. Pranjali; Anderson, David F.; Acharya, B. D.; Gupta, Purnima Zero-divisor labelings of graphs. *Comm. Algebra* 44 (2016), no. 11, 4697–4708.
12. Anderson, David F.; LaGrange, John D. Some remarks on the compressed zero-divisor graph. *J. Algebra* 447 (2016), 297–321.
13. Anderson, David F.; Chang, Gyu Whan Overrings as intersections of localizations of an integral domain. *Comm. Algebra* 43 (2015), no. 1, 225–235.
14. Anderson, David F.; LaGrange, John D. The semilattice of annihilator classes in a reduced commutative ring. *Comm. Algebra* 43 (2015), no. 1, 29–42.
15. Anderson, David F.; LaGrange, John D. Abian's poset and the ordered monoid of annihilator classes in a reduced commutative ring. *J. Algebra Appl.* 13 (2014), no. 8, 1450070, 18 pp.
16. Anderson, David F.; Atani, S. Ebrahimi; Kohan, M. Shajari; Sarvandi, Z. Ebrahimi The ideal-based zero-divisor graph of commutative chained rings. *Sarajevo J. Math.* 10(22)(2014), no. 1, 3–12.
17. Anderson, David F.; Shirinkam, Sara Some remarks on the graph ΓI(R). *Comm. Algebra* 42 (2014), no. 2, 545–562.
18. Anderson, David F.; Chang, Gyu Whan Graded integral domains and Nagata rings. *J. Algebra* 387 (2013), 169–184.
19. Anderson, David F.; Badawi, Ayman The generalized total graph of a commutative ring. *J. Algebra Appl.* 12 (2013), no. 5, 1250212, 18 pp.
20. Anderson, David F.; Ghalandarzadeh, Shaban; Shirinkam, Sara; Malakooti Rad, Parastoo On the diameter of the graph ΓAnn (M) (R). *Filomat* 26 (2012), no. 3, 623–629.
21. Anderson, David F.; Badawi, Ayman Von Neumann regular and related elements in commutative rings. *Algebra Colloq.* 19 (2012), Special Issue no. 1, 1017-1040.
22. Anderson, David F.; Fasteen, Jodi; LaGrange, John D. The subgroup graph of a group. *Arab. J. Math. (Springer)* 1 (2012), no. 1, 17–27.
23. Anderson, David F.; Badawi, Ayman On the total graph of a commutative ring without the zero element. *J. Algebra Appl.* 11 (2012), no. 4, 1250074, 18 pp.

24. Anderson, David F.; Chapman, Scott T. On bounding measures of primeness in integral domains. *Internat. J. Algebra Comput.* 22 (2012), no. 5, 1250040, 15 pp.
25. Anderson, David F.; Baghdadi, Said El; Zafrullah, Muhammad The v-operation in extensions of integral domains. *J. Algebra Appl.* 11 (2012), no. 1, 1250007, 18 pp.
26. Anderson, David F.; LaGrange, John D. Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. *J. Pure Appl. Algebra* 216 (2012), no. 7, 1626–1636.
27. Anderson, David F.; Badawi, Ayman On n-absorbing ideals of commutative rings. *Comm. Algebra* 39 (2011), no. 5, 1646–1672.
28. Anderson, David F.; Axtell, Michael C.; Stickles, Joe A., Jr. Zero-divisor graphs in commutative rings. *Commutative algebra—Noetherian and non-Noetherian perspectives,* 23–45, *Springer, New York,* 2011.
29. Anderson, David F.; Chapman, Scott T.; Kaplan, Nathan; Torkornoo. DesmondAn algorithm to compute ω-primality in a numerical monoid. *Semigroup Forum* 82 (2011), no. 1, 96–108.
30. Anderson, D. D.; Anderson, David F. Factorization in integral domains, IV. *Comm. Algebra* 38 (2010), no. 12, 4501–4513.
31. Anderson, David F.; Chapman, Scott T. How far is an element from being prime? *J. Algebra Appl.* 9 (2010), no. 5, 779–789.
32. Anderson, David F.; Badawi, Ayman; Dobbs, David E.; Shapiro, Jay Divisibility properties in ultrapowers of commutative rings. *Commutative algebra and its applications,* 9–19, *Walter de Gruyter, Berlin,* 2009.
33. Anderson, David F.; Chapman, Scott T.; Smith, William W. On a property of the division algorithm and its application to the theory of non-unique factorizations. *Elem. Math.* 64(2009), no. 4, 154–159.
34. Anderson, D. D.; Anderson, David F.; Fontana, Marco; Zafrullah, Muhammad On v-domains and star operations. *Comm. Algebra* 37 (2009), no. 9, 3018–3043.
35. Anderson, David F.; Badawi, Ayman The total graph of a commutative ring. *J. Algebra* 320 (2008), no. 7, 2706–2719.
36. Anderson, David F.; Badawi, Ayman On the zero-divisor graph of a ring. *Comm. Algebra* 36 (2008), no. 8, 3073–3092.
37. Anderson, David F. On the diameter and girth of a zero-divisor graph. II. *Houston J. Math.* 34 (2008), no. 2, 361–371.
38. Anderson, David F.; Fontana, Marco; Zafrullah, Muhammad Some remarks on Prüfer $\star$-multiplication domains and class groups. *J. Algebra* 319 (2008), no. 1, 272–295.
39. Anderson, David F.; Mulay, S. B. On the diameter and girth of a zero-divisor graph. *J. Pure Appl. Algebra* 210 (2007), no. 2, 543–550.
40. Anderson, David F.; Chang, Gyu Whan Almost splitting sets in integral domains. II. *J. Pure Appl. Algebra* 208 (2007), no. 1, 351–359.
41. Anderson, David F.; Chang, Gyu Whan The class group of D[X^2, X^3] [MR2056064]. *Focus on commutative rings research,* 1–7, *Nova Sci. Publ., New York,* 2006.

42. Anderson, David F. Robert Gilmer's work on semigroup rings. *Multiplicative ideal theory in commutative algebra,* 21–37, *Springer, New York,* 2006.
43. Anderson, David F.; Mclaughlin, Elizabeth V. How far is an HFD from a UFD'*Abelian groups, rings, modules, and homological algebra,* 15–22, Lect. Notes Pure Appl. Math., 249,*Chapman & Hall/CRC, Boca Raton, FL,* 2006.
44. Anderson, David F.; Chang, Gyu Whan; Park, Jeanam Weakly Krull and related domains of the form D+M, A+XB[X] and A+X2B[X]. *Rocky Mountain J. Math.* 36 (2006), no. 1,1–22.
45. Anderson, David F.; Chang, Gyu Whan; Park, Jeanam A general theory of splitting sets. *Comm. Algebra* 33 (2005), no. 12, 4693–4704.
46. Anderson, David F.; Badawi, Ayman On ϕ-Dedekind rings and ϕ-Krull rings. *Houston J. Math.* 31 (2005), no. 4, 1007–1022.
47. Anderson, David F.; Chang, Gyu Whan Homogeneous splitting sets of a graded integral domain. *J. Algebra* 288 (2005), no. 2, 527–544.
48. Anderson, David F.; Chang, Gyu Whan The m-complement of a multiplicative set. *Arithmetical properties of commutative rings and monoids,* 180–187, Lect. Notes Pure Appl. Math., 241, *Chapman & Hall/CRC, Boca Raton, FL,* 2005
49. Anderson, David F. Divisibility properties in graded integral domains. *Arithmetical properties of commutative rings and monoids,* 22–45, Lect. Notes Pure Appl. Math., 241,*Chapman & Hall/CRC, Boca Raton, FL,* 2005.
50. Anderson, David F.; Chang, Gyu Whan The class group of D[Γ] for D an integral domain and Γ a numerical semigroup. *Comm. Algebra* 32 (2004), no. 2, 787–792.
51. Anderson, David F.; Badawi, Ayman On ϕ-Prüfer rings and ϕ-Bezout rings. *Houston J. Math.* 30 (2004), no. 2, 331–343.
52. Anderson, David F.; Chang, Gyu Whan The class group of D[X2,X3]. *Int. J. Commut. Rings* 2 (2003), no. 1, 1–7.
53. Anderson, David F.; Chang, Gyu Whan; Park, Jeanam D[X2,X3] over an integral domain D. *Commutative ring theory and applications (Fez, 2001),* 1–14, Lecture Notes in Pure and Appl. Math., 231, *Dekker, New York,* 2003.
54. Anderson, David F.; Chang, Gyu Whan The class group of integral domains. *J. Algebra* 264 (2003), no. 2, 535–552.
55. Anderson, David F.; Levy, Ron; Shapiro, Jay Zero-divisor graphs, von Neumann regular rings, and Boolean algebras. *J. Pure Appl. Algebra* 180 (2003), no. 3, 221–241.
56. Anderson, David F.; Chang, Gyu Whan; Park, Jeanam Generalized weakly factorial domains. *Houston J. Math.* 29 (2003), no. 1, 1–13.
57. Anderson, David F.; El Baghdadi, Said; Kabbaj, Salah-Eddine The homogeneous class group of A+XB[X] domains. *Commutative rings,* 167–181, *Nova Sci. Publ., Hauppauge, NY,* 2002.
58. Anderson, David F.; Kim, Hwankoo; Park, Jeanam Factorable domains. *Comm. Algebra* 30 (2002), no. 9, 4113–4120.

59. Anderson, David F.; Badawi, Ayman Divisibility conditions in commutative rings with zerodivisors. *Comm. Algebra* 30 (2002), no. 8, 4031–4047.
60. Anderson, David F.; Badawi, Ayman On root closure in commutative rings. Commutative algebra. *Arab. J. Sci. Eng. Sect. C Theme Issues* 26 (2001), no. 1, 17–30.
61. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad The ring D +XDS[X]and t-splitting sets. Commutative algebra. *Arab. J. Sci. Eng. Sect. C Theme Issues* 26 (2001), no. 1, 3–16.
62. Anderson, David F.; Frazier, Andrea; Lauve, Aaron; Livingston, Philip S. The zero-divisor graph of a commutative ring. II. *Ideal theoretic methods in commutative algebra (Columbia, MO, 1999),* 61–72, Lecture Notes in Pure and Appl. Math., 220, *Dekker, New York,* 2001.
63. Anderson, David F.; Badawi, Ayman Conditions equivalent to seminormality in certain classes of commutative rings. *Ideal theoretic methods in commutative algebra (Columbia, MO, 1999),* 49–59, Lecture Notes in Pure and Appl. Math., 220, *Dekker, New York,* 2001.
64. Anderson, David F.; Park, Jeanam Splitting multiplicative sets in Dedekind domains. *Bull. Korean Math. Soc.* 38 (2001), no. 2, 389–398.
65. Anderson, David F.; El Abidine, Driss Nour The A + XB[X] and A+XB[[X]] constructions from GCD-domains. *J. Pure Appl. Algebra* 159 (2001), no. 1, 15–24.
66. Anderson, David F. The class group and local class group of an integral domain. *Non-Noetherian commutative ring theory,* 33–55, Math. Appl., 520, *Kluwer Acad. Publ., Dordrecht,* 2000.
67. Anderson, David F.; Badawi, Ayman; Dobbs, David E. Pseudo-valuation rings. II. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* 3 (2000), no. 2, 535–545.
68. Anderson, David F.; Chapman, Scott T. On the elasticities of Krull domains with finite cyclic divisor class group. *Comm. Algebra* 28 (2000), no. 5, 2543–2553.
69. Anderson, David F.; Park, Jeanam Rooty and root closed domains. *Advances in commutative ring theory (Fez, 1997),* 87–99, Lecture Notes in Pure and Appl. Math., 205, *Dekker, New York,* 1999.
70. Anderson, David F.; El Baghdadi, Said; Kabbaj, Salah-Eddine On the class group of A+XB[X] domains. *Advances in commutative ring theory (Fez, 1997),* 73–85, Lecture Notes in Pure and Appl. Math., 205, *Dekker, New York,* 1999.
71. Anderson, David F. Root closure in commutative rings: a survey. *Advances in commutative ring theory (Fez, 1997),* 55–71, Lecture Notes in Pure and Appl. Math., 205, *Dekker, New York,* 1999.
72. Anderson, David F.; Livingston, Philip S. The zero-divisor graph of a commutative ring. *J. Algebra* 217 (1999), no. 2, 434–447.
73. Anderson, D. D.; Anderson, David F.; Park, Jeanam GCD-sets in integral domains. *Houston J. Math.* 25 (1999), no. 1, 15–34.

74. Anderson, David F.; El Abidine, Driss Nour Factorization in integral domains. III. *J. Pure Appl. Algebra* 135 (1999), no. 2, 107–127. 13G05
75. Anderson, David F.; Dobbs, David E. Root closure in integral domains. III. *Canad. Math. Bull.* 41 (1998), no. 1, 3–9.
76. Anderson, David F.; Park, Jeanam; Kim, Gi-Ik; Oh, Heung-Joon Splitting multiplicative sets and elasticity. *Comm. Algebra* 26 (1998), no. 4, 1257–1276.
77. Anderson, David F.; Park, Jeanam Locally half-factorial domains. *Houston J. Math.* 23 (1997), no. 4, 617–630.
78. Anderson, David F.; Winner, Janice Factorization in K[[S]]. *Factorization in integral domains (Iowa City, IA, 1996),* 243–255, Lecture Notes in Pure and Appl. Math., 189, *Dekker, New York,* 1997.
79. Anderson, David F.; Park, Jaenam Factorization in subrings of K[X] or K [[X]]. *Factorization in integral domains (Iowa City, IA, 1996),* 227–241, Lecture Notes in Pure and Appl. Math., 189, *Dekker, New York,* 1997.
80. Anderson, David F. Elasticity of factorizations in integral domains: a survey. *Factorization in integral domains (Iowa City, IA, 1996),* 1–29, Lecture Notes in Pure and Appl. Math., 189, *Dekker, New York,* 1997.
81. Anderson, David F.; Dobbs, David E.; Roitman, Moshe When is a power series ring *n*-root closed? *J. Pure Appl. Algebra* 114 (1997), no. 2, 111–131.
82. Badawi, Ayman; Anderson, David F.; Dobbs, David E. Pseudo-valuation rings. *Commutative Ring Theory (Fès, 1995),* 57–67, Lecture Notes in Pure and Appl. Math., 185, *Dekker, New York,* 1997.
83. Anderson, David F.; Scherpenisse, Carla Factorization in K[S]. *Commutative ring theory (Fès, 1995),* 45–56, Lecture Notes in Pure and Appl. Math., 185, *Dekker, New York,* 1997.
84. Anderson, David F.; Nour El Abidine, D. Some remarks on the ring R#. *Commutative ring theory (Fès, 1995),* 33–44, Lecture Notes in Pure and Appl. Math., 185, *Dekker, New York,* 1997.
85. Anderson, David F.; Chapman, Scott T.; Smith, William W. Factorization sets and half-factorial sets in integral domains. *J. Algebra* 178 (1995), no. 1, 92–121.
86. Anderson, David F.; Chapman, Scott T.; Smith, William W. Overrings of half-factorial domains. II. *Comm. Algebra* 23 (1995), no. 11, 3961–3976.
87. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad A generalization of unique factorization. *Boll. Un. Mat. Ital. A (7)* 9 (1995), no. 2, 401–413.
88. Anderson, David F.; Dobbs, David E.; Fontana, Marco On the nonuniversality of weak normalization. *Zero-dimensional commutative rings (Knoxville, TN, 1994),* 143–154, Lecture Notes in Pure and Appl. Math., 171, *Dekker, New York,* 1995.
89. Anderson, David F.; Cahen, Paul-Jean; Chapman, Scott T.; Smith, William W. Some factorization properties of the ring of integer-valued polynomials. *Zero-dimensional commutative rings (Knoxville, TN, 1994),* 125–142, Lecture Notes in Pure and Appl. Math., 171, *Dekker, New York,* 1995.

90. Anderson, Daniel D.; Anderson, David F. The ring R[X,r/X]. *Zero-dimensional commutative rings (Knoxville, TN, 1994),* 95–113, Lecture Notes in Pure and Appl. Math., 171, *Dekker, New York,* 1995.
91. Anderson, David F.; Jenkens, Susanne Factorization in *K*[*Xn*, *Xn*+1,•••, *X*2*n* −1]. *Comm. Algebra* 23 (1995), no. 7, 2561–2576.
92. Anderson, David F.; Chapman, Scott T.; Smith, William W. On Krull half-factorial domains with infinite cyclic divisor class group. *Houston J. Math.* 20 (1994), no. 4, 561–570.
93. Anderson, David F.; Chapman, Scott T.; Smith, William W. Overrings of half-factorial domains. *Canad. Math. Bull.* 37 (1994), no. 4, 437–442.
94. Anderson, David F.; Chapman, Scott T.; Smith, William W. Some factorization properties of Krull domains with infinite cyclic divisor class group. *J. Pure Appl. Algebra* 96 (1994), no. 2, 97–112.
95. Anderson, D. D.; Anderson, David F. Elasticity of factorizations in integral domains. II. *Houston J. Math.* 20 (1994), no. 1, 1–15
96. Anderson, David F.; Chapman, Scott; Inman, Faith; Smith, W. W. Factorization in *K*[*X*2, *X*3]. *Arch. Math. (Basel)* 61 (1993), no. 6, 521–528.
97. Anderson, D. D.; Anderson, David F.; Chapman, Scott T.; Smith, William W. Rational elasticity of factorizations in Krull domains. *Proc. Amer. Math. Soc.* 117 (1993), no. 1, 37–43.
98. Anderson, David F. Star operations and the D+M construction. *Rend. Circ. Mat. Palermo (2)* 41 (1992), no. 2, 221–230.
99. Anderson, David F.; Dobbs, David E.; Fontana, Marco Hilbert rings arising as pullbacks. *Canad. Math. Bull.* 35 (1992), no. 4, 431–438.
100. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Factorization in integral domains. II. *J. Algebra* 152 (1992), no. 1, 78–93.
101. Anderson, D. D.; Anderson, David F. Elasticity of factorizations in integral domains. *J. Pure Appl. Algebra* 80 (1992), no. 3, 217–235.
102. Anderson, David F.; Dobbs, David E.; Fontana, Marco; Khalis, Mohammed. Catenarity of formal power series rings over a pullback. *J. Pure Appl. Algebra* 78 (1992), no. 2, 109–122.
103. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Atomic domains in which almost all atoms are prime. *Comm. Algebra* 20 (1992), no. 5, 1447–1462.
104. Anderson, David F. The kernel of Pic(R0)→Pic(R) for R a graded domain. *C. R. Math. Rep. Acad. Sci. Canada* 13 (1991), no. 6, 248–252.
105. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Splitting the t-class group. *J. Pure Appl. Algebra* 74 (1991), no. 1, 17–37.
106. Anderson, David F.; Houston, Evan G.; Zafrullah, Muhammad Pseudo-integrality. *Canad. Math. Bull.* 34 (1991), no. 1, 15–22.
107. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Rings between D[X]and K[X]. *Houston J. Math.* 17 (1991), no. 1, 109–129.
108. Anderson, David F.; Dobbs, David E.; Fontana, Marco Notes on N-dimension sequences. *Math. Japon.* 36 (1991), no. 1, 121–125.

109. Anderson, David F.; Pruis, Paula Length functions on integral domains. *Proc. Amer. Math. Soc.* 113 (1991), no. 4, 933–937.
110. Anderson, D. F.; Dobbs, D. E.; Fontana, M. Characterizing Kronecker function rings. *Ann. Univ. Ferrara Sez. VII (N.S.)* 36 (1990), 1–13 (1991).
111. Anderson, David F.; Dobbs, David E.; Roitman, Moshe Root closure in commutative rings. *Ann. Sci. Univ. Clermont-Ferrand II Math.* No. 26 (1990), 1–11.
112. Anderson, David F.; Dobbs, David E. Fields in which seminormality implies normality. *Houston J. Math.* 16 (1990), no. 2, 231–247.
113. Anderson, D. D.; Anderson, David F.; Zafrullah, Muhammad Factorization in integral domains. *J. Pure Appl. Algebra* 69 (1990), no. 1, 1–19.
114. Anderson, David F.; Dobbs, David E.; Eakin, Paul M., Jr.; Heinzer, William J. On the generalized principal ideal theorem and Krull domains. *Pacific J. Math.* 146 (1990), no. 2, 201–215.
115. Anderson, David F. The Picard group of a monoid domain. II. *Arch. Math. (Basel)*55 (1990), no. 2, 143–145.
116. Anderson, D. D.; Anderson, David F. Examples of star operations on integral domains. *Comm. Algebra* 18 (1990), no. 5, 1621–1643.
117. Anderson, D. D.; Anderson, D. F.; Costa, D. L.; Dobbs, D. E.; Mott, J. L.; Zafrullah, M. Some characterizations of v-domains and related properties. *Colloq. Math.* 58 (1989), no. 1, 1–9.
118. Anderson, David F.; Dobbs, David E.; Fontana, Marco On treed Nagata rings. *J. Pure Appl. Algebra* 61 (1989), no. 2, 107–122.
119. Anderson, David F.; Mulay, S. B. Noncatenary factorial domains. *Comm. Algebra*17 (1989), no. 5, 1179–1185.
120. Anderson, David F. Root closure in integral domains. II. *Glasgow Math. J.* 31 (1989), no. 1, 127–130.
121. Anderson, David F.; Dobbs, David E. Pairs of rings with the same prime ideals. II. *Canad. J. Math.* 40 (1988), no. 6, 1399–1409.
122. Anderson, David F.; Dobbs, David E.; Kabbaj, Salah; Mulay, S. B. Universally catenarian domains of D+M type. *Proc. Amer. Math. Soc.* 104 (1988), no. 2, 378–384.
123. Anderson, David F.; Ryckaert, Alain The class group of D+M. *J. Pure Appl. Algebra* 52 (1988), no. 3, 199–212.
124. Anderson, David F. The Picard group of a monoid domain. *J. Algebra* 115 (1988), no. 2, 342–351.
125. Anderson, D. D.; Anderson, David F. Some remarks on star operations and the class group. *J. Pure Appl. Algebra* 51 (1988), no. 1-2, 27–33.
126. Anderson, David F.; Bouvier, Alain; Dobbs, David E.; Fontana, Marco; Kabbaj, Salah On Jaffard domains. *Exposition. Math.* 6 (1988), no. 2, 145–175.
127. Anderson, David F. A general theory of class groups. *Comm. Algebra* 16 (1988), no. 4, 805–847.

128. Anderson, D. D.; Anderson, David F. Multiplicatively closed subsets of fields. *Houston J. Math.* 13 (1987), no. 1, 1–11
129. Barucci, Valentina; Anderson, David F.; Dobbs, David E. Coherent Mori domains and the principal ideal theorem. *Comm. Algebra* 15 (1987), no. 6, 1119–1156.
130. Anderson, D. D.; Anderson, David F. Locally almost factorial integral domains. *J. Algebra* 105 (1987), no. 2, 395–410.
131. Anderson, David F.; Dobbs, David E.; Fontana, Marco When is a Bezout domain a Kronecker function ring? *C. R. Math. Rep. Acad. Sci. Canada* 9 (1987), no. 1, 25–30.
132. Anderson, David F.; Bouvier, Alain Ideal transforms, and overrings of a quasilocal integral domain. *Ann. Univ. Ferrara Sez. VII (N.S.)* 32 (1986), 15–38 (1987).
133. Anderson, David F. Letter to the editor: "Solvability of infinite systems of polynomial equations over the field of complex numbers" [Amer. Math. Monthly **92** (1985), no. 2, 94–98; MR0777554] by A. Abian. *Amer. Math. Monthly* 93 (1986), no. 6, 496–497. 12E05
134. Anderson, D. D.; Anderson, David F.; Markanda, Raj The rings R(X) and R⟨X⟩. *J. Algebra* 95 (1985), no. 1, 96–115.
135. Anderson, David F.; Arnold, Jimmy T.; Dobbs, David E. Integrally closed condensed domains are Bezout. *Canad. Math. Bull.* 28 (1985), no. 1, 98–102.
136. Anderson, D. D.; Anderson, David F. Finite intersections of PID or factorial overrings. *Canad. Math. Bull.* 28 (1985), no. 1, 91–97.
137. Anderson, D. D.; Anderson, David F. Some remarks on cancellation ideals. *Math. Japon.* 29 (1984), no. 6, 879–886.
138. Anderson, D. D.; Anderson, David F. Locally factorial integral domains. *J. Algebra* 90 (1984), no. 1, 265–283
139. Anderson, D. D.; Anderson, David F.; Dobbs, David E.; Houston, Evan G. Some finiteness and divisibility conditions on the proper overrings of an integral domain. *Comm. Algebra* 12(1984), no. 13-14, 1689–1706.
140. Anderson, David F. When the dual of an ideal is a ring. *Houston J. Math.* 9 (1983), no. 3, 325–332.
141. Anderson, David F. Integrally closed domains. *Comment. Math. Univ. St. Paul.***32** (1983), no. 1, 37–38.
142. Anderson, D. D.; Anderson, David F. Grading integral domains. *Comm. Algebra***11** (1983), no. 1, 1–19.
143. Anderson, David F.; Dobbs, David E. On the product of ideals. *Canad. Math. Bull.* **26** (1983), no. 1, 106–114.
144. Anderson, David F. Root closure in integral domains. *J. Algebra* **79** (1982), no. 1, 51–59.
145. Anderson, David F.; Dobbs, David E.; Huckaba, James A. On seminormal overrings. *Comm. Algebra* **10** (1982), no. 13, 1421–1448.
146. Anderson, D. D.; Anderson, David F. Divisorial ideals and invertible ideals in a graded integral domain. *J. Algebra* **76** (1982), no. 2, 549–569.

147. Anderson, D. D.; Anderson, David F. Divisibility properties of graded domains. *Canad. J. Math.* **34** (1982), no. 1, 196–215.
148. Anderson, David F. Seminormal graded rings. II. *J. Pure Appl. Algebra* **23** (1982), no. 3, 221–226.
149. Anderson, D. D.; Anderson, D. F.; Johnson, E. W. Some ideal-theoretic conditions on a Noetherian ring. *Houston J. Math.* **7** (1981), no. 1, 1–10.
150. Anderson, David F. Integral V-ideals. *Glasgow Math. J.* **22** (1981), no. 2, 167–172.
151. Anderson, David F.; Ohm, Jack Valuations and semivaluations of graded domains. *Math. Ann.* **256** (1981), no. 2, 145–156.
152. Anderson, David F. Seminormal graded rings. *J. Pure Appl. Algebra* **21** (1981), no. 1, 1–7.
153. Anderson, D. D.; Anderson, D. F. Generalized GCD domains. *Comment. Math. Univ. St. Paul.* **28** (1980), no. 2, 215–221.
154. Anderson, David F.; Dobbs, David E. Pairs of rings with the same prime ideals.*Canad. J. Math.* **32** (1980), no. 2, 362–384.
155. Anderson, David F. The divisor class group of a semigroup ring. *Comm. Algebra***8** (1980), no. 5, 467–476.
156. Anderson, David F. Comparability of ideals and valuation overrings. *Houston J. Math.* **5** (1979), no. 4, 451–463.
157. Anderson, David F. Graded Krull domains. *Comm. Algebra* **7** (1979), no. 1, 79–106.
158. Anderson, David F. Projective modules over subrings of k[X,Y] generated by monomials. *Pacific J. Math.* **79** (1978), no. 1, 5–17.
159. Anderson, David F. The Chinese remainder theorem and the invariant basis property. *Canad. Math. Bull.* **21** (1978), no. 3, 361–362.
160. Anderson, David F. Projective modules over subrings of k[X,Y]. *Trans. Amer. Math. Soc.* **240** (1978), 317–328.
161. Anderson, David F. Subrings of k[X,Y] generated by monomials. *Canad. J. Math.***30** (1978), no. 1, 215–224.
162. Anderson, David F. Projective Modules Over Subrings of K[X,Y]: Thesis (Ph.D.)–The University of Chicago. 1976.

David Anderson and His Mathematics

D. D. Anderson

It is a pleasure to write a little about my twin brother David and his mathematical work. I will begin by reminiscing our growing up together.

We were born prematurely and were not expected to survive, but survive we did. We grew up in the small town of Gowrie, population just a little over 1,000, in rural North-Central Iowa. Our father was a rural mail carrier and our mother a grade school teacher. We have a younger sister Jane who never really liked math. David and I collected coins, flew model rockets, were in Boy Scouts, played tubas in the high school band, were on the football team, and ran track. David and I won a number of medals and ribbons on the sprint relay teams together—David once held the school record for the 100 yard dash. During the summer we were on the swim team, walked beans (for non-Iowans, this means weeding soybean fields), and worked at our Uncle Frank's hatchery.

I think we both knew we would be math professors by the time we were in junior high. Our parents always reminded us about two of our relatives Everett Lindquist (who invented the optical test scorer and founded ACT) and Uncle Art Arthur Wald who were professors. We first really got involved in mathematics through our Algebra I teacher Raymond Willis. No doubt part of this was due to him renting a room from our Uncle Frank. We had a very good Algebra II class from Lyle Knudson who went on to become a famous women's track coach. David and I wrote up classroom notes for an axiomatic treatment of the integers. Our eleventh grade math teacher Ron Warrick lent us his college abstract algebra book to read, the well known text by the ring theorist Neal McCoy. Between eleventh and twelfth grade we attended a summer NSF program at The University of Iowa. David had course work in group theory, graph theory, and computer programming.

At home David and I had a number of math books. I particularly remember our mother getting us a calculus book when she attended an NEA meeting in Des Moines. We had a CRC Handbook of Mathematics, some Barnes and Noble College Outline

D. D. Anderson (✉)
The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, USA
e-mail: dan-anderson@uiowa.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_1

Series books on college algebra, college mathematics, and calculus plus some Dover books including Davenport's *Higher Arithmetic* and several books from the New Mathematics Library including *Graphs and Their Uses* and *Hungarian Problem Book I* and *II*. We kind of owned the books together, although Davenport's book was mine and David had a book on algebraic number theory that I coveted. Funny thing, but I do not remember ever discussing mathematics with David at home.

After high school David went to Iowa State and I went to The University of Iowa. I remember giving David a copy of Zariski–Samuel *Commutative Algebra* for a Christmas present, probably during our sophomore year. During the summer of 1969 he attended an REU at St. Olaf College in Northfield. I believe it covered logic. Later that summer we took a cross country trip to attend the summer AMS meeting in Eugene, Oregon. Once again I do not really recall us discussing much mathematics.

David got a BS and MS degree from Iowa State in 1971 and went on to graduate school at the University of Chicago with an NSF Fellowship. I had arrived at the University of Chicago a year earlier. Once again I do not remember discussing much math with David. He might have been in the year-long commutative algebra course given by Murthy and I am pretty sure we were both in Murthy's course on projective modules. I graduated in 1974. David went on to write his dissertation *Projective Modules Over Subrings of* $K[X, Y]$ under M. Pavaman Murthy and graduated in 1976. I do remember going to some Cubs games together.

After graduation David went to the University of Tennessee where he spent his entire career. He was Director of Graduate Studies for a number of years. David had 11 Ph.D. students and authored more than 160 papers. He was always willing to share ideas with others as witnessed by his direction of graduate students, his work with REU students, and his 60 co-authors. His Ph.D. students were Hwanko Kim, Patricia Kiihne, Shane Redmond, Neal Smith, John LaGrange, Jason Laska, Ben Lynch, Ashley Rand, Jesse Smith, Elizabeth Lewis, and Darrin Webber. Two of his REU students that come to mind are Jim Coykendall (twice) and my Ph.D. student Andrea Frazier. Of his co-authors, he and I have the most papers together, over 30. I think the first time we really discussed mathematics was when we wrote our first joint paper in 1979 on generalized GCD domains. We continued working off and on together over the years with our last two papers together in 2017.

David's early work based on his dissertation concerned subrings of $K[X, Y]$ and projective modules over these subrings. I'll mention just one of his many results. He showed (*Projective modules over subrings of* $K[X, Y]$ *generated by monomials*, Pacific J. Math. **79** (1978), 5–17) that if A is an affine normal subring of $K[X, Y]$ generated by monomials, then any finitely generated projective A-module is free.

David is well known for his work on zero-divisor graphs and on factorization in integral domains. Let us first look at zero-divisor graphs.

Let R be a commutative ring. I. Beck (*Coloring of commutative rings*, J. Algebra **116**(1988), 208–226) made the set R into a simple graph by taking the elements of R as the vertices and two distinct elements x and y of R to be adjacent if $xy = 0$. Beck was mostly interested in the coloring of R. Let $\chi(R)$ be the chromatic number of (the graph) R and $c\ell(R)$ the clique number. Beck showed that the following are equivalent: (1) $\chi(R) < \infty$, (2) $c\ell(R) < \infty$, and (3) $\mathrm{nil}(R)$ is finite and R has only finitely many minimal prime ideals. He called a ring satisfying these conditions a coloring. Now $\chi(R) \geq c\ell(R)$ and let us call a coloring R a chromatic ring if $\chi(R) = c\ell(R)$. Beck showed that a coloring that was a finite direct product of reduced rings and principal ideal rings was a chromatic ring. He also showed that for $n = 2, 3,$ or 4, $\chi(R) = n \Leftrightarrow c\ell(R) = n$, and that for $\chi(R) = 5, c\ell(R) = 5$. Based on these positive results Beck conjectured that every coloring was a chromatic ring.

The area lay dormant for five years. Then my graduate student Muhammad Naseer and I (*Beck's coloring of a commutative ring*, J. Algebra **159** (1993), 500–514) gave an example of a local ring R with 32 elements for which $c\ell(R) = 5$, but $\chi(R) = 6$.

Again the area lay dormant, this time for 6 years, until the seminal paper *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447 by David and graduate student Philip Livingston. Let R be a commutative ring and $Z(R)$ the set of zero divisors of R. They considered the subgraph $\Gamma(R) = Z(R)\backslash\{0\}$, now called the zero-divisor graph of R, of Beck's graph. They first observed that $\Gamma(R)$ is finite if and only if R is finite or R is an integral domain, but as much as possible allowed R to be infinite. Many examples (and non-examples) were given. It was shown that $\Gamma(R)$ is connected and is relatively small in the sense that it has diameter $\mathrm{diam}(\Gamma(R)) \leq 3$ and if $\Gamma(R)$ contains a cycle, then has girth $g(\Gamma(R)) \leq 7$ (with even $g(\Gamma(R)) \leq 4$ if R is Artinian). They determined when $\Gamma(R)$ was complete or a star graph. Finally, the automorphism group of $\Gamma(R)$ was investigated. It is hard to overestimate the importance of this paper. A recent check of MathSciNet gave 317 citations for this paper and 603 papers containing the phrase zero-divisor graph (with many but not all of them actually related). A recent check also showed that this was the second most downloaded paper from Journal of Algebra for the last 90 days. In all, David has written 20 papers on zero-divisor graphs and their generalizations. David and co-authors M. Axtell and J. Stickles have an excellent survey article *Zero-divisor graphs of commutative rings*, *Commutative Algebra-Noetherian and Non-Noetherian Perspectives* (M. Fontana, S.E. Kabbaj, B. Olberding, and I. Swanson, eds.), 23–45, Springer, New York, 2011. A second excellent survey article is J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, and S. Spiroff, *On zero-divisor graphs*, *Progress in Commutative Algebra 2*: *Closure*, *Finiteness and Factorization* (C. Fransisco, L. Klinger, S. Sather-Wagstaff, and J. Vassilev, eds.), 241–299, Walter de Gruyter, Berlin 2012.

The notion of the zero-divisor graph has paved the way for a number of other graphs associated with rings, modules, or other algebraic systems. Let me name a few. First, with exactly the same definitions we can consider the zero-divisor graph for other algebraic structures such as semigroups or semirings. Two other graphs defined on $Z(R)\backslash\{0\}$ are the essential graph (resp., annihilator graph) where distinct x and y are adjacent if $\mathrm{ann}(xy)$ is an essential ideal of R (resp., $\mathrm{ann}(xy) \neq \mathrm{ann}(x) \cup \mathrm{ann}(y)$). We can modify the vertex set. The compressed zero-divisor graph is defined as follows for a commutative ring R. For $x, y \in R$ define $x \sim y \Leftrightarrow \mathrm{ann}(x) = \mathrm{ann}(y)$. Take the vertex set to be $R/{\sim}\backslash\{[0], [1]\}$ and define $[a]$ and $[b]$ to be adjacent $\Leftrightarrow ab = 0$. Or we can take the vertex set to be the set of nonzero ideals of R with nonzero annihilator where two distinct ideals I and J are adjacent $\Leftrightarrow IJ = 0$. This can be extended to modules. Replacing multiplication by addition and taking R as the vertex set we have the total graph (resp., clean graph, nil-clean graph) where distinct x and y are adjacent $\Leftrightarrow x + y \in Z(R)$ (resp., $x + y$ is clean, $x + y$ is nil-clean). And finally there is the co-zero-divisor graph. Here the vertex set is the set of nonzero nonunits of R and two distinct elements x and y of R are adjacent $\Leftrightarrow x \notin Ry$ and $y \notin Rx$.

David has also done important research concerning factorization in integral domains. This area too has received a lot of attention in recent years. I would like to think that our paper *Factorization in integral domains*, J. Pure Appl. Alg. **69** (1990), 1–19 with Muhammad Zafrullah had a lot to do with that. It contained the now familiar diagram

$$\begin{array}{ccccccc} & \text{HFD} & & & & & \\ \nearrow & & \searrow & & & & \\ \text{UFD} & & & \text{BFD} & \Longrightarrow & \text{ACCP} & \Longrightarrow \ \text{atomic.} \\ \searrow & & \nearrow & & & & \\ & \text{FFD} & & & & & \end{array}$$

I think this was the first paper to consider FFDs (every nonzero element has only finitely many factorizations up to order and associates) and BFDs (each nonzero element has a bound on the lengths of its factorizations). As an example of twin thinking, both David and I came up with the definition and name FFD on the same night. David has three additional papers Factorizations in integral domains, i, $i =$ 2, 3, and 4; the second with me and Muhammad, the third with Driss El Abidine, and the fourth with me.

In all, David has about 30 papers related to factorization. An area of factorization in which he has particularly done a lot of work involves elasticity. Let us recall the definition. Let R be an atomic integral domain and x a nonzero nonunit of R. The elasticity of x is $\rho(x) := \sup\{m/n \mid a_1 \cdots a_m = x = b_1 \cdots b_n,\ a_i, b_j \text{ irreducible}\}$ and $\rho(R) := \sup\{\rho(x) | x \text{ is a nonzero nonunit of } R\}$. So $\rho(R) = 1 \Leftrightarrow R$ is a HFD and $\rho(R)$ measures how far R is from being a HFD. He and I wrote two papers on this topic *Elasticity of factorizations in integral domains*, J. Pure Appl. Alg. **80** (1992), 217–235 and *Elasticity of factorizations in integral domains*, II, Houston J. Math. **20** (1994), 1–15 and had another joint paper with Scott Chapman and Bill Smith. Let me mention two results from our first paper on elasticity. (1) A one-dimensional local (Noetherian) domain has $\rho(R) < \infty$ if and only if R is analytically irreducible. (2) Let $r \geq 1$ be a real number or $r = \infty$. Then there is a Dedekind domain R with torsion class group such that $\rho(R) = r$. Moreover, if r is rational we may choose $C\ell(R)$ to be finite. David has several other papers on elasticity including the survey article *Elasticity of factorizations in integral domains*: *a survey*, *Factorization in Integral Domains* (D.D. Anderson, ed.), 1–29, Marcel Dekker, Inc., New York, 1997. David gave an invited hour address on elasticity at a sectional AMS meeting.

As previously mentioned David's dissertation involved subrings of $K[X, Y]$, generated by monomials. Of course these subrings are graded rings. David has always been interested in graded rings (usually integral domains graded by a torsionless grading monoid), especially in semigroup rings (usually monoid domains) or more generally twisted semigroup rings. He has also been interested in abelian groups associated to an integral domain (often graded) such as the Picard group, divisor class group, or group of divisibility. Two earlier papers of his in this direction are *Graded Krull domains*, Comm. Algebra **7** (1979), 79–106 and *The divisor class group of a semigroup ring*, Comm. Algebra, **8** (1980), 467–476. A neat result in the latter paper is that any abelian group of the form $F \oplus T$ where F is free and T is torsion is the divisor class group of a local Krull domain. In the early 80s David wrote two papers on seminormal graded rings (*Seminormal graded rings*, J. Pure Appl. Alg. **21** (1981), 1–7 and *Seminormal graded rings*, II, J. Pure Appl. Alg. **23** (1982), 221–226) and a paper with Jack Ohm (*Valuations and semivaluations of graded domain*, Math. Ann. **256** (1981), 145–146). I should also mention three of his papers involving the Picard group of a graded integral domain: *The Picard group of a monoid domain*, J. Algebra **115** (1988), 342–351, *The Picard group of a monoid domain*, II, Acta Math (Basel) **55** (1990), 143–145 and *The kernel of* $\mathrm{Pic}(R_o) \to \mathrm{Pic}(R)$ *for* R *a graded domain*, C. R. Math. Rep. Acad. Sci. Canada **13** (1991), 248–252. In all David has well over twenty papers involving these topics. I would be remiss not to mention his paper *A general theory of class groups*, Comm. Algebra **16** (1988), 805–847 which gives a thorough investigation of the $\star$-class group $C\ell_\star(D) := \star\text{-Inv}(D)/\mathrm{Prin}(D)$ where $\star$ is a star-operation on the integral domain D, $\star$-Inv(D) is the group of $\star$-invertible fractional ideals of D under the $\star$-product and Prin(D) is its subgroup of nonzero principal fractional ideals.

David has written a number of excellent survey articles in addition to the previously mentioned articles on the zero-divisor graph and elasticity: (1) *The class group and local class group of an integral domain*, 33–55, *Non-Noetherian Commutative Rings*, Math Appl. **520**, Kluwer Acad. Publ, Dordrecht, 2000, (2) *Root closure in commutative rings: a survey*, 55–71, Advances in Commutative Ring Theory (Fez, 1997), Lecture Notes in Pure and Appl. Math. **205**, Dekker, New York, 1999, (3) *Robert Gilmer's work on semigrouprings*, 21–37, *Multiplicative Ideal Theory in Commutative Algebra*, Springer, New York, 2006, and (4) *The zero-divisor graph of a commutative semigroup: a survey* (with Ayman Badawi), 23–29, *Groups, Modules, and Model Theory-Surveys and Developments*, Springer, New York, 2017.

David has had so many papers on so many different topics I just cannot list them all. He has done extensive work on star-operations, topics related to integral closure such as seminormality and root closure, and on rings of the form $A + XB[X]$. Let me mention three papers that I found particularly interesting. The first with his former colleague S.B. Mulay *Non-catenary factorial domains*, Comm. Algebra 17 (1989), 1179–1185 for any $d \geq 3$ constructs a non-Noetherian d-dimensional quasilocal non-catenary factorial domain whose maximal ideal is generated by two elements. The second and third papers involve condensed rings, that is, rings where the product of two ideals I and J is given by $IJ = \{ij \mid i \in I,\ j \in J\}$. Condensed rings were introduced by David and his former colleague David Dobbs (who together co-authored more than twenty papers) in *On the product of two ideals*, Canad. Math. Bull. **26** (1983), 106–114 and a second with the two Davids and Jimmy Arnold, *Integrally closed condensed domains are Bezout*, Canad. Math. Bull. **28** (1985), 908–102 that proved the result appearing in the title.

David has had an outstanding career. He is an AMS Fellow, he has influenced the area of commutative algebra through his research, exposition, and mentoring. He has been advisor and mentor to both undergraduate students, especially with his work on the REU's, and with graduate students as Director of Graduate Studies and Ph.D. advisor to his 11 students. We all wish him a happy retirement.

On ⋆-Semi-homogeneous Integral Domains

D. D. Anderson and Muhammad Zafrullah

Abstract Let $\star$ be a finite character star-operation defined on an integral domain D. A nonzero finitely generated ideal of D is $\star$-*homogeneous* if it is contained in a unique maximal $\star$-ideal. And D is called a $\star$-*semi-homogeneous* ($\star$-*SH*) *domain* if every proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals. Then D is a $\star$-semi-homogeneous domain if and only if the intersection $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$ is independent and locally finite where $\star$-Max(D) is the set of maximal $\star$-ideals of D. The $\star$-SH domains include h-local domains, weakly Krull domains, Krull domains, generalized Krull domains, and independent rings of Krull type. We show that by modifying the definition of a $\star$-homogeneous ideal we get a theory of each of these special cases of $\star$-SH domains.

1 Introduction

Many important types of integral domains have a representation of the form $D = \bigcap_{P \in \mathscr{F}} D_P$ where $\mathscr{F}$ is a set of prime ideals of D that is (1) independent, that is, two distinct elements of $\mathscr{F}$ do not contain a common nonzero prime ideal and (2) has finite character (or is locally finite), that is, each nonzero element of D is contained in at most finitely many elements of $\mathscr{F}$. These domains called $\mathscr{F}$-IFC domains were the subject of [10]. Suppose that D is an $\mathscr{F}$-IFC domain. If $\mathscr{F} = \mathrm{Max}(D)$, the set of maximal ideals of D, we get the h-local domains of Matlis [20] while if $\mathscr{F} = X^{(1)}(D)$, the set of height-one prime ideals of D, we get weakly Krull domains [5]. We can further put conditions on D_P for $P \in \mathscr{F}$. If each D_P is a valuation domain, we get the independent rings of Krull type (IRKT) of Griffin [15], generalized Krull domains if further $\mathscr{F} = X^{(1)}(D)$, and finally Krull domains when each D_P is a DVR.

D. D. Anderson (✉)
Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA
e-mail: dan-anderson@uiowa.edu

M. Zafrullah
Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA
e-mail: mzafrullah@usa.net

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_2

Now in [10] we began with a representation $D = \bigcap_{P \in \mathscr{F}} D_P$ and its induced star-operation $\star_{\mathscr{F}}$ given by $A^{\star_{\mathscr{F}}} = \bigcap_{P \in \mathscr{F}} AD_P$ for a nonzero fractional ideal A of D. (The definition of a star-operation and needed results about star-operations are reviewed in Sect. 2.) We showed that D is an $\mathscr{F}$-IFC domain if and only if each nonzero proper principal ideal of D (or equivalently, each nonzero proper ideal A of D with $A = A^{\star_{\mathscr{F}}}$) has a representation of the form $A = (I_1 \cdots I_n)^{\star_{\mathscr{F}}}$ where each I_i is contained in a unique element of $\mathscr{F}$. In this paper, we change the point of view. We begin with an integral domain D and $\star$ a finite character star-operation on D so $D = \bigcap_{P \in \star\text{-}\operatorname{Max}(D)} D_P$ where $\star$-Max(D) is the set of maximal $\star$-ideals of D. We define a nonzero finitely generated ideal I of D to be $\star$-homogeneous if I is contained in a unique element of $\star$-Max(D) and D to be a $\star$-semi-homogeneous ($\star$-SH) domain if each proper nonzero principal ideal Dx of D has a representation $Dx = (I_1 \cdots I_n)^\star$ where I_i is $\star$-homogeneous. We show (Theorem 4) that D is a $\star$-SH domain if and only if D is a $\star$-Max(D)-IFC domain, that is, the representation $D = \bigcap_{P \in \star\text{-}\operatorname{Max}(D)} D_P$ is independent and of finite character. In this case, each nonzero finitely generated ideal I with $I^\star \neq D$ has a representation $I^\star = (I_1 \cdots I_n)^\star$ where each I_i is a $\star$-homogeneous ideal (Theorem 6). We also show that for any domain D if a proper $\star$-ideal I has a representation as a $\star$-product of $\star$-homogeneous ideals, then I has a representation $I = (J_1 \cdots J_n)^\star$ where $J_1, \ldots, J_n$ are pairwise $\star$-comaximal $\star$-homogeneous ideals and that this representation is unique in the sense that if $I = (K_1 \cdots K_m)^\star$ where $K_1, \ldots, K_m$ are pairwise $\star$-comaximal $\star$-homogeneous ideals of D, then $n = m$ and after reordering $J_i^\star = K_i^\star$ for $i = 1, \ldots, n$.

Our approach in this paper is to add additional conditions to the definition of a $\star$-homogeneous ideal I (such as for each $\star$-homogeneous ideal $J \supseteq I$ (or perhaps just for I itself) $J^\star$ is $\star$-invertible or principal, or some $(J^n)^\star$ is principal) to get a "$\star$-β-homogeneous ideal." We then say that a $\star$-β-homogeneous ideal I has type 1 (resp., type 2) if $\sqrt{I} = M(I)$ where $M(I)$ is the unique $\star$-maximal ideal containing I (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). We define D to be a "$\star$-β-SH domain" (resp., $\star$-β-SH domain of type i, $i = 1, 2$) if each proper nonzero principal ideal of D is a $\star$-product of $\star$-β-homogeneous ideals (resp., $\star$-β-homogeneous ideals of type i, $i = 1, 2$). For example, we call the $\star$-homogeneous ideal I $\star$-super-homogeneous if for each $\star$-homogeneous ideal $J \supseteq I$, J is $\star$-invertible. We show (Theorem 10) that D is a $\star$-super-SH domain if and only if D is an $\star$-IRKT, that is, $D = \bigcap_{P \in \star\text{-}\operatorname{Max}(D)} D_P$ is independent and of finite character and each D_P is a valuation domain. As a second example, we show (Theorem 7) that D is a $\star$-SH domain of type 1 if and only if D is a $\star$-weakly Krull domain, that is, D is weakly Krull and $\star$-Max$(D) = X^{(1)}(D)$.

So here we define a class of integral domains by requiring that each proper nonzero principal ideal is a $\star$-product of a certain kind of $\star$-homogeneous ideal. As a bonus we get that if I is a finitely generated nonzero ideal with $I^\star \neq D$, then $I^\star$ is actually a $\star$-product of this kind of $\star$-homogeneous ideal. Moreover, if a proper $\star$-ideal I is a $\star$-product of this kind of $\star$-homogeneous ideal, we can write I as a $\star$-product of pairwise $\star$-comaximal $\star$-homogeneous ideals of that kind and this representation is unique in the sense previously mentioned. Also within this class of $\star$-β-SH domains, by slightly changing the definition of a $\star$-β-homogeneous ideal, we get $\star$-β-SH domains with trivial or torsion $\star$-class group $C\ell_\star(D)$.

Of course, we can also vary the star-operation. Two important star-operations are the d-operation $A \to A_d = A$ and the t-operation $A \to A_t = \bigcup\{J_v | J \subseteq A$ is a nonzero finitely generated ideal$\}$ where $J_v = (J^{-1})^{-1}$. A d-SH domain is just an h-local domain, while t-SH domains (not called that) were the subject of [7]. By varying the kind of $\star$-homogeneous ideal (and possibly adding a type) and varying the star-operation, we get a whole host of various important integral domains including h-local domains, weakly Krull domains, Krull domains, Dedekind domains, generalized Krull domains, independent rings of Krull, and these classes of domains that have trivial or torsion $\star$-class group.

2 Star-Operations and $\mathscr{F}$-IFC Domains

We begin with the definition of a star-operation.

Definition 1 Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. A *star-operation* $\star$ on D is a closure operation on $F(D)$ (i.e., $A \subseteq A^\star$, $(A^\star)^\star = A^\star$, and $A \subseteq B \Rightarrow A^\star \subseteq B^\star$ for $A, B \in F(D)$) that satisfies $D^\star = D$ and $(xA)^\star = xA^\star$ for $A \in F(D)$ and $x \in K^* := K \setminus \{0\}$.

With $\star$ we can associate a new star-operation $\star_s$ given by $A \to A^{\star_s} := \bigcup\{B^\star | B \subseteq A,\ B \in f(D)\}$ for $A \in F(D)$. We say that $\star$ has *finite character* if $\star = \star_s$. Three important star-operations are the *d-operation* $A \to A_d := A$, the *v-operation* $A \to A_v := (A^{-1})^{-1} = \bigcap\{Dx | Dx \supseteq A,\ x \in K^*\}$ where $A^{-1} = \{x \in K | xA \subseteq D\}$, and the *$t$-operation* $t := v_s$. Here d and t have finite character. A fractional ideal $A \in F(D)$ is a *$\star$-ideal* (resp., *finite type $\star$-ideal*) if $A = A^\star$ (resp., $A = A_1^\star$ for some $A_1 \in f(D)$). If $\star$ has finite character and $A^\star$ has finite type, then $A^\star = A_1^\star$ for some $A_1 \in f(D)$ with $A_1 \subseteq A$. A fractional ideal $A \in F(D)$ is *$\star$-invertible* if there exists a $B \in F(D)$ with $(AB)^\star = D$; in this case we can take $B = A^{-1}$. For any $\star$-invertible $A \in F(D)$, $A^\star = A_v$. If $\star$ has finite character and A is $\star$-invertible, then $A^\star$ is a finite type $\star$-ideal and $A^\star = A_t$. Given two fractional ideals $A, B \in F(D)$, $(AB)^\star$ is their *$\star$-product*. Note that $(AB)^\star = (A^\star B)^\star = (A^\star B^\star)^\star$. Given two star-operations $\star_1$ and $\star_2$ on D, we write $\star_1 \leq \star_2$ if $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. So $\star_1 \leq \star_2 \Leftrightarrow A^{\star_1\star_2} = A^{\star_2} \Leftrightarrow A^{\star_2\star_1} = A^{\star_2}$ for all $A \in F(D)$. For any finite character star-operation $\star$ on D we have $d \leq \star \leq t$. For an introduction to star-operations, the reader is referred to [14, Sect. 32]. For a more detailed treatment see [16, 18].

Suppose that $\star$ is a finite character star-operation on D. Then a proper $\star$-ideal is contained in a maximal $\star$-ideal and a maximal $\star$-ideal is prime. We denote the set of maximal $\star$-ideals of D by $\star$-$\mathrm{Max}(D)$, the set of maximal ideals of D by $\mathrm{Max}(D)$, and the set of height-one prime ideals of D by $X^{(1)}(D)$. We have $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$.

Let $\mathscr{F}$ be a nonempty collection of nonzero prime ideals of D. We say that $\mathscr{F}$ is a *defining family of primes for* D if $D = \bigcap_{P \in \mathscr{F}} D_P$. So for a finite character star-operation $\star$ on D, $\star$-Max(D) is a defining family of primes for D. We say that the intersection $D = \bigcap_{P \in \mathscr{F}} D_P$, or the set $\mathscr{F}$ of prime ideals itself, is of *finite character*, or is *locally finite*, if each nonzero element of D is in at most finitely many $P \in \mathscr{F}$. This is equivalent to each nonzero element of D (or of K) being a unit in almost all D_P, $P \in \mathscr{F}$. We will say that the finite character star-operation $\star$ is *locally finite* if $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$ is locally finite. The defining family of primes $\mathscr{F}$ is *independent* if for distinct $P, Q \in \mathscr{F}$, there does not exist a nonzero prime ideal m with $m \subseteq P \cap Q$. This is equivalent to $D_P D_Q = K$ [10, Lemma 4.1]. If $\mathscr{F}$ is independent, then $\mathscr{F}$ is an anti-chain. We say that a finite character star-operation $\star$ is *independent* if $\star$-Max(D) is independent. Note that if two prime $\star$-ideals contain a nonzero prime ideal, they actually contain a (nonzero) prime $\star$-ideal. Indeed, if P is a nonzero prime ideal and $0 \neq x \in P$, we can shrink P to a prime ideal P' minimal over Dx, and P' is a prime $\star$-ideal. For a finite character star-operation $\star$ on D, we call D a *$\star$-h-local domain* if $\star$ is independent and locally finite, that is, each proper principal ideal is contained in only finitely many maximal $\star$-ideals and each prime $\star$-ideal is contained in a unique maximal $\star$-ideal. For the case of $\star = d$, we just get the h-local domains of Matlis [20]. We say that D is a *$\mathscr{F}$-IFC domain* if $\mathscr{F}$ is an independent, finite character defining family of prime ideals for D. Thus for a finite character star-operation $\star$ on D, D being a $\star$-h-local domain is the same thing as D being a $\mathscr{F}$-IFC domain for $\mathscr{F} = \star$-Max(D).

Suppose that $\mathscr{F}$ is a defining family of primes for D. Then the operation $A \longrightarrow A^{\star_{\mathscr{F}}} := \bigcap_{P \in \mathscr{F}} AD_P$ is a star-operation on D which has finite character if $\mathscr{F}$ is locally finite [2, Theorem 1]. (However, $\star_{\mathscr{F}}$ may have finite character without $\mathscr{F}$ being locally finite. For example, for $\mathscr{F} = \mathrm{Max}(D)$, $\star_{\mathscr{F}}$ is just the d-operation which has finite character but $\mathscr{F}$ need not be locally finite.) Moreover, $A^{\star_{\mathscr{F}}} D_P = AD_P$ for $A \in F(D)$ and $P \in \mathscr{F}$. Thus if D is a $\mathscr{F}$-IFC domain, $\star_{\mathscr{F}}$ has finite character and $\star_{\mathscr{F}}$-Max$(D) = \mathscr{F}$. In the case where $\star$ is a finite character star-operation on D and $\mathscr{F} = \star$-Max(D), $\star_{\mathscr{F}} = \star_w$ where $\star_w$ is the star-operation defined by $A \to A^{\star_w} := \{x \in K | xJ \subseteq A \text{ for some } J \in f(D) \text{ with } J^\star = D\} = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} AD_P$ for $A \in F(D)$. Here $\star_w$ has finite character, $\star_w \leq \star$, and $(A \cap B)^{\star_w} = A^{\star_w} \cap B^{\star_w}$ for $A, B \in F(D)$. Also, $\star$-Max$(D) = \star_w$-Max(D) and hence $A \in F(D)$ is $\star$-invertible if and only if it is $\star_w$-invertible. Moreover, for a $\star$-invertible (or $\star_w$-invertible) ideal $A \in F(D)$, $A^\star = A^{\star_w} = A_t = A_v$. For results on the $\star_w$-operation, see [4].

We have the following result relating $\star$ and $\star_w$.

Theorem 1 *Let $\star_1$ and $\star_2$ be two finite character star-operations on an integral domain D. Then the following conditions are equivalent.*

1. $\star_{1w} = \star_{2w}$.
2. $\star_1$-Max$(D) = \star_2$-Max(D).
3. $A^{\star_1} = D \Leftrightarrow A^{\star_2} = D$ *for* $A \in F(D)$.

4. $A^{\star_1} = D \Leftrightarrow A^{\star_2} = D$ *for* $A \in f(D)$.
5. $P^{\star_{1w}} = P^{\star_{2w}}$ *for each nonzero prime ideal* P *of* D.

Proof (1) ⇒ (2) $\star_1$- Max$(D) = \star_{1w}$- Max$(D) = \star_{2w}$- Max$(D) = \star_2$- Max(D). (2) ⇒ (3) ⇒ (4) ⇒ (1) ⇒ (5) Clear. (5) ⇒ (2) We have $\star_{1w}$- Max$(D) = \star_{2w}$ - Max(D) and hence as in (1) ⇒ (2) we have $\star_1$- Max$(D) = \star_2$- Max(D).

We next briefly review some of the material from [10] concerning $\mathcal{F}$-IFC domains. So let D be an integral domain and $\mathcal{F}$ a defining family of primes for D. For an ideal A of D let $m(A) = \{P \in \mathcal{F} | A \subseteq P\}$ and call A *unidirectional* if $|m(A)| = 1$. Suppose that A is unidirectional. If P is the unique element of $\mathcal{F}$ containing A, we say that A is *unidirectional pointing to* P. The following theorem sums up some of the results from [10].

Theorem 2 *Let* $\mathcal{F}$ *be a defining family of prime ideals for the integral domain* D *and let* $\star_{\mathcal{F}}$ *be the star-operation given by* $A^{\star_{\mathcal{F}}} = \bigcap_{P \in \mathcal{F}} AD_P$ *for* $A \in F(D)$.

1. *If* A *is unidirectional pointing to* $P \in \mathcal{F}$, *then* $A^{\star_{\mathcal{F}}} = AD_P \cap D$. *Conversely, suppose that* $\mathcal{F}$ *is independent. Let* $P \in \mathcal{F}$. *Then for a nonzero ideal* $A \subseteq P$, $AD_P \cap D$ *is unidirectional pointing to* P.
2. *Two nonzero ideals* A *and* B *of* D *are* $\star_{\mathcal{F}}$*-comaximal* (*i.e.,* $(A + B)^{\star_{\mathcal{F}}} = D$) *if and only if* $m(A) \cap m(B) = \phi$.
3. *If a* $\star_{\mathcal{F}}$*-ideal* A *of* D *is expressible as a finite* $\star_{\mathcal{F}}$*-product of unidirectional ideals, then* A *is uniquely expressible* (*up to order*) *as a* $\star_{\mathcal{F}}$*-product of pairwise* $\star_{\mathcal{F}}$*-comaximal unidirectional* $\star_{\mathcal{F}}$*-ideals.*
4. *The following conditions are equivalent.*
   a. $\mathcal{F}$ *is an independent defining family of finite character, i.e.,* D *is a* $\mathcal{F}$*-IFC domain.*
   b. *Every proper integral* $\star_{\mathcal{F}}$*-ideal of* D *is* (*uniquely*) *expressible as a finite* $\star_{\mathcal{F}}$*-product of* (*pairwise* $\star_{\mathcal{F}}$*-comaximal*) *unidirectional* ($\star_{\mathcal{F}}$*-*) *ideals.*
   c. *Every proper integral principal ideal of* D *is* (*uniquely*) *expressible as a finite* $\star_{\mathcal{F}}$*-product of* (*pairwise* $\star_{\mathcal{F}}$*-comaximal*) *unidirectional* ($\star_{\mathcal{F}}$*-*) *ideals.*
   d. *Every nonzero prime ideal of* D *contains a nonzero element* x *such that* Dx *is* (*uniquely*) *expressible as a finite* $\star_{\mathcal{F}}$*-product of* (*pairwise* $\star_{\mathcal{F}}$*-comaximal*) *unidirectional* ($\star_{\mathcal{F}}$*-*) *ideals.*

Proof (1) [10, Lemma 2.3], (2) Clear, (3) [10, Lemma 2.6], (4) Combine [10, Proposition 2.7] and [10, Theorem 2.1].

3 ★-Homogeneous Ideals

For $\mathcal{F}$-IFC domains, we considered $\star_{\mathcal{F}}$-product representations of $\star_{\mathcal{F}}$-ideals. In this paper, we change our point of view. We begin with a finite character star-operation $\star$ on the integral domain D and consider $\star$-product representations of $\star$-ideals. We make the following fundamental definition.

Definition 2 Let $\star$ be finite character star-operation on the integral domain D. An ideal I of D is *$\star$-homogeneous* if I is a nonzero finitely generated ideal and I is contained in a unique maximal $\star$-ideal.

Suppose that I is a $\star$-homogeneous ideal of D. If P is the unique maximal $\star$-ideal containing I we say that I is *P-$\star$-homogeneous*. We will often denote the unique maximal $\star$-ideal containing I by $M(I)$. We say that two $\star$-homogeneous ideals I and J are *similar*, denoted $I \sim J$, if $M(I) = M(J)$.

Suppose that $\star$ is a finite character star-operation on the integral domain D. So $D = \bigcap_{P \in \star\text{-}\operatorname{Max}(D)} D_P$, that is, $\star\text{-}\operatorname{Max}(D)$ is a defining family of primes for D and hence for $\mathscr{F} = \star\text{-}\operatorname{Max}(D)$, the star-operation $\star_{\mathscr{F}}$ given by $A \longrightarrow A^{\star_{\mathscr{F}}} = \bigcap_{P \in \mathscr{F}} AD_P$ is just the $\star_w$-operation. So $\star_{\mathscr{F}} = \star_w$ is a finite character star-operation on D and $\star_w \leq \star$, that is, $A^{\star_w} \subseteq A^\star$ for all $A \in F(D)$. Note that I is P-$\star$-homogeneous if and only if I is P-$\star_w$-homogeneous if and only if I is a finitely generated unidirectional ideal pointing to P.

The next two propositions give some results concerning $\star$-homogeneous ideals.

Proposition 1 *Let D be an integral domain, I a nonzero finitely generated ideal of D, and $\star$ a finite character star-operation on D.*

1. *Suppose that $I^\star \neq D$. Then I is $\star$-homogeneous if and only if for (finitely generated) ideals J and K of D with $J, K \supseteq I$ and $J^\star, K^\star \neq D$, we have $(J + K)^\star \neq D$.*
2. *For I $\star$-homogeneous, $M(I) = \{x \in D | (I, x)^\star \neq D\}$.*
3. *If I is $\star$-homogeneous, $I^\star D_{M(I)} \cap D = I^\star$.*
4. *If I is $\star$-homogeneous and $A_1, \ldots, A_n$ are pairwise $\star$-comaximal ideals of D with $A_1 \cdots A_n \subseteq I^\star$, then some $A_i \subseteq I^\star$.*

Proof
1. First note that since $\star$ has finite character, if there are ideals $J, K \supseteq I$ with $J^\star, K^\star \neq D$, but $(J + K)^\star = D$, then there are finitely generated ideals J and K with this property. ($\Rightarrow$) Suppose that I is $\star$-homogeneous. If $J, K \supseteq I$ with $J^\star, K^\star \neq D$, then necessarily $J, K \subseteq M(I)$, so $(J + K)^\star \neq D$. ($\Leftarrow$) Let M_1 and M_2 be maximal $\star$-ideals containing I. Then $(M_1 + M_2)^\star \neq D$, so $M_1 = M_2$. Hence I is $\star$-homogeneous.
2. Here $M(I)$ is the unique maximal $\star$-ideal containing I. If $x \in M(I)$, then $(I, x) \subseteq M(I)$ and hence $(I, x)^\star \neq D$. Conversely, if $(I, x)^\star \neq D$, then (I, x) is contained in a maximal $\star$-ideal P that also contains I, so $P = M(I)$. Hence $x \in (I, x) \subseteq M(I)$.
3. Clearly $I^\star D_{M(I)} \cap D \supseteq I^\star$. Let $x \in I^\star D_{M(I)} \cap D$, so $x = i/s$ where $i \in I^\star$ and $s \notin M(I)$. So $xs \in I^\star$. Now $s \notin M(I)$ implies $(I, s)^\star = D$, so $Dx = (Ix, sx)^\star \subseteq I^\star$.
4. By induction it suffices to do the case $n = 2$. So suppose that A and B are $\star$-comaximal ideals of D with $AB \subseteq I^\star$. We cannot have both $A, B \subseteq M(I)$, so say $B \nsubseteq M(I)$. Then $A \subseteq AD_{M(I)} \cap D = ABD_{M(I)} \cap D \subseteq I^\star D_{M(I)} \cap D = I^\star$.

Proposition 2 *Let $\star$ be a finite character star-operation on the integral domain D. For $\star$-homogeneous ideals I and J of D, the following are equivalent.*

1. $I \sim J$.
2. $(I+J)^\star \neq D$.
3. IJ *is $\star$-homogeneous.*

If (1), (2), or (3) holds, then $IJ \sim I \sim J$. Thus if $I_1, \ldots, I_n$ are $\star$-homogeneous ideals of D with $I_1, \ldots, I_n$ all similar, then $I_1 \cdots I_n$ is $\star$-homogeneous and $I_1 \cdots I_n \sim I_1 \sim \cdots \sim I_n$.

Proof (1)⇒(2) $I, J \subseteq M(I) = M(J) \Rightarrow I + J \subseteq M(J)$ and hence $(I+J)^\star \neq D$. (2)⇒(1) Now $(I+J)^\star \neq D$ implies $I+J$ is contained in a maximal $\star$-ideal P. But since $I, J \subseteq P$ we must have $M(I) = P$ and $M(J) = P$, so $M(I) = M(J)$. (1)⇒(3) IJ is finitely generated and $(IJ)^\star \neq D$. Let P be a maximal $\star$-ideal containing IJ. Since P is prime, we have, say $I \subseteq P$. So $P = M(I)$. So IJ is $\star$-homogeneous with $M(IJ) = M(I)$. (3)⇒(1) Suppose that $I \nsim J$, so $M(I)$ and $M(J)$ are two distinct maximal $\star$-ideals containing IJ, a contradiction.

The last statement is now immediate.

We next give a uniqueness result for $\star$-products of $\star$-homogeneous ideals. Compare with Theorem 2(3) [10, Lemma 2.6].

Theorem 3 *Let D be an integral domain and $\star$ a finite character star-operation on D. Let I be an ideal of D. If I is a $\star$-product of $\star$-homogeneous ideals of D, then I is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-homogeneous.*

Proof Suppose $I = (I_1 \cdots I_n)^\star$ where I_i is $\star$-homogeneous. Let $M(I_{i_1}), \ldots, M(I_{i_s})$ be the distinct maximal $\star$-ideals among $M(I_1), \ldots, M(I_n)$. For $1 \leq \ell \leq s$, put $J_\ell := \prod\{I_j | I_j \sim I_{i_\ell}\}$. So $J_1, \ldots, J_s$ are $\star$-homogeneous ideals of D that are pairwise $\star$-comaximal and $I = (J_1 \cdots J_s)^\star = (J_1^\star \cdots J_s^\star)^\star$. Suppose that we have another representation $I = (K_1 \cdots K_t)^\star = (K_1^\star \cdots K_t^\star)^\star$ where $K_1, \ldots, K_t$ are pairwise $\star$-comaximal $\star$-homogeneous ideals of D. Now $K_1 \cdots K_t \subseteq (J_1 \cdots J_s)^\star \subseteq J_1^\star$, so by Proposition 1, some $K_i \subseteq J_1^\star$. Reordering, we can take $i = 1$, so $K_1 \subseteq J_1^\star$. Reversing the roles of the J_i's and K_i's, we have some $J_i \subseteq K_1^\star \subseteq J_1^\star$. By $\star$-comaximality, $i = 1$, so $J_1 \subseteq K_1^\star$ and hence $J_1^\star = K_1^\star$. Continuing we see that each J_i matches up to a K_j with $J_i^\star = K_j^\star$. Likewise each K_i matches up to a J_j with $K_i^\star = J_j^\star$. Thus $s = t$ and after reordering $J_i^\star = K_i^\star$ for $i = 1, \ldots, s$.

We next define $\star$-SH domains. We will see that a $\star$-SH domain is the same thing as a $\star$-h-local domain.

Definition 3 Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a *$\star$-semi-homogeneous ($\star$-SH) domain* if every proper nonzero principal ideal of D is a finite $\star$-product of $\star$-homogeneous ideals of D.

So by Theorem 3, D is a $\star$-SH domain if and only if each proper nonzero principal ideal Dx of D has a unique representation (up to order) as a finite $\star$-product of pairwise $\star$-comaximal $\star$-ideals $Dx = (J_1^\star \cdots J_s^\star)^\star$ $(= (J_1 \cdots J_s)^\star)$ where J_i is $\star$-homogeneous. We next use our results from [10] to get some characterizations of $\star$-SH domains.

Theorem 4 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-SH domain.*
2. *D is a $\star$-$\mathrm{Max}(D)$-IFC domain, that is, D is a $\star$-h-local domain.*
3. *D is a $\star_w$-SH domain.*

Proof (1)⇔(3) Since $\star\text{-}\mathrm{Max}(D) = \star_w\text{-}\mathrm{Max}(D)$, an ideal is $\star$-homogeneous if and only if it is $\star_w$-homogeneous. Let x be a nonzero nonunit of D. Now in a representation $Dx = (I_1 \cdots I_n)^\star$ (resp., $Dx = (J_1 \cdots J_m)^{\star_w}$) where each I_i (resp., J_i) is $\star$-homogeneous (resp., $\star_w$-homogeneous), $I_1 \cdots I_n$ (resp., $J_1 \cdots J_m$) is $\star$-invertible (resp., $\star_w$-invertible). But an ideal I is $\star$-invertible if and only if it is $\star_w$-invertible and in this case $I^\star = I_t = I^{\star_w}$. Thus $Dx = (I_1 \cdots I_n)^{\star_w}$ (resp., $(J_1 \cdots J_m)^\star$). So Dx is a $\star$-product of $\star$-homogeneous ideals if and only if it is a $\star_w$-product of $\star_w$-homogeneous ideals. (2)⇔(3) Let $\mathscr{F} = \star\text{-}\mathrm{Max}(D)$, so $\star_{\mathscr{F}} = \star_w$. By [10, Proposition 2.7], D is a $\mathscr{F}$-IFC domain if and only if for each nonzero nonunit $x \in D$, Dx is a $\star_{\mathscr{F}} = \star_w$-product of unidirectional ideals. Now (3)⇒(2) follows since a $\star_w$-homogeneous ideal is unidirectional. For (2)⇒(3) note that if $Dx = (I_1 \cdots I_n)^{\star_w}$ where each I_i is unidirectional, then I_i is $\star_w$-invertible and hence $I_i^{\star_w} = (I_i')^{\star_w}$ for some finitely generated ideal $I_i' \subseteq I_i$. So I_i' is $\star_w$-homogeneous and $Dx = (I_1' \cdots I_n')^{\star_w}$.

Theorem 5 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-SH domain.*
2. *$\star$ is locally finite and independent.*
3. *Every nonzero prime ideal of D contains a nonzero element x such that Dx is a $\star$-product of $\star$-homogeneous ideals.*
4. *Every nonzero prime ideal of D contains a $\star$-invertible $\star$-homogeneous ideal of D.*
5. *For $P \in \star\text{-}\mathrm{Max}(D)$ and $0 \neq x \in P$, $xD_P \cap D = I^\star$ for some $\star$-invertible P-$\star$-homogeneous ideal I.*
6. *$\star$ is independent and if A is a nonzero ideal of D with AD_P finitely generated for each $P \in \star\text{-}\mathrm{Max}(D)$, then $A^\star$ is a finite type $\star$-ideal.*

Proof (1)⇔(2) Theorem 4.

Note that for each i, $2 \le i \le 5$, (i) is equivalent to (i') where (i') is (i) with $\star$ replaced by $\star_w$. By [10, Theorem 3.3], $(2')$–$(5')$ are equivalent, and hence (2)–(5) are equivalent.

(2)⇒(6) Now by hypothesis, $\star$ is independent and by [10, Theorem 3.3] $A^{\star_w}$ is a finite type $\star_w$-ideal. Hence, $A^\star$ is a finite type $\star$-ideal. (6)⇒(5) Let $P \in \star\text{-}\mathrm{Max}(D)$

and $0 \neq x \in P$. Put $A := xD_P \cap D$. Let $Q \in \star\text{-}\operatorname{Max}(D)\backslash\{P\}$. Since $\star$ is independent, $D_P D_Q = K$, the quotient field of D. Thus $AD_Q = (xD_P \cap D)D_Q = xD_P D_Q \cap D_Q = xK \cap D_Q = D_Q$. So P is the only maximal $\star$-ideal containing A. Since AD_M is finitely generated for each $M \in \star\text{-}\operatorname{Max}(D)$, $A^\star = A_1^\star$ for some finitely generated ideal A_1 of D. Moreover, since $\star$ has finite character we can take $A_1 \subseteq A$. Since P is the only maximal $\star$-ideal containing A, the same is true for A_1 and $A_2 := (A_1, x)$. So A_2 is P-$\star$-homogeneous. Also, $AD_Q = D_Q = A_2 D_Q$ for $Q \in \star\text{-}\operatorname{Max}(D)\backslash\{P\}$ and $AD_P = xD_P \subseteq A_2 D_P$, so $AD_P = A_2 D_P$. Hence $A = AD_P \cap D = \bigcap_{Q \in \star\text{-}\operatorname{Max}(D)} AD_Q = \bigcap_{Q \in \star\text{-}\operatorname{Max}(D)} A_2 D_Q = A_2^{\star_w}$. As in the proof of (5)⇒(4) of [10, Theorem 3.3], A_2 is $\star_w$-invertible. Thus A_2 is $\star$-invertible and so $A = A_2^{\star_w} = A_2^\star$.

We next note that in a $\star$-SH domain every proper finite-type $\star$-ideal is a $\star$-product of $\star$-homogeneous ideals.

Theorem 6 *Let D be a $\star$-SH domain and I a nonzero finitely generated ideal of D with $I^\star \neq D$. Then $I^\star$ is uniquely expressible (up to order) as a $\star$-product $(J_1^\star \cdots J_n^\star)^\star$ of pairwise $\star$-comaximal $\star$-ideals $J_1^\star, \ldots, J_n^\star$ where each J_i is $\star$-homogeneous.*

Proof Since D is a $\star$-SH domain, $\star$ is locally finite by Theorem 5. Let $M_1, \ldots, M_n$ be the maximal $\star$-ideals containing I and put $I_i := ID_{M_i} \cap D$. So $I^{\star_w} = I_1 \cap \cdots \cap I_n$ and hence $I^\star = (I_1 \cap \cdots \cap I_n)^\star$. Since $\star$ is independent (Theorem 5) Theorem 2 gives that M_i is the unique maximal $\star$-ideal containing I_i. So $I_1, \ldots, I_n$ are pairwise $\star$-comaximal and thus $(I_1 \cap \cdots \cap I_n)^\star = (I_1 \cdots I_n)^\star$. By Theorem 5, $I_i^\star$ has $\star$-finite type, so $I_i^\star = J_i^\star$ where J_i is $\star$-homogeneous. Now $J_1, \ldots, J_n$ are pairwise $\star$-comaximal $\star$-homogeneous ideals with $I^\star = (J_1^\star \cdots J_n^\star)^\star$. Uniqueness follows from Theorem 3.

In [5] an integral domain D was defined to be *weakly Krull* if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ and the intersection is locally finite. Thus D is weakly Krull if D is a $\mathscr{F}$-IFC domain for $\mathscr{F} = X^{(1)}(D)$. We generalize this definition as follows.

Definition 4 Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a $\star$-*weakly Krull domain* ($\star$-*WKD*) if D is a $\star$-h-local domain for which $X^{(1)}(D) = \star\text{-}\operatorname{Max}(D)$.

Thus D is a $\star$-WKD if and only if D is weakly Krull and $X^{(1)}(D) = \star\text{-}\operatorname{Max}(D)$. Note that for D weakly Krull, $t\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Thus a weakly Krull domain is the same thing as a t-WKD. At the other extreme, D is a d-WKD if and only if $\dim D = 1$ and each nonzero element of D is in at most finitely many maximal ideals. If $\star_1$ and $\star_2$ be two finite character star-operations on D with $\star_1 \leq \star_2$, then D a $\star_1$-WKD implies that D is a $\star_2$-WKD. Evidently D is a $\star$-WKD if and only if it is a $\star_w$-WKD.

To give our characterization of $\star$-weakly Krull domains, we need the following definition.

Definition 5 Let D be an integral domain and $\star$ a finite character star-operation on D. We say that a $\star$-homogeneous ideal I of D has *type 1* if $M(I) = \sqrt{I^\star}$, and D is a *type 1* $\star$*-SH domain* if each nonzero proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals.

It is easy to see that a $\star$-homogeneous ideal I has type 1 if and only if for each $\star$-homogenous ideal $A \supseteq I$, there exists an $n \geq 1$ with $A^n \subseteq I^\star$.

Theorem 7 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-weakly Krull domain.*
2. *D is a $\star$-h-local domain and each $\star$-homogeneous ideal has type 1.*
3. *Every proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals, that is, D is a type 1 $\star$-SH domain.*
4. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of type 1 $\star$-homogeneous ideals.*

Proof (1)⇒(2) By definition a $\star$-weakly Krull domain is $\star$-h-local. Let I be a $\star$-homogeneous ideal of D. Since $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$, $M(I)$ is a minimal prime over $I^\star$ and as any prime ideal minimal over $I^\star$ is a $\star$-ideal, $M(I)$ is the unique prime ideal minimal over $I^\star$. Hence $M(I) = \sqrt{I^\star}$, so I has type 1.

(2)⇒(3) Clear since in a $\star$-h-local domain every proper principal ideal is a $\star$-product of $\star$-homogeneous ideals (Theorem 4).

(3)⇒(1) Certainly (3) gives that D is a $\star$-SH domain and hence $\star$-h-local (Theorem 4). We show $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Let M be a maximal $\star$-ideal. Suppose that there exists a nonzero prime ideal $Q \subsetneq M$. Let $0 \neq x \in Q$. Shrinking Q to a prime ideal minimal over Dx we can assume that Q is a $\star$-ideal. Now $Dx = (I_1 \cdots I_n)^\star$ where each I_i is a type 1 $\star$-homogeneous ideal. Now $I_1 \cdots I_n \subseteq Q$, so some $I_i \subseteq Q$ and hence $I_i^\star \subseteq Q$. But $M(I_i) = \sqrt{I_i^\star} \subseteq Q \subsetneq M$, a contradiction. Thus $\star\text{-}\operatorname{Max}(D) \subseteq X^{(1)}(D)$ and hence we have equality since each height-one prime ideal is a $\star$-ideal.

(4)⇒(3) Clear. (2)⇒(4) This follows from Theorem 6 since a $\star$-h-local domain is a $\star$-SH domain.

Invoking Theorem 3 we see that in a $\star$-weakly Krull domain a nonzero finitely generated ideal I with $I^\star \neq D$ has a unique representation (up to order) $I^\star = (J_1^\star \cdots J_n^\star)^\star$ where $J_1, \ldots, J_n$ are pairwise $\star$-comaximal type 1 $\star$-homogeneous ideals.

Now a Krull domain is a weakly Krull domain (or equivalently, a t-WKD) in which D_P is a DVR for each $P \in X^{(1)}(D)$. With this in mind, we make the following definition.

Definition 6 Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a $\star$*-Krull domain* if D is a $\star$-weakly Krull domain and D_P is a DVR for each $P \in \star\text{-}\operatorname{Max}(D)$.

Evidently D is a $\star$-Krull domain if and only if D is a Krull domain and $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Thus a Krull domain is the same thing as a t-Krull domain. At the other extreme, a d-Krull domain is a Dedekind domain. If $\star_1$ and $\star_2$ are finite character star-operations on D with $\star_1 \leq \star_2$, then D $\star_1$-Krull implies that D is $\star_2$-Krull.

Our characterization of $\star$-Krull domains requires the following definition.

Definition 7 Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D has *type 2* if $I^\star = (M(I)^n)^\star$ for some $n \geq 1$. And D is a *type 2 $\star$-SH domain* if each nonzero proper principal ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals.

Theorem 8 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-Krull domain.*
2. *Every proper $\star$-ideal of D is a $\star$-product of prime $\star$-ideals of D.*
3. *Every proper principal ideal of D is a $\star$-product of prime $\star$-ideals of D.*
4. *Every proper $\star$-ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals of D.*
5. *Every proper principal ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals of D, that is, D is a type 2 $\star$-SH domain.*

Proof (1)⇒(4) D is $\star$-Krull, so D is a Krull domain and $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. For $A \in F(D)$, $A^{\star_w} = \bigcap_{P \in X^{(1)}(D)} AD_P = A_t$, so $A^{\star_w} = A^\star = A_t$. Let $P \in X^{(1)}(D)$. Choose $x \in P \backslash P^2$. Let $Q_1, \ldots, Q_n$ be the other height-one primes containing x and choose $y \in P \backslash (Q_1 \cup \cdots \cup Q_n)$. So $(x, y)^\star = (x, y)^{\star_w} = \bigcap_{Q \in X^{(1)}(D)} (x, y)D_Q = P$. Put $H(P) := (x, y)$, so $H(P)$ is a type 2 $\star$-homogeneous ideal. Let A be a proper $\star$-ideal of D. Then $A = \bigcap_{P \in X^{(1)}(D)} AD_P = P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}$ where $P_1, \ldots, P_s$ are the height-one primes containing A and $P_i^{(n_i)} = P_i^{n_i} D_{P_i} \cap D$. But $P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)} = (P_1^{n_1} \cdots P_s^{n_s})_t = (P_1^{n_1} \cdots P_s^{n_s})^\star = ((H(P_1)^\star)^{n_1} \cdots (H(P_s)^\star)^{n_s})^\star = (H(P_1)^{n_1} \cdots H(P_s)^{n_s})^\star$.

(4)⇒(2)⇒(3), (4)⇒(5)⇒(3) Clear.

(3)⇒(1) Let x be a nonzero nonunit of D. So $Dx = (P_1 \cdots P_n)^\star$ where P_i is a prime $\star$-ideal of D. Then P_i is $\star$-invertible so $P_i = H(P_i)^\star$ where $H(P_i)$ is a finitely generated ideal contained in P_i. Thus $H(P_i)$ is a type 2 $\star$-homogeneous ideal and hence a type 1 $\star$-homogeneous ideal. So each proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals. By Theorem 7, D is a $\star$-WKD. Let $P \in X^{(1)}(D)$; we need to show that D_P is a DVR. Let $0 \neq x \in P$, so $Dx = (P_1 \cdots P_n)^\star$ where P_i is a prime $\star$-ideal which is $\star$-invertible. Now some $P_i \subseteq P$ and hence $P_i = P$, so P is $\star$-invertible. Thus $(PP^{-1}) \not\subseteq P$, so $PP^{-1}D_P = D_P$ and hence PD_P is invertible and therefore principal. Since ht $P = 1$, D_P is a DVR.

Once again we can invoke Theorem 3 to get the appropriate uniqueness result for pairwise $\star$-comaximal type 2 $\star$-homogeneous ideals in Theorem 8. We leave it to the reader to show that in a $\star$-Krull domain if $(P_1 \cdots P_n)^\star = (Q_1 \cdots Q_m)^\star$ where the

P_i's and Q_i's are maximal $\star$-ideals, then $n = m$ and after reordering $P_i = Q_i$ for each i.

The notion of a Krull domain can be generalized in a number of ways. We have already defined $\star$-Krull domains and $\star$-weakly Krull domains. An integral domain D is an *independent ring of Krull type* (*IRKT*) [15] if D is a $\mathscr{F}$-IFC domain for some defining family $\mathscr{F}$ of primes where D_P is a valuation domain for each $P \in \mathscr{F}$. For a finite character star-operation $\star$ on P, we call D a *$\star$-independent ring of Krull type* (*$\star$-IRKT*) if D is a $\mathscr{F}$-IFC domain for $\mathscr{F} = \star\text{-}\operatorname{Max}(D)$, that is, D is $\star$-h-local, and for each $P \in \star\text{-}\operatorname{Max}(D)$, D_P is a valuation domain. Thus D is a $\star$-IRKT if and only if D is an IRKT where $\mathscr{F} = \star\text{-}\operatorname{Max}(D)$. A d-IRKT is just a finite character, independent Prüfer domain. At the other extreme, a t-IRKT is just an IRKT. If $\star_1$ and $\star_2$ are finite character star-operations on D with $\star_1 \leq \star_2$ and D is a $\star_1$-IRKT, then D is a $\star_2$-IRKT, see Proposition 3 below. Recall that D is a P$\star$MD if each nonzero finitely generated ideal of D is $\star$-invertible, or equivalently, D_M is a valuation domain for each $M \in \star\text{-}\operatorname{Max}(D)$. Thus a $\star$-IRKT is a P$\star$MD. In fact, D is a $\star$-IRKT if and only if D is a $\star$-h-local P$\star$MD. A *PvMD* is usually defined to be a v-domain (each nonzero finitely generated ideal of D is v-invertible) in which A^{-1} is a finite-type v-ideal for each nonzero finitely generated ideal A of D. Thus a *PvMD* is just a *PtMD* and a P$\star$MD is a *PvMD*. Of course a *PdMD* is just a Prüfer domain.

The integral domain D is a *generalized Krull domain* (*GKD*) if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ is locally finite and for each $P \in X^{(1)}(D)$, D_P is a valuation domain, that is, D is weakly Krull and for each $P \in X^{(1)}(D)$, D_P is a valuation domain. Let $\star$ be a finite character star-operation on D. We call D a *$\star$-generalized Krull domain* (*$\star$-GKD*) if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ locally finite, $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$, and D_P is a valuation domain for each $P \in X^{(1)}(D)$, or equivalently, D is $\star$-weakly Krull and for each $P \in X^{(1)}(D)$, D_P is a valuation domain, that is, D is a $\star$-GKD if and only if D is a GKD and $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. So D is a d-GKD if and only if D is a one-dimensional finite character Prüfer domain. At the other extreme, a t-GKD is just a GKD. If $\star_1$ and $\star_2$ are two finite character star-operations on D with $\star_1 \leq \star_2$, then D a $\star_1$-GKD implies that D is a $\star_2$-GKD.

Proposition 3 *Let D be an integral domain and $\star_1$ and $\star_2$ be finite character star-operations on D with $\star_1 \leq \star_2$. If D is a $\star_1$-IRKT, then D is a $\star_2$-IRKT.*

Proof Let $P \in \star_2\text{-}\operatorname{Max}(D)$. Then $P^{\star_1} \subseteq P^{\star_2} = P$, so $P^{\star_1} \neq D$ and hence P is contained in a maximal $\star_1$-ideal Q. Moreover, Q is unique since $\star_1$ is independent. Also, D_Q is a valuation domain and hence so is $D_P = (D_Q)_{P_Q}$. Note that $\star_2$ is independent. Suppose that m is a nonzero prime ideal with $m \subseteq M_1, M_2$, two maximal $\star_2$-ideals. Then M_i is contained in a maximal $\star_1$-ideal M_i'. Since $m \subseteq M_1' \cap M_2'$, $M_1' = M_2'$ as $\star_1$ is independent. But then $M_1, M_2 \subseteq M_1'$ and $D_{M_1'}$ is a valuation domain. So M_1 and M_2 are comparable. Hence $M_1 = M_2$. So $\star_2$ is independent. We next show that $\star_2$ is locally finite. Suppose some $0 \neq x \in D$ is contained in an infinite number of maximal $\star_2$-ideals $\{Q_n\}_{n=1}^{\infty}$. Now each Q_n is contained in a maximal $\star_1$-ideal P_n.

Now if $P_n = P_m$, then Q_n and Q_m are comparable since D_{P_n} is a valuation domain, so $Q_n = Q_m$. Thus x is contained in infinitely many maximal $\star_1$-ideals, a contradiction.

The following diagram gives the various implications between the different generalizations of Krull domains.

Dedekind	$\Longrightarrow$	$\star$-Krull	$\Longrightarrow$	Krull
$\Downarrow$		$\Downarrow$		$\Downarrow$
one-dimensional finite character Prüfer	$\Longrightarrow$	$\star$-GKD	$\Longrightarrow$	GKD
		$\searrow$		$\swarrow$
$\Downarrow$		$\Downarrow$	$\star$-WKD$\Rightarrow$weakly Krull	$\Downarrow$
finite character independent Prüfer	$\Longrightarrow$	$\star$-IRKT	$\Longrightarrow$	IRKT
$\Downarrow$		$\Downarrow$		$\Downarrow$
Prüfer	$\Longrightarrow$	P$\star$MD	$\Longrightarrow$	PvMD

To characterize $\star$-IRKTs using $\star$-homogeneous ideals we need the following definition.

Definition 8 Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D is $\star$-*super-homogeneous* if each $\star$-homogeneous ideal containing I is $\star$-invertible. The $\star$-super-homogeneous ideal I has *type 1* (resp., *type 2*) if I has type 1 as a $\star$-homogeneous ideal, that is, $\sqrt{I^\star} = M(I)$ (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). The domain D is a $\star$-*super-SH domain* (resp., *type 1* $\star$-*super-SH domain*, *type 2* $\star$-*super-SH domain*) if every nonzero proper principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals (resp., of type 1, of type 2).

Note that if I is $\star$-super-homogeneous, then each finitely generated ideal containing I is $\star$-invertible. Now by [17, Theorem 1.11] a product of similar $\star$-super-homogeneous ideals is again $\star$-super-homogeneous. Thus the proof of Theorem 3 gives the corresponding uniqueness result for $\star$-products of $\star$-super-homogeneous ideals.

Theorem 9 *Let $\star$ be a finite character star-operation on the integral domain D and let $J_1, \ldots, J_n$ be a set of $\star$-super-homogeneous ideals of D. Then the $\star$-product $(J_1 \cdots J_n)^\star$ can be expressed uniquely, up to order, as a $\star$-product of pairwise $\star$-comaximal $\star$-super-homogeneous ideals.*

We next give several characterizations of $\star$-IRKTs.

Theorem 10 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-IRKT.*
2. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-invertible.*
3. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-super-homogeneous.*

4. *Every proper nonzero principal ideal is a $\star$-product of $\star$-super-homogeneous ideals, that is, D is a $\star$-super-SH domain.*
5. *If I is a nonzero finitely generated ideal with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-super-homogeneous ideals.*

Proof (1)⇒(2),(3) Let I be a $\star$-homogeneous ideal of D and let $J \supseteq I$ be a finitely generated ideal of D. Then JD_P is principal for each $P \in \star\text{-}\operatorname{Max}(D)$ since D_P is a valuation domain. Thus J is $\star$-invertible. (2)⇒(1) Let $P \in \star\text{-}\operatorname{Max}(D)$. We need to show that D_P is a valuation domain. It suffices to show that for $x, y \in P\backslash\{0\}$, $(x, y)D_P$ is principal. Let $A = (x, y)D_P \cap D$. By Theorem 5, $A^\star$ is a finite type $\star$-ideal. So $A^\star = A_1^\star$ where $A_1 \subseteq A$ is finitely generated. Now P is the unique maximal $\star$-ideal containing A and hence the unique maximal $\star$-ideal containing A_1. So by hypothesis A_1, and hence A, is $\star$-invertible. So $(x, y)D_P = AD_P$ is principal. (3)⇒(4) This is immediate since for a $\star$-h-local domain each proper nonzero principal ideal is a $\star$-product of $\star$-homogeneous ideals by Theorem 4. (4)⇒(1) Every proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals, so by Theorem 4, D is $\star$-h-local. Let $P \in \star\text{-}\operatorname{Max}(D)$. We need that D_P is a valuation domain. Let $0 \neq x \in P$, so $Dx = (I_1 \cdots I_n)^\star$ where I_i is $\star$-super-homogeneous. Let $I = \prod\{I_i | I_i$ is P $\star$-homogeneous$\}$. Then $xD_P \cap D = I^\star$. By [17, Theorem 1.11], I is $\star$-super-homogeneous. Let $0 \neq y \in P$. Then again $yD_P \cap D = J^\star$ for some $\star$-super-homogeneous ideal J of D. But by [17, Theorem 1.11] for two P-$\star$-super-homogeneous ideals I and J of D, $I^\star$ and $J^\star$ are comparable. Thus $xD_P \cap D$ and $yD_P \cap D$ are comparable, so D_P is a valuation domain. (5)⇒(4) Clear. (1)⇒(5) Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. By (1)⇒(3) it is enough to show $I^\star$ is a $\star$-product of $\star$-homogeneous ideals. But this follows from Theorem 6.

Using Theorems 9 and 10, we get the following result.

Proposition 4 *Let D be an integral domain and $\star$ a finite character star-operation on D. Suppose that D is a $\star$-IRKT. Let $a, b \in D^*$ with $(a, b)^\star \neq D$. Then $(a, b)^\star = (I_1 \cdots I_n)^\star$ where $I_1, \ldots, I_n$ are pairwise $\star$-comaximal $\star$-super-homogeneous ideals of D containing (a, b) such that $(a, b)D_{M(I_i)} = I_i D_{M(I_i)} = aD_{M(I_i)}$ or $bD_{M(I_i)}$.*

Proof Now by Theorems 9 and 10 $(a, b)^\star = (I_1 \cdots I_n)^\star$ where $I_1, \ldots, I_n$ are pairwise $\star$-comaximal $\star$-super-homogeneous ideals of D. Put $I_i' = I_i + (a, b)$. Then $M(I_i') = M(I_i)$, each I_i' is a $\star$-super-homogeneous ideal, and $I_i' \supseteq (a, b)$. Now $I_1 \cdots I_n \subseteq I_1' \cdots I_n' = (I_1 + (a, b)) \cdots (I_n + (a, b)) \subseteq I_1 \cdots I_n + (a, b)$, so $(I_1' \cdots I_n')^\star = (I_1 \cdots I_n)^\star$. Thus we can replace I_i by I_i' and hence assume that $(a, b) \subseteq I_i$. Since (a, b) and $I_1 \cdots I_n$ are $\star$-invertible we have $(a, b)^{\star_w} = (a, b)^\star = (I_1 \cdots I_n)^\star = (I_1 \cdots I_n)^{\star_w}$. So $(a, b)D_{M(I_i)} = (a, b)^{\star_w} D_{M(I_i)} = (I_1 \cdots I_n)^{\star_w} D_{M(I_i)} = I_1 \cdots I_n D_{M(I_i)} = I_i D_{M(I_i)}$. Now $D_{M(I_i)}$ is a valuation domain, so either $(a, b)D_{M(I_i)} = aD_{M(I_i)}$ or $(a, b)D_{M(I_i)} = bD_{M(I_i)}$.

Using Theorem 10, we get several characterizations of $\star$-GKDs.

Theorem 11 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-GKD.*
2. *D is a $\star$-IRKT and a $\star$-WKD.*
3. *D is a $\star$-IRKT and every $\star$-super-homogeneous ideal has type 1.*
4. *D is a $\star$-WKD and every $\star$-homogeneous ideal is $\star$-invertible.*
5. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-super-homogeneous and has type 1.*
6. *Every proper nonzero principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals of type 1, that is, D is a type 1 $\star$-super-SH domain.*
7. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of type 1 $\star$-super-homogeneous ideals.*

Proof (1)⇔(2) Clear. (2)⇔(3) First note that by Theorem 10, for a $\star$-IRKT the notions of $\star$-homogeneous and $\star$-super-homogeneous coincide. Then use Theorem 7. (2)⇔(4) Theorem 10. (4)⇔(5)⇔(6) Combine Theorems 7 and 10. (7)⇒(6) Clear. (5)⇒(7) Theorem 6.

Once again we can invoke Theorem 3 to get the appropriate uniqueness result for pairwise $\star$-comaximal type 1 $\star$-super-homogeneous ideals in Theorem 10.

By Theorem 8 D is a $\star$-Krull domain if and only if D is a type 2 $\star$-SH domain. Now in a $\star$-Krull domain a nonzero finitely generated ideal I is $\star$-homogeneous if and only if $I^\star = P^{(n)}$ for some $P \in X^1(D)$ and $n \geq 1$. Hence I is $\star$-homogeneous if and only if it is a type 2 $\star$-super-homogeneous ideal. Thus a type 2 $\star$-super-SH domain is the same thing as a $\star$-Krull domain and if I is a nonzero finitely generated ideal of D with $I^\star \neq D$, $I^\star$ is a $\star$-product of type 2 $\star$-super-homogeneous ideals.

Let $\star$ be a finite character star-operation on the integral domain D. We define D to be $\star$-*Bezout* if for $a, b \in D^*$, $(a, b)^\star$ is principal. It easily follows that D is $\star$-Bezout if and only if $A^\star$ is principal for each nonzero finitely generated (fractional) ideal A of D. If $\star_1$ and $\star_2$ are finite character star-operations on D, then D $\star_1$-Bezout implies that D is $\star_2$-Bezout. A d-Bezout domain is just a Bezout domain, while a t-Bezout domain is a GCD domain. We also define D to be a $\star$-*Prüfer domain* if for $a, b \in D^*$, $(a, b)^\star$ is invertible. Using [19, Exercise 22, p. 43], it is easy to see that D is $\star$-Prüfer if and only if $A^\star$ is invertible for each nonzero finitely generated (fractional) ideal A of D. Again if $\star_1 \leq \star_2$ are finite character star-operations on D, then D $\star_1$-Prüfer implies that D is $\star_2$-Prüfer. A d-Prüfer domain is just a Prüfer domain while a t-Prüfer domain is a generalized GCD domain (GGCD domain). GGCD domains were introduced in [1] and studied in more detail in [3]. We have $\star$-Bezout ⇒ $\star$-Prüfer ⇒ P$\star$MD.

Storch [21] defined a Krull domain D to be *almost factorial* if for $a, b \in D^*$ there exists an $n = n(a, b) \geq 1$ with $a^n D \cap b^n D$ principal. The second author initiated a general theory of almost factoriality in [22]. There he defined an integral domain D to be an *almost GCD domain* (*AGCD domain*) if for $a, b \in D^*$, there exists an $n = n(a, b) \geq 1$ with $a^n D \cap b^n D$ principal, or equivalently, $(a^n, b^n)_v$ $(= (a^n, b^n)_t)$ principal. This investigation was continued in [9]. In that paper an integral domain

D was defined to be an *almost Bezout domain* (*AB domain*) (resp., *almost Prüfer domain* (*AP domain*)) if for $a, b \in D^*$, there exists an $n = n(a, b) \geq 1$ with (a^n, b^n) principal (resp., invertible). It was shown that D is almost Bezout (resp., almost Prüfer) if and only if for $a_1, \ldots, a_s \in D^*$; there exists an $n = n(a_1, \ldots, a_s) \geq 1$ with $(a_1^n, \ldots, a_s^n)$ principal (resp., invertible). Briefly mentioned in [9] was the notion of an *almost generalized GCD domain* (*AGGCD domain*). Here D is a AGGCD domain if for $a, b \in D^*$ there exists an $n = n(a, b) \geq 1$ with $a^n D \cap b^n D$ invertible, or equivalently, $(a^n, b^n)_v$ $(= (a^n, b^n)_t)$ is invertible.

With the definitions in the previous two paragraphs in mind, we make the following definitions. Let D be an integral domain and $\star$ a finite character star-operation on D. We say the D is a $\star$-*almost Bezout domain* (resp., $\star$-*almost Prüfer domain*, *almost P$\star$MD*) if for $a, b \in D^*$, there exists an $n = n(a, b) \geq 1$ with $(a^n, b^n)^\star$ principal (resp., invertible, $\star$-invertible). (More generally, we could call D a $\star_2$-*almost P$\star_1$MD* if $(a^n, b^n)^{\star_2}$ is $\star_1$-invertible.) If $\star_1 \leq \star_2$ are finite character star-operations on D, then D $\star_1$-almost Bezout (resp., $\star_1$-almost Prüfer, almost P$\star_1$MD) implies D is $\star_2$-almost Bezout (resp., $\star_2$-almost Prüfer, almost P$\star_2$MD). A d-almost Bezout domain (resp., d-almost Prüfer domain) is just an almost Bezout domain (resp., almost Prüfer domain), while a t-almost Bezout domain (resp., t-almost Prüfer domain) is just an AGCD domain (resp., AGGCD domain).

We mention two useful results from [9]. First, let $\star$ be a finite character star-operation on D. Let $\{a_\alpha\} \subseteq D^*$ and $n \geq 1$. If $(\{a_\alpha\})$ is $\star$-invertible, then $(\{a_\alpha^n\})^\star = ((\{a_\alpha\})^n)^\star$. In particular, $(\{a_\alpha^n\})$ is also $\star$-invertible. This is stated for the case $\star = t$ in [9, Lemma 3.3]. The proof carries over mutatis mutandis for a general finite character star-operation $\star$. Next, for an integral domain D, the following conditions are equivalent [9, Theorem 6.8]: (1) D is n-root closed (i.e., for $x \in K$ with $x^n \in D$, $x \in D$), (2) for $\{a_\alpha\} \subseteq D^*$, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$, (3) for $\{a_\alpha\} \subseteq D^*$, $(\{a_\alpha^n\})_v = ((\{a_\alpha\})^n)_v$, and (4) for $a, b \in D^*$, $(a^n, b^n)_t = ((a, b)^n)_t$. Thus if D is integrally closed, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$ for all $\{a_\alpha\} \subseteq D^*$ and $n \geq 1$.

Using the first mentioned result of the previous paragraph, the proof of [9, Lemma 4.3] can easily be modified to show that for an integral domain D and finite character star-operation $\star$ on D, if D is $\star$-almost Bezout (resp., $\star$-almost Prüfer, almost P$\star$MD) and $a_1, \ldots, a_s \in D^*$, then there exits an $n = n(a_1, \ldots, a_s) \geq 1$ with $(a_1^n, \ldots, a_s^n)^\star$ principal (resp., invertible, $\star$-invertible). Thus for D integrally closed, D is $\star$-almost Bezout (resp., $\star$-almost Prüfer, almost P$\star$MD) if and only if for A a nonzero finitely generated (fractional) ideal of D, there exists an $n = n(A) \geq 1$ with $(A^n)^\star$ principal (resp., invertible, $\star$-invertible). The implication ($\Leftarrow$) does not require that D be integrally closed. Indeed, if $(A^n)^\star$ is $\star$-invertible, A is $\star$-invertible and hence for $A = (a, b)$, $(a^n, b^n)^\star = ((a, b)^n)^\star$. Conversely, suppose that D is integrally closed and let $A = (a_1, \ldots, a_s)$. Then for some $n \geq 1$, $(a_1^n, \ldots, a_s^n)$ is $\star$-invertible and hence $(a_1^n, \ldots, a_s^n)^\star = (a_1^n, \ldots, a_s^n)_t$. Thus $(A^n)_t \supseteq (a_1^n, \ldots, a_s^n)^\star = (a_1^n, \ldots, a_s^n)_t = (A^n)_t$.

Let $\star$ be a finite character star-operation on D. The set $\star$-Inv(D) of $\star$-invertible fractional $\star$-ideals forms a group under the $\star$-product $I \star J := (IJ)^\star$ with subgroup Prin(D), the set of nonzero principal fractional ideals of D. The quotient group $C\ell_\star(D) := \star$-Inv$(D)/$Prin(D) is called the $\star$-*class group of* D, see [11]. For $\star = d$, we have the usual class group $C(D)$, while for $\star = t$, we have the t-class group introduced by Bouvier [12] and further studied in [13]. For a Krull domain, $C\ell_t(D)$ is just the usual divisor class group. Suppose that $\star_1 \leq \star_2$ are finite character star-operations on D. Then we have natural inclusions $C(D) \subseteq C\ell_{\star_1}(D) \subseteq C\ell_{\star_2}(D) \subseteq C\ell_t(D)$. Let $\text{Inv}(D)$ be the subgroup of $\star$-$\text{Inv}(D)$ consisting of invertible ideals of D. The group $LC\ell_\star(D)$:=$\star$-$\text{Inv}(D)/\text{Inv}(D)$ is called the local $\star$-*class group of* D.

Proposition 5 *Suppose that D is a* $\star$*-IRKT. Then the following conditions are equivalent.*

1. *D is* $\star$*-almost Bezout (resp.,* $\star$*-almost Prüfer).*
2. $C\ell_\star(D)$ *is torsion (resp.,* $LC\ell_\star(D)$ *is torsion).*
3. *For each* $\star$*-super-homogeneous ideal A of D, there exists a natural number* $n = n(A)$ *with* $(A^n)^\star$ *principal (resp., invertible).*
4. *D is an AGCD (resp., AGGCD domain).*
5. $C\ell_t(D)$ *is torsion (resp.,* $LC\ell_t(D)$ *is torsion).*

Proof We do the $\star$-almost Bezout case, the $\star$-almost Prüfer case is similar. Now D being a $\star$-IRKT is integrally closed. Hence, D is $\star$-almost Bezout if and only if for each nonzero finitely generated ideal A of D, $(A^n)^\star$ is principal for some $n \geq 1$. Also, each nonzero finitely generated ideal of D is $\star$-invertible. So (1)⇒(2)⇒(3). (3)⇒(1) Let A be a nonzero finitely generated ideal of D. If $A^\star = D$, we can take $n = n(A) = 1$. So suppose that $A^\star \neq D$. Then by Theorem 10, $A^\star = (I_1 \cdots I_m)^\star$ where each I_i is $\star$-super-homogeneous. By hypothesis, there exists an n_i with $(I_i^{n_i})^\star$ is principal. Then for $n = n_1 \cdots n_m$, $(A^n)^\star = ((I_1^{n_1})^{n/n_1} \cdots (I_m^{n_m})^{n/n_m})^\star$ is principal. (1)⇒(4) Here D is $\star$-almost Bezout. Since $\star \leq t$, D is t-almost Bezout, that is, an AGCD domain. (4)⇔(5) This follows since D is integrally closed. (5)⇒(2) Here $C\ell_\star(D) \subseteq C\ell_t(D)$ so $C\ell_t(D)$ torsion gives that $C\ell_\star(D)$ is torsion.

Definition 9 Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D is a $\star$-*almost factorial-homogeneous ideal* ($\star$-*af-homogeneous ideal*) (resp., $\star$-*locally almost factorial-homogeneous ideal* ($\star$-*laf-homogeneous ideal*)) if for each $\star$-homogeneous ideal $J \supseteq I$, there exists an $n = n(J) \geq 1$ with $(J^n)^\star$ principal (resp., invertible). The integral domain D is a $\star$-*af-SH domain* (resp., $\star$-*laf-SH domain*) if for each nonzero nonunit $x \in D$, Dx is expressible as a $\star$-product of finitely many $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).

Thus a $\star$-homogeneous ideal I is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous) if and only if for each finitely generated (or equivalently, each finite type $\star$-ideal) $J \supseteq I$, some $(J^n)^\star$ is principal (resp., invertible). Note that a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) is actually $\star$-super-homogeneous. In the spirit of Theorems 3 and 9, we have the following uniqueness result for $\star$-products of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).

Theorem 12 *Let D be an integral domain and* $\star$ *a finite character star-operation on D. Let I be an ideal of D. If I is a* $\star$*-product of* $\star$*-af-homogeneous ideals (resp.,* $\star$*-laf-homogeneous ideals) of D, then I is uniquely expressible (up to order) as a* $\star$*-product* *of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous).*

Proof We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. The uniqueness of the product $(J_1^\star \cdots J_s^\star)^\star$ follows from Theorem 3. To show the existence of the product, the proof of Theorem 3 shows that it suffices to prove that the product IJ of two similar $\star$-af-homogeneous ideals I and J is again $\star$-af-homogeneous. Of course IJ is $\star$-homogeneous. Let $C \supseteq IJ$ be $\star$-homogeneous ideal of D. Then $E{:=}C + I$ is $\star$-homogeneous. So there exists a $n \geq 1$ with $(E^n)^\star$ principal. Thus E is $\star$-invertible. So $(CE^{-1} + IE^{-1})^\star = D$ where $C \subseteq CE^{-1} \subseteq D$ and $I \subseteq IE^{-1} \subseteq D$. Thus $(CE^{-1})^\star = D$ or $(IE^{-1})^\star = D$. In the first case, $C^\star = E^\star$ and hence $(C^n)^\star = (E^n)^\star$ is principal. So we can assume that $(IE^{-1})^\star = D$. Then $I^\star = E^\star \supseteq C \supseteq IJ$ so $D \supseteq (CI^{-1})^\star \supseteq J^\star$. Choose a finitely generated ideal $L \supseteq J$ with $(CI^{-1})^\star = L^\star$. So there exists an $m \geq 1$ with $(L^m)^\star$ principal. So $((CI^{-1})^m)^\star$ is principal. Choose n with $(I^n)^\star$ principal. Then $(C^{mn})^\star = (((CI^{-1})^m)^n(I^n)^m)^\star$ is principal.

We next give a characterization of AGCD $\star$-IRKTs (resp., AGGCD $\star$-IRKTs) using $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals). Of course we could enlarge the list of equivalences via Proposition 5.

Theorem 13 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-af-SH domain (resp., $\star$-laf-SH-domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).*
3. *D is an AGCD $\star$-IRKT (resp., AGGCD $\star$-IRKT).*
4. *D is an $\star$-SH domain and every $\star$-homogeneous ideal is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous).*
5. *D is a $\star$-IRKT with $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D)$ torsion) (equivalently, $C\ell_t(D)$ torsion (resp., $LC\ell_t(D)$ torsion)).*
6. *D is $\star$-h-local and for each $\star$-homogeneous ideal I of D there exists an $n \geq 1$ with $(I^n)^\star$ principal (resp., invertible).*

Proof We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (3)⇒(2) By Theorem 10 $I^\star$ is a $\star$-product of $\star$-super-homogeneous ideals. By Proposition 5 $C\ell_\star(D)$ is torsion. Hence each $\star$-super-homogeneous ideal is a $\star$-af-homogeneous ideal. So $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals. (2)⇒(1) Clear. (1)⇒(3) Since a $\star$-af-homogeneous ideal is $\star$-super-homogeneous, D is an $\star$-IRKT by Theorem 10. It remains to show that D is an AGCD domain. Let a be a nonzero nonunit of D. So $Da = (I_1 \cdots I_n)^\star$ where I_i is $\star$-af-homogeneous (and hence $\star$-super-homogeneous). By Theorem 12 we can take $I_1, \ldots, I_n$ to be pairwise $\star$-comaximal. Now for each i, $i = 1, \ldots, n$, there exists an $n_i \geq 1$ with $(I_i^{n_i})^\star$ principal. Hence, for a suitable $m \geq 1$ $Da^m = Da_1 \cdots Da_n$ where Da_i is $\star$-super-homogeneous and $Da_1, \ldots, Da_n$ are pairwise $\star$-comaximal. Thus $Da_1 \cdots Da_n = Da_1 \cap \cdots \cap$

Da_n. Let a, b be nonzero nonunits of D. By the previous remarks, there is an $m \geq 1$ with $Da^m = Da_1 \cdots Da_n = Da_1 \cap \cdots \cap Da_n$ and $Db^m = Db_1 \cdots Db_n = Db_1 \cap \cdots \cap Db_n$ where either Da_i and Db_i are similar $\star$-super-homogeneous ideals of D or exactly one of Da_i, Db_i is a $\star$-super-homogeneous ideal and the other is D, and $Da_1, \ldots, Da_n$ (resp., $Db_1, \ldots, Db_n$) are pairwise $\star$-comaximal. Now if Da_i and Db_i are both $\star$-super-homogeneous ideals, being similar, they are comparable [17, Theorem 1.11]. Thus in either case $Da_i \cap Db_i$ is a principal $\star$-super-homogeneous ideal. Thus $Da^m \cap Db^m = (Da_1 \cap Db_1) \cap \cdots \cap (Da_n \cap Db_n) = (Da_1 \cap Db_1) \cdots (Da_n \cap Db_n)$ is principal. So D is an AGCD. (4)⇒(1) Clear. (2)⇒(4) Let I be a $\star$-homogeneous ideal of D. Then $I^\star = (I_1 \cdots I_n)^\star$ where I_n is $\star$-af-homogeneous. Of course $I_1, \ldots, I_n$ must be similar. By the proof of Theorem 12, a product of similar $\star$-af-homogeneous ideals is again $\star$-af-homogeneous. Thus $I_1 \cdots I_n$ and hence I is $\star$-af-homogeneous. (3)⇔(5) Proposition 5. (6)⇔(3) Combine Theorem 10 and Proposition 5.

Recall that we defined a $\star$-homogeneous ideal I to be of type 1 (resp., type 2) if $M(I) = \sqrt{I^\star}$ (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). Thus by a *$\star$-af-homogeneous ideal of type 1* (resp., *type 2*), we mean a $\star$-af-homogeneous ideal that is type 1 (resp., type 2) as a $\star$-homogeneous ideal. And by a *$\star$-af-SH domain* of *type 1* (resp., *type 2*) we mean an integral domain in which each proper nonzero principal ideal is a $\star$-product of $\star$-af-homogeneous ideals of type 1 (resp., type 2). Of course we have the analogous definitions for $\star$-laf-homogeneous ideals. The next two theorems characterize these domains. Again we can invoke Theorem 3 to get the appropriate uniqueness results.

Theorem 14 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-af-SH domain of type 1 (resp., $\star$-laf-SH domain of type 1).*
2. *D is an AGCD $\star$-GKD (resp., AGGCD $\star$-GKD).*
3. *D is a $\star$-SH domain and each $\star$-homogeneous ideal is a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) of type 1.*
4. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals) of type 1.*
5. *D is a $\star$-GKD with $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D)$ torsion) or equivalently $C\ell_t(D)$ torsion (resp., $LC\ell_t(D)$ torsion).*

Proof We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (1)⇒(2) By Theorem 11 D is a $\star$-GKD since a $\star$-af-homogeneous ideal is $\star$-super-homogeneous. And by Theorem 13 D is an AGCD domain. (2)⇒(1) By Theorem 11 every nonzero proper principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals of type 1. Now a $\star$-GKD is a $\star$-IRKT and hence by Theorem 13 each $\star$-super-homogeneous ideal is $\star$-af-homogeneous. (3)⇒(1) Clear. (1)⇒(3) This follows from Theorem 13 once we observe that a product of similar type 1 $\star$-af-homogeneous ideals is again a $\star$-af-homogeneous ideal of type 1. (4)⇒(1) Clear. (3)⇒(4) Theorem 6 (2)⇔(5) Proposition 5.

Theorem 15 *Let D by an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-af-SH domain (resp., $\star$-laf-homogeneous-SH domain) of type 2.*
2. *D is an AGCD $\star$-Krull domain (resp., AGGCD $\star$-Krull domain).*
3. *D is a $\star$-SH domain and each $\star$-homogeneous ideal is a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) of type 2.*
4. *If I is a nonzero finitely generated ideal D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals) of type 2.*
5. *D is a $\star$-Krull domain with $C\ell_\star(D)$ torsion or equivalently $C\ell(D)$ torsion (resp., $LC\ell_\star(D)$ torsion or equivalently $LC\ell(D)$ torsion).*

Proof We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (1)⇒(2) By Theorem 8 D is $\star$-Krull. And since a $\star$-af-SH domain of type 2 is certainly a $\star$-af-SH domain of type 1, Theorem 14 gives that D is an AGCD domain. (2)⇒(1) By Theorem 8 each proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals of type 2. Now a $\star$-Krull domain is certainly a $\star$-GKD, so by Theorem 14 each $\star$-homogeneous ideal is actually $\star$-af-homogeneous. So each proper nonzero principal ideal of D is a $\star$-product of $\star$-af-homogeneous ideals of type 2. (3)⇒(1) Clear. (1)⇒(3) This follows from Theorem 13 once we observe that a product of similar type 2 $\star$-af-homogeneous ideals is again a $\star$-af-homogeneous ideal of type 2. (4)⇒(1) Clear. (3)⇒(4) Theorem 6. (2)⇔(5) Proposition 5.

To give GCD domain and GGCD domain versions of Theorems 13–15, we need the following definitions.

Definition 10 Let D be an integral domain and $\star$ a finite character star-operation on D. An ideal I of D is $\star$-*factorial* ($\star$-f)-*homogeneous* (resp., $\star$-*locally factorial* ($\star$-*lf*)-*homogeneous*) if I is $\star$-homogeneous and for each $\star$-homogeneous ideal $J \supseteq I$, $J^\star$ is principal (resp., invertible). We say the D is a $\star$-*f-SH domain* (resp., $\star$-*lf-SH domain*) if each nonzero proper principal ideal of D is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).

Let D be an integral domain and $\star$ a finite character star-operation on D. Let I be a nonzero ideal of D. Then we have I $\star$-f-homogeneous (resp., $\star$-lf-homogeneous) ⇒ I is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous) ⇒ I is $\star$-super-homogeneous ⇒ I is $\star$-homogeneous. Thus D a $\star$-f-SH domain ⇒ D is a $\star$-af-SH domain ⇒ D is a $\star$-super-SH domain ⇒ D is a SH domain with similar implications for the "locally" case. Also, I $\star$-f-homogeneous (resp., $\star$-af-homogeneous) ⇒ I is $\star$-lf-homogeneous (resp., $\star$-laf-homogeneous). So D a $\star$-f-SH domain (resp., $\star$-af-SH domain) ⇒ D is a $\star$-lf-SH domain (resp., $\star$-laf-SH domain). We have also shown that a product of similar $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous) ideals is again $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous). Using this, we showed that if an ideal I of D is a $\star$-product of $\star$-af-homogeneous (resp, $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous) ideals, then I is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous). Not surprisingly we have an analogous result for $\star$-f-homogeneous ideals and $\star$-lf-homogeneous ideals.

Theorem 16 *Let D be an integral domain and $\star$ a finite character star-operation on D.*

1. *If I and J are similar $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals) of D, then IJ is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*
2. *Let I be an ideal of D that is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals). Then $I^\star$ is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*

Proof We do the $\star$-f-homogeneous case, the $\star$-lf-homogeneous case is similar. Once we prove (1), the proof of (2) is similar to the proofs of the $\star$-af-homogeneous, $\star$-super-homogeneous, and $\star$-homogeneous cases (Theorems 3, 9 and 12, respectively). So let I and J be similar $\star$-f-homogeneous ideals. Let $C \supseteq IJ$ be a $\star$-homogeneous ideal. We need to show that $C^\star$ is principal. Since I and J are $\star$-super-homogeneous, so is their product IJ. Thus $I^\star$, $J^\star$, and $C^\star$ are comparable [17, Theorem 1.11]. If $C^\star \supseteq I^\star$, then $C + I \supseteq I$ is $\star$-homogeneous and hence $C^\star = (C + I)^\star$ is principal. Likewise $C^\star$ is principal when $C^\star \supseteq J^\star$. Thus without loss of generality we may assume that $I^\star \supseteq J^\star \supsetneq C^\star \supseteq C \supseteq IJ$. Now $D \supseteq I^\star I^{-1} \supseteq C^\star I^{-1} \supseteq J^\star$ where $I^{-1} = (I^\star)^{-1}$ is principal. So $CI^{-1} + J \supseteq J$ is $\star$-homogeneous and hence $(CI^{-1} + J)^\star$ is principal. But $(CI^{-1} + J)^\star = (CI^{-1})^\star = C^\star I^{-1}$ and hence $C^\star$ is principal since I^{-1} is.

We next give a characterization of GCD (resp., GGCD) $\star$-IRKTs using $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).

Theorem 17 *Let D be an integral domain and $\star$ a finite character star-operation on D. The the following conditions are equivalent.*

1. *D is a $\star$-f-SH domain (resp., $\star$-lf-SH domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).*
3. *D is a GCD (resp., GGCD) $\star$-IRKT.*
4. *D is a $\star$-Bezout (resp., $\star$-Prüfer) $\star$-IRKT.*
5. *D is a $\star$-SH domain and every $\star$-homogeneous ideal of D is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*
6. *D is a $\star$-IRKT with $C\ell_\star(D) = 0$, or equivalently, $C\ell_t(D) = 0$ (resp., $LC\ell_\star(D) = 0$, or equivalently, $LC\ell_t(D) = 0$).*

Proof We do the $\star$-f-homogeneous case, the $\star$-lf-homogeneous case is similar. (5)⇒(4) Since a $\star$-f-homogeneous ideal is $\star$-af-homogeneous, Theorem 13 gives that D is an AGCD $\star$-IRKT. Let I be a nonzero finitely generated ideal of D with

$I^\star \neq D$. By Theorem 13 $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals each of which is $\star$-f-homogeneous by hypothesis and hence principal. Thus for each nonzero finitely generated ideal I of D, $I^\star$ is principal. So D is $\star$-Bezout. (4)$\Rightarrow$(3) A $\star$-Bezout domain is a GCD domain. (3)$\Rightarrow$(2) Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. Since D is an AGCD $\star$-IRKT, $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals. But since D is a GCD domain, $C\ell_t(D) = 0$; so $C\ell_\star(D) \subseteq C\ell_t(D)$ gives each $\star$-invertible ideal is principal. Thus a $\star$-af-homogeneous ideal is $\star$-f-homogeneous. (2)$\Rightarrow$(1) Clear. (1)$\Rightarrow$(3) In the proof of (1)$\Rightarrow$(3) of Theorem 13, we can take $m = 1$ and get that $Da \cap Db$ is principal. Thus D is a GCD domain. (3)$\Rightarrow$(4) D a GCD domain gives $C\ell_t(D) = 0$ and hence $C\ell_\star(D) = 0$. So D is $\star$-Bezout. (4)$\Rightarrow$(5) A $\star$-IRKT is a $\star$-SH domain. Let I be a $\star$-homogeneous ideal. If $J \supseteq I$ is $\star$-homogeneous, then $J^\star$ is principal since D is $\star$-Bezout. Thus I is $\star$-f-homogeneous. (3)$\Rightarrow$(6) This follows since $C\ell_t(D) = 0$ for D a GCD domain. (6)$\Rightarrow$(4) Suppose that $C\ell_t(D) = 0$. Let I be a nonzero finitely generated ideal of D. By Theorem 10 I is $\star$-invertible. Since $C\ell_\star(D) = 0$, $I^\star$ is principal. So D is $\star$-Bezout.

Combining Theorem 17 with previous results, we have the following two theorems.

Theorem 18 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent:*

1. *D is a $\star$-f-SH domain of type 1 (resp., type 2).*
2. *D is a GCD $\star$-GKD (resp., GCD $\star$-Krull domain, or equivalently a UFD $\star$-Krull domain, or UFD $\star$-GKD).*
3. *D is a $\star$-GKD (resp., $\star$-Krull domain) with $C\ell_\star(D)=0$, or equivalently, $C\ell_t(D)=0$.*

Proof For the type 1 (resp., type 2) equivalences just combine Theorems 17 and 11 (resp., Theorem 8).

Recall that an integral domain D is *locally factorial* if D_M is a UFD for each maximal ideal M of D. And D is called a *π-domain* if each proper nonzero principal ideal of D is a product of (necessarily invertible) prime ideals. For an integral domain D the following are equivalent: (1) D is a π-domain, (2) D is a locally factorial Krull domain, and (3) D is a Krull domain with $LC\ell(D) = 0$ [1, Theorem 1].

Theorem 19 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-lf-SH domain of type 1 (resp., type 2).*
2. *D is a GGCD $\star$-GKD (resp., GGCD $\star$-Krull domain, or equivalently a locally factorial $\star$-Krull domain, or locally factorial $\star$-GKD).*
3. *D is a $\star$-GKD (resp., $\star$-Krull domain) with $LC\ell_\star(D) = 0$, or equivalently, $LC\ell_t(D) = 0$.*

Proof For the type 1 (resp., type 2) equivalence just combine Theorems 17 and 11 (resp., Theorem 8).

We next wish to characterize $\star$-SH domains with $C\ell_\star(D) = 0$ or $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D) = 0$ or $LC\ell_\star(D)$ torsion). For this we need to define yet more types of $\star$-homogeneous ideals.

Definition 11 Let D be an integral domain and $\star$ a finite character star-operation on D. An ideal of I of D is *$\star$-weakly factorial-($\star$-wf-) homogeneous* (resp., *$\star$-almost weakly factorial-($\star$-awf-) homogeneous*, *$\star$-weakly locally factorial ($\star$-wlf-) homogeneous*, *$\star$-weakly almost locally factorial ($\star$-walf-) homogeneous*) if (1) I is $\star$-homogeneous and (2) if I is $\star$-invertible, then $I^\star$ is principal (resp., $(I^n)^\star$ is principal for some $n \geq 1$, $I^\star$ is invertible, $(I^n)^\star$ is invertible for some $n \geq 1$). And D is called a *$\star$-wf-SH domain* (resp., *$\star$-awf-SH domain*, *$\star$-wlf-SH domain*,*$\star$-walf-SH domain*) if each proper nonzero principal ideal of D is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous, $\star$-wlf-homogeneous, $\star$-walf-homogeneous) ideals.

Theorem 20 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is an $\star$-wf-SH domain (resp., $\star$-awf-SH domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals.*
3. *D is a $\star$-SH domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion).*

Proof We do the case for $C\ell_\star(D) = 0$, the $C\ell_\star(D)$ torsion case is similar. (3)⇒(2) Since D is an $\star$-SH domain, by Theorem 6 $I^\star = (I_1 \cdots I_n)^\star$ where I_i is $\star$-homogeneous. Now if I_i is $\star$-invertible, then $I_i^\star$ is principal. Thus I_i is $\star$-wf-homogeneous. (2)⇒(1) Clear. (1)⇒(3) It suffices to show that if A is a finitely generated nonzero $\star$-invertible integral ideal with $A^\star \neq D$, then $A^\star$ is principal. As in the proof of Theorem 6, $A^\star = ((AD_{M_1} \cap D) \cdots (AD_{M_n} \cap D))^\star$ where $M_1, \ldots, M_n$ are the maximal $\star$-ideals containing A. Now $AD_{M_i} \cap D$ is $\star$-invertible, so $AD_{M_i} \cap D = (AD_{M_i} \cap D)^{\star_w} = (AD_{M_i} \cap D)^\star$. Hence $AD_{M_i} \cap D$ is a $\star$-invertible $\star$-ideal. So $(AD_{M_i} \cap D)_{M_i} = a_i D_{M_i}$ for some $a_i \in D$. Now by hypothesis $Da_i = (I_1 \cdots I_s)^\star$ where each I_j is $\star$-wf-homogeneous. Hence $I_j^\star = Dx_j$ for some $x_j \in D$. So $Da_i = Dx_1 \cdots Dx_s$ where Dx_j is $\star$-homogeneous. By combining similar factors, we can assume that $Dx_1, \ldots, Dx_s$ are pairwise $\star$-comaximal. Now some $M(Dx_j) = M_i$. By Proposition 1 $x_j D_{M_i} \cap D = x_j D$. Now $a_i D_{M_i} = x_j D_{M_i}$ and hence $AD_{M_i} \cap D = a_i D_{M_i} \cap D = x_j D_{M_i} \cap D = x_j D$. So $A^\star$ is principal.

We have a companion theorem for the "locally" case. The proof is left to the reader.

Theorem 21 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-wlf-SH domain (resp., $\star$-walf-SH domain).*
2. *If I is a nonzero finitely generated ideal with $I^\star \neq D$ then $I^\star$ is a $\star$-product of $\star$-wlf-homogeneous (resp., $\star$-walf-homogeneous) ideals.*
3. *D is a $\star$-SH domain with $LC\ell_\star(D) = 0$ (resp., $LC\ell_\star(D)$ torsion).*

Let D be an integral domain and $\star$ a finite character star-operation on D. It is evident that a $\star$-product of similar $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals is again $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous). Thus if an ideal is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals, it is a $\star$-product of pairwise $\star$-comaximal $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals. Similar results hold for the "locally" case. Let us call an element $x \in D$ $\star$-homogeneous if Dx is $\star$-homogeneous. We have the following element-wise characterization of $\star$-SH domains with $C\ell_\star(D) = 0$ or torsion.

Theorem 22 *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-SH domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion).*
2. *For each nonzero nonunit $x \in D$, x (resp., x^n for some $n = n(x) \geq 1$) is a product of $\star$-homogeneous elements.*
3. *For each nonzero nonunit $x \in D$, x (resp., x^n for some $n = n(x) \geq 1$) can be written uniquely up to order and associates as a product of pairwise $\star$-comaximal $\star$-homogeneous elements.*

Proof For both cases, it is clear that (2)⇔(3) and (1)⇒(2). And it is immediate from Theorem 20 that if each nonzero nonunit of D is a product of $\star$-homogeneous elements, then D is a $\star$-SH domain with $C\ell_\star(D) = 0$. So suppose that D is an integral domain with the property that for each nonzero nonunit x, some power of x is a product of $\star$-homogeneous elements. Let x be a nonzero nonunit of D. Then some x^n is a product of $\star$-homogeneous elements. Thus x^n, and hence x, is contained in only finitely many maximal $\star$-ideals. So $\star$ is locally finite. Suppose that M_1 and M_2 are distinct maximal $\star$-ideals and there is a nonzero prime ideal $P \subseteq M_1 \cap M_2$. Let $0 \neq x \in P$. So some x^n is a product of $\star$-homogeneous elements. Thus P contains a $\star$-homogeneous element which is absurd since $P \subseteq M_1 \cap M_2$. So $\star$ is independent. By Theorem 4, D is an $\star$-SH domain. Let A be a nonzero finitely generated integral $\star$-invertible ideal of D with $A^\star \neq D$. It suffices to show that for some $n \geq 1$, $(A^n)^\star$ is principal. But this follows from an easy modification of the proof of (1)⇒(3) of Theorem 20.

We note that the notions of type 2 $\star$-f-SH domain (resp., type 2 $\star$-af-SH domain) and type 2 $\star$-wf-SH domain (resp., type 2 $\star$-waf-SH domain) coincide, and they are both equivalent to being $\star$-Krull with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). Also, the notions of type 2 $\star$-lf-SH domain (resp., type 2 $\star$-laf-SH domain) and type 2 $\star$-wlf-SH domain (resp., type 2 $\star$-walf-SH domain) coincide, and they are both equivalent to being $\star$-Krull with $LC\ell_\star(D) = 0$ (resp., $LC\ell_\star(D)$ torsion). However, this is not the case for type 1. Now a type 1 $\star$-f-SH domain (resp., type 1 $\star$-af-SH domain) is a $\star$-GKD with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). And a type 1 $\star$-wf-SH domain (resp., type 1 $\star$-waf-SH domain) is a $\star$-weakly Krull domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). Finally, a type 1 $\star$-lf-SH domain (resp., type 1 $\star$-wlf-SH domain) is a $\star$-GKD with $LC\ell_\star(D) = 0$ (resp., $\star$-weakly Krull domain with $LC\ell_\star(D) = 0$) and a type 1 $\star$-laf-SH domain (resp., type 1 $\star$-walf-SH domain) is a $\star$-GKD domain (resp.,

⋆-Krull domain) with $LC\ell_\star(D)$ torsion. An integral domain is *weakly factorial* [6] if each nonzero nonunit is a product of primary elements. An integral domain D is weakly factorial if and only if D is weakly Krull and $C\ell_t(D) = 0$ [8, Theorem]. Also, the following are equivalent: (1) D is a weakly factorial GCD domain, (2) D is a weakly factorial GKD, and (3) D is a GCD GKD [6, Theorem 20]. For a Noetherian domain D, D is integrally closed weakly factorial if and only if D is factorial. For any field K, $K[[X^2, X^3]]$ is weakly factorial but not factorial and hence is a type 1 ⋆-wf-SH domain, but not a type 1 ⋆-f-SH domain (for $K[[X^2, X^3]]$, $d = t$).

References

1. D.D. Anderson, π-domains, overrings, and divisorial ideals. Glasgow Math. J. **19**, 199–203 (1978)
2. D.D. Anderson, Star-operations induced by overrings. Commun. Algebra **16**, 2535–2553 (1988)
3. D.D. Anderson, D.F. Anderson, Generalized GCD domains. Comment. Math. Univ. St. Pauli. **27**, 215–221 (1979)
4. D.D. Anderson, S.J. Cook, Two star operations and their induced lattices. Commun. Algebra. **28**, 2461–2476 (2000)
5. D.D. Anderson, E.G. Houston, M. Zafrullah, t-linked extensions, the t-class group, and Nagata's theorem. J. Pure Appl. Algebra **86**, 109–124 (1993)
6. D.D. Anderson, L.A. Mahaney, On primary factorization. J. Pure Appl. Algebra **54**, 141–154 (1988)
7. D.D. Anderson, J.L. Mott, M. Zafrullah, Finite character representations of integral domains. Boll. Un. Mat. Ital. **6**(B(7)), 613–630 (1992)
8. D.D. Anderson, M. Zafrullah, Weakly factorial domains and groups of divisibility. Proc. Am. Math. Soc. **109**, 907–913 (1990)
9. D.D. Anderson, M. Zafrullah, Almost Bezout domains. J. Algebra **142**, 285–309 (1991)
10. D.D. Anderson, M. Zafrullah, Independent locally-finite intersections of localizations. Houston J. Math. **25**, 109–124 (1999)
11. D.F. Anderson, A general theory of class groups. Commun. Algebra **16**, 805–847 (1988)
12. A. Bouvier, Le groupe de classes d'un anneau integre. 107 erne Congres National des Societe Savantes. Brest, France, Fasc. **IV**, 85–92 (1982)
13. A. Bouvier, M. Zafrullah, On some class groups of an integral domain. Bull. Soc. Math. Greece **29**, 45–49 (1988)
14. R. Gilmer, *Multiplicative Ideal Theory* (Marcel Dekker, New York, 1972)
15. M. Griffin, Rings of Krull type. J. Reine Angew. Math. **229**, 1–27 (1968)
16. F. Halter-Koch, *Ideal Systems - An Introduction to Multiplicative Ideal Theory* (Marcel Dekker, New York, 1998)
17. E.G. Houston, M. Zafrullah, ⋆ Super potent domains. J. Commut. Algebra, to appear
18. P. Jaffard, *Les Systèmes d' Ideáux* (Dunod, Paris, 1962)
19. I. Kaplansky, *Commutative Rings*, revised edn (Polygonal Publishing, Washington, 1994)
20. E. Matlis, *Torsion-Free Modules* (The University of Chicago Press, Chicago, 1972)
21. U. Storch, Fastfaktorielle Ringe. Schritenreiche Math. Inst. Univ. Munster, vol. 36, Munster (1967)
22. M. Zafrullah, A general theory of almost factoriality. Manuscripta Math. **51**, 29–62 (1985)

t-Local Domains and Valuation Domains

Marco Fontana and Muhammad Zafrullah

Dedicated to David F. Anderson

Abstract In a valuation domain (V, M), every nonzero finitely generated ideal J is principal and so, in particular, $J = J^t$; hence, the maximal ideal M is a t-ideal. Therefore, the t-local domains (i.e., the local domains, with maximal ideal being a t-ideal) are "cousins" of valuation domains, but, as we will see in detail, not so close. Indeed, for instance, a localization of a t-local domain is not necessarily t-local, but of course a localization of a valuation domain is a valuation domain. So it is natural to ask under what conditions is a t-local domain a valuation domain? The main purpose of the present paper is to address this question, surveying in part previous work by various authors containing useful properties for applying them to our goal.

1 Introduction

We begin by reviewing the notion of a t-local domain.

Let D be an integral domain with quotient field K, let $\boldsymbol{F}(D)$ be the set of nonzero fractional ideals of D, and let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K (obviously, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D)$). For $E \in \boldsymbol{F}(D)$, let $E^{-1} := \{x \in K \mid xE \subseteq D\}$. The functions on $\boldsymbol{F}(D)$ defined by $E \mapsto E^v := (E^{-1})^{-1}$ and $E \mapsto E^t := \bigcup\{F^v \mid 0 \neq F$ are a finitely generated subideal of $E\}$, called, respectively, the *v-operation* and the *t-operation* on the integral domain D, come under the

The first named author was partially supported by *GNSAGA* of *Istituto Nazionale di Alta Matematica.*

M. Fontana (✉)
Dipartimento di Matematica e Fisica, Università degli Studi, "Roma Tre", 00146 Rome, Italy
e-mail: fontana@mat.uniroma3.it

M. Zafrullah
Department of Mathematics, Idaho State University, Pocatello, ID 83209-8085, USA
e-mail: mzafrullah@usa.net

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_3

umbrella of star operations (briefly recalled in Sect. 2), discussed in Sects. 32 and 34 of [18], where the reader can find proofs of the basic statements made here about the v-, t- and, more generally, the star operations.

Recall that a nonzero fractional ideal E of D is a *v-ideal*, or a *divisorial ideal* (resp., a *t-ideal*), if $E = E^v$ (resp., $E = E^t$) and a *v-ideal* (resp., a *t-ideal*) *of finite type* if $E = E^v = F^v$ (resp., $E = E^t = F^t$) for some finitely generated $F \in \boldsymbol{f}(D)$ and, obviously, $F \subseteq E$. Next, the t-operation is a star operation of finite type on the integral domain D, in the sense that $E \in \boldsymbol{F}(D)$ is a t-ideal if and only if for each finitely generated nonzero subideal F of E we have $F^v = F^t \subseteq E$ and it is easy to see that if F is principal $F^v = F = F^t$.

An integral ideal of D maximal with respect to being an integral t-ideal is called a *maximal t-ideal* of D and it is always a prime ideal. We denote by $\text{Max}^t(D)$ the set of all the maximal t-ideals of D. This set is nonempty, since every t-ideal is contained in a maximal t-ideal, thanks to the definition of the t-operation and to Zorn's Lemma. An integral domain is called a *t-local domain* if it is local and its maximal ideal is a t-ideal.

The purpose of this article is to survey the notion indicating what t-local domains are, where they may or may not be found and what their uses are.

The first example of a t-local domain that comes to mind is a valuation domain, i.e., a local domain (V, M) in which every nonzero finitely generated ideal is principal. In this case, we can say that for each $F \in \boldsymbol{f}(V)$ with $F \subseteq M$ we have $F = (a) \in M$ and so $F^t = (a)^t = (a) \subseteq M$. But, of course, t-local domains are much more general than that. We can, for example, show that if P is a height one prime ideal of an integral domain D, then D_P is a t-local domain. We can show, as we will in more generality, that if $M = pD$ is a prime ideal generated by a prime element of a domain D then M is a maximal t-ideal and D_M is a t-local domain. However, we cannot just take a prime t-ideal P of D and claim that D_P is a t-local domain, as there are examples of some domains D with prime t-ideals P such that D_P is not a t-local domain. In Sect. 2, we discuss cases of prime t-ideals P with D_P a t-local domain and cases of domains that have prime t-ideals P with D_P non-t-local, indicating also that if D is t-local then, for some multiplicative set S of D, D_S the ring of fractions may not be a t-local domain.

Now localization may not always produce t-local domains, but there are elements of a special kind whose presence in a domain D ensures that D is a t-local domain. In Sect. 3, we record the results related to the fact that the presence of a nonzero nonunit comparable element (definition recalled later) in an integral domain D makes D into a t-local domain. The related results include, for instance, (1) the effects the presence of a nonzero nonunit comparable element on different kinds of domains, (2) the presence of a nonzero comparable element in some domains would make them into valuation domains, if D is Noetherian then the presence of a nonzero nonunit comparable element in D makes D a DVR (= discrete valuation ring), and (3) a t-local domain may not have a comparable element, and so on, the list continues.

Citing Krull, Cohn [10] showed that D is a valuation domain if and only if D is a Bézout domain and a local domain. (In fact, in this result "Bézout" can be replaced by "Prüfer"; here D is Bézout—respectively, Prüfer—if every nonzero finitely generated ideal of D is principal—respectively, invertible.) In Sect. 4, we show that D is a valuation domain if and only if D is a GCD domain and a t-local domain, and point out that if, in the above statement, we replace "GCD domain" by "PvMD" the result would still be a characterization of a valuation domain (here, D is a PvMD, if for each pair $0 \neq a, b \in D$ we have $((a,b)\frac{(a)\cap(b)}{ab})^t = D$). But of course we do not stop here, we point to situations where recognizing the fact that the domain in question is a t-local domain makes proving that it is a valuation domain easier.

Section 5 has to do with "applications" which are essentially more efficient proofs of known results. We follow the study of the ring called Shannon's quadratic extension in [27] and point out that it is indeed a t-local domain, thus providing a shorter, more efficient proof of Theorem 6.2 of [27]. We also point to examples of maximal t-ideals Q in a particular domain D such that D_Q is not t-local.

2 Background Results and t-Local Domains

We start with proving some important preliminary results. But, for that, we need to recall the formal definition of a star operation. A *star operation* on D is a map $* : \boldsymbol{F}(D) \to \boldsymbol{F}(D),\ E \mapsto E^*$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \boldsymbol{F}(D)$, the following properties hold:

($*_1$) $(xD)^* = xD$;
($*_2$) $E \subseteq F$ implies $E^* \subseteq F^*$;
($*_3$) $E \subseteq E^*$ and $E^{**} := (E^*)^* = E^*$;

[18, Sect. 32].

If $*$ is a star operation on D, then we can consider a map $*_f : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$ defined, for each $E \in \boldsymbol{F}(D)$, as follows:

$$E^{*_f} := \bigcup\{F^* \mid F \in \boldsymbol{f}(D) \text{ and } F \subseteq E\}.$$

It is easy to see that $*_f$ is a star operation on D, called *the finite-type star operation associated to* $*$ (or *the star operation of finite type associated to* $*$). A star operation $*$ is called a *finite-type star operation* (or, *star operation of finite type*) if $* = *_f$. It is easy to see that $(*_f)_f = *_f$ (that is, $*_f$ is of finite type).

If $*_1$ and $*_2$ are two star operations on D, we say that $*_1 \leq *_2$ if $E^{*_1} \subseteq E^{*_2}$, for each $E \in \boldsymbol{F}(D)$, equivalently, if $(E^{*_1})^{*_2} = E^{*_2} = (E^{*_2})^{*_1}$, for each $E \in \boldsymbol{F}(D)$. Obviously, for each star operation $*$, we have $*_f \leq *$. Clearly, $v_f = t$. Let d_D (or, simply, d) be the *identity star operation on D*. Clearly, $d \leq *$ and, moreover, $* \leq v$, for all star operations $*$ on D [18, Theorem 34.1(4)].

Recall that an integral domain D is called a *Prüfer v-multiplication domain* (for short, *PvMD*), if every nonzero finitely generated $F \in \boldsymbol{f}(D)$ is t-invertible, i.e., $(FF^{-1})^t = D$. Obviously, every Prüfer domain is a PvMD. It is well known (see, Griffin [22, Theorem 5]) that D is a PvMD if and only if D_Q is a valuation domain, for each maximal (or, equivalently, prime) t-ideal Q of D.

Any unexplained terminology is straightforward, well accepted, and usually comes from [33] or [18].

Lemma 2.1 (Hedstrom–Houston [25, Proposition 1.1]) *Let $*$ be a star operation on an integral domain D and let $*_f$ be the finite-type star operation on D canonically associated with $*$. If P is a minimal prime ideal over a $*_f$-ideal of D, then P is a $*_f$-ideal.*

Proof Let J be a finitely generated (integral) ideal contained in P, the conclusion will follow if we show that $J^* \subseteq P$. Since P is minimal over some (integral) ideal I, with $I = I^{*_f}$, then $\mathrm{rad}(ID_P) = PD_P$ and, since J is finitely generated, there exists an integer $m \geq 1$ such that $J^m D_P \subseteq ID_P$. Therefore, for some $s \in D \setminus P$, $sJ^m \subseteq I$. Thus, $s(J^*)^m \subseteq s(J^m)^* = s(J^m)^{*_f} \subseteq I^{*_f} = I \subseteq P$, and so $J^* \subseteq P$, since $s \notin P$. □

The next step is to apply this lemma for obtaining some sufficient conditions for a local domain to be a t-local domain (recall that an integral domain is a t-local domain if it is local and its maximal ideal is a t-ideal).

Remark 2.2 (1) Note that if D is an integral domain such that $\mathrm{Max}^t(D)$ contains only one element, then D is necessarily a t-local domain (and conversely). If not, let M be the unique t-maximal ideal of D and N be a maximal ideal of D with $N \neq M$. Let $x \in N \setminus M$, clearly, the t-ideal xD must be contained in some t-maximal ideal. In the present situation xD should be contained in M and this is a contradiction.

(2) Note that if D is a local domain with divisorial maximal ideal, then clearly D is t-local. The converse is not true: take, for instance, a valuation domain with nonprincipal maximal ideal (e.g., a 1-dimensional non-discrete valuation domain).

(3) In an integral domain D, the set of maximal divisorial ideals, $\mathrm{Max}^v(D)$, might be empty (e.g., take a 1-dimensional valuation domain with nonprincipal maximal ideal). However, if $\mathrm{Max}^v(D) \neq \emptyset$, a maximal divisorial ideal is a prime t-ideal, but it might be a nonmaximal t-ideal (for explicit examples see [17], where the problem of when a maximal divisorial ideal is a maximal t-ideal is investigated).

Corollary 2.3 *Let D be a local domain with maximal ideal M. Then, D is t-local in each of the following situations:*

(1) The maximal ideal M is minimal over (i.e., is the radical of) an integral t-ideal of D.
(2) The maximal ideal M is an associated prime over a principal ideal of D (i.e., there exist $a \in D$ and $b \in D \setminus aD$ such that M is minimal over $(aD :_D bD)$).
(3) The maximal ideal M is minimal over (i.e., is the radical of) a principal ideal of D.
(4) The maximal ideal M is principal.
(5) The integral domain D is 1-dimensional.

Proof (1) is a straightforward consequence of Lemma 2.1. (2) and (3) are obvious from (1), because a proper ideal of the type $(aD :_D bD)$ and a principal ideal are both t-ideals. (4) is trivial consequence of (3). Finally, (5) follows from the fact that, in this case, the maximal ideal is a minimal prime over every nonzero (principal) ideal contained in it. □

Proposition 2.4 *If* (D, M) *is a local domain and the prime ideals of* D *are comparable in pairs, i.e.,* $\operatorname{Spec}(D)$ *is linearly ordered under inclusion, then* D *is* t*-local.*

Proof Let $I = (x_1, x_2, \ldots, x_n)$ be a nonzero proper finitely generated ideal of D and let P be a minimal prime of I. The prime spectrum $\operatorname{Spec}(D)$ being linearly ordered forces P to be unique. Now let, for each $i = 1, 2, \ldots, n$, $P(x_i)$ be the minimal prime of the principal ideal (x_i). Again, by the linearity of order of $\operatorname{Spec}(D)$, for some $1 \leq k \leq n$, $P(x_k) \subseteq P(x_j)$ for all $j \neq k$. So $P(x_k) \supseteq I$ and so $P(x_k) \supseteq P$. But as $x_k \in P$, $P(x_k) \subseteq P$. Whence every proper nonzero finitely generated ideal of D is contained in a prime ideal of D that is minimal over a principal ideal and, hence, P is a t-ideal, by Corollary 2.3(1). Thus, $I^v = I^t \subseteq P \subseteq M$. Since I is arbitrary as a finitely generated proper ideal of D, M is a t-ideal. □

Remark 2.5 Note that, *mutatis mutandis*, from the proof of the previous proposition, if $\operatorname{Spec}(D)$ is linearly ordered under inclusion, we do not deduce only that D is t-local, but also that every prime ideal of D is a t-ideal (see also [32, Theorem 3.19]).

It is known that if J is a t-ideal of a ring of fractions D_S of an integral domain D with respect to a multiplicative subset S of D, then $J \cap D$ is a t-ideal of D [32, Lemma 3.17(1)]. However, I being a t-ideal of the integral domain D does not imply, in general, that ID_S is a t-ideal of D_S, even though $ID_S \cap D$ is a t-ideal of D [32, Lemma 3.17(2)]. In particular, as Example 2.6 will show, the prime t-ideals may have a "bad behavior", that is, if P is a prime t-ideal of D then PD_S may not be a prime t-ideal for some multiplicative set S disjoint with P.

The authors of [39] were led to this conclusion seeing an example given by Heinzer and Ohm [29] of an essential domain (i.e., an integral domain $D = \bigcap D_P$ where P ranges over prime ideals of D such that D_P is a valuation domain) that is not a PvMD. The reason for this conclusion came from the following observation. For each maximal ideal M of the Heinzer–Ohm example D, D_M is a unique factorization domain, meaning the Heinzer–Ohm example is a locally GCD domain. Now, if for each maximal t-ideal Q, QD_Q were a prime t-ideal of D_Q, and then D_Q would be a t-local domain and a GCD domain. But, as we shall see in Proposition 5.2, a t-local GCD domain is a valuation domain. So, we would have D_Q a valuation domain, for every maximal t-ideal Q of D, making D a PvMD. Therefore, since in this example D is not a PvMD, QD_Q might not be a t-ideal, for some maximal t-ideal Q of D. Indeed, an integral domain D which is locally a PvMD is a PvMD if and only if QD_Q is a t-ideal for every maximal t-ideal Q of D.

In [51], a *prime (t-ideal) P* in an integral domain D was called *well behaved* if PD_P is a prime t-ideal of D_P. We say that an *integral domain D* is *well behaved* if every prime (t-ideal) of D is well behaved. In [51], M. Zafrullah characterized well-behaved domains and showed that most of the known domains, including PvMDs, are well behaved. Furthermore, in the same paper, there is also an example of an integral domain D such that every $Q \in \operatorname{Max}^t(D)$ is well behaved, but D is not well behaved. This example is obtained by a pullback construction, as briefly recalled below (for the details of the proofs see [51]).

Example 2.6 Let (V, M) be a valuation domain with $\dim(V) \geq 2$ and let P be a nonzero nonmaximal prime ideal of V, set $D := V + XV_P[X]$. In [51, Lemma 2.3, 2.4, and Proposition 2.5], it is proved that

$$\mathrm{Max}^t(D) = \{fD \mid f \in D,\ f \text{ is a prime element of } D \text{ such that } f(0) \in V \setminus M\} \cup \{N\},$$

where $N := \{f \in D \mid f(0) \in M\} = M + XV_P[X]$ is a maximal ideal of D.

By the previous description of $\mathrm{Max}^t(D)$, it is not hard to see that, for each $Q \in \mathrm{Max}^t(D)$, QD_Q is a maximal t-ideal of D_Q. Now, we consider the prime ideal $\mathfrak{P} := P + XV_P[X]$ of D. Since $P = \bigcap\{aV \mid a \in M \setminus P\}$, a direct verification shows that $\mathfrak{P} = \bigcap\{aD \mid a \in M \setminus P\}$. Thus $\mathfrak{P}$ is a v-ideal and, in particular, a t-ideal of D. However, after observing that $\mathfrak{P} \cap (V \setminus P) = \emptyset$, and so $D_{\mathfrak{P}} = (V + XV_P[X])_{P+XV_P[X]} = (V_P[X])_{\mathfrak{P}V_P[X]}$ and $\mathfrak{P}V_P[X] = PV_P + XV_P[X]$, it can be shown that $\mathfrak{P}D_{\mathfrak{P}} = \mathfrak{P}V_P[X]_{\mathfrak{P}V_P[X]}$ is not a t-ideal of $D_{\mathfrak{P}}$.

By the previous observations and example, for each $P \in \mathrm{Spec}(D)$, if D_P is a t-local domain, then P is a t-prime ideal of D; on the other hand, if a prime ideal P is a t-ideal of D, it is not true, in general, that D_P is a t-local domain. We give next some sufficient conditions for the localizations of an integral domain to be t-local domains.

Proposition 2.7 *Let D be an integral domain.*

(1) If Q is an associated prime ideal over a principal ideal of D, then D_Q is a t-local domain.
(2) If $Q \in \mathrm{Max}^t(D)$ and Q is a potent ideal (i.e., it contains a nonzero finitely generated ideal that is not contained in any other maximal t-ideal), then D_Q is a t-local domain.
(3) If D has the finite t-character (i.e., every nonzero nonunit element of D belongs to at most a finite number of maximal t-ideals), then D_Q is a t-local domain, for each $Q \in \mathrm{Max}^t(D)$.

Proof (1) Since Q is minimal over a t-ideal of D of the type $(aD :_D bD)$, QD_Q is minimal over the ideal $(aD :_D bD)D_Q = (aD_Q :_{D_Q} bD_Q)$, which is a t-ideal of D_Q, and thus $QD_Q$is a t-ideal of D_Q (Corollary 2.3(2)).

(2) was proven in [3, Theorem 1.1(1)] and (3) follows from (2), since each maximal t-ideal in an integral domain with finite t-character is potent [3, Theorem 1.1(2)]. □

Remark 2.8 Recall that a prime t-ideal P of an integral domain D is said to be a *t-sharp ideal* if $\bigcap\{D_Q \mid Q \in \mathrm{Max}^t(D), P \nsubseteq Q\} \nsubseteq D_P$ [31, Sect. 3]. For a PvMD, it is known that a prime t-ideal P is t-sharp if and only if it is potent [31, Proposition 3.1].

If D has the finite t-character, then every maximal t-ideal is well behaved (Proposition 2.7(3)). It was observed in [3, Example 3.9] that the integral domain D, described in Example 2.6, has the finite t-character and so even an integral domain with the finite t-character might not be well behaved. We provide next another example of an integral domain which happens to be t-local (and so, trivially, with the finite t-character) and it is not well behaved (see, also, [3, Remark 3.2(2)]).

Example 2.9 Let $D_1 := \mathbb{Z}_{(p)}$ and so D_1 is a rank 1 discrete valuation domain of the field of rational numbers $K_1 := \mathbb{Q}$, with maximal principal ideal $N_1 := p\mathbb{Z}_{(p)}$.

Let $D_2 := \mathbb{Q}[\![X, Y]\!]$ be the power series ring in two variables with coefficients in the field $\mathbb{Q}$. Clearly, D_2 is an integrally closed local Noetherian 2-dimensional integral domain with maximal ideal $N_2 := (X, Y)\mathbb{Q}[\![X, Y]\!]$ and field of quotients $K_2 := \mathbb{Q}(\!(X, Y)\!)$. Let $D_3 = K_2[\![Z]\!] = \mathbb{Q}(\!(X, Y)\!)[\![Z]\!]$; D_3 is a rank 1 discrete valuation domain of the field $K_3 := K_2(\!(Z)\!)$, with maximal ideal $N_3 := ZK_2[\![Z]\!]$. Set

$$D := D_1 + N_2 + N_3 = \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!] + Z\mathbb{Q}(\!(X, Y)\!)[\![Z]\!]\,.$$

Clearly, $D \subset T := D_2 + N_3 = \mathbb{Q}[\![X, Y]\!] + Z\mathbb{Q}(\!(X, Y)\!)[\![Z]\!] \subset D_3 = K_2 + N_3 = \mathbb{Q}(\!(X, Y)\!)[\![Z]\!]$. By well-known properties of rings arising from pullback constructions, it is not hard to see that the following hold:

(1) T is a 3-dimensional local ring with maximal ideal $Q := N_2 + N_3$ and the localizations of T at each one of its infinitely many prime ideals of height 2 is a rank 2 discrete valuation domain.
(2) T has unique prime ideal of height 1, that is, N_3. More precisely, N_3 is a common prime ideal of T and D_3 and $N_3 = (T : D_3)$, since N_3 is the maximal ideal of the local domain D_3; therefore, N_3 is a t-ideal (in fact, a v-ideal) of T. Furthermore, $T_{N_3} = D_3$ is a rank 1 discrete valuation domain.
(3) D is a 4-dimensional local domain, with maximal ideal $M := N_1 + N_2 + N_3$.
(4) M is a t-ideal (in fact, a v-ideal) of D, since $M = pD$, and so D is a t-local domain.
(5) $Q = N_2 + N_3 = \bigcap\{p^n D \mid n \geq 0\}$ is the unique prime of height 3 in D and it is a t-ideal (in fact, a v-ideal) of D, since Q is a common ideal of D and T and, since it is the maximal ideal of T, $Q = (D : T)$.
(6) For each one of the infinitely many height 2 prime ideals P of D, there exist a unique prime ideal P' of T such that $P' \cap D = P$ and the canonical embedding homomorphism $D_P \subseteq T_{P'}$ is an isomorphism; thus D_P is a rank 2 discrete valuation domain.
(7) Set $S := \{p^n \mid n \geq 0\}$, clearly S is a multiplicative set of D and $D_S = \mathbb{Q} + N_2 + N_3 = \mathbb{Q} + (X, Y)\mathbb{Q}[\![X, Y]\!] + Z\mathbb{Q}(\!(X, Y)\!)[\![Z]\!] = D_Q = T$.
(8) $QD_S = QD_Q = QT = Q$ is not a t-ideal of $D_Q = T$, since the elements $X, Y \in QD_Q = Q$ are v-coprime (note that, if F is a nonzero finitely generated ideal in a t-ideal I, then $F^v \subseteq I$).
(9) By the previous properties, it follows that T is a local, but not t-local, PvMD, since the localization at all its nonzero nonmaximal prime ideals is a valuation domain and its maximal ideal Q is not a t-ideal of T. Moreover, T is not completely integrally closed and so it is not a Krull domain, since its complete integral closure is D_3, because $N_3 = (T : D_3)$. T does not have the finite

t-character, since each nonzero element inside its unique height 1 prime (t-)ideal N_3 is contained in all the infinitely many maximal t-ideals, which are all its prime ideals of height 2.

(10) Every nonzero prime ideal of D is a t-ideal and all of them are well behaved, except Q, its unique prime of height 3 (which is a t-ideal of D, but it is not a t-ideal in $D_Q = T$).

The following result was proved by Anderson et al. in [3, Proposition 1.12(1)].

Proposition 2.10 *Let D be a t-local domain, then the following hold:*

(1) Every t-invertible ideal (i.e., an ideal I such that $(II^{-1})^t = D$) is principal.
(2) If I is an ideal of D such that $(I^n)^t = D$ for some $n \geq 2$, then I is principal.

Proof (1) If I be a t-invertible ideal of D then II^{-1} is in no maximal t-ideals of D and this implies that $II^{-1}D_Q = D_Q$ for every $Q \in \mathrm{Max}^t(D)$. In this special situation, $\mathrm{Max}^t(D) = \mathrm{Max}(D) = \{M\}$, where M is the only maximal ideal of the t-local domain D. Thus, I is invertible in a local domain and hence it is principal.

(2) In this situation, I is t-invertible, hence, the conclusion follows from (1). □

Note that the set $\mathtt{TI}(D)$ of all the fractional t-invertible t-ideals of an integral domain D is a group with respect to the operation $I \cdot_t J := (IJ)^t$, having as subgroup the set $\mathtt{Princ}(D)$ of all nonzero fractional principal ideals of D. The quotient group $\mathtt{Cl}^t(D) := \mathtt{TI}(D)/\mathtt{Princ}(D)$ is called the *t-class group of D*. The previous Proposition 2.10 can be also stated by saying that *if D is a t-local domain then* $\mathtt{Cl}^t(D) = 0$.

3 *t*-Local Domains and Local DW-Domains

A nonzero ideal J of an integral domain D is called a *Glaz–Vasconcelos ideal* (for short, a *GV-ideal*) if J is finitely generated and $J^{-1} = D$. The set of Glaz–Vasconcelos ideals of D is denoted by $GV(D)$ [21]. Given a nonzero fractional ideal E of D, the *w-closure of E* is the fractional ideal $E^w := \{x \in K \mid xJ \subseteq E, \text{ for some } J \in GV(D)\}$. A nonzero fractional ideal E is called a *w-ideal* if $E = E^w$. The w-operation was introduced by Wang–McCasland in [46].

It is well known that w, like v, t, and the identity operation d are examples of star operations (respectively, w, like t, and d are examples of star operations of finite type) [25, Proposition 3.2] and also that $d \leq w \leq t \leq v$, this means that, for each $E \in \boldsymbol{F}(D)$, we have the following inclusions $E^d := E \subseteq E^w \subseteq E^t \subseteq E^v$. Furthermore, for each $E \in \boldsymbol{F}(D)$, $E^w = \bigcap\{ED_Q \mid Q \in \mathrm{Max}^t(D)\}$ and the set of maximal w-ideals of D, $\mathrm{Max}^w(D)$, coincide with the set of maximal t-ideals of D, $\mathrm{Max}^t(D)$ [44].

It is natural to ask what is the relation between a t-local domain and a *w-local domain*, i.e., a local domain such that its maximal ideal is a w-ideal. A t-local domain is necessarily a w-local domain, since $d \leq w \leq t$ and conversely, since as observed above, $\text{Max}^w(D) = \text{Max}^t(D)$. We will show that something more is true, that is, in a t-local domain, every nonzero ideal is a w-ideal. For showing this, we need some preliminaries.

Recall that a *DW-domain* is an integral domain D such that $d = w$, i.e., for each nonzero fractional ideal E of D, $E = E^w$; this is equivalent to requiring that every nonzero (integral) finitely generated ideal of D is a w-ideal. The following result is due to Wang [45, Proposition 1.3] (see also Mimouni [38, Proposition 2.2]).

Proposition 3.1 *Let D be an integral domain. The following are equivalent:*

(i) D *is a* DW*-domain.*
(ii) *Every nonzero prime ideal of* D *is a* w*-ideal.*
(iii) *Every maximal ideal of* D *is a* w*-ideal.*
(iv) *Every maximal ideal of* D *is a* t*-ideal.*
(v) $\text{GV}(D) = \{D\}$.

Proof Obviously, (i)⇒(ii)⇒(iii).

(iii)⇒(iv) is a consequence of the fact that $\text{Max}^w(D) = \text{Max}^t(D)$.

(iv)⇒(v) Let $J \in \text{GV}(D)$ and $J \subsetneq D$. Let $M \in \text{Max}^t(D)$ such that $J \subseteq M$, then $D = J^v = J^t \subseteq M^t = M$, which is a contradiction.

(v)⇒(i) Let I be a nonzero ideal of D and let $0 \neq x \in I^w$ then, for some $J \in \text{GV}(D)$, $xJ \subseteq I$. Since $\text{GV}(D) = \{D\}$, $xD \subseteq I$ and so $I^w \subseteq I$. □

From the previous proposition we deduce immediately the following.

Corollary 3.2 *Let D be an integral domain. The following are equivalent:*

(i) D *is a* t*-local.*
(ii) D *is a* w*-local.*
(iii) D *is a local* DW*-domain.*

Remark 3.3 Note that, for a t-local domain, it is not true that every nonzero ideal is a t-ideal, i.e., a domain such that $d = t$ or a *DT-domain*; even more, for a t-local domain, it may happen that every nonzero prime ideal is a t-ideal, without being a DT-domain (see Example 3.5). The DT-domains are also called *fgv-domains*, that is, domains such that every nonzero finitely generated ideal is a v-ideal since, for each nonzero ideal I, $I = I^t$ if and only if, for each nonzero finitely generated ideal J, $J^v = J^t = J$. Zafrullah in [48] studied the fgv-domains and he proved that an integrally closed fgv domain is a Prüfer domain. Note that, for a Noetherian domain, being a DT-domain is equivalent to being a domain such that each nonzero ideal is divisorial (i.e., a domain such that $d = v$). In particular, W. Heinzer has proven that, for a Noetherian domain D, if every nonzero ideal is divisorial, then $\dim(D) \leq 1$ [26, Corollary 4.3]; furthermore, for an integrally closed Noetherian domain (or, more generally, for any completely integrally closed domain) D, every nonzero ideal is divisorial if and only if D is Dedekind domain [26, Proposition 5.5].

Finally, note that DT-domains are exactly the DW-domains that are at the same time *TW-domains*, i.e., domains such that $w = t$ [37].

Lemma 3.4 *Let (T, N) be a local domain, let $\boldsymbol{k}(T) := T/N$, let $\varphi : T \to \boldsymbol{k}(T)$ be the canonical projection, and let R be a subring of the field $\boldsymbol{k}(T)$. Set $D := \varphi^{-1}(R)$. Then, D is a t-local domain with maximal ideal M if and only if R is a t-local domain (with maximal ideal $\varphi(M)$).*

Proof By the standard properties of the pullbacks constructions, D is a local domain with maximal ideal M if and only if R is a local domain (with maximal ideal $\varphi(M)$) [15, Corollary 1.5]. Moreover, for each $E \in \boldsymbol{F}(R)$, $\varphi^{-1}(E) \in \boldsymbol{F}(D)$ and $(\varphi^{-1}(E))^w = \varphi^{-1}(E^w)$ [37, Lemma 3.1]. Note that $M = \varphi^{-1}(\varphi(M))$, and thus $M = M^w$ if and only if $\varphi(M) = (\varphi(M))^w$. Therefore, (D, M) is w-local if and only if $(R, \varphi(M))$ is w-local. The conclusion follows from Corollary 3.2. □

Example 3.5 Example of a Noetherian t-local domain (hence, a local DW-domain) which is not a DT-domain, but each nonzero prime ideal is a t-ideal.

Consider the 2-dimensional Noetherian integrally closed domain $T := \mathbb{C}[X, Y]_{(X,Y)}$, which is clearly not a t-local domain, since its (finitely generated maximal) ideal $M := (X, Y)\mathbb{C}[X, Y]_{(X,Y)}$ is not a divisorial ideal of T (the only divisorial ideals of T are its height 1 prime ideals). However, by the previous lemma, the local 2-dimensional Noetherian domain $D := \mathbb{R} + (X, Y)\mathbb{C}[X, Y]_{(X,Y)}$ $(= \varphi^{-1}(\mathbb{R})$, where $\varphi : T \to T/M \cong \mathbb{C}$ is the canonical projection) is a t-local domain, since its maximal ideal $M = (X, Y)\mathbb{C}[X, Y]_{(X,Y)}$ is divisorial as an ideal of D, being $M = (D : T)$. Moreover, every nonzero prime ideal of D is a t-ideal. Indeed, for the well-known properties of the pullback constructions, every nonzero nonmaximal prime ideal P of D is such that $P = Q \cap D$, where Q is a nonzero nonmaximal prime ideal of T, and moreover D_P is canonically isomorphic to T_Q [15, Theorem 1.4 (part (c) of the proof)]. Since T_Q is a DVR, D_P is a DVR too and hence PD_P is a t-ideal of D_P and, in particular, P is a t-ideal of D.

Finally, D is not DT-domain or, equivalently for Noetherianity, D is not a divisorial domain, since $\dim(D) = 2$ (Remark 3.3). Explicitly, for instance, M^2 is not a divisorial ideal (or, equivalently, not a t-ideal) of D (and of T), since $(D : M^2) = ((D : M) : M) = (T : M) = T$ and so $(D : (D : M^2)) = (D : T) = M$.

Recall that an overring T of an integral domain D is called *t-linked over D* if, for each nonzero finitely generated ideal J of D such that $J^t = D$, then $(JT)^t = T$. An integral domain is *t-linkative* if every overring is t-linked [13].

Proposition 3.6 *Let D be an integral domain. Then, D is t-local domain if and only if D is a local t-linkative domain.*

The previous proposition is a straightforward consequence of the following theorem.

Theorem 3.7 (Dobbs–Houston–Lucas–Zafrullah, 1989 [13, Theorem 2.6] *Let D be an integral domain. The following are equivalent:*

(i) Every overring of D is t-linked over D.
(ii) Every valuation overring of D is t-linked over D.

(iii) Every maximal ideal of D is a t-ideal.
(iv) For each nonzero proper ideal I of D, $I^t \neq D$.
(v) For each nonzero proper finitely generated ideal J of D, $J^t \neq D$.
(vi) Each t-invertible ideal of D is invertible.

Finally, we introduce a construction for building new examples of t-local domains.

We recall that, given an integral domain D, the *Nagata ring of D* (see, for instance, [18, Sect. 33]) is defined as follows:

$$D(X) := \{f/g \mid f, g \in D[X],\ g \neq 0,\ \text{with } \boldsymbol{c}(g) = D\},$$

(where $\boldsymbol{c}(h)$ is the content of a polynomial $h \in D[X]$).

First in [32] and then in [16], the construction of the Nagata ring was extended to the case of an arbitrary chosen star (or, even semistar) operation. Given a star operation $*$ on D, set

$$\mathtt{Na}(D, *) := \{f/g \mid f, g \in D[X],\ g \neq 0,\ \text{with } \boldsymbol{c}(g)^* = D\}.$$

With this notation $\mathtt{Na}(D, d) = D(X)$. Moreover, it is clear that

$$\mathtt{Na}(D, v) = \mathtt{Na}(D, t) = \mathtt{Na}(D, w)$$

since, for each nonzero finitely generated ideal F of D, $F^v = F^t$ and, moreover, $F^t = D$ if and only if $F^w = D$, because $\mathrm{Max}^t(D) = \mathrm{Max}^w(D)$.

Proposition 3.8 *Let D be an integral domain.*

(1) The Nagata ring $\mathtt{Na}(D, v)$ is a DW-domain; in particular, if $\mathrm{Max}^t(D) = \{Q\}$ is a singleton, then $\mathtt{Na}(D, v)$ is a t-local-domain with maximal t-ideal $Q\mathtt{Na}(D, v)$.
(2) The following are equivalent:

(i) D is a t-local domain.
(ii) $\mathtt{Na}(D, v) = D(X)$ and $D(X)$ is local.
(iii) $D(X)$ is a t-local domain.

Proof (1) Recall that $\mathcal{N} := \{g \in D[X] \mid g \neq 0 \text{ and } \boldsymbol{c}(g)^* = D\}$ is a saturated multiplicatively closed subset of $D[X]$, $\mathcal{N} = D[X] \setminus \left(\bigcup\{QD[X] \mid Q \in \mathrm{Max}^{\star_f}(D)\}\right)$, $\mathtt{Na}(D, v) = D[X]_{\mathcal{N}}$, and $\mathrm{Max}(\mathtt{Na}(D, v)) = \{Q\mathtt{Na}(D, v) \mid Q \in \mathrm{Max}^t(D)\}$ (see [16, Proposition 3.1] or [32, Proposition 2.1]). Then, it is easy to see that $\mathtt{Na}(D, v)_{Q\mathtt{Na}(D,v)} = D[X]_{QD[X]} = D_Q(X)$ and $Q\mathtt{Na}(D, v) = QD_Q(X) \cap \mathtt{Na}(D, v)$, for each $Q \in \mathrm{Max}^t(D)$, and so:

$$\mathtt{Na}(D, v) = \bigcap\{D_Q(X) \mid Q \in \mathrm{Max}^t(D)\}.$$

Moreover, for each ideal I of D, $(I\mathtt{Na}(D, v))^t = I^t\mathtt{Na}(D, v)$ [32, Corollary 2.3]. Therefore, in particular, $Q\mathtt{Na}(D, v)$ is a t-ideal of $\mathtt{Na}(D, v)$ for each $Q \in \mathrm{Max}^t(D)$, i.e., $\mathrm{Max}(\mathtt{Na}(D, v)) = \mathrm{Max}^t(\mathtt{Na}(D, v))$.

(2) (i)⇒(ii). We already observed that $\texttt{Na}(D, v) = \texttt{Na}(D, t) = \texttt{Na}(D, w)$. In the present situation $d = w$ and so $\texttt{Na}(D, w) = \texttt{Na}(D, d) = D(X)$.

(ii)⇒(iii). Obviously, since we have shown in (1) that, when D is t-local, $\texttt{Na}(D, v)$ is t-local too.

(iii)⇒(i) Since the maximal ideals of $D(X)$ are exactly the ideals $M(X) := MD(X)$, with $M \in \text{Max}(D)$ [18, Proposition 33.1], and since $M(X)^t = M^t(X)$ [32, Corollary 2.3], the conclusion is straightforward. □

By the previous proposition, the Nagata ring can be used to give new examples of DW-domains and, in particular, of t-local domains. For instance, it is known that $D(X)$ is treed (i.e., the prime spectrum is a tree under the set theoretic inclusion $\subseteq$) if and only if D is treed and the integral closure $\overline{D}$ of D is a Prüfer domain [4, Theorem 2.10]. Thus, if we take a treed domain D such that $\overline{D}$ is not Prüfer, in this case $D(X)$ is a DW-domain, but not treed. For an explicit example, take $D := \mathbb{Q} + U\mathbb{Q}(V)[[U]]$, where U and V are two indeterminates, then $D = \overline{D}$ [4, Remark 2.11], D is a t-local (treed) integrally closed domain but not a valuation domain, and thus $D(X)$ is a t-local non treed integrally closed domain, since the integral closure $\overline{D(X)} = \overline{D}(X) = D(X)$ [4, Proposition 2.6].

4 Comparable Elements and t-Local Domains

A *nonzero* element $c \in D$ is called *comparable in* D if, for all $x \in D$, we have $cD \subseteq xD$ or $xD \subseteq cD$. It is easy to see that $c \in D$ is comparable if cD is comparable (under inclusion) with each ideal I of D. The following result is essentially Lemma 3.2 of [8].

Lemma 4.1 *Let α be a nonzero nonunit element of a local domain (D, M). If, for each $x \in D$, $\alpha D + xD = yD \subseteq M$, then α is a comparable element.*

Proof By the assumption, it follows that $(\alpha/y)D + (x/y)D = D$ and, since D is local, α/y or x/y is a unit of D. Thus, the element y is an associate of α or of x. In the first case, $y|x$ (or, equivalently, $\alpha|x$) and, in the second case, $y|\alpha$ (or, equivalently, $x|\alpha$). Therefore, α is a comparable element of D. □

Lemma 4.2 *Let c be a comparable element in an integral domain D. If h is a nonunit factor of c, then h is also a comparable element of D.*

Proof Let $c = hy$ and let $x \in D$. Then $cD + xyD = hyD + xyD = y(hD + xD)$ coincides with cD or xyD, since c is comparable. In the first case, $y(hD + xD) = cD = yhD$, thus $hD + xD = hD$, i.e., $x|h$. In the second case, $y(hD + xD) = xyD$ and thus $hD + xD = xD$, i.e., $h|x$. □

The comparable elements were introduced and studied in [5] to prove, in case of valuation domains, a Kaplansky-type theorem (recall that Kaplansky proved that an integral domain D is a UFD if and only if every nonzero prime ideal of D contains a prime element [33, Theorem 5]).

Lemma 4.3 (Anderson and Zafrullah [5, Theorem 3]) *An integral domain D is a valuation domain if and only if every nonzero prime ideal of D contains a comparable element.*

An important part of the result was the proof of the fact that the set of all comparable elements of D is a saturated multiplicative set.

We recall in the next lemma some of the consequences of the existence of a nonzero nonunit comparable element in an integral domain.

Lemma 4.4 (Gilmer–Mott–Zafrullah [20, Theorem 2.3]) *Suppose the integral domain D contains a nonzero nonunit comparable element and let $\mathscr{C}$ be the (nonempty) set of nonzero comparable elements of D. Then:*

(1) $P := \bigcap\{cD \mid c \in \mathscr{C}\}$ *is a prime ideal of D and $D \setminus P = \mathscr{C}$ (in particular, $\mathscr{C}$ is a saturated multiplicative set of D).*
(2) *D/P is a valuation domain.*
(3) $P = PD_P$.
(4) *D is local, P compares with every other ideal of D under inclusion, and* $\dim(D) = \dim(D/P) + \dim(D_P)$.
(5) *If T is any integral domain such that there is a nonmaximal prime ideal Q of T such that* (a) *T/Q is a valuation domain, and* (b) *$Q = QT_Q$, then each element of $T \setminus Q$ is comparable.*
(6) *If, in addition, Q is minimal in T with respect to properties* (5, a) *and* (5, b) *above, then $T \setminus Q$ is precisely the set of nonzero comparable elements of T.*

Of course, an integral domain D is a valuation domain if and only if every nonzero element of D is comparable. As an easy consequence of the previous lemma, we obtain immediately the following.

Corollary 4.5 *Suppose the integral domain D contains a nonzero nonunit comparable element and let $\mathscr{C}$ be the (nonempty) set of nonzero comparable elements of D. Then, D is a valuation domain if and only if $\cap\{cD \mid c \in \mathscr{C}\} = (0)$.*

Proof The statement follows from (1) and (2) of Lemma 4.4. □

Recall that E.D. Davis proved that, given a ring S and a subring R of S, if R is local then (R, S) is a normal pair (i.e., every ring T, $R \subseteq T \subseteq S$, is integrally closed in S) if and only if there is a prime ideal Q in R such that $S = R_Q$, $Q = QR_Q$, and R/Q is a valuation domain [12, Theorem 1]. From the previous remark and Lemma 4.4, we deduce immediately the following.

Corollary 4.6 *Suppose the integral domain D contains a nonzero nonunit comparable element. Let $\mathscr{C}$ be the set of nonzero comparable elements of D and $P := \bigcap\{cD \mid c \in \mathscr{C}\}$, as in Lemma 4.4(1). In this situation, (D, D_P) is a normal pair.*

In [20], a part of the following result was proved as a consequence of Lemma 4.4. We next prove, directly, that the existence of a nonzero nonunit comparable element in an integral domain is a sufficient but not necessary condition for being a t-local domain.

Proposition 4.7 *An integral domain D that contains a nonzero nonunit comparable element is a t-local domain, while a t-local domain may not contain a nonzero nonunit comparable element.*

Proof Let D be an integral domain and let c be a nonzero nonunit comparable element in D. We first show that D is local. Suppose, by way of contradiction, that there exist two co-maximal nonunit elements x, y in D, i.e., $rx + sy = 1$ for some $r, s \in D$. Now, as c is comparable, $c|rx$ or $rx|c$. So rx has a nonzero nonunit comparable factor c or, being a factor of c, rx is a nonzero nonunit comparable element. Thus, in both cases, rx has a nonzero nonunit comparable factor h. Similarly sy has a nonzero nonunit comparable factor k. Since h, k are comparable, $h|k$ or $k|h$, say $h|k$. Thus, assuming that $rx + sy = 1$, we get the contradictory conclusion that a nonunit divides a unit. So, D is local. We denote by M its maximal ideal.

Next, let $x_1, x_2, \ldots, x_n \in M$ and note that, as above, each of the x_i has a nonzero nonunit comparable factor h_i. Thus, $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n)$.

Now, consider h_1, h_2. They must have a nonzero nonunit common factor k_1 (which is equal to h_1 or h_2). So, $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n) \subseteq (k_1, h_3, \ldots, h_n)$. Continuing this process, we eventually get a nonzero nonunit comparable element k such that $(x_1, x_2, \ldots, x_n) \subseteq (h_1, h_2, \ldots, h_n) \subseteq (k) \subseteq M$. But, as $(x_1, x_2, \ldots, x_n) \subseteq (k)$ implies $(x_1, x_2, \ldots, x_n)^v \subseteq (k)$, we conclude that, for each finitely generated ideal $(x_1, x_2, \ldots, x_n) \subseteq M$, $(x_1, x_2, \ldots, x_n)^v \subseteq M$. Thus, D is a t-local domain.

For the converse, note that a 1-dimensional local domain has only one nonzero prime (=maximal) ideal and so it is a valuation ring if and only if it contains a nonunit comparable element, by the Kaplansky-type theorem mentioned above (Lemma 4.3). The proof is complete once we note that there do exist 1-dimensional (Noetherian t-)local domains that are not valuation domains (in fact, non-integrally closed domains) (e.g., $\mathbb{R} + X\mathbb{C}[\![X]\!]$).

Note also that there even exist 1-dimensional t-local integrally closed domains that are not valuation domains (e.g., $\overline{\mathbb{Q}} + X\mathbb{C}[\![X]\!]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$). □

Remark 4.8 Note that the previous example shows that a local domain with divisorial maximal ideal may not contain a nonzero nonunit comparable element. On the other hand, a valuation domain V with nonprincipal maximal ideal (in particular, $\dim(V) \geq 2$) is a domain containing a nonzero nonunit comparable element and so it is a t-local domain with nondivisorial maximal ideal.

Recall that an integral domain D with quotient field K is called a *pseudo-valuation domain* (for short, *PVD*) if D is local and the maximal ideal M of D is strongly prime (i.e., whenever elements x and y of K satisfy $xy \in M$, then either $x \in M$ or $y \in M$). From the proof of the previous Proposition 4.7, we give now a general class of t-local domains that do not contain nonzero nonunit comparable elements.

Example 4.9 Let (T, M) be any local domain, let $\boldsymbol{k}(T) := T/M$, let $\varphi : T \to \boldsymbol{k}(T)$ be the canonical projection, and let F be a proper subfield of $\boldsymbol{k}(T)$. Set $D := \varphi^{-1}(F)$. It is known that D is a local domain with maximal ideal M and $(M : M) = (D : M) = T$. Since $M = (D : T)$, it is easy to see that M is a divisorial ideal in D and, in particular, a t-ideal. Thus, (D, M) is a t-local domain. In particular, any PVD is a t-local domain [24, Theorem 2.10].

Remark 4.10 Note that the argument used in the previous example can be used to construct a more general class of t-local domains. Start from a (not necessarily local) integral domain T such that its Jacobson ideal $J(T)$ is nonzero and suppose that the ring $T/J(T)$ contains properly a field F. Let $\varphi : T \to T/J(T)$ be the canonical projection and let $D := \varphi^{-1}(F)$, then D is a t-local domain.

A fractional ideal $E \in \boldsymbol{F}(D)$ is said to be *v-invertible* (respectively, *t-invertible*) if there is $G \in \boldsymbol{F}(D)$ such that $((EG)^v = D$ (respectively, $(EG)^t = D$). Obviously, every invertible ideal is t-invertible.

Recall that a *GCD domain* is an integral domain D such that, for each $a, b \in D$, $aD \cap bD$ is principal or, equivalently, $(a, b)^v$ is principal. Therefore, a GCD domain (e.g., a Bézout domain) is a PvMD.

Corollary 4.11 *Let D be a PvMD, not a field. Then, D is a valuation domain if and only if D contains a nonzero nonunit comparable element.*

Proof The statement follows from Proposition 4.7, from the fact that a t-local PvMD is a valuation domain anyway and from the fact that a valuation domain that is not a field must contain many nonunit comparable elements (in fact, all nonunit elements are comparable). □

From the previous corollary it follows that every Krull domain (e.g., UFD) containing a nonzero nonunit comparable element is a DVR and that every GCD domain containing a nonzero nonunit comparable element is a valuation domain.

Now, here comes something more general and a tad surprising. Call an integral domain D *atomic* if every nonzero nonunit of D is expressible as a finite product irreducible elements. An irreducible element is called also *atom*. For instance, every Noetherian domain and every UFD is atomic.

Corollary 4.12 *An atomic domain that contains a nonzero nonunit comparable element is a DVR.*

Proof Let D be an atomic domain and let c be a nonzero nonunit comparable element in D. Then, by Proposition 4.7, D is t-local domain; denote by M its maximal ideal. Let h be an irreducible factor of c. Then h is a comparable element, being a factor of a comparable element (Lemma 4.2). So, for every x in D, either $h|x$ or $x|h$. Now, as h is irreducible, $x|h$ means that x is a unit or $x = h$. Thus, for all nonunits $x \in D$, necessarily $h|x$. That is $M = hD$ and so h is a prime element in D. Next, as $h|x$ for each nonzero nonunit $x \in D$, we have $x = x_1h$ and if x_1 is a nonunit then $x_1 = x_2h$ and so $x = h^2x_2$. Continuing this way, since D is atomic, for each nonzero nonunit

$x \in D$ there is an integer $n = n(x)$ (depending on x) such that $x = h^n x_n$ where x_n is a unit. But then we can conclude that D is a DVR and h is a uniformizing parameter of D. □

Corollary 4.12 was first proved for Noetherian domains; we thank Tiberiu Dumitrescu for suggesting the atomic domain assumption. With hindsight we can prove a more precise result.

Corollary 4.13 *Let D be a domain that contains a nonzero nonunit comparable element.*

(1) In this situation, D is local (Proposition 4.7) and the maximal ideal of D is generated by the nonunit comparable elements of D.
(2) The integral domain D contains an atom α if and only if α is the generator of the (unique) maximal ideal of D and, hence, α is a prime and comparable element.

Proof (1) By Proposition 4.7, D is t-local; let M denote the maximal ideal of D. With the notation of Lemma 4.4, M properly contains the comparable prime ideal P of D. If $(x_1, x_2, \ldots, x_n)$ is a finitely generated ideal and $P \subseteq (x_1, x_2, \ldots, x_n) \subseteq M$, since D/P is a valuation domain, then $(x_1, x_2, \ldots, x_n) = (x)$ for some $x \in \{x_1, x_2, \ldots, x_n\}$. Therefore, since $M = M^t$, M is generated by the nonunit comparable elements of D.

(2) Let α be an atom of D and let c be a nonzero nonunit comparable element of D. Then, either $c|\alpha$ or $\alpha|c$. If $c|\alpha$ then, as α is an atom and c a nonunit, c and α must be associate, so α is a comparable element. If, on the other hand, $\alpha|c$ then α is a comparable element, being a factor of a comparable element (Lemma 4.2). Thus, as above, $\alpha D = M$.

The converse is obvious, indeed if the maximal ideal M of a local domain D is principal and $M = \alpha D$ then, up to associates, α is the only atom in D. □

Note that if, instead of considering atoms (=irreducible elements), we consider prime elements, we can state a result analogous to the previous corollary in a more general setting, with a different proof.

Proposition 4.14 *Let D be a domain.*

(1) If a maximal t-ideal M of D contains a prime element p, then $M = pD$.
(2) If (D, M) is a t-local domain (e.g., if D contains a nonzero nonunit comparable element), then D contains a prime element p if and only if p is the generator of the maximal ideal of D and, hence, p is a comparable element.

Proof (1) Let p be a prime element of a domain D then, for each x in D, $pD \cap xD = xD$ or $pD \cap xD = pxD$.

So,

$$((p, x)D)^{-1} = \frac{pD \cap xD}{px} = \left(\frac{1}{p}\right) D \quad \text{or} \quad ((p, x)D)^{-1} = D.$$

But then $((p, x)D)^v = pD$ or $((p, x)D)^v = D$. So, if a prime element p belongs to a maximal t-ideal M then $M = pD$.

(2) If a prime element p belongs to a t-local ring (D, M) then $M = pD$, by (1) and consequently p is a comparable element of D. □

It is well known that, if p is a prime element in an integral domain D, then $\bigcap_{n\geq 0} p^n D$ is a prime ideal too (see, for instance, Kaplansky [33, Exercise 5, pages 7-8]).

Theorem 4.15 *If a domain D contains a nonzero nonunit comparable element then, for every nonzero nonunit comparable element x of D, we have that $Q := \bigcap_{n\geq 0} x^n D$ is a prime ideal such that D/Q is a valuation domain and $Q = QD_Q$.*

Conversely, if there is a nonzero element x in a domain D such that $Q := \bigcap_{n\geq 0} x^n D$ is a prime ideal, D/Q is a valuation domain, and $Q = QD_Q$, then D is t-local and x is a comparable element of D.

Proof Indeed Q is an ideal, being an intersection of ideals. Now, consider $S := D\backslash Q$ and let $a, b \in S$. Then $a \notin x^m D$ for some positive integer m and $b \notin x^n D$ for some positive integer n. Since x and hence x^m, x^n are comparable, we conclude that $aD \supsetneq x^m D$ and $bD \supsetneq x^n D$. Therefore, $abD \supsetneq ax^n D \supsetneq x^{n+m} D$ and so $ab \in S$ and Q is a prime ideal.

From the above proof, it follows that S consists of factors of powers of the comparable element x and so every element of S is comparable; this implies that D/Q is a valuation domain. Next, let $\alpha/\tau \in QD_Q$ where $\alpha \in Q$ and $\tau \in D\backslash Q$. In particular, τ divides some power of x and so τ is comparable. Hence, $\alpha D \subseteq Q \subsetneq \tau D$ which means that for some nonunit y we have $\alpha = \tau y$. As $\tau \notin Q$, then necessarily $y \in Q$. So $\alpha/\tau = y \in Q$. Thus $QD_Q \subseteq Q$, i.e., $Q = QD_Q$.

The converse follows from Lemma 4.4(5) and Proposition 4.7 (see also [20, Theorem 2.3]). □

Note that there are integral domains that may or may not be local, but have elements x such that $\cap x^n D =: Q$ is a prime ideal such that $Q = QD_Q$, but D/Q is not a valuation domain. Here are some examples using the $D + M$ construction studied by Gilmer [18, page 202].

We start from a valuation domain V, with quotient field K, expressible as $V = \boldsymbol{k} + M$, where $\boldsymbol{k}$ is a subfield of V (and K) and M is the maximal ideal of V; thus, in the present situation, the residue field V/M is canonically isomorphic to $\boldsymbol{k}$. Let D be a subring of $\boldsymbol{k}$. The ring $R := D + M$ (subring of V) with quotient field K (the same as V) has some interesting properties due to the mode of this construction, as indicated for instance in [7] (see also [15, Theorem 1.4]). Our concrete model for these examples would be $V := \boldsymbol{k}[\![X]\!] = \boldsymbol{k} + X\boldsymbol{k}[\![X]\!]$.

Example 4.16 Given a field $\boldsymbol{k}$, let D be a 1-dimensional local domain contained in $\boldsymbol{k}$, with quotient field F ($\subseteq \boldsymbol{k}$) and suppose that D is not a valuation domain. Then $R := D + X\boldsymbol{k}[\![X]\!]$ is a (local) 2-dimensional domain such that, for each nonzero nonunit x in D, we have $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$. Indeed, for a nonunit x in a 1-dimensional local domain D, we have $\bigcap_{n\geq 0} x^n D = (0)$ and so $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$. Moreover, since $R_{X\boldsymbol{k}[\![X]\!]} = F + X\boldsymbol{k}[\![X]\!]$, then $X\boldsymbol{k}[\![X]\!]R_{X\boldsymbol{k}[\![X]\!]} = X\boldsymbol{k}[\![X]\!](F + X\boldsymbol{k}[\![X]\!]) = X\boldsymbol{k}[\![X]\!]$. In this situation, $R/X\boldsymbol{k}[\![X]\!] = D$.

What makes the above example work is the fact that, for a nonunit x in a 1-dimensional local domain D, we have $\bigcap_{n\geq 0} x^n D = (0)$. Call an integral domain D an *Archimedean domain* if, for all nonunit elements x in D, we have $\bigcap_{n\geq 0} x^n D = (0)$ [43, Definition 3.6] (this class of domains was previously considered in [41] without naming them). By the Krull intersection theorem, every Noetherian domain is Archimedean. Since Mori domains satisfy the ascending chain condition on principal ideals, they are Archimedean; in particular, Krull domains are Archimedean. The class of Archimedean domains includes also completely integrally closed domains [19, Corollary 5] and 1-dimensional integral domains [41, Corollary 1.4].

An Archimedean (possibly nonlocal or any dimensional) version of the previous Example 4.16 is given next.

Example 4.17 Given a field $\boldsymbol{k}$, let D be an Archimedean domain contained in $\boldsymbol{k}$, with quotient field F $(\subseteq \boldsymbol{k})$ and suppose that D is not a valuation domain. Then, as above, $R := D + X\boldsymbol{k}[\![X]\!]$ is such that, for each nonzero nonunit x in D, we have $\bigcap_{n\geq 0} x^n R = X\boldsymbol{k}[\![X]\!]$, $X\boldsymbol{k}[\![X]\!] = X\boldsymbol{k}[\![X]\!]R_{X\boldsymbol{k}[\![X]\!]}$ and $R/X\boldsymbol{k}[\![X]\!] = D$. In the present situation, $\mathrm{Max}(R)$ has the same cardinality of $\mathrm{Max}(D)$ and $\dim(R) = \dim(D) + 1$.

Example 4.18 Let D be an integral domain and S a multiplicative subset of D. Following the construction $R := D + XD_S[X]$ of [11], if s is a nonunit element in S such that $\bigcap_{n\geq 0} s^n D = (0)$ then $\bigcap_{n\geq 0} s^n R = XD_S[X]$ a prime ideal of R. Also in this case $R/XD_S[X] = D$, which might not be a valuation domain. However, in the present situation, $XD_S[X] \subsetneq XD_S[X](R_{XD_S[X]}) = XD_S[X]_{(X)}$.

5 From *t*-Local Domains to Valuation Domains

Because in a valuation domain (V, M) every finitely generated ideal is principal, the maximal ideal M is obviously a t-ideal. So t-local domains are "cousins" of valuation domains, but sort of far removed. For instance, a localization of a t-local domain is not necessarily t-local (see, for instance, Example 2.9 or [51]), but of course a localization of a valuation domain is a valuation domain.

Explicitly, a more simple example is given by $R := \mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!]$. The integral domain R is local with maximal ideal $M := p\mathbb{Z}_{(p)} + (X, Y)\mathbb{Q}[\![X, Y]\!] = pR$, and so it is obviously a t-local domain. However, $R[1/p] = R_Q = \mathbb{Q}[\![X, Y]\!]$, where $Q := (X, Y)\mathbb{Q}[\![X, Y]\!]$, is a 2-dimensional local Noetherian Krull domain, and so it is far away from being t-local.

So it is legitimate to ask: Under what conditions is a t-local domain a valuation domain? Here we address this question.

The following is a simple result that hinges on the fact that if F is a nonzero finitely generated ideal in a t-ideal I then $F^v \subseteq I$.

Proposition 5.1 *For a finite set of elements $x_1, x_2, \ldots, x_n$, in a t-local domain (D, M), the following are equivalent:*

(i) $(x_1, x_2, \ldots, x_n)^v = D$.
(ii) At least one x_i is a unit.
(iii) $(x_1, x_2, \ldots, x_n) = D$.

Proof Clearly, (ii) ⇒ (iii) ⇒ (i).

(i) ⇒ (ii) By the previous observation $(x_1, x_2, \ldots, x_n) \nsubseteq M$, and so at least one $x_i \notin M$. □

Proposition 5.2 *For an integral domain D the following are equivalent:*

(i) D is a valuation domain
(ii) D is a t-local GCD domain (or, equivalently, a t-local Bézout domain).
(iii) D is a t-local PvMD (or, equivalently, a t-local Prüfer domain).

Proof (i) ⇒ (ii) ⇒ (iii) are straightforward.

For (iii) ⇒ (i) note for instance that, in a PvMD, every nonzero finitely generated ideal $(x_1, x_2, ..., x_n)$ is t-invertible. But, by [3, Proposition 1.12(1)], $(x_1, x_2, ..., x_n)$ is a principal ideal. □

Recall that a ring is *coherent* if every finitely generated ideal is finitely presented. It is well known that a commutative integral domain D is coherent if and only if the intersection of every pair of finitely generated ideals is finitely generated [9, Theorem 2.2].

Call a domain D *a finite conductor domain* (for short, *FC domain*; this name was used for the first time in [47]) if the intersection of every pair of principal ideals of D is finitely generated. Indeed, "finite conductor domain" is a generalization of "coherent domain."

Proposition 5.3 *For an integral domain D, the following are equivalent:*

(i) D is a valuation domain.
(ii) D is an integrally closed coherent t-local domain.
(iii) D is an integrally closed finite conductor t-local domain.

Proof (i) ⇒ (ii) ⇒ (iii) are all straightforward.

For (iii) ⇒ (i) note that an integrally closed FC domain is a PvMD [47, Theorem 2] (or, [18, Exercise 21, page 432]) and we already observed that a t-local PvMD is a valuation domain (Proposition 5.2((iii)⇒(i))). □

As an application of the previous proposition, we easily obtain the following result due to S. McAdam.

Corollary 5.4 *(S. McAdam [35, Theorem 1]) Let D be an integrally closed local domain whose primes are linearly ordered by inclusion. Assume that D is an FC domain, then D is a valuation domain.*

Proof By Proposition 2.4, D is t-local. The conclusion follows from Proposition 2.4((iii)⇒(i)). □

A nonzero element r of a domain D is called a *primal element* if for all $x, y \in D\backslash\{0\}$ $r|xy$ implies that $r = st$ where $s|x$ and $t|y$. A domain whose nonzero elements are all primal is called a *pre-Schreier domain*. An integrally closed pre-Schreier domain was called a *Schreier domain* by P.M. Cohn in his paper [10, page 254]. There, he showed that a GCD domain is a Schreier domain [10, Theorem 2.4].

Based on considerations initiated by McAdam and Rush [36], a module M is said to be *locally cyclic* if every finitely generated submodule of M is contained in a cyclic submodule of M. Thus, in particular, an ideal I of D is locally cyclic if, for any finite set of elements $x_1, x_2, \ldots, x_n \in I$, there is an element $d \in I$ such that $d|x_k$ for each k, $1 \leq k \leq n$.

In [50, Theorem 1.1], M. Zafrullah has shown that *an integral domain D is pre-Schreier if and only if for all $a, b \in D\backslash(0)$* and $x_1, x_2, \ldots, x_n \in (a) \cap (b)$ *there is* $d \in (a) \cap (b)$ such that $d|x_k$, *for each* k, $1 \leq k \leq n$.

Based on this, we easily obtain the following.

Lemma 5.5 *If D is a pre-Schreier domain and $a, b \in D\backslash\{0\}$, then the following are equivalent:*

(i) $(a) \cap (b)$ *is principal.*
(ii) $(a) \cap (b)$ *is finitely generated.*
(iii) $(a) \cap (b)$ *is a v-ideal of finite type.*

Proof Indeed (i) ⇒ (ii) ⇒ (iii) are all straightforward. All we need is to show (iii) ⇒ (i). For this note that if $(a) \cap (b) = (x_1, x_2, \ldots, x_n)^v$, then, $x_1, x_2, \ldots, x_n \in (a) \cap (b)$. Since D is pre-Schreier, there is an element $d \in (a) \cap (b)$ such that $d|x_k$, for each k, $1 \leq k \leq n$, i.e., $(x_1, x_2, \ldots, x_n) \subseteq (d)$. But then $(x_1, x_2, ...x_n)^v \subseteq (d)$, and so $(d) \subseteq (a) \cap (b) = (x_1, x_2, ...x_n)^v \subseteq (d)$. □

Call a domain D a *v-finite conductor* (for short, *v-FC*) *domain* if, for each pair $0 \neq a, b \in D$, the ideal $(a) \cap (b)$ is a v-ideal of finite type. Then, recalling that a GCD domain is integrally closed, from Lemma 5.5, we easily deduce the following.

Corollary 5.6 *Let D be an integral domain. The following are equivalent:*

(i) D is a GCD domain.
(ii) D is a Schreier and a v-FC domain.
(iii) D is a pre-Schreier and a v-FC domain.

With this preparation, we have the following result.

Corollary 5.7 *For an integral domain D, the following are equivalent:*

(i) D is a valuation domain.
(ii) D is a pre-Schreier t-local coherent domain.
(iii) D is a pre-Schreier t-local FC domain.
(iv) D is a pre-Schreier t-local v-FC domain.
(v) D is a GCD t-local domain.

Proof It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv); (iv) $\Leftrightarrow$ (v) by Corollary 5.6 and (v) $\Leftrightarrow$ (i) by Proposition 5.2. □

Obviously, the above are not the only situations in which a t-local integral domain becomes a valuation domain. We describe next another interesting situation of this phenomenon, in case of existence of a comparable element.

Proposition 5.8 *Suppose that an integral domain D contains a nonzero nonunit comparable element x and let $Q := \bigcap_{n\geq 0} x^n D$. Then, D is a valuation domain if and only if D_Q is a valuation domain.*

Proof Indeed, if D is a valuation domain, since Q is a prime ideal (Theorem 4.15), D_Q is also a valuation domain and so we have only to take care of its converse.

The presence of a nonzero nonunit comparable element makes D a t-local domain (Proposition 4.7). In order to prove that D is valuation domains, we consider the finitely generated ideals of D. We split the proper finitely generated ideals into two types: **(a)** ones that contain a nonunit factor of a power of x and **(b)** ones that do not contain a nonunit factor of a power of x.

Ones in part **(a)** are principal by [20, Theorem 2.4] and ones in part **(b)** are contained in Q and are principal proper ideals of the valuation domain D_Q and hence are in QD_Q. By Theorem 4.15 above, $QD_Q = Q$, so, for each y in Q, yD_Q is (also) an ideal of D, i.e., $yD_Q = yD$. Now, let $x_1, x_2, \dots, x_n \in Q$ and consider the ideal $(x_1, x_2, \dots, x_n)$. Since D_Q is a valuation domain, $(x_1, x_2, \dots, x_n)D_Q = dD_Q$ and we can assume that d is in D. So, for some $r_i \in D$ and $s_i \in D\backslash Q$ we have $x_i = \frac{r_i}{s_i}d$, for each i.

So $(x_1, x_2, \dots, x_n) = (\frac{r_1}{s_1}d, \frac{r_2}{s_2}d, \dots, \frac{r_n}{s_n}d)$. Removing the denominators, we get $s(x_1, x_2, \dots, x_n) = (t_1 d, t_2 d, \dots, t_n d) = (t_1, t_2, \dots, t_n)d$, for some $s \in D \setminus Q$, where $s_i | s$ and $t_i := \frac{s}{s_i} r_i$, for each i. As $dD_Q = (x_1, x_2, \dots, x_n)D_Q = s(x_1, x_2, \dots, x_n)D_Q = (t_1, t_2, \dots, t_n)dD_Q$, we conclude that $(t_1, t_2, \dots, t_n)D_Q = D_Q$. But that means that at least one of the t_i is in $D\backslash Q$ and hence is a comparable element (Lemma 4.4(5)). But then, by [20, Theorem 2.4], $(t_1, t_2, \dots, t_n)$ is principal generated by a comparable element t. Thus, $s(x_1, x_2, \dots, x_n) = (t_1, t_2, \dots, t_n)d = tdD$. Since s and t are comparable, we have two possibilities: ($\boldsymbol{\alpha}$) $u(x_1, x_2, \dots, x_n) = dD$ or ($\boldsymbol{\beta}$) $(x_1, x_2, \dots, x_n) = vdD$, for some $u, v \in D$. In both cases $(x_1, x_2, ...x_n)$ turns out to be a principal ideal of D (in case ($\boldsymbol{\alpha}$) because $d \in u(x_1, x_2, \dots, x_n)$ and so $u|d$ in D). □

6 Applications: Shannon's Quadratic Extension

A domain D is a *treed domain* if it has a treed spectrum, i.e., Spec(D) is a tree as a poset with respect to the set inclusion. Note that D is a treed domain if and only if any two incomparable primes of D are co-maximal. Indeed, if D is a treed then D_P is also a treed (more precisely, Spec(D_P) is linearly ordered) for every nonzero prime ideal P of D. So, by Proposition 2.4, D_P is a t-local domain and thus $P = PD_P \cap D$ is a t-ideal of D. Indeed, if F is a finitely generated ideal of D contained in P, then $F^t D_P = F^v D_P \subseteq (FD_P)^v = (FD_P)^t \subseteq (PD_P)^t = PD_P$ and so $F^t \subseteq (FD_P)^t \cap D \subseteq PD_P \cap D = P$ (see also [52, page 436]). Therefore, in a treed domain, every nonzero prime ideal is a t-ideal (Proposition 2.4), in particular every maximal ideal is a t-ideal, and moreover it is well behaved. However, a general t-local domain D may not have Spec(D) a tree as, for instance, Examples 2.9 and 4.17 indicate. So the class of treed domains is strictly contained in the class of domains whose maximal ideals are t-ideals. But, in the presence of some extra conditions, this distinction may disappear.

Proposition 6.1 *For a Prüfer v-multiplication domain D, the following conditions are equivalent:*

(i) Every maximal ideal of D is a t-ideal.
(ii) Every prime ideal of D is a t-ideal.
(iii) Spec(D) *is a tree.*
(iv) D is a Prüfer domain.

Proof (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) hold in general (without the PvMD assumption). More precisely, (iv) ⇒ (iii) is clear because in a Prüfer domain D, D_P is a valuation domain for every nonzero prime ideal P and so Spec(D) is a tree. (iii) ⇒ (ii) has been explained above.

(i) ⇒ (iv) For every prime t-ideal P of a PvMD D, we have D_P a valuation domain (see, for instance, [39, Corollary 4.3]) and if we assume that D_M is a valuation domain, for every maximal ideal M of D, then D is well known to be a Prüfer domain. □

The previous proposition leads to the following result for FC domains.

Corollary 6.2 *Let D be an integral domain. The following are equivalent:*

(i) D is an integrally closed finite conductor treed domain.
(ii) D is a treed PvMD.
(iii) D is Prüfer.

Proof (i) ⇒ (ii), since an integrally closed finite conductor domain is a PvMD by Proposition 5.3 and [39, Corollary 4.3]. (ii) ⇔ (iii) by Proposition 6.1 and (iii) ⇒ (i) because a Prüfer domain is an FC domain [47, Corollary 10]. □

Indeed, it is worth noting that a nonzero proper ideal I in an integral domain D is said to be an *ideal of grade 1* if I does not contain a set of elements forming a regular sequence of length ≥ 2. Recall that, if an ideal I of an integral domain D contains a regular sequence of length 2, then $I^{-1} = D$ [33, Exercise 1, page 102]. So, every t-ideal of an integral domain is a grade 1 ideal and every nonzero prime ideal in a treed domain is a grade 1 ideal. With this background, for the next application we need a little bit of preparation.

Let $(R, \mathfrak{m})$ be a regular local integral domain with quotient field F and $\mathfrak{p}$ a prime ideal of R so that $R/\mathfrak{p}$ is a regular local domain. A *monoidal transform of R with nonsingular center* $\mathfrak{p}$ is a local domain of the type $T := R[\mathfrak{p}x^{-1}]_Q$, where $0 \neq x \in \mathfrak{p}$ and Q is a prime ideal in $R[\mathfrak{p}x^{-1}]$ such that $\mathfrak{m} \subseteq Q$. In particular, assume that $\dim(R) = n$, and $\mathfrak{p} = \mathfrak{m} = (x_1, x_2, \ldots, x_n)R$, where $\{x_1, x_2, \ldots, x_n\}$ form a regular sequence in R. Choose $i \in \{1, 2, \ldots, n\}$, and consider the overring $R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]$ of R. Take any prime ideal Q of $R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]$ such that $Q \supseteq \mathfrak{m}$. The ring $R_1 := R[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]_Q$ is called a *local quadratic transform* (for short, *LQT*) of R, and, again, R_1 is a regular local integral domain with maximal ideal $\mathfrak{m}_1 := QR[x_1/x_i, x_2/x_i, \ldots, x_n/x_i]_Q$ [40, Corollary 38.2]. Assume that $\dim(R) \geq 2$ in order to have that $R \neq R_1$. By Cohen's dimension inequality formula $\dim(R_1) \leq n$ [34, Theorem 15.5] (and, more precisely, $\dim(R_1) = n$ if and only if $R_1/\mathfrak{m}_1$ is an algebraic extension of $R/\mathfrak{m}$) [2, (1.4)].

If we iterate the process, we obtain a sequence $R =: R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots$ of regular local overrings of R such that for each $j \geq 0$, R_{j+1} is a LQT of R_j. After a finite number of iterations, the sequence of nonincreasing integers $\dim(R_j)$ is necessarily bound to stabilize, and this process of iterating LQTs of the same Krull dimension (definitively, after a certain point) and ascending unions of the resulting regular sequences are of interest in algebraic geometry. For a description the reader may consult a couple of recent papers [23, 27]. So, let $R =: R_0 \subseteq R_1 \subseteq R_2 \subseteq \ldots$ be a sequence of LQTs from a regular local integral domain R with $\dim(R) \geq 2$ and $\dim(R_j) \geq 2$, for each $j \geq 1$, as described above. The ring $S := \bigcup_{j \geq 0} R_j$, dubbed in recent work as *Shannon's Quadratic Extension of R*, to honor David Shannon [43] for his interesting contribution, has drawn particular attention.

Briefly, before Shannon, Abhyankar [1, Lemma 12] had shown that, if the regular local ring R has dimension 2, then S is a valuation overring of R such that the maximal ideal $\mathfrak{m}_S$ of S contains the maximal ideal $\mathfrak{m}$ of R. David Shannon, one of Abhyankar's students, showed that if $\dim(R) > 2$, S need not to be a valuation ring [43, Examples 4.7 and 4.17].

Our purpose here is to look at S from a simple star-operation theoretic perspective, to provide some direct straightforward and brief proofs of some known results and point to known results that could simplify some of the considerations in recent work.

We start by gathering some information about the Shannon's Quadratic Extension S. Next two properties can be easily proved.

(1) $S (:= \bigcup_{j \geq 0} R_j)$, *as described above, is a local ring and, if* $\mathfrak{m}_S$ *denotes the maximal ideal of* S, $\mathfrak{m}_S = \bigcup_{j \geq 0} \mathfrak{m}_j$ where $\mathfrak{m}_j$ *is the maximal ideal of the* LQT R_j.

(2) *S is integrally closed, as being integrally closed a first-order property which is preserved by directed unions and hence, in particular, by ascending unions.*

Since S is directed union of regular local integral domains and, by the Auslander–Buchsbaum theorem [34, Theorem 20.3], each regular local integral domain is a UFD and hence, in particular, a GCD domain and so, *a fortiori*, a Schreier domain. This observation gives us the next property of S.

(3) *S is (at least) a Schreier domain.*

This follows from a direct verification that a direct union of (pre-)Schreier domains is a (pre-)Schreier domain.

Remark 6.3 Note that it is not true that a direct union of GCD domains is a GCD domain. An example can be given by an integral domain of the type $D^{(\Sigma)} := D + XD_{\Sigma}[X] = \bigcup\{D[X/s] \mid s \in \Sigma\}$, where D is a GCD domain and Σ is a saturated multiplicative subset D, since it is known that $D^{(\Sigma)}$ is not a GCD if Σ is not a splitting set, i.e., if Σ does not verify the condition that, for each $0 \neq d \in D$, $d = sa$ for some $s \in \Sigma$ and $a \in D$ with $aD \cap s'D = as'D$ for all $s' \in \Sigma$ [49, Corollary 1.5].

We give now an explicit example. Let $\mathcal{E}$ be the ring of entire functions. It is well known that $\mathcal{E}$ is a Bézout domain [18, Exercise 18, page 147] and that every nonzero nonunit x of $\mathcal{E}$ can be written uniquely as a countable product of finite powers of nonassociated primes, i.e., $x = u \prod_{\alpha \in A} p_{\alpha}^{n_{\alpha}}$ where A is a countable set, n_{α} are natural numbers and p_{α} are mutually nonassociated primes elements of $\mathcal{E}$ and u is a unit in $\mathcal{E}$. The last property follows from the fact that the set of zeros of a nontrivial entire function is discrete, including multiplicities, the multiplicity of a zero of an entire function is a positive integer and a zero of an entire function determines a principal prime in $\mathcal{E}$ [30, Theorem 6]. Clearly, each of these primes generates a height one maximal ideal of $\mathcal{E}$ [18, Exercise 19, page 147].

Let Σ be the multiplicative set generated by all of these principal, height one primes and let X be an indeterminate. Then, the ring $\mathcal{E}^{(\Sigma)} := \mathcal{E} + X\mathcal{E}_{\Sigma}[X] = \bigcup\{\mathcal{E}[X/s] \mid s \in \Sigma\}$ is not a GCD domain, even though $\mathcal{E}[X/s]$ is a GCD domain for each $s \in \Sigma$.

Indeed, if $x \in \mathcal{E}$ is an infinite product of primes then it is not possible to write $x = sx_1$ where $s \in \Sigma$ and x_1 is not divisible by any of the nonunits in Σ, since each s is a finite product of primes and x is a product of infinitely many primes from Σ. Thus, Σ is not a splitting set and so $\mathcal{E}^{(\Sigma)}$ cannot be a GCD domain.

However, we claim that $\mathcal{E}^{(\Sigma)}$ is a locally GCD domain. For proving the claim, we need some preliminaries. A prime ideal P of an integral domain D is said *to intersect in detail a multiplicative set* Σ of D if every nonzero prime ideal Q contained in P intersects Σ. It was shown [49, Proposition 4.1] that if D is a locally GCD domain and Σ is a multiplicative set of D such that every maximal ideal of D that intersects Σ, intersects Σ in detail, then $D^{(\Sigma)}$ is a locally GCD domain.

Indeed, clearly the Bézout domain $\mathcal{E}$ is a locally GCD domain. Moreover, as every maximal ideal of $\mathcal{E}$ that intersects Σ contains a finite product of principal primes and so must be a principal ideal. Thus, every maximal ideal of $\mathcal{E}$ that intersects Σ, intersects it in detail. Consequently $\mathcal{E}^{(\Sigma)}$ is a locally GCD domain; however, $\mathcal{E}^{(\Sigma)}$ is not a PvMD, since $\mathcal{E}^{(\Sigma)}$ is a Schreier domain and a PvMD which also is a Schreier domain is a GCD domain [6, Proposition 2.3].

As a final remark, we recall from [49, Proposition 4.3] that in a locally GCD non-PvMD D there always exists a maximal t-ideal Q of D such that QD_Q is not a t-deal of D_Q. More precisely, it can be shown that an integral domain D is a PvMD if and only if D is locally PvMD and, for every t-prime ideal P of D, PD_P is a (maximal) t-ideal of D_P [49, Corollary 4.4].

We now resume our study of Shannon's Quadratic Extension S.

(4) *There exists an element* $x \in \mathfrak{m}_S$ *such that* $\mathfrak{m}_S = \sqrt{xS}$ [27, Proposition 3.8].

The last property gives us, in light of Corollary 2.3(1), the following property that is of interest to us:

(5) *S is a t-local integral domain.*

This is enough information to provide very naturally the statements and easy new proof(s) of [23, Theorem 6.2].

Theorem 6.4 *(Guerrieri et al. [23, Theorem 6.2]) Let S be a quadratic Shannon extension of a regular local integral domain R. Then, the following are equivalent:*

(i) S is a valuation domain
(ii) S is coherent.
(iii) S is a finite conductor domain.
(iv) S is a GCD domain.
(v) S is a PvMD.
(vi) S is a v-finite conductor domain.

Proof The equivalence of (i) ⇔ (ii) ⇔ (iii) comes from Proposition 5.3. Now (i) ⇔ (iv) ⇔ (v) follow from Proposition 5.2 and, as S is Schreier (by (3)), (i) ⇔ (vi) by Corollary 5.7. □

From Lemma 5.5, Corollary 5.7, and Theorem 6.4, we easily deduce the following.

Corollary 6.5 *Let S be a quadratic Shannon extension of a regular local integral domain R. If S is not a valuation domain, then S contains a pair of elements a, b such that $aS \cap bS$ is not a v-ideal of finite type.*

Proof If, for each pair of elements $a, b \in S$, we had that $aS \cap bS$ is a v-ideal of finite type, then S would be a GCD domain by Corollary 5.6, since S is a Schreier domain (by point (3) above). Therefore, S would be a valuation domain by Theorem 6.4, which is not the case. □

This corollary is significant with reference to the proof of the previous theorem (Theorem 6.4) in that there are PvMDs D, such as Krull domains, that contain elements a, b such that $aS \cap bS$ a v-ideal of finite type, which may not be finitely generated.

From [27, Proposition 4.1], we conclude that S has another property of interest.

(5) *For each element* $x \in \mathfrak{m}_S$ *such that* $\mathfrak{m}_S = \sqrt{xS}$, *the integral domain* $T := S[1/x]$ *is a regular local ring with* $\dim(T) = \dim(S) - 1$.

So, if $\dim(S) = 2$ and $\mathfrak{m}_S$ contains a nonzero comparable element then we know that S is a valuation domain (Theorem 4.15 and (5)).

If $\dim(S) > 2$ then S cannot be a valuation domain, whether S contains a comparable element or not, because a regular local ring T, constructed from S as in (5), has $\dim(T) > 1$, and thus T may not be a valuation domain. However, if $\mathfrak{m}_S = pS$ is principal then, S is a non-valuation t-local domain that contains a comparable element, by Proposition 4.14(2). This fact, together with Proposition 5.8, provides a definitive criterion that can be used to construct examples of non-valuation t-local domains containing a comparable element, even in dimension two.

Example 6.6 Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ (resp., $\mathbb{R}$) the field of rational numbers (resp. real numbers) and p a prime element in $\mathbb{Z}$. Let P be the maximal ideal of the DVR $\mathbb{R}[\![X]\!]$ and set $D := \mathbb{Z}_{(p)} + X\mathbb{R}[\![X]\!] = \mathbb{Z}_{(p)} + P$. The integral domain D is local with principal maximal ideal $M := pD$ and $\bigcap_{n\geq 0} p^n D = X\mathbb{R}[\![X]\!] = P$. Clearly, p is a proper comparable element in D. Since $D_P = \mathbb{Q} + X\mathbb{R}[\![X]\!]$ is not a valuation domain, D is a 2-dimensional non-Noetherian non-valuation t-local integral domain with prime spectrum linearly ordered given by $\{M \supset P \supset (0)\}$.

In the same vein, and this is suggested by Tiberiu Dumitrescu, we have another example.

Example 6.7 Let $\mathbb{Z}$ be the ring of integers, $\mathbb{Q}$ the field of rational numbers and p a nonzero prime element in $\mathbb{Z}$. Let $D := \mathbb{Z}_{(p)} + P$ where P is the maximal ideal (X^2, X^3) of $\mathbb{Q}[\![X^2, X^3]\!]$. As above, D is a local domain with maximal ideal $M = p\mathbb{Z}_{(p)} + P = pD$ and $\bigcap_{n\geq 0} p^n D = P$. In this case, $D_P = \mathbb{Q}[\![X^2, X^3]\!]$ which is a well-known 2-dimensional Noetherian domain that is not a valuation domain (in fact, it is non-integrally closed). Thus, D is a 2-dimensional non-Noetherian non-valuation t-local integral domain, having a proper comparable element and prime spectrum linearly ordered given by $\{M := p\mathbb{Z}_{(p)} + (X^2, X^3)\mathbb{Q}[\![X^2, X^3]\!] \supset P \supset (0)\}$.

We can provide examples in any dimension. Let P be the maximal ideal of the n-dimensional regular local ring $\mathbb{Q}[\![X_1, X_2, \ldots, X_n]\!]$. Then $D := \mathbb{Z}_{(p)} + P$ is local with maximal ideal $M := pD$. In particular, D contains a proper comparable element, e.g., p, and, of course, D_P is far from being a valuation domain. Thus, D is an $(n+1)$-dimensional non-valuation t-local integral domain.

Note that a 1-dimensional domain that contains a nonzero nonunit comparable element is a valuation domain. This follows from the following two facts: (1) the presence of a comparable element forces the domain to be (1-dimensional) t-local and (2) a domain is a valuation domain if and only if every nonzero prime ideal contains a nonzero comparable element (Lemma 4.3).

From (5), we deduce another interesting property of S.

(6) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain), for each element* $x \in \mathfrak{m}_S$ such that $\mathfrak{m}_S = \sqrt{xS}$, *call the saturation of the multiplicative set* $\{x^n \mid n \in \mathbb{N}\}$, *span of x and denote it by* $\mathrm{span}(x)$. *Then,*

(6a) *for every nonunit h in* $\mathtt{span}(x)$ *we have* $\mathfrak{m}_S = \sqrt{hS}$ and
(6b) $\mathfrak{m}_S$ *is generated by nonunits in* $\mathtt{span}(x)$.

The saturated multiplicative set $\mathtt{span}(x)$ has been used before, by Dumitrescu et al. in [14], to determine the number of distinct maximal t-ideals that the element x belongs to. Here, the statement that the ideal $\mathfrak{m}_S$ is generated by nonunit members of $\mathtt{span}(x)$ is caused by the fact that there is only one maximal t-ideal (i.e., $\mathfrak{m}_S$) involved.

Note that, before introducing quadratic Shannon extensions of local regular rings, all examples of t-local domains that we have considered in the present paper were valuation domains or rings obtained by some pullback construction. At this point, it is natural to ask if the quadratic Shannon extensions, which are not valuation domains, could as well be obtained by some appropriate pullback construction. For this purpose, we start by recalling some other properties of the quadratic Shannon extensions.

(7) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain of dimension* > 2). *If S is Archimedean, then its complete integral closure* S^* coincides with $(\mathfrak{m}_S : \mathfrak{m}_S) = T \cap W$, *where* $\mathfrak{m}_S$ *is the maximal ideal of* S, $T = S[1/x]$ *is the local regular overring of S introduced in (5), and W is a uniquely determined valuation overring of S and if* $S \neq S^*$, S^* *is a generalized Krull domain* [27, Theorem 6.2].

In the previous situation, if $S \neq S^*$, $\mathfrak{m}_S$ is a height 1 prime ideal of S^*, since it is the center of the maximal ideal of the valuation overring W of S^* (see [27, Corollary 6.3] and [28, Theorem 7.4]). Therefore, S is the pullback of the residue field $S/\mathfrak{m}_S$ with respect to the canonical projection $S^* \to S^*/\mathfrak{m}_S$.

On the other hand, in the non-Archimedean case, we know the following fact:

(8) *Let S be as above (i.e., a quadratic Shannon extension of a regular local integral domain of dimension* > 2). *If S is non-Archimedean, then its complete integral closure* S^* *coincides with the overring* $T = S[1/x]$, $\bigcap\{x^n S \mid n \geq 0\} =: \mathfrak{p}$ *is a proper prime ideal of S and* $T = (\mathfrak{p} : \mathfrak{p})$ [27, Theorem 6.9 and Corollary 6.10].

In the previous situation, the integral domain $S/\mathfrak{p}$ is a DVR [27, Lemma 3.4], and $T = S_{\mathfrak{p}}$, since $T = S[1/x]$ is a ring of fractions of S and $\mathfrak{p}$ is disjoint from the multiplicative set $\{x^n \mid n \geq 0\}$. Therefore, S is the pullback of $S/\mathfrak{p}$ with respect to the canonical projection $T \to T/\mathfrak{p}$, where $T/\mathfrak{p}$ is a field, coinciding with the residue field $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}$ (isomorphic to the field of quotients of the integral domain $S/\mathfrak{p}$).

The last remaining case is when the quadratic Shannon extension S is (Archimedean and) completely integrally closed. An example is given in [28, Corollary 7.7]. In this situation S may not be obtained by a pullback construction of some of its overrings, since, if an integral domain A shares a nonzero ideal with one of its proper overrings B then A and B must have the same complete integral closure [19, Lemma 5].

We end with a classification of the t-local domains, which could be useful for detecting t-local domains that are not issued from a pullback construction.

The following proposition is a consequence of more general results concerning DT-domains, proved by G. Picozza and F. Tartarone in [42].

Proposition 6.8 *Let (D, M) be a local domain.*

(1) If $D \neq (M : M)$, then D is a t-local domain.
(2) If $D = (M : M)$ and M is finitely generated, then D is a t-local domain if and only if M is principal.
(3) If $D = (M : M)$, and M is not finitely generated, then D is a t-local domain if and only if M is not t-invertible.

Proof (1) If $D \neq (M : M)$, then necessarily the maximal ideal M is the conductor of the inclusion $D \hookrightarrow (M : M)$ and so M is a divisorial ideal of D.

(2) Assume that $D = (M : M)$, and M is finitely generated, clearly M is divisorial if and only if $(M : M) = D \neq M^{-1} = (D : M)$ and this happens if and only if $M \neq MM^{-1}(\subseteq D)$ or, equivalently, if and only if $MM^{-1} = D$. In a local domain, a nonzero ideal is invertible if and only if it is a principal ideal.

(3) Assume that $D = (M : M)$, M is not finitely generated and, moreover, M is not a t-invertible ideal. If M is not a t-ideal, then $M^t = D$ and thus $(MM^{-1})^t = M^t = D$, which is a contradiction.

Conversely, since M is not finitely generated, M is not invertible and, since D is t-local, M is not even t-invertible (Theorem 3.7 ((iii)⇒(vi)). □

Any pseudo-valuation non-valuation domain provides an example of case (1); a discrete valuation domain (for short, DVR) is an example of case (2) and a rank 1 non-DVR valuation domain is an example of case (3).

Acknowledgements The authors would like to thank Francesca Tartarone and Lorenzo Guerrieri for the useful conversations on some aspects of the present paper and the anonymous referee for several helpful suggestions which improved the quality of the manuscript.

References

1. S. Abhyankar, On the valuations centered in a local domain. Amer. J. Math. **78**, 321–348 (1956)
2. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces* (Academic Press, New York, 1966)
3. D.D. Anderson, G.W. Chang, M. Zafrullah, Integral domains of finite t-character. J. Algebra **396**, 169–183 (2013)
4. D.F. Anderson, D.E. Dobbs, M. Fontana, On treed nagata rings. J. Pure Appl. Alg. **261**, 107–122 (1989)
5. D.D. Anderson, M. Zafrullah, On a theorem of Kaplansky, Boll. Un. Mat. Ital. A(7), **8** (1994), 397–402
6. D.D. Anderson, M. Zafrullah, The Schreier property and Gauss' Lemma, Boll. Un. Mat. Ital. **10-B** (2007), 43–62
7. E. Bastida, R. Gilmer, Overrings and divisorial idels of rings of the form $D + M$. Michigan Math. J. **20**, 79–95 (1973)

8. G. Chang, T. Dumitrescu, M. Zafrullah, t-Splitting sets in integral domains. J. Pure Appl. Algebra **187**, 71–86 (2004)
9. S.U. Chase, Direct product of modules. Trans. Am. Math. Soc. **97**, 457–493 (1960)
10. P.M. Cohn, Bezout rings and their subrings. Proc. Cambridge Phil. Soc. **64**, 251–264 (1968)
11. D. Costa, J. Mott, M. Zafrullah, The construction $D + XD_S[X]$. J. Algebra **53**, 423–439 (1978)
12. E.D. Davis, Overrings of commutative rings, III. Trans. Am. Math. Soc. **182**, 175–185 (1973)
13. D. Dobbs, E. Houston, T. Lucas, M. Zafrullah, t-linked overrings and PVMDs. Comm. Algebra **17**, 2835–2852 (1989)
14. T. Dumitrescu, Y. Lequain, J. Mott, M. Zafrullah, Almost GCD domains of finite t-character. J. Algebra **245**, 161–181 (2001)
15. M. Fontana, Topologically defined classes of commutative rings. Ann. Mat. Pura Appl. **123**, 331–355 (1980)
16. M. Fontana, K.A. Loper, Nagata rings, Kronecker function rings, and related semistar operations. Comm. Algebra **31**, 4775–4805 (2003)
17. S. Gabelli, M. Roitman, Maximal divisorial ideals and t-maximal ideals, JP. J. Algebra Numb. Theor. Appl. **4**, 323–336 (2004)
18. R. Gilmer, *Multiplicative Ideal Theory* (Marcel Dekker, NewYork, 1972)
19. R.W. Gilmer, W.J. Heinzer, On the complete integral closure of an integral domain. J. Australian Math. Soc. **6**, 351–361 (1966)
20. R. Gilmer, J. Mott, M. Zafrullah, t-invertibility and comparability, in *Commutative Ring Theory*, ed. by P.-J. Cahen, D. Costa, M. Fontana, S.-E. Kabbaj (Marcel Dekker, New York, 1994), pp. 141–150
21. S. Glaz, W.V. Vasconcelos, Flat ideals, II. Manuscripta Math. **22**, 325–341 (1977)
22. M. Griffin, Some results on v-multiplication rings. Canad. J. Math. **19**, 710–722 (1967)
23. L. Guerrieri, W. Heinzer, B. Olberding, M. Toeniskoetter, Directed unions of local quadratic transforms of regular local rings and pullbacks, in *"Rings, Polynomials, and Modules"* Fontana, Marco; Frisch, Sophie; Glaz, Sarah; Tartarone, Francesca; Zanardo, Paolo (Editors), Springer International Publishing 2017
24. J.R. Hedstrom, E. Houston, Pseudo-valuation domains. Pacific J. Math. **75**, 137–147 (1978)
25. J.R. Hedstrom, E. Houston, Some remarks on star-operations. J. Pure Appl. Algebra **18**, 37–44 (1980)
26. W. Heinzer, Integral domains in which each non-zero ideal is divisorial. Mathematika **15**, 164–170 (1968)
27. W. Heinzer, K.A. Loper, B. Olberding, H. Schoutens, M. Toeniskoetter, Ideal theory of infinite directed unions of local quadratic transforms. J. Algebra **474**, 213–239 (2017)
28. W. Heinzer, B. Olberding, M.Toeniskoetter, Asymptotic properties of infinite directed unions of local quadratic transforms, J. Algebra **479** (2017), 216–243
29. W. Heinzer, J. Ohm, An essential ring which is not a v-multiplication ring. Canad. J. Math. **25**, 856–861 (1973)
30. O. Helmer, Divisibility properties of integral functions. Duke J. Math. **6**, 345–356 (1940)
31. E. Houston, M. Fontana, M.H. Park, Sharpness and semistar operations in Prüfer-like domains, Submitted
32. B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra **123**, 151–170 (1989)
33. I. Kaplansky, *Commutative Rings* (Allyn and Bacon, 1970)
34. H. Matsumura, *Commutative rings* (Cambridge University Press, Cambridge, 1986)
35. S. McAdam, Two conductor theorems. J. Algebra **23**, 239–240 (1972)
36. S. McAdam, D. Rush, Schreier rings. Bull. London Math. Soc. **10**, 77–80 (1978)
37. A. Mimouni, TW-domains and strong Mori domains. J. Pure Appl. Algebra **177**, 79–93 (2003)
38. A. Mimouni, Integral domains in which each ideal is a w-ideal. Comm. Algebra **33**, 1345–1355 (2005)
39. J. Mott, M. Zafrullah, On Prüfer v-multiplication domains. Manuscripta Math. **35**, 1–26 (1981)
40. M. Nagata, *Local Rings* (Interscience, New York, 1962)

41. J. Ohm, Some counterexamples related to integral closure in $D[\![x]\!]$. Trans. Amer. Math. Soc. **122**, 321–333 (1966)
42. G. Picozza, F. Tartarone, When the semistar operation $\widetilde{\star}$ is the identity. Comm. Algebra. **36**, 1954–1975 (2008)
43. D. Shannon, Monoidal transforms of regular local rings. Amer. J. Math. **95**, 294–320 (1973)
44. F. Wang, w-modules over a PVMD, In: *Proceedings of the International Symposium on Teaching and Applications of Engineering Mathematics*, Hong Kong, 2001, pp. 11–120
45. Fanggui Wang, w-dimension of domains, II. Comm. Algebra **29**, 224–28 (2001)
46. F. Wang, R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra **25** (1997), 1285–1306
47. M. Zafrullah, On finite conductor domains. Manuscripta Math. **24**, 191–203 (1978)
48. M. Zafrullah, The v-operation and intersections of quotient rings of integral domains. Comm. Algebra **13**, 1699–1712 (1985)
49. M. Zafrullah, The $D + XD_S[X]$ construction from GCD domains. J. Pure Appl. Algebra **50**, 93–107 (1988)
50. M. Zafrullah, On a property of pre-Schreier domains. Comm. Algebra **15**, 1895–1920 (1987)
51. M. Zafrullah, Well behaved prime t-ideals. J. Pure Appl. Algebra **65**, 199–207 (1990)
52. M. Zafrullah, Putting t-Invertibility to use, in *Non-Noetherian Commutative Ring Theory*, ed. by S.T. Chapman, S. Glaz (Kluwer, Dordrecht, 2000), pp. 429–457

Strongly Divided Pairs of Integral Domains

Ahmed Ayache and David E. Dobbs

Abstract This work generalizes the recent study of the class of strongly divided (commutative integral) domains. Let $R \subseteq T$ be domains with (R, m) quasi-local. Then (R, T) is said to be a strongly divided pair if, for each ring E such that $R \subseteq E \subseteq T$ and each $Q \in \text{Spec}(E)$ such that $Q \cap R \subset m$, one has $Q \subset R$. Let $\overline{R}$ be the integral closure of R in T. Then (R, T) is a strongly divided pair if and only if R and $\overline{R}$ have the same sets of nonmaximal prime ideals and, for each maximal ideal M of $\overline{R}$, $(\overline{R}_M, T_M)$ is a strongly divided pair. If R is integrally closed in T and R is treed, then (R, T) is a strongly divided pair if and only if $R[u]$ is a treed domain for each $u \in T$. If $mT = T$ and R is integrally closed in T, then (R, T) is a strongly divided pair if and only if $T = R_p$ for some divided prime ideal p of R and R/p is a strongly divided domain. Examples of strongly divided pairs $((R, m), T)$ such that $mT \neq T$ are given using pullbacks with data having prime spectra pinched at some nonmaximal prime ideal. Additional results and examples are given to illustrate the theory and its sharpness.

Keywords Integral domain · Overring · Prime ideal · Treed domain · Pullback · Integrality · Pseudo-valuation domain · Strongly divided domain · Krull dimension · Field

2010 Mathematics Subject Classification Primary: 13G05 · 13B99 · Secondary: 13A15 · 13B21

A. Ayache
Faculty of Science, Department of Mathematics, University of Bahrain, P. O. Box 32038, Sukhir, Bahrain
e-mail: aaayache@yahoo.com

D. E. Dobbs (✉)
Department of Mathematics, University of Tennessee, Knoxville, TN 37996-1320, USA
e-mail: dedobbs@comporium.net

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*, Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_4

1 Introduction

We begin with a paragraph collecting some conventions about notation and some standard definitions. All rings considered below are commutative with identity; nearly all these rings are (commutative integral) domains. If D is a domain, $\mathrm{qf}(D)$ denotes the quotient field of D. If A is a ring, $\mathrm{Spec}(A)$ denotes the set of all prime ideals of A, $\mathrm{Max}(A)$ denotes the set of all maximal ideals of A, $\mathcal{S}(A) := \mathrm{Spec}(A) \setminus \mathrm{Max}(A)$, and $\dim(A)$ denotes the Krull dimension of A. As in [9], a ring A is said to be *treed* if $\mathrm{Spec}(A)$, when viewed as a poset under inclusion, is a tree; that is, no maximal ideal of A can contain incomparable prime ideals of A. If D and E are domains, then $D \subseteq E$ is understood to mean that D is a (necessarily unital) subring of E, in which case the set of intermediate rings is denoted by $[D, E] := \{A \mid A \text{ is a ring and } D \subseteq A \subseteq E\}$. By an *overring* of a domain D, we mean any element $A \in [D, \mathrm{qf}(D)]$. By a *simple overring* of a domain D, we mean any overring of D of the form $D[u]$ where $u \in \mathrm{qf}(D)$. If $D \subset E$ are domains and $P \in \mathrm{Spec}(D)$, then $E_P := E_{D \setminus P}$. Also, for domains $D \subseteq E$, it will be convenient to denote the integral closure of D in E by $\overline{D}$; it should always be clear from the context which ring is intended to play the role of E. If I is an ideal of a domain D, then $\mathrm{Rad}_D(I)$ denotes the radical of I (in D), in the sense of [21, page 17]; that is, $\mathrm{Rad}_D(I) := \{d \in D \mid \text{there exists a positive integer } n \text{ such that } d^n \in I\}$. Following [21, page 28], we let INC, LO, and GU denote the incomparable, lying-over, and going-up properties, respectively, of ring extensions. As usual, $\subset$ denotes proper inclusion. Any unexplained material is standard, as in [19, 21].

Our purpose here is to initiate the study of the following concepts. Let (R, m) be a quasi-local domain and let T be a domain containing R as a subring. We say that $R \subseteq T$ is (*a*) *strongly divided extension* if, whenever $Q \in \mathrm{Spec}(T)$ satisfies $Q \cap R \in \mathcal{S}(R)$, then $Q = Q \cap R$ (that is, $Q \subseteq R$); some characterizations of strongly divided extensions are given in Proposition 2.5. We say that the pair (R, T) is (*a*) *strongly divided pair* if the extension $R \subseteq E$ is a strongly divided extension for each $E \in [R, T]$. To motivate these definitions, note that if $T = \mathrm{qf}(R)$, then (R, T) is a strongly divided pair if and only if R is a strongly divided domain (in the sense of [3]). It was shown in [3] that the class of strongly divided domains fits properly between the class of pseudo-valuation domains (or PVDs, in the sense of [20]) and the class of divided domains (in the sense of [10], also known as the AV-domains introduced in [1]).

Caution is needed when seeking generalizations of results concerning strongly divided domains. Indeed, although any strongly divided domain is a divided domain [3, Proposition 1(a)], the analogous ring-theoretic concept was recently shown to behave more pathologically. Indeed, [4, Example 3.23] gave an example of an idealization which is a “strongly divided ring in the first sense” but not a divided ring (in the sense of [7]). The existence of this example and several other examples in [4] has convinced us that one should focus on domains, rather than on more general rings, when seeking pair-theoretic generalizations of results concerning the “strongly divided ring” concepts from [3, 4].

In that regard, recall that [3, Proposition 3] gave a very useful characterization that essentially reduced the study of strongly divided domains D to studying $\mathcal{S}(D)$ and the case in which D is integrally closed (in its quotient field). In Proposition 2.10, we generalize [3, Proposition 3] by giving the following characterization of strongly divided pairs: if $R \subseteq T$ are domains such that R is quasi-local, then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $(\overline{R}_M, T_M)$ is a strongly divided pair for each $M \in \mathrm{Max}(\overline{R})$. As a consequence, it is shown in Corollary 2.14 that if $R \subseteq T$ are domains such that R and $\overline{R}$ are quasi-local, then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $\overline{R}[u]$ is a treed domain for each $u \in T$. This result generalizes a characterization of strongly divided domains that was given in [3, Theorem 1].

This paragraph summarizes some other noteworthy results from Sect. 2. For this paragraph, fix domains $R \subseteq T$ such that R is quasi-local. We begin with some facts from Proposition 2.1: if (R, T) is a strongly divided pair and R is not a field, then T is algebraic over R (in fact, if it is also the case that $\dim(R) > 1$, then T is an overring of R); and if T is integral over R, then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(\overline{R})$. A sufficient (but not necessary) condition for (R, T) to be a strongly divided pair is that R is PV in T, in the sense of [5] (for instance, if (R, T) is a normal pair): see Proposition 2.6. (For a partial converse, see Proposition 3.7.) Strongly divided pairs are stable under localization at nonmaximal prime ideals (Proposition 2.8) and homomorphic images (Proposition 2.4) but need not be stable under juxtaposition (Example 2.2(a)). For some situations where juxtaposition preserves the "strongly divided pair" property, see Proposition 2.10 (which was discussed in the preceding paragraph) and Proposition 2.11.

Section 3 is devoted to the study of strongly divided pairs $((R, m), T)$ that also satisfy $mT = T$. Since the latter property holds in case R is a quasi-local domain which is distinct from its quotient field T, it may be expected that the strongly divided pairs $((R, m), T)$ for which $mT = T$ admit a rich theory. In fact, Sect. 3 gives some characterizations of such pairs that make use of the "strongly divided domain" concept (see Proposition 3.1 and Theorem 3.3). One consequence is Corollary 3.5: if $R \subseteq T$ are domains such that (R, m) is quasi-local, $mT = T$ and $\overline{R} = R$, then (R, T) is a strongly divided pair if and only if $T = R_p$ for some divided prime ideal p of R and R/p is a strongly divided domain. (Recall from [10] that if D is a domain and $P \in \mathrm{Spec}(D)$, then P is called a *divided prime ideal* (*of*) D if $P = PD_P$, that is, if P is comparable (under inclusion) with each (resp., with each principal) ideal of D.) Treed-theoretic work in Sect. 3 produces results such as the following. If $((R, m), T)$ is a strongly divided pair and $mT = T$, then R is treed $\Leftrightarrow$ T is treed $\Leftrightarrow$ $R[u]$ is treed for each $u \in T$ (Proposition 3.9). Example 4.1 shows that the conclusion of Proposition 3.9 fails if one deletes the hypothesis that $mT = T$.

Many of the examples of strongly divided pairs given below are constructed via pullback-theoretic methods. In this regard, we would point to Remark 3.10, Examples 3.11, 4.1, 4.4 and Proposition 4.5. The proofs of all these results use the fundamental "gluing" description of the prime spectrum of a pullback [18, Theorem 1.4].

Section 4 concerns the strongly divided pairs $((R, m), T)$ satisfying $mT \neq T$. Several examples of such pairs are given in this case, often using pullbacks whose data include a domain whose prime spectrum is pinched at some nonmaximal prime ideal. Such examples can be integral, integrally closed or neither: see Example 4.4 and Proposition 4.5. An analogous result for the "$mT = T$" context is given in Remark 4.6.

2 The General Case

For any quasi-local domain D, it is clear that (D, D) is a strongly divided pair. Proposition 2.1 collects some less trivial facts concerning strongly divided pairs and strongly divided extensions. To motivate the formulation of parts (d), (e), and (f) of Proposition 2.1, note that if (R, m) is a quasi-local domain then R is not a field $\Leftrightarrow$ $m \neq \{0\}$ $\Leftrightarrow$ $\dim(R) > 0$.

Proposition 2.1 *(a) If F is a field and T is a domain containing F as a subring, then (F, T) is a strongly divided pair.*

(b) Let $R \subseteq T$ be an integral extension of domains such that R is quasi-local. Then (R, T) is a strongly divided pair $\Leftrightarrow$ $R \subseteq T$ is a strongly divided extension $\Leftrightarrow$ $\mathcal{S}(R) = \mathcal{S}(T)$.

(c) Let $R \subseteq T$ be domains such that R is quasi-local and $R \subseteq T$ satisfies INC, LO *and* GU. *Then $R \subseteq T$ is a strongly divided extension if and only if $\mathcal{S}(R) = \mathcal{S}(T)$.*

(d) Let $R \subseteq T$ be domains such that R is quasi-local but not a field. If $R \subseteq T$ is a strongly divided extension, then T is algebraic over R. In particular, if (R, T) is a strongly divided pair (and R is not a field), then T is algebraic over R

(e) Let $R \subseteq T$ be domains such that R is quasi-local and $\dim(R) = 1$. *Then (R, T) is a strongly divided pair if and only if T is algebraic over R.*

(f) Let $R \subseteq T$ be domains such that R is quasi-local, $\dim(R) > 1$ *and (R, T) is a strongly divided pair. Then T is (R-algebra isomorphic to) an overring of R.*

Proof (a) Since each ring $E \in [F, T]$ is a domain, it is enough to prove that $F \subseteq T$ is a strongly divided extension. This, in turn, holds since $\mathcal{S}(F) = \emptyset$.

(b) We will use the facts that any integral ring extension satisfies INC, LO, and GU (cf. [21, Theorem 44]). We claim that if $\mathcal{S}(R) = \mathcal{S}(T)$, then $\mathcal{S}(R) = \mathcal{S}(E)$ for each $E \in [R, T]$. To see this, suppose first that $p \in \mathcal{S}(R)$. Then $p \in \mathcal{S}(T)$, and so $p = p \cap E \in \mathrm{Spec}(E)$. In fact, since $E \subseteq T$ satisfies INC (being an integral extension), we get that $p \in \mathcal{S}(E)$. Thus $\mathcal{S}(R) \subseteq \mathcal{S}(E)$. For the reverse inclusion, consider any $Q \in \mathcal{S}(E)$. Then, since $E \subseteq T$ satisfies LO and GU (being an integral extension), there exists $\mathfrak{Q} \in \mathcal{S}(T) = \mathcal{S}(R)$ such that $\mathfrak{Q} \cap E = Q$. As $\mathfrak{Q} \subseteq R$, we get $Q \subseteq \mathfrak{Q} = \mathfrak{Q} \cap R = \mathfrak{Q} \cap (E \cap R) = (\mathfrak{Q} \cap E) \cap R = Q \cap R \subseteq Q$. It follows that $Q = \mathfrak{Q} \in \mathcal{S}(R)$. Thus $\mathcal{S}(E) \subseteq \mathcal{S}(R)$. This completes the proof of the claim.

By the above claim, it suffices to show that (R, T) is a strongly divided extension if and only if $\mathcal{S}(R) = \mathcal{S}(T)$. Suppose first that (R, T) is a strongly divided extension. Since $R \subseteq T$ satisfies INC, it follows that if $Q \in \mathcal{S}(T)$, then $Q \cap R \in \mathcal{S}(R)$. As $R \subseteq T$ is a strongly divided extension, we get that $Q = Q \cap R$. Thus $\mathcal{S}(T) \subseteq \mathcal{S}(R)$.

For the reverse inclusion, consider any $p \in \mathcal{S}(R)$. Since $R \subseteq T$ satisfies LO and GU, there exists $P \in \mathcal{S}(T)$ such that $P \cap R = p$. As $R \subseteq T$ is a strongly divided extension, we get that $p = P \in \mathcal{S}(T)$. Thus $\mathcal{S}(R) \subseteq \mathcal{S}(T)$. This completes the proof of the "only if" assertion.

Suppose next that $\mathcal{S}(R) = \mathcal{S}(T)$. Let $Q \in \text{Spec}(T)$ satisfy $Q \cap R \in \mathcal{S}(R)$. As $R \subseteq T$ satisfies INC, we get $Q \in \mathcal{S}(T)$. Then, by arguing as in the second half of the first paragraph of this proof, we see that $Q = Q \cap R$. Thus (R, T) is a strongly divided extension.

(c) The proof of (c) can be extracted from the above proof of (b).

(d) Suppose the assertion fails. Then $R \subseteq T$ is a strongly divided extension and there exists $z \in T$ such that z is transcendental over R. Put $E := R[z] \in [R, T]$ and $Q := zE \in \text{Spec}(E)$. Note that $Q \cap R = \{0\}$ by transcendentality, and $\{0\} \in \mathcal{S}(R)$ since R is not a field. As $z \in Q \setminus R$ and $R \subset E$ is a strongly divided extension, we have the desired contradiction.

(e) By (d), it suffices to show that if (R is quasi-local with $\dim(R) = 1$ and) $R \subseteq T$ is an algebraic extension, then (R, T) is a strongly divided pair. As each $E \in [R, T]$ is an algebraic extension of R, it suffices to prove that $R \subseteq T$ is a strongly divided extension. Our task is to show that if $Q \in \text{Spec}(T)$ satisfies $Q \cap R \in \mathcal{S}(R)$, then $Q \subseteq R$. Since $\dim(R) = 1$ and R is a domain, the only element of $\mathcal{S}(R)$ is $\{0\}$. Thus $Q \cap R = \{0\}$. By [19, Lemma 11.1], it follows from the algebraicity of the extension $R \subseteq T$ that $Q = \{0\}$, and so $Q \subseteq R$, as desired.

(f) By (d), T is algebraic over R. Hence, *a fortiori*, T is algebraic over $\overline{R}$. Since $\overline{R}$ is integrally closed in T, it follows from [21, Exercise 35, page 44] that T is an overring of $\overline{R}$. Thus, it suffices to prove that $\overline{R}$ is an overring of R. Therefore, by a harmless *abus de language*, we may assume, without loss of generality, that T is integral over R (that is, that $\overline{R} = T$). Consequently, by (b), $\mathcal{S}(R) = \mathcal{S}(T)$. As $\dim(R) > 1$, there exists a nonzero prime ideal $p \in \mathcal{S}(R)$. Then $p \in \mathcal{S}(T)$, whence $pT = p$. Pick a nonzero element $r \in p$. If $t \in T$, we have $rt \in pT = p \subseteq R$, and so $t = (rt)/r \in \text{qf}(R)$. Therefore, T is an overring of R. □

The statement of Proposition 2.1(c) could be simplified by deleting the "LO" hypothesis, since GU ⇒ LO [21, Theorem 42]. We used an explicit "LO" hypothesis in that result for two reasons: the statement can be read more smoothly and the applications will be to integral extensions (where the fact that both LO and GU hold can go without saying).

Some familiar examples of transitive binary relations on the class of ring extensions include INC, LO, GU, "integral extension," and "integrally closed extension." Despite expectations that may have been raised by Proposition 2.1(b), Example 2.2(a) will show that "strongly divided extension" cannot be added to that list. Example 2.2(b) presents a counterpoint to Proposition 2.1(a) by examining a ring extension whose larger ring is a field.

Example 2.2 (a) There exist domains $A \subseteq B \subseteq C$ such that $A \subseteq B$ and $B \subseteq C$ are strongly divided extensions but $A \subseteq C$ is not a strongly divided extension. (By Proposition 2.1(b), it cannot be the case that C is integral over A.) One way to construct such data is to take $A := \mathbb{Z}_{2\mathbb{Z}}$, $B := \mathbb{Q}$ and $C := \mathbb{Q}[X]$, where X is transcendental over $\mathbb{Q}$.

(b) There exist domains $A \subseteq B$ such that $A \subseteq B$ is a strongly divided extension but (A, B) is not a strongly divided pair. One way to construct such data is to take A to be any quasi-local domain which is not a strongly divided domain and $B := \mathrm{qf}(A)$.

Proof (a) Note that $A \subseteq B$ are domains, with A quasi-local and $\dim(A) = 1$, such that B is algebraic over A. Hence, by Proposition 2.1(e), (A, B) is a strongly divided pair, and so $A \subseteq B$ is a strongly divided extension. Moreover, Proposition 2.1(a) ensures that $B \subseteq C$ is a strongly divided extension. However, $A \subseteq C$ is not a strongly divided extension. To see this, note that $Q := XB \in \mathrm{Spec}(C)$ satisfies

$$Q \cap A = Q \cap (B \cap A) = (Q \cap B) \cap A = \{0\} \cap A = \{0\} \in \mathcal{S}(A),$$

although $Q \nsubseteq A$ (since $X \in Q \setminus A$).

(b) If D is any quasi-local domain and F is a field that contains D as a subring, then $D \subseteq F$ is a strongly divided extension. (Indeed, if $Q \in \mathrm{Spec}(F)$, then $Q = \{0\} \subseteq D$.) Thus, $A \subseteq B$ is a strongly divided extension. But (A, B) is not a strongly divided pair precisely because A is not a strongly divided domain. For an example of a divided (hence quasi-local) domain which is not a strongly divided domain, see [3, Example 3]. □

Remark 2.3 (a) Parts (b) and (c) of Proposition 2.1 can be motivated by Proposition 2.1(a), since one can use (a) to prove that both (b) and (c) hold in case R is a field. But the conclusion of Proposition 2.1(d) would fail if we deleted the hypothesis that R is not a field. To see this, consider the ring extension $F \subset F[X]$ where F is a field and X is transcendental over F. Then F is a quasi-local domain, $F[X]$ is a domain and $(F, F[X])$ is a strongly divided pair by Proposition 2.1(a), but $F[X]$ is not algebraic over F.

(b) Proposition 2.1(e) is best-possible as one cannot add "T is an overring of R" as an equivalent condition. For what may be the easiest example showing this, take R to be the one-dimensional quasi-local domain $\mathbb{Z}_{2\mathbb{Z}}$ and take T to be the algebraic (in fact, integral) extension $\mathbb{Z}[i]_{2\mathbb{Z}}$, where $i := \sqrt{-1} \in \mathbb{C}$. By Proposition 2.1(e), (R, T) is a strongly divided pair but, of course, T is not an overring of R. This example also shows that the equivalent conditions in Proposition 2.1(b) do not imply that T is an overring of R.

(c) By [12, Corollary 3.2], any LO-pair is a GU-pair. Therefore, by using the proof of Proposition 2.1(b), we can conclude that if $R \subseteq T$ are domains and R is quasi-local such that (R, T) is both an INC-pair and an LO-pair, then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(T)$. However, this conclusion is not new, as it is part of Proposition 2.1(b), the point being that if $A \subseteq B$ are commutative rings (with the same identity), then B is integral over A if (and only if) (A, B) is both an INC-pair and an LO-pair (cf. [12, Theorem 2.1, Corollary 2.4 (bis)]).

The interplay between strongly divided extensions (or strongly divided pairs) and strongly divided domains goes far beyond what we saw in Example 2.2(b). One soupçon of this theme is given in the next easy result.

Proposition 2.4 *Let $R \subset T$ be domains such that R is quasi-local and (R, T) is a strongly divided pair (resp., a strongly divided extension). If $P \in \text{Spec}(T)$ and $p := P \cap R \in \text{Spec}(R)$, then $(R/p, T/P)$ is a strongly divided pair (resp., a strongly divided extension).*

Proof We will prove the assertion concerning pairs, leaving the similar proof of the parenthetical assertion to the reader. Let m denote the maximal ideal of R. If $p = m$, the assertion follows from Proposition 2.1(a). Hence, without loss of generality, $p \subset m$; that is, $p \in \mathcal{S}(R)$. Then, since the hypotheses ensure that $R \subseteq T$ is a strongly divided extension, $P = p$ is a common prime ideal of R and T. Let $E' \in [R/p, T/P] = [R/p, T/p]$. Then $E' = E/p$ for a (uniquely determined) ring $E \in [R, T]$. Let $Q' \in \text{Spec}(E')$ satisfy $Q' \cap (R/p) \in \mathcal{S}(R/p)$. Our task is to show that $Q' \subseteq R/p$. We have $Q' = Q/p$ for a (uniquely determined) $Q \in \text{Spec}(E)$ such that $p \subseteq Q$. Then

$$(Q \cap R)/p = (Q/p) \cap (R/p) = Q' \cap (R/p) \subset m/p,$$

and so $Q \cap R \subset m$; that is, $Q \cap R \in \mathcal{S}(R)$. As the hypotheses also ensure that $R \subseteq E$ is a strongly divided extension, we have $Q \subseteq R$, whence $Q' = Q/p \subseteq R/p$, as desired. □

The following characterizations of strongly divided extensions will lead to deeper results on strongly divided pairs.

Proposition 2.5 *Let $(R, m) \subseteq T$ be domains such that R is quasi-local. Then the following conditions are equivalent:*

(1) $R \subseteq T$ is a strongly divided extension;
(2) For every $u \in T \setminus R$, $m \subseteq \text{Rad}_T(uT)$;
(3) For every ideal I of T, either $I \subseteq m$ or $m \subseteq \text{Rad}_T(I)$;
(4) Every prime ideal of T is comparable to m.

Proof (1) ⇒ (2): Assume (1). If u is a unit of T, then $uT = T$, whence $\text{Rad}_T(uT) = \{t \in T \mid \text{there exists a positive integer } n \text{ such that } t^n \in T\} = T$, and so $m \subset Rad_T(uT)$. Thus, it now suffices to prove that if $u \in T \setminus R$ and u is not a unit of T, then $m \subseteq Rad_T(uT)$. Since u is a nonunit of T, we can pick $Q \in \text{Spec}(T)$ such that $u \in Q$. If $Q \cap R \subset m$, then $Q \cap R \in \mathcal{S}(R)$, whence $Q \cap R = Q$ via (1), and so $u \in Q \cap R \subseteq R$, a contradiction (since $u \notin R$). Therefore, $Q \cap R = m$ for each prime ideal Q of T that contains u. As $\text{Rad}_T(uT)$ is the intersection of all such Q, we get $m \subseteq \text{Rad}_T(uT)$.

(2) ⇒ (3): Assume (2). We will prove that if I is an ideal of T such that $I \nsubseteq m$, then $m \subseteq \text{Rad}_T(I)$. Pick an element $u \in I \setminus m$. By (2), $m \subseteq \text{Rad}_T(uT)$. The assertion follows since $uT \subseteq I$ gives $\text{Rad}_T(uT) \subseteq \text{Rad}_T(I)$.

(3) ⇒ (4): This is clear since $\text{Rad}_T(I) = I$ for all $I \in \text{Spec}(T)$.

(4) ⇒ (1): Suppose the assertion fails. Then there exists $Q \in \text{Spec}(T)$ such that $Q \cap R \in \mathcal{S}(R)$ (that is, $Q \cap R \subset m$) and $Q \nsubseteq m$. Then (4) gives $m \subset Q$, whence $m \subseteq Q \cap R$, a contradiction (to $Q \cap R \subset m$). □

We pause to recall some background from [5]. Let $A \subseteq B$ be domains. We say that A is VD *in* B if $u^{-1} \in A$ whenever $u \in B \setminus A$. Also, we say that A is PV *in* B if $u^{-1}a \in A$ whenever $u \in B \setminus A$ and a is a nonunit of A. (For our purposes, it is useful to note that if, in addition, (A, m) is quasi-local, then A is PV in B if and only if $u^{-1}m \subseteq m$ for all $u \in B \setminus A$.) It is clear that if A is VD in B, then A is PV in B. The converse fails. Indeed, for any domain A, A is VD (resp., PV) in $\mathrm{qf}(A)$ if and only if A is a valuation domain (resp., a pseudo-valuation domain). In general, if A is PV in B, then A is quasi-local [5, Proposition 1.7]. If A is VD in B, then both A and B are quasi-local [5, Corollary 1.6].

Proposition 2.6 shows that the properties whose definitions were just recalled each give a sufficient condition for (R, T) to be a strongly divided pair.

Proposition 2.6 *Let $R \subseteq T$ be domains such that R is quasi-local. If R is* PV *in T (for instance, if R is* VD *in T), then (R, T) is a strongly divided pair.*

Proof We will show that if $E \in [R, T]$, then $R \subseteq E$ is a strongly divided extension. By Proposition 2.5, it is enough to prove that if $u \in E \setminus R$, then $m \subseteq \mathrm{Rad}_E(uE)$. Since R is PV in T, it is clear that R is PV in E. Therefore, by [5, Proposition 1.8], $u^{-1}m \subseteq m$. Hence, $m \subseteq um \subseteq uE \subseteq Rad_E(uE)$, as desired. □

The converse of Proposition 2.6 is false. The most natural way to show this would take R to be a strongly divided domain that is not a pseudo-valuation domain and $T := \mathrm{qf}(R)$. Examples of such R can be found in [3, Examples 1 and 2, Remark 4].

We next recall more background. Let $A \subseteq B$ be domains. Then (A, B) is said to be a *normal pair* if each $E \in [A, B]$ is integrally closed in B. It is a classic result of E. D. Davis (cf [19, Theorems 24.3 and 26.2]) that if D is a domain, then $(D, \mathrm{qf}(D))$ is a normal pair if and only if D is a Prüfer domain. For a systematic study of normal pairs (of domains) by Davis, see [8]. Several authors have studied normal pairs for certain types of rings that need not be domains. These studies include [16], where the rings A and B were usually taken to be complemented rings (note that any domain is complemented); and [22], where "(A, B) is a normal pair" was dubbed "A is a B-Prüfer ring" and A and B were taken to be arbitrary (commutative unital) rings. Although Davis assumed as a riding hypothesis that B is an overring of the domain A, it can be shown that if (A, B) is a normal pair, then B is (A-algebra isomorphic to) an overring of A (that is, a ring extension of A that is contained in the total quotient ring of A) for complemented rings (and hence for domains): see [16, Proposition 3.4]. It is easy to see but important (and apparently not noted until [16, Proposition 3.1]) that being a normal pair is a local property: for rings $A \subseteq B$, (A, B) is a normal pair if and only if (A_M, B_M) is a normal pair for each $M \in \mathrm{Max}(A)$. Thus, the following result of Davis [8, Proposition 1] is particularly relevant: if $A \subseteq B$ are domains such that A is quasi-local, then (A, B) is a normal pair if and only if each $u \in B \setminus A$ satisfies $u^{-1} \in A$. When one combines this result with the material recalled above from [5], one gets Corollary 2.7 which connects the two sets of recently recalled materials.

Corollary 2.7 *Let $R \subseteq T$ be domains such that R is quasi-local. Then:*

(*a*) *(R, T) is a normal pair if and only if R is a* VD *in T.*
(*b*) *If (R, T) is a normal pair, then (R, T) is a strongly divided pair.*

Proof (a) Combine [8, Proposition 1] with [5, Definition 0.1].

(b) Combine (a) with the parenthetical assertion in Proposition 2.6. □

By combining Proposition 2.6 with Corollary 2.7(a), we see that the converse of Corollary 2.7(b) is false. In other words, a strongly divided pair (R, T) need not be a normal pair, even if R is (quasi-local and) integrally closed in T. The most natural example showing this would take R to be an integrally closed pseudo-valuation domain that is not a valuation domain and $T := \mathrm{qf}(R)$; one such R is $k + Yk(X)[[Y]]$, where k is a field and X and Y are independent indeterminates over k.

The next result examines the behavior of some of the above properties under localization.

Proposition 2.8 *Let $R \subseteq T$ be domains such that (R, T) is a strongly divided pair (necessarily with (R, m) quasi-local). Let $p \in \mathcal{S}(R)$. Then:*

(a) (R_p, T_p) is a strongly divided pair.
(b) $\overline{R}_p := \overline{R}_{R\setminus p} = \overline{R}_{\overline{R}\setminus p}$ is the integral closure of R_p in T_p.
(c) $\overline{R}_p$ is VD in T_p.
(d) R_p is PV in T_p.

Proof It will be convenient in this proof to let $N := R \setminus p$. In view of Proposition 2.6, (a) will follow from (d). However, we believe that the direct proof of (a) given below is of some interest.

(a) Let $E' \in [R_p, T_p]$ and $Q' \in \mathrm{Spec}(E')$ such that $Q' \cap R_p \in \mathcal{S}(R_p)$, that is, such that $Q' \cap R_p \subset pR_p$. Our task is to show that $Q' \subseteq R_p$. Note that there exists a (uniquely determined) $E \in [R, T]$ such that $E' = E_N$. Put $Q := Q' \cap E$ and $q := Q \cap R$. Then $Q' = QE_N$. In addition, since the formation of rings of quotients commutes with finite intersections,

$$Q' \cap R_p = (Q')_N \cap R_N = (Q' \cap R)_N = q_N = qR_p.$$

As $q \subseteq p \subset m$ and $R \subseteq E$ is a strongly divided extension, $Q = q$. Since R and E share the ideal Q, it follows that R_N $(= R_p)$ and E_N $(= E')$ share the ideal $QR_N = QE_N = Q'$. Thus $Q' \subseteq R_p$, as desired.

(b) Since $R \subseteq \overline{R}$ is a strongly divided extension and $p \in \mathcal{S}(R)$, it follows from Proposition 2.1(b) that $p \in \mathcal{S}(\overline{R})$. In particular, p is a common prime ideal of R and $\overline{R}$. Therefore, [3, Lemma 1] gives that $\overline{R}_{R\setminus p} = \overline{R}_{\overline{R}\setminus p}$. The final assertion follows because integral closure commutes with the formation of rings of quotients [19, Proposition 10.2].

(c) As $\overline{R}_p$ is integrally closed in T_N by (b), it follows from [6, Theorems 2.5 and 2.3] that it will suffice to prove that $\overline{R}_p \subseteq E'$ satisfies INC for each $E' \in [\overline{R}_p, T_N]$. We will show even more, namely, that if P' and Q' are prime ideals of E' such that $P' \cap \overline{R}_p = Q' \cap \overline{R}_p$, then $P' = Q'$. Note that there exists a (uniquely determined) $E \in [\overline{R}, T]$ such that $E' = E_N$. We have $P' = PE_N$ and $Q' = QE_N$ for some (uniquely determined) prime ideals P and Q of E such that $P \cap N = \emptyset = Q \cap N$, that is, such that $P \cap R \subseteq p$ and $Q \cap R \subseteq p$. In fact, $P \cap R = Q \cap R$ since

$$P \cap R = (P' \cap E) \cap R = (P' \cap E) \cap (R_p \cap R) = (P' \cap R_p) \cap R$$

and, similarly, $Q \cap R = (Q' \cap R_p) \cap R$. As $R \subseteq E$ is a strongly divided extension and $q := P \cap R = Q \cap R$, we get $P = q = Q$, whence $P' = PE_N = QE_N = Q'$, as desired.

(d) We must show that if $u \in T_N \setminus R_p$ and a is a nonunit of R_p, then $au^{-1} \in R_p$. Recall from the proof of (b) that p is a common nonmaximal prime ideal of R and $\overline{R}$. As the (integral) extension $R_p \subseteq \overline{R}_p$ satisfies INC, it follows that R_p and $\overline{R}_p$ share the common maximal ideal $pR_p = p\overline{R}_p$. Since R_p is quasi-local with unique maximal ideal pR_p, it follows from [2, Proposition 3.8] that the same is true of $\overline{R}_p$. (The fact that $\overline{R}_p$ is quasi-local can also be seen by combining (c) and [5, Corollary 1.6].) Thus, if $u \in \overline{R}_p$, then u must be a unit of $\overline{R}_p$ (that is, $u^{-1} \in \overline{R}_p$) since $u \notin pR_p$. On the other hand, if $u \notin \overline{R}_p$, then it follows from (c) that $u^{-1} \in \overline{R}_p$. Thus, in all cases, $u^{-1} \in \overline{R}_p$. As $a \in pR_p$, we have

$$au^{-1} \in (pR_p)\overline{R}_p = p\overline{R}_p = pR_p \subseteq R_p,$$

as desired. □

It is possible to use the above background material to give the following quick proof of Proposition 2.1(f). Since $\dim(R) > 1$, there exists a nonzero prime ideal $p \in \mathcal{S}(R)$. By Proposition 2.8(d), R_p is PV in T_p. Then $\mathrm{qf}(R_p) = \mathrm{qf}(T_p)$. In other words, $\mathrm{qf}(R) = \mathrm{qf}(T)$; that is, T is an overring of R.

The converse of Proposition 2.8(d) fails severely. In other words, it is possible for (R, T) not to be a strongly divided pair even if $R \subseteq T$ are domains such that R is quasi-local and R_p is PV in T_p for every $p \in \mathcal{S}(R)$: see Remark 2.9(a).

Remark 2.9 (a) If $2 \leq d \leq \infty$, there exist domains $R \subseteq T$ such that $\dim(R) = d$, R is quasi-local, R is integrally closed, and R_p is PV in T_p for each $p \in \mathcal{S}(R)$, but (R, T) is not a strongly divided pair.

For a proof, consider the domain R constructed in [3, Example 3] and put $T := \mathrm{qf}(R)$. It was shown in [3] that R is a d-dimensional integrally closed divided (hence quasi-local) domain, but not a strongly divided domain, such that R_p is a PVD for each $p \in \mathcal{S}(R)$. The final two assertions hold since $T_p = \mathrm{qf}(R)$ for each $p \in \mathcal{S}(R)$.

(b) We have seen that the example in (a) satisfies the conclusion (d) in Proposition 2.8. Hence, by Proposition 2.6, the example in (a) also satisfies the conclusion (a) in Proposition 2.8. Since R is integrally closed, conclusion (b) in Proposition 2.8 is trivially satisfied. We leave open the question whether R satisfies the conclusion (c) in Proposition 2.8.

(c) We wish to mention two interesting features of the example R in (a) that are not explicitly connected to Proposition 2.8. The first of these states the following. If $p \in \mathrm{Spec}(R)$, then R/p is a divided domain. A proof of this assertion is immediate via [10, Lemma 2.2(c)], which applies since R is a divided domain.

The second feature that we note here is that (R, T) fails to exhibit a characteristic property of normal pairs having a quasi-local base. (This is perhaps not surprising in view of Corollary 2.7(b) and the fact that R is not a strongly divided domain.) A result of Davis [8, Theorem 1] asserts that if $A \subseteq B$ are domains with A quasi-local, then (A, B) is a normal pair if and only if there exists a divided prime ideal P of A such that $B = A_P$ and A/P is a valuation domain. Notice that when these equivalent conditions hold, A is the pullback $A_P \times_L A/P$, where $L = \mathrm{qf}(A/P)$ is the residue field of B. On the other hand, since each $p \in \mathrm{Spec}(R)$ is a divided prime ideal of R, there is a natural pullback description of R, namely, as $R_p \times_F R/p$, where $F = \mathrm{qf}(A/P)$ is the residue field of R_p.

It may not be apparent whether one should expect the existence of a (divided) nonzero prime ideal $p \in \mathcal{S}(R)$ such that (the integrally closed divided domain) R/p is a valuation domain. For the case $d = 2$, where there is only one nonzero prime ideal $p \in \mathcal{S}(R)$, the data in [3, Example 3] lead to $R/p \cong k[X]_{(X)}$ (where X is an indeterminate over a field k). So this R/p is a one-dimensional valuation domain, and Davis' result implies that (R, R_p) is a normal pair. However, (R, T) is not a normal pair, for $T = R_{(0)}$ but $R/(0)$ ($\cong R$) is not a valuation domain.

(d) A glance at [3, Example 3] reveals that the domain R in (a) was constructed using pullbacks. Several relevant examples later in this paper will also involve pullback constructions. For the sake of completeness, we next sketch an example with the properties announced in (a) that is distinct from the example given in (a). Like that earlier example, the example given below is also constructed using pullbacks.

Another way to construct data (R, T) with the properties announced in (a) is as follows. Let K be a field, with X and Y algebraically independent indeterminates over K. Put $V := K + M_1$ and $T := K(X) + N_1$, where $M_1 = XK[X]_{(X)} + YK(X)[Y]_{(Y)}$ and $N_1 = (Y+1)K(X)[Y]_{(Y+1)}$. Then put $E := V \cap T$, $M := M_1 \cap E$, $N := N_1 \cap E$, $m := M \cap N$ and $R := K + m$.

We next outline a proof that (R, T) has the asserted properties, omitting some explanatory details when they are classical. Note that V and T are incomparable valuation domains, with maximal ideals M_1 and N_1, respectively. Set $P := YK(X)[Y]_{(Y)}$. By standard facts about pullbacks (as in [18]), we have $\dim(V) = 2$, $\dim(T) = 1$, $\mathrm{Spec}(V)$ consists of the chain $(0) \subset P \subset M_1$, and $\mathrm{Spec}(T)$ consists of the chain $(0) \subset N_1$. Set $E := V \cap T$. As V and T have the same quotient field, E is a semi–quasi-local Prüfer domain with exactly two maximal ideals, $M := M_1 \cap E$ and $N := N_1 \cap E$, such that $E/M \cong V/M_1 \cong K$ and $E/N \cong T/N_1 \cong K(X)$ (cf. [21, Theorem 107]). Set $Q := Q_1 \cap E$. Then $\mathrm{Spec}(E)$ consists of two chains, $(0) \subset Q \subset M$ and $(0) \subset N$. Set $m := M \cap N$ and $R := K + m$. Then (R, m) is a quasi-local domain, and $\mathrm{Spec}(R)$ consists of the chain $(0) \subset q \subset m$, where $q := Q \cap R$. (Identifying $\mathrm{Spec}(R)$ is most easily done by first viewing R as the pullback of the diagonal map $K \to K \times K(X)$ and the projection map $E \to E/m \cong E/(M \cap N) \cong E/M \times E/N \cong K \times K(X)$ and then using the fundamental gluing result for pullbacks [18, Theorem 1.4].) Note that $q \neq Q$, since $q \subset m \subseteq N$ while $Q \not\subseteq N$. Thus $R \subset E$ is not a strongly divided extension, and so (R, T) is not a strongly divided pair. It remains only to verify that R_p is PV in T_p for each $p \in \mathcal{S}(R) = \{(0), q\}$. This is clear if $p = (0)$, since $R_{(0)} = K(X, Y) = T_{(0)}$; and it also holds if $p = q$, since $R_q = D_q$ is a (pseudo-)valuation domain with quotient field $K(X, Y) = T_q$.

Next, we give a helpful result that permits one to reduce certain questions concerning a (possibly strongly divided) pair (R, T) to the corresponding questions when (R, T) is replaced by a family of pairs of the form (D, E) such that D is integrally closed in E. Note that Proposition 2.10 is a generalization of [3, Proposition 3].

Proposition 2.10 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local. Then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $(\overline{R}_M, T_M)$ is a strongly divided pair for every $M \in \text{Max}(\overline{R})$.*

Proof Suppose that (R, T) is a strongly divided pair. As $(R, \overline{R})$ then inherits the "strongly divided pair" property, $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ by Proposition 2.1(b). Next, we must show that if $M \in \text{Max}(\overline{R})$ and $E' \in [\overline{R}_M, T_M]$, with $Q' \in \text{Spec}(E')$ such that $Q' \cap \overline{R}_M \in \mathcal{S}(\overline{R}_M)$, then $Q' \subseteq \overline{R}_M$. There exists (a uniquely determined) $E \in [\overline{R}, T]$ such that $E_M = E'$. There exists (a uniquely determined) $Q \in \text{Spec}(E)$ such that $Q' = QE_M$. As $Q' \cap \overline{R}_M \subset M\overline{R}_M$, we get

$$P := Q \cap \overline{R} = (Q' \cap E) \cap \overline{R} = (Q' \cap \overline{R}_M) \cap \overline{R} \subset M;$$

that is, $P \in \mathcal{S}(\overline{R})$. Since $\mathcal{S}(R) = \mathcal{S}(\overline{R})$, we get $P \in \mathcal{S}(R)$. Hence $P = P \cap R$. Then

$$Q \cap R = (Q \cap \overline{R}) \cap R = P \cap R = P \in \mathcal{S}(R).$$

As $R \subseteq E$ is a strongly divided extension, $Q = Q \cap R$, and so $Q = P$. Hence $Q' = QE_M = Q_M = P_M = P\overline{R}_M \subseteq \overline{R}_M$.

Conversely, suppose that $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $(\overline{R}_M, T_M)$ is a strongly divided pair for every $M \in \text{Max}(\overline{R})$. We must prove that if $E \in [R, T]$ and $P \in \text{Spec}(E)$ such that $p := P \cap R \in \mathcal{S}(R)$, then $P \subseteq R$ or equivalently, that $P \subseteq p$. Let $\overline{E}$ denote the integral closure of E in T. Since integrality ensures that $E \subseteq \overline{E}$ satisfies LO, we can pick $P' \in \text{Spec}(\overline{E})$ such that $P' \cap E = P$. Put $p' := P' \cap \overline{R}$. Then $p' \cap R = P' \cap R = P' \cap E \cap R = P \cap R = p \in \mathcal{S}(R) = \mathcal{S}(\overline{R})$. It follows that $p' = p$. Next, let $M \in \text{Max}(\overline{R})$. By integrality (more precisely, the fact that $R \subseteq \overline{R}$ satisfies GU), $M \cap R = m$. As $p \subset m \subseteq M$, we get that $p\overline{R}_M$ is a prime ideal of $\overline{R}_M$ that is properly contained in $M\overline{R}_M$; that is, $p\overline{R}_M \in \mathcal{S}(\overline{R}_M)$. Note that $P'\overline{E}_M \cap \overline{R}_M = p\overline{R}_M$. As $\overline{R}_M \subseteq \overline{E}_M$ is a strongly divided extension, $P'\overline{E}_M = p\overline{R}_M$. Consequently $P' \subseteq p\overline{R}_M$. As $M \in \text{Max}(\overline{R})$ was arbitrary, $P' \subseteq \bigcap_{M \in \text{Max}(\overline{R})} p\overline{R}_M$. By globalization, this intersection coincides with p. Thus $P' \subseteq p$. As $P \subseteq P'$, we get $P \subseteq p$, as desired. □

After the proof of Corollary 2.7, we showed that a strongly divided pair (A, B) need not be a normal pair. Another proof of this fact can be given by using Example 2.2(a), as that example showed that being a strongly divided pair is not a transitive property (whereas being a normal pair is a transitive property: see [22, Theorem 5.6] or [16, Proposition 3.9(a)]). To show that the emerging theory of strongly divided pairs shares at least some of the flavor of the well-established theory of normal pairs, it would be desirable to show that certain chains $A \subset B \subset C$ *do* exhibit transitivity of the "strongly divided pair" property. We will do so in Proposition 2.11. The setting for that result should, of course, not mirror that of Example 2.2(a). In that example, (A, B) was assumed to be a normal pair but (B, C) was not. The hypotheses of Proposition 2.11 will reverse the roles/nature of those subextensions.

Proposition 2.11 *Let $A \subseteq B \subseteq C$ be domains such that $A \subseteq B$ is an integral strongly divided extension (necessarily with A quasi-local) and (B, C) is a normal pair. Then (A, C) is a strongly divided pair.*

Proof By Proposition 2.1(b), $\mathcal{S}(A) = \mathcal{S}(B)$. Let $\overline{A}$ denote the integral closure of A in C. As B is integrally closed in C, we have $\overline{A} = B$. Thus, by Proposition 2.10, we need only to prove that (B_M, C_M) is a strongly divided pair for every $M \in \text{Max}(B)$. This, in turn, follows from Corollary 2.7(b), since (B_M, C_M) inherits the "normal pair" property from (B, C) [16, Proposition 3.1]. □

The next goal in this section is to generalize a result from [3] which showed that a quasi-local integrally closed domain is a strongly divided domain if and only if all its simple overrings are treed. First, for motivation and reference purposes, we give an easy result identifying some roles for the simple or 2-generated intermediate rings of a ring extension $A \subseteq B$ in studying whether (A, B) is a strongly divided pair or whether all the rings in $[A, B]$ are treed.

Proposition 2.12 *(a) Let $R \subseteq T$ be domains such that R is quasi-local. Then (R, T) is a strongly divided pair if and only if $R \subset R[u]$ is a strongly divided extension for each $u \in T \setminus R$.*

(b) Let $R \subseteq T$ be domains such that (R, T) is a strongly divided pair (and, necessarily, R is quasi-local, say with maximal ideal m). Then $R[u]$ is a treed domain for each $u \in T$ if and only if R is a treed domain.

(c) If $A \subseteq B$ are rings, then each ring in $[A, B]$ is treed if and only if $A[u, v]$ is treed for all $u, v \in B$.

Proof (a) The "only if" assertion is trivial. We will prove the contrapositive of the converse. Suppose, then, that there exists $Q \in \text{Spec}(T)$ such that $q := Q \cap R \in \mathcal{S}(R)$ and $q \subset Q$. Pick $u \in Q \setminus q$. Then $R \subset R[u]$ is not a strongly divided extension, as $Q' := Q \cap R[u] \in \text{Spec}(R[u])$ satisfies $Q' \cap R = Q \cap (R[u] \cap R) = Q \cap R = q$ and $u \in Q' \setminus q$.

(b) The "only if" assertion is trivial. We will suppose that the converse fails and seeks a contradiction. Thus, we are supposing that, for some $u \in T$ there exists a maximal ideal M of $R[u]$ which contains incomparable prime ideals Q_1 and Q_2 of $R[u]$. For each $i = 1, 2$, consider $p_i := Q_i \cap R$. As R is assumed to be treed, p_1 and p_2 are comparable. Thus, we need to only consider the following five cases.

Case 1: $p_1 = p_2 = m$. Then $M \cap R = m$ as well, and so $R \subseteq R[u]$ does not satisfy INC. So, by [11, Theorem], u is not primitive over R; that is, u is not the root of a polynomial over R with a unit coefficient. Let X be a transcendental element over R and let $\varphi : R[X] \to R[u]$ be the (surjective) R-algebra homomorphism sending X to u. We have $\ker(\varphi) \subseteq mR[X]$. Hence, by a standard isomorphism theorem, there are

R-algebra isomorphisms $R[u]/mR[u] \cong R[X]/mR[X] \cong (R/m)[X]$. Note that if $P \in \mathrm{Spec}(R[u])$ satisfies $P \cap R = m$, then $mR[u] \subseteq PR[u] = P$. Thus (again by a standard homomorphism theorem), either $P = mR[u]$ or P is one of the infinitely many incomparable prime ideals of $R[u]$ that properly contain $mR[u]$ (cf. [21, page 25]). However, since each of M, Q_1, and Q_2 must meet R in m, we get that $mR[u] \subset Q_1 \subset M$, the desired contradiction.

Case 2: $p_1 \subset p_2 = m$. Then $Q_1 = p_1$ since $R \subseteq R[u]$ is a strongly divided extension, and so $Q_1 \subset p_2 \subseteq Q_2$. Then $Q_1 \subset Q_2$, the desired contradiction.

Case 3: $p_2 \subset p_1 = m$. This case can be handled as in Case 2 by interchanging the subscripts "1" and "2."

Case 4: $p_1 \subseteq p_2 \subset m$. Then $Q_1 = p_1$ and $Q_2 = p_2$ since $R \subseteq R[u]$ is a strongly divided extension, and so $Q_1 \subseteq Q_2$, the desired contradiction.

Case 5: $p_2 \subseteq p_1 \subset m$. This case can be handled as in Case 4 by interchanging the subscripts "1" and "2."

(c) The "only if" assertion is trivial. We will prove the contrapositive of the converse. Suppose, then, that there exist $C \in [A, B]$ and $Q_1, Q_2, Q_3 \in \mathrm{Spec}(C)$ such that $Q_1 \subseteq Q_3$ and $Q_2 \subseteq Q_3$, while Q_1 and Q_2 are incomparable. Pick $u \in Q_1 \setminus Q_2$ and $v \in Q_2 \setminus Q_1$. Set $Q'_i := Q_i \cap A[u, v] \in \mathrm{Spec}(A[u, v])$ for $i = 1, 2, 3$. Then $A[u, v]$ is not treed. Indeed, $Q'_1 \subseteq Q'_3$ and $Q'_2 \subseteq Q'_3$, but Q'_1 and Q'_2 are incomparable, since $u \in Q'_1 \setminus Q'_2$ and $v \in Q'_2 \setminus Q'_1$. □

It may be useful for future reference purposes to record that the proof of Proposition 2.12 did not make significant use of the hypothesis that the rings of interest are domains.

We can now give a treed-theoretic characterization of the integrally closed strongly divided pairs whose base ring is treed. Theorem 2.13 generalizes part of [3, Corollary 3].

Theorem 2.13 *Let $R \subseteq T$ be domains such that R is integrally closed in T and (R, m) is quasi-local and treed. Then (R, T) is a strongly divided pair if and only if $R[u]$ is treed for each $u \in T$.*

Proof Suppose first that R is a field. Then (R, T) is a strongly divided pair by Proposition 2.1(a); and if $u \in T$, then the domain $R[u]$ is treed because $\dim(R[u]) \leq 1$. Thus, we can assume, without loss of generality, that R is not a field.

Next, suppose that (R, T) is a strongly divided pair. Then, since R is treed by assumption, Proposition 2.12(b) ensures that $R[u]$ is treed for each $u \in T$.

Conversely, suppose that $R[u]$ is treed for each $u \in T$. We will show that (R, T) is a strongly divided pair by adapting most of the proof of the implication (3) $\Rightarrow$ (1) in [3, Theorem 1, page 137]. To do so, we must verify two facts: first, that T is algebraic over R; and then, second, that T is an overring of R (inside $\mathrm{qf}(R)$). We get the first of these facts, for if $\xi \in T$ were transcendental over R, then $R[\xi]$ would not be treed (because R is not a field), contrary to hypothesis; and then we get the second fact via [21, Exercise 35, page 44]. One can now check that the argument on [3, page 137] carries over, with the following four changes: choose $u \in Q'' \setminus P'$ (rather than $u \in Q' \setminus P'$), to ensure that $u \in T$; then define Q as $Q'' \cap R[u]$ (rather than as $Q' \cap R[u]$); get $u^{-1} \notin R$ (because one would otherwise have $1 = uu^{-1} \in M'$, a contradiction); and when enlarging M, ensure that $mR[u]$ is properly contained in M. The proof is complete □

We can now generalize part of a main result on strongly divided domains [3, Theorem 1].

Corollary 2.14 *Let $R \subseteq T$ be domains such that (R, m) is a quasi-local treed domain and $\overline{R}$ is also treed. Then (R, T) is a strongly divided pair if and only if $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $\overline{R}[u]$ is a treed domain for each $u \in T$.*

Proof By Proposition 2.10, it is enough to show that $(\overline{R}_M, T_M)$ is a strongly divided pair for each $M \in \mathrm{Max}(\overline{R})$ if and only if $\overline{R}[u]$ is treed for each $u \in T$. For the moment, fix $M \in \mathrm{Max}(\overline{R})$ and an element $u \in T_M$. We can write $u = v/z$ for some $v \in T$ and $z \in \overline{R} \setminus M$. Then $\overline{R}_M[u] = \overline{R}_M[v/1, 1/z] = \overline{R}_M[v/1] = \overline{R}[v]_{\overline{R} \setminus M} =: \overline{R}[v]_M$. Hence, for each $M \in \mathrm{Max}(\overline{R})$, we have that $\overline{R}_M[u]$ is treed for each $u \in T_M$ if and only if $\overline{R}_M[v]$ is treed for each $v \in T$. On the other hand, if $v \in T$, then $\overline{R}[v]$ is treed if and only if $\overline{R}[v]_M$ is treed for each $M \in \mathrm{Max}(\overline{R})$; that is, if and only if $\overline{R}_M[v]$ is treed for each $M \in \mathrm{Max}(\overline{R})$. Thus, $\overline{R}[v]$ is treed for each $v \in T$ if and only if $\overline{R}_M[v]$ is treed for each $M \in \mathrm{Max}(\overline{R})$ and for each $v \in T$; that is, if and only if $\overline{R}_M[w]$ is treed for each $M \in \mathrm{Max}(\overline{R})$ and for each $w \in T_M$. It now suffices to prove the following: if $M \in \mathrm{Max}(\overline{R})$, then $(\overline{R}_M, T_M)$ is a strongly divided pair if and only if $\overline{R}_M[w]$ is treed for each $w \in T_M$. Note that $\overline{R}_M$ is a quasi-local domain which is integrally closed in T_M. By Theorem 2.13, it will suffice to have that $\overline{R}_M$ is treed. This fact, in turn, holds because of the hypothesis that $\overline{R}$ is treed. □

In view of Corollary 2.14, it is natural to ask if there exist examples of strongly divided pairs (R, T) such that R is treed (resp., not treed) and T is not treed. The answer is in the affirmative: see Example 4.1 (resp., Example 3.11). On the other hand, Proposition 4.2 will show that there does not exist a strongly divided pair (R, T) such that R is not treed and T is treed.

3 The Case $mT = T$

This section starts by characterizing a subcase that is admittedly special and rather easy, but it is also interesting and very useful. Note that if $p = \{0\}$ in Proposition 3.1, we recover the archetypical situation that was studied in [3].

Proposition 3.1 *Let (R, m) be a quasi-local domain and let $p \in \mathcal{S}(R)$. Then (R, R_p) is a strongly divided pair if and only if p is a divided prime ideal of R and R/p is a strongly divided domain.*

Proof Suppose first that (R, R_p) is a strongly divided pair. Since $R \subseteq R_p$ is a strongly divided extension and $pR_p \in \mathrm{Spec}(R_p)$ with $pR_p \cap R = p \in \mathcal{S}(R)$, it must be the case that $pR_p = p$; that is, p is a divided prime ideal of R. Furthermore, by Proposition 2.4, $(R/p, R_p/pR_p)$ is a strongly divided pair. As $R_p/pR_p = \mathrm{qf}(R/p)$, this means that R/p is a strongly divided domain.

Next, for the converse, suppose that p is a divided prime ideal of R and R/p is a strongly divided domain. Let $E \in [R, R_p]$ and $Q \in \mathrm{Spec}(E)$ such that $q := Q \cap R \in \mathcal{S}(R)$; that is, $q \subset m$. Our task is to prove that $Q = q$ (or, equivalently, that $Q \subseteq R$). Since p is a divided prime ideal of R, p and q are comparable (with respect to inclusion). We proceed to consider the two cases that can occur.

Case 1: $p \subseteq q$. Then $R/p \subseteq E/p \subseteq R_p/pR_p = \mathrm{qf}(R/p)$. As R/p is a strongly divided domain with maximal ideal m/p and $Q/p \cap R/p = (Q \cap R)/p = q/p \subset m/p \in \mathcal{S}(R/p)$, we get $Q/p = q/p$, whence $Q = q$.

Case 2: $q \subset p$. Then QE_p is a prime ideal of E_p. Since $R \subseteq E \subseteq R_p$ leads to $R_p \subseteq E_p \subseteq (R_p)_p = R_p$ (in general), we get $E_p = R_p$. Hence, $QE_p = QR_p \in \mathrm{Spec}(R_p)$ satisfies $QE_p \subseteq pR_p$. But $pR_p = p$ since p is a divided prime ideal of R, and so $QE_p \subseteq p \subseteq R$. Thus $Q \subseteq QE_p \subseteq R$, and so $Q \subseteq R$, as desired. □

We can now augment Remark 2.9(c) by noting another interesting feature of the d-dimensional domain studied there (that is, the domain R from [3, Example 3]) in case $(2 \leq) d < \infty$. It is the following concrete example. If $2 \leq d < \infty$, then R is a two-dimensional divided domain which is not a strongly divided domain but has a proper overring S such that (R, S) is a strongly divided pair. For a proof, one can take $S := R_q$, where q is the unique prime ideal of R of height $d - 1$, and apply Proposition 3.1.

Remark 2.9(d), as well as some examples in this section and the next section, makes explicit use of pullbacks. One should also note that pullbacks occasionally appear implicitly in this work. For instance, when the equivalent conditions in Proposition 3.1 hold, one has that $R \cong R/p \times_{R_p/pR_p} R_p$.

Lemma 3.2 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local. Then the following conditions are equivalent:*

(1) For each $u \in T \setminus R$, one has $u^{-1} \in T$;
(2) T is quasi-local and its maximal ideal is a divided prime ideal of R.

Proof If $R = T$, then both (1) and (2) hold. Hence, without loss of generality, $R \subset T$. It follows that the conductor $p := (R :_T T) := \{u \in T \mid uT \subseteq R\}$, the largest common ideal of R and T, is a proper ideal of both R and T (since $R \subset T$ ensures that $1 \notin (R :_T T)$).

(1) ⇒ (2): Assume (1). Let $Q \in \mathrm{Max}(T)$. If $0 \neq u \in Q$, then $u^{-1} \notin T$ (for otherwise, $1 = uu^{-1} \in QT = Q$, a contradiction). Hence, by (1), $(Q \setminus \{0\}) \cap (T \setminus R) = \emptyset$. It follows that $Q \subseteq R$. Thus, Q is a common ideal of R and T, and so $Q \subseteq (R :_T T) = p \subset R$. Therefore, T is quasi-local with unique maximal ideal p, which must therefore be a prime ideal of R. It remains only to show that p is a divided prime ideal of R. This, in turn, holds since $pR_p \subseteq pT_{R\setminus p} \subseteq pT_{T\setminus p} = pT = p$.

(2) ⇒ (1): Assume (2); that is, T is quasi-local with maximal ideal $P \in \mathrm{Spec}(R)$. Then the set of units of T is $T \setminus P$. To finish the proof, it is enough to observe that $T \setminus R \subseteq T \setminus P$. □

We can now give some characterizations of the strongly divided pairs $((R, m), T)$ such that $mT = T$.

Theorem 3.3 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local and the maximal ideal m of R satisfies $mT = T$. Then $R \subset T$ and the following three conditions are equivalent:*

(1) (R, T) is a strongly divided pair;
(2) T is quasi-local, the maximal ideal p of T is a divided prime ideal of R, and $(R/p, T/p)$ is a strongly divided pair;
(3) T is quasi-local, the maximal ideal p of T is a divided prime ideal of R, and either (i) $\dim(R/p) = 1$ and T/p is algebraic over R/p or (ii) $\dim(R/p) > 1$, $T = R_p$, and R/p is a strongly divided domain.

It is necessary in (2) and (3) that $p \in \mathcal{S}(R)$; and it is also necessary in (3)(ii) that $\mathrm{qf}(R/p) = T/p$.

Proof Of course, $R \subset T$ since $1 \in T = mT$ and $1 \notin m = mR$. We next establish the "It is necessary" assertion. If p is as in (2) or (3), then $p \in \mathcal{S}(R)$ (for otherwise, $p = m$ and then $1 \in T = mT = pT = p$, a contradiction). If (3) and its condition (ii) hold, then $\mathrm{qf}(R/p) = R_p/pR_p = T/p$ since $R_p = T$ and $pR_p = p$. We turn to proving that (1), (2), and (3) are equivalent.

(1) $\Rightarrow$ (2): Assume (1). Let $u \in T \setminus R$. If $u \in Q$ for some $Q \in \mathrm{Spec}(T)$ and $q := Q \cap R$ satisfies $q \in \mathcal{S}(R)$, then it follows from (1) that $Q = q \subseteq R$, so that $u \in \mathrm{Rad}_T(uT) \subseteq Q \subseteq R$, contrary to the choice of u. Therefore, if $u \in \mathfrak{Q}$ for some $\mathfrak{Q} \in \mathrm{Spec}(T)$, it must be the case that $\mathfrak{Q} \cap R = m$. As $mT = T$, no such $\mathfrak{Q}$ exists, and so $uT = T$; that is, $u^{-1} \in T$. Hence, by Lemma 3.2, T is quasi-local and its maximal ideal p is a divided prime ideal of R. Also, since $1 \in T = mT$ and $1 \notin p = pT$, we get $p \neq m$; that is, $p \in \mathcal{S}(R)$. Thus, by Proposition 2.4, $(R/p, T/p)$ is a strongly divided pair.

(2) $\Rightarrow$ (3): Assume (2). Then $R \subseteq R_p \subseteq T_p \subseteq T_{T\setminus p} = T$ since (T, p) is quasi-local. Hence, as p is a divided prime ideal of R, we get $R/p \subseteq R_p/p = R_p/pR_p = \mathrm{qf}(R/p) \subseteq T/p$. Since $(R/p, T/p)$ is a strongly divided pair, so is $(R/p, \mathrm{qf}(R/p))$; that is, R/p is a strongly divided domain. Also, as shown above, $p \in \mathcal{S}(R)$. Thus R/p is not a field. Therefore, since $(R/p, T/p)$ is a strongly divided pair, it follows from Proposition 2.1(d) that T/p is algebraic over R/p. It remains to prove that if $\dim(R/p) > 1$, then $T = R_p$. As we saw that $R_p \subseteq T$, it will suffice to prove the reverse inclusion. By Proposition 2.1(f), T/p is an overring of R/p; that is, $T/p \subseteq \mathrm{qf}(R/p) = R_p/p$. Hence $T \subseteq R_p$.

(3) $\Rightarrow$ (1): Assume (3). Let $E \in [R, T]$ and $Q \in \mathrm{Spec}(E)$ such that $q := Q \cap R \in \mathcal{S}(R)$; that is, $q \subset m$. Our task is to prove that $Q \subseteq R$ (or, equivalently, $Q = q$). Note that p is a common prime ideal of R, E, and T. Also, p and q are comparable because p is a divided prime ideal of R. There are two cases.

Case 1: $p \nsubseteq q$. Then $p \nsubseteq Q$. In addition, $E_Q = T_Q := T_{E\setminus Q}$. (This can be shown as in the hint for [21, Exercise 41 (a), page 46].) Thus, there exists $Q' \in \mathrm{Spec}(T)$ such that $Q' \cap E = Q$. As (T, p) is quasi-local, $Q' \subseteq p$. Hence, $Q \subseteq Q' \subseteq p \subseteq R$, so that $Q \subseteq R$, as desired.

Case 2: $p \subseteq q$. Then $Q/p \in \mathrm{Spec}(E/p)$ satisfies $(Q/p) \cap (R/p) = (Q \cap R)/p = q/p \subset m/p$ since $q \in \mathcal{S}(R)$. Suppose first that $\dim(R/p) = 1$. Then $q/p = \{0\}$; that is, $p = q$. As T/p is algebraic over R/p, [19, Lemma 11.1] yields that $Q/p = \{0\}$; that is, $Q = p$. Hence $Q = q$, as desired. In the remaining subcase, $\dim(R/p) > 1$. Then, as noted above, $\mathrm{qf}(R/p) = T/p$. Since R/p is a strongly divided domain, $(R/p, T/p)$ is a strongly divided pair. As $E/p \in [R/p, T/p]$, it follows that $R/p \subseteq E/p$ is a strongly divided extension. As $q/p \in \mathcal{S}(R/p)$, we get $Q/p = q/p$, whence $Q = q$, as desired. □

The proof that (3) ⇒ (1) in Theorem 3.3 did not use the hypothesis that $mT = T$. We record this fact and slightly more in the next corollary.

Corollary 3.4 *Let $R \subseteq T$ be domains such that R and T are quasi-local, the maximal ideal p of T is a divided prime ideal of R, and either* (*i*) $\dim(R/p) \leq 1$ *and T/p is algebraic over R/p or* (*ii*) $\dim(R/p) > 1$*, $T = R_p$, and R/p is a strongly divided domain. Then (R, T) is a strongly divided pair.*

Proof It clearly suffices to show that $\mathrm{Spec}(E) = \mathrm{Spec}(R)$ (as sets) for all $E \in [R, T]$. Without loss of generality, we can assume that $R \neq T$. By the preceding comment, we can also assume that (i) holds with $\dim(R/p) = 0$. Then $p \in \mathrm{Max}(R)$; that is, p is the maximal ideal of R. As $\mathrm{Max}(R) = \{p\} = \mathrm{Max}(T)$, [2, Theorem 3.10] gives $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ (as sets). Hence, by [2, Corollary 3.26], it will suffice to have that T/p is algebraic over R/p. As this algebraicity was assumed in (i), the proof is complete. □

In case $\overline{R} = T$, condition (3) of Theorem 3.3 admits a more elegant formulation that is evocative of Proposition 3.1 and the archetypical context from [3]: see Corollary 3.5.

Corollary 3.5 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local, the maximal ideal m of R satisfies $mT = T$, and R is integrally closed in T. Then $R \subset T$ and the following three conditions are equivalent:*

(1) (R, T) is a strongly divided pair;
(2) T is quasi-local, the maximal ideal p of T is a divided prime ideal of R, and $(R/p, T/p)$ is a strongly divided pair;
(3) $T = R_p$ for some divided prime ideal p of R and R/p is a strongly divided domain.

It is necessary in (2) and (3) that $p \in \mathcal{S}(R)$; and it is also necessary in (3) that (T, p) is quasi-local and $\mathrm{qf}(R/p) = T/p$.

Proof Any domain of Krull dimension 1 is a strongly divided domain [3, Proposition 1(b)]. Hence, by Theorem 3.3, it suffices to prove that if (2) holds with $\dim(R/p) = 1$, then $T = R_p$. Assume (2). Then, as noted above (and in Theorem 3.3), $p \in \mathcal{S}(R)$, and so R/p is not a field. Since $(R/p, T/p)$ is a strongly divided pair, it follows from Proposition 2.1(d) that T/p is algebraic over R/p. However, as is well known (cf. [18, Corollary 1.5 (5)]), the hypothesis that R is integrally closed in T implies that R/p is integrally closed in T/p. Therefore, by [21, Exercise 35, page 44], T/p is an overring of R/p. Since $pR_p = p$, it follows that $T \subseteq R_p$. To get the reverse inclusion (and thus complete the proof), one can revisit the beginning of the proof that (2) ⇒ (3) in Theorem 3.3. □

Propositions 3.7 and 3.8 will deepen the comments in Sect. 2 about the failure of the converse of Proposition 2.6. It will be convenient to use the following notation. If $R \subseteq T$ are domains and T is quasi-local with maximal ideal p, let $p^* := \bigcup\{q \in \mathcal{S}(R) \mid p \subseteq q\}$.

Lemma 3.6 *Let $R \subseteq T$ be domains, with (R, m) quasi-local, such that (R, T) is a strongly divided pair and $mT = T$. Let p denote the maximal ideal of T. (T is quasi-local by Theorem 3.3.) Then:*

(a) If $q \in \mathcal{S}(R)$ and $p \subseteq q$, then q is a divided prime ideal of R.
(b) p^* is a divided prime ideal of R.

Proof (a) As $R \subseteq R_q \subseteq R_p \subseteq T_p \subseteq T$, we get that (R, R_q) inherits the "strongly divided pair" property from (R, T). Since $q \subset m$, (a) now follows from Proposition 3.1. (The preceding clause can be replaced with an appeal to Theorem 3.3, since $mR_q = R_q$.)

(b) Since $pT = p \subset T$ and $mT = T$, we get $p \subset m$; that is, $p \in \mathcal{S}(R)$. As R is quasi-local, it follows from (a) that p^* is the (directed) direct limit, with nonempty index set, of the set of all the (necessarily divided prime ideals) $q \in \mathcal{S}(R)$ such that $p \subseteq q$. Now, one need to only observe that the proof of [14, Proposition 2.5(b)] shows that any (directed) direct limit of a nonempty set of divided prime ideals of a domain D is a divided prime ideal of D. □

Proposition 3.7 *Let $R \subseteq T$ be domains, with both (R, m) and (T, p) quasi-local, such that $mT = T$ and $p^* = m$. Then (R, T) is a strongly divided pair if and only if R is* PV *in T.*

Proof By Proposition 2.6, it remains to show that if (R, T) is a strongly divided pair, then R is PV in T. Let $u \in T \setminus R$. As $u \notin m$, u is not an element of any $q \in \mathcal{S}(R)$ such that $p \subseteq q$. Fix $a \in m$. We must show that $u^{-1}a \in m$. As $p^* = m$, there exists $q_o \in \mathcal{S}(R)$ such that $p \subseteq q_o$ and $a \in q_o$. Since $T \subseteq T_{q_o} \subseteq T_p = T$, we get $T_{q_o} = T$, and so we have the inclusions $R \subset R_{q_o} \subseteq T_{q_o} = T$. Furthermore, (R, R_{q_o}) inherits the "strongly divided pair" property from (R, T) and $mR_{q_o} = R_{q_o}$ (since $q_o \subset m$). Also, since $p \subseteq q_o$, Lemma 3.6(a) ensures that $q_oR_{q_o} = q_o$. There are two possible cases.

Case 1: $u \in R_{q_o} \setminus R$. Then, by applying Lemma 3.2 to (R, R_{q_o}), we get $u^{-1} \in R_{q_o}$. Hence, $u^{-1}a \in q_oR_{q_o} = q_o \subseteq m$.

Case 2: $u \in T \setminus R_{q_o}$. Then, since R_{q_o} is PV in $T_{q_o} = T$ (by Proposition 2.8(d)) and a is a nonunit of R_{q_o}, we get $u^{-1}a \in q_oR_{q_o} = q_o \subseteq m$. □

If one deletes the hypothesis that $p^* = m$ in Proposition 3.7, the above "only if" assertion is false. For the case $p^* \neq m$, the following result provides necessary and sufficient conditions for R to be PV in T.

Proposition 3.8 *Let $R \subseteq T$ be domains, with (R, m) quasi-local, such that (R, T) is a strongly divided pair and $mT = T$. Suppose also that $p^* \subset m$, where p denotes the maximal ideal of T. Then R is* PV *in T if and only if R/p^* is a* PVD *and R_{p^*} is* VD *in T.*

Proof By Lemma 3.6(b), p^* is a divided prime ideal of R. Also, since $pT = p \subset T$ and $mT = T$, we get $p \subset m$; that is, $p \in \mathcal{S}(R)$. Consequently $p \subseteq p^*$, by the definition of p^*. Thus $T_{p^*} \subseteq T_p = T$, and so $T_{p^*} = T$. We now have $R \subset R_{p^*} \subseteq T$.

Suppose first that R is PV in T. As $p^* \subset m$, [5, Corollary 1.9] gives that R_{p^*} is VD in $T_{p^*} = T$. Moreover, R is PV in R_{p^*} (since R is PV in T and $R_{p^*} \in [R, T]$) and $mR_{p^*} = R_{p^*}$. Hence, by [5, Theorem 3.7], R/p^* is a PVD.

Conversely, suppose that R/p^* is a PVD and R_{p^*} is VD in T. We must show that if $u \in T \setminus R$, then $u^{-1}m \subseteq m$. There are two possible cases.

Case 1: $u \in R_{p^*}$. As R/p^* is a PVD and p^* is a divided prime ideal of R (by Lemma 3.6(b)), [5, Theorem 3.7] gives that R is PV in R_{p^*}. Hence, since $u \in R_{p^*} \setminus R$, we have $u^{-1}m \subseteq m$.

Case 2: $u \in T \setminus R_{p^*}$. Then $u^{-1} \in R_{p^*}$ since R_{p^*} is VD in T. Hence $u^{-1} \in p^*R_{p^*} = p^* \subseteq R$, and so $u^{-1}m \subseteq m$. □

To close the section, we build on the results from Sect. 2 concerning the interplay between strongly divided pairs and treed simple intermediate rings, working now in the case where $mT = T$.

Proposition 3.9 *Let $R \subseteq T$ be domains such that (R, T) is a strongly divided pair and $mT = T$, where m denotes the unique maximal ideal of R. Let p denote the unique maximal ideal of T. (Note that T is quasi-local by Theorem 3.3.) Then the following conditions are equivalent:*

(1) R is treed;
(2) T is treed;
(3) $\{Q \in \mathrm{Spec}(R) \mid Q \subseteq p\}$ is linearly ordered under inclusion;
(4) $R[u]$ is treed for each $u \in T$.

Proof We will prove that (3) ⇔ (2) and that (1) ⇔ (2). It will be convenient to let $\mathcal{U} := \{Q \in \mathrm{Spec}(R) \mid Q \subseteq p\}$.

By Theorem 3.3, T is quasi-local, its maximal ideal p is a divided prime ideal of R, and $(R/p, T/p)$ is a strongly divided pair. In addition, since any one-dimensional domain is a strongly divided domain [3, Proposition 1(b)], it also follows from Theorem 3.3 that R/p is a strongly divided domain, hence a divided domain [3, Proposition 1(a)], hence a quasi-local treed domain. Thus, as a poset under inclusion, $\mathrm{Spec}(R/p)$ is a linearly ordered set. This is relevant because R is the pullback $R/p \times_{T/p} T$. Then $\mathrm{Spec}(R)$ (with the Zariski topology) can be described up to homeomorphism by a fundamental gluing result [18, Theorem 1.4]. The order-theoretic upshot of the information from that gluing is that, as a poset, $\mathrm{Spec}(R)$ is order-isomorphic to the result of placing $\mathrm{Spec}(R/p)$ atop $\mathrm{Spec}(T)$ with $0 \in \mathrm{Spec}(R/p)$ identified with $p \in \mathrm{Spec}(T)$. This order-isomorphism has two important consequences. First, $\mathcal{U}$ is order-isomorphic to Spec(T), whence (3) $\Leftrightarrow$ (2). Second, since we have seen that Spec(R/p) is linearly ordered, R is (quasi-local and) treed $\Leftrightarrow$ Spec(R) is linearly ordered $\Leftrightarrow$ Spec(T) is linearly ordered; that is, (1) $\Leftrightarrow$ (2).

Finally, (1) $\Rightarrow$ (4) by Propositions 2.12(b); and (4) $\Rightarrow$ (1) trivially. □

The constructions in Examples 3.11 and 4.1 will depend in part on the example in the following remark.

Remark 3.10 We will use classical methods, similar to those in Remark 2.9(d), to construct a useful quasi-local domain which is not treed. Let k be a field and let X and Y be (commuting) algebraically independent indeterminates over k. Consider

$$V := k[Y]_{(Y)} + Xk(Y)[X]_{(X)} = k + Yk[Y]_{(Y)} + Xk(Y)[X]_{(X)} \text{ and}$$

$W := k[X]_{(X+1)} + Yk(X)[Y]_{(Y)} = k + (X+1)k[X]_{(X+1)} + Yk(X)[Y]_{(Y)}$. Then V and W are two incomparable valuations domains with maximal ideals $M' = Yk[Y]_{(Y)} + Xk(Y)[X]_{(X)}$ and $N' = (X+1)k[X]_{(X+1)} + Yk(X)[Y]_{(Y)}$, respectively. Set $P' := Xk(Y)[X]_{(X)}$ and $Q' := Yk(X)[Y]_{(Y)}$. Since V and W are incomparable valuation domains with the same quotient field, $H := V \cap W$ is a quasi–semi-local Prüfer domain whose prime spectrum consists of two chains, $(0) \subset P \subset M$ and $(0) \subset Q \subset N$, where $P := P' \cap H$, $Q := Q' \cap H$, $M := M' \cap H$ and $N := N' \cap H$. Moreover, P and Q are incomparable. Finally set $T := k + I$, where $I := M \cap N$. Then T is a quasi-local two-dimensional domain which is not treed. Indeed,

$$\text{Spec}(T) = \{(0), P \cap T, Q \cap T, I\},$$

where $(0) \subset P \cap T \subset I$, $(0) \subset Q \cap T \subset I$, and $P \cap T$ and $Q \cap T$ are incomparable.

Examples of strongly divided pairs $((R.m), T)$ with $mT = T$ such that both R and T are treed are commonplace: consider, for instance, $(R, \text{qf}(R))$ where R is any strongly divided domain which is not a field. In light of Proposition 3.9, it is natural to ask if there also exist examples of strongly divided pairs $((R.m), T)$ with $mT = T$ such that neither R nor T is treed. Example 3.11 will answer that question in the affirmative.

Example 3.11 There exists a strongly divided pair (R, T) of domains such that (R, m) is quasi-local but not treed, $mT = T$, and T is not treed. One way to construct such data is to take $F \subseteq L$ as an algebraic extension of fields; X an indeterminate over L; $k := L(X)$; T and I as in Remark 3.10; $D := F[X]_{(X)}$; and R the pullback $R := D \times_k T$.

Proof Recall that T is a quasi-local two-dimensional domain with unique maximal ideal I and exactly two (incomparable) height 1 prime ideals. In particular, T is not treed. Next, consider the pullback description that was used to define R. (That description makes sense since $T/I \cong k$.) An application of [18, Theorem 1.4] reveals that as a poset, Spec(R) is order-isomorphic to the result of placing Spec(D) atop Spec(T) with $(0) \in$ Spec(D) identified with $I \in$ Spec(T). As (D, XD) is a one-dimensional (quasi-)local domain, it follows that R is a three-dimensional quasi-local domain, say with maximal ideal m; the unique height 2 prime ideal of R is I; and the incomparable height 1 prime ideals of T intersect with R to give incomparable height 1 prime ideals of R. Thus R is not treed. Moreover, $mT = T$ (for otherwise, $mT \subseteq I$, a contradiction since $m \subseteq mT$ and $I \subset m$). It remains only to prove that (R, T) is a strongly divided pair.

Since $A := F(X) + I$ is the pullback $F(X) \times_k T$, an application of [18, Theorem 1.4] reveals that Spec$(A) =$ Spec(T). In particular, A is quasi-local, with unique maximal ideal I. Observe that $A/I \cong F(X)$ canonically. As R is the pullback $D \times_{F(X)} A$ and qf$(D) = F(X)$, it follows from [13, Lemma 2.5(v)] that I is a divided prime ideal of R. We claim that $(R/I, T/I)$ is a strongly divided pair. Indeed, this pair can be identified with (D, k), and *this* pair is strongly divided by Proposition 2.1(e), since $\dim(D) = 1$ and k is algebraic over D. This proves the claim. Therefore, by (verifying condition (2) in) Theorem 3.3, (R, T) is a strongly divided pair. At this point, the proof is complete, but we can now offer a second proof that R is not treed: combine Proposition 3.9 with the fact that T is not treed. □

4 The Case $mT \neq T$

In light of Proposition 3.9 and Example 3.11, it is natural to ask if there exists an example of a strongly divided pair $((R, m), T)$ such that exactly one of R, T is treed. Example 4.1 will answer this question in the affirmative. By Proposition 3.9, any such example must satisfy $mT \neq T$, and so its placement here seems a fitting way to begin this section.

Example 4.1 There exists a strongly divided pair (R, T_1) of domains such that (R, m) is a quasi-local treed domain, $mT_1 \neq T_1$, and the domain T_1 is not treed. One way to construct such data is to take k, I and T as in Remark 3.10; $K =$ qf(T); $V_1 := K[Z]_{(Z)} = K + m$, where Z is an indeterminate over K and $m = ZV_1$ is the unique maximal ideal of V_1; $T_1 := T \times_K V_1 = T + m = k + I + m$; and R the pullback $R = k \times_T T_1$.

Proof The pullback giving T_1 is well-defined since $V_1/m = K$ canonically. An application of [18, Theorem 1.4] to the above pullback description of T_1 reveals that as a poset, Spec(T_1) is order-isomorphic to the result of placing Spec(T) atop Spec(V_1) with $(0) \in$ Spec(T) identified with $m \in$ Spec(V_1). As (V_1, m) is a one-dimensional (quasi-)local domain, it follows that T_1 is a three-dimensional quasi-local domain with maximal ideal I; the (two) incomparable height 1 prime ideals of T become (after the gluing identification process) incomparable height 2 prime ideals of T_1; and the unique height 1 prime ideal of T_1 is m. In particular, the domain T_1 is not treed.

Note that the pullback giving R is well-defined since $T_1/m = T$ canonically and $k \subset k + I = T$. An application of [18, Theorem 1.4] to this pullback reveals that as a poset, $\mathrm{Spec}(R) = \{(0), m\}$, since the gluing process identifies each prime ideal of T_1 that contains m with $(0) \in \mathrm{Spec}(k)$. It follows that R is a one-dimensional quasi-local domain with maximal ideal m. In particular, R is a treed domain. Moreover, $mT_1 = m \neq T_1$. It remains only to show that (R, T_1) is a strongly divided pair. Since $\dim(R) = 1$, it follows from Proposition 2.1(e) that it suffices to prove that T_1 is algebraic over R. This, in turn, holds, since T_1 is an overring of R, the point being that R and T_1 share the nonzero common ideal m. □

Example 4.1 has shown that the condition on 2-generated intermediate rings in Proposition 2.12(c) cannot be weakened to the corresponding condition on 1-generated intermediate rings. Example 4.1 has also shown that the conclusion of Proposition 3.9 does not hold in general for the context studied in this section. More generally, this result indicates that the strongly divided pairs $((R, m), T)$ with $mT \neq T$ cannot be expected to behave as in the theory developed in Sect. 3. Thus, one should perhaps cautiously begin the theoretical development in this section by examining its simplest subcase. For domains $(R, m) \subseteq T$ such that R is quasi-local but not a field, it seems clear to us that the simplest sufficient condition for mT to be unequal to T is that m is an ideal of T (that is, that $mT = m$). Proposition 4.2 records some consequences of this sufficient condition.

Proposition 4.2 *Let $(R, m) \subseteq T$ be domains such that R is quasi-local. Assume also that m is an ideal of T. Then:*

(a) (R, T) is a strongly divided pair if and only if $R \subseteq T$ is a strongly divided extension.
(b) m is a maximal ideal of T if and only if $\mathrm{Spec}(R) = \mathrm{Spec}(T)$ (as sets).
(c) If m is a maximal ideal of T, then (R, T) is a strongly divided pair.
(d) Let $K := \mathrm{qf}(R)$, let $(m :_K m) := \{u \in K \mid um \subseteq m\}$, and let $\pi : (m :_K m) \to (m :_K m)/m$ be the canonical projection map. Then for any field $F \in [R/m, (m :_K m)/m]$, $E := \pi^{-1}(F)$ is a quasi-local domain with maximal ideal m and (R, E) is a strongly divided pair. Furthermore, $E \in [R, T]$ if and only $F \subseteq T/m$.

Proof Each of (a)–(d) holds if R is a field (that is, if $m = (0)$. Indeed, to see this for (a) and (c), use Proposition 2.1(a); for (b), note that if D is a domain, then (0) is the only prime ideal of D if and only if D is a field; and for (d), note that $m = (0)$ forces $(m :_K m) = K = R = F = E$ (where we do not distinguish between a ring A and $A/(0)$). Thus, we can assume henceforth that R is not a field.

(a) The "only if" assertion is trivial. It remains to prove that if $R \subseteq T$ is a strongly divided extension and $H \in [R, T]$, then $R \subseteq H$ is a strongly divided extension. Suppose that $Q \in \mathrm{Spec}(H)$ satisfies $p := Q \cap R \in \mathcal{S}(R)$; that is, $p \subset m$. Our task is to show that $Q \subseteq R$. Observe that $m \nsubseteq Q$ (for otherwise, $m = m \cap R \subseteq Q \cap R = p \subset m$, a contradiction). Since $m \neq (0)$, we can conclude that $H_Q = T_Q$ $(:= T_{H \setminus Q})$ by reasoning as in the hint given for [21, Exercise 41 (b), page 46]. (In detail, if $t \in T$, $z \in H \setminus Q$ and $a \in m \setminus Q$, then $at \in mT = m \subseteq R \subseteq H$ and $az \in H \setminus Q$, whence $t/z = (at)/(az) \in H_Q$.) It follows that $Q' := QH_Q \cap T$ is a (in fact, the unique) prime ideal of T such that $Q' \cap H = Q$. Hence $Q' \cap R = Q' \cap (H \cap R) = (Q' \cap H) \cap R = Q \cap R = q \in \mathcal{S}(R)$. Therefore $Q' \subseteq R$, since $R \subseteq T$ is a strongly divided extension, and *a fortiori*, $Q \subseteq R$, as desired.

(b) This is a restatement of [2, Proposition 3.8].

(c) By hypothesis, $m \in \text{Max}(T)$. Thus, by (b), $\text{Spec}(R) = \text{Spec}(T)$. As (R, m) is quasi-local, it follows that each prime ideal of T is comparable to m (with respect to inclusion). Therefore, by Proposition 2.5, $R \subseteq T$ is a strongly divided extension. Hence, by (a), $R \subseteq T$ is a strongly divided pair.

(d) We will first prove the "Furthermore" assertion. Since m is a nonzero common ideal of ideal of R and T, it follows that T is an overring of R and, in fact, $T \subseteq (m :_K m)$. As $R = \pi^{-1}(R/m) \subseteq \pi^{-1}(F) = E$ and $E/m = F$, we get that $E \in [R, T] \Leftrightarrow E \subseteq T \Leftrightarrow E/m \subseteq T/m \Leftrightarrow F \subseteq T/m$. Next, we will prove the first assertion in (d). By (c), it suffices to show that $m \in \text{Max}(E)$. Hence, by (b), it is enough to prove that $\text{Spec}(R) = \text{Spec}(E)$. This, in turn, follows from [2, Theorem 3.25] since F is a field. Finally, we offer the following sketch of an alternate way to see that $\text{Spec}(R) = \text{Spec}(E)$. View R as the pullback $R = E \times_F R/m$, apply the fundamental gluing result [18, Theorem 1.4] to this pullback to get a description (up to homeomorphism) of $\text{Spec}(R)$ with the Zariski topology, and consider the order-theoretic implications of that description. □

The next result introduces a condition that is reminiscent of, but distinct from, condition (4) in Proposition 2.5.

Proposition 4.3 *If (R, m) is a quasi-local domain and $R \subseteq T$ is a strongly divided extension such that $mT \neq T$, then each prime ideal of T is comparable to mT (with respect to inclusion).*

Proof Let $Q \in \text{Spec}(T)$ and put $q := Q \cap R$. If $q \subset m$, then $Q = q$ (by the "strongly divided" hypothesis), and so $Q \subset m \subseteq mT$, whence $Q \subset mT$. In the remaining case, $q = m$, and so $mT = qT \subseteq Q$. □

Recall that if A is a ring and $P \in \text{Spec}(A)$, then $\text{Spec}(A)$ is said to be *pinched at* P if each $Q \in \text{Spec}(A)$ is comparable to P (with respect to inclusion). Two trivial examples are the following: if (A, m) is a quasi-local ring, then $\text{Spec}(A)$ is pinched at m; if D is a domain, then $\text{Spec}(D)$ is pinched at (0). A less trivial example, which follows from Proposition 4.3, is the following: if (R, m) is a quasi-local domain and $R \subseteq T$ is a strongly divided extension such that $mT \in \text{Spec}(T)$, then $\text{Spec}(T)$ is pinched at mT. Combining this last example with Proposition 4.2(c) gives the following: if $R \subseteq E$ are domains such that (R, m) is quasi-local and $m \in \text{Max}(E)$, then (R, E) is a strongly divided pair and $\text{Spec}(R) = \text{Spec}(E)$ is pinched at m. Exactly the same conclusion holds for the data R, E from Proposition 4.2(d). As the ring R in Proposition 4.2(d) is the pullback $R/m \times_F E$, these examples suggest that it may be fruitful to build additional examples of strongly divided pairs $((R, m), T)$ such that $mT \neq T$ by using pullbacks that are pinched at some prime ideal. This is done in the next two results. Example 4.4 treats the case of integral extensions. Proposition 4.5 treats the remaining cases and is of some greater interest because the prime ideals at which the given spectra are pinched are not maximal ideals.

Example 4.4 Let $0 \le d \le \infty$ and let (T, m) be a quasi-local domain of Krull dimension d. Put $k := T/m$. Let F be a subfield of k such that k is algebraic over F. Let R be the pullback $R = F \times_k T$. Then:

(a) T is integral over R.
(b) For all $H \in [R, T]$, one has $H = H/m \times_k T$ and $\mathrm{Spec}(R) = \mathrm{Spec}(H) = \mathrm{Spec}(T)$, whence (H, T) is a strongly divided pair and $\mathrm{Spec}(H)$ is pinched at m.
(c) $\dim(H) = d$ for all $H \in [R, T]$.

Proof (a) Since k is algebraic over F, the conclusion follows from a standard fact about pullbacks [18, Corollary 1.5 (5)].

(b) As $H/m \in [F, k]$, H/m is a domain that is algebraic over the field F, and so H/m is a field. Then, since $H = H/m \times_k T$, applying [18, Theorem 1.4] to this pullback description leads to $\mathrm{Spec}(H) = \mathrm{Spec}(T)$, from which the rest of the assertion follows easily.

(c) Apply the fact, from (b), that $\mathrm{Spec}(H) = \mathrm{Spec}(T)$; for an alternate proof of (c), combine (a) with [21, Theorem 48]. □

Proposition 4.5 *Let T be a domain and let Q be a nonmaximal prime ideal of T such that* $\mathrm{Spec}(T)$ *is pinched at Q. Put $D := T/Q$. Let F be a field such that $F \subseteq D$. (Then $F \subset D$ since D is not a field.) Let R be the pullback $R = F \times_D T$. Next, let L be the integral closure of F in D and, if $\pi : T \to D$ is the canonical projection map, let $S := \pi^{-1}(L)$. Then:*

(*a*) *(R,T) is a strongly divided pair and T is not integral over R. In addition, R is integrally closed in T if and only if F is integrally closed in D. Moreover, Q is the unique maximal ideal of R and $QT = Q \neq T$.*
(*b*) *(S,T) is a strongly divided pair and S is integrally closed in T. Moreover, Q is the unique maximal ideal of S and $QT = Q \neq T$.*
(*c*) *(S,T) is not a normal pair.*

Proof (a) In the "gluing" identification of $\mathrm{Spec}(R)$ via [18, Theorem 1.4], all the prime ideals of T that contain Q are identified with $(0) \in \mathrm{Spec}(F)$. The upshot is that $\mathrm{Spec}(R)$ is order-isomorphic to $\{P \in \mathrm{Spec}(T) \mid P \subseteq Q\}$. In particular, R is quasi-local with unique maximal ideal Q. Plainly, $QT = Q \neq T$. In addition, [18, Corollary 1.5 (5)] ensures that T is not integral over R, the point being that D is not integral over F (since the domain D is not a field); and that R is integrally closed in $T \Leftrightarrow F = L$ ($\Leftrightarrow R = S$) $\Leftrightarrow F$ is integrally closed in D.

To prove that (R, T) is a strongly divided pair, we will show that if $H \in [R, T]$, then $R \subseteq H$ is a strongly divided extension. Observe that $H = H/Q \times_D T$. By applying [18, Theorem 1.4] to this pullback description, we get that $\mathrm{Spec}(H)$ is order-isomorphic to the disjoint union of $\{P \in \mathrm{Spec}(T) \mid P \subset Q\}$ and some set $\mathcal{T} \subseteq \{\mathfrak{q} \in \mathrm{Spec}(H) \mid Q \subseteq \mathfrak{q}\}$ (with the obvious partial order). Identify $\mathcal{S}(R)$ with $\{P \in \mathrm{Spec}(T) \mid P \subset Q\}$. It is clear that if $\mathfrak{p} \in \mathrm{Spec}(H)$ with $\mathfrak{p} \cap R \in \mathcal{S}(R)$, then $\mathfrak{p}$ has been identified with some element of $\mathcal{S}(R)$, and so $\mathfrak{p} = \mathfrak{p} \cap R$; that is, $R \subseteq H$ is a strongly divided extension.

(b) We have $S = L \times_D T$. Also, L is integrally closed in D. Thus, it suffices to repeat the proof of (a), with L (resp., S) playing the earlier role of F (resp., R).

(c) By the proof of (b), $\text{Spec}(S)$ is order-isomorphic to $\{P \in \text{Spec}(T) \mid P \subseteq Q\}$. Since Q is not a maximal ideal of T, we can choose $N \in \text{Max}(T)$ such that $Q \subset N$. As Q is a (in fact, the) maximal ideal of S, we have $Q \cap S = Q = N \cap S$. Thus $S \subseteq T$ does not satisfy INC. Hence (S, T) is not an INC-pair. Consequently, by [22, Theorem 5.2], (S, T) is not a normal pair. □

In view of Example 4.4 and Proposition 4.5, it is natural to pause and ask whether the "$mT = T$" context of Sect. 3 admits examples of strongly divided pairs (R, T) such that R is integrally closed in T (resp., such that R is not integrally closed in T). In view of the lying-over theorem [19, Theorem 11.5], the condition $mT = T$ ensures that T cannot be integral over R. Remark 4.6 indicates some examples that answer these questions.

Remark 4.6 There exist strongly divided pairs $((A, n), B)$ such that $nB = B$ and, as desired, either (i) A is integrally closed in B or (ii) A is not integrally closed in B. One way to construct such data is the following. Let (R, m) be a PVD which is not integrally closed. (In particular, R is not a valuation domain.) Let H be the canonically associated valuation overring of R and let $T := \text{qf}(R)$. Then (H, T) illustrates (i) and (R, T) illustrates (ii).

For a proof, we begin by showing that $mT = T$. As all fields are valuation domains by convention, neither R nor H is a field, so that $m \neq (0)$, whence $mT = T$, as desired. Next, we will show that (R, T) (resp., (H, T)) is a strongly divided pair. This follows via Proposition 2.6 since R (resp., H) is quasi-local and is PV in (its quotient field) T. To complete the proof, recall that R is not integrally closed (in T) and observe that H (being a valuation domain) is integrally closed (in T).

There are at most four types of strongly divided pairs (R, T), as R and T could each either be treed or fail to be treed. Thanks to Example 3.11 (and the comments preceding it) and Example 4.1, we have seen that three of these four possibilities can be realized by specific examples. The next two results address the question of whether there exists an example that realizes the fourth possibility. Their upshot is that if such an example exists, it can be arranged to have a number of additional features: see Proposition 4.8(d).

Lemma 4.7 *Let $R \subseteq T$ be domains such that $R \subseteq T$ is a strongly divided extension and R is (quasi-local and) not treed. Suppose also that $R \subseteq T$ satisfies* GU *and* INC *(for instance, suppose that T is integral over R). Then T is not treed.*

Proof As R is not treed, the maximal ideal m of R contains incomparable prime ideals p and q of R. Note that $p, q \in \mathcal{S}(R)$. As $R \subseteq T$ is a strongly divided extension that satisfies LO (since GU implies LO [21, Theorem 42]), it follows that both p and q are prime ideals of T. Since $R \subseteq T$ satisfies GU, there exists a prime ideal M (resp., N) of T that contains p (resp., q) and lies over m. As $R \subseteq T$ satisfies INC, both M and N are maximal ideals of T.

Now suppose the assertion fails. Then $M \neq N$ (for otherwise, the "treed" property of T would force p and q to be comparable, a contradiction). Similarly, one sees that $q \not\subseteq M$. Hence, by the maximality of M, we have $(M, q) = T$. Thus, there exist $a \in M$ and $b \in q$ such that $a + b = 1$. Then $a = 1 - b \in R \cap M = m$, so that $1 = a + b \in m + q \subseteq m + m = m$, the desired contradiction. □

Proposition 4.8 *Suppose that there exist domains $R \subseteq T$ such that (R, T) is a strongly divided pair, R is (quasi-local with maximal ideal m and) not treed, and T is treed. Then:*

(a) $mT \neq T$.
(b) T is not integral over R; that is, $\overline{R} \neq T$.
(c) $\overline{R}$ is not a locally Noetherian ring; in particular, $\overline{R}$ is not a Noetherian ring.
(d) There exist domains $A \subseteq B$ such that (A, B) is a strongly divided pair, A is (quasi-local with maximal ideal n and) not treed, B is treed, A is integrally closed in B (that is, $\overline{A} = A$), $nB \neq B$, and A is not a Noetherian ring.

Proof (a) Apply Proposition 3.9.

(b) Apply Lemma 4.7.

(c), (d) Note that $(R, \overline{R})$ inherits the "strongly divided pair" property from (R, T). So, by (b) (or Lemma 4.7), $\overline{R}$ is not treed. Hence, there exists $M \in \text{Max}(\overline{R})$ such that $\overline{R}_M$ is not treed. On the other hand, T_M inherits the "treed" property from T. Moreover, Proposition 2.10 ensures that $(\overline{R}_M, T_M)$ is a strongly divided pair. Therefore, by (a) (or Proposition 3.9), $(M\overline{R}_M)T_M \neq T_M$. In addition, $\overline{R}_M$ is integrally closed in T_M. Thus, to establish (d) by taking $A := \overline{R}_M$ and $B := T_M$, it will suffice to show that $\overline{R}_M$ is not a Noetherian ring. Once we will have shown this, (c) will also have been established.

The above analysis shows that, by replacing (R, T) with $(\overline{R}_M, T_M)$, we may also assume that R is integrally closed in T and need to only derive a contradiction from the assumption that R is Noetherian. To sum up, we have that $R \subseteq T$ are domains such that (R, m) is (quasi-)local Noetherian and integrally closed in T, R is not treed, T is treed and, by (a), $mT \neq T$. Thus $mT \subset T$, and so we can choose $N \in \text{Max}(T)$ such that $mT \subseteq N$. It follows that $N \cap R = m$. If $N = m$, then $\text{Max}(R) \subseteq \text{Max}(T)$, so that [2, Proposition 3.8] yields that $\text{Spec}(R) = \text{Spec}(T)$, a contradiction (since R is not treed and T is treed). Thus $N \neq m$, and so we can choose $u \in N \setminus m$ ($= N \setminus R$). In addition, by considering the order-theoretic impact of applying the gluing result [18, Theorem 1.4] to the pullback $R + N = R/m \times_{T/N} T$, we see that $\text{Spec}(R + N) = \text{Spec}(T)$, and so we can further change notation and assume that $T = R + N$. (In so doing, we also retain the condition that $N \in \text{Max}(T)$ since the former data satisfied $(R + N)/N \cong R/(R \cap N) = R/m$, which is a field.) Observe also that $u^{-1} \notin R$ (for otherwise, $u^{-1} \in m \subset N \subseteq T$, so that $1 = uu^{-1} \in NT = N$, a contradiction). So, as in the proof of [3, Lemma 3] (that is, essentially by the proof of the (u, u^{-1})-Lemma [21, Theorem 67]), $mR[u]$ is the unique nonmaximal prime ideal of $R[u]$ that lies over m.

Since u is a nonunit of (T and hence a nonunit of) $R[u]$, we can choose $Q \in \text{Spec}(R[u])$ such that Q is minimal among the prime ideals of $R[u]$ that contain u.

As $u \neq 0$ and $R[u]$ is a Noetherian ring, it follows from the Principal Ideal Theorem [21, Theorem 142] that the height of Q (in $R[u]$) is 1. However, since $u \notin R$ and $u^{-1} \notin R$ (and R is assumed integrally closed in T), the (u, u^{-1})-Lemma ensures that $u \notin mR[u]$, and so $Q \nsubseteq mR[u]$. As $R \subseteq R[u]$ is a strongly divided extension and $mR[u] \neq R[u]$, Proposition 4.2 yields $mR[u] \subset Q$. Since Q has height 1, we then get $mR[u] = (0)$, whence $m = mR[u] \cap R = (0)$. But R is not a field (since R is not treed), and so we have the desired contradiction, thus completing the proof. □

We will close with a couple of results that probe more deeply into the setting of Proposition 2.11. Proposition 4.9 will show that if one adds a few more assumptions, the normal pair in that setting becomes trivial. First, we need to recall the following definition from [12]. Let $A \subseteq B$ be rings. Then $A \subseteq B$ is said to satisfy MINC if, whenever $Q_1 \subseteq Q_2$ in $\text{Spec}(B)$ with $Q_1 \cap A = Q_2 \cap A \in \text{Max}(A)$, then $Q_1 = Q_2$. To appreciate the influence of the MINC property, observe that the data in Proposition 4.5(c) are such that $S \subseteq T$ does not satisfy MINC.

Proposition 4.9 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local and $mT \neq T$. Suppose also the following three conditions:*

(*i*) *For all $E \in [R, T]$, the extension $R \subseteq E$ satisfies* MINC*; and*
(*ii*) *$\overline{R}$ is quasi-local, say with maximal ideal M; and*
(*iii*) $\text{Spec}(R)$ *is finite.*
Then (R, T) is a strongly divided pair if and only if T is integral over R with $\mathcal{S}(R) = \mathcal{S}(T)$.

Proof By Proposition 2.1(b), we need only to prove the "only if" assertion. Assume, then, that (R, T) is a strongly divided pair. By Proposition 2.1(b), it suffices to prove that T is integral over R. Let $E \in [R, T]$ and $Q_1 \subseteq Q_2$ in $\text{Spec}(E)$ with $q := Q_1 \cap R = Q_2 \cap R$. If $q \in \mathcal{S}(R)$, then $Q_1 = q = Q_2$ since $R \subseteq E$ is a strongly divided extension. On the other hand, if $q \in \text{Max}(R)$, then $Q_1 = Q_2$ since $R \subseteq E$ satisfies MINC by (i). Hence $Q_1 = Q_2$. Consequently, $R \subseteq E$ satisfies INC for all $E \in [R, T]$; that is, (R, T) is an INC-pair. As usual, let $\overline{R}$ denote the integral closure of R in T. It follows that $(\overline{R}, T)$ is a normal pair (cf. [22, Theorem 5.2, (9′) ⇒ (4)]). It suffices to prove that this normal pair collapses, in the sense that $\overline{R} = T$.

Suppose the assertion fails. Then $\overline{R}_M = \overline{R} \subset T = T_M$. Next, recall that if $A \subseteq B$ are rings, the support $\text{Supp}_A(B) := \{P \in \text{Spec}(A) \mid A_P \subset B_P\}$. We have just seen that $M \in \text{Supp}_{\overline{R}}(T)$. In any case, $\text{Supp}_{\overline{R}}(T) \setminus \{M\} \subseteq \mathcal{S}(\overline{R})$. However, since the (integral) extension $R \subseteq \overline{R}$ satisfies INC, each $\mathfrak{Q} \in \mathcal{S}(\overline{R})$ satisfies $\mathfrak{Q} \cap R \in \mathcal{S}(R)$ and so, since $R \subseteq \overline{R}$ is a strongly divided extension, we get $\mathfrak{Q} = \mathfrak{Q} \cap R \in \mathcal{S}(R) \subseteq \text{Spec}(R)$. Therefore, by (iii), $\mathcal{S}(\overline{R})$ is finite. Hence $\text{Supp}_{\overline{R}}(T)$ is finite. Then, as $(\overline{R}, T)$ is a normal pair, it follows from [15, Proposition 6.9] that $[\overline{R}, T]$ is finite. Consequently, since $\overline{R} \subset T$, there exists $E \in [\overline{R}, T]$ such that $\overline{R} \subset E$ is a minimal ring extension (in the sense of [17]). Let $\mathcal{M}$ denote the crucial maximal ideal of this minimal ring extension; that is, the element of $\text{Max}(\overline{R})$ such that $\overline{R}_{\mathcal{P}} = E_{\mathcal{P}}$ canonically for all $\mathcal{P} \in \text{Spec}(\overline{R}) \setminus \{\mathcal{M}\}$ [17, Théorème 2.2 (i)]. It is clear that if $\mathcal{P} \in \text{Spec}(\overline{R}) \setminus \{\mathcal{M}\}$, then there exists a prime ideal of E that meets $\overline{R}$ in $\mathcal{P}$. Thus, since $\overline{R}$ is integrally closed in E (and E is not integral over $\overline{R}$), [17, Théorème 2.2 (ii)] ensures that no prime ideal of E can meet $\overline{R}$ in $\mathcal{M}$.

As $mT \neq T$, there exists $N \in \mathrm{Max}(T)$ such that $mT \subseteq N$. As $m \in \mathrm{Max}(R)$, it must be the case that $N \cap R = m$. Consider $\mathcal{N} := N \cap \overline{R} \in \mathrm{Spec}(\overline{R})$. We then have $\mathcal{N} \cap R = m$. Since the (integral) ring extension $R \subseteq \overline{R}$ satisfies INC and $m \in \mathrm{Max}(R)$, it follows that $\mathcal{N} \in \mathrm{Max}(\overline{R})$. Hence, by (ii), $\mathcal{N} = M$. But (ii) also ensures that $\mathcal{M} = M$. Therefore, $N \cap \overline{R} = \mathcal{N} = M = \mathcal{M}$. Thus, $N \cap E \in \mathrm{Spec}(E)$ satisfies $(N \cap E) \cap \overline{R} = N \cap \overline{R} = \mathcal{M}$, the desired contradiction. □

It is natural to ask what can be said if one deletes the hypothesis that $mT \neq T$ in Proposition 4.9. Corollary 4.10 shows how to further illuminate that situation by using the ideas behind the first paragraph of the proof of Proposition 4.9.

Corollary 4.10 *Let $R \subseteq T$ be domains such that (R, m) is quasi-local. Then the following conditions are equivalent:*

(1) (R, T) is a strongly divided pair and $\overline{R} \subseteq E$ satisfies MINC *for all $E \in [\overline{R}, T]$;*
(2) $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ and $(\overline{R}, T)$ is a normal pair.

Proof (2) ⇒ (1): If $(\overline{R}, T)$ is a normal pair, then $\overline{R} \subseteq E$ satisfies (M)INC for all $E \in [\overline{R}, T]$ (cf. [22, Theorem 5.2, (4) ⇒ (9)′]). Then to complete the proof that (2) ⇒ (1), one need only combine Propositions 2.10 and 2.1(b) with the above-noted fact that being a normal pair is a local property and Corollary 2.7(b).

(1) ⇒ (2): Assume (1). Then $\mathcal{S}(R) = \mathcal{S}(\overline{R})$ by Proposition 2.10. It remains only to prove that $(\overline{R}, T)$ is a normal pair, or equivalently, that $(\overline{R}_M, T_M)$ is a normal pair for each $M \in \mathrm{Max}(\overline{R})$. Fix any such M. By Proposition 2.10, $(\overline{R}_M, T_M)$ is a strongly divided pair. Next, consider any $\mathcal{E} \in [\overline{R}_M, T_M]$. There exists a (uniquely determined) ring $E \in [\overline{R}, T]$ such that $E_M = \mathcal{E}$. As $\overline{R} \subseteq E$ satisfies MINC by hypothesis, it follows easily that $\overline{R}_M \subseteq E_M$ (that is, $\overline{R}_M \subseteq \mathcal{E}$) satisfies MINC. Then, by adapting the reasoning in the first paragraph of the proof of Proposition 4.9, we get that $(\overline{R}_M, T_M)$ is an INC-pair. As $\overline{R}_M$ is integrally closed in T_M, it follows that $(\overline{R}_M, T_M)$ is a normal pair (cf. [22, Theorem 5.2, (9)′ ⇒ (4)]). □

Finally, note that the pair (S, T) in Proposition 4.5 illustrates the fact that the implication (1) ⇒ (2) in Corollary 4.10 would fail if one deleted the hypothesis that $\overline{R} \subseteq E$ satisfies MINC for all $E \in [\overline{R}, T]$.

References

1. T. Akiba, A note on AV-domains. Bull. Kyoto Univ. Educ. Ser. B **31**, 1–3 (1967)
2. D.F. Anderson, D.E. Dobbs, Pairs of rings with the same prime ideals. Can. J. Math. **32**, 362–384 (1980)
3. A. Ayache, D.E. Dobbs, Strongly divided domains. Ric. Mat. **65**, 127–154 (2016)
4. A. Ayache, D.E. Dobbs, Strongly divided rings with zero-divisors. Palest. J. Math. **6**, 380–395 (2017)

5. A. Ayache, O. Echi, Valuation and pseudovaluation subrings of an integral domain. Commun. Algebra **34**, 2467–2483 (2006)
6. A. Ayache, A. Jaballah, Residually algebraic pairs of rings. Math. Z. **225**, 49–65 (1997)
7. A. Badawi, On divided commutative rings. Commun. Algebra **27**, 1465–1474 (1999)
8. E.D. Davis, Overrings of commutative rings. III. Normal pairs. Trans. Am. Math. Soc. **182**, 175–185 (1973)
9. D.E. Dobbs, On going-down for simple overrings, II. Commun. Algebra **1**, 439–458 (1974)
10. D.E. Dobbs, Divided rings and going down. Pac. J. Math. **67**, 353–363 (1976)
11. D.E. Dobbs, On INC-extensions and polynomials with unit content. Can. Math. Bull. **23**, 37–42 (1980)
12. D.E. Dobbs, Lying-over pairs of commutative rings. Can. J. Math. **33**, 454–475 (1981)
13. D.E. Dobbs, M. Fontana, J.A. Huckaba, I.J. Papick, Strong ring extensions and pseudo-valuation domains. Houst. J. Math. **8**, 167–184 (1982)
14. D.E. Dobbs, M. Fontana, I.J. Papick, Direct limits and going-down. Comment. Math. Univ. St. Paul. **31**, 129–135 (1982)
15. D.E. Dobbs, G. Picavet, M. Picavet-L'Hermitte, Characterizing the ring extensions that satisfy FIP or FCP. J. Algebra **371**, 391–429 (2012)
16. D.E. Dobbs, J. Shapiro, Normal pairs with zero-divisors. J. Algebra Appl. **10**, 335–356 (2011)
17. D. Ferrand, J.-P. Olivier, Homomorphismes minimaux d'anneaux. J. Algebra **16**, 461–471 (1970)
18. M. Fontana, Topologically defined classes of commutative rings. Ann. Mat. Pura Appl. **123**, 331–355 (1980)
19. R. Gilmer, *Multiplicative Ideal Theory* (Dekker, New York, 1972)
20. J.R. Hedstrom, E.G. Houston, Pseudo-valuation domains. Pac. J. Math. **75**, 137–147 (1978)
21. I. Kaplansky, *Commutative Rings*, rev edn. (University Chicago Press, Chicago, 1974)
22. M. Knebusch, D. Zhang, Manis Valuations and Prüfer Extensions I. Lecture Notes in Mathematics, vol. 1791. Springer, Berlin (2002)

Finite Intersections of Prüfer Overrings

Bruce Olberding

Abstract This article is motivated by the open question of whether every integrally closed domain is an intersection of finitely many Prüfer overrings. We survey the work of Dan Anderson and David Anderson on this and related questions, and we show that an integrally closed domain that is a finitely generated algebra over a Dedekind domain or a field is a finite intersection of Dedekind overrings. We also discuss how recent work on the intersections of valuation rings has implications for this question.

1 Introduction

An integral domain R is a *Prüfer domain* if every nonzero finitely generated ideal of R is invertible; equivalently, R_M is a valuation domain for each maximal ideal M of R. The only Noetherian Prüfer domains are the Dedekind domains. From this point of view, the class of Prüfer domains is rather special, and thus it might be expected that the class of integral domains that are finite intersections of Prüfer overrings is also somewhat special. (By an *overring* of an integral domain, we mean a ring between the domain and its quotient field.) One elementary observation, perhaps the *least* nontrivial thing that can be said about such an intersection, is that since an intersection of valuation rings is integrally closed, an intersection of Prüfer domains is necessarily an integrally closed domain. Strikingly, it remains an open question whether this is also the *most* that can be said for a finite intersection of Prüfer overrings. In other words, it is unknown whether every integrally closed domain is an intersection of finitely many Prüfer overrings, or even finitely many Bézout overrings. (An integral domain is a *Bézout domain* if every finitely generated ideal is principal.)

This question has its origins in the work of Dan Anderson and David Anderson. In their 1985 article [3, p. 97], the following questions are posed.

B. Olberding (✉)
New Mexico State University, Las Cruces, NM 88003-8001, USA
e-mail: bruce@nmsu.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*, Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_5

Questions 1.1

(1) Is each integrally closed (resp., completely integrally closed) domain an intersection of two Bézout (resp., completely integrally closed Bézout) overrings?
(2) Is each Krull domain an intersection of a finite number of PID or factorial overrings?
(3) If a domain R is a finite intersection of PID or factorial overrings, is R an intersection of two such overrings?

To these questions, Paul-Jean Cahen added a weaker version of (1):

(4) Is every integrally closed domain an intersection of two Prüfer overrings?

Each of these questions, which are stated also in [10, Sect. 6], remains open, and this survey article does not resolve any of them. (In fact, at the end of Sect. 4, we add three more questions to this list.) Instead, the aim of this article is to use these questions and the related work of Anderson and Anderson as motivation for highlighting the intricacies involved in studying intersections of pairs of PIDs, Bézout domains or Prüfer domains. Even the case of understanding the intersection of a PID and a DVR proves quite subtle and technically complicated, as Sect. 5 discusses.

In Sect. 2, we discuss a number of positive results from the work of Anderson and Anderson related to Questions 1.1, and in Sect. 3 we use one of these results to answer Question 1.1(4) in the special case of an integrally closed domain that is a finitely generated algebra over a Dedekind domain or a field. More generally, we give a positive answer to Question 1.1(4) for some classes of Krull domains.

The goal of Sect. 4 is to reverse the representation question and to ask what can be said about the Prüfer domains in a finite representation of a completely integrally closed local domain R? If these Prüfer domains are G-domains (i.e., the intersection of their nonzero prime ideals is nonzero), then they must satisfy several strong conditions, as illustrated by Theorem 4.6. Demanding G-domains for the representation is quite restrictive here, and by Lemma 4.4 this forces R itself to be a G-domain, which rules out many interesting choices for R. However, the real objective of Sect. 4 is a case in which the G-domain requirement is automatically satisfied for R and its overrings, that in which R is a one-dimensional completely integrally closed local domain. Section 4 is thus ultimately motivated by what can be said about the Prüfer overrings in a representation of such a choice of the ring R. We narrow this even further to a specific example due to Nagata that can serve as an interesting test case for Questions 1.1(1) and (4).

With the exception of Anderson and Anderson's articles [3, 4], which are the subject of Sect. 2 and motivate Sect. 3, the results in Sects. 4 and 5 are extracted from various contexts that are motivated otherwise than by the above questions. Most of these contexts involve a technical step, somewhere amidst other considerations, in which the intersection of valuation rings must be considered and described. A secondary goal of the article is to highlight a few examples of some of this recent work.

Throughout the article, we assume all rings are commutative and have an identity. We denote the Jacobson radical of a ring R by $J(R)$.

2 Positive Results: The Work of Anderson and Anderson

Questions 1.1 concern the representation of integral domains by intersections of *overrings* from specified classes of rings. It is the restriction to overrings that proves to be the crux of the problem. Without the restriction to overrings, these questions can be answered in the affirmative using the Kronecker function ring construction.

Theorem 2.1 (Anderson–Anderson [3, Theorems 1.1 and 1.3 and Remark 1.2])

(1) Every Krull domain is an intersection of a PID (in fact, a Euclidean domain) and a field.
(2) Every completely integrally domain is an intersection of a completely integrally closed Bézout domain and a field.
(3) Every integrally closed domain is an intersection of a Bézout domain and a field.

In each case, the domain R in question is written as an intersection of its quotient field and the Kronecker function ring of R with respect to an appropriate e.a.b. star operation. (For general definitions and background, see [9].) In the case of Theorem 2.1(1), R is an intersection of its quotient field F and the Kronecker function ring R^v of R with respect to the v-operation:

$$R^v := \{f/g : f, g \in R[X], g \neq 0, c(f) \subseteq (R :_F (R :_F c(g)))\},$$

where X is an indeterminate for R and $c(-)$ denotes the content of the polynomial. Since R is a Krull domain, the ring R^v is a Euclidean domain [3, Remark 1.2]. Standard properties of the Kronecker function ring construction yield that $R = R^v \cap F$, so (1) follows. Statement (2) is proved similarly using the fact that if R is completely integrally closed, then so is R^v.

Instead of R^v, statement (3) requires the Kronecker function ring with respect to the b-operation:

$$R^b := \{f/g : f, g \in R[X], g \neq 0, c(f) \subseteq \overline{c(g)}\},$$

where $\overline{c(g)}$ denotes the integral closure of the ideal $c(g)$ in R. If R is integrally closed, then R^b is a Bézout domain. Since $R = R^b \cap F$, statement (3) now follows.

The converses to each of statements (1)–(3) of Theorem 2.1 are also valid, as noted in [3]; i.e., the intersection of two PIDs is a Krull domain; the intersection of two Bézout domains is an integrally closed domain; and the intersection of two completely integrally closed Bézout domains is a completely integrally closed domain.

Thus, the version of Questions 1.1 in which the restriction to overrings is removed is completely solved by Theorem 2.1.

A similar approach resolves Questions 1.1 for polynomial rings.

Theorem 2.2 (Anderson–Anderson [3, Theorem 1.4]) *Let R be an integral domain.*

(1) If R is integrally closed (resp., completely integrally closed), then $R[X]$ is an intersection of two Bézout (resp., completely integrally closed Bézout) overrings.

(2) *If R is a Krull domain, then $R[X]$ is an intersection of two PID overrings. Moreover, each PID overring may be chosen to be a Euclidean domain which is a localization of $R[X]$.*

As in the proof of Theorem 2.1, the Kronecker function ring construction is the key. With F the quotient field of R, $R[X] = R^b \cap F[X]$, and if R is a completely integrally closed domain, then $R[X] = R^v \cap F[X]$. (More generally, $R[X] = R^* \cap F[X]$ for any e.a.b. star operation $*$.) Since $F[X]$ is a Euclidean overring of $R[X]$, statements (1) and (2) of Theorem 2.2 follow as in the argument for Theorem 2.1.

While the Kronecker function ring construction provides an elegant solution to the situations considered in Theorems 2.1, 2.2 and 3.3, what makes it so useful for these cases is precisely what prevents its applicability to more general settings, namely, that $R = R^* \cap F$ for an e.a.b. star operation $*$. Except in special cases such as that of $R[X]$ in Theorem 2.2, this feature of R^* limits its usefulness when seeking representation by rings inside the quotient field of the base domain.

Thus, different methods are generally needed for dealing with Questions 1.1. Anderson and Anderson develop one such method for classes of Krull domains whose divisor class groups are constrained in various ways. The first such class of rings is considered in a 1984 paper whose themes anticipate Questions 1.1. A domain R is *locally factorial* if R_x is factorial for each nonunit $x \in R$.

Theorem 2.3 (Anderson–Anderson [2, Theorem 2.3 and Proposition 6.1]) *If R is a locally factorial Krull domain, then R is an intersection of two factorial overrings.*

Here the representation of R by two factorial rings is accomplished by showing that $R = R_x \cap R_y$ for an appropriate choice of $x, y \in R$. If x and y are nonunits, then R_x and R_y are factorial since R is locally almost factorial. The proof thus depends on the choice of x and y. If R is local and has Krull dimension one, then R itself is factorial [2, Proposition 6.1]. If R is local and has Krull dimension greater than one, then R has a regular sequence x, y of length 2 [2, Proposition 6.1]. Since R is a Krull domain, it follows that $R = R_x \cap R_y$. If R is not local, then any choice of nonunits x and y such that $(x, y)R = R$ works, since for such a choice, $R = R_x \cap R_y$ [2, Corollary 2.2].

In [3], Anderson and Anderson show precisely how far this method of representation of Krull domains via factorial localizations R_x and R_y can be extended. In Theorem 2.4, $\mathrm{Cl}(R)$ denotes the divisor class group of the Krull domain R. A *subintersection* of a domain R is an overring of the form $\bigcap_{P \in X} R_P$, where X is a set of prime ideals of R.

Theorem 2.4 (Anderson–Anderson [3, Theorems 2.3 and 2.10]) *The following are equivalent for a domain R:*

(1) *R is a Krull domain and $\mathrm{Cl}(R)$ is finitely generated.*
(2) *There are nonzero $x, y \in R$ such that $R = R_x \cap R_y$ and R_x and R_y are each factorial.*
(3) *There are nonzero $x_1, \ldots, x_n \in R$ such that $R = R_{x_1} \cap \cdots \cap R_{x_n}$ and each R_{x_i} is factorial.*

(4) $R = R_1 \cap R_2$, *where* R_1 *and* R_2 *are subintersections with* R_1 *factorial and* R_2 *a semilocal PID.*

It is also noted [3, Theorem 2.10] that Theorem 2.4 implies that if R is a Dedekind domain, then $\mathrm{Cl}(R)$ is finitely generated if and only if $R = R_1 \cap R_2$, where R_1 and R_2 are subintersections with R_1 a PID and R_2 a semilocal PID. Thus, a positive answer to question (2) from the introduction is obtained in the case where R is a Dedekind domain with finitely generated class group.

Theorem 2.4 shows that the representation of a Krull domain R as $R = R_x \cap R_y$ for factorial overrings R_x and R_y is limited to the case where R has finitely generated divisor class group. Consequently, if a Krull domain without finitely generated divisor class group is to be represented as an intersection of two (or finitely many) factorial overrings, these overrings must be sought elsewhere than the localizations of the form R_x.

Moving beyond Krull domains with finitely generated divisor class group, Anderson and Anderson consider the case of countable divisor class group.

Theorem 2.5 (Anderson–Anderson [3, Theorem 2.7]) *A Krull domain with countable divisor class group is an intersection of two factorial subintersections.*

Countability is used in a strong way here to show that the set of height-one prime ideals of the Krull domain in question can be written as a disjoint union of two sets X_1 and X_2, where each $\bigcap_{P \in X_i} R_P$ is a factorial subintersection.

The emphasis on the divisor class group in Theorems 2.4 and 2.5 suggests that the question of whether a Krull domain can be represented by an intersection of two factorial overrings might depend on the size or structure of the divisor class group. Interestingly, this is not the case: Given any abelian group G and positive integer n, there is a Krull domain of Krull dimension n having class group G such that R is an intersection of two PID overrings [3, Example 2.9]. The construction of the Krull domain here is via Claborn's theorem on the existence of a Krull domain R of Krull dimension n with divisor class group G. This existence theorem is coupled with a certain localization of $R[X]$ and Theorem 2.2 to produce the desired example.

The treatment of Krull domains so far has focused on the version of Question 1.1(2) that asks for representation by factorial overrings. The method of proof outlined for special cases of Krull domains only produces in general a representation as an intersection of factorial overrings, not PID overrings. In the case in which R has Krull dimension 2, PID overrings can be used instead.

Theorem 2.6 (Anderson–Anderson [3, Theorem 2.11]) *Let R be a two-dimensional Krull domain with only a finite number of height-two maximal ideals:*

(1) *If* $\mathrm{Cl}(R)$ *is finitely generated, then R is an intersection of a PID overring and a semilocal PID overring.*
(2) *If* $\mathrm{Cl}(R)$ *is countable, then R is an intersection of two PID overrings and a semilocal PID overring.*

Theorem 2.6 gives sufficient conditions for a two-dimensional semilocal Krull domain to decompose into an intersection of a PID overring and a semilocal PID overring. The following theorem from [19] is in a similar spirit, although one of the components of the intersection is allowed to be a Noetherian domain instead of a PID. Recall that a collection X of valuation overrings has *finite character* if each nonzero element of the quotient field is a unit in all but at most finitely members of X. For example, the collection of localizations of a Krull domain at height-one prime ideals is a finite character collection of DVR overrings.

Theorem 2.7 [19, Proposition 9.1] *Let R be an overring of a two-dimensional Noetherian domain such that $J(R) \neq 0$. Then there exists a finite character collection X of valuation overrings of R such that $R = \bigcap_{V \in X} V$ if and only if $R = A \cap B$, where A is an integrally closed Noetherian overring and B is a semilocal Prüfer overring.*

Weakening the requirement in Theorem 2.6 that the Krull domain R is represented by two factorial overrings, Anderson and Anderson consider in a 1987 paper when a Krull domain can be represented by an intersection of *almost factorial* Krull domains, where a Krull domain is almost factorial if it has torsion divisor class group (equivalently, each subintersection of R is a localization of R [4, p. 395]).

Theorem 2.8 (Anderson–Anderson [4, Theorem 4.1]) *The following statements are equivalent for an integral domain R:*

(1) R is a Krull domain and $Cl(R)$ has finite rank.
(2) R is a Krull domain and R_x is almost factorial for some nonzero $x \in R$.
(3) There are nonzero x and y in R such that $R = R_x \cap R_y$ and R_x and R_y are each almost factorial Krull domains.
(4) There are nonzero $x_1, \ldots, x_n \in R$ such that $R = R_{x_1} \cap \cdots \cap R_{x_n}$ and each R_{x_i} is an almost factorial Krull domain.

Theorem 2.8 is the "almost factorial" analogue of Theorem 2.4. Similarly, the following theorem is analogous to Theorem 2.5.

Theorem 2.9 (Anderson–Anderson [4, Theorem 4.2]) *Let R be a Krull domain whose divisor class group has countable rank. Then R is an intersection of two almost factorial subintersections.*

Summary. The most general positive result for Question 1.1(2) from the introduction that has been obtained is that for which the Krull domain R has countable divisor class group. In this case, R is an intersection of two factorial overrings, but it is not known whether R can always be written as an intersection of two PID overrings. Weakening Question 1.1(2) so that the components of the intersection representation are only required to be almost factorial permits extension to divisor class groups of countable rank. In either case, the Krull domain is an intersection of two almost factorial *subintersections*. This raises another open question [4, p. 406]: Can every Krull domain be written as an intersection two factorial (or even almost factorial) subintersections?

3 Krull Domains

In this section, we add to the list of examples of Krull domains that can be represented as a finite intersection of Prüfer overrings. For example, we show in Corollary 3.4 that every integrally closed domain that is a finitely generated algebra over a Dedekind domain or a field has this property.

We begin with the observation that for Krull domains, representation as a finite intersection of Prüfer overrings implies representation as a finite intersection of Dedekind overrings. (In what follows, when we write that a domain is an intersection of n overrings for some positive integer n, then we do not necessarily assume these overrings are distinct.)

Lemma 3.1 *Let R be a Krull domain that is not a field. If R is an intersection of a finite number of n Prüfer overrings (resp., Bézout overrings), then R is an intersection of n Dedekind subintersections (resp., PID subintersections).*

Proof Suppose that $R = A_1 \cap \cdots \cap A_n$, where each A_i is a Prüfer overring of R. If P is a height-one prime ideal of R, then $R_P = A_1 R_P \cap \cdots \cap A_n R_P$. Since R is a Krull domain and P has height one, R_P is a valuation ring, and hence, its overrings form a chain under inclusion, so it follows that $A_i \subseteq R_P$ for some i. Thus, each localization of R at a height-one prime ideal contains one of the A_i.

For each i, let $B_i = \bigcap_{A_i \subseteq R_P} R_P$, where P ranges over the height-one prime ideals of R with $A_i \subseteq R_P$. (If there is no height-one prime P with $A_i \subseteq R_P$, then let B_i be the quotient field of R.) Since R is a Krull domain and hence the intersection of the localizations of R at its height-one prime ideals, we have $R = B_1 \cap \cdots \cap B_n$. Moreover, for each i such that B_i is not the quotient field of R, we have that as a Krull overring of the Prüfer domain A_i, the ring B_i is a Prüfer Krull domain, hence a Dedekind domain. If also A_i is a Bézout overring, then B_i is a Bézout Krull domain, hence a PID. Since the B_i are subintersections, the lemma follows. □

Lemma 3.2 *Let $D \subseteq R$ be an integral extension of Krull domains. If D is an intersection of a finite number of n Prüfer overrings, then R is an intersection of n Dedekind overrings.*

Proof Let $A_1, \ldots, A_n$ be Prüfer overrings of D such that $D = A_1 \cap \cdots \cap A_n$. For each i, let $\mathscr{F}_i$ be the set of height-one prime ideals P of R such that $A_i \subseteq R_P$. We claim that $\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_n$ is the set of all height-one prime ideals of R. Indeed, let P be a height-one prime ideal of R. As R is integral over D and a domain, $P \cap D$ is nonzero since no two comparable prime ideals of R lie over the same prime ideal of D. Since D is an integrally closed domain and R is a domain that is integral over D, Going Down holds for the extension $D \subseteq R$ [27, 13.41, p. 261]. Consequently, the height of $(P \cap D)R$ is equal to the height of $P \cap D$ [29, Proposition B.2.4, p. 398]. Since P has height one and $P \cap D \neq 0$, it follows that $(P \cap D)R$ has height one, and hence, $P \cap D$ has height one. Therefore, $D_{P \cap D}$ is a DVR since D is a Krull domain. Also, since $D_{P\cap D} = A_1 D_{P\cap D} \cap \cdots \cap A_n D_{P\cap D}$ and the overrings of a valuation ring form a chain under inclusion, we conclude that $A_i \subseteq D_{P\cap D} \subseteq R_P$ for some i. This proves that $\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_n$ is the set of height-one prime ideals of R.

For each i, let $B_i = \bigcap_{P\in\mathscr{F}_i} R_P$. Then $A_i \subseteq B_i$. As an intersection of DVRs, B_i is integrally closed, so the integral closure $\overline{A_i}$ of A_i in the quotient field F of R is contained in B_i. Since F is an algebraic extension of the quotient field of D, $\overline{A_i}$ is a Prüfer domain [14, Theorem 101, p. 71], and hence, as an overring of $\overline{A_i}$, B_i is also a Prüfer domain. Finally, since $\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_n$ is the set of height-one prime ideals of R, we have $R = B_1 \cap \cdots \cap B_n$, proving that R is an intersection of n Prüfer overrings. By Lemma 3.1, R is an intersection of n Dedekind overrings. □

We now use Theorem 2.2, Lemma 3.2 and Noether normalization to exhibit classes of Krull domains that can be represented as a finite intersection of Dedekind overrings.

Theorem 3.3 *Let R be a Krull domain with quotient field $Q(R)$, let D be a subring of R with quotient field $Q(D)$, and suppose that R is a finitely generated D-algebra:*

(1) If D is a Krull domain and the extension $Q(D) \subseteq Q(R)$ is not algebraic, then R is the intersection of two Dedekind overrings and a semilocal PID overring.
(2) If D is a Krull domain that is the intersection of n Prüfer overrings and the extension $Q(D) \subseteq Q(R)$ is algebraic, then R is the intersection of n Dedekind overrings and a semilocal PID overring.
(3) If D is a Dedekind domain, then R is the intersection of two Dedekind overrings and a semilocal PID overring.
(4) If D is a field, then R is either a field or the intersection of two Dedekind overrings.

Proof (1) By Noether normalization for rings [28, Tag 07NA], there is a nonzero element $d \in D$ and algebraically independent elements $x_1, \ldots, x_m \in R[1/d]$ such that $R[1/d]$ is a module-finite extension of $D[1/d][x_1, \ldots, x_m]$, where the latter ring is isomorphic to a polynomial ring in m variables over $D[1/d]$. (Since $Q(D) \subseteq Q(R)$ is not algebraic, we have $m \geq 1$.) By Theorem 2.2, there exist PID overrings A_1 and A_2 of $D[1/d][x_1, \ldots, x_n]$ such that $D[1/d][x_1, \ldots, x_n] = A_1 \cap A_2$. Therefore, Lemma 3.2 implies that $R[1/d] = B_1 \cap B_2$ for some Dedekind overrings B_1, B_2 of $R[1/d]$. If d is a unit in R and hence $R[1/d] = R$, then we may choose any DVR overring of B_1 to be the semilocal PID in the statement of (1) and the claim is proved. On the other hand, suppose d is not a unit in R. Since R is a Krull domain, there are only finitely many height-one prime ideals $P_1, \ldots, P_n$ of R that contain d and we have $R = R[1/d] \cap R_{P_1} \cap \cdots \cap R_{P_n}$. The ring $C = R_{P_1} \cap \cdots \cap R_{P_n}$ is a semilocal PID (see [18, (11.11), p. 38]), so since $R = B_1 \cap B_2 \cap C$, statement (1) is proved.

(2) Again by Noether normalization and the fact that $Q(R)$ is algebraic over $Q(D)$, we have $0 \neq d \in D$ such that $R[1/d]$ is finite over $D[1/d]$. By Lemma 3.2, $R[1/d]$ is an intersection of n Dedekind overrings. With $P_1, \ldots, P_n$ the height-one prime ideals of R that contain d, we have as in (1) that $R = R[1/d] \cap (R_{P_1} \cap \cdots \cap R_{P_n})$ is an intersection of n Dedekind overrings and a semilocal PID overring.

(3) If $Q(R)$ is transcendental over $Q(D)$, then this follows from (1), while if $Q(R)$ is algebraic over $Q(D)$, we may apply (2).

(4) Since D is a field and R has dimension at least 1, Noether normalization implies there exist algebraically independent elements $x_1, \ldots, x_m \in R$ such that R is a module-finite extension of $D[x_1, \ldots, x_m]$. By Theorem 2.2, $D[x_1, \ldots, x_m]$ is an intersection of two PID overrings, and so by Lemma 3.2, R is an intersection of two Dedekind overrings. □

Corollary 3.4 *Let R be an integrally closed domain that is not a field. If R is a finitely generated algebra over a Dedekind domain or a field, then R is an intersection of two Dedekind overrings and a semilocal PID overring.*

Proof Let D be a ring, and let R be an integrally closed domain that is a finitely generated D-algebra. Then R is a Noetherian domain, and hence a Krull domain. If D is a field, then D is isomorphic to a subring of R and we may apply Theorem 3.3(4) to obtain that R is an intersection of two Dedekind overrings. Since any local overring of one of these Dedekind overrings is a local PID, the claim is proved in the case in which D is a field.

Now suppose that D is a Dedekind domain. Let $f : D \to R$ denote the structure map given by $f(d) = d \cdot 1$ for each $d \in D$. If f is not injective, then since R is a domain and D has Krull dimension one, the image of f in R is a field. Since R is a finitely generated algebra over this field, we are once more in the previous case, and the claim is proved. Otherwise, if f is injective, then the claim follows from Theorem 3.3(3). □

4 One-Dimensional Completely Integrally Closed Domains

This section is motivated by a very narrow version of Questions 1.1(1) and (4): Is every completely integrally closed local domain R of Krull dimension one an intersection of finitely many Prüfer overrings? While we have no answer to this question, we show that as long as R is not a valuation domain, then any Prüfer overrings in such a representation of R must meet several strong requirements that in effect assert that they are each the intersection of a large and complicated set of valuation overrings, both quantitatively (in terms of cardinality) and qualitatively (in terms of the intersection having no sharp prime ideals; see the discussion before Lemma 4.5 for the definition of a sharp prime ideal).

Pulling back from the one-dimensional case, our approach works in a more general situation, that in which R is a completely integrally closed domain for which $R = A_1 \cap \cdots \cap A_n \cap D$, where each A_i is a proper Prüfer overring and D is a proper integrally closed overring. We examine in Theorem 4.6 what such a decomposition implies about R in the case in which the A_i are also G-domains, where a domain is a *G-domain* (in the sense of Goldman [14, Sect. 1.3]) if the intersection of all its nonzero prime ideals is nonzero. The relevance here to the one-dimensional case is that every overring of a one-dimensional local domain is a G-domain. This allows us in Theorem 4.6 to draw out consequences for the one-dimensional case. As the arguments indicate, the preoccupation with G-domains is because we rely on a number of results from elsewhere that require various weaker constraints on intersections of prime ideals, constraints that are encompassed by this assumption.

Thus, the theme of this section is: *if* a completely integrally closed local domain can be represented as the intersection of finitely many Prüfer G-overrings, then what can be said about these Prüfer domains? The hope is that the observations made in examining this question can be brought to bear on the one-dimensional completely integrally local version of Questions 1.1(1) and (4). As we discuss at the end of the section, an example due to Nagata is a particularly relevant test case for this question.

Most of the results in this section rely on recent work on intersections of valuation domains. We often consider subsets X of the set of all valuation overrings of a domain R. We denote by $J(X)$ the intersection of the maximal ideals of the valuation rings in X. If $R = \bigcap_{V \in X} V$, then $J(X)$ is contained in the Jacobson radical $J(R)$ of R; see [12, Remark 1.4].

Our first results concern the number of maximal ideals of the components in an irredundant intersection of finitely many Prüfer G-overrings. For this, we rely on the following lemma that gives a sufficient condition for when countably many valuation rings can be omitted from a representation of a completely integrally closed local domain.

Lemma 4.1 ([24, Corollary 3.7]) *Let R be a completely integrally closed local domain, and let X be a set of valuation overrings of R with $J(X) \neq 0$ and $R = \bigcap_{V \in X} V$. Then either R is a rank one valuation ring or X is uncountable and $R = \bigcap_{V \in X \setminus Y} V$ for every countable subset Y of X.*

A collection $\mathscr{F}$ of prime ideals of a domain R is a *defining family* for R if $R = \bigcap_{P \in \mathscr{F}} R_P$. For example, the set of maximal ideals of R is a defining family for R, while if R is a Krull domain, the set of height-one prime ideals is also a defining family.

Lemma 4.2 *Let R be a completely integrally closed local domain, and suppose that $R = A \cap D_1 \cap \cdots \cap D_n$, where A is a proper Prüfer overring of R and each D_i is a proper integrally closed overring of R with nonzero Jacobson radical. If A cannot be omitted from this representation, then each defining family $\mathscr{G}$ of prime ideals of A having nonzero intersection is uncountable. Moreover, for each nonempty co-countable subset $\mathscr{F}$ of $\mathscr{G}$ we have*

$$R = \left(\bigcap_{P \in \mathscr{F}} A_P \right) \cap D_1 \cap \cdots \cap D_n.$$

In particular, if $J(A) \neq 0$, then A has uncountably many maximal ideals.

Proof Since the overrings of a valuation domain are ordered by inclusion and A and all the D_i are proper overrings of R, the ring R is not a valuation domain.

We use this below when we appeal to Lemma 4.1. Let $\mathscr{G}$ be a defining family of prime ideals of A having nonzero intersection. Then $A = \bigcap_{P \in \mathscr{G}} A_P$, and since A is a Prüfer domain, each ring A_P is a valuation ring. Moreover, the intersection $\bigcap_{P \in \mathscr{G}} PA_P$ of the maximal ideals of the valuation rings A_P, where P ranges over $\mathscr{G}$, is nonzero since the intersection of the prime ideals in $\mathscr{G}$ is nonzero.

Let X be the set of all the valuation overrings V of R such that V is minimal among the valuation overrings of D_i for some i. Then $D_1 \cap \cdots \cap D_n = \bigcap_{V \in X} V$ and $J(D_1) \cap \cdots \cap J(D_n) \subseteq J(X)$, since the minimality of each $V \in X$ implies it is centered on some maximal ideal of one of the D_i. Because $D_1 \cap \cdots \cap D_n$ has the same quotient field as each of the D_i and each D_i has nonzero Jacobson radical, it follows that $J(D_1) \cap \cdots \cap J(D_n) \neq 0$. This shows $J(X) \neq 0$, and similarly $J(X) \cap (\bigcap_{P \in \mathscr{G}} PA_P) \neq 0$. Thus, if $\mathscr{G}$ is a countable set, Lemma 4.1 implies that $R = D_1 \cap \cdots \cap D_n$, contrary to the assumption that A cannot be omitted from the given representation of R. We conclude that every defining family of prime ideals of A having nonzero intersection is uncountable. Similarly, if $\mathscr{F}$ is a co-countable nonempty subset of $\mathscr{G}$, Lemma 4.1 implies that

$$R = \left(\bigcap_{P \in \mathscr{F}} A_P \right) \cap D_1 \cap \cdots \cap D_n.$$

The last claim of the lemma is now clear. □

The relevance of the next lemma is that it will allow us to replace a Prüfer overring in a representation of a domain with one that is "maximal" in the representation.

Lemma 4.3 *Let R be a domain, and suppose that $R = A_1 \cap \cdots \cap A_n \cap D$, where D and each A_i is an overring of R. Then there exist overrings $B_1, \ldots, B_n$ of R such that $A_i \subseteq B_i$ for each i, $R = B_1 \cap \cdots \cap B_n \cap D$ and no B_i can be replaced in this intersection by one of its proper overrings. Moreover, if R is completely integrally closed, then so is each B_i.*

Proof Let $\mathscr{F}$ be the set of n-tuples $(B_1, \ldots, B_n)$ such that each B_i is an overring of A_i and $R = B_1 \cap \cdots \cap B_n \cap D$. Then $\mathscr{F}$ is nonempty since $(A_1, \ldots, A_n) \in \mathscr{F}$. Define a partial order $\leq$ on $\mathscr{F}$ by $(B_1, \ldots, B_n) \leq (C_1, \ldots, C_n)$ if $B_i \subseteq C_i$ for each i. To justify an application of Zorn's lemma, suppose that $\{(B_{i1}, \ldots, B_{in}) : i \in I\}$ is a chain in $\mathscr{F}$. Let $B_1 = \bigcup_i B_{i1}, \ldots, B_n = \bigcup_i B_{in}$. The fact that for each j the rings in $\{B_{ij} : i \in I\}$ form a chain implies that B_j is a ring. Moreover, $R = B_1 \cap \cdots \cap B_n \cap D$, so $(B_1, \ldots, B_n) \in \mathscr{F}$. By Zorn's lemma, there is a maximal member $(C_1, \ldots, C_n)$ of $\mathscr{F}$. Thus, $R = C_1 \cap \cdots \cap C_n \cap D$ and maximality implies that no C_i can be replaced by one of its proper overrings.

Suppose now that R is completely integrally closed. Let $1 \leq i \leq n$. To prove that C_i is completely integrally closed it suffices to show that no proper overring of C_i is a fractional ideal of C_i. Suppose D_i is an overring of C_i and there is $0 \neq r \in R$ such that $rD_i \subseteq C_i$. Then $r(D_i \cap (\bigcap_{j \neq i} C_j) \cap D) \subseteq R$, so since R is completely integrally closed, we have $R = D_i \cap (\bigcap_{j \neq i} C_j) \cap D$. Since C_i cannot be replaced by one of its proper overrings, we conclude that $C_i = D_i$, thus proving that C_i is completely integrally closed. □

For lack of a reference, we mention the following routine observation that will be needed in the proof of Theorem 4.6.

Lemma 4.4 *Let R be a domain, and let A and B be overrings of R. Then A and B are G-domains if and only if $A \cap B$ is a G-domain.*

Proof Suppose A and B are G-domains, and let F denote the quotient field of R. Since A and B are overrings of R and G-domains, there exists $0 \neq r \in R$ such that r is in the intersection of all the nonzero prime ideals of A as well as those of B. Thus,

$$(A \cap B)[1/r] = A[1/r] \cap B[1/r] = F.$$

Consequently, the nonzero element r is contained in every prime ideal of $A \cap B$, proving that $A \cap B$ is a G-domain.

Conversely, if $A \cap B$ is a G-domain, then there is $0 \neq r \in A \cap B$ such that r is contained in every prime ideal of $A \cap B$. Thus, $F = (A \cap B)[1/r]$, so that $A[1/r] = F = B[1/r]$, from which it follows that A and B are G-domains. □

A prime ideal P of a Prüfer domain R is *sharp* if there is a finitely generated ideal contained in P but not in any maximal ideal of R that does not contain P; equivalently, $\bigcap_{P \not\subseteq M} R_M \not\subseteq R_P$, where M ranges over the maximal ideals of R not containing P; see [7, Proposition 2.2]. A prime ideal of a Prüfer domain that is not the union of the prime ideals properly contained in it is sharp if and only if it is the radical of a finitely generated ideal; see the discussion before Proposition 2.2 in [7].

Lemma 4.5 *Let R be a Prüfer domain, and let A and B be Prüfer overrings of R such that $R = A \cap B$. If some nonzero prime ideal of R is sharp, then so is some nonzero prime ideal of A or B.*

Proof Let P be a nonzero sharp prime ideal of R. Since R is a Prüfer domain, $P = PA \cap PB$ [9, Theorem 26.1] and so we may assume without loss of generality that $PA \neq A$ and hence (again since R is a Prüfer domain) that PA is a prime ideal of A [9, Theorem 26.1]. Since P is sharp there is a finitely generated ideal I of R contained in P such that the only maximal ideals containing I are those that contain P. To see that IA plays the same role for PA as I does for P, suppose that N is a maximal ideal of A that contains IA. Since $I \subseteq N \cap R$, the choice of I implies that every maximal ideal of R containing $N \cap R$ contains P. Moreover, since R is a Prüfer domain, no two incomparable prime ideals of R are contained in the same maximal ideal of R. Therefore, $N \cap R$ and P are comparable, so that PA and $(N \cap R)A$ are comparable. Since R is a Prüfer domain, $N = (N \cap R)A$ [9, Theorem 26.1]. As N is a maximal ideal of A, we thus have $PA \subseteq N$, which proves that PA is a sharp prime ideal of A. □

We do not have an example for which the hypotheses of the next theorem and corollary are satisfied. Instead, the point of these two results is to derive some consequences of such hypotheses that may be useful in verifying whether the hypotheses themselves can be obtained.

Theorem 4.6 *Let R be a completely integrally closed local domain, and suppose there are proper overrings $A_1, \ldots, A_n, D$ of R such that $R = A_1 \cap \cdots \cap A_n \cap D$, each A_i is a Prüfer G-overring of R, and D is an integrally closed domain that either has nonzero Jacobson radical or is the quotient field of R. Then there is a positive integer $m \leq n$ and Prüfer G-overrings $B_1, \ldots, B_m$ of R such that:*

(a) $R = B_1 \cap \cdots \cap B_m \cap D$,
(b) *each B_i is an overring of one of the A_j,*
(c) *no B_i has a sharp prime ideal,*
(d) *each B_i has uncountably many height-one prime ideals (and hence uncountably many maximal ideals),*
(e) *each of $B_2, \ldots, B_m$ has at least as many height-one prime ideals as the cardinality of the residue field of R, and*
(f) *at most one of the B_i has Krull dimension one.*

Proof Since the overrings of a valuation domain are ordered by inclusion and D and all the A_i are proper overrings of R, the ring R is not a valuation domain. We use this fact later in the proof when we appeal to Lemmas 4.1 and 4.2.

We first prove (a), (b) and (c). By Lemma 4.3, there are completely integrally closed overrings $B_1, \ldots, B_n$ of R such that $A_i \subseteq B_i$ for each i; $R = B_1 \cap \cdots \cap B_n \cap D$; and no B_i can be replaced in this intersection by one of its proper overrings. Since an overring of a G-domain is a G-domain by Lemma 4.4, and an overring of a Prüfer domain is a Prüfer domain, each B_i is a Prüfer G-domain. After relabeling we may assume there is $m \leq n$ such that $R = B_1 \cap \cdots \cap B_m \cap D$ and each B_i is irredundant in this representation. (Since D is a proper overring of R, at least one of the B_i is needed to represent R.)

We claim that for each i, B_i has no nonzero prime ideals that are sharp. Fix a positive integer. $i \leq m$, and let P be a nonzero prime ideal of B_i. Let $\mathscr{F}$ be the set of maximal ideals of B_i not containing P. Since B_i is a Prüfer domain, we have by [8, Theorem 3.2.6, p. 49] that if $\mathscr{F}$ is empty, then $(P :_F P) = (B_i)_P$, while if $\mathscr{F}$ is nonempty, then

$$(P :_F P) = (B_i)_P \cap \left(\bigcap_{M \in \mathscr{F}} (B_i)_M \right).$$

Since B_i is completely integrally closed, $B_i = (P :_F P)$ so that in the former case $B_i = (B_i)_P$, while in the latter case

$$B_i = (B_i)_P \cap \left(\bigcap_{M \in \mathscr{F}} (B_i)_M \right).$$

If $B_i = (B_i)_P$, then B_i is local with maximal ideal P, contrary to Lemma 4.2, which requires B_i to have uncountably many maximal ideals. Therefore, $\mathscr{F}$ is nonempty. The intersection of all the ideals in the defining family $\{P\} \cup \mathscr{F}$ of B_i is nonzero since B_i is a G-domain. Since B_i cannot be omitted from the representation in (a), Lemma 4.2 implies that

$$R = \left(\bigcap_{M \in \mathscr{F}} (B_i)_M \right) \cap \left(\bigcap_{j \neq i} B_j \right) \cap D.$$

Since B_i cannot be replaced in the representation in (a) with one of its overrings, we conclude that

$$\bigcap_{M \in \mathscr{F}} (B_i)_M = B_i \subseteq (B_i)_P,$$

proving that P is not a sharp prime ideal of B_i.

We have verified (a), (b), and (c). However, we wish to make an additional reduction to the case that $B_i \cap B_j$ is not a Prüfer domain for any $i \neq j$. Such a reduction is possible since if there are i, j such that $B_i \cap B_j$ is a Prüfer domain, then by Lemmas 4.4 and 4.5, $B_i \cap B_j$ is a completely integrally closed Prüfer G-domain with the property that no nonzero prime ideal is sharp. Therefore, we can replace B_i and B_j in the representation of R with the intersection $B_i \cap B_j$ to obtain a representation $R = B_1 \cap \cdots \cap B_n \cap D$ which satisfies (a), (b) and (c) and has the additional property that no intersection of two distinct B_i is a Prüfer domain. We use this last property in what follows.

(d) Fix $1 \leq i \leq m$. Since B_i is a completely integrally closed G-domain, B_i is in the intersection of the rank one valuation rings containing B_i; cf. [11, p. 359]. Since B_i is a Prüfer domain, the rank one valuation overrings of B_i are precisely the localizations of B_i at height-one prime ideals. Therefore, the height-one prime ideals of B_i constitute a defining family for B_i. Since B_i is a G-domain, the set of height-one prime ideals has nonzero intersection, so Lemma 4.2 implies that B_i has uncountably many height-one prime ideals. Since a Prüfer domain has the property that no two incomparable prime ideals are contained in the same maximal ideal, it follows that B_i has uncountably many maximal ideals.

(e) Suppose there are two distinct indices i, j such that B_i and B_j each has fewer height-one prime ideals than the cardinality of the residue field of R. As in (d), B_i and B_j are each intersections of the valuation rings that are localizations at height-one primes. Hence $B_i \cap B_j$ is an intersection of a set of valuation rings whose cardinality is less than that of the cardinality of the residue field of R. By [22, Corollary 3.8], this implies that $B_i \cap B_j$ is a Prüfer domain, contrary to assumption. Therefore, no more than one of the B_i can have fewer height-one primes than the cardinality of the residue field of R. After relabeling we may assume only B_1 can have this property.

(f) Suppose that there are distinct i, j such that B_i, B_j have Krull dimension one. The intersection of finitely many one-dimensional Prüfer overrings of a domain, each with nonzero Jacobson radical, is a one-dimensional Prüfer domain [23, Corollary 5.11]. Therefore, $B_i \cap B_j$ is a one-dimensional Prüfer domain contradicting the choice of B_i, B_j. We conclude that at most one of $B_1, \ldots, B_m$ has Krull dimension one. □

Remark 4.7 A theme of Theorem 4.6 is that the Prüfer overrings B_i that comprise the representation of R must meet somewhat stringent conditions. This ultimately is because the resulting representation by valuation overrings must also meet stringent conditions. To explain this topologically, we recall the notion of the *Zariski–Riemann space* Zar(R) of a domain R. As a set, Zar(R) consists of the valuation overrings of R. The *Zariski topology* on Zar(R) has as a basis of open sets the sets of the form

$$\mathscr{U}(x_1, \ldots, x_n) = \{V \in \text{Zar}(R) : x_1, \ldots, x_n \in V\},$$

where $x_1, \ldots, x_n$ are in the quotient field F of R. For each $x \in F$, let

$$\mathscr{V}(x) = \{V \in \text{Zar}(R) : x \notin V\}.$$

The *patch topology* on Zar(R) has a basis of open sets the sets of the form $\mathscr{U}(x_1, \ldots, x_n) \cap \mathscr{V}(y_1) \cap \cdots \cap \mathscr{V}(y_m)$, where $x_1, \ldots, x_n, y_1, \ldots, y_m \in F$. With respect to the patch topology, Zar(R) is a compact Hausdorff space having a basis of clopen sets. For more background on the patch topology in the context of the Zariski–Riemann space (considerably more background than will be needed for our present purposes), see, for example, [6, 21, 23, 24] and their references. A topological space is *perfect* if every point in the space is a limit point; i.e., the space has no open subsets consisting of one element only. In [24, Theorem 3.5] it is proved that if R is a completely integrally closed local domain that is not a valuation domain, and X is a set of valuation overrings of R such that $R = \bigcap_{V \in X} V$ and $J(X) \neq 0$, then there is a subset Y of the set of patch limit points of X such that $R = \bigcap_{V \in Y} V$ and Y is perfect and closed in the patch topology. The fact that X can thus be replaced by a perfect space has several strong consequences that are examined in [24]. Related ideas lead to the assertion that X can also be replaced by a collection of valuation rings whose residue fields are transcendental over the residue field of R [24, Theorem 3.11].

Theorem 4.6 specializes in dimension one to the following corollary, which is the main objective of this section. The ostensible difference here with Theorem 4.6 is that we do not need to assume the original representation of R consists of G-domains. This is simply because every overring of a one-dimensional local domain is a G-domain by Lemma 4.4.

Corollary 4.8 *Let R be a completely integrally closed one-dimensional local domain. If $R = A_1 \cap \cdots \cap A_n$ for some proper Prüfer overrings $A_1, \ldots, A_n$ of R, then there is $m \leq n$ and Prüfer overrings $B_1, \ldots, B_m$ of R such that $R = B_1 \cap \cdots \cap B_m$ and statements (b)–(f) of Theorem 4.6 hold for the B_i.*

Proof With D the quotient field of R, this is a consequence of Theorem 4.6 and the fact that every overring of a one-dimensional local domain is a G-domain. □

Despite an early conjecture by Krull to the contrary, there do exist completely integrally closed one-dimensional local domains that are not valuation rings. Nagata constructed such an example in [16, 17] by intersecting certain rank one valuation rings, and Ribenboim gave a different perspective on this example in [26]. This construction can even be adapted to produce examples over the polynomial ring $k[X, Y]$, where k is a field [20, Proposition 4.4].

We formalize some of the questions that are implicit in this section.

Questions 4.9

(1) Does there exist a completely integrally closed local domain R that is an intersection of finitely many proper Prüfer G-overrings? By Lemma 4.4, R must itself be a G-domain.
(2) Is every one-dimensional completely integrally closed local domain an intersection of finitely many Prüfer overrings? This is a special case of Question 1.1(4) from the introduction.
(3) Is the one-dimensional completely integrally closed local domain constructed by Nagata an intersection of finitely many Prüfer overrings?

Loosening the requirement that the local domain in (1) be completely integrally closed to the assumption that it is integrally closed, the answer to question (1) is affirmative. Such an example can be found in [10, Example 12, Sect. 32], where a one-dimensional local domain R is constructed as a pullback of a DVR and the field of rational functions in one variable over the residue field of R. This ring has the property that if $X \cup Y$ is a disjoint partition into nonempty subsets of the set of rank two valuation overrings that dominate R, then $R = (\bigcap_{V \in X} V) \cap (\bigcap_{V \in Y} V)$ and both $\bigcap_{V \in X} V$ and $\bigcap_{V \in Y} V$ are proper Prüfer G-overrings of R. For a close analysis of this example, see [5, Sect. 4].

5 Special Case: The Intersection of a Prüfer Domain and a Valuation Domain

As a final illustration of some of the subtleties involved in intersecting Prüfer overrings, we briefly discuss the special case of an intersection of a Prüfer overring and finitely many valuation overrings. To contrast this with the setting of the previous section, we begin with the following observation.

Proposition 5.1 *A completely integrally closed local domain cannot be written as an intersection of finitely many proper valuation overrings and a proper integrally closed overring with nonzero Jacobson radical.*

Proof Suppose R is a completely integrally closed domain and $R = V_1 \cap \cdots \cap V_n \cap D$, where the V_i are proper valuation overrings and D is a proper integrally closed overring with nonzero Jacobson radical. Since D is a proper overring of R, we may assume without loss of generality that V_1 cannot be omitted from this representation. However, Lemma 4.2 implies then that V_1 is not local, contrary to assumption. Therefore, no such representation of R is possible. □

Since any overring of a one-dimensional local domain is either equal to the quotient field of R or is a G-domain, Proposition 5.1 implies that one-dimensional completely integrally closed local domains are excluded from the setting of this section. We state this more precisely.

Corollary 5.2 *A one-dimensional completely integrally closed local domain cannot be written as an intersection of finitely many proper valuation overrings and a proper integrally closed overring.*

In the case in which the valuation rings have rank one, another explanation for Proposition 5.1 is given by the following general result due to Heinzer.

Theorem 5.3 (Heinzer [12, Corollary 1.16]) *Let D be a domain with nonzero Jacobson radical. If $V_1, \ldots, V_n$ are rank one valuation rings of the quotient field of D, then D is a localization of $R := V_1 \cap \cdots \cap V_n \cap D$. Moreover, irredundant V_i are centered on maximal ideals of R and are localizations of R.*

If in Theorem 5.3 the valuation rings are assumed to have rational rank one, then regardless of whether D has nonzero Jacobson radical, the irredundant V_i are localizations of R, though not necessarily at a maximal ideal, see [13, Lemma 1.3].

If in Theorem 5.3, D is assumed to be a Prüfer G-domain, then the structure of R is disposed of by the following theorem, which is a consequence of some of the topological methods from [24] discussed in Remark 4.7.

Theorem 5.4 [24, Corollary 4.5] *Let $V_1, \ldots, V_n$ be rank one valuation rings of F. If R is a Prüfer G-domain with quotient field F, then $V_1 \cap \cdots \cap V_n \cap R$ is a Prüfer G-domain with quotient field F.*

The crux of the theorem is that R is a G-domain and that the valuation rings have rank one. If one of the valuation rings V_i does not have rank one, then R need not be a Prüfer domain, see [24, Example 4.6]. Similarly, without the requirement that D is a G-domain, the structure of an intersection of a Prüfer domain and finitely many rank one valuation rings can be much more complicated, and certainly not a Prüfer domain. Theorem 2.6 illustrates this well: Every semilocal two-dimensional Krull domain with finitely generated class group is an intersection of a PID and finitely many DVRs.

Motivated by Theorem 2.6, we consider more closely the intersection of a PID and a valuation ring. We restrict even further to the case of the intersection of a polynomial ring over a field and a valuation ring. Two elementary observations are that such an intersection is an integrally closed domain, and if the valuation ring in question has rank one, then the intersection is completely integrally closed. To close this section we discuss some results that show how sensitive the structure of the intersection is to the choice of valuation overring. A theorem of Abhyankar, Eakin, and Heinzer provides a good example of this sensitivity. We state the theorem in a form that is similar to the way it is given in [15].

Theorem 5.5 (Abhyankar–Eakin–Heinzer [1, Theorem 5.7]) *Let V be a DVR with quotient field F, and let $V_1, V_2, \ldots, V_n$ be DVR overrings of $V[X]$ such that $V_i \cap F = V$ for each i. Let $R = V_1 \cap V_2 \cap \cdots \cap V_n \cap F[X]$. Then R is a Dedekind domain provided the residue field of each V_i is algebraic over the residue field of V. If one of the residue fields is not algebraic over the residue field of V, then R is a two-dimensional Noetherian domain.*

Motivated by this theorem, Loper and Tartarone explore in [15] the structure of the integrally closed rings between $V[X]$ and $F[X]$ in the case where $V = \mathbb{Z}_p$ with p a prime integer. Each such ring R is an intersection of $F[X]$ and (in general, infinitely many) valuation overrings of $V[X]$. Loper and Tartarone prove however that if P is a prime ideal of R, then R_P is an intersection of a single valuation ring and a localization of $F[X]$. Thus, the structure of the ring R depends locally on the question of the structure of the intersection of a valuation overring of $V[X]$ and a PID overring of $F[X]$. The case in which the valuation overring is a DVR is settled by Theorem 5.5. The remaining cases in which the valuation overring need not be a DVR or even have rank one are dealt with in [15], and it is shown which choices of valuation ring produce Prüfer domains, PvMDs, Mori domains, and Krull-type domains.

Some of this framework is extended to regular local rings of Krull dimension two in [25]. Let (D, m) be a two-dimensional integrally closed local Noetherian domain, let $0 \neq f \in m$ such that $\sqrt{fD}$ is a prime ideal of D, and let n be a positive integer. A criterion [25, Theorem 1.1] is given for when every integrally closed ring R between D and D_f and maximal ideal M of R, there is a representation $R_M = V_1 \cap \cdots \cap V_n \cap (D_f)_M$, for some not necessarily distinct valuation overrings $V_1, \ldots, V_n$ of R. This criterion is shown to be satisfied for $n = 1$ if D is a two-dimensional regular local ring and either D is equicharacteristic or D has mixed characteristic and f is a prime integer in D. In this setting, too, the structure of each integrally closed ring between D and D_f depends locally on the structure of the intersection of a PID overring and a valuation overring.

Acknowledgements I thank the referee for the helpful comments.

References

1. S. Abhyankar, W. Heinzer, P. Eakin, On the uniqueness of the coefficient ring in a polynomial ring. J. Algebra **23**, 310–342 (1972)
2. D.D. Anderson, D.F. Anderson, Locally factorial integral domains. J. Algebra **90**, 265–283 (1984)
3. D.D. Anderson, D.F. Anderson, Finite intersections of PID or factorial overrings. Canad. Math. Bull. **28**, 91–97 (1985)
4. D.D. Anderson, D.F. Anderson, Locally almost factorial integral domains. J. Algebra **105**, 395–410 (1987)
5. A. Fabbri, Integral domains having a unique Kronecker function ring. J. Pure Appl. Algebra **215**, 1069–1084 (2011)

6. C. Finochario, M. Fontana, K.A. Loper, The constructible topology on spaces of valuation domains. Trans. Am. Math. Soc. **365**, 6199–6216 (2013)
7. M. Fontana, E. Houston, T. Lucas, Toward a classification of prime ideals in Prüfer domains. Forum Math. **22**, 741–766 (2010)
8. M. Fontana, J. Huckaba, I. Papick, *Prüfer Domains* (Marcel Dekker, New York, 1997)
9. R. Gilmer, *Multiplicative Ideal Theory*. Queen's Papers in Pure and Applied Mathematics, vol. 12 (Queen's University, Kingston, 1968)
10. R. Gilmer, Some questions for further research, *Multiplicative Ideal Theory in Commutative Algebra* (Springer, New York, 2006), pp. 405–415
11. R. Gilmer, W. Heinzer, On the complete integral closure of an integral domain. J. Aust. Math. Soc. **6**, 351–361 (1966)
12. W. Heinzer, Noetherian intersections of integral domains. II, in *Conference on Commutative Algebra (University of Kansas, Lawrence, 1972)*. Lecture Notes in Mathematics, vol. 311 (Springer, Berlin, 1973), pp. 107–119
13. W. Heinzer, J. Ohm, Noetherian intersections of integral domains. Trans. Am. Math. Soc. **167**, 291–308 (1972)
14. I. Kaplansky, *Commutative Rings* (Allyn and Bacon, Boston, 1970)
15. K.A. Loper, F. Tartarone, A classification of the integrally closed rings of polynomials containing $\mathbb{Z}[x]$. J. Commut. Algebra **1**, 91–157 (2009)
16. M. Nagata, On Krull's conjecture concerning valuation rings. Nagoya Math. J. **4**, 29–33 (1952)
17. M. Nagata, Corrections to my paper "On Krull's conjecture concerning valuation rings". Nagoya Math. J. **9**, 209–212 (1955)
18. M. Nagata, Local rings, *Interscience Tracts in Pure and Applied Mathematics*, vol. 13 (Wiley, New York, 1962)
19. B. Olberding, Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings. J. Pure Appl. Algebra **212**, 1797–1821 (2008)
20. B. Olberding, Intersections of valuation overrings of two-dimensional Noetherian domains, *Commutative Algebra: Noetherian and non-Noetherian Perspectives* (Springer, Berlin, 2010), pp. 335–362
21. B. Olberding, Affine schemes and topological closures in the Zariski-Riemann space of valuation rings. J. Pure Appl. Algebra **219**, 1720–1741 (2015)
22. B. Olberding, On the geometry of Prüfer intersections of valuation rings. Pac. J. Math. **273**, 353–368 (2015)
23. B. Olberding, A principal ideal theorem for compact sets of rank one valuation rings. J. Algebra **489**, 399–426 (2017)
24. B. Olberding, On the topology of valuation-theoretic representations of integrally closed domains. J. Pure Appl. Algebra **222**, 2267–2287 (2018)
25. B. Olberding, F. Tartarone, Integrally closed rings in birational extensions of two-dimensional regular local rings. Math. Proc. Camb. Philos. Soc. **155**, 101–127 (2013)
26. P. Ribenboim, Sur une note de Nagata relative à un problème de Krull. Math. Z. **64**, 159–168 (1956)
27. R. Sharp, *Steps in Commutative Algebra (London Mathematical Society Student Texts)* (Cambridge University Press, Cambridge, 2001)
28. Stacks Project Authors, The stacks project (2018), https://stacks.math.columbia.edu
29. I. Swanson, C. Huneke, *Integral Closure of Ideals, Rings, and Modules*, vol. 336, London Mathematical Society Lecture Note Series (Cambridge University Press, Cambridge, 1994)

Strongly Additively Regular Rings and Graphs

Thomas G. Lucas

Abstract A commutative ring R is said to be additively regular if for each pair of elements $f, g \in R$ with f regular, there is an element $t \in R$ such that $g + ft$ is regular. For any commutative ring R, the polynomial ring $R[\text{x}]$ is additively regular, moreover if $deg(g) < n$, then $g + f\text{x}^n$ is regular when $f \in R[x]$ is regular. We introduce several stronger types of additively regular rings where the choice for t is restricted: R is strongly additively regular if for each pair of elements $f, g \in R$ with f regular and g a zero divisor, there is a regular element $t \in R$ such that $g + ft$ is regular; R is very strongly additively regular if for each pair of elements $h, k \in R$ with h regular, there is a regular element $s \in R$ such that $k + hs$ is regular. Even stronger are strongly u-additively regular and very strongly u-additively regular, for these the "t" is further restricted to being a unit of R.

1 Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity. For a ring R, we let $Z(R)$ denote the set of zero divisors of R and let $T(R)$ denote the total quotient ring of R.

Recall that for a ring R, R is *additively regular* if for each regular element $f \in R$ and each $g \in R$, there is an element $t \in R$ such that $g + ft$ is regular (this equivalent to the original definition in [4], see, for example [4, Lemma 7]). Note that if g is regular, we can simply choose $t = 0$. We introduce four stronger conditions, two where the choice of t is restricted to being regular and two where the choice of t is restricted to being a unit. We say that R is *strongly additively regular* (*strongly u-additively regular*) if for each regular $f \in R$ and each zero divisor $g \in R$, there is a regular element (unit) $t \in R$ such that $g + ft$ is regular. Also, R is *very strongly additively regular* (*very strongly u-additively regular*) if for each regular element $f \in R$ and each $g \in R$, there is a regular element (unit) $t \in R$ such that $g + ft$ is

T. G. Lucas (✉)
Department of Mathematics and Statistics, University of North Carolina Charlotte,
Charlotte, NC 28223, USA
e-mail: tglucas@uncc.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_6

regular. The field $\mathbb{Z}_2$ is strongly u-additively regular but not very strongly u-additively regular. All other fields are very strongly u-additively regular. Also, for $\mathbb{Z}_2$, $\mathbb{Z}_2[\mathrm{x}]$ is both strongly u-additively regular and very strongly additively regular but not very strongly u-additively regular. In contrast, for any ring R, the corresponding Nagata ring $R(\mathrm{x})$ is very strongly u-additively regular (for the "unit" $t \in R(\mathrm{x})$ one can simply choose a sufficiently large power of x^n). In the case $R = T(R)$ is its own total quotient ring, then regular = unit, so in this case, we are starting with the regular f as a unit and trying to find a unit t such that $g + ft$ is a unit of R. Finally, note that in the general case for a ring R, if $g = 0$, then we simply choose $t = 1$ to have $g + ft$ regular.

To save a few twigs (and quite a bit of typing), we introduce the following abbreviations: strongly additively regular = sAR, strongly u-additively regular = suAR, very strongly additively regular = vsAR, and very strongly u-additively regular = vsuAR.

The second section is devoted to the study of the various strongly additively regular properties. One of the main results there is that if $R = S \times T$ for a pair of rings S and T and R is sAR, then all three of R, S, and T are vsAR. Conversely, if both S and T are vsAR, then R is vsAR (Theorem 2.3). In the third section, we concentrate on the case R is a von Neumann regular ring. One of the main results there is that if R is a von Neumann regular ring that is not a field, then it is vsAR if and only if there is a unit $u \in R$ such that $1 + u$ is a unit (Theorem 3.3). Another characterization is that R is vsAR if and only if it has no prime ideal P such that $R/P = \mathbb{Z}_2$ (Corollary 3.10).

The fourth section introduces four new types of graphs associated with the ring R. In 1988, Istvan Beck introduced the zero-divisor graph associated to the ring R where the vertices are the elements of R and for $f \neq g$, (f, g) is an edge if and only if $fg = 0$ [3]. In 1999, David F. Anderson and Philip S. Livingston refined the definition by limiting the vertex set to the set of nonzero zero divisors of R, they denoted the resulting graph as $\Gamma(R)$ [2]. They proved that $\Gamma(R)$ is connected and has diameter less than or equal to 3. Another graph associated with David Anderson (together with Ayman Badawi) is the *total graph* of R [1]. For this graph the vertices include all elements of R and for $f \neq g$, (f, g) is an edge if and only if $f + g$ is a zero divisor. Unlike the zero-divisor graph, this graph need not be connected. Another graph associated to a ring is the comaximal graph (as in [6, 7]) where (r, s) is an edge if $rR + sR = R$ (in [7] the units of R are included in the set of vertices, while in [6] only nonunits of R are vertices). The graphs we introduce in Sect. 4 include all elements of R as the vertices. The starting point is the graph $\mathfrak{A}(R)$ where for $f \neq g$, (f, g) is an edge if and only if there is a regular element $p \in R$ such that at least one of $f + gp$ and $fp + g$ is regular. A subgraph of $\mathfrak{A}(R)$ is the graph $\mathfrak{A}_u(R)$ where (b, c) is an edge if and only if there is a unit $u \in R$ such that $b + cu$ is regular (equivalently, $bu^{-1} + c$ is regular). Both $\mathfrak{A}(R)$ and $\mathfrak{A}_u(R)$ exist for all rings (and have at least two vertices, 1 and 0, and at least one edge, namely, $(1, 0)$). For several of the results in this section we define two other graphs which need not exist for a given ring R. For existence, we require that for each regular element g of R, there is a regular element (unit) p such that $g + gp$ is regular, which makes (g, g) a loop in this new graph. It is an easy exercise to show that there is a loop at each regular element if and only if there is a loop at 1 if and only if there is a loop at some regular element f. We define the set of edges of $\mathfrak{A}_\ell(R)$ to be the union of the set of edges of $\mathfrak{A}(R)$ and the set of loops at regular elements provided there is a loop at each regular element. Similarly, the set of edges of $\mathfrak{A}_{u\ell}(R)$ is the union of the set of edges of $\mathfrak{A}_u(R)$ and the set of loops at regular elements provided there is a loop at each regular element.

2 Strongly Additively Regular Rings

We start with a simple example of a strongly additively regular ring that is not a very strongly additively regular ring.

Example 2.1 For the field $\mathbb{Z}_2$, the only regular element is 1 so there is no regular element t such that $1 + 1 \cdot t$ is regular. Thus, $\mathbb{Z}_2$ is not vsAR, but $\mathbb{Z}_2[\mathrm{x}]$ is vsAR (see Theorem 2.2 below). Also, for a field F, the regular elements of the ring $R = \mathbb{Z}_2 \times F$ all have the form $(1, x)$ for some (each) nonzero $x \in F$. The element $(1, 0)$ is a zero divisor for which there is no regular element $(b, c) \in R$ such that $(1, 0) + (1, x)(b, c)$ is regular since b must be 1.

For a polynomial $f(\mathrm{x}) = f_0 + f_1\mathrm{x} + \cdots + f_n\mathrm{x}^n \in R[\mathrm{x}]$, the ideal $C(f) = f_0R + f_1R + \cdots + f_nR$ is referred to as the "content" of $f(\mathrm{x})$. In the event $C(f) = R$, $f(\mathrm{x})$ is said to have unit content. Using the Dedekind–Mertens content formula it is easy to see that the set of all unit content polynomials is multiplicatively closed. The Nagata ring corresponding to R is the ring $R(\mathrm{x}) = R[\mathrm{x}]_{\mathscr{U}(R)}$ where $\mathscr{U}(R) = \{f(\mathrm{x}) \in R[\mathrm{x}] \mid C(f) = R\}$.

Theorem 2.2 *Let R be a commutative ring.*

1. *$R[\mathrm{x}]$ is vsAR.*
2. *$R(\mathrm{x})$ is vsuAR.*

Proof Let $f(\mathrm{x}), g(\mathrm{x}) \in R[\mathrm{x}]$ with $f(\mathrm{x})$ regular. Then $C(f)$ has no nonzero annihilator in R. For $n > deg(g)$, the content of the polynomial $d(\mathrm{x}) = g(\mathrm{x}) + f(\mathrm{x})\mathrm{x}^n$ contains $C(f)$. Thus, $C(d)$ has no nonzero annihilator in R, and therefore, $d(\mathrm{x})$ is regular.

Next, suppose (instead) that $f(\mathrm{x}), g(\mathrm{x}) \in R(\mathrm{x})$ with $f(\mathrm{x})$ regular. Then there are polynomials $h(\mathrm{x}), k(\mathrm{x}) \in R[\mathrm{x}]$ and $u(\mathrm{x}), v(\mathrm{x}) \in \mathscr{U}(R)$ such that $f(\mathrm{x}) = h(\mathrm{x})/u(\mathrm{x})$ and $g(\mathrm{x}) = k(\mathrm{x})/v(\mathrm{x})$. The polynomial $h(\mathrm{x})$ is a regular element of $R[\mathrm{x}]$ and thus $C(h)$ has no nonzero annihilator in R. For $n > deg(k(x)) + deg(u(\mathrm{x}))$, $g(\mathrm{x}) + f(\mathrm{x})\mathrm{x}^n = (k(\mathrm{x})u(\mathrm{x}) + h(\mathrm{x})v(\mathrm{x})\mathrm{x}^n)/u(\mathrm{x})v(\mathrm{x})$ is regular as the content of $k(\mathrm{x})u(\mathrm{x}) + h(\mathrm{x})v(\mathrm{x})\mathrm{x}^n$ contains $C(hv) = C(h)$. Hence, $R(\mathrm{x})$ is vsuAR. □

Theorem 2.3 *For a pair of rings S and T, let $R = S \times T$.*

1. *The following are equivalent:*
   a. *Both S and T are vsAR.*
   b. *R is vsAR.*
   c. *R is sAR.*
2. *The following are equivalent:*
   a. *Both S and T are vsuAR.*
   b. *R is vsuAR.*
   c. *R is suAR.*

Proof Recall that an element $p = (x, y) \in R$ is regular if and only if x is a regular element of S and y is a regular element of T. Also, p is a unit if and only if x is a unit of S and y is a unit of T.

First suppose both S and T are vsAR (vsuAR) and let $f = (b, c)$ be regular and $g = (d, e) \in R$. Since S and T are vsAR (vsuAR), there are regular elements (units) $v \in S$ and $w \in T$ such that $d + bv$ and $e + cw$ are regular. It follows that $z = (v, w)$ is a regular element (unit) of R such that $g + fz$ is a regular element of R. Thus, R is vsAR (vsuAR).

It is clear that (b) implies (c) in both cases, so we assume R is sAR (suAR). We will show S is vsAR (vsuAR). Let $b \in S$ be regular and let d be in S. Then $j = (b, 1)$ is a regular element of R and $k = (d, 0)$ is a zero divisor of R. Thus, there is regular element (unit) $q = (u, v)$ of R such that $k + jq = (d + bu, 0 + v) = (d + bu, v)$ is regular element of R. It follows that $d + bu$ is a regular element of S, and therefore, S is vsAR (vsuAR). The analogous proof involving T shows that T is vsAR (vsuAR). □

In [5], we introduced the notion of an additively regular family of prime ideals as a nonempty set of primes $\mathscr{P} = \{P_\alpha\}$ of a ring R such that for each $f \in R\backslash \bigcup P_\alpha$ and each $g \in R$, there is an element $h \in R$ such that $g + fh \in R\backslash \bigcup P_\alpha$. If $\bigcup P_\alpha = Z(R)$, then $\mathscr{P}$ is an additively regular family if and only if R is additively regular (a trivial generalization of [5, Theorem 2.1]). We say that $\mathscr{P}$ is a *strongly additively regular family* if for each $f \in R\backslash \bigcup P_\alpha$ and each $g \in \bigcup P_\alpha$, there is an $h \in R\backslash \bigcup P_\alpha$ such that $g + fh \in R\backslash \bigcup P_\alpha$. Similarly, $\mathscr{P}$ is a *very strongly additively regular family* if for each $f \in R\backslash \bigcup P_\alpha$ and each $g \in R$, there is an $h \in R\backslash \bigcup P_\alpha$ such that $g + fh \in R\backslash \bigcup P_\alpha$. If one can choose h to be a unit, then $\mathscr{P}$ is a *(very) strongly additively regular u-family*. There is no requirement that f and/or h be regular, only that neither is in $\bigcup P_\alpha$. Next, we show that if there is a $P_\beta \in \mathscr{P}$ such that $R/P_\beta = \mathbb{Z}_2$, then $\mathscr{P}$ is never a very strongly additively regular family, and it is not a strongly additively regular family except in the rare case that P_β contains each P_α.

Lemma 2.4 *Let R be a ring and let $\mathscr{P} = \{P_\alpha\}$ be a nonempty family of prime ideals of R such that there is a prime $P_\beta \in \mathscr{P}$ such that $R/P_\beta = \mathbb{Z}_2$.*

1. *$\mathscr{P}$ is not a very strongly additively regular family.*
2. *If there is a $g \in \bigcup P_\alpha$ that is not in P_β, then for each pair $f, h \in R\backslash \bigcup P_\alpha$, $g + fh \in P_\beta$. Thus, in this case, $\mathscr{P}$ is not a strongly additively regular family.*
3. *If P_β contains each P_α (equivalently, $P_\beta = \bigcup P_\alpha$), then $\mathscr{P}$ is a strongly additively regular u-family.*

Proof Let $f, h, k \in R\backslash \bigcup P_\alpha$. Then all are congruent to 1 mod P_β and thus $k + fh \in P_\beta$. Hence, $\mathscr{P}$ is not a very strongly additively regular family. Similarly, if there is a g in $\bigcup P_\alpha$ but not in P_β, we have $g + fh \in P_\beta$ (since both g and fh are congruent to 1 mod P_β) and thus $\mathscr{P}$ is not a strongly additively regular family in this case.

Finally, if P_β contains each P_α, then each $g \in \bigcup P_\alpha$ is contained in P_β. Thus, we have $g + fh \in R\backslash \bigcup P_\alpha$ and so $\mathscr{P}$ is a strongly additively regular family in this (very special) case. Moreover since $g \in P_\beta$ and $f \notin P_\beta$, $g + f \in D\backslash P_\beta$ so we can simply choose $h = 1$. Therefore, $\mathscr{P}$ is a strongly additively regular u-family. □

Lemma 2.5 *Let $\mathscr{P} = \{P_\alpha\}$ be a nonempty family of primes of a ring R such that $\bigcup P_\alpha = Z(R)$.*

1. *R is sAR (suAR) if and only if $\mathscr{P}$ is a strongly additively regular family (u-family).*
2. *R is vsAR (vsuAR) if and only if $\mathscr{P}$ is a very strongly additively regular family (u-family).*

Proof Since $\bigcup P_\alpha = Z(R)$, an element is in $R\backslash \bigcup P_\alpha$ if and only if it is regular. □

A useful method for constructing reduced rings with/without certain properties is referred to as the *$A + B$ construction*. For a domain D (that is not a field), let $\mathscr{P} = \{P_\alpha\}$ be a nonempty set of prime ideals and let $\mathscr{I} = \mathscr{A} \times \mathbb{N}$ where $\mathscr{A}$ is an index set for $\mathscr{P}$. For each $i = (\alpha, n) \in \mathscr{I}$, let K_i denote the quotient field of D/P_α. Next, let $B = \sum K_i$ and form a ring $R = D + B$ from $D \times B$ by setting $(r, b) + (s, c) = (r + s, b + c)$ and $(r, b)(s, c) = (rs, rc + sb + bc)$. The ring R is referred to as the $A + B$ ring corresponding to D and $\mathscr{P}$. It is known that R is additively regular if and only if $\mathscr{P}$ is an additively regular family [5, Theorem 4.4]. We first consider the case where there is no $P_\beta \in \mathscr{P}$ such that $D/P_\beta = \mathbb{Z}_2$.

Theorem 2.6 *Let D be a domain that is not a field and let $\mathscr{P} = \{P_\alpha\}$ be a nonempty set of prime ideals of D for which there is no $P_\beta \in \mathscr{P}$ such that $D/P_\beta = \mathbb{Z}_2$. Also, let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathscr{P}$.*

1. *R is sAR (suAR) if and only if $\mathscr{P}$ is a strongly additively regular family (strongly additively regular u-family).*
2. *R is vsAR (vsuAR) if and only if $\mathscr{P}$ is a very strongly additively regular family (very strongly additively regular u-family).*

Proof For each $i = (\alpha, n)$, $r \in D$ and $b \in B$, we let r_i denote the image of r in K_i and let b_i denote the ith coordinate of b. Also, it is convenient to let e_i denote the element in B whose ith coordinate is 1 and all others are 0. If $r_i = -b_i$ for some i, then $(r, b)(0, e_i) = (0, 0)$. For the case $r \in P_\alpha$ for some α, $r_i = 0$ for each $i = (\alpha, n)$. Hence, in this case, $(r, b) \in Z(R)$ for each $b \in B$. We also have that $(1, -e_i)$ is a zero divisor for each i (with $(0, e_i)$ a nonzero annihilator). It follows easily that $(s, c) \in R$ is a zero divisor if and only if there is an $i \in \mathscr{I}$ such that $s_i = -c_i$.

It is clear that the D component of a unit of R must be a unit of D. So start with $u \in D$ a unit. As above, if $b \in B$ is such that $u_i = -b_i$ for some i, then (u, b) is a zero divisor. So assume there is no i such that $u_i = -b_i$. Define an element $c \in B$ by setting $c_i = 0$ when $b_i = 0$ and $c_i = -u^{-1}b_i/(u_i + b_i)$ when $b_i \neq 0$. We have $(u, b)(u^{-1}, c) = (1, 0)$ and thus (u, b) is a unit in this case.

Let $f = (r, b)$ and $g = (s, c)$ be elements of R with f regular. Also, let $h = (t, d)$ be a regular element of R. Then $t \in D\backslash \bigcup P_\alpha$. A necessary condition to have $g + fh$ regular is that $s + rt \notin \bigcup P_\alpha$. We also need no $i \in \mathscr{I}$ such that $s_i + r_i t_i = -(c_i + r_i d_i + t_i b_i + b_i d_i)$. We know $r_i + b_i$ is never zero, and t and d must be such that $t_i + d_i$ is never zero.

If R is sAR/vsAR (suAR/vsuAR), then we have such a regular (unit) h such that $g + fh$ is regular and so $s + rt \in D\backslash \bigcup P_\alpha$. Thus, the family is a strongly additively regular (u-)family in the sAR (suAR) case, and a very strongly regular (u-)family in the vsAR (vsuAR) case.

For the converses, suppose $\mathscr{P}$ is a [very] strongly additively regular (u-)family. Then for r and s, we have an element (unit) $t \in D\backslash \bigcup P_\alpha$ such that $s + rt \in D\backslash \bigcup P_\alpha$. The goal is show there is a $d \in B$ such that both (t, d) and $(s, c) + (r, b)(t, d)$ are regular.

Since $s + rt \in D\backslash \bigcup P_\alpha$, there is no i such that $s_i + r_i t_i = 0$. For a suitable $d \in B$, we set $d_i = 0$ when both b_i and c_i are 0. For those finitely many i where at least one of b_i and c_i is not 0, we make use of knowing that $r_i + b_i$ is never 0 to see that the equation $s_i + r_i t_i = -(c_i + t_i b_i) + (r_i + b_i)x$ has a unique solution for x in K_i. Since K_i has at least three elements, there is a $d_i \in K_i$ such that $t_i + d_i \neq 0$ and $s_i + r_i t_i \neq -(c_i + r_i d_i + t_i b_i + b_i d_i)$. It follows that both (t, d) and $(s, c) + (r, b)(t, d)$ are regular.

Thus, $\mathscr{P}$ is a [very] strongly additively regular (u-)family if and only if R is a [very] strongly (u-)additively regular ring. □

From Lemma 2.4, if there is a $P_\beta \in \mathscr{P}$ such that $D/P_\beta = \mathbb{Z}_2$, then $\mathscr{P}$ is not a very strongly additively regular family, and thus, the corresponding ring $R = D + B$ is not a very strongly additively regular ring. Similarly, $R = D + B$ is not strongly additively regular if there is an element $s \in \bigcup P_\alpha$ that is not in P_β. The remaining case is when P_β contains $\bigcup P_\alpha$ and $D/P_\beta = \mathbb{Z}_2$.

Theorem 2.7 *Let D be a domain and let $\mathscr{P} = \{P_\alpha\}$ be a nonempty family of prime ideals such that there is a $P_\beta \in \mathscr{P}$ where $P_\beta \supseteq \bigcup P_\alpha$ and $D/P_\beta = \mathbb{Z}_2$. Also, let $R = D + B$ be the $A + B$ ring corresponding to D and $\mathscr{P}$. Then R is not sAR.*

Proof Let $f = (r, b)$ be a regular element of R. Then $r \in D\backslash P_\beta$. Let $j = (\beta, 1)$ and define an element $e \in B$ by setting $e_j = 1$ and $e_i = 0$ for all other $i \in \mathscr{I}$. Since f is regular, $r \in D\backslash P_\beta$ and thus $r_j = 1$ and $b_j = 0$. Similarly, for each regular $h = (t, d) \in R$, $h_j = 1$ and $d_j = 0$. For $g = (0, e)$, $g + fh = (rt, e + rd + tb + bd)$ and so for j we have $(rt)_j = 1 = e_j + r_j d_j + t_j b_j + b_j d_j$. Thus, $g + fh$ is a zero divisor of R, and therefore, R is not sAR. □

3 Von Neumann Regular Rings

In this section, we assume R is von Neumann regular ring that is not a field. We are interested in determining when R is vsAR (equivalently, vsuAR).

For a nonzero idempotent $e \in R$, we say that $x = ex$ is an *e-unit* if $x \in R$ is a unit of the ring eR. A unit of R is a 1-unit, but will simply say unit. Also, we say that a pair of (not necessarily distinct) e-units $u = eu$ and $v = ev$ is a *unit sum pair* if $u + v = e(u + v)$ is an e-unit. The corresponding ring $S = eR$ is said to *admit unit sum pairs* if at least one pair of e-units forms a unit sum pair.

Let $\mathscr{S} = \{S = eR \mid e \in \mathscr{E}$ and S admits unit sum pairs$\}$ where $\mathscr{E}$ is the set of nonzero idempotents of R.

Lemma 3.1 *Let R be von Neumann regular ring such that the set $\mathscr{S}$ is nonempty.*

1. *For $S = eR, T = fR \in \mathscr{S}$ with $S \subsetneq T$, a unit sum pair $u = eu$ and $v = ev$ can be extended to a unit sum pair $x = fx$, $y = fy$ of T where $ex = u$ and $ey = v$.*
2. *For $S = eR \in \mathscr{S}$, if $h \in \mathscr{E}$ is such that $eh = h$, then $W = hR = hS \subseteq S$ is in the set $\mathscr{S}$.*

Proof Since $S \subsetneq T, ef = e$ and $g = f - ef = f - e$ are nonzero idempotents of T. Let u, v be e-units that form a unit sum pair. Next, let w, z be f-units that form a unit sum pair. Then we have $u = eu, v = ev, w = fw$ and $z = fz$ with $u + v = e(u + v)$ a unit of S and $w + z = f(w + z)$ a unit of T. The ring T is the internal direct sum of $S = eR = eT$ and $S' = gR = gT$. Also, gw and gz are units of S' with $egw = 0 = egz$ and $gu = geu = 0 = gev = gv$. Let $x = u + gw$ and $y = v + gz$. Both x and y are f-units. The sum $x + y = u + gw + v + gz = (u + v) + g(w + z)$ is a unit of T and we have $ex = u$ and $ey = v$.

For the second statement, we may assume $h \neq e$ so that $j = e - eh = e - h$ is a nonzero idempotent in S. Let $W' = jS = jR$. Since $hj = 0$, S is the internal direct sum of W and W'. As above, let u, v be e-units that form a unit sum pair. Then $s = hu$ and $t = hv$ are h-units (of W), and $p = ju$ and $q = jv$ are j-units (of W'). We have $s + p = u$ and $t + q = v$. As $u + v$ is an e-unit, $h(u + v) = s + t$ is an h-unit and $j(u + v) = p + q$ is a j-unit. Thus, s and t form a unit sum pair as do p and q. Therefore, W and W' are in $\mathscr{S}$. □

From statement (2) in the previous lemma, if R has a unit sum pair, then $\mathscr{S} = \{eR \mid e \in \mathscr{E}\}$. Continuing using (1): every unit sum pair u and v of $S = eR$ for some $e \in \mathscr{E}$ can be extended to a unit sum pair x and y of R (so that $ex = u$ and $ey = v$).

Lemma 3.2 *The following are equivalent for a von Neumann regular ring R:*

1. *For each unit $u \in R$, there is a unit $v \in R$ such that $u + v$ is a unit of R.*
2. *There is a unit w such that $1 + w$ is a unit of R.*
3. *R has a unit sum pair.*

Proof It is clear that (1) implies (2) and that (2) implies (3). To complete the proof we show (3) implies (1) (it is quite simple). Suppose $z, t \in R$ are units such that

$z+t$ is a unit of R and let u be a unit of R. Then all three of uz^{-1}, $uz^{-1}t$ and $uz^{-1}(z+t) = u + uz^{-1}t$ are units of R. □

The next result connects the property of being vsAR with the existence of unit pair sums (when R is von Neumann regular).

Theorem 3.3 *Let R be von Neumann regular ring that is not a field. Then the following are equivalent:*

1. *R is vsAR (=vsuAR).*
2. *eR is vsAR for each nonzero idempotent e.*
3. *There is a nonzero idempotent $e \in R\backslash\{1\}$, such that both eR and $(1-e)R$ are vsAR.*
4. *eR is sAR for each nonzero idempotent e.*
5. *R is sAR (=suAR).*
6. *For every unit $u \in R$, there is a unit $v \in R$ such that $u+v$ is a unit.*
7. *There is a unit $q \in R$ such that $1+q$ is a unit.*
8. *R admits unit sum pairs.*

Proof Since R is a total quotient ring, each regular element is a unit. Hence, R is vsAR if and only if it is vsuAR, and it is sAR if and only if it is suAR. The equivalence of (1) through (5) follows from Theorem 2.3 and the fact that R is the internal direct sum of eR and $(1-e)R$ for each nonzero idempotent $e \neq 1$.

The equivalence of (6), (7) and (8) is from Lemma 3.2.

To see that (1) implies (7), we start with $1 \in R$ viewed both as the regular element f and the arbitrary element g from the definition of "vsAR." Then we have a unit $q \in R$ such that $g + fq = 1 + q$ is a unit.

To complete the proof we show that (8) implies (1). Let $f \in R$ be a unit and let $g \in R\backslash\{0\}$. Since R is von Neumann regular ring, there is a nonzero idempotent $e \in R$ (perhaps $e = 1$) and a unit $w \in R$ such that $g = ew$. Both g and ef are e-units. By Lemma 3.1 (and the assumption that R admits unit pair sums), the ring eR admits unit sum pairs. Hence, by Lemma 3.2, there is an e-unit $k = ek$ such that $g+k$ is an e-unit. For k, there is a unit $t \in R$ such that $k = et$. The element $q = f^{-1}t$ is a unit of R and $g + k = g + fqe = g + feqe$ where both fe and qe are e-units. We are done if $e = 1$, otherwise $f(1-e)$ is a $(1-e)$-unit of $T = (1-e)R$ with R the internal direct sum of S and T. It follows that $g + fqe + f(1-e) = g + f(qe + (1-e))$ is a unit of R with $qe + (1-e)$ a unit of R. Hence, (8) implies (1). □

Corollary 3.4 *Let R be a von Neumann regular ring. If R is vsAR, then there is no nonzero idempotent e such that eR is Boolean.*

Proof We prove the contrapositive. Suppose there is a nonzero idempotent e such that $S = eR$ is Boolean. Then e is the only unit of S, so clearly, S has no unit sum pair. By Lemma 3.1, it must be that R has no unit sum pair, and therefore, R is not vsAR in this case. □

Corollary 3.5 *Let R be a von Neumann regular ring. If R is vsAR, then there is no prime ideal P such that R/P is $\mathbb{Z}_2$ (equivalently, cannot have $R_P = \mathbb{Z}_2$).*

Proof We prove the contrapositive. Suppose there is a prime P such that $R/P = \mathbb{Z}_2$ and let u and v be units of R. Then $u + P = 1 + P = v + P$ and thus $(u + v) + P = 0 + P$ and so $u + v$ is not a unit and R has no unit sum pairs. □

In the next example, the ring R is a von Neumann regular ring with characteristic 2 which has no unit sum pair. In addition, there is no nonzero idempotent e such that eR is Boolean. However, there is a prime P such that $R/P = \mathbb{Z}_2$.

Example 3.6 Let F_4 be the field with four elements and let $T = \prod_{n\in\mathbb{Z}} K_n$ with $K_n = F_4$ for each n. Next, let $\mathscr{U}$ be a nonprincipal ultrafilter on $\mathbb{Z}$ and let $\mathscr{X} = \{q \in T \mid q_m = 1 \text{ for all } m \in U, \text{ for some } U \in \mathscr{U}\}$ and $\mathscr{Y} = \{p \in T \mid p_m = 0 \text{ for all } m \in U, \text{ for some } U \in \mathscr{U}\}$.

1. $\mathscr{X} \cup \mathscr{Y}$ is a subring R of T.
2. R is von Neumann regular with characteristic 2.
3. There is no nonzero idempotent e of R such that eR is Boolean.
4. For each pair of units u and v of R, $u + v$ is not a unit.
5. The set $\mathscr{Y}$ is a prime ideal M of R such that $R/M = \mathbb{Z}_2$.
6. R is not vsAR.

Proof Note that $1 \in \mathscr{X}$ (as the tuple with all coordinates 1). Since $\mathscr{U}$ is an ultrafilter, it is closed to finite intersections. Hence, $\mathscr{X}$ is closed to finite products and $\mathscr{Y}$ is closed to finite sums. In addition, $p + q \in \mathscr{X}$ for each $p \in \mathscr{X}$ and $q \in \mathscr{Y}$. It is also the case that if $p \in \mathscr{X} \cup \mathscr{Y}$ and $q \in \mathscr{Y}$, then $pq \in \mathscr{Y}$. On the other hand, if $q_1, q_2, \ldots, q_n \in \mathscr{X}$, then $q = \sum q_i$ is in $\mathscr{X}$ if n is odd and it is in $\mathscr{Y}$ if n is even. In particular, $1 + q \in \mathscr{Y}$ for each $q \in \mathscr{X}$. Thus, $\mathscr{X} \cup \mathscr{Y}$ is a subring R of T with $\mathscr{Y}$ an ideal, in fact a prime ideal. Also, since T has characteristic 2, so does R.

For nonzero $x \in R$, if there is no k such that $x_k = 0$, then it must be that there is a set $U \in \mathscr{U}$ such that $x_m = 1$ for all $m \in U$. We may define an element $z \in R$ by setting $z_m = 1(= x_m^{-1})$ for all $m \in U$ and $z_n = x_n^{-1}$ for all $n \notin U$. Then $xz = 1$. On the other hand, if $x_k = 0$ for some k, then we may define a nonzero annihilator for x. Suppose $x \in \mathscr{Y}$. Then the set $V = \{m \in \mathbb{Z} \mid x_m = 0\}$ is in $\mathscr{U}$. Define $q \in \mathscr{X}$ such that $q_m = 1$ for all $m \in V$ and $q_n = 0$ for all $n \notin V$. Clearly, $qx = 0$. We could also simply choose an m such that $x_m = 0$, and define a nonzero idempotent $e \in \mathscr{Y}$ by setting $e_m = 1$ and $e_n = 0$ for all $n \neq m$. On the other hand, if $x \in \mathscr{X}$, then an annihilator must be in $\mathscr{Y}$ as $x_m = 1$ for all $m \in U$ for some $U \in \mathscr{U}$. For such an x, we define $z \in \mathscr{Y}$ by setting $z_n = 0$ when $x_n \neq 0$ and $z_n = 1$ when $x_n = 0$. We will have $z_m = 0$ for all $m \in U$, and $z_k = 1$ for at least one $k \notin U$. So $z \neq 0$, but $zx = 0$. We have also shown that each unit of R is in the set $\mathscr{X}$.

Next, we seek an idempotent e and unit u such that $x = eu$. There is nothing to prove if x is a unit. Thus, from the discussion above, there is a k such that $x_k = 0$ and an n such that $x_n \neq 0$. Let $S = \{k \in \mathbb{Z} \mid x_k = 0\}$ and $S' = \{n \in \mathbb{Z} \mid x_n \neq 0\}$. Neither set is empty, and exactly one is in $\mathscr{U}$. Note that if S is not in $\mathscr{U}$, then it must be that the set $\{m \in \mathbb{Z} \mid x_m = 1\}$ is in $\mathscr{U}$ (and clearly, it is a subset of S'). Thus, we may define a nonzero idempotent e such that $e_k = 0$ for all $k \in S$ and $e_n = 1$ for all $n \in S'$. To define the unit u, we simply set $u_k = 1$ for all $k \in S$ and $u_n = x_n$ for all $n \in S'$. Clearly, there is no m such that $u_m = 0$. Thus, u is a unit with $x = eu$. Therefore, R is von Neumann regular ring and there is no pair of units whose sum is a unit. By Theorem 3.3 R is not vsAR.

For a nonzero idempotent f, fR contains elements that are not idempotent. Hence, there is no nonzero idempotent e such that eR is Boolean. However, as we noted above, $M = \mathscr{Y}$ is a prime ideal of R, necessarily both minimal and maximal since R is von Neumann regular ring. For each $q \in R \backslash M$ (equivalently, $q \in \mathscr{X}$) we have $1 + q \in M$ and thus $R/M = \mathbb{Z}_2$. □

Our final goal of this section is to establish the converse of Corollary 3.5. We start by doing this for the case R has characteristic 2. As a first step, we record the following corollary of Theorem 3.3 which holds in the general case.

Corollary 3.7 *Let R be a von Neumann regular ring that is not a field. Then the following are equivalent:*

1. *R is not vsAR.*
2. *Each unit u can be written as $u = 1 + p$ for some zero divisor p.*
3. *For units u and v, $u + v$ is a zero divisor.*

Next, we record the following elementary result dealing with idempotents of a von Neumann regular ring that has characteristic 2 (so that $1 + e = 1 - e$ is idempotent for each idempotent e). Combining this observation with Theorem 2.3 will be very useful.

Lemma 3.8 *Let R be a von Neumann regular ring with characteristic 2. Also, let e and h be nonzero idempotents.*

1. *If eR and hR are incomparable, then the ring $eR + hR$ can be realized as a nontrivial internal direct sum as $eR + (h + eh)R$ and as $hR + (e + eh)R$.*
2. *If $eR \subsetneq hR$, then $hR = eR + hR$ and $(h + eh)R + eR$ is a nontrivial internal direct sum decomposition of hR.*

Proof Since R has characteristic 2, $e + h + eh$ is an idempotent. If $eh = 0$, then $eR + hR$ is an internal direct sum. Otherwise, we have $(e + eh)R + ehR + (h + eh)R$ is an internal direct sum representation of $eR + hR$. In addition, $eR = (e + eh)R + ehR$ and $hR = ehR + (h + eh)R$ are both internal direct sums. So we have $eR + hR = eR + (h + eh)R = (e + eh)R + hR$ with both $eR + (h + eh)R$ and $(e + eh)R + hR$ as internal direct sum representation.

For (2), suppose $eR \subsetneq hR$. Then $eh = e$, and $h + eh$ generates the annihilator of eR in hR. Thus, we have that $hR = (h + eh)R + eR$ is an internal direct sum decomposition of hR. □

Theorem 3.9 *Let R be a von Neumann regular ring with characteristic 2. Then R is vsAR if and only if there is no prime ideal Q such that $R/Q = \mathbb{Z}_2$.*

Proof The assumption that R has characteristic 2 means that $r = -r$ for each $r \in R$.
Corollary 3.5 takes care of showing that the existence of a prime Q such $R/Q = \mathbb{Z}_2$ implies R is not vsAR. Next, we assume R is not vsAR. By Corollary 3.7, each unit u can be written as a sum $u = 1 + p$ for some (unique) zero divisor p. Let $Y = \{q \in Z(R) \mid 1 + q \text{ is a unit of } R\}$.

If the set Y generates a proper ideal of R, then there is a prime M containing Y such that $R/M = \mathbb{Z}_2$. To see this, note that it is clear that each unit is congruent to 1 mod M. For a zero divisor $g \notin M$, we have $g = fw$ for some unit w and nontrivial idempotent f with $1 + f \in M$. The element $z = g + (1 + f)$ is a unit of R. Note that $z = 1 + (g + f)$ and $g + f$ is a zero divisor, so $g + f$ is in the set Y. We also have z congruent to both 1 and g mod M. Hence, $R/M = \mathbb{Z}_2$ (provided Y generates a proper ideal of R).

To see that Y generates a proper ideal of R, we first show that if $s, t \in Y$, then $sR + tR$ is a proper ideal of R. In the process, we will see that if $s_1, s_2, \ldots, s_n \in Y$, then $\sum s_i R$ is a proper ideal of R.

Let $u = 1 + p$ and $v = 1 + q$ be distinct units with p and q nonzero zero divisors. For p and q, there are units x and y and nonzero idempotents e and h such that $p = ex$, $q = hy$, $pR = eR$ and $qR = hR$.

For p suppose there is an idempotent $f \in eR$ such that $pf = f$ (clearly true for the case $f = 0$). Then f annihilates $e + p$. But as $u = (1 + e) + (e + p)$ is a unit of R and $(1 + e)R + eR$ is an internal direct sum decomposition of R, $e + p$ is a unit of eR. Thus, $f = 0$. Hence, we have that e, p, and $e + p$ are units of eR and so by Theorem 3.3, eR is vsAR.

Similarly, hR is vsAR.

By Lemma 3.8 and Theorem 2.3, $eR + hR$ is a von Neumann regular ring that is vsAR. Since R is not vsAR, $eR + hR$ is a proper ideal of R.

For finitely many $p_i \in Y$, we have corresponding idempotents e_i and units w_i such that $p_i = e_i w_i$ and $p_i R = e_i R$. As rings, each $e_i R$ is von Neumann regular ring that is vsAR. So as with $eR + hR$, the sum $\sum e_i R$ is a von Neumann regular ring that is vsAR. Thus, it is a proper ideal of R. It follows that $\sum p_i R$ is a proper ideal of R. Hence, Y generates a proper ideal of R. From the argument above, if M is a prime that contains Y, then $R/M = \mathbb{Z}_2$. So by Corollary 3.5, R is not vsAR in this case. □

Corollary 3.10 *Let R be a von Neumann regular ring. Then R is vsAR if and only if there is no prime P such that $R/P = \mathbb{Z}_2$.*

Proof We may assume R does not have characteristic 2. By Corollary 3.5, if there is a prime P such that $R/P = \mathbb{Z}_2$, then R does not admit unit sum pairs. So all we need to prove is that such a prime exists whenever R does not admit unit sum pairs. Hence, we have may assume $0 \neq 1 + 1 = eu = e + e =$ for some nonzero idempotent e and unit u where $e \neq 1$. Since $R = eR + (1 - e)R$ and $e + e = eu$ is a unit of eR, Theorem 3.3 implies that $(1 - e)R$ does not admit unit sum pairs. We have $2(1 - e) = (1 + 1) - (e + e) = 0$. Thus, $(1 - e)R$ has characteristic 2 and therefore there is a prime ideal P of $(1 - e)R$ such that $(1 - e)R/P = \mathbb{Z}_2$. It follows that $Q = eR + P$ is a prime ideal of R such that $R/Q = (1 - e)R/P = \mathbb{Z}_2$. □

It may be helpful to consider the ring R from Example 3.6 with regard to the proof that each zero divisor $g \in R \backslash M$ is congruent to 1 mod M. In the example, Y and M coincide as the set of elements $p \in T$ such that $p_n = 0$ for all $n \in U$ for some set U in the ultrafilter $\mathscr{U}$. For g we have $g_m = 1$ for all m in some set $V \in \mathscr{U}$. We may assume V contains each m where this occurs and thus $g_n \neq 1$ when $n \notin V$ (and $g_n = 0$ for at least one such n). For the corresponding idempotent f, $f_m = 1$ for all $m \in V$ (and all $n \notin V$ where $g_n \neq 0$) so that $(g + f)_m = g_m + f_m = 0$ for all such m, and thus, we see that $g + f$ is in $Y = M$.

Example 3.11 Let $S = R \oplus R$ where R is the von Neumann regular ring of Example 3.6 (that is not vsAR). Then S is a von Neumann regular ring that is not vsAR and the set $Y = \{t \in Z(S) \mid 1 + t \text{ is a unit of } S\}$ generates a proper ideal (in fact, is a proper ideal) that is contained in more than one prime ideal.

Proof Using the notation from Example 3.6, we have that $M = \{t \in T \mid t_n(x) = 0 \text{ for all } x \in U \text{ for some } U \in \mathscr{U}\}$ is a maximal ideal of R such that $R/M = \mathbb{Z}_2$. In S, both $N_1 = M \oplus R$ and $N_2 = R \oplus M$ are maximal ideals such that $S/N_1 = \mathbb{Z}_2 = S/N_2$. Note that for each nonzero $p \in M$, $(p, 1)$ and $(1, p)$ are zero divisors of S such that neither $(p, 1) + (1, 1) = (p + 1, 0)$ nor $(1, p) + (1, 1) = (0, p + 1)$ is a unit of S (even though $p + 1$ is a unit of R). For S, we have $Y = M \oplus M$ is an ideal that is contained in both N_1 and N_2. □

4 AR Graphs

We introduce four graphs whose vertices are the elements of a ring R. For distinct $f, g \in R$, (f, g) is an edge if there is a regular element $t \in R$ such that at least one of $f + gt$ and $ft + g$ is regular. We denote this graph by $\mathfrak{A}(R)$ and refer to it as the *AR graph* of R. We may augment the graph by adding a loop at f when there is a regular element $s \in R$ such that $f + fs$ is regular. A necessary condition for there to be a loop at f is that f is regular. When there is a loop at each regular $f \in R$, we denote the augmented graph by $\mathfrak{A}_\ell(R)$, and refer to it as the *looped AR graph* of R. The other two graphs are subgraphs of $\mathfrak{A}(R)$ and $\mathfrak{A}_\ell(R)$. For these, we have an edge (f, g) (loop when $g = f$) when there is a unit u of R such that $f + gu$ is regular (equivalently, $fu^{-1} + g$ is regular). The graph with no loops allowed is denoted by $\mathfrak{A}_u(R)$ and the one with loops included (when they exist) is $\mathfrak{A}_{u\ell}(R)$, these are the *u-AR graph* of R and the *looped u-AR graph* of R, respectively. Unlike the zero-divisor graph of R (and many others associated with R), an AR graph need not be connected. However, when R is sAR, then the AR graph is connected and the diameter of $\mathfrak{A}(R)$ is bounded above by 2. Lemma 4.6 provides a way to construct a ring whose AR graph is connected with diameter 3. Also, the ring R in Example 4.17 is such that $\mathfrak{A}(R)$ is connected, and for each positive integer n, there is a nonzero element $r_n \in R$ such that the distance between r_n and 0 is n. Hence, the diameter of the graph is unbounded. For $n > 3$, we do not know of a ring R such that $\mathfrak{A}(R)$ is connected and has diameter n.

If we remove the restriction of having a regular element r such that at least one of $fr + g$ and $f + gr$ is regular and replace it with just requiring the existence of a nonzero s such that at least one of $fs + g$ and $f + gs$ is regular, the resulting graph will always be connected with diameter less than or equal to 2: (h, g) is an edge for each regular element h and each zero divisor g, simply choose s to be a nonzero annihilator of g to get $h + gs = h$ regular.

The first lemma in this section is a generalized version of Lemma 3.2.

Lemma 4.1 *Let R be a commutative ring.*

1. *The following are equivalent:*
   a. *For each regular element $g \in R$, there is a regular element $p \in R$ such that $g + gp$ is regular.*
   b. *There is a regular element $q \in R$ such that $1 + q$ is regular.*
   c. *There is a pair of regular elements f and q (not necessarily distinct) such that $f + fq$ is regular.*
2. *The following are equivalent:*
   a. *For each regular element $g \in R$, there is a unit $u \in R$ such that $g + gu$ is regular.*
   b. *There is a unit $v \in R$ such that $1 + v$ is regular.*
   c. *There is a pair of regular elements f and w (not necessarily distinct) with w a unit such that $f + fw$ is regular.*

Proof For both equivalences, it suffices to prove (c) implies (a). So assume there is regular element $f \in R$ and a regular element (unit) $w \in R$ such that $f + fw$ is regular. Then for each regular $g \in R$, $g(f + fw) = gf + gfw = f(g + gw)$ is regular and thus $g + gw$ is regular. □

The ring $R = \mathbb{Z}_2[x]$ is such that there is a loop at 1 in $\mathfrak{A}_\ell(R)$, but since 1 is the only unit, the graph $\mathfrak{A}_{u\ell}(R)$ does not exist.

Lemma 4.2 *If D is an integral domain, then $\mathfrak{A}(D)$ is the complete graph on $|D|$ vertices. Moreover, if D has at least three elements, then $\mathfrak{A}_\ell(D)$ exists.*

Proof A special case is $D = \mathbb{Z}_2$. In this case, we have only 1 and 0. Clearly, $1 \cdot 1 + 0 = 1$ is regular and so $(1, 0)$ is an edge. Thus, $\mathfrak{A}(D) = K_2$, the complete graph on 2 vertices. However, there is no loop at 1.

Next, suppose D has at least three elements and let f, g be distinct elements of D. We may assume $g \neq 0$. If $f \neq -g$, then $f + g \cdot 1 = f + g \neq 0$ so that (f, g) is an edge. If $f = -g$, then we know $-g \neq g$ (since f and g are assumed to be distinct). In this case, $f \cdot (-1) + g = 2g \neq 0$ so again (f, g) is an edge. Therefore, $\mathfrak{A}(D)$ is the complete graph on $|D|$ vertices. Also, for $f \neq 0$, there is a nonzero $c \in D$ such that $f + fc$ is not zero. Hence, there is a loop at f, and therefore, $\mathfrak{A}_\ell(D)$ exists. □

Lemma 4.3 *If R is a ring such that $\mathfrak{A}(R)$ is a complete graph, then R is an integral domain.*

Proof If R has a nonzero zero divisor b, then $(b, 0)$ is not an edge, and thus, $\mathfrak{A}(R)$ is not complete. □

Lemma 4.4 *If R is sAR and not a domain, then $\mathfrak{A}(R)$ is connected with diameter 2. Moreover, if R is suAR, then $\mathfrak{A}_u(R)$ is connected with diameter equal to 2.*

Proof If R is not a domain, then it has a nonzero zero divisor b. Since $(b, 0)$ is not an edge, the diameter (if it exists) is not 1. Since R is sAR, for each regular element f, there is a regular element r such that $b + fr$ is regular. Hence, both (f, b) and $(f, 0)$ are edges so we have a path $b - f - 0$. For distinct zero divisors c and d, we have a path $c - f - d$. So the distance between c and d is at most 2. Also, for a pair of distinct regular elements g and h, $g - b - h$ is a path. Hence, $\mathfrak{A}(R)$ is connected with diameter 2. In the case R is suAR, $\mathfrak{A}_u(R)$ is connected with diameter 2. □

Example 4.5 For each $n \geq 2$, let $R_n = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ be the direct sum of n copies of $\mathbb{Z}_2$. Then the (u-)AR graph $\mathfrak{A}(R_n)$ is the disjoint union of 2^{n-1} copies of K_2.

Proof Since $1_n = (1, 1, \ldots, 1)$ is the only regular element of R_n, $\mathfrak{A}(R_n) = \mathfrak{A}_u(R_n)$. Also, for $b, c \in R_n$, $b + c$ is regular if and only if $b + c = 1_n$. Hence, each edge has the form $(e, 1 - e)$ for some (idempotent) $e \in R_n$. It follows that $\mathfrak{A}(R_n)$ is the disjoint union of 2^{n-1} copies of K_2. □

Lemma 4.6 *Let $R = \mathbb{Z}_2 \times D$ where D is an integral domain with at least three elements. Then the graph $\mathfrak{A}(R)$ is connected with diameter 3 but $\mathfrak{A}_\ell(R)$ does not exist.*

Proof The regular elements of R are those of the form $(1, r)$ for some nonzero $r \in D$. We have an edge $((1, r), (0, 0))$ for each such r and there are no other edges at $(0, 0)$. In particular $((1, 0), (0, 0))$ is not an edge. Moreover, for $x \in \mathbb{Z}_2$, there are no edges of the form $((x, t), (x, s))$ since $1 \cdot x + x = 0$. Hence, there are no loops and no pair of distinct regular elements forms an edge.

Let $a \in D\backslash\{0\}$ and $b \in D$. Since D has at least three element, there is a nonzero $d \in D$ such that $ad + b \neq 0$. For such a d, both $(1, d)$ and $(1, a)(1, d) + (0, b) = (1, ad + b)$ are regular elements of R. Hence, $((1, a), (0, b))$ is an edge. Also, note that $(1, 0) \cdot (1, 1) + (0, a) = (1, 0) + (0, a) = (1, a)$ is regular so $((1, 0), (0, a))$ is also an edge. Thus, the graph $\mathfrak{A}(R)$ is connected. Since there is no edge of the form $((1, 0), (1, a))$, there is no path of length 2 or less between $(1, 0)$ and $(0, 0)$. We do have a path of length 3: $(1, 0) - (0, 1) - (1, 1) - (0, 0)$ is one such path. Finally, for each $c \in D\backslash\{a\}$, we have paths $(1, c) - (0, 1) - (1, a)$ and $(0, c) - (1, 1) - (0, a)$. Thus, $\mathfrak{A}(R)$ has diameter 3. □

Lemma 4.7 *Let $R = T \times D$ where T is the direct sum of $n > 1$ copies of $\mathbb{Z}_2$ and D is an integral domain with at least three elements. Then $\mathfrak{A}(R)$ is the disjoint union of 2^{n-1} copies of the connected graph $\mathfrak{A}(R_0)$ where $R_0 = \mathbb{Z}_2 \times D$.*

Proof We let $1_n = (1, 1, \ldots, 1) \in T$. Then the regular elements of R are those of the form $(1_n, g)$ for some nonzero $g \in D$.

Let $e, f \in T$ and let $b, c, g \in D$ with $g \neq 0$, for the sum $(e, b) + (f, c)(1_n, g) = (e + f, b + cg)$ to be regular we must have $f = 1 - e$ and $b + cg \neq 0$. Thus, $((e, b), (f, c))$ is an edge if and only if $f = 1 - e$ and at least one of b and c is nonzero. Hence, $((e, b), (f, c))$ is an edge in $\mathfrak{A}(R)$ if and only if $((1, b), (0, c))$ is an edge in $\mathfrak{A}(R_0)$. As there are 2^{n-1} pairs of idempotents $\{e, 1 - e\}$ in T, $\mathfrak{A}(R)$ is the disjoint union of 2^{n-1} copies of $\mathfrak{A}(R_0)$. □

Theorem 4.8 *Let $R = \mathbb{Z}_2 \times S$ for some ring S. Then $\mathfrak{A}(R)$ is connected if and only if $\mathfrak{A}(S)$ is connected and for all $f, g \in S\backslash\{0\}$ there are paths of odd length between f and g and when $f \neq g$, paths of even length between f and g. In the case $\mathfrak{A}(R)$ is connected and there are elements $s \neq t$ in $\mathfrak{A}(S)$ where the distance between them is m, then there are points in $\mathfrak{A}(R)$ where the distance between them is at least $m + 1$.*

Proof First, we note that for $x \in \mathbb{Z}_2$ and $f, g \in S$, $(x, f) + (x, g) = (0, f + g)$ a zero divisor of R. On the other hand, $(x, f) + (x + 1, g) = (1, f + g)$ is regular if and only if $f + g$ is regular. Thus, a (nontrivial) path between (x, f) and (x, g) must have even length and a path between (x, f) and $(x + 1, g)$ must have odd length.

Assume $\mathfrak{A}(R)$ is connected and suppose f, g are in $S\backslash\{0\}$. In $\mathfrak{A}(R)$, we have a path (of odd length) $(1, f) - (0, g_1) - (1, g_2) - \cdots - (0, g_{2n+1}) = (0, g)$. Thus, in $\mathfrak{A}(S)$, we have a path $f - g_1 - g_2 - \cdots - g_{2n+1} = g$ (perhaps with loops) of odd length. In the case $f \neq g$ we also have a path $(1, f) - (0, f_1) - (1, f_2) - \cdots - (1, f_{2m}) = (1, g)$ which has even length. In $\mathfrak{A}(S)$, $f - f_1 - f_2 - \cdots - f_{2m} = g$ is a path of even length.

Next, suppose that for all $f, g \in S\backslash\{0\}$, there is a path $f - g_1 - g_2 - \cdots - g_{2n+1} = g$ in $\mathfrak{A}(S)$. Also, when $f \neq g$ we have a path $f - f_1 - f_2 - \cdots - f_{2m} = g$. For $x \in \mathbb{Z}_2$, we get paths $(x, f) - (x + 1, g_1) - (x, g_2) - \cdots - (x + 1, g_{2n+1}) = (x + 1, g)$ for each g and $(x, f) - (x + 1, f_1) - (x, f_2) - \cdots - (x, f_{2m}) = (x, g)$ when $f \neq g$. Thus, $\mathfrak{A}(R)$ is connected.

Finally, assume $\mathfrak{A}(R)$ is connected and let $s \neq t$ in S be such that the distance between them is m. Suppose $s - s_1 - s_2 - \cdots - s_m = t$ is a (shortest) path between s and t. In the case m is odd, we have a path $(1, s) - (0, s_1) - (1, s_2) - \cdots - (0, s_m) = (0, t)$ of length m. But to have a path between $(1, s)$ and $(1, t)$ we must have a path of even length which must have length at least $m + 1$. On the other hand, if m is even, a path between $(1, s)$ and $(0, t)$ must have length at least $m + 1$. So the diameter of $\mathfrak{A}(R)$ must be least one more than the diameter of $\mathfrak{A}(S)$ (in the case the diameter of $\mathfrak{A}(S)$ is finite). □

For a pair of distinct nonzero elements $f, g \in R$, if there is a unit u such that $f + gu$ is regular, then we also have $fu^{-1} + g$ regular. On the other hand, it may be that there is a regular element s such that $f + gs$ but no regular element r such that $fr + g$ is regular. For example, in the ring R of [5, Example 5.6], $(8, 0)$ is regular while $(3, 0)$ and each element of the form $(8k + 3, b)$ is a zero divisor (for k an integer). Thus, there is no regular element (k, c) such that $(8, 0)(k, c) + (3, 0)$ is regular. On the other hand, $(8, 0) + (3, 0)(8, 0) = (32, 0)$ is regular.

Theorem 4.9 *Let $R = S \times T$ for rings S and T. Then $\mathfrak{A}_{u\ell}(R)$ exists and is connected if and only if both $\mathfrak{A}_{u\ell}(S)$ and $\mathfrak{A}_{u\ell}(T)$ exist and are connected.*

Proof The regular elements of R are those of the form (f, g) for regular $f \in S$ and regular $g \in T$. If there is a unit $(p, q) \in R$ such that $(f, g)(p, q) + (f, g) = (fp + f, gq + g)$ is regular, then both $fp + f$ and $gq + g$ are regular. Hence, if there is a loop at (f, g), then there are loops at both f and g. We may easily reverse the argument: if $p \in S$ and $q \in T$ are units with $f + fp$ and $g + gq$ regular, then (p, q) and $(f, g) + (f, g)(p, q)$ are both regular and so we have a loop at (f, g). Therefore, loops exist at each regular element of R if and only if loops exist at each regular element of S and each regular element of T.

Suppose $\mathfrak{A}_{u\ell}(R)$ exists and is connected. We have loops at each regular element of S and at each regular element of T. We will show that there is a path between each pair of distinct elements $b, d \in S$ and each pair of distinct elements $c, e \in T$. We can do these simultaneously. Since $\mathfrak{A}_{u\ell}(R)$ is connected, there is a finite path from (b, c) to (d, e). Hence, there are elements $(b, c) = a_0 = (b_0, c_0)$, $a_1 = (b_1, c_1)$, $a_2 = (b_2, c_2), \ldots, a_n = (b_n, c_n)$ and $(d, e) = a_{n+1} = (b_{n+1}, c_{n+1})$ such that $a_0 - a_1 - a_2 - \cdots - a_n - a_{n+1}$ is a path. It follows that we have units $u_0 = (v_0, w_0)$, $u_1 = (v_1, w_1), \ldots, u_n = (v_n, w_n)$ such that $a_i + a_{i+1}u_i$ is a regular for each $0 \le i \le n$. Hence, $b_i + b_{i+1}v_i$ and $c_i + c_{i+1}w_i$ are regular for each $0 \le i \le n$ with each v_i a unit in S and each w_i a unit in T. It follows that $b = b_0 - b_1 - b_2 - \cdots - b_n - b_{n+1} = d$ is a path in $\mathfrak{A}_{u\ell}(S)$ and $c = c_0 - c_1 - c_2 - \cdots - c_n - c_{n+1} = e$ is a path in $\mathfrak{A}_{u\ell}(T)$. Thus, both $\mathfrak{A}_{u\ell}(S)$ and $\mathfrak{A}_{u\ell}(T)$ exist and are connected. Note that for both $b_0 - b_1 - b_2 - \cdots - b_{n+1}$ and $c_0 - c_1 - c_2 - \cdots - c_n - c_{n+1}$ some $b_i - b_{i+1}$ and/or some $c_j - c_{j+1}$ may be loops. For example in $(\mathbb{Z} \times \mathbb{Z})$, $(1, 1) - (2, 1) - (2, 3)$ is a path with three distinct vertices, but both extracted paths have loops: $1 - 2 - 2$ and $1 - 1 - 3$, respectively.

For the converse, we assume both $\mathfrak{A}_{u\ell}(S)$ and $\mathfrak{A}_{u\ell}(T)$ exist and are connected. From the argument in the first paragraph, R has loops at each regular element. To see that $\mathfrak{A}_{u\ell}(R)$ is connected it suffices to show that for each $p = (b, c) \in R \backslash \{(1, 1)\}$ there is a path between p and $(1, 1)$. Let $b = b_0 - b_1 - b_2 - \cdots - b_n = 1$ be a shortest path between b and 1, including the possibility that $b = 1$ and so $n = 1$ and we simply have a loop. Also, let $c = c_0 - c_1 - c_2 - \cdots - c_m = 1$ be a shortest path between c and 1, including the possibility that $c = 1$ and so $m = 1$. Without loss of generality, we may assume $n \le m$ (simply reverse roles in the following argument when $m < n$). For $0 \le i \le n$, there are units $v_0, v_1, \ldots, v_n \in S$ such that $b_i + b_{i+1}v_i$ is regular (in particular, $b_{n-1} + v_n$ is regular). Also, for $0 \le j \le m$, there are units $w_0, w_1, \ldots, w_m \in T$ such that $c_j + c_{j+1}w_j$ ($c_{m-1} + w_m$ for $j = m$) is regular. In the event $n < m$, we let $v' = v_{n+1} = v_{n+1} = \cdots = v_m$ be a unit of S such that $1 + v'$ is regular. The elements $u_j = (v_j, w_j)$ are units of R with $(b_j, c_j) + (b_{j+1}, c_{j+1})u_j$ regular for $0 \le j \le n$ and $(1, c_j) + (1, c_{j+1})u_j$ regular for $n < j \le m$. Thus, $(b, c) - (b_1, c_1) - (b_2, c_2) - \cdots - (b_n, c_n) = (1, c_n) - (1, c_{n+1}) - \cdots - (1, c_m) = (1, 1)$ is a path in $\mathfrak{A}_{u\ell}(R)$. Therefore, $\mathfrak{A}_{u\ell}(R)$ exists and is connected. □

We break down the proof of the analogous result for the graphs $\mathfrak{A}_\ell(R)$, $\mathfrak{A}_\ell(S)$, and $\mathfrak{A}_\ell(T)$. We start by showing that if $\mathfrak{A}_\ell(R)$ exists and is connected, then both $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$ exist and are connected.

Theorem 4.10 *Let $R = S \times T$ for a pair of rings S and T. If $\mathfrak{A}_\ell(R)$ exists and is connected, then both $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$ exist and are connected.*

Proof Assume $\mathfrak{A}_\ell(R)$ exists and is connected. To show the same conclusions hold for $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$, it suffices to show $\mathfrak{A}_\ell(S)$ exists and is connected.

We first show that there is a loop at each regular $r \in S$. Consider the element $(r, 1)$. This a regular element of R. Hence, there is a regular element $(b, c) \in R$ such that $(r, 1) + (r, 1)(b, c) = (r + rb, 1 + c)$ is regular. We have $r + rb$ a regular element of S. Hence, there is a loop at r.

Next, let $f, g \in S$ with $f \neq g$ and choose any h in T. Then we have a path from (f, h) to (g, h). There is a nonnegative integer n, regular elements $p_0 = (s_0, t_0)$, $p_1 = (s_1, t_1), \ldots, p_n = (s_n, t_n)$ and elements $a_0 = (f, h)$, $a_1 = (b_1, c_1)$, $a_2 = (b_2, c_2), \ldots, a_n = (b_n, c_n)$ and $a_{n+1} = (g, h)$ such that at least one in each pair $\{a_0 + a_1 p_0, a_0 p_0 + a_1\}, \{a_1 + a_2 p_2, a_1 p_2 + a_2\}, \ldots, \{a_n + a_{n+1} p_n, a_n p_n + a_{n+1}\}$ is regular. The elements $s_0, s_1, \ldots, s_n$ are regular elements of S and we have that there is at least one regular element in each pair $\{f + b_1 s_0, f s_0 + b_1\}, \{b_1 + b_2 s_1, b_1 s_1 + b_2\}, \ldots, \{b_{n-1} + b_n s_{n-1}, b_{n-1} s_{n-1} + b_n\}, \{b_n + g s_n, b_n s_n g\}$. Hence, there is a path from f to g in $\mathfrak{A}_\ell(S)$. Therefore, $\mathfrak{A}_\ell(S)$ exists and is connected. The same conclusions hold for $\mathfrak{A}_\ell(T)$. $\square$

The converse is more difficult to establish so we include it as a separate result. First a few definitions and a pair of lemmas. For a directed path $a_0 - a_1 - a_2 - \cdots - a_n$ in $\mathfrak{A}(R)$, a list of regular elements $q_0, q_1, \ldots, q_{n-1}$ is a *corresponding list for the path* if for each i, at least one of $a_i + a_{i+1} q_i$ and $a_i q_i + a_{i+1}$ is regular. Also, we say we have a *neutral step* at i if $q_i = 1$ ($a_i + a_{i+1}$ is regular), a *left step* at i if both $q_i \neq 1$ and $a_i q_i + a_{i+1}$ is regular, and a *right step* at i if both $q_i \neq 1$ and $a_i + a_{i+1} q_i$ is regular. For a particular i, it may be that we have both a left step at i and a right step at i for the same q_i. Also, for a different corresponding list $p_0, p_1, \ldots, p_n$, it may be that there is a left step at i with respect to q_i and a right step (or neutral step) at i for p_i.

Lemma 4.11 *For a ring R, if $b_0 - b_1 - b_2 - \cdots - b_n = 0$ is a path in $\mathfrak{A}(R)$ with corresponding list of regular elements $q_0, q_1, \ldots, q_{n-1}$ such that there is a right shift at some $i < n$, then there is an alternate path $b_0 - b_1 - \cdots - b_i - q_i b_{i+1} - q_i b_{i+2} - \cdots - q_i b_n = 0 = b_n$ with corresponding list of regular elements $q_0, q_1, \ldots, q_{i-1}, 1, q_{i+1}, \ldots, q_{n-1}$ with the same step pattern except at i which is now a neutral step.*

Proof Since $b_i + b_{i+1} q_i$ is regular, $(b_i, q_i b_{i+1})$ is an edge with a neutral step. For $j > i$, at least one of $b_j + b_{j+1} q_j$ and $b_j q_j + b_{j+1}$ is regular. Since q_i is regular, $q_i b_j + q_i b_{j+1} q_j$ is regular when $b_j + b_{j+1} q_j$ is regular, and $q_i b_j q_j + q_i b_{j+1}$ is regular when $b_j q_j + b_{j+1}$ is regular. Moreover, if $q_j = 1$, we still have a neutral step at j. On the other hand, if $q_j \neq 1$, then $q_ib_j - q_ib_{j+1}$ with q_j is a left step when $b_j - b_{j+1}$ with q_j is a left step, and $q_ib_j - q_ib_{j+1}$ with q_j is a right step when $b_j - b_{j+1}$ with q_j is a right step. □

By iterating the process in the previous lemma we obtain a way to modify a path between a given $b \in R$ and 0 (assuming one exists).

Lemma 4.12 *For a ring R, if $b_0 - b_1 - b_2 - \cdots - b_n = 0$ is a path in $\mathfrak{A}(R)$ with corresponding list of regular elements $q_0, q_1, \ldots, q_{n-1}$ such that there is a right step at some smallest $h < n$, then there is an alternate path $b_0 - b'_1 - b'_2 - \cdots - b'_{n-1} - b'_n = 0 = b_n$ with corresponding list of regular elements $q'_0, q'_1, \ldots, q'_{n-1}$ such that*

(i) $b'_i = b_i$ for all $i \leq h$,
(ii) for $j > h$, b'_j is the product of b_j and all q_is with $i < j$ such that $b_iq_i + b_{i+1}$ is not regular,
(iii $q'_i = 1$ when $q_i = 1$,
(iv) $q'_i = 1$ for each i such that $b_i + b_{i+1}q_i$ is regular (and $q_i \neq 1$), and
(v) $q'_i = q_i (\neq 1)$ for each i such that $b_i + b_{i+1}q_i$ is not regular,

this resulting path $b_0 - b'_1 - b'_2 - \cdots - b'_{n-1} - b'_n = 0 = b_n$ with corresponding list of regular elements $q'_0, q'_1, \ldots, q'_{n-1}$ has the property that there are no right steps.

Proof Apply Lemma 4.11 and the rules (i)–(v) to get the new path $b_0 - b'_1 - b'_2 - \cdots - b'_{n-1} - b'_n = 0 = b_n$ with corresponding list of regular elements $q'_0, q'_1, \ldots, q'_{n-1}$. Each step that is not neutral is a left step (and not a right step). □

Theorem 4.13 *Let $R = S \times T$. If both $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$ exist and are connected, then $\mathfrak{A}_\ell(R)$ exists and is connected.*

Proof Assume both $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$ exist and are connected.

First, we show there is a loop at each regular element of R. Let $r = (f, g)$ be regular. Then f is a regular element of S and g is a regular element of T. Since $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$ have loops at each regular element, there are regular elements $f' \in S$ and $g' \in T$ such that $f + ff'$ and $g + gg'$ are regular. The element $r' = (f', g')$ is a regular element of R as is $r + rr' = (f + ff', g + gg')$. Hence, there is a loop at r.

Let $(f, g) \in R\backslash\{(0, 0)\}$. Next, let $f = f_0 - f_1 - f_2 - \cdots - f_n = 0$ and $g = g_0 - g_1 - g_2 - \cdots - g_m = 0$ be paths in $\mathfrak{A}_\ell(S)$ and $\mathfrak{A}_\ell(T)$, respectively. Note that f_{n-1} is a regular element of S and g_{m-1} is a regular element of T. If $n < m$, we may insert additional loops at f_{n-1} to get a path of length m between f and 0. Similarly, we may insert loops at g_{m-1} when $m < n$ to get a new path of length n between g and 0. Hence, we may assume $n = m$. By Lemma 4.12, we may further assume that there are corresponding lists of regular elements $p_0, p_1, \ldots, p_{n-2}, p_{n-1} = f_{n-1}$ in S and $q_0, q_1, \ldots, q_{n-1} = g_{n-1}$ in T such that both $f_ip_i + f_{i+1}$ and $g_jq_j + g_{j+1}$ are regular for each i. Thus, for each $0 \leq i \leq n - 1$, the elements (p_j, q_j) and $(f_j, g_j)(p_j, q_j) + (f_{j+1}, g_{j+1})$ are regular in R. Hence, we have a path between (f, g) and $(0, 0)$, and therefore, $\mathfrak{A}_\ell(R)$ (exists and) is connected. □

Corollary 4.14 *Suppose D and E are domains with at least three elements each. Then for $R = D \times E$, $\mathfrak{A}_\ell(R)$ exists and both $\mathfrak{A}(R)$ and $\mathfrak{A}_\ell(R)$ have diameter 2. Moreover, for each regular element $r \in R$, (r, t) is an edge for each $t \in R \backslash \{r\}$.*

Proof By Lemma 4.2 and Theorem 4.13, $\mathfrak{A}(R)$ is connected and $\mathfrak{A}_\ell(R)$ both exists and is connected. Also, since R is not an integral domain, $\mathfrak{A}(R)$ is not a complete graph and so the diameter (if it exists) is greater than 1. Let $r = (a, b)$ be a regular element of R and let $t = (c, d)$ be any element other than r. Since both D and E have at least three elements, there are nonzero elements $f \in D$ and $g \in E$ such that $a + cf$ and $b + dg$ are nonzero. Thus, both (f, g) and $(a, b) + (c, d)(f, g)$ are regular. Hence, (r, t) is an edge. It follows that both $\mathfrak{A}(R)$ and $\mathfrak{A}_\ell(R)$ have diameter 2. □

The proof of the next result is adapted from the one given for Theorem 4.13.

Corollary 4.15 *Let $R = \mathbb{Z}_2 \times T$. If $\mathfrak{A}_\ell(T)$ exists and is connected, then $\mathfrak{A}(R)$ is connected, but loops do not exist.*

Proof Each regular element in R has the form $(1, t)$ for some regular element t of T. So the sum of two regular elements is a zero divisor. Thus, there are no loops for R. While there is no loop at $(1, t)$, there is a loop at t since $\mathfrak{A}_\ell(T)$ exists. For regular $q \in T$ such that $t + tq$ is regular, both $(1, q)$ and $(1, t) + (0, t)(1, q)$ are regular elements of R, and thus, $((1, t), (0, t))$ is an edge in $\mathfrak{A}(R)$. Also, note that $((1, 0), (0, 1))$ is an edge in $\mathfrak{A}(R)$.

Let $f \in T\backslash\{0\}$. Since there is a loop at 1 in $\mathfrak{A}_\ell(T)$, T has at least three elements (and at least two regular elements). Since $\mathfrak{A}(T)$ is connected, there is a path $f = f_0 - f_1 - f_2 - \cdots - f_{n-1} - f_n = 0$, necessarily with f_{n-1} a regular element of T. There is a loop at f_{n-1} and thus we also have a path $f_{n-1} - f_{n-1} - 0$ in $\mathfrak{A}_\ell(T)$. By replacing $f_{n-1} - 0$ with $f_{n-1} - f_{n-1} - 0$, we have a pair of paths $f = g_0 - g_1 - g_2 - \cdots - g_{2k} - g_{2k+1} = 0$ and $f = h_0 - h_1 - h_2 - \cdots - h_{2m-1} - h_{2m} = 0$ with $g_i = f_i = h_i$ when $0 \le i \le n - 1$. We then have paths $(1, f) = (1, g_0) - (0, g_1) - (1, g_2) - \cdots - (0, g_{2k+1}) = (0, 0)$ and $(0, f) - (1, h_1) - (0, h_2) - \cdots - (0, h_{2m}) = (0, 0)$ in $\mathfrak{A}(R)$. Since $((1, 0), (0, 1))$ is an edge, $\mathfrak{A}(R)$ is connected. □

Theorem 4.16 *Let $R = T(R)$ be a von Neumann regular ring that is not a field. Then the following are equivalent:*

1. *$\mathfrak{A}_\ell(R)$ exists and is connected.*
2. *$\mathfrak{A}_\ell(R)$ exists and is connected with diameter 2.*
3. *For each nonzero idempotent $e \neq 1$, there is a loop at e in the ring eR.*
4. *There is a loop at 1 in R.*
5. *R is vsAR.*
6. *There is a nonzero idempotent $e \neq 1$ such that there is a loop at e in the ring eR and a loop at $1 - e$ in the ring $(1 - e)R$.*

Proof It is clear that (1) implies (4), and (2) implies (1). Since R is von Neumann regular, each regular element is a unit and each nonzero element f is the product of a unique idempotent e and a unit v. The element $1-e$ annihilates f and $Ann(f) = (1-e)R$. Since $e + (1-e) = 1$, $(e, 1-e)$ is an edge. Also, it is clear that $(1, 0)$ is an edge in $\mathfrak{A}(R)$.

Since each regular element of R is a unit, having a loop at 1 is equivalent to having a unit p such that $1 + p$ is regular. Thus, (4) and (5) are equivalent by Theorem 3.3. In addition, for a nonzero idempotent $e \neq 1$, there is a loop at e in the ring eR if and only if eR is vsAR. Thus, another application of Theorem 3.3 establishes the equivalence of (3), (4), (5) and (6).

All that is left is to show that (3) implies (2). Since R is not a field, the diameter of $\mathfrak{A}_\ell(R)$ is not 1. To see that $\mathfrak{A}(R)$ is connected with diameter 2, it suffices to show that $(f, 1)$ is an edge for each $f \in R\backslash\{0, 1\}$. Assume that for each nonzero idempotent $e \neq 1$, there is a loop at e in the ring eR. Thus, there is an e-unit $g \in eR$ such that $e + eg = e + g$ is an e-unit. Also, we have a unit $p \in R$ such that $1 + p$ is regular (thus a unit).

To start, we consider the case that f is a unit. Then $f^{-1}p$ is also a unit and we have $f(f^{-1}p) + 1 = p + 1$, a regular element. Hence, $(f, 1)$ is an edge when f is a unit.

Next, suppose f is not a unit. Then there is a nonzero idempotent $e \neq 1$ such that $eR = fR$. In the ring eR, there is an element $h = he \in eR$ such that $fh = e$. We then have $v = h + (1-e)$ is a unit of R such that $fv = e$. Also, we have an e-unit $g \in eR$ such that $e + eg = e + g$ is an e-unit. The elements $e + g + (1-e)$ and $q = vg + (1-e)$ are units of R such that $fq + 1 = f(vg) + 1 = eg + e + (1-e)$. Hence, $(f, 1)$ is an edge. It follows that $\mathfrak{A}_\ell(R)$ exists and is connected. Also, both $\mathfrak{A}_\ell(R)$ and $\mathfrak{A}(R)$ have diameter 2. □

The ring R in our last example is the one promised above: $\mathfrak{A}(R)$ is connected and for each positive integer n, there is a nonzero element $r_n \in R$ such that the distance between r_n and 0 is n.

Example 4.17 Let $D = K[\mathrm{x}]$ for some field K that does not have characteristic 2 and let $R = D + B$ be the $A + B$ ring corresponding to D and the set $\mathscr{P} = \mathrm{Max}(D)\backslash\{\mathrm{x}D\}$. Also, let $S = \{a\mathrm{x}^m \mid a \in K\backslash\{0\}, m \geq 0\}$. Then $\mathscr{A}(R)$ has the following properties:

1. For each nonzero $b \in B$, there is a path of length 2 between $(0, b)$ and $(0, 0)$.
2. For each $f \in S$ and each $b \in B$, there is a path of length at most 2 between (f, b) and $(0, 0)$.
3. For each nonzero $h \in D\backslash S$ and each $b \in B$, there is a path of length n between (h, b) and $(0, 0)$ where n is the number of nonzero terms of h and there is no shorter path.
4. $\mathscr{A}(R)$ is connected with infinite diameter and for each positive integer m there is a nonzero element $p \in D$ such that the distance between $(p, 0)$ and $(0, 0)$ is m.

Proof Let $f \in S$. Since K does not have characteristic 2, $2f$ is not zero and there is no i such that $2f_i$ is 0. In addition, for $b \in B\backslash\{0\}$, $2b_i = 0$ if and only if $b_i = 0$.

Suppose (r, b) and (s, c) form an edge in $\mathfrak{A}(R)$. Since each $h \in S$ is a nonzero monomial, hr and r have the same number of nonzero terms. In addition, a necessary condition for $hr + s$ to be in S is that the number of nonzero terms of r must be within one of the number of nonzero terms of s. In R, having both (g, d) and $(g, d)(r, b) + (s, c)$ regular requires that $g \in S$ and the number of nonzero terms of r must be within one of the number of nonzero terms of s. For d, we must have that there is no i where $d_i = -g_i$ and no i where $(gb + rd + db + c)_i = -(gr + s)_i$.

For nonzero $b \in B$, $(0, b)$ is a zero divisor of R and so $((0, b), (0, 0))$ is not an edge. However, we may define $c \in B$ by $c_i = 0$ when $b_i + 1 \neq 0$ (necessarily the case when $b_i = 0$) and $c_i = 1$ when $b_i + 1 = 0$. For such a c both $(1, c)$ and $(1, c + b)$ are regular since both c_i and $c_i + b_i$ are never -1. Thus, we have a path $(0, b) - (1, c) - (0, 0)$.

Next, let $f \in S$ and $b \in B$. If (f, b) is regular, then $((f, b), (0, 0))$ is an edge. If (f, b) is a zero divisor, then $((f, b), (0, 0))$ is not an edge. We have a slightly different definition for a c to get a regular element (f, c) such that $(f, b) + (f, c)$ is also regular. As above, we set $c_i = 0$ when $b_i + 2f_i \neq 0$ (so $c_i = 0$ whenever $b_i = 0$) and $c_i = f_i$ when $b_i + 2f_i = 0$ (necessarily with $b_i \neq 0$). There is no i such that $c_i + f_i = 0$ so (f, c) is regular. Also, there is no i such that $c_i + b_i + 2f_i = 0$ so that $(f, b) + (f, c) = (2f, b + c)$ is regular. Thus, $(f, b) - (f, c) - (0, 0)$ is a path of length 2.

Finally, we consider the case of an element (h, b) where $h = \sum h_j \mathrm{x}^j$ is a nonzero element in $D \backslash S$ of degree m (with $m > 0$). The element (h, b) is a zero divisor no matter the choice of b. Let n be the number of nonzero terms of h. Then $n \geq 2$. We use induction on the number of nonzero terms of h. For $n = 2$, we have $h = h_k\mathrm{x}^k + h_m\mathrm{x}^m$ with $k < m$ and neither h_k nor h_m equal to 0. Let $h' = -h_k\mathrm{x}^k$ $(= h_m\mathrm{x}^m - h)$. We define an element $b' \in B$ by setting $b'_i = 0$ when $b_i + (h_m\mathrm{x}^m)_i \neq 0$ (which includes when $b_i = 0$) and $b'_i = -(h_k\mathrm{x}^k)_i (\neq 0)$ when $b_i + (h_m\mathrm{x}^m)_i = 0$. There is no i such that $b'_i = (h_k\mathrm{x}^k)_i$, so (h', b') is regular. Also, there is no i such that $b'_i + b_i + (h_m\mathrm{x}^m)_i = 0$. Thus, both $(h_m\mathrm{x}^m, b + b') = (h, b) + (h', b')$ and (h', b') are regular and so we have a path $(h, b) - (h', b') - (0, 0)$ of length 2.

Next, assume that for all $2 \leq j < n$ and $g_0 \in D$ with j nonzero terms and arbitrary $c \in B$, we have a path (of length j) $(g_0, c_0) - (g_1, c_1) - \cdots - (g_{j-1}, c_{j-1}) - (0, 0)$, necessarily with each g_s having one fewer nonzero term than g_{s-1} and with (g_{j-1}, c_{j-1}) a regular element of R. For (h, b) we simply need to make an edge $((h, b), (h', b'))$ where h' has $n - 1 (\geq 2)$ nonzero terms (in this case, (h', b') is not regular). As above, let $h' = h_m x^m - h$ (where m is the degree of h and h_m is its leading term) and define b' by $b'_i = 0$ when $b_i + (h_m x^m)_i \neq 0$ and $b'_i = 1$ when $b_i + (h_m x^m)_i = 0$. Then $(h_m x^m, b + b') = (h, b) + (h', b')$ is regular, and therefore, $((h, b), (h', b'))$ is an edge. By the induction hypothesis, we have a path of length $n - 1$ between (h', b') and $(0, 0)$. Thus, we have a path of length n between (h, b) and $(0, 0)$. From the argument above, there can be no shorter path between (h, b) and $(0, 0)$ so the distance between these two points is n.

We conclude that $\mathscr{A}(R)$ is connected, but there is no upper bound on the distance between vertices. Hence, the diameter is infinite. Moreover, for each positive integer m, there is a nonzero element $p \in D$ such that the distance between $(p, 0)$ and $(0, 0)$ is m. $\square$

References

1. D.F. Anderson, A. Badawi, The total graph of a commutative ring. J. Algebra **320**, 2706–2719 (2008)
2. D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring. J. Algebra **217**, 434–447 (1999)
3. I. Beck, Coloring of commutative rings. J. Algebra **116**, 208–226 (1988)
4. R. Gilmer, J. Huckaba, Δ-Rings. J. Algebra **28**, 414–432 (1974)
5. T.G. Lucas, Weakly additively regular rings and special families of prime ideals. Palest. J. Math. **7**, 14–31 (2018)
6. H.R. Maimani, M. Salimi, A. Sattari, S. Yassemi, Comaximal graph of commutative rings. J. Algebra **319**, 1801–1808 (2008)
7. P.K. Sharma, S.M. Bhatwadekar, A note on graphical representation of rings. J. Algebra **176**, 124–127 (1995)

On t-Reduction and t-Integral Closure of Ideals in Integral Domains

Salah Kabbaj

Dedicated to David F. Anderson

Abstract Let R be an integral domain and I a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some $n \geq 0$. An element x of R is t-integral over I if there is an equation $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, \ldots, n$. The set of all elements that are t-integral over I is called the t-integral closure of I. This paper surveys recent literature which studies t-reductions and t-integral closure of ideals in arbitrary domains as well as in special contexts such as Prüfer v-multiplication domains, Noetherian domains, and pullback constructions.

1 Introduction

Throughout, all rings considered are commutative with identity. Let R be a domain with quotient field K, I a nonzero fractional ideal of R, and let $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$ the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. Recall that factorial domains, Krull domains, GCDs, and PvMDs can be regarded as t-analogues of the principal domains, Dedekind domains, Bézout domains, and Prüfer domains, respectively. For instance, a domain is Prüfer (resp., a PvMD) if every nonzero finitely generated ideal is invertible (resp., t-invertible).

S. Kabbaj (supported by KFUPM under DSR Research Grant # RG1328").

S. Kabbaj (✉)
Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia
e-mail: kabbaj@kfupm.edu.sa

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_7

We also recall the w-closure of I defined by $I_w := \bigcup(I : J)$, where the union is taken over all finitely generated ideals J of R that satisfy $J_v = R$; equivalently, $I_w = \bigcap IR_M$, where M ranges over the maximal t-ideals of R. We always have $I \subseteq I_w \subseteq I_t \subseteq I_v$. For ample details on the v-, t-, and w-operations, we refer the reader to David Anderson's papers [1–16] and also [21, 23, 26, 35, 46, 53–55, 57, 60, 62, 63, 65–67].

Let R be a ring and I an ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [38, 39, 59]. The notion of reduction was introduced by Northcott and Rees and its usefulness resides mainly in two facts: "First, it defines a relationship between two ideals which is preserved under homomorphisms and ring extensions; secondly, what we may term the reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it" [59]. The main purpose of their paper was to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions. An element $x \in R$ is integral over I if there is an equation $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ with $a_i \in I^i$ for $i = 1, \ldots, n$. The set of all elements that are integral over I is called the integral closure of I and is denoted by $\overline{I}$. Reductions happened to be a very useful tool for the theory of integral dependence over ideals. For a full treatment of these topics, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [48].

Let R be a domain and I a nonzero ideal. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some $n \geq 0$, and $x \in R$ is t-integral over I if there is an equation $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, \ldots, n$. The set of all elements that are t-integral over I is called the t-integral closure of I. This paper surveys recent literature which studies t-reductions and t-integral closure of ideals in arbitrary domains as well as in special contexts such as Prüfer v-multiplication domains (PvMDs), Noetherian domains, and pullback constructions. The four papers involved in this survey are [50] (co-authored with A. Kadri), [44] (co-authored with E. Houston and A. Mimouni), and [51, 52] (co-authored with A. Kadri and A. Mimouni). In this survey, we present and discuss the results without proofs and provide most of the examples with full details (from the original papers).

2 The General Case of Integral Domains

This part covers [50] which deals with t-reductions and t-integral closure of ideals in arbitrary domains. The aim is to provide t-analogues of well-known results on the integral closure of ideals and the correlations with reductions. Namely, Sect. 2.1 identifies basic properties of t-reductions of ideals and gives explicit examples discriminating between the notions of reduction and t-reduction. Section 2.2 examines the concept of t-integral closure of ideals as well as its correlation with t-reductions. Section 2.3 studies the persistence and contraction of t-integral closure of ideals under ring homomorphisms. All along this part, the main results are illustrated with original examples.

2.1 *t*-Reductions of Ideals

This section identifies basic ideal-theoretic properties of the notion of t-reduction including its behavior under localizations. We first provide an example (with full details) discriminating between the notions of reduction and t-reduction. Recall that, in a ring R, a subideal J of an ideal I is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n [59]. An ideal which has no reduction other than itself is called a basic ideal [38, 39].

Definition 2.1 Let R be a domain and $J \subseteq I$ nonzero [fractional] ideals of R.

- J is a *trivial t-reduction* of I if $J_t = I_t$.
- J is a *t-reduction* of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$.
- I is *t-basic* if it has no t-reduction other than the trivial t-reductions.
- R has the t-basic (resp., finite t-basic) ideal property if every nonzero (resp., finitely generated) [fractional] ideal of R is t-basic.

This is not to be confused with the identically named notion of Epstein [28–30], which generalizes the original notion of reduction in a different way and was studied in different settings. Namely, let c be a closure operation. An ideal $J \subseteq I$ is a c-reduction of I if $J^c = I^c$. Thus, Epstein's c-reduction coincides with our trivial c-reduction.

Recall a basic property of the t-operation (which, in fact, holds for any arbitrary star operation): for any two nonzero ideals I and J, we have $(IJ)_t = (I_tJ)_t = (IJ_t)_t = (I_tJ_t)_t$. So, for nonzero ideals $J \subseteq I$, J is a t-reduction of I if and only if J is a t-reduction of I_t if and only if J_t is a t-reduction of I_t. Notice also that any reduction is also a t-reduction, and the converse is not true, in general, as shown by the next example which exhibits a domain R with two t-ideals $J \subsetneq I$ such that J is a t-reduction but not a reduction of I.

Example 2.2 ([50, Example 2.2]) We use a construction from [49]. Let x be an indeterminate over $\mathbb{Z}$ and let $R := \mathbb{Z}[3x, x^2, x^3]$, $I := (3x, x^2, x^3)$, and $J := (3x, 3x^2, x^3, x^4)$. Then $J \subsetneq I$ are two finitely generated t-ideals of R such that $JI^n \subsetneq I^{n+1} \forall\, n \in \mathbb{N}$ and $(JI)_t = (I^2)_t$.

Proof I is a height-one prime ideal and, hence, a t-ideal of R [49]. Next, we prove that J is a t-ideal. We first claim that $J^{-1} = \frac{1}{x}\mathbb{Z}[x]$. Indeed, notice that $\mathbb{Q}(x)$ is the quotient field of R and since $3x \subseteq J$, then $J^{-1} \subseteq \frac{1}{3x}R$. So, let $f := \frac{g}{3x} \in J^{-1}$ where $g = \sum_{i=0}^{m} a_i x^i \in \mathbb{Z}[x]$ with $a_1 \in 3\mathbb{Z}$. Then the fact that $x^3 f \in R$ implies that $a_i \in 3\mathbb{Z}$ for $i = 0, 2, \ldots, m$; i.e., $g \in 3\mathbb{Z}[x]$. Hence, $f \in \frac{1}{x}\mathbb{Z}[x]$, whence $J^{-1} \subseteq \frac{1}{x}\mathbb{Z}[x]$. The reverse inclusion holds since $\frac{1}{x}J\mathbb{Z}[x] = (3, 3x, x^2, x^3)\mathbb{Z}[x] \subseteq R$, proving the claim. Next, let $h \in (R : \mathbb{Z}[x]) \subseteq R$. Then $xh \in R$ forcing $h(0) \in 3\mathbb{Z}$ and thus $h \in (3, 3x, x^2, x^3)$. So, $(R : \mathbb{Z}[x]) \subseteq (3, 3x, x^2, x^3)$, hence $(R : \mathbb{Z}[x]) = \frac{1}{x}J$. It follows that $J_t = J_v = \left(R : \frac{1}{x}\mathbb{Z}[x]\right) = x(R : \mathbb{Z}[x]) = J$, as desired. Next, let $n \in \mathbb{N}$. It is to see that $x^3x^{2n} = x^{2n+3}$ is the monic monomial with the smallest degree in JI^n. Therefore, $x^{2(n+1)} = x^{2n+2} \in I^{n+1} \setminus JI^n$. That is, J is not a reduction of I. It remains to prove $(JI)_t = (I^2)_t$. We first claim that $(JI)^{-1} = \frac{1}{x^2}\mathbb{Z}[x]$. Indeed, $(JI)^{-1} \subseteq (J^{-1})^2 = \frac{1}{x^2}\mathbb{Z}[x]$ and the reverse inclusion holds since $\frac{1}{x^2}JI\mathbb{Z}[x] = (3, 3x, x^2, x^3)(3, x, x^2)\mathbb{Z}[x] \subseteq R$, proving the claim. Now, observe that $I^2 = (9x^2, 3x^3, x^4, x^5)$. It follows that $(IJ)_t = (IJ)_v = \left(R : \frac{1}{x^2}\mathbb{Z}[x]\right) = x^2(R : \mathbb{Z}[x]) = xJ \supseteq I^2$. Thus, $(IJ)_t \supseteq (I^2)_t$, as desired.

In the above example, the domain R is not integrally closed. In fact, there is a class of integrally closed domains where the notions of reduction and t-reduction are always distinct. Indeed, in [50, Example 2.3], we show that if R is any integrally closed Mori domain that is not completely integrally closed, then there always exist nonzero ideals $J \subsetneqq I$ in R such that J is a t-reduction but not a reduction of I. Another crucial fact concerns reductions of t-ideals. That is, if J is a reduction of a t-ideal, then so is J_t, and the converse is not true, in general, as shown by [50, Example 2.4] which features a domain R with a t-ideal I and an ideal $J \subseteq I$ such that J_t is a reduction but J is not a reduction of I.

In the rest of this section, we provide basic ideal-theoretic properties of t-reduction. Let R be an arbitrary domain. Recall that, for any nonzero ideals I, J, H of R, the equality $(IJ + H)_t = (I_tJ + H)_t$ always holds. This property allowed us to prove the next basic result which examines the t-reduction of the sum and product of ideals.

Lemma 2.3 *Let $J \subseteq I$ and $J' \subseteq I'$ be nonzero ideals of R. If J and J' are t-reductions of I and I', respectively, then $J + J'$ and JJ' are t-reductions of $I + I'$ and II', respectively.*

The next basic result examines the transitivity for t-reduction.

Lemma 2.4 *Let $K \subseteq J \subseteq I$ be nonzero ideals of R. Then:*

(1) If K is a t-reduction of J and J is a t-reduction of I, then K is a t-reduction of I.
(2) If K is a t-reduction of I, then J is a t-reduction of I.

The next basic result examines the t-reduction of the power of an ideal.

Lemma 2.5 *Let $J \subseteq I$ be nonzero ideals of R and let n be a positive integer. Then:*

(1) J is a t-reduction of $I \Leftrightarrow J^n$ is a t-reduction of I^n.
(2) If $J = (a_1, \ldots, a_k)$, then: J is a t-reduction of $I \Leftrightarrow (a_1^n, \ldots, a_k^n)$ is a t-reduction of I^n.

The next basic result examines the t-reduction of localizations.

Lemma 2.6 *Let $J \subseteq I$ be nonzero ideals of R and let S be a multiplicatively closed subset of R. If J is a t-reduction of I, then $S^{-1}J$ is a t-reduction of $S^{-1}I$.*

Note that, in a PvMD, J is a t-reduction of I if and only if J is t-locally a reduction of I (Lemma 3.9).

2.2 t-Integral Closure of Ideals

This section investigates the concept of t-integral closure of ideals and its correlation with t-reductions. Our objective is to establish satisfactory t-analogues of (and in some cases generalize) well-known results, in the literature, on the integral closure of ideals and its correlation with reductions.

Definition 2.7 Let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \text{ with } a_i \in (I^i)_t \ \forall i = 1, \ldots, n.$$

The set of all elements that are t-integral over I is called the t-integral closure of I and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is called t-integrally closed.

The t-integral closure of the ideal R is always R, whereas the t-integral closure of the ring R (also called pseudo-integral closure) may be larger than R. Also, we have $J \subseteq I \Rightarrow \widetilde{J} \subseteq \widetilde{I}$. More properties are listed in Remark 2.14. It is well known that the integral closure of an ideal is an ideal which is integrally closed. The next theorem provides a t-analogue for this result.

Theorem 2.8 *The t-integral closure of an ideal is an integrally closed ideal. In general, it is not t-closed and, a fortiori, not t-integrally closed.*

The proof of the first statement of this theorem relied on the following lemma which sets a t-analogue for the notion of Rees algebra of an ideal [48, Chap. 5]. The Rees algebra of an ideal I (in a ring R) is the graded subring of $R[x]$ given by $R[Ix] := \bigoplus_{n\geq 0} I^n x^n$ [48, Definition 5.1.1] and whose integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0} \overline{I^n}\, x^n$ [48, Proposition 5.2.1].

Lemma 2.9 *Let R be a domain, I a t-ideal of R, and x an indeterminate over R. The t-Rees algebra of I is given by $R_t[Ix] := \bigoplus_{n\geq 0}(I^n)_t x^n$, and it is a graded subring of $R[x]$ and its integral closure in $R[x]$ is the graded ring $\bigoplus_{n\geq 0} \widetilde{I^n} x^n$.*

The proof of the last statement of the above theorem is handled by the next example, which provides a domain with an ideal I such that $\widetilde{I}$ is not a t-ideal and, hence, not t-integrally closed since $(\,\widetilde{I}\,)_t \subseteq \widetilde{\widetilde{I}}$ always holds.

Example 2.10 ([50, Example 3.10]) Let $R := \mathbb{Z} + x\mathbb{Q}(\sqrt{2})[x]$, $I := (\frac{x}{\sqrt{2}})$, and $a := \frac{x}{2}$, where x is an indeterminate over $\mathbb{Q}$. Then:

(1) I is a t-reduction of $I + aR$ and $a \notin \widetilde{I}$.
(2) $\widetilde{I} \subsetneq (\,\widetilde{I}\,)_t$ and hence $\widetilde{I} \subsetneq \widetilde{\widetilde{I}}$.

Proof (1) First, we prove that $(I(I + aR))_t = ((I + aR)^2)_t$. It suffices to show that $a^2 \in (I(I + aR))_t$. For this purpose, let $f \in (I(I + aR))^{-1} = (\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})^{-1} \subseteq$

$(\frac{x^2}{2})^{-1} = \frac{2}{x^2}R$. Then, $f = \frac{2}{x^2}(a_0 + a_1x + \ldots + a_nx^n)$, for some $n \geq 0$, $a_0 \in \mathbb{Z}$, and $a_i \in \mathbb{Q}(\sqrt{2})$ for $i \geq 1$. Since $\frac{x^2}{2\sqrt{2}}f \in R$, $a_0 = 0$. It follows that $(I(I + aR))^{-1} \subseteq \frac{1}{x}\mathbb{Q}(\sqrt{2})[x]$. On the other hand, $(I(I + aR))(\frac{1}{x}\mathbb{Q}(\sqrt{2})[x]) \subseteq R$. So, we have

$$\big(I(I + aR)\big)^{-1} = \left(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}}\right)^{-1} = \frac{1}{x}\mathbb{Q}(\sqrt{2})[x]. \qquad (1)$$

Now, clearly, $a^2(I(I + aR))^{-1} \subseteq R$. Therefore, $a^2 \in (I(I + aR))_v = (I(I + aR))_t$, as desired. Next, we prove that $a \notin \widetilde{I} = \overline{I}$. By [48, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + aR$. Deny and suppose that $I(I + aR)^n = (I + aR)^{n+1}$, for some positive integer n. Then $a^{n+1} = (\frac{x}{2})^{n+1} \in I(I + aR)^n = \frac{x}{\sqrt{2}}(\frac{x}{\sqrt{2}}, \frac{x}{2})^n$. One can check that this yields $1 \in \sqrt{2}(\sqrt{2}, 1)^n \subseteq (\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$, the desired contradiction.

(2) We claim that $a \in (\widetilde{I})_t$. Notice first that $x \in \widetilde{I}$ as $x^2 \in I^2 = (I^2)_t$. Therefore, $A := (x, \frac{x}{\sqrt{2}}) \subseteq \widetilde{I}$. Clearly, $A = \frac{2}{x}(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}})$. Hence, by (1), $A^{-1} = \mathbb{Q}(\sqrt{2})[x]$. However, $aA^{-1} \subseteq R$. Whence, $a \in A_v = A_t \subseteq (\widetilde{I})_t$. Consequently, $a \in (\widetilde{I})_t \setminus \widetilde{I}$.

The next result shows that the t-integral closure coincides with the t-closure in the class of integrally closed domains. It also completes two existing results in the literature on the integral closure of ideals (Gilmer [37] and Mimouni [57]).

Theorem 2.11 *Let R be a domain. The following assertions are equivalent:*

(1) R is integrally closed;
(2) Every principal ideal of R is integrally closed;
(3) Every t-ideal of R is integrally closed;
(4) $\overline{I} \subseteq I_t$ for each nonzero ideal I of R;
(5) Every principal ideal of R is t-integrally closed;
(6) Every t-ideal of R is t-integrally closed; and
(7) $\widetilde{I} = I_t$ for each nonzero ideal I of R.

If all ideals of a domain are t-integrally closed, then it must be Prüfer. This is a well-known result in the literature.

Corollary 2.12 ([37, Theorem 24.7]) *A domain R is Prüfer if and only if every ideal of R is (t-)integrally closed.*

Now, we examine the correlation between the t-integral closure and t-reductions of ideals. In this vein, recall that, for the trivial operation, two crucial results assert that $x \in \overline{I} \Leftrightarrow$ *I is a reduction of $I + Rx$* [48, Corollary 1.2.2] and *if I is finitely generated and $J \subseteq I$, then: $I \subseteq \overline{J} \Leftrightarrow J$ is a reduction of I* [48, Corollary 1.2.5]. Here are the t-analogues of these two results.

Proposition 2.13 *Let R be a domain and let $J \subseteq I$ be nonzero ideals of R.*

(1) If $x \in \widetilde{I}$, then I is a t-reduction of $I + Rx$.

(2) If I is finitely generated with $I \subseteq \widetilde{J}$, then J is a t-reduction of I.

Moreover, both implications are irreversible in general.

The next remark collects some basic properties of the t-integral closure.

Remark 2.14 Let R be a domain and let I, J be nonzero ideals of R. Then:

(1) $\forall\, x \in R, x\,\widetilde{I} \subseteq \widetilde{xI}$.
(2) $\widetilde{I \cap J} \subseteq \widetilde{I} \cap \widetilde{J}$. The inclusion can be strict, see Example 2.15(3).
(3) $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. These inclusions can be strict, see Example 2.15(1).
(4) $\forall\, n \geq 1$, $(\widetilde{I})^n \subseteq \widetilde{I^n}$. The inclusion can be strict, see Example 2.15(2).
(5) $\widetilde{I} + \widetilde{J} \subseteq \widetilde{I + J}$. The inclusion can be strict. For instance, in $\mathbb{Z}[x]$, we have $\widetilde{(2)} + \widetilde{(x)} = (2, x)$ and $\widetilde{(2, x)} = (2, x)_t = \mathbb{Z}[x]$ (via Theorem 2.11).

Example 2.15 ([50, Example 3.9]) Let $R := \mathbb{Z}[\sqrt{-3}][2x, x^2, x^3]$. Let $J := (x^3)$ and $I := (2x^2, 2x^3, x^4, x^5)$, where x is an indeterminate over $\mathbb{Z}$. Then I is a t-ideal such that

(1) $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$.

(2) $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$.

(3) $\widetilde{J \cap I} \subsetneqq \widetilde{J} \cap \widetilde{I}$.

Proof We first show that I is a t-ideal. Clearly, $\frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x] \subseteq I^{-1}$. For the reverse inclusion, let $f \in I^{-1} \subseteq x^{-4}R$. Then $f = \frac{1}{x^4}(a_0 + a_1x + \cdots + a_nx^n)$ for some $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}[\sqrt{-3}]$, $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$, and $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \geq 2$. Since $2x^2 f \in R$, then $a_0 = a_1 = 0$. It follows that $f \in \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Therefore, $I^{-1} = \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]$. Next, let $g \in (R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq R$. Then $xg \in R$, forcing $g(0) \in 2\mathbb{Z}[\sqrt{-3}]$ and hence $g \in (2, 2x, x^2, x^3)$. So $(R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq (2, 2x, x^2, x^3)$. The reverse inclusion is obvious. Thus, $(R : \mathbb{Z}[\sqrt{-3}][x]) = (2, 2x, x^2, x^3)$. Consequently, we obtain $I_t = I_v = (R : \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x]) = x^2(R : \mathbb{Z}[\sqrt{-3}][x]) = I$.

(1) Next, we prove the strict inclusions $I \subsetneqq \overline{I} \subsetneqq \widetilde{I} \subsetneqq \sqrt{I}$. For $I \subsetneqq \overline{I}$, notice that $(1 + \sqrt{-3})x^2 \in \overline{I} \setminus I$ as $\big((1 + \sqrt{-3})x^2\big)^3 = -8x^6 \in I^3$ and $1 + \sqrt{-3} \notin 2\mathbb{Z}[\sqrt{-3}]$.

For $\overline{I} \subsetneqq \widetilde{I}$, we claim that $x^3 \in \widetilde{I} \setminus \overline{I}$. Indeed, let $f \in (I^2)^{-1} \subseteq x^{-8}R$. Then there are $n \in \mathbb{N}$, $a_i \in \mathbb{Z}[\sqrt{-3}]$ for $i \in \{0, 2, \ldots, n\}$, and $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$ such that $f = \frac{1}{x^8}(a_0 + a_1x + \cdots + a_nx^n)$. Since $4x^4 f \in R$, then $a_0 = a_1 = a_2 = a_3 = 0$. Therefore, $(I^2)^{-1} \subseteq \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. The reverse inclusion is obvious. Hence, $(I^2)^{-1} = \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]$. It follows that $(I^2)_t = (I^2)_v = (R : \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x]) = x^4 (R : \mathbb{Z}[\sqrt{-3}][x]) = x^2 I$. Hence, $x^6 \in (I^2)_t$ and thus $x^3 \in \widetilde{I}$. It remains to show that $x^3 \notin \overline{I}$. By [48, Corollary 1.2.2], it suffices to show that I is not a reduction of $I + (x^3)$. Let $n \in \mathbb{N}$. It is easy to see that x^4x^{3n} is the monic monomial with the smallest degree in $I\big(I + (x^3)\big)^n$. Therefore, $x^{3(n+1)} = x^{3n+3} \in \big(I + (x^3)\big)^{n+1} \setminus I\big(I + (x^3)\big)^n$. Hence, I is not a reduction of $I + (x^3)$, as desired.

For $\widetilde{I} \subsetneqq \sqrt{I}$, we claim that $x^2 \in \sqrt{I} \setminus \widetilde{I}$. Obviously, $x^2 \in \sqrt{I}$. In order to prove that $x^2 \notin \widetilde{I}$, it suffices by Proposition 2.13 to show that I is not a t-reduction of $I +$

(x^2). To this purpose, notice that $I + (x^2) = (x^2)$. Suppose by way of contradiction that $(I(I + (x^2))^n)_t = ((I + (x^2))^{n+1})_t$ for some $n \in \mathbb{N}$. Then $(x^2)^{n+1} = x^{2n+2} \in (I(I + (x^2))^n)_t = x^{2n}I$. Consequently, $x^2 \in I$, absurd.

(2) We first prove that $\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4)$. In view of (1) and its proof, we have $(2x^2, (1+\sqrt{-3})x^2, x^3, x^4) \subseteq \widetilde{I}$. Next, let $\alpha := (a + b\sqrt{-3})x^2 \in \widetilde{I}$ where $a, b \in \mathbb{Z}$. If $b = 0$, then $a \neq 1$ as $x^2 \notin \widetilde{I}$. Moreover, since $2x^2 \in \widetilde{I}$, a must be even; that is, $\alpha \in (2x^2)$. Now assume $b \neq 0$. If $a = 0$, then $b \neq 1$ as $\sqrt{-3}x^2 \notin \widetilde{I}$. Moreover, since $2\sqrt{-3}x^2 \in \widetilde{I}$, b must be even; that is, $\alpha \in (2x^2)$. So suppose $a \neq 0$. Then similar arguments force a and b to be of the same parity. Further, if a and b are even, then $\alpha \in (2x^2)$; and if a and b are odd, then $\alpha \in (2x^2, (1+\sqrt{-3})x^2)$. Finally, we claim that $\widetilde{I}$ contains no monomials of degree 1. Deny and let $ax \in \widetilde{I}$, for some nonzero $a \in 2\mathbb{Z}[\sqrt{-3}]$. Then, by [48, Remark 1.1.3(7)], $ax \in \widetilde{I} \subseteq \widetilde{(x^2)} = \overline{(x^2)} \subseteq \overline{x^2\mathbb{Z}[\sqrt{-3}][x]}$. By [48, Corollary 1.2.2], (x^2) is a reduction of (ax, x^2) in $\mathbb{Z}[\sqrt{-3}][x]$, absurd. Consequently, $\widetilde{I} = (2x^2, (1+\sqrt{-3})x^2, x^3, x^4)$. Now, we are ready to check that $(\widetilde{I})^2 \subsetneqq \widetilde{I^2}$. For this purpose, recall that $(I^2)_t = x^2I$. So, $2x^4 \in \widetilde{I^2}$. We claim that $2x^4 \notin (\widetilde{I})^2$. Deny. Then, $2x^4 \in (4x^4, 2(1+\sqrt{-3})x^4)$. So $x^2 \in (2x^2, (1+\sqrt{-3})x^2) \subseteq \widetilde{I}$, absurd.

(3) We claim that $x^3 \in \widetilde{I} \cap \widetilde{J} \setminus \widetilde{I \cap J}$. We proved in (1) that $x^3 \in \widetilde{I}$. So, $x^3 \in \widetilde{I} \cap \widetilde{J}$. Now, observe that $I \cap J = xI$ and assume, by way of contradiction, that $x^3 \in \widetilde{I \cap J} = \widetilde{xI}$. Then x^3 satisfies an equation of the form $(x^3)^n + a_1(x^3)^{n-1} + \cdots + a_n = 0$ with $a_i \in ((xI)^i)_t = x^i(I^i)_t$, $i = 1, \ldots, n$. For each i, let $a_i = x^ib_i$, for some $b_i \in (I^i)_t$. Therefore, $(x^2)^n + b_1(x^2)^{n-1} + \cdots + b_n = 0$. It follows that $x^2 \in \widetilde{I}$, the desired contradiction.

2.3 Persistence and Contraction of *t*-Integral Closure

For any ring homomorphism, $\varphi : R \to T$, the persistence of integral closure describes the fact $\varphi(\overline{I}) \subseteq \overline{\varphi(I)T}$ for every ideal I of R, and the contraction of integral closure describes the fact $\overline{\varphi^{-1}(J)} = \varphi^{-1}(J)$ for every integrally closed ideal J of T. This section deals with the persistence and contraction of t-integral closure. For this purpose, we first need to introduce the concept of t-compatible homomorphism (which extends the well-known notion of t-compatible extension [13]). Throughout, t (resp., t_1) and v (resp., v_1) denote the t- and v- closures in R (resp., T).

Lemma 2.16 *Let $\varphi : R \longrightarrow T$ be a homomorphism of domains. Then, the following statements are equivalent:*

(1) $\varphi(I_v)T \subseteq \big(\varphi(I)T\big)_{v_1}$, for each nonzero finitely generated ideal I of R;
(2) $\varphi(I_t)T \subseteq \big(\varphi(I)T\big)_{t_1}$, for each nonzero ideal I of R; and
(3) $\varphi^{-1}(J)$ is a t-ideal of R for each t_1-ideal J of T such that $\varphi^{-1}(J) \neq 0$.

Definition 2.17 A homomorphism of domains $\varphi : R \longrightarrow T$ is called t-compatible if it satisfies the equivalent conditions of Lemma 2.16.

Under the embedding $R \subseteq T$, this definition matches the notion of t-compatible extension (i.e., $I_t T \subseteq (IT)_{t_1}$) well studied in the literature (cf. [13, 18, 27, 31]). Next, the main result of this section establishes persistence and contraction of t-integral closure under t-compatible homomorphisms.

Proposition 2.18 *Let $\varphi : R \longrightarrow T$ be a t-compatible homomorphism of domains. Let I be an ideal of R and J an ideal of T. Then:*

(1) $\varphi(\widetilde{I})T \subseteq \widetilde{\varphi(I)T}$.
(2) $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$*; and if J is t-integrally closed, then* $\widetilde{\varphi^{-1}(J)} = \varphi^{-1}(J)$.

If both R and T are integrally closed, then persistence of t-integral closure coincides with t-compatibility by Theorem 2.11. So the t-compatibility assumption in Proposition 2.18 is imperative.

Corollary 2.19 *Let $R \subseteq T$ be a t-compatible extension of domains and let I be an ideal of R. Then:*

(1) $\widetilde{I}T \subseteq \widetilde{IT}$.
(2) $\widetilde{I} \subseteq \widetilde{IT} \cap R \subseteq \widetilde{IT} \cap R$.

Moreover, the above inclusions are strict in general.

Corollary 2.20 *Let R be a domain, I an ideal of R, and S a multiplicatively closed subset of R. Then* $S^{-1}\widetilde{I} \subseteq \widetilde{S^{-1}I}$.

Recall that, for the integral closure, we have $S^{-1}\overline{I} = \overline{S^{-1}I}$ [48, Proposition 1.1.4], whereas in the above corollary the inclusion can be strict, as shown by the following example.

Example 2.21 We use a construction due to Zafrullah [65]. Let E be the ring of entire functions and x an indeterminate over E. Let S denote the set generated by the principal primes of E. Then, we claim that $R := E + xS^{-1}E[x]$ contains a prime ideal P such that $S^{-1}\widetilde{P} \subsetneq \widetilde{S^{-1}P}$. Indeed, R is a P-domain that is not a PvMD [65, Example 2.6]. By [66, Proposition 3.3], there exists a prime t-ideal P in R such that PR_P is not a t-ideal of R_P. By Theorem 2.11, we have $\widetilde{P}R_P = PR_P \subsetneq R_P = (PR_P)_t = \widetilde{PR_P}$ since R is integrally closed. Also notice that $P = \widetilde{PR_P} \cap R \subsetneq \widetilde{PR_P} \cap R = R$.

Corollary 2.22 *Let R be a domain and I a t-ideal that is t-locally t-integrally closed (i.e., I_M is t-integrally closed in R_M for every maximal t-ideal M of R). Then I is t-integrally closed.*

3 The Case of Prüfer v-multiplication Domains

In [38, 39], Hays investigated reductions of ideals in commutative rings with a particular focus on Prüfer domains. He studied the notion of basic ideal and examined domains subject to the basic ideal property. He showed that this class of domains is strictly contained in the class of Prüfer domains; namely, a domain is Prüfer if and only if it has the finite basic ideal property [38, Theorem 6.5]. The second main result of these two papers characterizes domains with the basic ideal property as one-dimensional Prüfer domains ([38, Theorem 6.1] and [39, Theorem 10]).

This part covers [44] which deals with the extension of Hays' aforementioned results on Prüfer domains to Prüfer v-multiplication domains (PvMDs). In Sect. 3.1 we first extend the definition of t-reduction to $\star$-reduction, for any arbitrary $\star$-operation, and then discuss the notion of $\star$-basic ideals and prove that a domain with the finite $\star$-basic ideal property (resp., $\star$-basic ideal property) must be integrally closed (resp., completely integrally closed). We also observe that a domain has the v-basic ideal property if and only if it is completely integrally closed. Section 3.2 is devoted to generalizing Hays' results; we show that a domain has the finite w-basic ideal property (resp., w-basic ideal property) if and only if it is a PvMD (resp., PvMD of t-dimension one). In Sect. 3.3, we present a diagram of implications among domains having various $\star$-basic properties and provide examples showing that most of the implications are not reversible.

3.1 $\star$-Basic Ideals

Let R be a domain with quotient field K and let $F(R)$ denote the set of nonzero fractional ideals of R. A map $\star : F(R) \to F(R)$, $I \mapsto I^\star$, is called a *star operation on* R if the following conditions hold for every $0 \neq a \in K$ and $I, J \in F(R)$:

- $R^\star = R$ and $(aI)^\star = aI^\star$,
- $I \subseteq J \Rightarrow I^\star \subseteq J^\star$, and
- $I \subseteq I^\star$ and $I^{\star\star} = I^\star$.

The next definition extends the notion of t-reduction and related concepts to an arbitrary star operation $\star$ on R.

Definition 3.1 Let $J \subseteq I$ be nonzero [fractional] ideals of R.

- J is a *trivial $\star$-reduction* of I if $J^\star = I^\star$.
- J is a *$\star$-reduction* of I if $(JI^n)^\star = (I^{n+1})^\star$ for some integer $n \geq 0$.
- I is *$\star$-basic* if it has no $\star$-reduction other than the trivial $\star$-reductions.
- R has the $\star$-basic (resp., finite $\star$-basic) ideal property if every nonzero (resp., finitely generated) [fractional] ideal of R is $\star$-basic.

If $\star_1$ and $\star_2$ are two star operations on R with $I^{\star_1} \subseteq I^{\star_2}$ for each ideal I, then any $\star_1$ reduction is also a $\star_2$-reduction, and the converse is not true in general, since a t-reduction may not be a reduction (see also Example 2.2).

The next results provide elementary properties and natural examples of $\star$-basic ideals and domains with the (finite) $\star$-basic ideal property.

Lemma 3.2 *$\star$-invertible ideals and $\star$-idempotent ideals are $\star$-basic.*

Recall that R is *completely integrally closed* (resp., a *v-domain*) if every nonzero ideal (resp., finitely generated ideal) of R is v-invertible.

Proposition 3.3 *The following assertions always hold:*

(1) If R has the finite $\star$-basic ideal property, then R is integrally closed.
(2) If R has the $\star$-basic ideal property, then R is completely integrally closed.
(3) R has the v-basic ideal property if and only if R is completely integrally closed.
(4) If R is a v-domain, then R has the finite v-basic ideal property.

The next example features a Noetherian domain with two t-ideals I, J such that J is a t-reduction, but not a reduction, of I. Since the v- and t-operations coincide under Noetherianess, such domain is not (completely) integrally closed by Proposition 3.3.

Example 3.4 Let k be a field, x, y two indeterminates over k, and $T := k[x, y]$. Consider the Noetherian domain $R = k + M^2$, where $M := (x, y)T$ (cf. [22]). As an ideal of T, M is basic [38, Theorem 2.3]. In particular, M^2 is not a reduction of M in T, and hence, it is not a reduction of M as a fractional ideal of R. However, M^2 is a nontrivial t-reduction of M in R. Indeed, we have $(T : M) = T$. It follows that $M \subseteq M^{-1}$ $(= (R : M)) \subseteq T$. On the other hand, if $f \in T$ satisfies $fM \subseteq R$, then, writing $f = a + m$ with $a \in k$ and $m \in M$, we immediately obtain that $aM \subseteq R$, whence $a = 0$, i.e., $f \in M$. Thus, $M^{-1} = M$, whence also $M_t = M_v = M$. However, $(R : T) = M^2$, whence $(M^2)^{-1} = ((R : M) : M) = (M : M) = T$ and then $(M^2)_t = (M^2)_v = (R : T) = M^2$, where the t- and v-operations are taken in R. A similar argument yields $(M^n)_t = M^2$ for $n \geq 2$. Hence, $M^2 = (M^3)_t = (M^2M)_t$. Consequently, $J := xM^2 \subseteq I := xM$ are two (integral) t-ideals of R, where J is a nontrivial t-reduction, but not a reduction, of I.

Recall that, to the star operation $\star$, we may define an associated star operation $\star_f$ by setting, for each $I \in F(R)$, $I^{\star_f} = \bigcup J^{\star}$, where J ranges over all finitely generated subideals of I; and then $\star$ is of *finite type* if $\star = \star_f$. In this case, minimal primes of $\star$-ideals are necessarily $\star$-ideals and each $\star$-ideal is contained in a maximal $\star$-ideal. For instance, $v_f = t$ and $t_f = t$.

Lemma 3.5 *Assume that $\star$ is of finite type. If I is a finitely generated ideal of R and J is a $\star$-reduction of I, then there is a finitely generated ideal $K \subseteq J$ such that K is a $\star$-reduction of I.*

This lemma allows to prove the following result.

Proposition 3.6 *If R has the finite $\star$-basic ideal property, then R also has the finite $\star_f$-basic ideal property. In particular, if R has the finite v-basic ideal property, then R also has the finite t-basic ideal property.*

Corollary 3.7 *A v-domain has the finite t-basic ideal property.*

3.2 Characterizations

At this point, we recall Kang's result [55, Theorem 3.5] that a PvMD is an integrally closed domain in which the t- and w-operations coincide. The next theorem features an analogue of Hays' first result that "*a domain is Prüfer if and only if it has the finite basic ideal property*" [38, Theorem 6.5].

Theorem 3.8 *A domain is a PvMD if and only if it has the finite w-basic ideal property.*

Hays proved that, in a Prüfer domain, the definition of a reduction can be restricted; namely, $J \subseteq I$ is a reduction if and only if $JI = I^2$ [39, Proposition 1]. The next lemma establishes a similar property for t-reductions and shows that this notion is local in the class of PvMDs.

Lemma 3.9 *Let R be a PvMD and $J \subseteq I$ nonzero ideals of R. Then, the following assertions are equivalent:*

(1) $(JI)_t = (I^2)_t$;
(2) *J is a t-reduction of I; and*
(3) $JR_M IR_M = (IR_M)^2$ *for each maximal t-ideal M of R.*

It is useful to note if J is a t-reduction of an ideal I, then a prime t-ideal of R contains I if and only if it contains J. We also recall that if I is a nonzero ideal of a domain R and S is a multiplicatively closed subset of R, then $(I_t R_S)_{t_{R_S}} = (IR_S)_{t_{R_S}}$ (this fact follows from [64, Lemma 4] and is stated explicitly in [55, Lemma 3.4]).

Lemma 3.10 *Let R be a PvMD and let $0 \neq x \in R$. Let P be a minimal prime of xR and set $I := xR_P \cap R$. Then:*

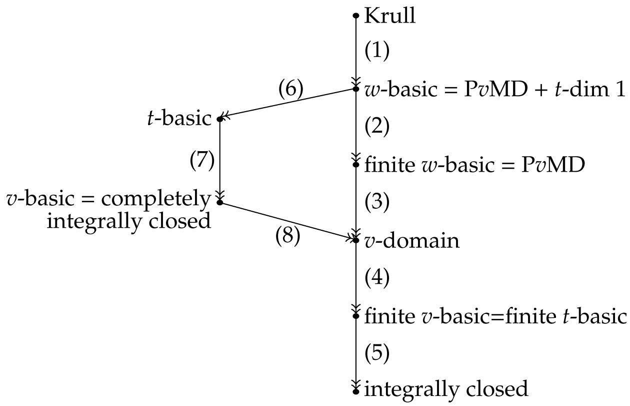
(1) *I is a w-ideal of R.*
(2) *$xR + I^2$ is a w-reduction of I.*
(3) *If I is w-basic, then P is a maximal t-ideal of R.*

The above two lemmas allowed us to prove the next theorem, which features an analogue of Hays' second result that "*a domain has the basic ideal property if and only if it is a Prüfer domain of dimension 1*" [39, Theorem 10].

Theorem 3.11 *A domain has the w-basic ideal property if and only if it is a PvMD of t-dimension 1.*

3.3 Examples

Consider the following diagram of implications putting in perspective the (finite) v-, t-, and w-basic ideal properties.



Notice that the implications (1)–(3) and (8) are well known, and (4)–(7) follow from Propositions 3.3, 3.6, Theorem 3.11, and the fact that the w- and t-operations coincide in a PvMD. Also, it is well known that (1)–(3) and (8) are irreversible in general. Moreover, the finite v-basic ideal property obviously implies the finite t-basic ideal property, and in Sect. 5.2 we will see that in fact, they are equivalent (Theorem 5.5).

Next, we provide examples with full details, from [44], proving that the remaining implications in the diagram are, too, irreversible in general.

Example 3.12 ([44, Example 3.1]) Implication (4) is irreversible.

Proof Let k be a field and X, Y, Z indeterminates over k. Let $T := k((X)) + M$ and $R := k[[X]] + M$, where $M := (Y, Z)k((X))[[Y, Z]]$. Let A be an ideal of R. Then A is comparable to M. Suppose $A \subseteq M$ and A is not invertible. If $AA^{-1} \supsetneq M$, then AA^{-1} is principal, and hence, A is invertible, contrary to assumption. Hence, $AA^{-1} \subseteq M$. We claim that $(AA^{-1})_v = M$. To verify this, first recall that M is divisorial in R. Then, since AA^{-1} is a trace ideal, that is, $(AA^{-1})^{-1} = (AA^{-1} : AA^{-1})$, we have $(AA^{-1})^{-1} \subseteq (AA^{-1}T : AA^{-1}T) = T = M^{-1}$ (the first equality holding since T is Noetherian and integrally closed). This forces $(AA^{-1})^{-1} = M^{-1}$, whence $(AA^{-1})_v = M_v = M$, as claimed. Now let I be a finitely generated ideal of R and J a v-reduction of I, so that $(JI^n)_v = (I^{n+1})_v$ for some positive integer n. We shall show that $J^{-1} = I^{-1}$ (and hence that $J_v = I_v$), and for this, we may assume that I is not invertible. Suppose, by way of contradiction, that $IT(T : IT) = T$, i.e., that IT is invertible in T. Then, since T is local, IT is principal and, in fact, $IT = aT$ for some $a \in I$. We then have $R \subseteq a^{-1}I \subseteq T$. Then $k[[X]] \cong R/M \subseteq a^{-1}I/M \subseteq T/M \cong k((X))$, from which it follows that $a^{-1}I/M$ must be a cyclic $k[[X]]$-module. However, this is easily seen to imply that $a^{-1}I$, hence I, is principal, the desired contradiction. We therefore have $(T : IT)I \subseteq M$, whence $(IM)^{-1} = (R : IM) = ((R : M) : I) = (T : I) = (M : I) \subseteq I^{-1}$. This immediately yields $I^{-1} = (IM)^{-1}$.

Now set $Q = I^n(I^n)^{-1}$. From above (setting $A = I^n$), we have $Q_v = M$. Therefore, $I^{-1} \subseteq J^{-1} \subseteq (JM)^{-1} = (JQ)^{-1} = (IQ)^{-1} = (IM)^{-1} = I^{-1}$, which yields $J^{-1} = I^{-1}$, as desired. Hence, R has the finite v-basic property. Finally, again from above, we have $((y, z)(y, z)^{-1})_v = M$, so that R is not a v-domain.

Example 3.13 ([44, Example 3.2]) Implication (5) is irreversible.

Proof Let k be a field and X, Y indeterminates over k. Let $V = k(X)[[Y]]$ and $R = k + M$, where $M = Yk(X)[[Y]]$. Clearly, R is an integrally closed domain. Of course, M is divisorial in R. Also, $(M^2)^{-1} = ((R : M) : M) = (V : M) = Y^{-1}V$, and so $(M^2)_v = (R : Y^{-1}V) = Y(R : V) = YM = M^2$, i.e., M^2 is also divisorial. We claim that R does not have the finite t-basic ideal property. Indeed, let $W := k + Xk$ and consider the finitely generated ideal I of R given by $I = Y(W + M)$. We have $(k : W) = (0)$; otherwise, we have $0 \neq f \in (k : W)$, and both f and $fX \in k$, whence $X \in k$, a contradiction. Therefore, $I^{-1} = Y^{-1}M$ and thus $I_t = I_v = YM^{-1} = M$. Now, let $J = YR$. Then $J_t = YR \subsetneq M = I_t$. However, $(JI)_t = (YI)_t = YI_t = YM = M^2 = ((I_t)^2)_t = (I^2)_t$, and so R does not have the finite t-basic ideal property.

Example 3.14 ([44, Example 3.3]) Implication (6) is irreversible.

Proof In [42], Heinzer and Ohm give an example of an essential domain that is not a PvMD. In that example, k is a field, y, z, and $\{x_i\}_{i=1}^{\infty}$ are indeterminates over k, and $D = R \cap (\bigcap_{i=1}^{\infty} V_i)$, where $R = k(\{x_i\})[y, z]_{(y,z)k(\{x_i\})[y,z]}$ and V_i is the rank-one discrete valuation ring on $k(\{x_j\}_{j=1}^{\infty}, y, z)$ with x_i, y, z all having value 1 and x_j having value 0 for $j \neq i$ (using the "infimum" valuation). As further described in [58, Example 2.1], we have $\mathrm{Max}(D) = \{M\} \cup \{P_i\}$, where M is the contraction of $(y, z)R$ to D and the P_i are the centers of the maximal ideals of the V_i; moreover, $D_M = R$ and $V_i = D_{P_i}$.

It was pointed out in [35, Example 1.7] that each finitely generated ideal of D is contained in almost all of the V_i. If fact, one can say more. Let a be an element of D. We may represent a as a quotient f/g with $f, g \in T := k[\{x_i\}, y, z]_{(y,z)k[\{x_i\},y,z]}$ and $g \notin (y, z)T$ (and hence $g \notin M$). Since f and g involve only finitely many x_j and $g \notin M$, the sequence $\{v_i(a)\}$ must be eventually constant, where v_i is the valuation corresponding to V_i. We denote this constant value by $w(a)$. A similar statement holds for finitely generated ideals of D.

Let K be a nonzero ideal of D. Then $K_t D_{P_i} \supseteq KD_{P_i} = (KD_{P_i})_{t_{D_{P_i}}} = (K_t D_{P_i})_{t_{D_{P_i}}} \supseteq K_t D_{P_i}$, whence $K_t D_{P_i} = KD_{P_i}$. Now suppose that we have nonzero ideals $J \subseteq I$ of D with $(JI^n)_t = (I^{n+1})_t$. Let $a \in I$, and choose $a_0 \in I$ so that $w(a_0)$ is minimal. Then $aa_0^n \in I^{n+1} \subseteq (JI^n)_t$, and so $aa_0^n \in (BA^n)_v$ for finitely generated ideals $B \subseteq J$ and $A \subseteq I$. With the observation in the preceding paragraph, we then have $aa_0^n \in BA^n D_{P_i}$ for each i. However, since $w(a_0) \leq w(A)$, it must be the case that $w(a) \geq w(B)$; i.e., for some integer k, $a \in BD_{P_i}$ for all $i > k$. Since the equality $(JI^n)_t = (I^{n+1})_t$ yields $JD_{P_i} = ID_{P_i}$ for each i, we may choose elements $b_j \in J$

for which $v_j(a) = v_j(b_j)$, $j = 1, \ldots, k$. With $B' = (B, b_1, \ldots, b_k)$, we then have $a \in B'D_{P_i}$ for each i. This yields $a(B')^{-1} \subseteq \bigcap D_{P_i}$.

Next, we consider extensions to D_M. From $(JI^n)_t = (I^{n+1})_t$, we obtain $(JI^nD_M)_{t_{D_M}} = (I^{n+1}D_M)_{t_{D_M}}$. Since D_M is a regular local ring, each nonzero ideal of D_M is t-invertible, and we may cancel to obtain $(ID_M)_{t_{D_M}} = (JD_M)_{t_{D_M}}$. There is a finitely generated subideal B_1 of J with $B_1D_M = JD_M$. We then have $IB_1^{-1} \subseteq ID_MB_1^{-1}D_M = ID_M(B_1D_M)^{-1} \subseteq (JD_M(JD_M)^{-1})_{t_{D_M}} \subseteq D_M$. Now let $B_2 = B' + B_1$. Then $a(B_2)^{-1} \subseteq D_M \cap \bigcap D_{P_i} = D$, whence $a \in (B_2)_v \subseteq J_t$. It follows that D has the t-basic property. However, since D is not a PvMD, D cannot have the (finite) w-basic property.

Example 3.15 ([44, Example 3.4]) Implication (7) is irreversible. For instance, the ring of entire functions is a completely integrally closed Prüfer domain with infinite Krull dimension, and hence, it does not have the (t-) basic ideal property by [39, Theorem 10].

4 The Case of Noetherian Domains

This part covers [52], which studies t-reductions and t-integral closure of ideals in Noetherian domains. The main objective is to establish t-analogues for well-known results on reductions and integral closure of ideals in Noetherian rings. Section 4.1 investigates t-reductions of ideals subject to t-invertibility and localization in arbitrary Noetherian domains. Section 4.2 investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one.

4.1 t-Reductions Subject to t-Invertibility and Localization

This section deals with t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. The first main result establishes a t-analogue for Hays' result on the correlation between invertible reductions and the Krull dimension [38, Theorem 4.4]; and the second main result establishes a t-analogue for Hays' global–local result on the basic ideal property [38, Theorem 3.6]. In 1973, Hays proved the following result:

Theorem 4.1 ([38, Theorem 4.4]) *Let R be a Noetherian domain such that R/M is infinite for every maximal ideal M of R. Then, each nonzero ideal has an invertible reduction if and only if* $\dim(R) \leq 1$.

The t-dimension of a domain R, denoted t-$\dim(R)$, is the supremum of the lengths of chains of prime t-ideals in R (here (0) is considered as a prime t-ideal although technically it is not); and the inequality t-$\dim(R) \leq \dim(R)$ always holds [43]. Here is a t-analogue of the above result.

Theorem 4.2 *Let R be a Noetherian domain such that the residue field of each maximal t-ideal is infinite. Then, the following statements are equivalent:*

(1) *Each t-ideal of R has a t-invertible t-reduction;*
(2) *Each maximal t-ideal of R has a t-invertible t-reduction; and*
(3) $t\text{-}\dim(R) \leq 1$.

The next lemma handles the implication (2) $\Rightarrow$ (3) without the infinite residue field assumption.

Lemma 4.3 *Let R be a Noetherian domain. If every maximal t-ideal of R has a t-invertible t-reduction, then* $t\text{-}\dim(R) \leq 1$.

Observe that, in general, the converse of Lemma 4.3 is not true. For instance, consider an almost Dedekind domain R which is not Dedekind. Then R is a one-dimensional locally Noetherian Prüfer domain. Hence, R has the basic ideal property [38, Theorem 6.1]. Since R is not Dedekind, it has a non-invertible maximal ideal which has no proper reduction.

Next, we move to the global–local transfer of the t-basic ideal property. For this purpose, recall that an ideal I is locally basic (resp., t-locally t-basic) if IR_M is basic (resp., t-basic) for each maximal ideal (resp., maximal t-ideal) M of R containing I. In 1973, Hays proved the following result.

Theorem 4.4 ([38, Theorem 3.6]) *In a Noetherian ring, an ideal is basic if and only if it is locally basic.*

Here is a t-analogue for the "if" assertion of this result.

Theorem 4.5 *In a Noetherian domain, if an ideal is t-locally t-basic, then it is t-basic.*

Now, note that, in his proof of the implication "basic $\Rightarrow$ locally basic," Hays used two basic facts. The first one states that, in an arbitrary ring R, if $J \subseteq I$ and JR_M is a reduction of IR_M, then $(J \cap I) + IM$ is a reduction of I; and here is a t-analogue for this result.

Proposition 4.6 *Let R be a domain, M a maximal t-ideal of R, and $I \subseteq M$ a nonzero ideal of R. If J is an ideal of R such that JR_M is a t-reduction of IR_M, then $(J \cap I) + IM$ is a t-reduction of I.*

However, the second fact was Nakayama's lemma, which ensures that $J \subseteq I \subseteq J + IM$ in a local Noetherian ring (R, M) forces $J = I$, and a t-analogue for this Nakayama property is not true in general. For example, consider the local Noetherian ring $R := k + M^2 \subseteq k[x, y]$, where $M = (x, y)$ and $(M^2)_t = (M^3)_t$ [44, Example 1.5].

4.2 *t*-Reductions and *t*-Integral Closure in One-Dimensional Noetherian Domains

This section deals with the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. The objective is to establish t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions of ideals in Noetherian settings.

Recall from Sect. 2.2 that "*$\widetilde{I}$ is an integrally closed ideal which is not t-integrally closed in general*." Several ideal-theoretic properties of $\widetilde{I}$ are collected in Remark 2.14, including the inclusions $I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}$. Consider the two sets related to the (trivial) d-operation and t-operation, respectively:

$$\widehat{I}^{\,d} := \left\{x \in R \mid I \text{ is a reduction of } (I, x)\right\}$$

$$\widehat{I}^{\,t} := \left\{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\right\}.$$

For the trivial operation, it is well known that the equality $\overline{I} = \widehat{I}^{\,d}$ always holds [48, Corollary 1.2.2]. This fact which was used to show that $\overline{I}$ is an ideal [48, Corollary 1.3.1]. However, it is still an open problem of whether $\widehat{I}^{\,t}$ is an ideal [51, Question 3.5]. We always have $I_t \subseteq \widetilde{I} \subseteq \widehat{I}^{\,t}$ where the second containment is proved by [50, Proposition 3.7] and can be strict as shown by [50, Example 3.10(a)]. Moreover, "*$I_t = \widetilde{I}$ for each nonzero ideal I if and only if R is integrally closed*" [50, Theorem 3.5], and "*$I_t = \widehat{I}^{\,t}$ for each nonzero ideal I if and only if R has the finite t-basic ideal property*" [51, Theorem 3.2].

The class of Prüfer domains is the only known class of domains, where the two notions of reduction and t-reduction coincide (since the trivial and t- operations are the same). The next result shows that the same happens in one-dimensional Noetherian domains (where the trivial and t- operations are not necessarily the same).

Theorem 4.7 *In a one-dimensional Noetherian domain, the notions of reduction and t-reduction coincide. Moreover, $\overline{I} = \widetilde{I} = \widehat{I}^{\,t}$ for any nonzero ideal I.*

As illustrative examples, consider one-dimensional Noetherian domains which are not divisorial (i.e., t-operation is not trivial), as shown below.

Example 4.8 Let $R := \mathbb{Q} + x\mathbb{Q}(\sqrt{2}, \sqrt{3})[[x]]$, where $\mathbb{Q}$ is the field of rational numbers and x is an indeterminate over $\mathbb{Q}$. Then, R is a pseudo-valuation domain (see definition in Sect. 5.1) issued from the DVR $\mathbb{Q}(\sqrt{2}, \sqrt{3})[[x]]$ and hence is a one-dimensional Noetherian domain. Further, R is not a divisorial domain by [40, Theorem 3.5] or [45, Theorem 2.4] since $[V/M : R/M] \neq 2$.

One wonders whether there exist Noetherian domains of dimension > 1 where the notions of reduction and t-reduction coincide. Next, we show this cannot happen in a large class of Noetherian domains.

Proposition 4.9 *Let R be a Noetherian domain with* $(R : \overline{R}) \neq 0$*. Then, the notions of reduction and t-reduction coincide in R if and only if* $\dim(R) = 1$.

5 The Case of Pullbacks

This part covers [51], which investigates t-reductions of ideals in pullback constructions (defined in Sect. 5.3). Section 5.1 examines the correlation between the notions of reduction and t-reduction in pseudo-valuation domains. Section 5.2 solves an open problem raised in [44] on whether the finite t-basic and v-basic ideal properties are distinct. In fact, these two notions coincide in any arbitrary domain (Theorem 5.5). Section 5.3 features the main result, which establishes the transfer of the finite t-basic ideal property to pullbacks in line with Fontana–Gabelli's result on PvMDs [31, Theorem 4.1] and Gabelli–Houston's result on v-domains [34, Theorem 4.15]. This allows us to enrich the literature with new examples, putting the class of domains subject to the finite t-basic ideal property strictly between the two classes of v-domains and integrally closed domains.

5.1 t-Reductions in Pseudo-Valuation Domains

Recall that a pseudo-valuation domain (PVD) R is a special pullback issued from the following diagram:

$$\begin{array}{ccc} R = \varphi^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & K := V/M, \end{array}$$

where (V, M) is a valuation domain with residue field K and k is a subfield of K. We say that R is a PVD issued from (V, M, k). For more details on pseudo-valuation domains, see [40, 41] and also [17, 19, 24, 25, 61].

Note that a reduction is necessarily a t-reduction; and the converse is not true in general. The next result investigates necessary and sufficient conditions for the notions of reduction and t-reduction to coincide in PVDs. This result can be used readily to provide examples discriminating between the two notions of reduction and t-reduction.

Theorem 5.1 *Let R be a PVD issued from* (V, M, k) *with* $K := V/M$*. Then, the following statements are equivalent:*

(1) For every nonzero ideals $J \subseteq I$*, J is a t-reduction of I* $\Leftrightarrow$ *J is a reduction of I.*
(2) For each k-vector subspace W of K containing k, W^n *is a field for some* $n \geq 0$.

Note that Condition (2) in the above result forces K to be algebraic over k, and so this fact can be used to build examples where the two notions of reduction and t-reduction are the same or distinct, as shown below.

Example 5.2 ([51, Example 2.3]) Let R be a PVD issued from (V, M, k) with $K := V/M$.

(1) Assume K is a transcendental over k. Then, the notions of reduction and t-reduction are distinct in R. For example, pick a transcendental element $\lambda \in K$ over k and let $W := k + k\lambda$, $I := a\varphi^{-1}(W)$ and $J =: aR$. Then, J is a proper t-reduction of I, whereas I is basic in R.
(2) Assume $[K : k]$ is finite. Then for every k-submodule W of K with $k \subseteq W \subseteq K$, some power of W is a field, and hence, the notions of reduction and t-reduction coincide in R.

Given nonzero ideals $J \subseteq I$, if J_t is a reduction of I_t, then J is a t-reduction of I. The converse is not true in general as shown by Example 2.2. The next result provides a class of (integrally closed) pullbacks where the two assumptions are always equivalent.

Proposition 5.3 *Let R be a PVD and let $J \subseteq I$ be nonzero ideals of R. Then, J is a t-reduction of I if and only if J_t is a reduction of I_t.*

The class of Prüfer domains is, so far, the only known class of domains where the two notions of reduction and t-reduction coincide. We close this section with the next result, which features necessary conditions for such a coincidence. For this purpose, recall that a domain where the trivial and w-operations are the same is said to be a DW domain [36, 47, 57]. Common examples of DW domains are pseudo-valuation domains, Prüfer domains, and quasi-Prüfer domains (i.e., domains with Prüfer integral closure) [32, p. 190].

Proposition 5.4 *Let R be a domain where the notions of reduction and t-reduction coincide for all ideals of R. Then:*

(1) Every nonzero prime ideal of R is a t-ideal. In particular, R is a DW domain.
(2) R is integrally closed if and only if R has the finite t-basic ideal property.
(3) R is a PvMD if and only if R is a Prüfer domain.

5.2 Equivalence of the Finite t- and v-Basic Ideal Properties

A domain is called a v-domain if all its nonzero finitely generated ideals are v-invertible; a comprehensive reference for v-domains is Fontana and Zafrullah's survey paper [33]. Also, recall the finite v-basic ideal property obviously implies the finite t-basic ideal property, and the question of whether this implication is reversible was left open in [44, Sect. 3]. The main result of this section (Theorem 5.5) solves this open question. For this purpose, recall from Sect. 4.2 the following objects $\widetilde{I} := \{x \in R \mid x \text{ is } t\text{-integral over } I\}$ and $\widehat{I}^t := \{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\}$ along with the basic inclusions $I_t \subseteq \widetilde{I} \subseteq \widehat{I}^t$. Finally, in order to put the main result into perspective, recall the important result that "*a domain R is integrally closed if and only if $I_t = \widetilde{I}$ for each nonzero (finitely generated) ideal I of R*" (Theorem 2.11).

Here is the main result of this section.

Theorem 5.5 *For a domain R, the following assertions are equivalent:*

(1) $I_t = \widehat{I}$ for each nonzero (finitely generated) ideal I of R;
(2) R has the finite t-basic ideal property; and
(3) R has the finite v-basic ideal property.

The proof of this result required the following two elementary lemmas.

Lemma 5.6 (cf. Lemma *3.5*) *Let R be a domain and let I be a finitely generated ideal of R. If $J \subseteq I$ is a t-reduction of I, then there exists a finitely generated ideal $K \subseteq J$ such that K is a t-reduction of I.*

Lemma 5.7 *For a domain R, let $K \subseteq J \subseteq I$ and $J' \subseteq I'$ be nonzero fractional ideals, and let n and k be positive integers.*

(1) If J and J' are $\star$-reductions of I and I', respectively, then $J + J'$ and JJ' are $\star$-reductions of $I + I'$ and II', respectively.
(2) Assume K is a $\star$-reduction of J. If J is a $\star$-reduction of I, then so is K.
(3) If K is a $\star$-reduction of I, then J is a $\star$-reduction of I.
(4) J is a $\star$-reduction of I if and only if J^n is a $\star$-reduction of I^n.
(5) $J = (a_1, \ldots, a_k)$ is a $\star$-reduction of $I \Leftrightarrow (a_1^n, \ldots, a_k^n)$ is a $\star$-reduction of I^n.

New examples of domains subject to the finite t-basic (equiv., v-basic) ideal property will be provided in the next section. We close this section with the following open question:

Question 5.8 ([51, Question 3.5]) Let I be a nonzero ideal, is $\widehat{I}$ always an ideal?

5.3 Transfer of the Finite t-Basic Ideal Property to Pullbacks

Throughout, R will be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M, \end{array}$$

where T is a domain, M is a maximal ideal of T with residue field K, $\varphi : T \longrightarrow K$ is the canonical surjection, and D is a proper subring of K with quotient field k. So, $R := \varphi^{-1}(D) \subsetneqq T$. First, note that Proposition 3.3 ensures that a domain with the t-basic ideal property is necessarily completely integrally closed, and so, by [37, Lemma 26.5], R never has the t-basic ideal property. This section investigates conditions for R to inherit the finite t-basic (or, equivalently, v-basic) ideal property when T is local.

Recall from Sects. 3.3 and 5.2 that the finite t-basic ideal property lies between the two notions of v-domain and integrally closed domain; and that the finite w-basic ideal property coincides with the PvMD notion. Also, at this point, it is worthwhile recalling Fontana and Gabelli's [31] and Gabelli and Houston's [34] well-known results, which establish the transfer of the notions of PvMD and v-domain to pullbacks, respectively, and which summarize as follows.

Theorem 5.9 ([31, Theorem 4.1] and [34, Theorem 4.15]) *R is a PvMD (resp., v-domain) if and only if T and D are PvMDs (resp., v-domains), T_M is a valuation domain, and $k = K$.*

Here is the main result of this section.

Theorem 5.10 *Assume that T is local. Then, R has the finite t-basic ideal property if and only if T and D have the finite t-basic ideal property and $k = K$.*

This result enables the construction of new examples, which put the finite t-basic ideal property strictly between the two notions of integrally closed domain and v-domain. Follow some examples with full details from [51].

Example 5.11 ([51, Example 4.3]) Consider any nontrivial pseudo-valuation domain R issued from (V, M, k) with k algebraically closed in $K := V/M$. Then, R is an integrally closed domain by [20, Theorem 2.1], which does not have the finite t-basic ideal property by Theorem 5.10. Moreover, the two notions of reduction and t-reduction are distinct in R by Proposition 5.4.

Example 5.12 ([51, Example 4.4]) Consider a pullback R issued from (T, M, D), where T is a non-valuation local v-domain and D is a v-domain with quotient field T/M. Then, R has the finite t-basic ideal property by [44, Proposition 1.6] and Theorem 5.5 and Theorem 5.10, which is not a v-domain by [34, Theorem 4.15]. One can easily build non-valuation local v-domains via pullbacks through [34, Theorem 4.15].

Example 5.13 ([51, Example 4.5]) Let $T := \mathbb{Q}(X)[[Y, Z]] = \mathbb{Q}(X) + M$ and $R := \mathbb{Z}[X] + M$. Clearly, T and $D := \mathbb{Z}[X]$ have the finite t-basic property (since they are Noetherian Krull domains). By Theorem 5.10, R has the finite t-basic property. Also, R is not a v-domain since T is a non-valuation local domain. Next, let $0 \neq a \in \mathbb{Z}$ and consider the finitely generated ideal of R given by $I := (a, X)\mathbb{Z}[X] + M = aR + XR$. Clearly $I^{-1} = R$ and so $(I^s)^{-1} = R$, for every positive integer s. In particular, we have $(I^2I)_t = (I^3)_t = (I^3)_v = R = (I^2)_v = (I^2)_t$, and hence, I^2 is a t-reduction of I. However, I^2 is not a reduction of I; otherwise, if $I^{n+2} = I^2 I^n = I^{n+1}$, for some $n \geq 1$, this would contradict [56, Theorem 76]. It follows that the notions of reduction and t-reduction are distinct in R.

We close this section with the following two open questions from [51].

Question 5.14 ([51, Question 4.6]) Is Theorem 5.10 valid for the classical pullbacks $R = D + M$ issued from $T := K + M$ not necessarily local? The idea here is that (since $k = K$, then) $T = S^{-1}R$ for $S := D \setminus \{0\}$. Clearly, the current proof of the "only if" assertion works for this context.

Question 5.15 ([51, Question 4.7]) Is Theorem 5.10 valid for the nonlocal case through an additional assumption on T_M? The idea here is that "($k = K$ and hence) $R_M = T_M$" is a necessity for the finite t-basic property, and for the PvMD and v-domain notions, $R_M = T_M$ is a valuation domain. So, one needs to investigate this localization for the t-basic ideal property in this context.

References

1. D.D. Anderson, D.F. Anderson, Some remarks on star operations and the class group. J. Pure Appl. Algebra **51**, 27–33 (1988)
2. D.D. Anderson, D.F. Anderson, D.L. Costa, D.E. Dobbs, J.L. Mott, M. Zafrullah, Some characterizations of v-domains and related properties. Colloq. Math. **58**, 1–9 (1989)
3. D.D. Anderson, D.F. Anderson, Examples of star operations on integral domains. Commun. Algebra **18**, 1621–1643 (1990)
4. D.D. Anderson, D.F. Anderson, M. Fontana, M. Zafrullah, On v-domains and star operations. Commun. Algebra **37**, 3018–3043 (2009)
5. D.D. Anderson, D.F. Anderson, M. Zafrullah, Splitting the t-class group. J. Pure Appl. Algebra **74**, 17–37 (1991)
6. D.D. Anderson, D.F. Anderson, M. Zafrullah, The ring $D + XD_S[X]$ and t-splitting sets. Arab. J. Sci. Eng. Sect. C **26**, 3–16 (2001)
7. D.D. Anderson, D.F. Anderson, M. Zafrullah, Completely integrally closed Prüfer v-multiplication domains. Commun. Algebra **45**, 5264–5282 (2017)
8. D.F. Anderson, Integral v-ideals. Glasgow Math. J. **22**, 167–172 (1981)
9. D.F. Anderson, A general theory of class groups. Commun. Algebra **16**, 805–847 (1988)
10. D.F. Anderson, Star operations and the $D + M$ construction. Rend. Circ. Mat. Palermo **2**(41), 221–230 (1992)
11. D.F. Anderson, G.W. Chang, The class group of integral domains. J. Algebra **264**, 535–552 (2003)
12. D.F. Anderson, S. El Baghdadi, S. Kabbaj, On the class group of $A + XB[X]$ domains. Lect. Notes Pure Appl. Math. Dekker **205**, 73–85 (1999)
13. D.F. Anderson, S. El Baghdadi, M. Zafrullah, The v-operation in extensions of integral domains. J. Algebra Appl. **11**, 1250007, p. 18 (2012)
14. D.F. Anderson, M. Fontana, M. Zafrullah, Some remarks on Prüfer $\star$-multiplication domains and class groups. J. Algebra **319**, 272–295 (2008)
15. D.F. Anderson, E. Houston, M. Zafrullah, Pseudo-integrality. Can. Math. Bull. **34**, 15–22 (1991)
16. D.F. Anderson, A. Ryckaert, The class group of $D + M$. J. Pure Appl. Algebra **52**, 199–212 (1988)

17. A. Badawi, E. Houston, Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains. Commun. Algebra **30**, 1591–1606 (2002)
18. V. Barucci, S. Gabelli, M. Roitman, The class group of a strongly Mori domain. Commun. Algebra **22**, 173–211 (1994)
19. V. Barucci, E. Houston, T.G. Lucas, I.J. Papick, m-canonical ideals in integral domains II. Lect. Notes Pure Appl. Math. Dekker **220**, 89–108 (2001)
20. E. Bastida, R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$. Michigan Math. J. **20**, 79–95 (1973)
21. A. Bouvier, M. Zafrullah, On some class groups of an integral domain. Bull. Soc. Math. Grèce (N.S.) **29**, 45–59 (1988)
22. J. Boynton, S. Sather-Wagstaff, Regular pullbacks, in *Progress in Commutative Algebra 2* (Walter de Gruyter, Berlin, 2012), pp. 145–169
23. G.M. Chang, M. Zafrullah, The w-integral closure of integral domains. J. Algebra **295**, 195–210 (2006)
24. R.D. Chatham, D.E. Dobbs, On pseudo-valuation domains whose overrings are going-down domains. Houston J. Math. **28**, 13–19 (2002)
25. D.E. Dobbs, Coherent pseudo-valuation domains have Noetherian completions. Houston J. Math. **14**, 481–486 (1988)
26. S. El Baghdadi, S. Gabelli, w-divisorial domains. J. Algebra **285**, 335–355 (2005)
27. J. Elliott, Functorial properties of star operations. Commun. Algebra **38**, 1466–1490 (2010)
28. N. Epstein, A tight closure analogue of analytic spread. Proc. Camb. Philos. Soc. **139**, 371–383 (2005)
29. N. Epstein, Reductions and special parts of closures. J. Algebra **323**, 2209–2225 (2010)
30. N. Epstein, A guide to closure operations in commutative algebra, in *Progress in Commutative Algebra 2* (Walter de Gruyter, Berlin, 2012), pp. 1–37
31. M. Fontana, S. Gabelli, On the class group and the local class group of a pullback. J. Algebra **181**, 803–835 (1996)
32. M. Fontana, G. Picozza, On some classes of integral domains defined by Krull's a.b. operations. J. Algebra **341**, 179–197 (2011)
33. M. Fontana, M. Zafrullah, On v-domains: a survey, in *Commutative Algebra Noetherian and Non-Noetherian Perspectives* (Springer, New York, 2011), pp. 145–179
34. S. Gabelli, E. Houston, Coherentlike conditions in pullbacks. Michigan Math. J. **44**, 99–123 (1997)
35. S. Gabelli, E. Houston, T.G. Lucas, The t#-property for integral domains. J. Pure Appl. Algebra **194**, 281–298 (2004)
36. S. Gabelli, G. Picozza, w-stability and Clifford w-regularity of polynomial rings. Commun. Algebra **43**, 59–75 (2015)
37. R. Gilmer, *Multiplicative Ideal Theory* (Dekker, New York, 1972)
38. J. Hays, Reductions of ideals in commutative rings. Trans. Am. Math. Soc. **177**, 51–63 (1973)
39. J. Hays, Reductions of ideals in Prüfer domains. Proc. Am. Math. Soc. **52**, 81–84 (1975)
40. J. Hedstrom, E. Houston, Pseudo-valuation domains. Pac. J. Math. **75**, 137–147 (1978)
41. J. Hedstrom, E. Houston, Pseudo-valuation domains II. Houston J. Math. **4**, 199–207 (1978)
42. W. Heinzer, J. Ohm, An essential ring which is not a v-multiplication ring. Can. J. Math. **25**, 856–861 (1973)
43. E. Houston, Prime t-ideals in $R[X]$. Lect. Notes Pure Appl. Math. Dekker **153**, 163–170 (1994)
44. E. Houston, S. Kabbaj, M. Mimouni, $\star$-reductions of ideals and Prüfer v-multiplication domains. J. Commut. Algebra **9**, 491–505 (2017)
45. E. Houston, M. Mimouni, M.H. Park, Noetherian domains which admit only finitely many star operations. J. Algebra **366**, 78–93 (2012)
46. E. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial. Michigan Math. J. **35**, 291–300 (1988)
47. E. Houston, M. Zafrullah, Integral domains in which any two v-coprime elements are comaximal. J. Algebra **423**, 93–113 (2015)

48. C. Huneke, I. Swanson, *Integral Closure of Ideals, Rings, and Modules* (Cambridge University Press, Cambridge, 2012)
49. H.C. Hutchins, *Examples of Commutative Rings* (Polygonal Publishing House, Washington, 1981)
50. S. Kabbaj, A. Kadri, t-Reductions and t-integral closure of ideals. Rocky Mt. J. Math. **47**, 1875–1899 (2017)
51. S. Kabbaj, A. Kadri, M. Mimouni, On t-reductions of ideals in pullbacks. N. Y. J. Math. **22**, 875–889 (2016)
52. S. Kabbaj, A. Kadri, M. Mimouni, t-Reductions and t-integral closure of ideals in Noetherian domains. N. Y. J. Math. **24**, 404–418 (2018)
53. S. Kabbaj, M. Mimouni, t-Class semigroups of integral domains. J. Reine Angew. Math. **612**, 213–229 (2007)
54. S. Kabbaj, M. Mimouni, Constituent groups of Clifford semigroups arising from t-closure. J. Algebra **321**, 1443–1452 (2009)
55. B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra **123**, 151–170 (1989)
56. I. Kaplansky, *Commutative Rings* (University of Chicago Press, Chicago, 1974)
57. A. Mimouni, Integral domains in which each ideal is a w-ideal. Commun. Algebra **33**, 1345–1355 (2005)
58. J.L. Mott, M. Zafrullah, On Prüfer v-multiplication domains. Manuscr. Math. **35**, 1–26 (1981)
59. D.G. Northcott, D. Rees, Reductions of ideals in local rings. Proc. Camb. Philos. Soc. **50**, 145–158 (1954)
60. M.H. Park, Prime t-ideals in power series rings over a discrete valuation domain. J. Algebra **324**, 3401–3407 (2010)
61. M.H. Park, Krull dimension of power series rings over a globalized pseudo-valuation domain. J. Pure Appl. Algebra **214**, 862–866 (2010)
62. F. Wang, w-dimension of domains. Commun. Algebra **27**, 2267–2276 (1999)
63. F. Wang, w-dimension of domains II. Commun. Algebra **29**, 2419–2428 (2001)
64. M. Zafrullah, On finite-conductor domains. Manuscr. Math. **24**, 191–204 (1978)
65. M. Zafrullah, The $D + XD_S[X]$ construction from GCD-domains. J. Pure Appl. Algebra **50**, 93–107 (1988)
66. M. Zafrullah, Well behaved prime t-ideals. J. Pure Appl. Algebra **65**, 199–207 (1990)
67. M. Zafrullah, t-invertibility and Bazzoni-like statements. J. Pure Appl. Algebra **214**, 654–657 (2010)

Local Types of Classical Rings

L. Klingler and W. Wm. McGovern

Abstract Motivated by recent results on commutative rings with zero divisors [2, 11], we investigate the difference between the three notions of locally classical, maximally classical, and classical rings. Motivated also by results in [12], we explore these notions when restricted to certain subsets of the prime spectrum of the ring. As an application, we examine the case of locally classical rings of continuous functions, the case of maximally classical and classical rings having already been considered [1, 14].

1 Introduction and Main Results

Throughout, we shall assume that R denotes a commutative ring with identity. We denote the classical ring of quotients (also known as the total quotient ring) of R by $q(R)$. When a ring equals its classical ring of quotients, the ring is said to be *classical*. Classical rings are characterized by the simple condition that all regular elements are units. For any ring R, $q(R)$ is a classical ring.

A ring-theoretic property is said to be *local* if an arbitrary ring R satisfies the property if and only if the localization R_P satisfies the property for every prime ideal P of R. For example, "reduced" is a well-known local property, where a ring is *reduced* if it has no nonzero nilpotent elements. As a second example, we note the useful fact that "regular" is also a local property, which we show in the following lemma.

L. Klingler (✉)
Department of Mathematical Sciences, Florida Atlantic University,
Boca Raton, FL 33431, USA
e-mail: klingler@fau.edu

W. Wm. McGovern
Wilkes Honors College, Florida Atlantic University, Jupiter, FL 33458, USA
e-mail: warren.mcgovern@fau.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_8

Lemma 1 *For an element $r \in R$, the following statements are equivalent.*

1. *r is regular.*
2. *$\frac{r}{1}$ is regular in R_P for every prime ideal $P \subset R$.*
3. *$\frac{r}{1}$ is regular in R_M for every maximal ideal $M \subset R$.*

Proof For (1) implies (2), if $\frac{r}{1} \cdot \frac{a}{s} = 0$ in R_P, then $ra \cdot t = 0$ for some $t \in R \setminus P$; if r is regular, then $at = 0$ in R forces $\frac{a}{s} = 0$ in R_P. (2) implies (3) is trivial. For (3) implies (1), if $ra = 0$ for some $a \in R$, then $\frac{r}{1} \cdot \frac{a}{1} = 0$ and $\frac{r}{1}$ regular in R_M for every maximal ideal $M \subset R$ implies that the annihilator $\mathrm{Ann}_R(a)$ of a in R is contained in no maximal ideal of R; that is, R annihilates a, so that $a = 0$.

The goal of this paper is to investigate the extent to which "classical" is a local property. The first motivation comes from [2, 11], in which it is shown that "Prüfer " is not a local property. (Recall that a ring R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible. Note that a classical ring is Prüfer, because it has no proper regular ideals.) The ring R is said to be *locally* (respectively, *maximally*) *Prüfer* if the localization R_P is a Prüfer ring for every prime (respectively, maximal) ideal $P \subset R$. Boynton showed in [2] that locally Prüfer implies Prüfer but not conversely. Klingler, Lucas, and Sharma showed in [11] that maximally Prüfer implies Prüfer but not conversely, and that maximally Prüfer does not imply locally Prüfer. These results are summarized by the diagram

$$\text{locally Prüfer} \Rightarrow \text{maximally Prüfer} \Rightarrow \text{Prüfer}$$

in which neither arrow is reversible. Moreover, examples show that these implications are not reversible even under the extra hypothesis that the ring is reduced.

In this paper, we establish the corresponding results for "classical." We shall say that a ring R is *locally* (respectively, *maximally*) *classical* if the localization R_P is a classical ring for every prime (respectively, maximal) ideal $P \subset R$. Clearly, locally classical implies maximally classical; our main theorem of this section establishes the second implication.

Theorem 1 *If R is maximally classical, then it is classical.*

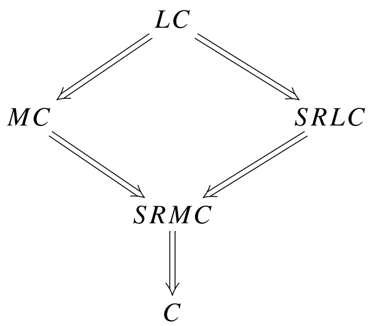
Proof It is easier to prove the contrapositive, so suppose that R is not classical, say $r \in R$ is regular but not a unit. Then there is a maximal ideal $M \subset R$ such that $r \in M$, so $\frac{r}{1}$ is not a unit in the localization R_M. By Lemma 1, $\frac{r}{1}$ is regular, so R_M is not classical, and hence R is not maximally classical.

Examples 2 and 3 below show that these two implications are not reversible, even under the additional hypothesis that the ring is reduced. That is, Example 2 gives a (reduced) ring R which is classical but not maximally classical, and Example 3 gives a (reduced) ring R which is maximally classical but not locally classical.

The second motivation for this paper comes from [12], in which the authors considered restricting the Prüfer property locally to only the regular or semiregular, prime or maximal ideals. (Recall that an ideal is called *semiregular* if it contains a finitely generated dense ideal, that is, a finitely generated ideal whose annihilator is zero.) Restricting the locally (or maximally) classical property to regular prime (or maximal) ideals would be pointless, however, since by Lemma 1, a regular nonunit in a prime ideal will remain a regular nonunit in the localization at that prime ideal. Thus, "regular locally classical" and "regular maximally classical" are equivalent to classical; that is, a ring is classical if and only if it has no regular prime ideals if and only if it has no regular maximal ideals. On the other hand, restricting the locally (or maximally) classical property to semiregular prime (or maximal) ideals does produce a weaker condition, as we shall show. Therefore, we shall say that a ring R is *semiregular locally* (respectively *semiregular maximally*) *classical* if the localization R_P is a classical ring for every semiregular prime (respectively semiregular maximal) ideal $P \subset R$.

Obviously locally classical implies semiregular locally classical, and semiregular locally classical and maximally classical each implies semiregular maximally classical. Moreover, a regular element generates a finitely generated ideal with zero annihilator, so the proof of Theorem 1 shows that, if the ring R is not classical, then it is not semiregular maximally classical. Therefore, semiregular maximally classical implies classical, and we obtain the following diagram of implications.

Diagram 2



where LC abbreviates locally classical, SRLC abbreviates semiregular locally classical, etc. Examples 5 and 6 below (together with the examples already mentioned) show that none of the arrows is reversible, and indeed, that there are no other implications than those implied by the diagram, even under the added hypothesis that the ring is reduced.

The third motivation for this paper comes from the theory of rings of (real-valued) continuous functions. For a topological space X, let $C(X)$ denote the set of continuous functions from the space X to the real field $\mathbb{R}$; $C(X)$ is a commutative ring with identity under pointwise addition and multiplication. One area of particular research interest has been determining conditions on the space X equivalent to some desired property of the ring $C(X)$. (See [8] as a good general reference for the theory of rings of continuous functions.) For example, Levy [14] gave necessary and sufficient conditions on X that $C(X)$ be classical, and Banerjee, Ghosh, and Henriksen [1] gave necessary and sufficient conditions on X that $C(X)$ be maximally classical. We devote Sect. 2 of this paper to reviewing the necessary terminology for these results from the theory of rings of continuous functions, and to constructing two examples to sharpen Theorem 1. We also characterize the topological spaces X for which the ring $C(X)$ is locally classical (Theorem 7).

Note that the ring $C(X)$ has *Property A*, that is, every semiregular ideal of $C(X)$ is regular. This follows from the fact that in $C(X)$, the annihilator of f, g equals the annihilator of $f^2 + g^2$. Thus, for $C(X)$, semiregular locally (respectively maximally) classical is equivalent to regular locally (respectively maximally) classical, both of which, as noted above, are equivalent to classical. Therefore, to establish the claim made about the completeness of the implications in Diagram 2, we need to look for examples beyond rings of continuous functions. This we do in Sect. 3, where we also collect some miscellaneous results on classical rings.

We finish the current section by developing a useful condition on the prime ideal P which, for a reduced ring R, is equivalent to the localization R_P being classical.

For prime ideal $P \subset R$, we denote by $O(P)$ the set of elements of R annihilated by an element of $R \setminus P$:

$$O(P) = \{a \in R : \text{ there is an } x \in R \setminus P \text{ such that } ax = 0\}$$

Note that $O(P) \subseteq P$; we easily obtain the following alternative description of R_P, since $O(P)$ is the kernel of the natural map from R to the localization R_P. (See Sect. 4 of [9] for details.)

Proposition 1 *Let P be a prime ideal of R, and set $\bar{R} = R/O(P)$ and $\bar{P} = P/O(P)$. Then $\bar{R}_{\bar{P}} \cong R_P$.*

If the ring R is reduced, the minimal prime ideals of R play a crucial role in determining whether or not R is locally or maximally classical. The following characterization of minimal primes in reduced rings will prove useful.

Lemma 2 ([10, Corollary 2.2]) *Let R be a reduced ring and suppose P is a prime ideal of R. Then P is a minimal prime ideal if and only if for each $a \in P$ there exists an element $x \in R \setminus P$ such that $ax = 0$.*

For a reduced ring R and prime ideal $P \subset R$, the following proposition gives an elegant description of the ideal $O(P)$.

Proposition 2 *If R is a reduced ring and P is a prime ideal of R, then*

$$O(P) = \cap\{Q \subseteq P : Q \text{ is a minimal prime}\},$$

the intersection of the minimal primes of R contained in P.

Proof Let $a \in O(P)$ and Q be a minimal prime contained in P. By definition, there is an element $x \in R \setminus P$ such that $ax = 0$. Since $Q \subseteq P$, it follows that $x \notin Q$, and hence $a \in Q$. This demonstrates one containment.

For the opposite containment, if a is in all of the minimal primes contained in P, then $\frac{a}{1}$ is in all of the minimal primes of the localization R_P, so in the nilradical of R_P. Since R is assumed to be reduced, R_P is also reduced (as noted above), and hence $\frac{a}{1} = 0$ in R_P. This implies that the annihilator of a is not contained in P, so $a \in O(P)$.

For a reduced ring R and prime ideal $P \subset R$, we can now determine conditions on P which guarantee that the localization R_P is classical.

Theorem 3 *Let R be a reduced ring and P be a prime ideal of R. The following statements are equivalent.*

1. *The localization R_P is classical.*
2. *For every $a \in P$, the annihilator of a in R is not contained in $O(P)$.*
3. *$P = \cup\{Q \subseteq P : Q$ is a minimal prime$\}$, the union of the minimal primes of R contained in P.*

Proof (1) implies (2). Let $a \in P$. If $a \in O(P)$, then by definition there is an element $x \in R \setminus P$ such that $ax = 0$, so the annihilator of a is not contained in P, and so not contained in $O(P)$. Assume instead that $a \in P \setminus O(P)$. By hypothesis, the localization R_P is classical, so $\frac{a}{1}$ must be a zero divisor, and hence there is a nonzero $\frac{b}{s} \in R_P$ such that $\frac{a}{1} \cdot \frac{b}{s} = 0$ in R_P. This means that there is some $t \notin P$ such that $abt = 0$. If $bt \in O(P)$, then $btx = 0$ for some $x \in R \setminus P$, but then $tx \in R \setminus P$ would imply $\frac{b}{s} = 0$ in R_P, contrary to assumption. Therefore, $abt = 0$ with $bt \notin O(P)$, as required.

(2) implies (3). Clearly, $\cup\{Q \subseteq P : Q$ is a minimal prime$\} \subseteq P$. Conversely, suppose $a \in P$, so that, by hypothesis, $ab = 0$ for some $b \notin O(P)$. By Proposition 2, there is a minimal prime $Q \subseteq P$ such that $b \notin Q$. Then $ab = 0$ and Q prime implies $a \in Q$, proving the opposite containment.

(3) implies (1). Let $\frac{a}{s} \in PR_P$, so that $a \in P$. By hypothesis, $a \in Q$ for some minimal prime $Q \subseteq P$. By Lemma 2, there is an element $x \notin Q$ such that $ax = 0$. By Proposition 2, $x \notin O(P)$, so $\frac{x}{1}$ is a nonzero element of R_P, whence $\frac{a}{s}$ is a zero divisor of R_P.

Quantifying overall prime or all maximal ideals of the reduced ring R, Theorem 3 yields criteria for R to be locally or maximally classical.

Corollary 1 *If R is reduced, then:*

1. *R is locally classical if and only if every prime ideal is a union of minimal prime ideals.*
2. *R is maximally classical if and only if every maximal ideal is a union of minimal prime ideals.*

2 Rings of Continuous Functions

We recall a useful classification of local Prüfer rings. First, recall that a ring R is called a *Bézout ring* if every finitely generated ideal is principally generated. In the weaker case that every finitely generated regular ideal is principally generated we say R is *quasi-Bézout*. Observe that a quasi-Bézout ring is a Prüfer ring. Of course, there are Prüfer domains which are not Bézout (and hence not quasi-Bézout). Theorem 2 of [15] states that for rings of continuous functions the notions of quasi-Bézout and Prüfer are equivalent. It is also the case that these conditions are equivalent for local rings. We let $Z(R)$ denote the set of zero divisors of the ring R.

Theorem 4 ([15, Proposition]) *Let R be a local ring. The following statements are equivalent.*

1. *R is a Prüfer ring.*
2. *R is a quasi-Bézout ring.*
3. *$Z(R)$ is an (prime) ideal of R and $R/Z(R)$ is a valuation domain.*

Remark 1 Recall the domain is a valuation domain if and only if its ideals are linearly ordered, i.e., the domain is a chained ring. When the ring has zero divisors the distinction between valuation rings and chained rings becomes slightly more delicate. For more information on this, we suggest the reader peruse [13].

Next, we give a brief account of the theory of rings of continuous functions. For a topological space X, we let $C(X)$ denote the ring of real-valued continuous functions on X. The subring of bounded continuous functions on X is denoted by $C^*(X)$. We shall assume that X is a Tychonoff space, that is, completely regular and Hausdorff.

Next, recall the following subsets of X that are useful in describing algebraic properties of $C(X)$. For $f \in C(X)$, we denote its *zeroset* by $Z(f) = \{x \in X : f(x) = 0\}$. The set-theoretic complement of $Z(f)$ in X is denoted by $\operatorname{coz}(f)$ and is called the *cozeroset* of f. A subset $V \subseteq X$ is called a zeroset (respectively, cozeroset) if there is some $f \in C(X)$ such that $V = Z(f)$ (respectively, $V = \operatorname{coz}(f)$). We shall use $\operatorname{cl}_X V$ and $\operatorname{int}_X V$ to denote the closure and interior of V in X, respectively. We shall also feel free to drop the subscripts on these operators when it is clear which space is being discussed.

For a Tychonoff space X, the Stone–Čech compactification of X, denoted βX, is the unique compact space (up to homeomorphism) containing X densely and C^*-embedded. Recall that a subspace Y of a space X is said to be *C^*-embedded* in X if every bounded continuous function on Y has a continuous extension to X. Our main reference for $C(X)$ is [8].

For $p \in \beta X$ we form two ideals of $C(X)$:

$$M^p = \{f \in C(X) : p \in \operatorname{cl}_{\beta X} Z(f)\}$$

and

$$\begin{aligned} O^p &= \{f \in C(X) : \mathrm{cl}_{\beta X} Z(f) \text{ is a neighborhood of } p\} \\ &= \{f \in C(X) : \text{ there is a } \beta X\text{-neighborhood } V \text{ of } p \text{ such that } V \cap X \subseteq Z(f)\}. \end{aligned}$$

It is known that each M^p is a maximal ideal of $C(X)$ and that every maximal ideal of $C(X)$ is of the form M^p for some (unique) $p \in \beta X$; this is known as the Gelfand–Kolmogoroff Theorem. The ring $C(X)$ is a *pm*-ring, that is, every prime ideal is contained in a unique maximal ideal. Furthermore, for any prime $P \in \mathrm{Spec}(C(X))$ the set of prime ideals containing P forms a chain, i.e., $\mathrm{Spec}(C(X))$ is a root system. Since O^p is a radical ideal, it follows that O^p is the intersection of the minimal prime ideals contained in M^p. (When $p \in X$ we instead write M_p and O_p and notice that $M_p = \{f \in C(X) : f(p) = 0\}$ and $O_p = \{f \in C(X) : p \in \mathrm{int}_X Z(f)\}$.)

Chronologically, Gilman and Henriksen [6] characterized when $C(X)$ is a von Neumann regular ring, calling such a space X a *P-space*. It is known that X is a P-space if and only if the topology of open sets is closed under countable intersections if and only if every zeroset is open. Next, the authors classified in [7] when $C(X)$ is a Bézout ring, calling such a space an *F-space*. X is an F-space if and only if every cozeroset is C^*-embedded if and only if $C(X)$ is an arithmetical ring. Then, in [4], the authors classified when $C(X)$ is quasi-Bézout calling such a space X a *quasi F-space*. X is a quasi F-space if and only if every dense cozeroset is C^*-embedded. In [15] the authors proved that for $C(X)$ (and in a more general situation), $C(X)$ is a quasi-Bézout ring if and only if $C(X)$ is a Prüfer ring. Formally:

Theorem 5 ([15, Theorem 2], [4, Theorem 5.1]) *For a space X the following statements are equivalent.*

1. *$C(X)$ is a Prüfer ring.*
2. *Every dense cozeroset of X is C^*-embedded, that is, X is a quasi F-space.*
3. *βX is a quasi F-space.*
4. *$C^*(X)$ is a Prüfer ring.*

Levy [14] called a space X an *almost P-space* if the interior of every nonempty zeroset is nonempty. It follows that every P-space is an almost P-space. It also follows that in an almost P-space, there are no nontrivial dense cozerosets of X. Since $f \in C(X)$ is regular precisely when $\mathrm{coz}(f)$ is dense we obtain:

Theorem 6 *The space X is an almost P-space if and only if $C(X)$ is a classical ring. In particular, an almost P-space is a quasi F-space.*

Example 1 The one-point compactification of an uncountable discrete space, αD, is an example of an almost P-space. If X is locally compact and real compact but not compact, then $\beta X \setminus X$ is an almost P-space [5, Lemma 3.1].

The question of when $C(X)_M$ is classical for every maximal ideal was addressed in [1], though not in these terms. The authors call X a *UMP-space* (pronounced U - M - P - space) if every maximal ideal of $C(X)$ is the union of minimal prime ideals. An application of Theorem 3 yields the following result.

Corollary 2 ([1, Theorem 2.2]) *The space X is a UMP-space if and only if $C(X)$ is maximally classical.*

Example 2 It is pointed out in [1] that a UMP-space is an almost P-space. This also follows from the fact that a maximally classical ring is classical. In Observation 1.6 of [1], it is pointed out that the space $\beta\mathbb{N} \setminus \mathbb{N}$ is an example of an almost P-space which is not a UMP-space. Therefore, $C(\beta\mathbb{N} \setminus \mathbb{N})$ is classical but not maximally classical.

Example 3 Let D be an uncountable discrete space and let αD denote its one-point compactification. The ring $C(\alpha D)$ is a maximally classical ring, equivalently, X is a UMP-space (see [1, Example 1.8]). However, $C(\alpha D)$ is not a locally classical ring. In $C(\alpha D)$ every non-maximal prime ideal (necessarily lying beneath M_α) has a unique minimal prime ideal beneath it [3, Proposition 3]. Therefore, if P is any non-maximal prime, then $O(P)$ is a prime ideal. Thus, $C(X)_P$ is a domain. To be classical, we would need $C(X)_P$ to be a field, which occurs precisely when $P = O(P)$, i.e., P is a minimal prime ideal. Since there are primes which are both non-maximal and non-minimal prime ideals of $C(\alpha D)$ it follows that $C(X)$ is not locally classical.

A recap is in order. $C(X)$ is classical if and only if X is an almost P-space, and $C(X)$ is maximally classical if and only if X is a UMP-space. We now come to the main theorem in this section, characterizing when $C(X)$ is locally classical.

Theorem 7 *Let X be a space. The following statements are equivalent.*

1. *The ring $C(X)$ is a locally classical ring.*
2. *The ring $C(X)$ is a von Neumann regular ring.*
3. *X is a P-space.*

Before we supply a proof, we recall a needed definition. Recall that an ideal $I \subset C(X)$ is called a *z-ideal* if $f \in I$ and $Z(f) = Z(g)$ implies that $g \in I$. For example, each maximal ideal and each minimal prime ideal of $C(X)$ is a z-ideal.

Lemma 3 *If $\{I_\sigma\}_{\sigma\in\tau}$ is a collection of z-ideals such that the union $I = \cup_{\sigma\in\tau} I_\sigma$ forms an ideal, then I is a z-ideal.*

Proof Let $f \in I$ and $Z(f) = Z(g)$. By hypothesis there is a $\sigma \in \tau$ such that $f \in I_\sigma$. Since I_σ is a z-ideal it follows that $g \in I_\sigma$, whence $g \in I$.

Remark 2 Lemma 3 can be generalized to the join of z-ideals being a z-ideal using [8, Lemma 14.8]. However, we do not need the full version here, and so we provided a proof for completeness sake.

We can now prove Theorem 7.

Proof That (2) and (3) are equivalent has already been pointed out. If $C(X)$ is von Neumann regular, then it is locally a field, hence locally classical, so (2) implies (1).

For (1) implies (2), suppose that $C(X)$ is locally classical; we show that each point in X is a P-point. Let $p \in X$. If $O_p \subsetneq M_p$, then it is well known that there exists a prime ideal P beneath M_p which is not a z-ideal [8, Sect. 14.13]. Since $C(X)_P$ is classical, by Proposition 3 we know that P is the union of the minimal prime ideals beneath it. But minimal prime ideals are z-ideals, so it follows from Lemma 3 that P is a z-ideal, contradiction. Therefore, $O_p = M_p$, and hence p is a P-point. Consequently, X is a P-space.

For reduced rings, being von Neumann regular is equivalent to being zero-dimensional, so Theorem 7 means that a ring $C(X)$ is locally classical if and only if it is zero-dimensional. The next example shows that, for rings in general, this is not the case, even for reduced rings.

Example 4 Let k be a field and $D = k[[x]]$ a power series ring in one indeterminate over k; so D is a discrete valuation domain with unique maximal ideal $M = xk[[x]]$. Set $B = \oplus_{n\in\mathbb{N}} M$, the direct sum of countably many copies of M. One can define a ring structure on the cartesian product $R = D \times B$ using coordinatewise addition, and multiplication defined by $(a, b)(c, d) = (ac, ad + bc + bd)$; in [12], R is called a *ring of form* $A + ZB[[Z]]$. By [12, Theorem 3.7 (8)], R is a local ring of Krull dimension 1, and by [12, Theorem 3.7 (1) and (2)], R is a classical ring. One easily checks that R is reduced, so R is locally a field at all minimal primes and hence at all non-maximal primes. It follows that R is a locally classical, reduced ring with Krull dimension equal to 1.

3 Additional Examples and Further Results

The ring $R = C(\beta\mathbb{N} \setminus \mathbb{N})$ of Example 2 is classical but not maximally classical. As noted in the introduction, since R has property A, it is also semiregular locally (and hence semiregular maximally) classical. Thus, in the notation of Diagram 2, no member of {C, SRMC, SRLC} implies a member of {MC, LC}. Similarly, the ring $R = C(\alpha D)$ of Example 3 is maximally classical but not locally classical; that is, MC does not imply LC. Moreover, both examples are reduced, so none of these implications holds even under the additional assumption that the ring is reduced.

To complete the claim following Diagram 2 that no implications hold other than those implied by the diagram, we give additional (reduced) examples showing that C implies neither SRMC nor SRLC (Example 5), and that neither MC nor SRMC implies SRLC (Example 6). It is then straightforward to verify that the only necessary implications are the (downward) directed paths in Diagram 2.

Example 5 The ring R of the form $A + B$ in [12, Example 3.5] is classical ($R = q(R)$, the total quotient ring of R), while the maximal ideal $N + B$ of R is semiregular, and $R_{N+B} = D$ is an integral domain but not a field. Therefore, R is classical but not semiregular maximally classical, and hence (in the notation of Diagram 2), C implies neither SRMC nor SRLC. Moreover, the ring R is reduced.

Example 6 The ring $Q(R) = R_{N+B} = \hat{D} + B$ (where $R = D + B$) is a ring of the form $A + B[[Z]]$ in [12, Example 3.11]. The ring $\hat{D} + B$ is classical (because $Q(R)$ is the total quotient ring of R), and $N + B$ is the unique maximal ideal of $\hat{D} + B$ by [12, Theorem 3.7 (8)] (because N is the unique maximal ideal of $\hat{D}$). Therefore, $\hat{D} + B$ is a local classical ring and hence maximally (and semiregular maximally) classical. The ideal $P + B$ (where $P = (X_2, X_3)\hat{D}$) is a semiregular prime ideal of both R and $\hat{D} + B$ (so that, incidentally, the maximal ideal $N + B$ of $\hat{D} + B$ is semiregular as well), and $(\hat{D} + B)_{P+B} = R_{P+B} = \hat{D}_P$ is an integral domain but not a field. Therefore, $\hat{D} + B$ is (semiregular) maximally classical but not semiregular locally classical, and hence (in the notation of Diagram 2), neither MC nor SRMC implies SRLC. Again, the ring $\hat{D} + B$ is reduced.

We conclude this section with a few miscellaneous results and examples concerning classical rings. We start by noting that "classical" *lifts* modulo the nilradical but does not *pass* modulo the nilradical.

Proposition 3 *If $R/\mathfrak{N}$ is classical, where $\mathfrak{N}$ is the nilradical of R, then R is also classical.*

Proof Let r be a regular element of R. We claim that $r + \mathfrak{N}$ is a regular element of $R/\mathfrak{N}$. If $(r + \mathfrak{N})(s + \mathfrak{N}) = 0 + \mathfrak{N}$ for some $s \in R$, then $(rs)^n = 0$ for some $n \in \mathbb{N}$, so regularity of r (and thus of r^n) implies that $s \in \mathfrak{N}$, proving the claim. Now by hypothesis, $r + \mathfrak{N}$ is a unit in $R/\mathfrak{N}$, from which it follows that r is a unit in R. Therefore, R is classical.

Example 7 The converse of Proposition 3 is not true. Let K be a field and set $R = K[x, y]_{(x,y)}/(xy, y^2)$. Observe that R is a local ring whose maximal ideal consists of zero divisors, whence R is classical. Moreover, the nilradical of R is $\mathfrak{N} = (y + (xy, y^2))$, and $R/\mathfrak{N}$ is isomorphic to $K[x]_{(x)}$, which is a domain but not a field. Consequently, $R/\mathfrak{N}$ is not classical.

Finally, we show that a "trivial extension" of a classical ring is classical. Recall that, for ring R and R-module M, we can form the *trivial extension* $R \propto M$ (also called the *idealization*) starting with the additive group $R \times M$, and defining multiplication by $(r, m)(s, n) = (rs, rn + sm)$ (see [10, Sect. 25] for details). In the following theorem, we collect together some important (known) facts about trivial extensions.

Theorem 8 *If R is a ring and M is an R-module, then:*

1. *$(r, m) \in R \propto M$ is a unit if and only if $r \in R$ is a unit.*
2. *$(r, m) \in R \propto M$ is regular if and only if $r \in R$ is regular and r acts faithfully on M (that is, $rn = 0$ implies $n = 0$ for $n \in M$).*
3. *J is a prime ideal of $R \propto M$ if and only if $J = P \propto M$ for some prime ideal P of R, in which case $(R \propto M)_J \cong R_P \propto M_P$.*
4. *If P is a prime ideal of R such that $P \propto M$ is a semiregular prime ideal of $R \propto M$, then P is a semiregular prime ideal of R.*

Proof (1) is [10, Theorem 25.1 (6)]; (2) is [10, Theorem 25.3]; and (3) is [10, Theorem 25.1 (3) and Corollary 25.5 (2)].

To prove (4), suppose that $P \subset R$ is a prime ideal and $(r_1, m_1), \ldots, (r_t, m_t) \in P \propto M$ generate a subideal J with zero annihilator. If $x \in R$ annihilates the subideal I of P generated by $r_1, \ldots, r_t$, then for each index i, $(0, xm_i)$ annihilates J, so that $xm_i = 0$. Thus, $(x, 0)$ annihilates J, which forces $x = 0$, and hence I has zero annihilator. Therefore, P is also semiregular.

As an immediate consequence, we get the following characterization of "classical" for trivial extensions.

Corollary 3 *For a ring R and R-module M, $R \propto M$ is classical if and only if every regular element of R that acts faithfully on M is a unit in R.*

Note that $(0) \propto M$ is a nilpotent ideal of $R \propto M$ and hence contained in the nilradical. Although $(0) \propto M$ need not equal the nilradical of $R \propto M$, the result of Proposition 3 still holds.

Corollary 4 *If R is a classical ring and M is an R-module, then $R \propto M$ is classical.*

Proof If $(r, m) \in R \propto M$ is regular, then $r \in R$ is regular by Theorem 8(2), so r is a unit by assumption, and hence (r, m) is a unit by Theorem 8(1).

In fact, we can extend this result to both "locally classical" and"maximally classical," and to their semiregular analogs.

Corollary 5 *Let R be a ring and M an R-module.*

1. *If R is (semiregular) locally classical, then so is $R \propto M$.*
2. *If R is (semiregular) maximally classical, then so is $R \propto M$.*

Proof By Theorem 8(3), the prime (respectively maximal) ideals of $R \propto M$ have the form $P \propto M$ as P ranges over the prime (respectively maximal) ideals of R, and $(R \propto M)_{P\propto M} \cong R_P \propto M_P$. Moreover, by Theorem 8(4), if $P \propto M$ is a semiregular prime ideal of $R \propto M$, then P is a semiregular prime ideal of R, so both statements (and their semiregular analogs) follow immediately from Corollary 4.

Note that the converse to Theorem 8(4) does not hold. For example, if $p \in \mathbb{Z}$ is prime, then $p\mathbb{Z} \propto \mathbb{Z}/p\mathbb{Z}$ is not a semiregular ideal of $\mathbb{Z} \propto \mathbb{Z}/p\mathbb{Z}$ (being annihilated by $(0, 1)$), even though $p\mathbb{Z}$ is a regular prime ideal of $\mathbb{Z}$. We conclude by adapting this example to show that the converses of the statements in Corollaries 4 and 5 do not hold either.

Example 8 If $p \in \mathbb{Z}$ is prime, then $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is a classical ring by Corollary 3, because only the elements of $\mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$ act faithfully on $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$. On the other hand, clearly $\mathbb{Z}_{(p)}$ is not classical, so that the converse of Corollary 4 fails. Since $\mathbb{Z}_{(p)}$ (and hence also $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$) is local, the converse of Corollary 5(2) also fails. In fact, $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is also locally classical, because, by Theorem 8(3), its only non-maximal prime ideal is $(0) \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$, and

$$(\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)})_{(0)\propto\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}} \cong (\mathbb{Z}_{(p)})_{(0)} \propto (\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)})_{(0)} \cong \mathbb{Q}.$$

Thus, Corollary 5(1) fails as well. Finally, since $\mathbb{Z}_{(p)} \propto \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is locally classical, it is semiregular locally and semiregular maximally classical, but $\mathbb{Z}_{(p)}$ is neither, so the converse of the semiregular variations of Corollary 5 fail too.

References

1. B. Banerjee, S. Ghosh, M. Henriksen, Unions of minimal prime ideals in rings of continuous functions on compact spaces. Algebr. Univers. **62**(2–3), 239–246 (2009)
2. J. Boynton, Prüfer conditions and the total quotient ring. Commun. Algebra **39**(5), 1624–1630 (2011)
3. G. Cherlin, M. Dickmann, Real closed rings I. Residue rings of rings of continuous functions. Fund. Math. **126**(2), 147–183 (1986)
4. F. Dashiell, A. Hager, M. Henriksen, Order-Cauchy completions of rings and vector lattices of continuous functions. Canad. J. Math. **32**(3), 657–685 (1980)
5. N. Fine, L. Gillman, Extensions of continuous functions in $\beta\mathbb{N}$. Bull. Am. Math. Soc. **66**, 376–381 (1960)
6. L. Gillman, M. Henriksen, Concerning rings of continuous functions. Trans. Am. Math. Soc. **77**, 340–362 (1954)
7. L. Gillman, M. Henriksen, Rings of continuous functions in which every finitely generated ideal is principal. Trans. Am. Math. Soc. **82**, 366–391 (1956)
8. L. Gillman, M. Jerison, *Rings of Continuous Functions*. Graduate Texts in Mathemetics, vol. 43 (Springer, Berlin, 1976)
9. R. Gilmer, *Multiplicative Ideal Theory*. Queen's Papers in Pure and Applied Mathematics, vol. 90 (Queen's University, Kingston, 1992)
10. J. Huckaba, *Commutative Rings with Zero Divisors*. Monographs and Textbooks in Pure and Applied Mathematics, vol. 117 (Marcel Dekker, New York, 1988)
11. L. Klingler, T. Lucas, M. Sharma, Maximally Prüfer rings. Commun. Algebra **43**, 120–129 (2015)
12. L. Klingler, T. Lucas, M. Sharma, Local types of Prüfer rings. J. Algebra Appl. To appear
13. M. Knebusch, D. Zhang, *Manis Valuations and Prüfer Extensions I. A New Chapter in Commutative Algebra*. Lecture Notes in Mathematics, vol. 1791 (Springer, Berlin, 2002)
14. R. Levy, Almost P-spaces. Canad. J. Math. **29**(2), 284–288 (1977)
15. J. Martinez, S. Woodward, Bézout and Prüfer f-rings. Commun. Algebra **20**(10), 2975–2989 (1992)

How Do Elements Really Factor in $\mathbb{Z}[\sqrt{-5}]$?

Scott T. Chapman, Felix Gotti and Marly Gotti

Dedicated to David F. Anderson on the occasion of his retirement.

Abstract Most undergraduate level abstract algebra texts use $\mathbb{Z}[\sqrt{-5}]$ as an example of an integral domain which is not a unique factorization domain (or UFD) by exhibiting two distinct irreducible factorizations of a nonzero element. But such a brief example, which requires merely an understanding of basic norms, only scratches the surface of how elements actually factor in this ring of algebraic integers. We offer here an interactive framework which shows that while $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, it does satisfy a slightly weaker factorization condition, known as half-factoriality. The arguments involved revolve around the Fundamental Theorem of Ideal Theory in algebraic number fields.

2010 AMS Mathematics Subject Classification Primary 11R11 · 13F15 · 20M13

S. T. Chapman (✉)
Sam Houston State University, Huntsville, TX 77341, USA
e-mail: scott.chapman@shsu.edu

F. Gotti
UC Berkeley, Berkeley, CA 94720, USA
e-mail: felixgotti@berkeley.edu

M. Gotti
University of Florida, Gainesville, FL 32611, USA
e-mail: marlycormar@ufl.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_9

1 Introduction

Consider the integral domain

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}.$$

Your undergraduate abstract algebra text probably used it as the base example of an integral domain that is not a unique factorization domain (or UFD). The Fundamental Theorem of Arithmetic fails in $\mathbb{Z}[\sqrt{-5}]$ as this domain contains elements with multiple factorizations into irreducibles; for example,

$$6 = 2 \cdot 3 = (1 - \sqrt{-5})(1 + \sqrt{-5}) \tag{1}$$

even though $2, 3, 1 - \sqrt{-5}$, and $1 + \sqrt{-5}$ are pairwise non-associate irreducible elements in $\mathbb{Z}[\sqrt{-5}]$. To argue this, the norm on $\mathbb{Z}[\sqrt{-5}]$, i.e.,

$$N(a + b\sqrt{-5}) = a^2 + 5b^2, \tag{2}$$

plays an important role, as it is a multiplicative function satisfying the following properties:

- $N(\alpha) = 0$ if and only if $\alpha = 0$;
- $N(\alpha\beta) = N(\alpha)N(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$;
- α is a unit if and only if $N(\alpha) = 1$ (i.e., ± 1 are the only units of $\mathbb{Z}[\sqrt{-5}]$);
- if $N(\alpha)$ is prime, then α is irreducible.

However, introductory abstract algebra books seldom dig deeper than what Equation (1) does. The goal of this paper is to use ideal theory to describe exactly how elements in $\mathbb{Z}[\sqrt{-5}]$ factor into products of irreducibles. In doing so, we will show that $\mathbb{Z}[\sqrt{-5}]$ satisfies a nice factorization property, which is known as *half-factoriality*. Thus, we say that $\mathbb{Z}[\sqrt{-5}]$ is a *half-factorial domain* (or HFD). Our journey will require nothing more than elementary algebra, but will give the reader a glimpse of how The Fundamental Theorem of Ideal Theory resolves the nonunique factorizations of $\mathbb{Z}[\sqrt{-5}]$. The notion that unique factorization in rings of integers could be recovered via ideals was important in the late 1800s in attempts to prove Fermat's Last Theorem (see [9, Chap. 11]).

Our presentation is somewhat interactive, as many steps that follow from standard techniques of basic algebra are left to the reader as exercises. The only background we expect from the reader is introductory courses in linear algebra and abstract algebra. Assuming such prerequisites, we have tried to present here a self-contained and friendly approach to the phenomenon of nonuniqueness of factorizations occurring in $\mathbb{Z}[\sqrt{-5}]$. More advanced and general arguments (which apply to any ring of integers) can be found in [8, 9].

2 Integral Bases and Discriminants

Although in this paper we are primarily concerned with the phenomenon of nonunique factorizations in the particular ring of integers $\mathbb{Z}[\sqrt{-5}]$, it is more enlightening from an algebraic perspective to introduce our needed concepts for arbitrary commutative rings with identity, rings of integers, or quadratic rings of integers, depending on the most appropriate context for each concept being introduced. In what follows, we shall proceed in this manner while trying, by all means, to keep the exposition as elementary as possible.

An element $\alpha \in \mathbb{C}$ is said to be *algebraic* provided that it is a root of a nonzero polynomial with rational coefficients, while α is said to be an *algebraic integer* provided that it is a root of a monic polynomial with integer coefficients. It is not hard to argue that every subfield of $\mathbb{C}$ contains $\mathbb{Q}$ and is a $\mathbb{Q}$-vector space.

Definition 2.1 A subfield K of $\mathbb{C}$ is called an *algebraic number field* provided that it has finite dimension as a vector space over $\mathbb{Q}$. The subset

$$\mathscr{O}_K := \{\alpha \in K \mid \alpha \text{ is an algebraic integer}\}$$

of K is called the *ring of integers* of K.

The ring of integers of any algebraic number field is, indeed, a ring. The reader is invited to verify this observation. If α is a complex number, then $\mathbb{Q}(\alpha)$ denotes the smallest subfield of $\mathbb{C}$ containing α. It is well known that a subfield K of $\mathbb{C}$ is an algebraic number field if and only if there exists an algebraic number $\alpha \in \mathbb{C}$ such that $K = \mathbb{Q}(\alpha)$ (see, for example, [6, Theorem 2.17]). Among all algebraic number fields, we are primarily interested in those that are two-dimensional vector spaces over $\mathbb{Q}$.

Definition 2.2 An algebraic number field that is a two-dimensional vector space over $\mathbb{Q}$ is called a *quadratic number field*. If K is a quadratic number field, then $\mathscr{O}_K$ is called a *quadratic ring of integers*.

For $\alpha \in \mathbb{C}$, let $\mathbb{Z}[\alpha]$ denote the set of all polynomial expressions in α having integer coefficients. Clearly, $\mathbb{Z}[\alpha]$ is a subring of $\mathbb{Q}(\alpha)$. It is also clear that, for $d \in \mathbb{Z}$, the field $\mathbb{Q}(\sqrt{d})$ has dimension at most two as a $\mathbb{Q}$-vector space and, therefore, it is an algebraic number field. Moreover, if $d \notin \{0, 1\}$ and d is squarefree (i.e., d is not divisible by the square of any prime), then it immediately follows that $\mathbb{Q}(\sqrt{d})$ is a two-dimensional vector space over $\mathbb{Q}$ and, as a result, a quadratic number field. As we are mainly interested in the case when $d = -5$, we propose the following exercise.

Exercise 2.3 Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer such that $d \equiv 2, 3 \pmod 4$. Prove that $\mathbb{Z}[\sqrt{d}]$ is the ring of integers of the quadratic number field $\mathbb{Q}(\sqrt{d})$.

Remark When $d \in \mathbb{Z} \setminus \{0, 1\}$ is a squarefree integer satisfying $d \equiv 1 \pmod 4$, it is not hard to argue that the ring of integers of $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$. However, we will not be concerned with this case as our case of interest is $d = -5$.

For d as specified in Exercise 2.3, the elements of $\mathbb{Z}[\sqrt{d}]$ can be written in the form $a + b\sqrt{d}$ for $a, b \in \mathbb{Z}$. The *norm* N on $\mathbb{Z}[\sqrt{d}]$ is defined by

$$N(a + b\sqrt{d}) = a^2 - db^2$$

(cf. Eq. (2)). The norm N on $\mathbb{Z}[\sqrt{d}]$ also satisfies the four properties listed in the introduction.

Let us now take a look at the structure of an algebraic number field K with linear algebra in mind. For $\alpha \in K$ consider the function $m_\alpha : K \to K$ defined via multiplication by α, i.e., $m_\alpha(x) = \alpha x$ for all $x \in K$. One can easily see that m_α is a linear transformation of $\mathbb{Q}$-vector spaces. Therefore, after fixing a basis for the $\mathbb{Q}$-vector space K, we can represent m_α by a matrix M. The *trace* of α, which is denoted by $\mathrm{Tr}(\alpha)$, is defined to be the trace of the matrix M. It is worth noting that $\mathrm{Tr}(\alpha)$ does not depend on the chosen basis for K. Also, notice that $\mathrm{Tr}(\alpha) \in \mathbb{Q}$. Furthermore, if $\alpha \in \mathscr{O}_K$, then $\mathrm{Tr}(\alpha) \in \mathbb{Z}$ (see [10, Lemma 4.1.1], or Exercise 2.5 for the case when $K = \mathbb{Q}(\sqrt{d})$).

Definition 2.4 Let K be an algebraic number field that has dimension n as a $\mathbb{Q}$-vector space. The *discriminant* of a subset $\{\omega_1, \ldots, \omega_n\}$ of K, which is denoted by $\Delta[\omega_1, \ldots, \omega_n]$, is $\det T$, where T is the $n \times n$ matrix $\big(\mathrm{Tr}(\omega_i\omega_j)\big)_{1 \le i, j \le n}$.

With K as introduced above, if $\{\omega_1, \ldots, \omega_n\}$ is a subset of $\mathscr{O}_K$, then it follows that $\Delta[\omega_1, \ldots, \omega_n] \in \mathbb{Z}$ (see Exercise 2.5 for the case when $K = \mathbb{Q}(\sqrt{d})$). In addition, the discriminant of any basis for the $\mathbb{Q}$-vector space K is nonzero; we will prove this for $K = \mathbb{Q}(\sqrt{d})$ in Proposition 2.10.

Exercise 2.5 Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer such that $d \equiv 2, 3 \pmod 4$.

1. If $\alpha = a_1 + a_2\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, then $\mathrm{Tr}(\alpha) = 2a_1$.
2. If, in addition, $\beta = b_1 + b_2\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, then

$$\Delta[\alpha, \beta] = \left(\det \begin{bmatrix} \alpha & \sigma(\alpha) \\ \beta & \sigma(\beta) \end{bmatrix}\right)^2 = 4d(a_1b_2 - a_2b_1)^2,$$

where $\sigma(x + y\sqrt{d}) = x - y\sqrt{d}$ for all $x, y \in \mathbb{Z}$.

Example 2.6 Let $d \notin \{0, 1\}$ be a squarefree integer such that $d \equiv 2, 3 \pmod 4$. It follows from Exercise 2.5 that the subset $\{1, \sqrt{d}\}$ of the ring of integers $\mathbb{Z}[\sqrt{d}]$ satisfies that $\Delta[1, \sqrt{d}] = 4d$.

We proceed to introduce the concept of integral basis.

Definition 2.7 Let K be an algebraic number field of dimension n as a vector space over $\mathbb{Q}$. The elements $\omega_1, \ldots, \omega_n \in \mathscr{O}_K$ form an *integral basis* for $\mathscr{O}_K$ if for each $\beta \in \mathscr{O}_K$ there are unique $z_1, \ldots, z_n \in \mathbb{Z}$ satisfying $\beta = z_1\omega_1 + \cdots + z_n\omega_n$.

Example 2.8 Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer such that $d \equiv 2, 3 \pmod 4$. Clearly, every element in $\mathbb{Z}[\sqrt{d}]$ is an integral linear combination of 1 and $\sqrt{d}$. Suppose, on the other hand, that $a_1 + a_2\sqrt{d} = b_1 + b_2\sqrt{d}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{Z}$. Note that $a_2 = b_2$; otherwise $\sqrt{d} = \frac{a_1 - b_1}{b_2 - a_2}$ would be a rational number. As a result, $a_1 = b_1$. Thus, we have verified that every element of $\mathbb{Z}[\sqrt{d}]$ can be uniquely written as an integral linear combination of 1 and $\sqrt{d}$. Hence, the set $\{1, \sqrt{d}\}$ is an integral basis for $\mathbb{Z}[\sqrt{d}]$.

In general, the ring of integers of any algebraic number field has an integral basis (see [6, Theorem 3.27]). On the other hand, although integral bases are not unique, any two integral bases for the same ring of integers have the same discriminant. We shall prove this for $\mathbb{Z}[\sqrt{d}]$ in Theorem 2.13.

Notation: If S is a subset of the complex numbers, then we let $S^\bullet$ denote $S \setminus \{0\}$.

Lemma 2.9 *Let K be an algebraic number field of dimension n as a $\mathbb{Q}$-vector space. An integral basis for $\mathcal{O}_K$ is a basis for K as a vector space over $\mathbb{Q}$.*

Proof Suppose that $\{\omega_1, \ldots, \omega_n\}$ is an integral basis for $\mathcal{O}_K$, and take rational coefficients $q_1, \ldots, q_n$ such that

$$q_1\omega_1 + \cdots + q_n\omega_n = 0.$$

Multiplying the above equality by the common denominator of the nonzero q_i's and using the fact that $\{\omega_1, \ldots, \omega_n\}$ is an integral basis for the ring of integers $\mathcal{O}_K$, we obtain that $q_1 = \cdots = q_n = 0$. Hence, $\{\omega_1, \ldots, \omega_n\}$ is a linearly independent set of the $\mathbb{Q}$-vector space K. As K has dimension n over $\mathbb{Q}$, the set $\{\omega_1, \ldots, \omega_n\}$ is a basis for the vector space K over $\mathbb{Q}$. $\square$

Proposition 2.10 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. If $\{\alpha_1, \alpha_2\}$ is a vector space basis for $\mathbb{Q}(\sqrt{d})$ contained in $\mathbb{Z}[\sqrt{d}]$, then $\Delta[\alpha_1, \alpha_2] \in \mathbb{Z}^\bullet$.*

Proof From the fact that $\{\alpha_1, \alpha_2\} \subseteq \mathbb{Z}[\sqrt{d}]$, it follows that $\Delta[\alpha_1, \alpha_2] \in \mathbb{Z}$. So suppose, by way of contradiction, that $\Delta[\alpha_1, \alpha_2] = 0$. Taking $\{\omega_1, \omega_2\}$ to be an integral basis for $\mathbb{Z}[\sqrt{d}]$, one has that

$$\begin{aligned} \alpha_1 &= z_{1,1}\omega_1 + z_{1,2}\omega_2 \\ \alpha_2 &= z_{2,1}\omega_1 + z_{2,2}\omega_2, \end{aligned}$$

for some $z_{i,j} \in \mathbb{Z}$. Using Exercise 2.5, we obtain

$$\Delta[\alpha_1, \alpha_2] = \left(\det \begin{bmatrix} \alpha_1 & \sigma(\alpha_1) \\ \alpha_2 & \sigma(\alpha_2) \end{bmatrix}\right)^2 = \left(\det \left(\begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} \begin{bmatrix} \omega_1 & \sigma(\omega_1) \\ \omega_2 & \sigma(\omega_2) \end{bmatrix}\right)\right)^2$$
$$= \left(\det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}\right)^2 \left(\det \begin{bmatrix} \omega_1 & \sigma(\omega_1) \\ \omega_2 & \sigma(\omega_2) \end{bmatrix}\right)^2 = \left(\det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}\right)^2 \Delta[\omega_1, \omega_2], \tag{3}$$

where $\sigma(x + y\sqrt{d}) = x - y\sqrt{d}$ for all $x, y \in \mathbb{Z}$. If $\omega_1 = 1$ and $\omega_2 = \sqrt{d}$, then

$$\det \begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \end{bmatrix} = \det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} = 0,$$

and so there are elements $q_1, q_2 \in \mathbb{Q}$ not both zero with

$$\begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} 0 &= \omega_1(q_1 z_{1,1} + q_2 z_{2,1}) + \omega_2(q_1 z_{1,2} + q_2 z_{2,2}) \\ &= q_1(z_{1,1}\omega_1 + z_{1,2}\omega_2) + q_2(z_{2,1}\omega_1 + z_{2,2}\omega_2) \\ &= q_1\alpha_1 + q_2\alpha_2, \end{aligned}$$

which is a contradiction because the set $\{\alpha_1, \alpha_2\}$ is linearly independent in the vector space $\mathbb{Q}(\sqrt{d})$. Thus, $\Delta[\alpha_1, \alpha_2] \neq 0$, as desired. □

Exercise 2.11 Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Show that $\Delta[\alpha_1, \alpha_2] \neq 0$ whenever $\{\alpha_1, \alpha_2\}$ is a basis for the $\mathbb{Q}$-vector space $\mathbb{Q}(\sqrt{d})$.

Using Lemma 2.9 and Exercise 2.5, we obtain the following important result.

Corollary 2.12 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. The discriminant of each integral basis for $\mathbb{Z}[\sqrt{d}]$ is in $\mathbb{Z}^\bullet$.*

Notation: Let $\mathbb{N}$ denote the set of positive integers, and set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Theorem 2.13 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Any two integral bases for $\mathbb{Z}[\sqrt{d}]$ have the same discriminant.*

Proof Let $\{\alpha_1, \alpha_2\}$ and $\{\omega_1, \omega_2\}$ be integral bases for $\mathbb{Z}[\sqrt{d}]$, and let $z_{i,j}$ be defined as in the proof of Proposition 2.10. Since $\Delta[\alpha_1, \alpha_2]$ and $\Delta[\omega_1, \omega_2]$ are both integers, Eq. (3) in the proof of Proposition 2.10, along with the fact that $\left(\det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}\right)^2 \in \mathbb{N}$, implies that $\Delta[\omega_1, \omega_2]$ divides $\Delta[\alpha_1, \alpha_2]$. Using a similar argument, we can show that $\Delta[\alpha_1, \alpha_2]$ divides $\Delta[\omega_1, \omega_2]$. As both discriminants have the same sign, $\Delta[\alpha_1, \alpha_2] = \Delta[\omega_1, \omega_2]$. □

Using Example 2.6 and Example 2.8, we obtain the following corollary.

Corollary 2.14 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Every integral basis for $\mathbb{Z}[\sqrt{d}]$ has discriminant $4d$.*

3 General Properties of Ideals

Let R be a commutative ring with identity. In most beginning algebra classes, the units, irreducibles, and associate elements in R are standard concepts of interest. Recall that the units of R are precisely the invertible elements, while nonunit elements $x, y \in R$ are associates if $a = ub$ for a unit u of R. A nonunit $x \in R^{\bullet} := R \setminus \{0\}$ is irreducible if whenever $x = uv$ in R, then either u or v is a unit.

To truly understand factorizations in $\mathbb{Z}[\sqrt{-5}]$, we will need to know first how ideals of $\mathbb{Z}[\sqrt{-5}]$ are generated. Recall that a subset I of a commutative ring R with identity is called an ideal of R provided that I is a subring with the property that $rI \subseteq I$ for all $r \in R$. It follows immediately that if $x_1, \ldots, x_k \in R$, then the set

$$I = \langle x_1, \ldots, x_k \rangle = \{r_1 x_1 + \cdots + r_k x_k \mid \text{ each } r_i \in R\}$$

is an ideal of R, that is, the ideal generated by $x_1, \ldots, x_k$. Recall that I is said to be principal if $I = \langle x \rangle$ for some $x \in R$, and R is said to be a principal ideal domain (or a PID) if each ideal of R is principal. The zero ideal $\langle 0 \rangle$ and the entire ring $R = \langle 1 \rangle$ are principal ideals. May it be that all the ideals of $\mathbb{Z}[\sqrt{-5}]$ are principal? It turns out that the answer is "no" as we shall see in the next example.

Example 3.1 The ring of integers $\mathbb{Z}[\sqrt{-5}]$ is not a PID. We argue that the ideal

$$I = \langle 2, 1 + \sqrt{-5} \rangle$$

is not principal. If $I = \langle \alpha \rangle$, then α divides both 2 and $1 + \sqrt{-5}$. The reader will verify in Exercise 3.2 below that both of these elements are irreducible and non-associates. Hence, $\alpha = \pm 1$ and $I = \langle \pm 1 \rangle = \mathbb{Z}[\sqrt{-5}]$. Now we show that $3 \notin I$. Suppose there exist $a, b, c, d \in \mathbb{Z}$ so that

$$(a + b\sqrt{-5})2 + (c + d\sqrt{-5})(1 + \sqrt{-5}) = 3.$$

Expanding the previous equality, we obtain

$$\begin{aligned} 2a + c - 5d &= 3 \\ 2b + c + d &= 0. \end{aligned} \tag{4}$$

After subtracting, we are left with $2(a - b) - 6d = 3$, which implies that 2 divides 3 in $\mathbb{Z}$, a contradiction.

Exercise 3.2 Show that the elements 2 and $1 + \sqrt{-5}$ are irreducible and non-associates in $\mathbb{Z}[\sqrt{-5}]$. (Hint: use the norm function.)

Let us recall that a proper ideal I of a commutative ring R with identity is said to be prime if whenever $xy \in I$ for $x, y \in R$, then either $x \in I$ or $y \in I$. In addition, we know that an element $p \in R \setminus \{0\}$ is said to be prime provided that the principal ideal $\langle p\rangle$ is prime. It follows immediately that, in any integral domain, every prime element is irreducible.

Exercise 3.3 Let P be an ideal of a commutative ring R with identity. Show that P is prime if and only if the containment $IJ \subseteq P$ for ideals I and J of R implies that either $I \subseteq P$ or $J \subseteq P$.

Example 3.4 We argue that the ideal $I = \langle 2\rangle$ is not prime in $\mathbb{Z}[\sqrt{-5}]$ and will in fact use Eq. (1). Since $(1-\sqrt{-5})(1+\sqrt{-5}) = 2\cdot 3$, it follows that

$$(1-\sqrt{-5})(1+\sqrt{-5}) \in \langle 2\rangle.$$

Now if $1-\sqrt{-5} \in \langle 2\rangle$, then there is an element $\alpha \in \mathbb{Z}[\sqrt{-5}]$ with $1-\sqrt{-5} = 2\alpha$. But then $\alpha = \frac{1}{2} - \frac{\sqrt{-5}}{2} \notin \mathbb{Z}[\sqrt{-5}]$, a contradiction. A similar argument works with $1+\sqrt{-5}$. Hence, $\langle 2\rangle$ is not a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.

We remind the reader that a proper ideal I of a commutative ring R with identity is called maximal if for each ideal J the containment $I \subseteq J \subseteq R$ implies that either $J = I$ or $J = R$. What we ask the reader to verify in the next exercise is a well-known result from basic abstract algebra.

Exercise 3.5 Let I be a proper ideal of a commutative ring R with identity, and let $R/I = \{r + I \mid r \in R\}$ be the quotient ring of R by I.

1. Show that I is prime if and only if R/I is an integral domain.
2. Show that I is maximal if and only if R/I is a field. Deduce that maximal ideals are prime.

Example 3.6 We expand our analysis of $I = \langle 2, 1+\sqrt{-5}\rangle$ in Example 3.1 by showing that I is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$. To do this, we first argue that an element $\alpha = z_1 + z_2\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is contained in I if and only if z_1 and z_2 have the same parity. If $\alpha \in I$, then there are integers a, b, c, and d so that

$$z_1 + z_2\sqrt{-5} = (a + b\sqrt{-5})2 + (c + d\sqrt{-5})(1+\sqrt{-5}).$$

Adjusting the equations from (4) yields

$$\begin{aligned} 2a + c - 5d &= z_1 \\ 2b + c + d &= z_2. \end{aligned} \tag{5}$$

Notice that if $c \equiv d \pmod 2$, then both z_1 and z_2 are even, while $c \not\equiv d \pmod 2$ implies that both z_1 and z_2 are odd. Hence, z_1 and z_2 must have the same parity. Conversely, suppose that z_1 and z_2 have the same parity. As, clearly, every element of the form $2k_1 + 2k_2\sqrt{-5} = 2(k_1 + k_2\sqrt{-5})$ is in I, let us assume that z_1 and z_2 are both odd. The equations in (5) form a linear system that obviously has solutions over $\mathbb{Q}$ for any choice of z_1 and z_2 in $\mathbb{Z}$. By solving this system, we find that a and b are dependent variables and

$$a = \frac{z_1 - c + 5d}{2} \quad \text{and} \quad b = \frac{z_2 - c - d}{2}.$$

Letting c be any even integer and d any odd integer now yields a solution with both a and b integers. Thus, $z_1 + z_2\sqrt{-5} \in I$.

Now consider $\mathbb{Z}[\sqrt{-5}]/I$. As I is not principal (Example 3.1), $1 \notin I$. Therefore $1 + I \neq 0 + I$. If $c_1 + c_2\sqrt{-5} \notin I$, then c_1 and c_2 have opposite parity. If c_1 is odd and c_2 even, then $((c_1 - 1) + c_2\sqrt{-5}) + I = 0 + I$ implies that $(c_1 + c_2\sqrt{-5}) + I = 1 + I$. If c_1 is even and c_2 odd, then $((c_1 - 1) + c_2\sqrt{-5}) + I = 0 + I$ again implies that $(c_1 + c_2\sqrt{-5}) + I = 1 + I$. Hence, $\mathbb{Z}[\sqrt{-5}]/I \cong \{0 + I, 1 + I\} \cong \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a field, I is a maximal ideal and thus prime (by Exercise 3.5).

Exercise 3.7 Show that $\langle 3, 1 - 2\sqrt{-5}\rangle$ and $\langle 3, 1 + 2\sqrt{-5}\rangle$ are prime ideals in the ring of integers $\mathbb{Z}[\sqrt{-5}]$.

Let R be a commutative ring with identity. If every ideal of R is finitely generated, then R is called a *Noetherian ring*. In addition, R satisfies the *ascending chain condition on ideals* (ACC) if every increasing (under inclusion) sequence of ideals of R eventually stabilizes.

Exercise 3.8 Let R be a commutative ring with identity. Show that R is Noetherian if and only if it satisfies the ACC.

We shall see in Theorem 4.3 that the rings of integers $\mathbb{Z}[\sqrt{d}]$ are Noetherian and, therefore, satisfy the ACC.

4 Ideals in $\mathbb{Z}[\sqrt{-5}]$

In this section we explore the algebraic structure of all ideals of $\mathbb{Z}[\sqrt{-5}]$ under ideal multiplication, encapsulating the basic properties of multiplication of ideals. Let us begin by generalizing the notion of an integral basis, which also plays an important role in ideal theory.

Definition 4.1 Let K be an algebraic number field of dimension n as a vector space over $\mathbb{Q}$, and let I be a proper ideal of the ring of integers $\mathcal{O}_K$. The elements $\omega_1, \ldots, \omega_n \in I$ form an *integral basis* for I provided that for each $\beta \in I$ there exist unique $z_1, \ldots, z_n \in \mathbb{Z}$ satisfying that $\beta = z_1\omega_1 + \cdots + z_n\omega_n$.

With notation as in the above definition, notice that if $\{\omega_1, \ldots, \omega_n\}$ is an integral basis for I, then $I = \langle \omega_1, \ldots, \omega_n\rangle$. Care is needed here as the converse is not necessarily true. For instance, $\{3\}$ is not an integral basis for the ideal $I = \langle 3\rangle$ of $\mathbb{Z}[\sqrt{-5}]$ (note that $3\sqrt{-5} \in I$).

Exercise 4.2 Argue that $\{3, 3\sqrt{-5}\}$ is an integral basis for the ideal $I = \langle 3\rangle$ of the ring of integers $\mathbb{Z}[\sqrt{-5}]$.

We now show that every proper ideal of $\mathbb{Z}[\sqrt{-5}]$ has an integral basis.

Theorem 4.3 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Every nonzero proper ideal of $\mathbb{Z}[\sqrt{d}]$ has an integral basis. Hence, every ideal of $\mathbb{Z}[\sqrt{d}]$ is finitely generated.*

Proof Let I be a nonzero proper ideal of $\mathbb{Z}[\sqrt{d}]$. To find an integral basis for I consider the collection $\mathscr{B}$ of all subsets of I which form a vector space basis for $\mathbb{Q}(\sqrt{d})$. Note that if $\{\omega_1, \omega_2\}$ is an integral basis for $\mathbb{Z}[\sqrt{d}]$ and $\alpha \in I^\bullet$, then the subset $\{\alpha\omega_1, \alpha\omega_2\}$ of I is also a linearly independent subset inside the vector space $\mathbb{Q}(\sqrt{d})$. As a result, the collection $\mathscr{B}$ is nonempty. As $I \subseteq \mathbb{Z}[\sqrt{d}]$, Proposition 2.10 ensures that $\Delta[\delta_1, \delta_2] \in \mathbb{Z}^\bullet$ for every member $\{\delta_1, \delta_2\}$ of $\mathscr{B}$. Then we can take a pair $\{\delta_1, \delta_2\}$ in $\mathscr{B}$ and assume that the absolute value of its discriminant, i.e., $|\Delta[\delta_1, \delta_2]|$, is as small as possible. We argue now that $\{\delta_1, \delta_2\}$ is an integral basis for I.

Assume, by way of contradiction, that $\{\delta_1, \delta_2\}$ is not an integral basis for I. Since $\{\delta_1, \delta_2\}$ is a basis for $\mathbb{Q}(\sqrt{d})$ as a vector space over $\mathbb{Q}$, there must exist $\beta \in I$ and $q_1, q_2 \in \mathbb{Q}$ such that $\beta = q_1\delta_1 + q_2\delta_2$, where not both q_1 and q_2 are in $\mathbb{Z}$. Without loss of generality, we can assume that $q_1 \in \mathbb{Q} \setminus \mathbb{Z}$. Write $q_1 = z + r$, where $z \in \mathbb{Z}$ and $0 < r < 1$. Let

$$\begin{aligned} \delta_1^* &= \beta - z\delta_1 = (q_1 - z)\delta_1 + q_2\delta_2 \\ \delta_2^* &= \delta_2. \end{aligned}$$

It is easy to verify that $\{\delta_1^*, \delta_2^*\}$ is linearly independent and thus is another vector space basis for $\mathbb{Q}(\sqrt{d})$ which consists of elements of I, that is, $\{\delta_1^*, \delta_2^*\}$ is a member of $\mathscr{B}$. Proceeding as we did in the proof of Proposition 2.10, we find that

$$\Delta[\delta_1^*, \delta_2^*] = r^2\Delta[\delta_1, \delta_2];$$

this is because $\left(\det \begin{bmatrix} q_1 - z & q_2 \\ 0 & 1 \end{bmatrix}\right)^2 = r^2$. It immediately follows from $0 < r < 1$ that $|\Delta[\delta_1^*, \delta_2^*]| < |\Delta[\delta_1, \delta_2]|$, contradicting the minimality of $|\Delta[\delta_1, \delta_2]|$. Hence, $\{\delta_1, \delta_2\}$ is an integral basis for I, which completes the proof. □

Theorem 4.3 yields the next important corollary.

Corollary 4.4 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. If I is a proper ideal of the ring of integers $\mathbb{Z}[\sqrt{d}]$, then there exist elements $\alpha_1, \alpha_2 \in I$ such that $I = \langle \alpha_1, \alpha_2 \rangle$. Thus, $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring.*

Remark One can actually say much more. For d as in Corollary 4.4, the following stronger statement is true: if I is a nonzero proper ideal of $\mathbb{Z}[\sqrt{d}]$ and $\alpha_1 \in I^\bullet$, then there exists $\alpha_2 \in I$ satisfying that $I = \langle \alpha_1, \alpha_2 \rangle$. This condition is known as the $1\frac{1}{2}$-*generator property*. The interested reader can find a proof of this result in [9, Theorem 9.3].

Definition 4.5 A pair $(M, *)$, where M is a set and $*$ is a binary operation on M, is called a *monoid* if $*$ is associative and there exists $e \in M$ satisfying that $e * x = x * e = x$ for all $x \in M$. The element e is called the *identity element*. The monoid M is called *commutative* if the operation $*$ is commutative.

Let R be a commutative ring with identity. Recall that we have a natural multiplication on the collection consisting of all ideals of R, that is, for any two ideals I and J of R, the product

$$IJ = \Big\{ \sum_{i=1}^{k} a_i b_i \;\Big|\; k \in \mathbb{N},\ a_1, \dots, a_k \in I, \text{ and } b_1, \dots, b_k \in J \Big\} \tag{6}$$

is again an ideal. It is not hard to check that ideal multiplication is both associative and commutative, and satisfies that $RI = I$ for each ideal I of R. This amounts to arguing the following exercise.

Exercise 4.6 Let R be a commutative ring with identity. Show that the set of all ideals of R is a commutative monoid under ideal multiplication.

Example 4.7 To give the reader a notion of how ideal multiplication works, we show that

$$\langle 2, 1 + \sqrt{-5} \rangle^2 = \langle 2 \rangle.$$

It follows by (6) that ideal multiplication can be achieved by merely multiplying generators. For instance,

$$\begin{aligned} \langle 2, 1 + \sqrt{-5} \rangle^2 &= \langle 2, 1 + \sqrt{-5} \rangle \langle 2, 1 + \sqrt{-5} \rangle \\ &= \langle 4, 2(1 + \sqrt{-5}), 2(1 + \sqrt{-5}), -2(2 - \sqrt{-5}) \rangle. \end{aligned}$$

Since 2 divides each of the generators of $\langle 2, 1 + \sqrt{-5} \rangle^2$ in $\mathbb{Z}[\sqrt{-5}]$, we clearly have that $\langle 2, 1 + \sqrt{-5} \rangle^2 \subseteq \langle 2 \rangle$. To verify the reverse inclusion, let us first observe that $2\sqrt{-5} = 4 - 2(2 - \sqrt{-5}) \in \langle 2, 1 + \sqrt{-5} \rangle^2$. As $2\sqrt{-5} \in \langle 2, 1 + \sqrt{-5} \rangle^2$, one immediately sees that $2 = 2(1 + \sqrt{-5}) - 2\sqrt{-5} \in \langle 2, 1 + \sqrt{-5} \rangle^2$. Hence, the inclusion $\langle 2 \rangle \subseteq \langle 2, 1 + \sqrt{-5} \rangle^2$ holds, and equality follows.

Exercise 4.8 Verify that the next equalities hold:

$$\begin{aligned} \langle 3 \rangle &= \langle 3, 1 - 2\sqrt{-5} \rangle \langle 3, 1 + 2\sqrt{-5} \rangle, \\ \langle 1 - \sqrt{-5} \rangle &= \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 + 2\sqrt{-5} \rangle, \\ \langle 1 + \sqrt{-5} \rangle &= \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 - 2\sqrt{-5} \rangle. \end{aligned}$$

Example 4.8 is no accident. Indeed, every nonprincipal ideal of $\mathbb{Z}[\sqrt{-5}]$ has a multiple which is a principal ideal as it is established in the following theorem.

Theorem 4.9 *Let I be an ideal of $\mathbb{Z}[\sqrt{-5}]$. Then there exists a nonzero ideal J of $\mathbb{Z}[\sqrt{-5}]$ such that IJ is principal.*

Proof If I is a principal ideal, then the result follows by letting $J = \langle 1 \rangle$. So suppose $I = \langle \alpha, \beta \rangle$ is not a principal ideal of $\mathbb{Z}[\sqrt{-5}]$, where $\alpha = a + b\sqrt{-5}$ and $\beta = c + d\sqrt{-5}$. Notice that it is enough to verify the existence of such an ideal J when $\gcd(a, b, c, d) = 1$, and we make this assumption. It is easy to check that $\alpha\overline{\beta} + \overline{\alpha}\beta = 2ac + 10bd \in \mathbb{Z}$. Hence, $\alpha\overline{\alpha}$, $\alpha\overline{\beta} + \overline{\alpha}\beta$, and $\beta\overline{\beta}$ are all integers. Let

$$\begin{aligned} f &= \gcd(\alpha\overline{\alpha}, \alpha\overline{\beta} + \overline{\alpha}\beta, \beta\bar{\beta}) \\ &= \gcd(a^2 + 5b^2, 2ac + 10bd, c^2 + 5d^2). \end{aligned}$$

Take $J = \langle \overline{\alpha}, \overline{\beta} \rangle$. We claim that $IJ = \langle f \rangle$. Since $f = \gcd(\alpha\overline{\alpha}, \alpha\overline{\beta} + \overline{\alpha}\beta, \beta\overline{\beta})$, there are integers z_1, z_2, and z_3 so that

$$f = z_1\alpha\overline{\alpha} + z_2\beta\overline{\beta} + z_3(\alpha\overline{\beta} + \overline{\alpha}\beta).$$

Because $IJ = \langle \alpha\overline{\alpha}, \alpha\overline{\beta}, \beta\overline{\alpha}, \beta\overline{\beta} \rangle$, we have that f is a linear combination of the generating elements. Thus, $f \in IJ$ and, therefore, $\langle f \rangle \subseteq IJ$.

To prove the reverse containment, we first show that f divides $bc - ad$. Suppose, by way of contradiction, that this is not the case. Notice that $25 \nmid f$; otherwise $25 \mid a^2 + 5b^2$ and $25 \mid c^2 + 5d^2$ would imply that $5 \mid \gcd(a, b, c, d)$. On the other hand, $4 \mid f$ would imply $4 \mid a^2 + 5b^2$ and $4 \mid c^2 + 5d^2$, forcing a, b, c, and d to be even, which is not possible as $\gcd(a, b, c, d) = 1$. Hence, $4 \nmid f$ and $25 \nmid f$. Because

$$\begin{aligned} 2c(a^2 + 5b^2) - a(2ac + 10bd) &= 10b(bc - ad) \\ 2a(c^2 + 5d^2) - c(2ac + 10bd) &= 10d(ad - bc), \end{aligned}$$

f must divide both $10b(bc - ad)$ and $10d(bc - ad)$. As, by assumption, $f \nmid bc - ad$, there must be a prime p and a natural n such that $p^n \mid f$ but $p^n \nmid bc - ad$. If $p = 2$, then $4 \nmid f$ forces $n = 1$. In this case, both $a^2 + 5b^2$ and $c^2 + 5d^2$ would be even, and so $2 \mid a - b$ and $2 \mid c - d$, which implies that $2 \mid bc - ad$, a contradiction. Thus, $p \neq 2$. On the other hand, if $p = 5$, then again $n = 1$. In this case, $5 \mid a^2 + 5b^2$ and $5 \mid c^2 + 5d^2$ and so 5 would divide both a and c, contradicting that $5 \nmid bc - ad$. Then, we can assume that $p \notin \{2, 5\}$. As $p^n \mid 10b(bc - ad)$ but $p^n \nmid bc - ad$, we have that $p \mid 10b$. Similarly, $p \mid 10d$. Since $p \notin \{2, 5\}$, it follows that $p \mid b$ and $p \mid d$. Now the fact that p divides both $a^2 + 5b^2$ and $c^2 + 5d^2$ yields that $p \mid a$ and $p \mid c$, contradicting that $\gcd(a, b, c, d) = 1$. Hence, $f \mid bc - ad$.

Let us verify now that $f \mid ac + 5bd$. If f is odd, then $f \mid ac + 5bd$. Assume, therefore, that $f = 2f_1$, where $f_1 \in \mathbb{Z}$. As $4 \nmid f$, the integer f_1 is odd. Now, $f \mid a^2 + 5b^2$ implies that a and b have the same parity. Similarly, one sees that c and d have the same parity. As a consequence, $ac + 5bd$ is even. Since f_1 is odd, it must divide $(ac + 5bd)/2$, which means that f divides $ac + 5bd$, as desired.

Because f divides both $\alpha\overline{\alpha}$ and $\beta\overline{\beta}$ in $\mathbb{Z}$, proving that $IJ \subseteq \langle f \rangle$ amounts to verifying that f divides both $\alpha\overline{\beta}$ and $\overline{\alpha}\beta$ in $\mathbb{Z}[\sqrt{-5}]$. Since f divides both $ac + 5bd$ and $bc - ad$ in $\mathbb{Z}$, one has that

$$x = \frac{ac + 5bd}{f} \in \mathbb{Z} \quad \text{and} \quad y = \frac{bc - ad}{f} \in \mathbb{Z}.$$

Therefore

$$\alpha\overline{\beta} = ac + 5bd + (bc - ad)\sqrt{-5} = (x + y\sqrt{-5})f \in \langle f \rangle.$$

Also, $\overline{\alpha}\beta = \overline{\alpha\overline{\beta}} = (x - y\sqrt{-5})f \in \langle f \rangle$. Hence, the reverse inclusion $IJ \subseteq \langle f \rangle$ also holds, which completes the proof. □

A commutative monoid $(M, *)$ is said to be *cancellative* if for all $a, b, c \in M$, the equality $a * b = a * c$ implies that $b = c$. By Exercise 4.6, the set

$$\mathscr{I} := \{I \mid I \text{ is an ideal of } \mathbb{Z}[\sqrt{-5}]\}$$

is a commutative monoid. As the next corollary states, the set $\mathscr{I}^\bullet := \mathscr{I} \setminus \{\langle 0 \rangle\}$ is indeed a commutative cancellative monoid.

Corollary 4.10 *The set $\mathscr{I}^\bullet$ under ideal multiplication is a commutative cancellative monoid.*

Proof Because $\mathscr{I}$ is a commutative monoid under ideal multiplication, it immediately follows that $\mathscr{I}^\bullet$ is also a commutative monoid. To prove that $\mathscr{I}^\bullet$ is cancellative, take $I, J, K \in \mathscr{I}^\bullet$ such that $IJ = IK$. By Theorem 4.9, there exists an ideal I' of $\mathbb{Z}[\sqrt{-5}]$ and $x \in \mathbb{Z}[\sqrt{-5}]^\bullet$ with $I'I = \langle x \rangle$. Then

$$\langle x \rangle J = I'IJ = I'IK = \langle x \rangle K.$$

As $x \neq 0$ and the product in $\mathbb{Z}[\sqrt{-5}]^\bullet$ is cancellative, $J = K$. □

5 The Fundamental Theorem of Ideal Theory

We devote this section to prove a version of the Fundamental Theorem of Ideal Theory for the ring of integers $\mathbb{Z}[\sqrt{-5}]$. To do this, we need to develop a few tools. In particular, we introduce the concept of a fractional ideal of $\mathbb{Z}[\sqrt{-5}]$ and show that the set of such fractional ideals is an abelian group.

Let us begin by exploring the relationship between the concepts of prime and maximal ideals. We recall that every proper ideal of a commutative ring R with identity is contained in a maximal ideal, which implies, in particular, that maximal ideals always exist.

Exercise 5.1 Show that every maximal ideal of a commutative ring with identity is prime.

Prime ideals, however, are not necessarily maximal. The following example sheds some light upon this observation.

Example 5.2 Let $\mathbb{Z}[X]$ denote the ring of polynomials with integer coefficients. Clearly, $\mathbb{Z}[X]$ is an integral domain. It is not hard to verify that the ideal $\langle X \rangle$ of $\mathbb{Z}[X]$ is prime. Because $2 \notin \langle X \rangle$, one obtains that $\langle X \rangle \subsetneq \langle 2, X \rangle$. It is left to the reader to argue that $\langle 2, X \rangle$ is a proper ideal of $\mathbb{Z}[X]$. Since $\langle X \rangle \subsetneq \langle 2, X \rangle \subsetneq \mathbb{Z}[X]$, it follows that $\langle X \rangle$ is not a maximal ideal of $\mathbb{Z}[X]$. (An alternate argument can easily be given using Exercise 3.5.)

In the ring of integers $\mathscr{O}_K$ of any algebraic number field K, every nonzero prime ideal is maximal (see, for instance, [6, Proposition 5.21]). Let us establish this result here for our case of interest.

Proposition 5.3 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Then every nonzero prime ideal of $\mathbb{Z}[\sqrt{d}]$ is maximal.*

Proof Let P be a nonzero prime ideal in $\mathbb{Z}[\sqrt{d}]$, and let $\{\omega_1, \omega_2\}$ be an integral basis for $\mathbb{Z}[\sqrt{d}]$. Fix $\beta \in P^\bullet$. Note that $n := N(\beta) = \beta\bar{\beta} \in P \cap \mathbb{N}$. Consider the finite subset

$$S = \big\{n_1\omega_1 + n_2\omega_2 + P \mid n_1, n_2 \in \{0, 1, \dots, n-1\}\big\}$$

of $\mathbb{Z}[\sqrt{d}]/P$. Take $x \in \mathbb{Z}[\sqrt{d}]$. As $\{\omega_1, \omega_2\}$ is an integral basis, there exist $z_1, z_2 \in \mathbb{Z}$ such that $x = z_1\omega_1 + z_2\omega_2$ and, therefore, $x + P = n_1\omega_1 + n_2\omega_2 + P \in S$, where $n_i \in \{0, \dots, n-1\}$ and $n_i \equiv z_i \pmod n$. Hence, $\mathbb{Z}[\sqrt{d}]/P = S$, which implies that $\mathbb{Z}[\sqrt{d}]/P$ is finite. It follows by Exercise 3.5(1) that $\mathbb{Z}[\sqrt{d}]/P$ is an integral domain. As a result, $\mathbb{Z}[\sqrt{d}]/P$ is a field (see Exercise 5.4 below). Thus, Exercise 3.5(2) guarantees that P is a maximal ideal. □

Exercise 5.4 Let R be a finite integral domain. Show that R is a field.

Although the concepts of (nonzero) prime and maximal ideals coincide in $\mathbb{Z}[\sqrt{d}]$, we will use both terms depending on the ideal property we are willing to apply.

Lemma 5.5 *If I is a nonzero ideal of a Noetherian ring R, then there exist nonzero prime ideals $P_1, \dots, P_n$ of R such that $P_1 \cdots P_n \subseteq I$.*

Proof Assume, by way of contradiction, that the statement of the lemma does not hold. Because R is a Noetherian ring and, therefore, satisfies the ACC, there exists an ideal I of R that is maximal among all the ideals failing to satisfy the statement of the lemma. Clearly, I cannot be prime. By Exercise 3.3, there exist ideals J and K of R such that $JK \subseteq I$ but neither $J \subseteq I$ nor $K \subseteq I$. Now notice that the ideals $J' = I + J$ and $K' = I + K$ both strictly contain I. The maximality of I implies that both J' and K' contain products of nonzero prime ideals. Now the fact that $J'K' \subseteq I$ would also imply that I contains a product of nonzero prime ideals, a contradiction. □

Recall that if R is an integral domain contained in a field F, then the field of fractions of R is the smallest subfield of F containing R. If K is an algebraic number field, then it is not hard to argue that the field of fractions of $\mathcal{O}_K$ is precisely K.

Definition 5.6 Let R be an integral domain with field of fractions F. A *fractional ideal* of R is a subset of F of the form $\alpha^{-1}I$, where $\alpha \in R^\bullet$ and I is an ideal of R.

With notation as in the previous definition, it is clear that every ideal of R is a fractional ideal. However, fractional ideals are not necessarily ideals. The product of fractional ideals is defined similarly to the product of standard ideals. Therefore it is easily seen that the product of two fractional ideals is again a fractional ideal. Indeed, for elements α and β of $R^\bullet$ and for ideals I and J of R, we only need to observe that $(\alpha^{-1}I)(\beta^{-1}J) = (\alpha\beta)^{-1}IJ$.

Notation: Let $\mathscr{F}$ denote the set of all fractional ideals of $\mathbb{Z}[\sqrt{-5}]$, and set $\mathscr{F}^\bullet := \mathscr{F} \setminus \{\langle 0 \rangle\}$.

Definition 5.7 Let R be an integral domain with field of fractions F. For a fractional ideal I of R, the set

$$I^{-1} := \{\alpha \in F \mid \alpha I \subseteq R\}$$

is called the *inverse* of I.

Exercise 5.8 Show that the inverse of a fractional ideal is again a fractional ideal.

Lemma 5.9 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. If I is a proper ideal of the ring of integers $\mathbb{Z}[\sqrt{d}]$, then $\mathbb{Z}[\sqrt{d}]$ is strictly contained in the fractional ideal I^{-1}.*

Proof Since I is a proper ideal of $\mathbb{Z}[\sqrt{d}]$, then there exists a maximal ideal M of $\mathbb{Z}[\sqrt{d}]$ containing I. Fix $\alpha \in M^\bullet$. By the definition of the inverse of an ideal, $\mathbb{Z}[\sqrt{d}] \subseteq M^{-1}$. Since $\mathbb{Z}[\sqrt{d}]$ is a Noetherian ring, Lemma 5.5 ensures the existence of $m \in \mathbb{N}$ and prime ideals $P_1, \dots, P_m$ in $\mathbb{Z}[\sqrt{d}]$ such that $P_1 \cdots P_m \subseteq \langle \alpha \rangle \subseteq M$. Assume that m is the minimum natural number satisfying this property. Since M is a prime ideal (Exercise 5.1), by Exercise 3.3 there exists $P \in \{P_1, \dots, P_m\}$ such that $P \subseteq M$. There is no loss of generality in assuming that $P = P_1$. Now, by Proposition 5.3, the ideal P_1 is maximal, which implies that $P_1 = M$. By the minimality of m, there exists $\alpha' \in P_2 \cdots P_m \setminus \langle \alpha \rangle$. Therefore, we find that $\alpha^{-1}\alpha' \notin \mathbb{Z}[\sqrt{d}]$ and $\alpha' M = \alpha' P_1 \subseteq P_1 \cdots P_m \subseteq \langle \alpha \rangle$, that is $\alpha^{-1}\alpha' M \subseteq \langle 1 \rangle = \mathbb{Z}[\sqrt{d}]$. As a result, $\alpha^{-1}\alpha' \in M^{-1} \setminus \mathbb{Z}[\sqrt{d}]$. Hence, we find that $\mathbb{Z}[\sqrt{d}] \subsetneq M^{-1} \subseteq I^{-1}$, and the proof follows. $\square$

We focus throughout the remainder of our work on the ring of integers $\mathbb{Z}[\sqrt{-5}]$. This, via Theorem 4.9, will substantially simplify our remaining arguments.

Lemma 5.10 *If $I \in \mathscr{F}^\bullet$ and $\alpha \in \mathbb{Q}(\sqrt{-5})$, then $\alpha I \subseteq I$ implies $\alpha \in \mathbb{Z}[\sqrt{-5}]$.*

Proof Let I and α be as in the statement of the lemma. By Theorem 4.9, there exists a nonzero ideal J of $\mathbb{Z}[\sqrt{-5}]$ such that $IJ = \langle \beta \rangle$ for some $\beta \in \mathbb{Z}[\sqrt{-5}]$. Then $\alpha\langle\beta\rangle = \alpha IJ \subseteq IJ = \langle\beta\rangle$, which means that $\alpha\beta = \sigma\beta$ for some $\sigma \in \mathbb{Z}[\sqrt{-5}]$. As $\beta \neq 0$, it follows that $\alpha = \sigma \in \mathbb{Z}[\sqrt{-5}]$. □

Theorem 5.11 *The set $\mathscr{F}^\bullet$ is an abelian group under multiplication of fractional ideals.*

Proof Clearly, the multiplication of fractional ideals is associative. In addition, it immediately follows that the fractional ideal $\mathbb{Z}[\sqrt{-5}] = 1^{-1}\langle 1 \rangle$ is the identity. The most involved part of the proof consists in arguing that each fractional ideal is invertible.

Let $M \in \mathscr{I}^\bullet$ be a maximal ideal of $\mathbb{Z}[\sqrt{-5}]$. By definition of M^{-1}, we have that $MM^{-1} \subseteq \mathbb{Z}[\sqrt{-5}]$, which implies that $MM^{-1} \in \mathscr{I}^\bullet$. As $M = M\mathbb{Z}[\sqrt{-5}] \subseteq MM^{-1}$ and M is maximal, $MM^{-1} = M$ or $MM^{-1} = \mathbb{Z}[\sqrt{-5}]$. As M is proper, Lemma 5.9 ensures that M^{-1} strictly contains $\mathbb{Z}[\sqrt{-5}]$, which implies, by Lemma 5.10, that $MM^{-1} \neq M$. So $MM^{-1} = \mathbb{Z}[\sqrt{-5}]$. As a result, each maximal ideal of $\mathbb{Z}[\sqrt{-5}]$ is invertible.

Now suppose, by way of contradiction, that not every ideal in $\mathscr{I}^\bullet$ is invertible. Among all the nonzero non-invertible ideals take one, say J, maximal under inclusion (this is possible because $\mathbb{Z}[\sqrt{-5}]$ satisfies the ACC). Because $\mathbb{Z}[\sqrt{-5}]$ is an invertible fractional ideal, $J \subsetneq \mathbb{Z}[\sqrt{-5}]$. Let M be a maximal ideal containing J. By Lemma 5.9, one has that $\mathbb{Z}[\sqrt{-5}] \subsetneq M^{-1} \subseteq J^{-1}$. This, along with Lemma 5.10, yields $J \subsetneq JM^{-1} \subseteq JJ^{-1} \subseteq \mathbb{Z}[\sqrt{-5}]$. Thus, JM^{-1} is an ideal of $\mathbb{Z}[\sqrt{-5}]$ strictly containing J. The maximality of J now implies that $JM^{-1}(JM^{-1})^{-1} = \mathbb{Z}[\sqrt{-5}]$ and, therefore, $M^{-1}(JM^{-1})^{-1} \subseteq J^{-1}$. Then

$$\mathbb{Z}[\sqrt{-5}] = JM^{-1}(JM^{-1})^{-1} \subseteq JJ^{-1} \subseteq \mathbb{Z}[\sqrt{-5}],$$

which forces $JJ^{-1} = \mathbb{Z}[\sqrt{-5}]$, a contradiction.

Finally, take $F \in \mathscr{F}^\bullet$. Then there exist an ideal $I \in \mathscr{I}^\bullet$ and $\alpha \in \mathbb{Z}[\sqrt{-5}]^\bullet$ such that $F = \alpha^{-1}I$. So one obtains that

$$(\alpha I^{-1})F = (\alpha I^{-1})(\alpha^{-1}I) = I^{-1}I = \mathbb{Z}[\sqrt{-5}].$$

As a consequence, the fractional ideal αI^{-1} is the inverse of F in $\mathscr{F}^\bullet$. Because each nonzero fractional ideal of $\mathbb{Z}[\sqrt{-5}]$ is invertible, $\mathscr{F}^\bullet$ is a group. Since the multiplication of fractional ideals is commutative, $\mathscr{F}^\bullet$ is abelian. □

Corollary 5.12 *If $I \in \mathscr{I}^\bullet$ and $\alpha \in I^\bullet$, then $IJ = \langle\alpha\rangle$ for some $J \in \mathscr{I}^\bullet$.*

Proof Let I and α be as in the statement of the corollary. As $\alpha^{-1}I$ is a nonzero fractional ideal, there exists a nonzero fractional ideal J such that $\alpha^{-1}IJ = \mathbb{Z}[\sqrt{-5}]$, that is $IJ = \langle\alpha\rangle$. Since $\beta I \subseteq JI = \langle\alpha\rangle \subseteq I$ for all $\beta \in J$, Lemma 5.10 guarantees that $J \subseteq \mathbb{Z}[\sqrt{-5}]$. Hence, J is a nonzero ideal of $\mathbb{Z}[\sqrt{-5}]$. □

Theorem 5.13 *[The Fundamental Theorem of Ideal Theory] Let I be a nonzero proper ideal of $\mathbb{Z}[\sqrt{-5}]$. There exists a unique (up to order) list of prime ideals $P_1, \ldots, P_k$ of $\mathbb{Z}[\sqrt{-5}]$ such that $I = P_1 \cdots P_k$.*

Proof Suppose, by way of contradiction, that not every ideal in $\mathscr{I}^\bullet$ can be written as the product of prime ideals. From the set of ideals of $\mathbb{Z}[\sqrt{-5}]$ which are not the product of primes ideals, take one, say I, maximal under inclusion. Clearly, I is not prime. Therefore I is contained in a maximal ideal P_1, and such containment must be strict by Exercise 5.1. By Lemma 5.9, one has that $\mathbb{Z}[\sqrt{-5}] \subsetneq P_1^{-1}$ and so $I \subseteq IP_1^{-1}$. Now Lemma 5.10 ensures that the latter inclusion is strict. The maximality of I now implies that $IP_1^{-1} = P_2 \cdots P_k$ for some prime ideals $P_2, \ldots, P_k$. This, along with Theorem 5.11, ensures that $I = P_1 \cdots P_k$, a contradiction.

To argue uniqueness, let us assume, by contradiction, that there exists an ideal having two distinct prime factorizations. Let m be the minimum natural number such that there exists $I \in \mathscr{I}$ with two distinct factorizations into prime ideals, one of them containing m factors. Suppose that

$$I = P_1 \cdots P_m = Q_1 \cdots Q_n. \tag{7}$$

Because $Q_1 \cdots Q_n \subseteq P_m$, there exists $Q \in \{Q_1, \ldots, Q_n\}$ such that $Q \subseteq P_m$ (Exercise 3.3). By Proposition 5.3, both Q and P_m are maximal ideals, which implies that $P_m = Q$. As $IQ^{-1} \subseteq II^{-1} \subseteq \mathbb{Z}[\sqrt{-5}]$, it follows that $IQ^{-1} \in \mathscr{I}$. Multiplying the equality (7) by the fractional ideal Q^{-1}, we obtain that IQ^{-1} is an ideal of $\mathbb{Z}[\sqrt{-5}]$ with two distinct factorizations into prime ideals such that one of them, namely $P_1 \cdots P_{m-1}$, contains less than m factors. As this contradicts the minimality of m, uniqueness follows. □

An element a of a commutative monoid M is said to be an *atom* if for all $x, y \in M$ such that $a = xy$, either x is a unit or y is a unit (i.e., has an inverse). A commutative cancellative monoid is called *atomic* if every nonzero nonunit element can be factored into atoms.

Corollary 5.14 *The monoid $\mathscr{I}^\bullet$ is atomic.*

6 The Class Group

To understand the phenomenon of nonunique factorization in $\mathbb{Z}[\sqrt{-5}]$, we first need to understand certain classes of ideals of $\mathbb{Z}[\sqrt{-5}]$. Let

$$\mathscr{P} := \{I \in \mathscr{I} \mid I \text{ is a principal ideal of } \mathbb{Z}[\sqrt{-5}]\}.$$

Two ideals $I, J \in \mathscr{I}$ are *equivalent* if $\langle\alpha\rangle I = \langle\beta\rangle J$ for some $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]^\bullet$. In this case, we write $I \sim J$. It is clear that $\sim$ defines an equivalence relation on $\mathbb{Z}[\sqrt{-5}]$. The equivalence classes of $\sim$ are called *ideal classes*. Let $I\mathscr{P}$ denote the ideal class of I, and we also let $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ denote the set of all nonzero ideal classes. Now define a binary operation $*$ on $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ by

$$I\mathscr{P} * J\mathscr{P} = (IJ)\mathscr{P}.$$

It turns out that $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ is, indeed, a group under the $*$ operation.

Theorem 6.1 *The set of ideal classes $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ is an abelian group under $*$.*

Proof Because the product of ideals is associative and commutative, so is $*$. Also, it follows immediately that $\langle 1\rangle\mathscr{P} * I\mathscr{P} = (\langle 1\rangle I)\mathscr{P} = I\mathscr{P}$ for each $I \in \mathscr{I}^\bullet$, which means that $\mathscr{P} = \langle 1\rangle\mathscr{P}$ is the identity element of $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$. In addition, as any two nonzero principal ideals are in the same ideal class, Theorem 4.9 ensures that, for any $I\mathscr{P} \in \mathscr{C}(\mathbb{Z}[\sqrt{-5}])$, there exists $J \in \mathscr{I}^\bullet$ such that $I\mathscr{P} * J\mathscr{P} = IJ \in \mathscr{P} = \langle 1\rangle\mathscr{P}$. So $J\mathscr{P}$ is the inverse of $I\mathscr{P}$ in $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$. Hence, $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ is an abelian group. □

Definition 6.2 The group $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ is called the *class group* of $\mathbb{Z}[\sqrt{-5}]$, and the order of $\mathscr{C}(\mathbb{Z}[\sqrt{-5}])$ is called the *class number* of $\mathbb{Z}[\sqrt{-5}]$.

Recall that if $\theta\colon R \to S$ is a ring homomorphism, then $\ker\theta = \{r \in R \mid \theta(r) = 0\}$ is an ideal of R. Moreover, the First Isomorphism Theorem for rings states that $R/\ker\theta \cong \theta(R)$.

Definition 6.3 Let K be an algebraic number field. For any nonzero ideal I of $\mathscr{O}_K$, the cardinality $|\mathscr{O}_K/I|$ is called the *norm* of I and is denoted by $N(I)$.

Proposition 6.4 *Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer with $d \equiv 2, 3 \pmod 4$. Then $N(I)$ is finite for all nonzero ideals I of $\mathbb{Z}[\sqrt{d}]$.*

Proof Take $n = \alpha\bar{\alpha}$ for any nonzero $\alpha \in I$. Then $n \in I \cap \mathbb{N}$. As $\langle n\rangle \subseteq I$, it follows that $|\mathbb{Z}[\sqrt{d}]/I| \le |\mathbb{Z}[\sqrt{d}]/\langle n\rangle|$. In addition, each element of $\mathbb{Z}[\sqrt{d}]/\langle n\rangle$ has a representative $n_1 + n_2\sqrt{d}$ with $n_1, n_2 \in \{0, 1, \dots, n-1\}$. Hence, $\mathbb{Z}[\sqrt{d}]/\langle n\rangle$ is finite and, therefore, $N(I) = |\mathbb{Z}[\sqrt{d}]/I| < \infty$. □

As ideal norms generalize the notion of standard norms given in (2), we expect they satisfy some similar properties. Indeed, this is the case.

Exercise 6.5 Let I and P be a nonzero ideal and a nonzero prime ideal of $\mathbb{Z}[\sqrt{-5}]$, respectively. Show that $|\mathbb{Z}[\sqrt{-5}]/P| = |I/IP|$.

Proposition 6.6 $N(IJ) = N(I)N(J)$ *for all $I, J \in \mathscr{I}^\bullet$.*

Proof By factoring J as the product of prime ideals (Theorem 5.13) and applying induction on the number of factors, we can assume that J is a prime ideal. Consider the ring homomorphism $\theta\colon \mathbb{Z}[\sqrt{-5}]/IJ \to \mathbb{Z}[\sqrt{-5}]/I$ defined by $\theta(\alpha + IJ) = \alpha + I$. It follows immediately that θ is surjective and $\ker\theta = \{\alpha + IJ \mid \alpha \in I\}$. Therefore

$$\frac{\mathbb{Z}[\sqrt{-5}]/IJ}{I/IJ} \cong \mathbb{Z}[\sqrt{-5}]/I$$

by the First Isomorphism Theorem. As IJ is nonzero, $|\mathbb{Z}[\sqrt{-5}]/IJ| = N(IJ)$ is finite and so $|\mathbb{Z}[\sqrt{-5}]/IJ| = |\mathbb{Z}[\sqrt{-5}]/I| \cdot |I/IJ|$. Since J is prime, we can use Exercise 6.5 to conclude that

$$\begin{aligned} N(IJ) &= |\mathbb{Z}[\sqrt{-5}]/IJ| = |\mathbb{Z}[\sqrt{-5}]/I| \cdot |I/IJ| \\ &= |\mathbb{Z}[\sqrt{-5}]/I| \cdot |\mathbb{Z}[\sqrt{-5}]/J| = N(I)N(J). \end{aligned}$$

□

Corollary 6.7 *If $N(I)$ is prime for some $I \in \mathscr{I}^\bullet$, then I is a prime ideal.*

Let us verify now that the ideal norm is consistent with the standard norm on principal ideals.

Proposition 6.8 $N(\langle\alpha\rangle) = N(\alpha)$ *for all* $\alpha \in \mathbb{Z}[\sqrt{-5}]^\bullet$.

Proof Set $S = \{a + b\sqrt{-5} \mid a, b \in \{0, 1, \ldots, n-1\}\}$. Clearly, $|S| = n^2$. In addition,

$$\mathbb{Z}[\sqrt{-5}]/\langle n\rangle = \{s + \langle n\rangle \mid s \in S\}.$$

Note that if $s + \langle n\rangle = s' + \langle n\rangle$ for $s, s' \in S$, then we have $s = s'$. As a consequence, $N(\langle n\rangle) = n^2 = N(n)$ for each $n \in \mathbb{N}$. It is also easily seen that the map $\theta\colon \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}]/\langle\bar{\alpha}\rangle$ defined by $\theta(x) = \bar{x} + \langle\bar{\alpha}\rangle$ is a surjective ring homomorphism with $\ker\theta = \langle\alpha\rangle$. Therefore the rings $\mathbb{Z}[\sqrt{-5}]/\langle\alpha\rangle$ and $\mathbb{Z}[\sqrt{-5}]/\langle\bar{\alpha}\rangle$ are isomorphic by the First Isomorphism Theorem. This implies that $N(\langle\alpha\rangle) = N(\langle\bar{\alpha}\rangle)$. Because $\alpha\bar{\alpha} \in \mathbb{N}$, using Proposition 6.6, one obtains

$$N(\langle\alpha\rangle) = \sqrt{N(\langle\alpha\rangle)N(\langle\bar{\alpha}\rangle)} = \sqrt{N(\langle\alpha\bar{\alpha}\rangle)} = \alpha\bar{\alpha} = N(\alpha).$$

□

Lemma 6.9 *If P is a nonzero prime ideal in $\mathbb{Z}[\sqrt{-5}]$, then P divides exactly one ideal $\langle p\rangle$, where p is a prime number.*

Proof For $\alpha \in P^\bullet$, it follows that $z = \alpha\bar{\alpha} \in P \cap \mathbb{N}$. Then, writing $z = p_1 \cdots p_k$ for some prime numbers $p_1, \ldots, p_k$, we get $\langle z\rangle = \langle p_1\rangle \cdots \langle p_k\rangle$. As $\langle p_1\rangle \cdots \langle p_k\rangle \subseteq P$, we have that $\langle p_i\rangle \subseteq P$ for some $i \in \{1, \ldots, k\}$ (Exercise 3.3). As $p_i \in P^\bullet$, Corollary 5.12 ensures that P divides $\langle p_i\rangle$. For the uniqueness, note that if P divides $\langle p\rangle$ and $\langle p'\rangle$ for distinct primes p and p', then the fact that $mp + np' = 1$ for some $m, n \in \mathbb{Z}$ would imply that P divides the full ideal $\langle 1\rangle = \mathbb{Z}[\sqrt{-5}]$, a contradiction.

□

Theorem 6.10 *The class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$.*

Proof First, we verify that every nonzero ideal I of $\mathbb{Z}[\sqrt{-5}]$ contains a nonzero element α with $N(\alpha) \leq 6N(I)$. For $I \in \mathscr{I}^\bullet$, take $B = \lfloor\sqrt{N(I)}\rfloor$ and define

$$S_I := \left\{a + b\sqrt{-5} \mid a, b \in \{0, 1, \ldots, B\}\right\} \subsetneq \mathbb{Z}[\sqrt{-5}].$$

Observe that $|S_I| = (B+1)^2 > N(I)$. Thus, there exist $\alpha_1 = a_1 + b_1\sqrt{-5} \in S_I$ and $\alpha_2 = a_2 + b_2\sqrt{-5} \in S_I$ such that $\alpha = \alpha_1 - \alpha_2 \in I \setminus \{0\}$ and

$$N(\alpha) = (a_1 - a_2)^2 + 5(b_1 - b_2)^2 \leq 6B^2 \leq 6N(I).$$

Now, let $I\mathscr{P}$ be a nonzero ideal class of $\mathbb{Z}[\sqrt{-5}]$. Take $J \in \mathscr{I}^\bullet$ satisfying $IJ\mathscr{P} = \mathscr{P}$. By the argument given in the previous paragraph, there exists $\beta \in J^\bullet$ such that $N(\beta) \leq 6N(J)$. By Corollary 5.12, there exists an ideal $K \in \mathscr{I}^\bullet$ such that $JK = \langle\beta\rangle$. Using Propositions 6.6 and 6.8, one obtains

$$N(J)N(K) = N(\langle\beta\rangle) = N(\beta) \leq 6N(J),$$

which implies that $N(K) \leq 6$. Because $KJ \sim IJ$ (they are both principal), it follows that $K \in I\mathscr{P}$. Hence, every nonzero ideal class of $\mathbb{Z}[\sqrt{-5}]$ contains an ideal whose norm is at most 6.

To show that the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, let us first determine the congruence relations among ideals of norm at most 6. Every ideal P of norm $p \in \{2, 3, 5\}$ must be prime by Corollary 6.7. Moreover, by Lemma 6.9, Theorem 5.13, and Proposition 6.6, the ideal P must show in the prime factorization

$$\langle p\rangle = P_1^{n_1} \cdots P_k^{n_k} \tag{8}$$

of the ideal $\langle p\rangle$. The following ideal factorizations have been already verified in Example 4.7 and Exercise 4.8:

$$\begin{aligned} \langle 2\rangle &= \langle 2, 1 + \sqrt{-5}\rangle^2, \\ \langle 3\rangle &= \langle 3, 1 - 2\sqrt{-5}\rangle\langle 3, 1 + 2\sqrt{-5}\rangle, \\ \langle 5\rangle &= \langle\sqrt{-5}\rangle^2. \end{aligned} \tag{9}$$

In addition, we have proved in Example 3.6 and Exercise 3.7 that the ideals on the right-hand side of the first two equalities in (9) are prime. Also, the fact that $N(\langle\sqrt{-5}\rangle) = N(\sqrt{-5}) = 5$ implies that the ideal $\langle\sqrt{-5}\rangle$ is prime. It follows now by the uniqueness of Theorem 5.13 that the ideals on the right-hand side of the equalities (9) are the only ideals of $\mathbb{Z}[\sqrt{-5}]$ having norm in the set $\{2, 3, 5\}$. Once again, combining Lemma 6.9, Theorem 5.13, and Proposition 6.6, we obtain that any ideal I whose norm is 4 must be a product of prime ideals dividing $\langle 2\rangle$, which forces $I = \langle 2\rangle$. Similarly, any ideal J with norm 6 must be the product of ideals dividing the ideals $\langle 2\rangle$ and $\langle 3\rangle$. The reader can readily verify that,

$$\langle 1-\sqrt{-5}\rangle = \langle 2, 1+\sqrt{-5}\rangle\langle 3, 1+2\sqrt{-5}\rangle \tag{10}$$

$$\langle 1+\sqrt{-5}\rangle = \langle 2, 1+\sqrt{-5}\rangle\langle 3, 1-2\sqrt{-5}\rangle. \tag{11}$$

Therefore $\langle 1-\sqrt{-5}\rangle$ and $\langle 1+\sqrt{-5}\rangle$ are the only two ideals having norm 6. Now since we know all ideals of $\mathbb{Z}[\sqrt{-5}]$ with norm at most 6, it is not difficult to check that $|\mathscr{C}(\mathbb{Z}[\sqrt{-5}])| = 2$. Because each principal ideal of $\mathbb{Z}[\sqrt{-5}]$ represents the identity ideal class $\mathscr{P}$, we find that

$$\langle 1\rangle\mathscr{P} = \langle 2\rangle\mathscr{P} = \langle\sqrt{-5}\rangle\mathscr{P} = \langle 1-\sqrt{-5}\rangle\mathscr{P}.$$

On the other hand, we have seen that the product of $\langle 2, 1+\sqrt{-5}\rangle$ and each of the three nonprincipal ideals with norm at most 6 is a principal ideal. Thus,

$$\langle 2, 1+\sqrt{-5}\rangle\mathscr{P} = \langle 3, 1+2\sqrt{-5}\rangle\mathscr{P} = \langle 3, 1-2\sqrt{-5}\rangle\mathscr{P}.$$

Since there are only two ideal classes, $\mathscr{C}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}_2$. □

Exercise 6.11 Verify the equalities (10), and (11).

From this observation, we deduce an important property of the ideals of $\mathbb{Z}[\sqrt{-5}]$.

Corollary 6.12 *If $I, J \in \mathscr{I}^\bullet$ are not principal, then IJ is principal.*

7 Half-factoriality

The class group, in tandem with The Fundamental Theorem of Ideal Theory, will allow us to determine exactly what elements of $\mathbb{Z}[\sqrt{-5}]$ are irreducible.

Proposition 7.1 *Let α be a nonzero nonunit element in $\mathbb{Z}[\sqrt{-5}]$. Then α is irreducible in $\mathbb{Z}[\sqrt{-5}]$ if and only if*

1. *$\langle\alpha\rangle$ is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$ (and hence α is a prime element), or*
2. *$\langle\alpha\rangle = P_1P_2$ where P_1 and P_2 are nonprincipal prime ideals of $\mathbb{Z}[\sqrt{-5}]$.*

Proof ($\Rightarrow$) Suppose α is irreducible in $\mathbb{Z}[\sqrt{-5}]$. If $\langle\alpha\rangle$ is a prime ideal, then we are done. Assume $\langle\alpha\rangle$ is not a prime ideal. Then by Theorem 5.13 there are prime ideals $P_1, \ldots, P_k$ of $\mathbb{Z}[\sqrt{-5}]$ with $\langle\alpha\rangle = P_1 \cdots P_k$ for some $k \geq 2$. Suppose that one of the P_i's is a principal ideal. Without loss of generality, assume that $P_1 = \langle\beta\rangle$ for some prime β in $\mathbb{Z}[\sqrt{-5}]$. Using the class group, $P_2 \cdots P_k = \langle\gamma\rangle$, where γ is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$. Thus, $\langle\alpha\rangle = \langle\beta\rangle\langle\gamma\rangle$ implies that $\alpha = (u\beta)\gamma$ for some unit u of $\mathbb{Z}[\sqrt{-5}]$. This contradicts the irreducibility of α in $\mathbb{Z}[\sqrt{-5}]$. Therefore all the P_i's are nonprincipal. Since the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, it follows that k is even. Now suppose that $k > 2$. Using Corollary 6.12 and proceeding in a manner similar to the previous argument, $P_1P_2 = \langle\beta\rangle$ and $P_3 \ldots P_k = \langle\gamma\rangle$, and again $\alpha = u\beta\gamma$ for some unit u, which contradicts the irreducibility of α. Hence, either $k = 1$ and α is a prime element, or $k = 2$.

($\Leftarrow$) If $\langle \alpha \rangle$ is a prime ideal, then α is prime and so irreducible. Then suppose that $\langle \alpha \rangle = P_1 P_2$, where P_1 and P_2 are nonprincipal prime ideals of $\mathbb{Z}[\sqrt{-5}]$. Let $\alpha = \beta\gamma$ for some $\beta, \gamma \in \mathbb{Z}[\sqrt{-5}]$, and assume, without loss of generality, that β is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$. Notice that $\langle \beta\gamma \rangle = \langle \beta \rangle \langle \gamma \rangle = P_1 P_2$. Because P_1 and P_2 are nonprincipal ideals, $\langle \beta \rangle \notin \{P_1, P_2\}$. As a consequence of Theorem 5.13, we have that $\langle \beta \rangle = P_1 P_2$. This forces $\langle \gamma \rangle = \langle 1 \rangle$, which implies that $\gamma \in \{\pm 1\}$. Thus, α is irreducible. □

Let us use Proposition 7.1 to analyze the factorizations presented in (1) at the beginning of the exposition. As the product of any two nonprincipal ideals of $\mathbb{Z}[\sqrt{-5}]$ is a principal ideal, the decompositions

$$\begin{aligned}
\langle 6 \rangle = \langle 2 \rangle \langle 3 \rangle &= \langle 2,\ 1 + \sqrt{-5} \rangle^2 \langle 3,\ 1 - \sqrt{-5} \rangle \langle 3,\ 1 + \sqrt{-5} \rangle \\
&= \langle 2,\ 1 + \sqrt{-5} \rangle \langle 3,\ 1 + \sqrt{-5} \rangle \langle 2,\ 1 + \sqrt{-5} \rangle \langle 3,\ 1 - \sqrt{-5} \rangle \\
&= \langle 1 + \sqrt{-5} \rangle \langle 1 - \sqrt{-5} \rangle
\end{aligned}$$

yield that $2 \cdot 3$ and $(1 + \sqrt{-5})(1 - \sqrt{-5})$ are the only two irreducible factorizations of 6 in $\mathbb{Z}[\sqrt{-5}]$. Thus, any two irreducible factorizations of 6 in $\mathbb{Z}[\sqrt{-5}]$ have the same factorization length. We can take this observation a step further.

Theorem 7.2 *If α is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$ and $\beta_1, \dots, \beta_s, \gamma_1, \dots, \gamma_t$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$ with $\alpha = \beta_1 \cdots \beta_s = \gamma_1 \cdots \gamma_t$, then $s = t$.*

Proof Let $\alpha = \omega_1 \cdots \omega_m$ be a factorization into irreducibles of α in $\mathbb{Z}[\sqrt{-5}]$. By Theorem 5.13, there are unique prime ideals $P_1, \dots, P_k$ in $\mathbb{Z}[\sqrt{-5}]$ satisfying that $\langle \alpha \rangle = P_1 \cdots P_k$. Suppose that exactly d of these prime ideals are principal and assume, without loss, that $P_i = \langle \alpha_i \rangle$ for all $i \in \{1, \dots, d\}$, where each α_i is prime in $\mathbb{Z}[\sqrt{-5}]$. Since the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, there exists $n \in \mathbb{N}$ such that $k - d = 2n$. Hence,

$$\langle \alpha \rangle = (P_1 \cdots P_d)(P_{d+1} \cdots P_k) = \langle \alpha_1 \cdots \alpha_d \rangle (P_{d+1} \cdots P_k),$$

and any factorization into irreducibles of α will be of the form $u\alpha_1 \cdots \alpha_d \cdot \beta_1 \cdots \beta_n$, where each ideal $\langle \beta_j \rangle$ is the product of two ideals chosen from $P_{d+1}, \dots, P_k$. As a result, $m = d + n$ and, clearly, $s = t = m$, completing the proof. □

Thus, while some elements of $\mathbb{Z}[\sqrt{-5}]$ admit many factorizations into irreducibles, the number of irreducible factors in any two factorizations of a given element is the same. As we mentioned in the introduction, this phenomenon is called half-factoriality. Since the concept of half-factoriality does not involve the addition of $\mathbb{Z}[\sqrt{-5}]$, it can also be defined for commutative monoids.

Definition 7.3 An atomic monoid M is called *half-factorial* if any two factorizations of each nonzero nonunit element of M have the same number of irreducible factors.

Half-factorial domains and monoids have been systematically studied since the 1950s, when Carlitz gave a characterization theorem of half-factorial rings of integers, which generalizes the case of $\mathbb{Z}[\sqrt{-5}]$ considered in this exposition.

Theorem 7.4 (Carlitz [1]) *Let R be the ring of integers in a finite extension field of $\mathbb{Q}$. Then R is half-factorial if and only if R has class number less than or equal to two.*

A list of factorization inspired characterizations of class number two can be found in [3]. In addition, a few families of half-factorial domains in a more general setting are presented in [7]. We will conclude this paper by exhibiting two simple examples of half-factorial monoids, using the second one to illustrate how to compute the number of factorizations in $\mathbb{Z}[\sqrt{-5}]$ of a given element.

Example 7.5 (*Hilbert monoid*) It is easily seen that

$$H = \{1 + 4k \mid k \in \mathbb{N}_0\}$$

is a multiplicative submonoid of $\mathbb{N}$. The monoid H is called *Hilbert monoid*. It is not hard to verify (Exercise 7.6) that the irreducible elements of H are

1. the prime numbers p satisfying $p \equiv 1 \pmod 4$ and
2. $p_1 p_2$, where p_1 and p_2 are prime numbers satisfying $p_i \equiv 3 \pmod 4$.

Therefore every element of H is a product of irreducibles. Also, in the factorization of any element of H into primes, there must be an even number of prime factors congruent to 3 modulo 4. Hence, any factorization of an element $x \in H$ comes from pairing the prime factors of x that are congruent to 3 modulo 4. This implies that H is half-factorial. For instance, $x = 5^2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 19$ has exactly two factorizations into irreducibles, each of them contains five factors:

$$x = 5^2 \cdot 13 \cdot (3^2) \cdot (11 \cdot 19) = 5^2 \cdot 13 \cdot (3 \cdot 11) \cdot (3 \cdot 19).$$

Exercise 7.6 Argue that the irreducible elements of the Hilbert monoid are precisely those described in Example 7.5.

Definition 7.7 Let p be a prime number.

1. We say that p is *inert* if $\langle p \rangle$ is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.
2. We say that p is *ramified* if $\langle p \rangle = P^2$ for some prime ideal P in $\mathbb{Z}[\sqrt{-5}]$.
3. We say that p *splits* if $\langle p \rangle = PP'$ for two distinct prime ideals in $\mathbb{Z}[\sqrt{-5}]$.

Prime numbers p can be classified according to the above definition. Indeed, we have seen that p is ramified when $p \in \{2, 5\}$. It is also known that p splits if $p \equiv 1, 3, 7, 9 \pmod{20}$ and is inert if $p \not\equiv 1, 3, 7, 9 \pmod{20}$ (except 2 and 5). A proof of this result is given in [8].

Example 7.8 When $n \geq 2$, the submonoid $\mathbb{X}_n$ of the additive monoid $\mathbb{N}_0^{n+1}$ given by

$$\mathbb{X}_n = \{(x_1, \ldots, x_{n+1}) \mid x_i \in \mathbb{N}_0 \text{ and } x_1 + \cdots + x_n = x_{n+1}\}$$

is a half-factorial Krull monoid with divisor class group $\mathbb{Z}_2$ (see [5, Sect. 2] for more details). Following [4], we will use $\mathbb{X}_n$ to count the number of distinct factorizations into irreducibles of a given nonzero nonunit $\alpha \in \mathbb{Z}[\sqrt{-5}]$. Let

$$\langle \alpha \rangle = P_1^{n_1} \cdots P_k^{n_k} Q_1^{m_1} \cdots Q_t^{m_t},$$

where the P_i's are distinct prime ideals in the trivial class ideal of $\mathbb{Z}[\sqrt{-5}]$, the Q_j's are distinct prime ideals in the nontrivial class ideal of $\mathbb{Z}[\sqrt{-5}]$, and $m_1 \leq \cdots \leq m_t$. Then the desired number of factorizations $\eta(\alpha)$ of α in $\mathbb{Z}[\sqrt{-5}]$ is given by

$$\eta(\alpha) = \eta_{\mathbb{X}_t}\left(m_1, \ldots, m_t, \frac{m_1 + \cdots + m_t}{2}\right),$$

which, when $t = 3$, can be computed by the formula

$$\eta_{\mathbb{X}_3}(x_1, x_2, x_3, x_4) = \sum_{j=0}^{\lfloor x_1/2 \rfloor} \sum_{k=0}^{x_1 - 2j} \left(\left\lfloor \frac{\min\{x_2 - k, x_3 - x_1 + 2j + k\}}{2} \right\rfloor + 1 \right).$$

For instance, let us find how many factorizations $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$ has in $\mathbb{Z}[\sqrt{-5}]$. We have seen that 5 ramifies as $\langle 5 \rangle = P_1^2$, where P_1 is principal. As 11 is inert, $P_2 = \langle 11 \rangle$ is prime. In addition, 3 splits as $\langle 3 \rangle = Q_1 Q_2$, where Q_1 and Q_2 are nonprincipal. Finally, 2 ramifies as $\langle 2 \rangle = Q_3^2$, where Q_3 is nonprincipal. Therefore one has that $\langle 1980 \rangle = P_1^2 P_2 Q_1^2 Q_2^2 Q_3^4$, and so

$$\eta(1980) = \eta_{\mathbb{X}_3}(2, 2, 4, 4) = \sum_{j=0}^{1} \sum_{k=0}^{2-2j} \left(\left\lfloor \frac{\min\{2 - k, 2 + 2j + k\}}{2} \right\rfloor + 1 \right) = 6.$$

Acknowledgements It is a pleasure for the authors to thank the referee, whose helpful suggestions vastly improved the final version of this paper.

References

1. L. Carlitz, A characterization of algebraic number fields with class number two. Proc. Am. Math. Soc. **11**, 391–392 (1960)
2. S.T. Chapman, A tale of two monoids: A friendly introduction to nonunique factorizations. Math. Mag. **87**, 163–173 (2014)
3. S.T. Chapman, So what is class number 2? Am. Math. Mon. to appear
4. S.T. Chapman, J. Herr, N. Rooney, A factorization formula for class number two. J. Number Theory **79**, 58–66 (1999)

5. S.T. Chapman, U. Krause, E. Oeljeklaus, Monoids determined by a homogeneous linear diophantine equation and the half-factorial property. J. Pure Appl. Algebra **151**, 107–133 (2000)
6. F. Jarvis, *Algebraic Number Theory*, Springer Undergraduate Mathematics Series (Springer, New York, 2014)
7. H. Kim, Examples of half-factorial domains. Canad. Math. Bull. **43**, 362–367 (2000)
8. D. Marcus, *Number Fields* (Springer, New York, 1977)
9. H. Pollard, H.G. Diamond, *The Theory of Algebraic Numbers* (Courier Corporation, New York, 1998)
10. M. Ram Murty, J. Esmonde, *Problems in Algebraic Number Theory*, vol. 190, Graduate Text in Mathematics (Springer, New York, 2005)

David Anderson's Work on Graded Integral Domains

Gyu Whan Chang and Hwankoo Kim

Dedicated to David F. Anderson

Abstract In this paper, we survey David Anderson's work on graded integral domains, with emphasis on Picard groups of graded integral domains, graded Krull domains, graded Prüfer v-multiplication domains, graded Prüfer domains, Nagata rings, and Kronecker function rings.

1 Introduction

Graded rings, as a generalization of polynomial rings and semigroup rings, have played an important role in modern algebra and geometry. In this survey paper, we are interested in David Anderson's seminal work on graded integral domains. This includes at least a dozen of his papers solely or with some collaborators: Daniel Anderson, G.W. Chang, and M. Zafrullah.

In the introduction, we give terminologies and notation related to star operations on an integral domain and graded integral domains. In the second section, we discuss normality and (homogeneous) Picard groups of graded integral domains. In the third section, we give some characterizations of graded Krull domain, graded factorial domains, and graded Dedekind domains. In the forth section, we discuss Anderson's work with Daniel Anderson on graded Prüfer v-multiplication domains. In the fifth section, we discuss Anderson's two papers with Daniel Anderson and Chang on graded-valuation domains and with Chang and Zafrullah on graded Prüfer domains.

G. W. Chang
Department of Mathematics Education,
Incheon National University, Incheon 22012, Republic of Korea
e-mail: whan@inu.ac.kr

H. Kim (✉)
Division of Computer and Information Engineering, Hoseo University,
Asan 31499, Republic of Korea
e-mail: hkkim@hoseo.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_10

In the final section, we cover Anderson's work with Chang on Nagata rings and Kronecker function rings of graded integral domains.

1.1 Star Operations

Let D be an integral domain with quotient field K. An overring of D means a subring of K containing D. Let $\mathbf{F}(D)$ (resp., $\mathbf{f}(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. A map $* : \mathbf{F}(D) \to \mathbf{F}(D)$, $I \mapsto I_*$, is called a *star operation* on D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in \mathbf{F}(D)$: (i) $(aD)_* = aD$ and $(aI)_* = aI_*$, (ii) $I \subseteq I_*$, and if $I \subseteq J$, then $I_* \subseteq J_*$, and (iii) $(I_*)_* = I_*$. Given a star operation $*$ on D, we can construct a new star operation $*_f$ by setting $I_{*_f} = \bigcup\{J_* \mid J \in \mathbf{f}(D) \text{ and } J \subseteq I\}$ for all $I \in \mathbf{F}(D)$. Clearly we have $(*_f)_f = *_f$; $I_* = I_{*_f}$ for all $I \in \mathbf{f}(D)$; $I \subseteq I_{*_f} \subseteq I_*$ for all $I \in \mathbf{F}(D)$.

An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I_* = I$; a $*$-ideal $I \in \mathbf{F}(D)$ is said to be of *finite type* if $I = J_*$ for some $J \in \mathbf{f}(D)$; a $*$-ideal is called a *maximal $*$-ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. It is well known that $*_f$-Max$(D) \neq \emptyset$ if D is not a field; each maximal $*_f$-ideal is a prime ideal; each proper $*_f$-ideal is contained in a maximal $*_f$-ideal; each prime ideal minimal over a $*_f$-ideal is a $*_f$-ideal. Examples of the most well-known star operations include the v-operation, the t-operation, and the d-operation. The v-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} = \{x \in K \mid xI \subseteq D\}$. The t-operation is defined by $t = v_f$. The d-operation is just the identity function on $\mathbf{F}(D)$, i.e., $I_d = I$ for all $I \in \mathbf{F}(D)$; so $d_f = d$. It is known that v-Max$(D) = \emptyset$ even though D is not a field as in case of a rank-one nondiscrete valuation domain.

An $I \in \mathbf{F}(D)$ is said to be $*$-*invertible* if $(II^{-1})_* = D$. Note that if $I \in \mathbf{F}(D)$ is invertible, then I is a $*$-invertible $*$-ideal for any star operation $*$ on D. A domain D is called a *Prüfer $*$-multiplication domain* (P$*$MD) if each nonzero finitely generated ideal of D is $*_f$-invertible, equivalently, D_P is a valuation domain for all $P \in *_f$-Max(D) [26, Theorem 1.1]. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under $I * J = (IJ)_t$, $Inv(D)$ be the group of invertible fractional ideals of D, and $Prin(D)$ be the group of nonzero principal fractional ideals of D. It is easy to see that $Inv(D)$ is a subgroup of $T(D)$ and $Prin(D)$ is a subgroup of $Inv(D)$. Then the factor group $\mathrm{Cl}(D) := T(D)/Prin(D)$, called the ($t$-)class group of D, is an abelian group, and $\mathrm{Pic}(D) := Inv(D)/Prin(D)$, the Picard group of D, is a subgroup of $\mathrm{Cl}(D)$. If D is a Prüfer domain, then every t-invertible t-ideal of D is invertible; so $\mathrm{Cl}(D) = \mathrm{Pic}(D)$. Also, if D is a Krull domain, then $Cl(D)$ is the usual divisor class group of D.

Let S be a saturated multiplicative set of D and $N(S) = \{a \in D \mid (a, s)_v = D$ for all $s \in S\}$. Then S is called a *splitting set* if, for each $0 \neq d \in D$, there are $s \in S$ and $a \in N(S)$ so that $d = sa$. A splitting set S of D is an *lcm splitting set* if $sD \cap dD$ is principal for all $s \in S$ and $d \in D$. A multiplicative subset S of D is called a *t-splitting set* if each $0 \neq d \in D$ can be written as $dD = (AB)_t$, where A and B are integral ideals of D such that $A_t \cap sD = sA_t$ (equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. By definition, splitting sets are t-splitting sets, and if $\mathrm{Cl}(D) = \{0\}$, then t-splitting sets are splitting sets.

1.2 Graded Integral Domains

The following notation is fixed throughout this paper. Let Γ be a nonzero torsionless grading monoid, i.e., a nonzero torsionless commutative cancellative monoid, $\langle \Gamma \rangle$ the quotient group of Γ (so $\langle \Gamma \rangle$ is a torsion-free abelian group), $\mathrm{U}(\Gamma) := \Gamma \cap -\Gamma$ the group of units of Γ, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ a Γ-graded integral domain with $R_\alpha \neq \{0\}$ for all $\alpha \in \Gamma$, and K_0 the quotient field of R_0. Then Γ can be given a total order compatible with the monoid operation [28, p. 123]. Hence, each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $x_{\alpha_i} \in R_{\alpha_i}$ and $\alpha_1 < \cdots < \alpha_n$. Each nonzero $x \in R_\alpha$ is called a *homogeneous element* of degree α, i.e., $\deg(x) = \alpha$, and $\deg(0) = 0$. Let $C(f)$ be the ideal of R generated by the homogeneous components of $f \in R$ and $C(I) = \sum_{f \in I} C(f)$ for an ideal I of R. Let H be the multiplicative set of nonzero homogeneous elements of R. Then H is a saturated multiplicative subset of R and R_H is a $\langle \Gamma \rangle$-graded quotient ring of R, where $R_H = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H)_\alpha$ with $(R_H)_\alpha = \{\frac{a}{b} \mid a \in R_\beta, 0 \neq b \in R_\gamma,$ and $\beta - \gamma = \alpha\}$. Clearly, each nonzero homogeneous element of R_H is a unit. Also, $R \neq R_H$ if and only if R contains a nonzero nonunit homogeneous element. The next result is very useful when we study the divisibility properties of graded integral domains.

Theorem 1.1 ([3, Proposition 2.1]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and H be the set of nonzero homogeneous elements of R. Then R_H is a completely integrally closed GCD domain.*

Proof This can be proved by the fact that each nonzero homogeneous element of R_H is a unit.

An ideal A of R is said to be *homogeneous* if $A = \bigoplus_{\alpha \in \Gamma} (A \cap R_\alpha)$; so A is homogeneous if and only if A is generated by homogeneous elements. A fractional ideal A of R is said to be *homogeneous* if there is an $0 \neq x \in H$ such that $xA \subseteq R$ and xA is homogeneous. A homogeneous ideal of R is a *maximal homogeneous ideal* if it is maximal among proper homogeneous ideals. Let h-Max(R) be the set of maximal homogeneous ideals of R. It is easy to show that every maximal homogeneous ideal is a prime ideal and every homogeneous ideal is contained in a maximal homogeneous ideal. We mean by h-dim$(R) = 1$ that each nonzero homogeneous prime ideal of R is a maximal homogeneous ideal. A subring T of R_H containing R is called a *homogeneous overring* of R if $T = \bigoplus_{\alpha \in \langle \Gamma \rangle} (R_H \cap T)_\alpha$. Hence, a homogeneous overring of R is a $\langle \Gamma \rangle$-graded integral domain. Clearly, if $T = R_S$ for a multiplicative set S of nonzero homogeneous elements of R, then T is a homogeneous overring of R.

It is easy to construct a graded integral domain R with a maximal ideal M such that $M \cap H \neq \emptyset$ but M is not homogeneous. The next lemma shows that each maximal t-ideal of R containing an element of H is homogeneous. Hence, t-Max(R) is a disjoint union of $\{Q \in t\text{-Max}(R) \mid Q \cap H = \emptyset\}$ and $\{Q \in t\text{-Max}(R) \mid Q$ is homogeneous$\}$.

Lemma 1.2 ([14, Lemmas 1.2]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then each maximal t-ideal of R intersecting H is homogeneous.*

Let $N(H) = \{g \in R \mid (g, h)_v = R$ for all $h \in H\}$. Then it is shown that $N(H) = \{g \in R \mid g \neq 0$ and $C(g)_v = R\}$ [14, Lemmas 1.1]. Moreover, recall that if $f, g \in R$, then $C(f)^{m+1}C(g) = C(f)^m C(fg)$ for some integer $m \geq 1$ [27]. Hence, $N(H)$ is a saturated multiplicative set of R. It is easy to see that if R contains a unit of nonzero degree, then $I \cap N(H) \neq \emptyset$ for all nonzero ideals I of R with $C(I)_t = R$ [15, Example 1.6]. More generally, we have

Lemma 1.3 ([15, Proposition 1.4]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then Max$(R_{N(H)}) = \{Q_{N(H)} \mid Q \in t$-Max$(R)$ and Q is homogeneous$\}$ if and only if $I \cap N(H) \neq \emptyset$ for all nonzero ideals I of R with $C(I)_t = R$.*

We say that R is a *graded-valuation domain* (*gr-valuation domain*) if either $x \in R$ or $x^{-1} \in R$ for every nonzero homogeneous element $x \in R_H$. It is known that if V is a gr-valuation homogeneous overring of R, then $\widehat{V} := \{\frac{f}{g} \mid f, g \in R, g \neq 0,$ and $C(f) \subseteq C(g)V\}$ is a valuation overring of R such that $\widehat{V} \cap R_H = V$ [22, Theorem 4.3].

Theorem 1.4 *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

1. *R is a gr-valuation domain.*
2. *Either $a|b$ or $b|a$ for every nonzero homogeneous elements $a, b \in R$.*
3. *The set of principal homogeneous ideals of R is totally ordered under inclusion.*
4. *The set of homogeneous (fractional) ideals of R is totally ordered under inclusion.*

Proof This is an easy exercise (see, for example, [4, Theorem 2.2]).

As in [17], we say that R is a *graded Prüfer domain* if each nonzero finitely generated homogeneous ideal of R is invertible. Clearly, gr-valuation domains are graded Prüfer domains.

Let HT(R) be the group of homogeneous t-invertible t-ideals of a graded integral domain R under t-multiplication, HInv(R) be its subgroup of homogeneous invertible ideals of R, and HPrin(R) its subgroup of principal homogeneous ideals. Then HCl(R) := HT(R)/HPrin(R) is an abelian group, called the *homogeneous (t-)class group* of R and HPic(R) := HInv(R)/HPrin(R), called the *homogeneous Picard group* of R, is a subgroup of HCl(R). Also, HCl(R) and HPic(R) can be considered as subgroups of Cl(R) and Pic(R), respectively. Similarly to the ungraded case, one may define the *homogeneous group of divisibility* of R, denoted by HG(R), to be A/B, where A is the set of nonzero homogeneous elements of R_H and $B = H \cap$ U(R), the set of (homogeneous) units of R.

2 Normality and (Homogeneous) Picard Groups

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and H be the set of nonzero homogeneous elements of R. In this section, we survey the David Anderson's work on when $\text{Pic}(R) = \text{HPic}(R)$ (resp., $\text{Pic}(R_0) = \text{Pic}(R)$, $\text{Pic}(R_0) = \text{HPic}(R)$). The results of this section are based on [2, 8, 9, 12].

Lemma 2.1 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then* $\text{Pic}(R) = \text{HPic}(R)$ *if and only if for each integral invertible ideal I of R, $I = xJ$ for some $x \in R_H$ and some homogeneous integral invertible ideal J of R.*

Proof This is an easy exercise.

Let I be a fractional ideal of a graded integral domain R. Then I^h will denote the (homogeneous) fractional ideal of R generated by the homogeneous elements of I. Thus, I is homogeneous if and only if $I = I^h$. It is well known that if P is a prime ideal, then P^h is also a prime ideal. The next result gives several conditions equivalent to $(r) : (x) = \{f \in R \mid fx \in (r)\}$ being homogeneous for every $r \in H$ and $x \in R$.

Theorem 2.2 *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

1. *For $r \in H$ and $x \in R$, $(r) : (x)$ is homogeneous.*
2. *If I is an integral v-ideal of R with I^h nonzero, then I is homogeneous.*
3. *If I is an integral v-ideal of R of finite type with I^h nonzero, then I is homogeneous.*
4. *$C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$.*
5. *For each nonzero $x \in R$, $xR_H \cap R = xC(x)^{-1}$.*
6. *If I is an integral v-ideal of R of finite type, then $I = qJ$ for some $q \in R_H$ and some homogeneous integral v-ideal J of R of finite type.*

Moreover, if R_H is a PID, then each of the above conditions is equivalent to the following statement.

7. *If I is an integral v-ideal of R, then $I = xJ$ for some $x \in R$ and some homogeneous integral v-ideal J of R.*

Proof The equivalences of (1)–(6) are in [2, Theorem 3.2], while the equivalence of (1) and (7) is just [2, Corollary 3.4].

Following [2], a graded integral domain R is said to be *almost normal* if every homogeneous element $x \in R_H$ of nonzero degree which is integral over R is actually in R. Let $A \subseteq B$ be an extension of commutative rings with identity. Following Cohn [23], we say that $A \subseteq B$ is *inert* if whenever $xy \in A$ for some $x, y \in B$, then $x = ru$ and $y = su^{-1}$ for some $r, s \in A$ and u a unit of B. The next result examines relationship between (almost) normality of a graded integral domain R and the equality $C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$.

Theorem 2.3 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain.*

1. *If $C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$, then R is almost normal.*
2. *R is integrally closed if and only if R_0 is integrally closed in $(R_H)_0$ and $C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$.*
3. *If R contains a (homogeneous) unit of nonzero degree, then R is integrally closed if and only if R is almost normal. Thus R satisfies $C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$ if and only if R is integrally closed.*
4. *If $R_0 \subseteq R$ is inert, then R satisfies $C(xy)_v = (C(x)C(y))_v$ for all nonzero $x, y \in R$ if and only if R is almost normal.*

Proof See [2, Theorem 3.5] for (1), [2, Corollary 3.6] for (2), and [2, Theorem 3.7] for (3) and (4).

Following [2], a graded integral domain R is said to be *almost seminormal* if whenever $x^2, x^3 \in R$ for all homogeneous $x \in R_H$ of nonzero degree, then $x \in R$. The next result gives some sufficient conditions for a graded domain R to be almost seminormal.

Theorem 2.4 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Consider the following conditions.*

1. *If I is an integral invertible ideal of R with I^h nonzero, then I is homogeneous.*
2. *If I is an integral invertible ideal of R, then $I = xJ$ for some $x \in R_H$ and some homogeneous integral invertible ideal J of R.*
3. $\mathrm{HPic}(R) = \mathrm{Pic}(R)$.
4. *R is almost seminormal.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4).

Proof (1) $\Leftrightarrow$ (2) [2, Theorem 4.1]. (2) $\Leftrightarrow$ (3) [2, Theorem 4.3]. (2) $\Rightarrow$ (4) [2, Theorem 4.3].

Recall that an integral domain D with quotient field K is *seminormal* if $x^2, x^3 \in D$ for $x \in K$ implies $x \in D$. Equivalently (following Swan), D is seminormal if and only if for $x, y \in D$ with $x^2 = y^3$, there exists $z \in D$ so that $z^3 = x$ and $z^2 = y$. The next result is the graded analog of seminomality.

Theorem 2.5 ([2, Theorem 6.1]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

1. *R is seminormal.*
2. *For $x \in R_H$, $x^2, x^3 \in R$ implies $x \in R$.*
3. *For $x, y \in H$ with $x^2 = y^3$, there exists $z \in R$ (necessarily homogeneous) so that $z^3 = x$ and $z^2 = y$.*

Let $\mathbb{N}_0$ denote the additive monoid of all nonnegative integers and $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be an $\mathbb{N}_0$-graded integral domain. In [7, Corollary 6.2], Anderson showed that if R is a Krull domain, then $\mathrm{Pic}(R_0) = \mathrm{Pic}(R)$. The following theorem is an answer to a question: When does the inclusion $R_0 \hookrightarrow R$ induce an isomorphism $\varphi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$?

Theorem 2.6 *Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be an $\mathbb{N}_0$-graded integral domain. Then the following conditions are equivalent.*

1. *R is seminormal if and only if R_0 is seminormal and the natural map $\varphi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ is an isomorphism.*
2. *R is almost seminormal if and only if the natural map $\varphi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ is an isomorphism.*

Proof (1) [8, Theorem 3]. (2) [9, Theorem 1].

Clearly, if K is a field, then K is seminormal and $\mathrm{Pic}(K) = \{0\}$. Hence, by Theorem 2.6, we have

Corollary 2.7 ([8, Corollary 4]) *Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ be an $\mathbb{N}_0$-graded integral domain with R_0 a field. Then R is seminormal if and only if $\mathrm{Pic}(R) = \{0\}$.*

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. It is easy to see that $R_0 \subset R$ is inert if and only if R has a (homogeneous) unit of degree α for all $\alpha \in \mathrm{U}(\Gamma)$. There are three important cases when $R_0 \subsetneq R$ is an inert extension: (i) $R = R_0[\Gamma]$ is a semigroup ring, (ii) R_0 is a field, and (iii) $\mathrm{U}(\Gamma) = \{0\}$. Hence, if $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is an $\mathbb{N}_0$-graded integral domain, then $R_0 \subset R$ is an inert extension.

Proposition 2.8 ([2, Proposition 5.2]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

1. *$R_0 \subset R$ is an inert extension.*
2. *Let $a \in R$ with $aR \cap R_0$ nonzero. Then $aR \cap R_0$ is principal and $aR = (aR \cap R_0)R$.*
3. *Let I be an integral invertible ideal of R with $I_0 = I \cap R$ nonzero. Then I_0 is R_0-invertible, $I = I_0R$, and the natural homomorphism $\phi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ is injective.*

Let $\Gamma_1 = \{\alpha \in \mathrm{U}(\Gamma) \mid R_\alpha R_{-\alpha} = R_0\}$. Clearly, Γ_1 is a subgroup of $\mathrm{U}(\Gamma)$, $R_\alpha R_\beta = R_{\alpha+\beta}$ and $R_\alpha^{-1} = R_{-\alpha}$ for all $\alpha, \beta \in \Gamma_1$. Let $\Gamma_2 = \{\alpha \in \Gamma \mid R_\alpha \cap \mathrm{U}(R) \neq \emptyset\}$. Then Γ_2 is a subgroup of Γ_1, and moreover, if $\alpha \in \Gamma_2$, then $R_\alpha = a_\alpha R_0$ for all $a_\alpha \in R_\alpha \cap \mathrm{U}(R)$.

Theorem 2.9 *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with $R_0 \subset R$ an inert extension.*

1. *The natural homomorphism $\phi : \mathrm{Pic}(R_0) \to \mathrm{HPic}(R)$ is an isomorphism.*
2. *If R is almost normal, then the natural homomorphism $\phi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ is an isomorphism.*
3. *If $\phi : \mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ is the natural homomorphism, then $\ker(\phi) = \Gamma_1/\Gamma_2$, the factor group of Γ_1 modulo Γ_2.*

Proof See [2, Theorem 5.4] for (1), [2, Theorem 5.5] for (2), and [12, Theorem] for (3).

Anderson also studied the Picard group of monoid domains [10, 11]. The next theorem completely characterizes when $\mathrm{HCl}(R) = \mathrm{Cl}(R)$.

Theorem 2.10 ([2, Theorem 5.9]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integrally closed graded integral domain. Then $\mathrm{HCl}(R) = \mathrm{Cl}(R)$ if and only if for each integral v-ideal I of R, IR_H is principal in R_H. In particular, $\mathrm{HCl}(R) = \mathrm{Cl}(R)$ if either R is a Krull domain or R is $\mathbb{Z}$- or $\mathbb{N}_0$-graded.*

3 Graded Krull Domains

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. We say that R is a *graded Krull domain* if it is completely integrally closed with respect to homogeneous elements and satisfies the ascending chain condition on homogeneous integral v-ideals [3, Definition 5.1]. Clearly, Krull domains are graded Krull domains, while graded Krull domains need not be Krull domains (e.g., let $R = \mathbb{Z}[\mathbb{Q}]$ be the semigroup ring of the additive group $\mathbb{Q}$ of rational numbers over the ring $\mathbb{Z}$ of integers). The results of this section are based on [2, 3, 14, 17].

First we characterize when a graded domain R is completely integrally closed (resp., integrally closed).

Proposition 3.1 *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

1. *R is completely integrally closed (resp., integrally closed).*
2. *R is completely integrally closed (integrally closed) with respect to homogeneous elements.*
3. *$I : I = R$ for each nonzero (resp., nonzero finitely generated) homogeneous fractional ideal I of R.*
4. *R is completely integrally closed (resp., integrally closed) in R_H.*

Proof The completely integrally closed property appears in [3, Proposition 5.2] and the integrally closed property is in [3, Proposition 5.4].

The next result gives some nice properties of graded Krull domains.

Theorem 3.2 *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Krull domain. Then*

1. *R_0 is a Krull domain.*
2. *$\mathrm{Cl}(R)$ is generated by the classes of the homogeneous height-one prime ideals.*
3. *In addition, if $\mathrm{U}(\Gamma) = \{0\}$, then $\varphi : \mathrm{Cl}(R_0) \to \mathrm{Cl}(R)$ is injective. (Note that φ need not be a homomorphism.)*

Proof See [7, Proposition 1.1] for (1), [7, Theorem 4.2] for (2), and [7, Theorem 6.4] for (3).

A gr-valuation ring R is said to be *discrete* if each homogenous primary ideal of R is a power of its radical. Hence, if R is a discrete gr-valuation domain with h-dim$(R) = 1$, which is called a *graded DVR*, then R is a graded Krull domain. Let V be a gr-valuation homogeneous overring of R and $\widehat{V} = \{\frac{f}{g} \mid f, g \in R, g \neq 0$, and $C(f) \subseteq C(g)V\}$. Then V is discrete as a gr-valuation domain if and only if $\widehat{V}$ is discrete [22, Theorem 4.3].

Theorem 3.3 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent.*

1. *R is a graded Krull domain.*
2. *$R = R_H \cap (\bigcap_{P \in X_H} R_P)$, the intersection is locally finite, and each R_P is a DVR, where X_H is the set of height-one homogeneous prime ideals of R.*
3. *$R = R_H \cap (\bigcap_\alpha V_\alpha)$, each nonzero nonunit of R is a unit in V_α except finitely many α's, and each V_α is a rank-one DVR.*
4. *$R = \bigcap_\alpha V_\alpha$, each nonzero nonunit homogeneous element of R is a unit in V_α except finitely many α's, and each V_α is a homogeneous overring of R which is a graded DVR.*
5. *Every nonzero homogeneous ideal of R is t-invertible.*
6. *Every nonzero homogeneous prime (t-)ideal of R is t-invertible.*
7. *$R_{N(H)}$ is a Dedekind domain.*
8. *$R_{N(H)}$ is a PID.*

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) [3, Theorem 5.12]. (1) $\Leftrightarrow$ (4) [3, Theorem 5.15]. (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) [14, Theorem 2.4]. (1) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) [15, Theorem 2.3].

Corollary 3.4 ([3, Theorem 5.8]) *A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a Krull domain if and only if R is a graded Krull domain and R_H is a Krull domain.*

It is known that S is a t-splitting set of an integral domain D if and only if $dD_S \cap D$ is t-invertible for all $0 \neq d \in D$ [5, Corollary 2.3].

Theorem 3.5 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then:*

1. *H is a t-splitting set of R if and only if $C(Q)_t = R$ for each prime t-ideal Q of R with $Q \cap H = \emptyset$.*
2. *Suppose that R_H is a UFD. Then H is a t-splitting set if and only if Q is t-invertible for each prime t-ideal Q of R with $Q \cap H = \emptyset$. In this case, $ht(Q) = 1$.*

Proof (1) [14, Theorem 2.1]. (2) [14, Corollary 2.2].

Recall that a t-splitting set S of D is called a *Krull t-splitting set* if sD is a t-product of (height-one) prime ideals of D for all nonunits $s \in S$, equivalently, every integral ideal of D intersecting S is t-invertible (see [21, Theorem 2.2]).

Theorem 3.6 ([14, Theorem 2.4]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

1. *H is a Krull t-splitting set.*

2. *R is a graded Krull domain.*
3. *Every nonzero homogeneous prime ideal of R contains a t-invertible prime t-ideal.*
4. *Every nonzero homogeneous prime ideal of R contains a t-invertible prime ideal.*
5. *Every prime ideal of R minimal over a nonzero homogeneous principal ideal is t-invertible.*

An integral domain is a *π-domain* if each nonzero principal ideal is a product of prime ideals. Then an integral domain D is a π-domain if and only if D is a Krull domain with $\mathrm{Cl}(D) = \mathrm{Pic}(D)$. Analogously to the ungraded case, a graded integral domain R is called a *graded π-domain* if each nonzero principal homogeneous ideal is a product of (necessarily homogeneous invertible) prime ideals. Hence, a graded π-domain R is just a graded Krull domain with $\mathrm{HCl}(R) = \mathrm{HPic}(R)$.

Theorem 3.7 ([3, Theorem 6.2]) *A graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a π-domain if and only if R is a graded π-domain and R_H is a π-domain.*

A graded integral domain R is called a *graded GCD-domain* if each pair of nonzero homogenous elements has a (necessarily homogeneous) GCD, equivalently if each pair of nonzero homogenous elements has a (necessarily homogeneous) LCM [3, Lemma 3.2]. The next theorem shows that the GCD domain property of R is completely determined by the nonzero homogeneous elements of R.

Theorem 3.8 ([3, Theorem 3.4]) *If $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded GCD domain, then R is a GCD domain.*

A graded integral domain R is called a *graded UFD* (or *graded factorial*) if each nonzero nonunit homogenous element of R is a product of (necessarily homogeneous) principal primes. Hence, by Theorem 3.6, a graded UFD is a graded Krull domain.

Theorem 3.9 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent.*

1. *R is a graded UFD.*
2. *R is a GCD domain and R satisfies the ascending chain condition on homogeneous principal ideals.*
3. $\mathrm{HG}(R)$*, the homogeneous group of divisibility of R, is order isomorphic to a direct sum of copies of $\mathbb{Z}$ with the usual product order.*
4. *Each nonzero homogeneous prime ideal of R contains a nonzero homogeneous principal prime ideal.*
5. *H is a splitting set and $R_{N(H)}$ is a PID.*
6. *R is a GCD domain and $R_{N(H)}$ is a PID.*

Proof (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) [3, Proposition 4.2]. (1) $\Leftrightarrow$ (5) [15, Corollary 2.5]. (1) $\Leftrightarrow$ (6) [15, Corollary 2.6].

It is well known that a Krull domain is a UFD if and only if it is a GCD domain. Hence, by Corollary 3.4 and Theorem 3.9, we have

Corollary 3.10 ([3, Theorem 4.4]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a UFD if and only if R is a graded UFD and R_H is a UFD.*

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an $\mathbb{N}_0$- or a $\mathbb{Z}$-graded integral domain. Then $R_H \cong k[y, y^{-1}]$ for some field k and an indeterminate y over k, and hence R_H is a UFD, Thus, by Corollaries 3.4, 3.10 and Theorem 3.7, we have

Corollary 3.11 ([3, Corollaries 4.6, 5.10, and 6.4]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an $\mathbb{N}_0$- or a $\mathbb{Z}$-graded integral domain. Then R is a Krull domain (resp., π-domain, UFD) if and only if R is a graded Krull domain (resp., graded π-domain, graded UFD).*

If a graded integral domain R satisfies certain extra properties, then we can say more about characterizations of graded Prüfer domains. Let

$$S(H) = \{f \in R \mid C(f) = R\}.$$

Then $S(H)$ is a saturated multiplicative set of R such that $S(H) \subseteq N(H)$, and equality holds when each maximal homogeneous ideal of R is a t-ideal (e.g., h-dim$(R) = 1$ or R is a graded Prüfer domain).

Lemma 3.12 *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and I be a nonzero homogeneous ideal of R. Then we have*

1. $I = \bigcap_{Q\in h\text{-Max}(R)} IR_{H\setminus Q} = (\bigcap_{Q\in h\text{-Max}(R)} IR_Q) \cap R_H = IR_{S(H)} \cap R_H$.
2. $(IR_{S(H)})^{-1} = I^{-1}R_{S(H)}$.
3. *I is invertible if and only if $IR_{S(H)}$ is invertible.*

Proof See [17, Corollary 2.5] for (1) and [16, Lemma 2] for (2) and (3).

Following [17, p. 801], we say that R satisfies property $(*)$ if whenever A is a nonzero ideal of R with $C(A) = R$, then $A \cap S(H) \neq \emptyset$. Let $R = D[X, X^{-1}]$ be the Laurent polynomial ring over an integral domain D. Then R is a $\mathbb{Z}$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in D$ and $n \in \mathbb{Z}$. Clearly, R satisfies property $(*)$ and $R_{S(H)} = D(X)$, the Nagata ring of D. More generally, if R contains a unit of nonzero degree, then R satisfies property $(*)$ [17, Example 4.2]. The following lemma is very useful.

Lemma 3.13 ([17, Lemma 4.1]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

1. $\text{Max}(R_{S(H)}) = \{Q_{S(H)} \mid Q \in h\text{-Max}(R)\}$.
2. *R satisfies property $(*)$.*

The next corollary can be easily proved by Lemmas 3.12(3) and 3.13.

Corollary 3.14 ([17, Corollary 4.4]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain that satisfies property* $(*)$ *and $0 \neq f \in R$. Then $C(f)$ is invertible if and only if $C(f)R_{S(H)} = fR_{S(H)}$.*

A graded Noetherian domain is a graded integral domain whose homogeneous ideals are finitely generated. We say that R is a *graded Dedekind domain* (resp., *graded PID*) if every nonzero homogeneous ideal of R is invertible (resp., principal). Note that invertible ideals are finitely generated; hence, graded PID $\Rightarrow$ graded Dedekind domain $\Rightarrow$ graded Noetherian domain. The following result characterizes graded Dedekind domains.

Theorem 3.15 ([16, Theorem 4]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with $R \neq R_H$. Then the following statements are equivalent.*

1. *R is a graded Dedekind domain.*
2. *$R_{S(H)}$ is a Dedekind domain.*
3. *$R_{S(H)}$ is a PID.*
4. *R is a graded Krull domain and h-$\dim(R) = 1$.*
5. *R is an integrally closed graded Noetherian domain and h-$\dim(R) = 1$.*

Moreover, if the above statements hold, then $\mathrm{Max}(R_{S(H)}) = \{Q_{S(H)} \mid Q \in h\text{-}\mathrm{Max}(R)\}$, *and hence R satisfies property* $(*)$.

A graded Dedekind domain (resp., graded PID) need not be a Dedekind domain (resp., PID) [16, Example 1]. But, the next result shows that if $\mathrm{U}(\Gamma) = \{0\}$, then a graded Dedekind domain is a PID.

Corollary 3.16 ([16, Corollary 7]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain such that* $\mathrm{U}(\Gamma) = \{0\}$. *Then the following statements are equivalent.*

1. *R is a graded Dedekind domain.*
2. *R is a graded PID.*
3. *R_0 is a field and $R \cong R_0[X]$, where X is an indeterminate over R_0.*
4. *R is a PID.*
5. *R is a Dedekind domain.*

In [3], Anderson also studied when $D[\Gamma]$ is a Krull domain (resp., π-domain, UFD). For example, $D[\Gamma]$ is a UFD if and only if D is a UFD, Γ is a unique factorization semigroup, and $U(\Gamma)$ satisfies the ascending chain condition on cyclic subgroups [3, Proposition 4.7].

4 Graded Prüfer *v*-Multiplication Domains

A graded Prüfer v-multiplication domain (PvMD) $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a graded integral domain in which every nonzero finitely generated homogeneous ideal of R is t-invertible, equivalently, the monoid of homogeneous v-ideals of finite type forms a group under v-multiplication. By definition, almost all graded integral domains considered in this paper are graded PvMDs. The next theorem shows that it is enough to consider the homogeneous ideals in order to study the PvMD property of graded integral domains.

Theorem 4.1 ([2, Theorem 6.4]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is a graded PvMD if and only if R is a PvMD.*

By Theorem 3.3, graded Krull domains are graded PvMDs, and thus by Theorem 4.1, graded Krull domains are PvMDs.

Theorem 4.2 ([15, Theorem 3.4]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with a unit of nonzero degree.*

1. *R is a PvMD.*
2. *Every ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
3. *Every principal ideal of $R_{N(H)}$ is extended from a homogeneous ideal of R.*
4. *R is integrally closed and $I_t = IR_{N(H)} \cap R$ for every nonzero homogeneous ideal I of R.*
5. *$R_{N(H)}$ is a Prüfer domain.*
6. *$R_{N(H)}$ is a Bézout domain.*

Recall that a t-splitting set S of a domain D is called a *t-lcm t-splitting set* if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$.

Theorem 4.3 ([14, Theorem 2.6]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then H is a t-lcm t-splitting set of R if and only if R is a PvMD.*

Let $A \subseteq B$ be an extension of integral domains, $\text{qf}(A)$ be the quotient field of A, and X be an indeterminate over B. It is clear that $A + XB[X]$ is an $\mathbb{N}_0$-graded integral domain with $\deg(aX^n) = n$ for $0 \neq a \in B$ and $n \in \mathbb{N}_0$. The notion of t-splitting sets was introduced by Anderson, Anderson, and Zafrullah [5] in order to study the PvMD property of $A + XB[X]$.

Theorem 4.4 ([5, Theorem 2.5]) *Let D be an integral domain, S be a multiplicative set of D, X be an indeterminate over D, and $R = D + XD_S[X]$. Then R is a PvMD if and only if D is a PvMD and S is a t-splitting set.*

In [14, Corollary 2.7], it is characterized when H is a Krull (resp., t-lcm) t-splitting set in the group ring $D[G]$, where G is a torsion-free abelian group. Also, [14, Corollary 2.8] (resp., [14, Corollary 2.9]) characterizes when H is a t-lcm t-splitting set in $A + XB[X]$ (resp., $A + XA_S[X]$), where $\text{qf}(A) \subseteq B$ (resp., S is a multiplicative subset of A).

Theorem 4.5 ([14, Theorem 1.4]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

1. *H is an lcm splitting set.*

2. *R is a GCD domain.*

Moreover, if R contains a (homogeneous) unit of nonzero degree, then the above statements are equivalent to

3. *H is a splitting set.*

Theorem 4.5 can be applied to some special cases: [14, Corollary 1.5] (resp., [14, Corollary 1.6]) characterizes when H is a (an lcm) splitting set (resp., a splitting set generated by principal primes) in semigroup rings, while [14, Corollary 1.7] (resp., [14, Theorem 1.8]) characterizes when H is an lcm splitting set (resp., a splitting set) in $A + B[X]$ for an extension $A \subseteq B$ (resp., $A \subseteq B \subseteq \mathrm{qf}(A)$) of domains.

Theorem 4.6 ([18, Theorem 2.10]) *Let $A \subseteq B$ be an extension of integral domains, X be an indeterminate over B, and $R = A + XB[X]$. Then R is a GCD domain if and only if A is a GCD domain and $B = A_S$ for a splitting set S of A.*

Following [1], an integral domain D is a *G-GCD domain* (generalized GCD domain) if the intersection of two invertible (equivalently, principal) ideals of D is invertible. Equivalently, D is a G-GCD domain if and only if every finite type v-ideal of D is invertible [1, Theorem 1]. In [2], Anderson and Anderson defined a graded integral domain R to be a *graded G-GCD domain* if the intersection of two homogeneous invertible ideals of R is invertible. As mentioned in [2], this is clearly equivalent to each homogeneous v-ideal of R of finite type being invertible.

Proposition 4.7 ([2, Proposition 6.6]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is a G-GCD domain if and only if R is a graded G-GCD domain.*

In [2, 3], Anderson also used some results of this section to completely characterize when $D[\Gamma]$ is a PvMD (resp., GCD domain, G-GCD domain).

5 Graded Prüfer Domains

In this section, we discuss Anderson's two papers with Daniel Anderson and Chang [4] on gr-valuation domains and with Chang and Zafrullah [17] on graded Prüfer domains. The main goal of [4] is to characterize graded-valuation domains, while the purpose of [17] is to generalize several results on Prüfer domains to graded Prüfer domains.

We begin this section with a theorem which gives some very useful properties of gr-valuation domains.

Theorem 5.1 *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a gr-valuation domain. Then we have:*

1. *R is a GCD domain.*
2. *R is integrally closed.*
3. *R has a unique maximal homogeneous ideal M.*

4. *Let T be a homogeneous overring of R. Then T is a gr-valuation domain and $T = R_S$ for some multiplicative set $S \subseteq H$. In fact, we can take $S = H \setminus (P \cap R)$, where P is the unique maximal homogeneous ideal of T.*
5. *$R_M = \widehat{R}$ is a valuation domain and $R_M \cap R_H = R$.*
6. *Let Q be a prime ideal of R with $Q \subseteq M$. Then Q is homogeneous and $QR_{H \setminus Q} = Q$.*
7. *R is completely integrally closed if and only if $ht(M) \leq 1$.*
8. *R_0 is a valuation domain and $K_0 = (R_H)_0$.*
9. *Γ is a valuation monoid.*
10. *Every R_α is a torsion-free R_0-module.*
11. *If $\alpha \in \Gamma$ is not a unit, then $R_\alpha = K_0 x$ for every $0 \neq x \in R_\alpha$.*

Proof (5) [22, Theorem 4.2], while the others are in [4, Theorem 2.3 and Lemma 3.1].

The following is a nice characterization of gr-valuation domains.

Theorem 5.2 ([4, Theorem 3.2]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then R is a gr-valuation domain if and only if the following conditions hold:*

1. *Γ is a valuation monoid,*
2. *$R_\alpha = K_0 x$ for every $0 \neq x \in R_\alpha$ whenever α is not a unit of Γ, and*
3. *$T = \bigoplus_{\alpha \in \mathrm{U}(\Gamma)} R_\alpha$ is a gr-valuation domain.*

Theorem 5.2 has some interesting corollaries. See [4, Corollaries 3.4, 3.5, 3.6] for details. In [19], Anderson also studied valuations of graded integral domains. Next, we give some characterizations of graded Prüfer domains.

Theorem 5.3 ([17, Theorem 3.1]) *The following statements are equivalent for a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$.*

1. *R is a graded Prüfer domain.*
2. *R is a PvMD and every nonzero finitely generated homogeneous ideal of R is a t-ideal.*
3. *R is a PvMD and every nonzero homogeneous ideal of R is a t-ideal.*
4. *R_Q is a valuation domain for every $Q \in h\text{-Max}(R)$.*
5. *R_Q is a valuation domain for every homogeneous prime ideal Q of R.*
6. *$R_{H \setminus Q}$ is a gr-valuation domain for every $Q \in h\text{-Max}(R)$.*
7. *$R_{H \setminus Q}$ is a gr-valuation domain for every homogeneous prime ideal Q of R.*

As a corollary of Theorem 5.3, we have the following result.

Corollary 5.4 ([17, Corollary 4.5]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain that satisfies property $(*)$.*

1. *If R is a graded Prüfer domain, then every ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R.*
2. *$R_{S(H)}$ is a Prüfer domain if and only if R is a graded Prüfer domain.*

We next give some properties of homogeneous overrings of a graded Prüfer domain which is the graded Prüfer domain analog of [24, Theorem 26.1].

Theorem 5.5 ([17, Theorem 3.5]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Prüfer domain, T be a homogeneous overring of R, and Ω be the set of nonzero homogeneous prime ideals P of R such that $PT \subsetneq T$.*

1. *If M is a proper homogeneous prime ideal of T and $P = M \cap R$, then $R_{H\setminus P} = T_{H\setminus M}$, $R_P = T_M$, and $M = PR_P \cap T = PR_{H\setminus P} \cap T$.*
2. *T is a graded Prüfer domain.*
3. *If P is a nonzero homogeneous prime ideal of R, then $P \in \Omega$ if and only if $T \subseteq R_{H\setminus P}$. Moreover, $T = \bigcap_{P\in\Omega} R_{H\setminus P} = \bigcap_{M\in h\text{-Max}(T)} T_{H\setminus M}$.*
4. *If A is a homogeneous ideal of T, then $A = (A \cap R)T$.*
5. *$\{PT \mid P \in \Omega\}$ is the set of nonzero homogeneous prime ideals of T.*

An overring B of an integral domain A is said to be t*-flat* over A if $B_M = A_{A\cap M}$ for every maximal t-ideal M of B. Hence, flat overrings are t-flat. Let I be an ideal of A and $\{V_\lambda\}$ be the set of valuation overrings of A. Then $I^* = \bigcap_\lambda IV_\lambda$ is called the *completion* of I, and I is said to be *complete* if $I^* = I$.

Corollary 5.6 *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then:*

1. *The following statements are equivalent:*
   a. *R is a graded Prüfer domain.*
   b. *Every homogeneous overring of R is t-flat over R.*
   c. *Every homogeneous overring of R is integrally closed.*
   d. *Every homogeneous ideal of R is complete.*
2. *Let T be a homogeneous overring of a graded Prüfer domain R. Then $T = \bigcap_{Q\in t\text{-Max}(T)} R_{Q\cap R} = \bigcap_{M\in h\text{-Max}(T)} R_{H\setminus M}$.*

Proof (1) [17, Corollaries 3.6, 3.8, 3.9]. (2) [17, Corollary 3.7].

Recall that an integral domain D is said to have the *QR-property* if every overring of D is a quotient ring of D. Analogously a graded integral domain R has the *h-QR-property* if every homogeneous overring of R is a quotient ring of R. The following result is the h-QR-property analog of [24, p. 334] that an integral domain with the QR-property is a Prüfer domain.

Proposition 5.7 ([17, Lemma 3.10]) *If a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ has the h-QR-property, then R is a graded Prüfer domain.*

In order to give a complete characterization of graded integral domains with the h-QR-property, we need the following lemma, which is a graded analog of [24, Lemma 27.1] that an integral domain D has the QR-property if and only if $D[\frac{a}{b}]$ is a quotient ring of D for every $0 \neq a, b \in D$.

Lemma 5.8 ([17, Lemma 3.11]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and T a homogeneous overring of R. If $R[x]$ is a quotient ring of R for every homogeneous element $x \in T$, then T is a quotient ring of R. Thus, R has the h-QR-property if and only if $R[y]$ is a quotient ring of R for every homogeneous element $y \in R_H$.*

The following result is a generalization of [24, Theorem 27.5] that an integral domain D has the QR-property if and only if for a finitely generated ideal A of D, there is an element $a \in A$ and an integer $n \geq 1$ such that $A^n \subseteq (a) \subseteq A$.

Theorem 5.9 ([17, Theorem 3.12]) *The following statements are equivalent for a graded Prüfer domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$.*

1. *R has the h-QR-property.*
2. *If A is a finitely generated homogeneous ideal of R, then there exist an $a \in A \cap H$ and an integer $n \geq 1$ such that $A^n \subseteq (a) \subseteq A$.*

It is clear that D is a Bézout domain if and only if D is a Prüfer domain with $\mathrm{Cl}(D) = \{0\}$. The following result is [17, Corollary 3.13].

Corollary 5.10 *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Prüfer domain with $\mathrm{Cl}(R)$ torsion. Then R has the h-QR-property.*

We say that an integral domain D is *of finite character* if the intersection $D = \bigcap_{M\in\mathrm{Max}(D)} D_M$ is locally finite. An ideal I of D is said to be *locally* (resp., *t-locally*) *principal* if ID_P is principal for every maximal ideal (resp., maximal t-ideal) P of D. As in [6], D is called an *LPI domain* if every nonzero locally principal ideal of D is invertible. It is known that a Prüfer domain D is an LPI domain if and only if D is of finite character [25, Theorem 10]. The following result gives the graded Prüfer domain analog.

Theorem 5.11 ([17, Corollary 4.6]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Prüfer domain that satisfies property $(*)$. Then the following statements are equivalent.*

1. *If I is a nonzero homogeneous ideal of R such that IR_M is principal for every $M \in h\text{-}\mathrm{Max}(R)$, then I is invertible.*
2. *The intersection $\bigcap_{M\in h\text{-}\mathrm{Max}(R)} R_M$ is locally finite.*
3. *$R_{S(H)}$ is of finite character.*
4. *Every nonzero locally principal ideal of $R_{S(H)}$ is invertible.*
5. *Every nonzero t-locally principal homogeneous ideal of R is invertible.*

The next result can be obtained by combining [15, Theorems 3.3, 3.4, and 3.7] and [29, Theorems 4.5 and 4.7].

Theorem 5.12 ([17, Proposition 4.9]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then the following statements are equivalent.*

1. *R is a graded Prüfer domain.*
2. *Every ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R.*
3. *Every principal ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R.*
4. *$R_{S(H)}$ is a Prüfer domain.*
5. *$R_{S(H)}$ is a Bézout domain.*
6. *$C(fg) = C(f)C(g)$ for every $f, g \in R$.*

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. If $\mathrm{U}(\Gamma) = \{0\}$, i.e., $\Gamma \cap (-\Gamma) = \{0\}$, then R satisfies property $(*)$ [17, Example 4.2], and hence $\mathrm{Max}(R_{S(H)}) = \{Q_{S(H)} \mid Q \in h\text{-Max}(R)\}$ by Lemma 3.13. The following result characterizes graded Prüfer domains R with $\mathrm{U}(\Gamma) = \{0\}$.

Theorem 5.13 ([17, Theorem 5.2]) *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with $\mathrm{U}(\Gamma) = \{0\}$. Then the following statements are equivalent.*

1. *R is a graded Prüfer domain.*
2. *R_0 is a Prüfer domain, and if $x, y \in H$, then $x|y$ or $y|x$ in R when $\deg(x) \neq \deg(y)$, and $dx, dy \in R_0$ for some $0 \neq d \in R_H$ when $\deg(x) = \deg(y)$.*
3. *R_0 is a Prüfer domain, every principal ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R, and if $x, y \in H$ with $\deg(x) = \deg(y)$, then there is a $0 \neq d \in R_H$ such that $dx, dy \in R_0$.*
4. *R_0 is a Prüfer domain, every ideal of $R_{S(H)}$ is extended from a homogeneous ideal of R, and if $x, y \in H$ with $\deg(x) = \deg(y)$, then there is a $0 \neq d \in R_H$ such that $dx, dy \in R_0$.*
5. *$R_{S(H)}$ is a Prüfer domain.*

Clearly, if $\Gamma = \mathbb{N}_0$, then $\mathrm{U}(\Gamma) = \{0\}$, and in this case, the graded Prüfer domain R has a simple form $D + XK[X]$ for an integral domain with quotient field K. The next result can be proved by Theorem 5.13.

Corollary 5.14 ([17, Corollary 5.3]) *Let $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ be an $\mathbb{N}_0$-graded integral domain. Then the following statements are equivalent.*

1. *R is a graded Prüfer domain.*
2. *R_0 is a Prüfer domain and $R = R_0 + XK[X]$, where K is the quotient field of R_0 and X is an indeterminate over R_0.*
3. *R is a Prüfer domain.*

6 Nagata Rings and Kronecker Function Rings

In this section, we cover Anderson's work [15] with Chang on Nagata rings and Kronecker function rings of graded integral domains. A star operation $*$ on a domain D is said to be an *endlich arithmetisch brauchbar* (e.a.b.) star operation if $(AB)_* \subseteq (AC)_*$ implies $B_* \subseteq C_*$ for all $A, B, C \in \mathbf{f}(D)$. Clearly $*$ is an e.a.b. star operation if and only if $*_f$ is an e.a.b. star operation. It is known that if D admits an e.a.b. star operation, then D is integrally closed [24, Corollary 32.8]. Conversely, suppose that D is integrally closed, and define

$$I_b := \bigcap\{IV \mid V \text{ is a valuation overring of } D\}$$

for every $I \in \mathbf{F}(D)$. Then the mapping $b : \mathbf{F}(D) \to \mathbf{F}(D)$, given by $I \mapsto I_b$, is an e.a.b. star operation on D [24, Theorem 32.5].

Let $D[X]$ be the polynomial ring over D, A_f be the ideal of D generated by the coefficients of $f \in D[X]$, and $*$ be an e.a.b. star operation on D. Then $\text{Kr}(D, *) = \{\frac{f}{g} \mid f, g \in D[X], g \neq 0 \text{ and } A_f \subseteq (A_g)_*\}$, called the Kronecker function ring of D with respect to $*$, is a Bézout domain. The next result is a generalization of Kronecker function rings to graded integral domains.

Theorem 6.1 ([15, Theorem 2.9]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $*$ an e.a.b. star operation on R, and*

$$\text{Kr}(R, *) := \left\{\frac{f}{g} \mid f, g \in R, g \neq 0, \text{ and } C(f) \subseteq C(g)_*\right\}.$$

Then

1. $\text{Kr}(R, *)$ *is an integral domain.*
2. $\text{Kr}(R, *) \cap R_H = R$.
3. *If $f, g \in R$ are nonzero such that $C(f + g)_* = (C(f) + C(g))_*$, then $(f, g)\text{Kr}(R, *) = (f + g)\text{Kr}(R, *)$. In particular, $f\text{Kr}(R, *) = C(f)\text{Kr}(R, *)$ for all $f \in R$.*

Corollary 6.2 ([15, Corollary 2.10]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded Krull domain. Then $R_{N(H)} = \text{Kr}(R, t)$.*

Theorem 6.3 ([15, Theorem 3.3]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then $\text{Cl}(R_{N(H)}) = \text{Pic}(R_{N(H)}) = \{0\}$.*

Theorem 6.4 ([15, Theorem 3.5]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree and $*$ an e.a.b. star operation on R.*

1. $\text{Kr}(R, *)$ *is a Bézout domain.*
2. *$I\text{Kr}(R, *) \cap R_H = I_*$ for every nonzero finitely generated homogeneous ideal I of R.*
3. *If $*_f = t$, then R is a PvMD if and only if $R_{N(H)} = \text{Kr}(R, t)$.*

Theorem 6.5 ([15, Theorem 3.7]) *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. If R is integrally closed, then R is a graded Prüfer domain if and only if $R_{S(H)} = \text{Kr}(R, b)$.*

Notice that the results of Theorems 6.3, 6.4 and 6.5 are not true without the assumption that R has a unit of nonzero degree (for concrete examples, see [20]). For more on divisibility properties of graded integral domains, the reader can refer to Anderson's survey article [13].

References

1. D.D. Anderson, D.F. Anderson, Generalized GCD domain. Comment. Math. Univ. St. Paul. **28**, 215–221 (1979)
2. D.D. Anderson, D.F. Anderson, Divisorial ideals and invertible ideals in a graded integral domain. J. Algebra **76**, 549–569 (1982)
3. D.D. Anderson, D.F. Anderson, Divisibility properties of graded domains. Canad. J. Math. **34**, 196–215 (1982)
4. D.D. Anderson, D.F. Anderson, G.W. Chang, Graded-valuation domains. Commun. Algebra **45**, 4018–4029 (2017)
5. D.D. Anderson, D.F. Anderson, M. Zafrullah, The ring $D+XD_S[X]$ and t-splitting sets. Commut. Algebra Arab. J. Sci. Eng. Sect. C Theme Issues **26**, 3–16 (2001)
6. D.D. Anderson, M. Zafrullah, Integral domains in which nonzero locally principal ideals are invertible. Commun. Algebra **39**, 933–941 (2011)
7. D.F. Anderson, Graded Krull domains. Commun. Algebra **7**, 79–106 (1979)
8. D.F. Anderson, Seminormal graded rings. J. Pure Appl. Algebra **21**, 1–7 (1981)
9. D.F. Anderson, Seminormal graded rings II. J. Pure Appl. Algebra **23**, 221–226 (1982)
10. D.F. Anderson, The Picard group of a monoid domain. J. Algebra **115**, 342–351 (1988)
11. D.F. Anderson, The Picard group of a monoid domain II. Arch. Math. **55**, 143–145 (1990)
12. D.F. Anderson, The kernel of $\mathrm{Pic}(R_0) \to \mathrm{Pic}(R)$ for R a graded domain. C. R. Math. Rep. Acad. Sci. Can. **13**, 248–252 (1991)
13. D.F. Anderson, Divisibility properties in graded integral domains, in *Arithmetical Properties of Commutative Rings*, ed. by S.T. Chapman (Chapman & Hall, Boca Raton, 2005), pp. 22–45
14. D.F. Anderson, G.W. Chang, Homogeneous splitting sets of a graded integral domain. J. Algebra **288**, 527–544 (2005)
15. D.F. Anderson, G.W. Chang, Graded integral domains and Nagata rings. J. Algebra **387**, 169–184 (2013)
16. D.F. Anderson, G.W. Chang, Graded integral domains whose nonzero homogeneous ideals are invertible. Int. J. Algebra Comput. **26**, 1361–1368 (2016)
17. D.F. Anderson, G.W. Chang, M. Zafrullah, Graded Prüfer domains. Commun. Algebra **46**, 792–809 (2018)
18. D.F. Anderson, D.N. El Abidine, The $A + XB[X]$ and $A + XB[\![X]\!]$ constructions from GCD-domains. J. Pure Appl. Algebra **159**, 15–24 (2001)
19. D.F. Anderson, J. Ohm, Valuations and semi-valuations of graded domains. Math. Ann. **256**, 145–156 (1981)
20. G.W. Chang, Graded integral domains and Nagata rings, II. Korean J. Math. **25**, 215–227 (2017)
21. G.W. Chang, T. Dumitrescu, M. Zafrullah, t-splitting sets in integral domains. J. Pure Appl. Algebra **187**, 71–86 (2004)
22. G.W. Chang, D.Y. Oh, Discrete valuation overrings of a graded Noetherian domain. J. Commut. Algebra **10**, 45–61 (2018)
23. P.M. Cohn, Bézout rings and their subrings. Proc. Camb. Philos. Soc. **64**, 251–264 (1968)
24. R. Gilmer, *Multiplicative Ideal Theory* (Marcel Dekker, New York, 1972)
25. W.C. Holland, J. Martinez, W.W. McGovern, M. Tesemma, Bazzoni's conjecture. J. Algebra **320**, 1764–1768 (2008)
26. E.G. Houston, S.B. Malik, J.L. Mott, Characterizations of $*$-multiplication domains. Canad. Math. Bull. **27**, 48–52 (1984)
27. D.G. Northcott, A generalization of a theorem on the content of polynomials. Proc. Camb. Philos. Soc. **55**, 282–288 (1959)
28. D.G. Northcott, *Lessons on Rings, Modules, and Multiplicities* (Cambridge University Press, Cambridge, 1968)
29. P. Sahandi, Characterizations of graded Prüfer $*$-multiplication domains. Korean J. Math. **22**, 181–206 (2014)

Divisor Graphs of a Commutative Ring

John D. LaGrange

This work is dedicated to David Fenimore Anderson, whose guidance and leadership has fostered multitudes of mathematical pursuits, and whose friendship has inspirited a lighthearted approach to all of them.

Abstract If x is an element of a commutative ring R then define the *x-divisor graph* $\Gamma_x(R)$ to be the graph, whose vertices are the elements of $d(x) = \{r \in R \mid rs = x$ for some $s \in R\}$ such that two distinct vertices r and s are adjacent if and only if $rs = x$. In this chapter, the components of $\Gamma_x(R)$ are completely characterized when R is a von Neumann regular ring. Various other types of "divisor graphs" are considered as well. For example, if x is a nonzero element of an integral domain R with group of units $U(R)$ then the *compressed divisor graph* $(\Gamma_E)_x^{d^\times}(R)$ associated with x is defined to be the graph, whose vertices are the associate-equivalence classes $\overline{r} = rU(R)$ of elements $r \in d(x)^\times = d(x) \setminus (xU(R) \cup U(R))$ such that two distinct vertices $\overline{r}$ and $\overline{s}$ are adjacent if and only if $rs \in d(x)$. Alternatively, by letting M be the positive cone of the group of divisibility of R, every $(\Gamma_E)_x^{d^\times}(R)$ is a member of the class of graphs $\Gamma_{\leq x}(M)$ defined by picking an element x of a partially ordered commutative monoid M with least element equal to its identity 1, and letting the vertices of $\Gamma_{\leq x}(M)$ be the elements of $\{m \in M \mid 1 < m < x\}$ such that two distinct vertices m and n are adjacent if and only if $mn \leq x$. Other aspects of the chapter include the exploration of graph-theoretic criteria that reveal when two elements of an integral domain are associates, and it is proved that R is a unique factorization domain if and only if $(\Gamma_E)_x^{d^\times}(R)$ is either null or finite with a dominant clique for every $x \in R \setminus \{0\}$. Throughout, emphasis is placed on similarities with zero-divisor graphs. For example, it is proved that if R is von Neumann regular and G is a component of $\Gamma_x(R)$ that contains a square root of x then $G \cong \Gamma_0(\mathrm{ann}_R(x))$ (in particular, if

J. D. LaGrange (✉)
Lindsey Wilson College, Columbia, KY 42728, USA
e-mail: lagrangej@lindsey.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_11

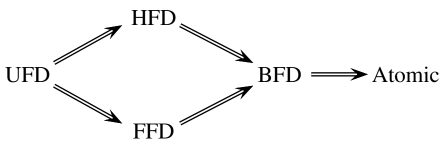
$x = 0$ then we have the tautology $G \cong \Gamma_0(R)$), and if x is a square-free element of a unique factorization domain then $(\Gamma_E)_x^{d^\times}(R)$ is isomorphic to a zero-divisor graph of a finite Boolean ring.

1 Introduction

Factorization in integral domains, and zero-divisor structure in nonintegral domains are important parts of the work on divisibility pioneered by David F. Anderson. In this chapter, some of the methods that he used to study zero-divisors are expanded to include more general elements of commutative rings, and these ideas are applied to provide new perspectives on divisibility in integral domains. In particular, the zero-divisor structure exploited in his seminal paper with Livingston [11] on zero-divisor graphs is linked with general divisor structure in commutative von Neumann regular rings, and then factorization in integral domains (especially unique factorization domains, and also the finite factorization domains that were introduced in his paper with Anderson and Zafrullah [2]) is considered from a graph-theoretic point of view.

Throughout, $\mathrm{irr}(R)$ will denote the set of irreducible elements of an integral domain R. As in [24], R is called *atomic* if for every nonzero nonunit $r \in R$ there exists a finite multiset $S \subseteq \mathrm{irr}(R)$ such that $r = \prod S$. Thus, a unique factorization domain (UFD) is an atomic integral domain such that the multiset S is unique (up to associates) for every nonzero nonunit r. If R is atomic and $|S_1| = |S_2|$ whenever $S_1, S_2 \subseteq \mathrm{irr}(R)$ are multisets with $\prod S_1 = \prod S_2$ then R is called a *half-factorial domain* (HFD) [45]. Moreover, following [2], an atomic integral domain R is called a *finite factorization domain* (FFD) if every nonzero nonunit $r \in R$ has only a finite number of nonassociate irreducible divisors, and it is a *bounded factorization domain* (BFD) if for every nonzero nonunit $r \in R$ there exists a positive integer n such that if $S \subseteq \mathrm{irr}(R)$ is a multiset with $r = \prod S$ then $|S| \leq n$. Examples are given in [2] to show that no additional implications can be inserted into the following diagram.

All monoids and rings will be assumed commutative. The set of positive integers, the ring of integers, the ring integers modulo n, the field of rational numbers, and



the field of complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, and $\mathbb{C}$, respectively. The set of vertices of a graph G will be denoted by $V(G)$, and the group of units of a commutative ring R with identity will be denoted by $U(R)$. If v is a vertex of a graph G then let $N(v)$ be the set of all vertices of G that are adjacent to v, and if r is an element of a commutative ring R then set $\mathrm{ann}_R(r) = \{s \in R \mid rs = 0\}$. There will be no risk of confusion by letting the graph-theoretic and ring-theoretic isomorphism relations both be denoted by $\cong$. See [23, 29] for clarification of any undefined terms from graph theory and ring theory, respectively.

1.1 Divisor Graphs

Let R be a commutative ring. The *zero-divisor graph* $\Gamma_0(R)$, whose vertices are the elements of R such that two distinct vertices r and s are adjacent if and only if $rs = 0$, was introduced by I. Beck in 1988 [21], and then investigated further by D.D. Anderson and M. Naseer in 1993 [1]. In 1999, D.F. Anderson and P.S. Livingston began the study of the subgraph $\Gamma(R)$ of $\Gamma_0(R)$ induced by the nonzero zero-divisors of R, and this work has been the basis for much of the research on zero-divisor graphs over the past 20 years. Similar constructions have subsequently been defined for nonassociative binary structures [36] and semigroups [28], partially ordered sets [32, 38], and multiplicative lattices [34], and the idea has been modified in various ways to examine other aspects of algebraic structure (e.g., [5, 7, 9, 10, 15, 16, 19, 20, 22, 25, 33, 35, 40–44]). Some surveys on zero-divisor graphs include [6, 14, 27].

One of the first variations of the zero-divisor graph concept was introduced in 2002 by S.B. Mulay in [42]. Here, a (multiplicative) congruence relation $\sim$ was defined on a commutative ring R with identity by $r \sim s$ if and only if r and s have equal annihilators (i.e., $\mathrm{ann}_R(r) = \mathrm{ann}_R(s)$), and the "zero-divisor graph" $\Gamma_E(R)$ of the multiplicative monoid $R/\!\sim$ of equivalence classes $r^* = \{s \in R \mid r \sim s\}$ was defined (using different notation) to be the graph, whose vertices are the elements of $(R/\!\sim) \setminus \{0^*, 1^*\}$ such that two distinct vertices r^* and s^* are adjacent if and only if $r^*s^* = 0^*$, if and only if $rs = 0$.

In [42, p. 3551], it was noted that several graph-theoretic properties of $\Gamma(R)$ remain valid for $\Gamma_E(R)$ (for example, each is connected with diameter at most three; cf. Theorem 4(2)). Later, the notation $\Gamma_E(R)$ was used by S. Spiroff and C. Wickham in [44] (with the subscript emphasizing that it is a graph of "equivalence" classes), where *associated primes* (that is, prime annihilator ideals) were among the main objects of interest. The graph $\Gamma_E(R)$ was termed the *compressed zero-divisor graph of* R by D.F. Anderson and the author in [7–10, 12, 18, 27, 33, 39] are examples of references where the continued study of $\Gamma_E(R)$ can be found.

Another offshoot of the zero-divisor graph was introduced in 2007 by J. Coykendall and J. Maney in [25]. Here, for an integral domain R and an element $x \in R$ that can be factored into irreducibles, the *(reduced) irreducible divisor graph* of x was defined to be the graph $\overline{G}(x)$ whose vertex-set $V(\overline{G}(x))$ is a collection of irreducible divisors of x such that $r \in \mathrm{irr}(R)$ divides x if and only if $V(\overline{G}(x))$ contains exactly one associate of r, and two distinct vertices r and s are adjacent if and only if rs divides x (although $V(\overline{G}(x))$ is not unique, it is clear that $\overline{G}(x)$ is well defined up to graph-isomorphism). Among their findings is the following delightful result.

Theorem 1 *If R is an atomic integral domain then R is a UFD if and only if $\overline{G}(x)$ is connected for every nonzero nonunit $x \in R$, if and only if $\overline{G}(x)$ is a complete graph for every nonzero nonunit $x \in R$.*

Proof The result is verified in [25, Theorem 5.1]. □

It was also observed that an atomic integral domain R is an FFD if and only if $\overline{G}(x)$ is finite [25, Proposition 3.1] (cf. Theorem 13). Irreducible divisor graphs were studied further in [16, 40], and they were extended to rings with zero-divisors in [15]. Moreover, they were defined in terms of some generalized-factorization properties in [41]. A survey on irreducible divisor graphs is given in [17].

Recently, a more direct extension of the zero-divisor graph idea has been considered. If x is an element of a commutative ring R then define the *x-divisor graph* $\Gamma_x(R)$ (or, Γ_x if there is no risk of confusion) to be the graph whose vertices are the elements of $d(x) = \{r \in R \mid rs = x \text{ for some } s \in R\}$ such that two distinct vertices r and s are adjacent if and only if $rs = x$. The graphs Γ_x were studied in [35] for the special case when x is an idempotent element of a commutative ring R. A similar construction was used in [36], where R was a commutative, not necessarily associative, binary structure. In the present exposition, the graphs $\Gamma_x(R)$ associated with general elements x of commutative *von Neumann regular* rings R (i.e., rings R such that for every $r \in R$ there exists $t \in R$ such that $r = r^2t$) are linked closely with zero-divisor graphs. These results will be motivated in Sect. 1.3, and they are the focus of Sect. 2.

Of course, if $x = 0$ then Γ_x is the graph $\Gamma_0(R)$ introduced by Beck in [21]. Also, if x is not a zero-divisor then Γ_x is either null (e.g., consider $\Gamma_2(2\mathbb{Z})$), or it is a disjoint union of complete graphs K_1 and K_2 of orders one and two, respectively. More precisely, we have the following characterization.

Proposition 1 *Let x be an element of a commutative ring R. If x is not a zero-divisor then $\Gamma_x \cong \bigsqcup_{i \in I} G_i$, where $G_i \in \{K_1, K_2\}$ for every element i of some (possibly empty if R does not have identity) indexing set I.*

Proof If x is not a zero-divisor and $r, s, t \in V(\Gamma_x)$ such that $rs = rt$ then $s = t$ (indeed, if x is not a zero-divisor then neither is any element of $d(x)$). Hence, no vertex of Γ_x has degree greater than one, and the result follows. □

Proposition 1 suggests that x-divisor graphs are more interesting when R is not an integral domain. To obtain a graph that is more appropriate for studying integral domains, we begin by defining $\Gamma_x^d(R)$ (for any commutative ring R and $x \in R$) to be the graph whose vertices are the elements of $d(x)$ such that two distinct vertices r and s are adjacent if and only if $rs \in d(x)$. If R has identity then $\Gamma_x^d(R) = \bigcup_{a|x} \Gamma_a(R)$, and $rs \in d(x)$ if and only if $d(rs) \subseteq d(x)$ (on the other hand, $\bigcup_{a|4} \Gamma_a(2\mathbb{Z})$ is null even though $d(4) = \{-2, 2\}$).

Note that, for an integral domain R and an element $x \in R$ that can be factored into irreducibles, $\overline{G}(x)$ is an induced subgraph of $\Gamma_x^d(R)$. In contrast to irreducible divisor graphs, $\Gamma_x^d(R)$ is defined even if x cannot be factored into irreducibles. For example, the next observation is easily verified.

Proposition 2 *The following statements hold for a commutative ring R.*

1. *$\Gamma_0^d(R)$ is the complete graph on R.*
2. *If R has identity and $u \in U(R)$ then $\Gamma_u^d(R)$ is the complete graph on $U(R)$.*

Let x be a nonzero element of an integral domain R. The relations in $\Gamma_x^d(R)$ involving elements of $xU(R) \cup U(R)$ are trivially understood (every element of $U(R)$ is adjacent to every other vertex of $\Gamma_x^d(R)$, and if $u \in U(R)$ then the vertices of $\Gamma_x^d(R)$ that are adjacent to ux are precisely the elements of $U(R) \setminus \{ux\}$), so divisibility properties of x are illuminated more efficiently by the subgraph $\Gamma_x^{d^\times}(R)$ (or, $\Gamma_x^{d^\times}$ if there is no risk of confusion) of $\Gamma_x^d(R)$ induced by $d(x)^\times = d(x) \setminus (xU(R) \cup U(R))$. Note that $\Gamma_x^{d^\times}$ is null if and only if $x \in \text{irr}(R) \cup U(R)$ (recall that x is being assumed nonzero, so this claim is valid even if R is a field).

Actually, the graphs $\Gamma_x^{d^\times}$ can be simplified even further without losing too much essential structure. In analogy with the multiplicative monoid $R/\sim$ of equivalence classes r^* and the compressed zero-divisor graph $\Gamma_E(R)$ defined above, if x is a nonzero element of an integral domain R then let the *compressed divisor graph* $(\Gamma_E)_x^{d^\times}(R)$ (or, $(\Gamma_E)_x^{d^\times}$ if there is no risk of confusion) associated with x be the graph whose vertices are the associate-equivalence classes $\overline{r} = rU(R)$ of elements $r \in d(x)^\times$ such that two distinct vertices $\overline{r}$ and $\overline{s}$ are adjacent if and only if $rs \in d(x)$.[1]

Like $\Gamma_E(R)$, the graphs $(\Gamma_E)_x^{d^\times}$ can be described as divisor graphs of the induced quotient monoid. Indeed, consider the partially ordered multiplicative monoid $G(R)^+ = (R \setminus \{0\})/U(R)$, where $rU(R) \leq sU(R)$ in $G(R)^+$ if and only if r divides s in R.[2] Note that $(\Gamma_E)_x^{d^\times}$ can be defined by extending (in the obvious way) the definition of $\Gamma_x^{d^\times}(R)$ from rings to include more general commutative monoids; in particular, it is straightforward to check that we have $(\Gamma_E)_x^{d^\times}(R) = \Gamma_{xU(R)}^{d^\times}(G(R)^+)$.

Moreover, since $rs \in d(x)$ if and only if $(rU(R))(sU(R)) \leq xU(R)$, every $(\Gamma_E)_x^{d^\times}$ belongs to the family of graphs $\Gamma_{\leq x}(M)$ (or, $\Gamma_{\leq x}$ if there is no risk of confusion) defined by picking an element x of a partially ordered commutative monoid M with least element equal to its identity 1, and letting the vertices of $\Gamma_{\leq x}(M)$ be the elements of $(1, x) = \{m \in M \mid 1 < m < x\}$ such that two distinct vertices m and n are adjacent if and only if $mn \leq x$. Specifically, we now have the following interpretations of $(\Gamma_E)_x^{d^\times}$.

Proposition 3 *Suppose that R is an integral domain. If $x \in R \setminus \{0\}$ then the equalities $(\Gamma_E)_x^{d^\times}(R) = \Gamma_{xU(R)}^{d^\times}(G(R)^+) = \Gamma_{\leq xU(R)}(G(R)^+)$ hold.*

As with compressed zero-divisor graphs, an advantage of working with the graph $(\Gamma_E)_x^{d^\times}$ is that it provides a more succinct model of the divisor structure of x. Of course, additional algebraic information (namely, the "associate" relations in R) is used to refine $\Gamma_x^{d^\times}$ into the compressed graph $(\Gamma_E)_x^{d^\times}$. Thus, if purely graph-theoretic characterizations are desirable, then it is natural to seek non-algebraic criteria from which $(\Gamma_E)_x^{d^\times}$ can be obtained from $\Gamma_x^{d^\times}$.

Let G be a graph. An equivalence relation on $V(G)$ is defined by $v \equiv w$ if and only if $N(v) \setminus \{w\} = N(w) \setminus \{v\}$ [9, Theorem 2.1], and we define $\widetilde{G}$ to be the graph

[1] "Compressed irreducible divisor graphs" were defined in [16, Sect. 5] using coarser equivalence relations on $\text{irr}(R)$.

[2] For an integral domain R with quotient field K, the *group of divisibility* $(K \setminus \{0\})/U(R)$ of R is often denoted by $G(R)$; it is a partially ordered abelian group under the relation $rU(R) \leq sU(R)$ if and only if $sr^{-1} \in R$, and then $G(R)^+$ is the positive cone of $G(R)$.

whose vertices are the equivalence classes $[v] = \{w \in V(G) \mid v \equiv w\}$ of elements $v \in V(G)$ such that two distinct vertices $[v]$ and $[w]$ are adjacent in $\widetilde{G}$ if and only if some (and hence every) element of $[v]$ is adjacent to some (and hence every) element of $[w]$ in G. Applications of this construction to zero-divisor graphs are given in [9], where the following result was provided to show that $\Gamma_E(R)$ can usually be obtained from $\Gamma(R)$ using only graph-theoretic information.

Theorem 2 *Let R be a commutative ring with identity. If $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\widetilde{\Gamma(R)} = \Gamma_E(R)$.*

Proof The desired equality is verified in the proof of [9, Theorem 2.5].

Regarding the graphs $\Gamma_x^{d^\times}$, the decision to neglect adjacency between v and w in the definition of $\equiv$ is based on the premise that if v is a vertex of $\Gamma_x^{d^\times}$ and $u \in U(R)$ then the vertices v and $w = uv$ should be "equivalent" even if v^2 divides x. The utility of this convention will become evident as the forthcoming arguments unfold. In fact, it will be shown in Sect. 3.1 that if $U(R) \neq \{1\}$ (in particular, if the characteristic of R is not equal to 2) then two vertices v and w of $\Gamma_x^{d^\times}$ are equivalent under $\equiv$ if and only if v and w are associates in R (Corollary 2) . Hence, in analogy with Theorem 2, we have the following result.

To ease notation, if $G = \Gamma_x^{d^\times}(R)$ then $\widetilde{G}$ will be denoted by $\widetilde{\Gamma}_x^{d^\times}(R)$ (or, $\widetilde{\Gamma}_x^{d^\times}$ if there is no risk of confusion).

Theorem 3 *Suppose that R is an integral domain such that $U(R) \neq \{1\}$, and let $x \in R \setminus \{0\}$. Then $\widetilde{\Gamma}_x^{d^\times}(R) = (\Gamma_E)_x^{d^\times}(R)$. In particular, this is the case if the characteristic of R is not equal to 2.*

Proof The graphs $\widetilde{\Gamma}_x^{d^\times}$ and $(\Gamma_E)_x^{d^\times}$ are null if and only if $x \in \mathrm{irr}(R) \cup U(R)$, so the result follows immediately by Corollary 2 in Sect. 3.1.

As we shall see, $\widetilde{\Gamma}_x^{d^\times}$ and $(\Gamma_E)_x^{d^\times}$ carry enough information so that a purely graph-theoretic characterization of UFDs (without the "atomic" condition that was assumed in Theorem 1) can be obtained. This result will be motivated in Sect. 1.4, and its justification is the primary focus of Sect. 3.2.

1.2 Idempotent Divisors

While the earlier papers [1, 21] on zero-divisor graphs were mainly concerned with colorings, the Anderson–Livingston paper [11] deliberately exploited the interplay between algebraic and graph-theoretic properties. The following theorem illustrates a couple of the fundamental results from [11], along with an important finding that appeared in Anderson's paper [13] with R. Levy and J. Shapiro on the relationship between $\Gamma(R)$ and the zero-divisor graph of the total quotient ring $T(R)$ of R. (Also, for the case when R is not assumed to have identity, the analogous results are provided in [12, Theorems 2.2 and 4.1].)

Theorem 4 *Let R be a commutative ring with identity that contains nonzero zero-divisors. The following statements hold.*

1. *$\Gamma(R)$ is finite if and only if R is finite.*
2. *$\Gamma(R)$ is connected with diameter at most three.*
3. *$\Gamma(T(R)) \cong \Gamma(R)$.*

Proof The first two statements follow from [11, Theorems 2.2 and 2.3], and (3) holds by [13, Theorem 2.2]. □

In contrast to zero-divisor graphs, $\Gamma_x(R)$ need not be connected even if x is idempotent (e.g., if R has a nontrivial group of units then certainly $\Gamma_1(R)$ is disconnected). On the other hand, the statements in Theorem 4 were generalized in [35] as follows (note the similarities with Theorem 4 in the case where $x = 0$).

Theorem 5 *Let x be an idempotent element of a commutative ring R.*

1. *If R has identity then $\Gamma_x(R)$ is finite if and only if $(1-x)R$ and $U(R)$ are finite.*
2. *If xR is Boolean then $\Gamma_x(R)$ is connected, and the converse holds if R is von Neumann regular.*
3. *Suppose that R has identity. If $\Gamma_x(T(R))$ is connected then $\Gamma_x(T(R)) \cong \Gamma_x(R)$. More generally, if v is a vertex of a component G of $\Gamma_x(R)$, and if H is the component of $\Gamma_x(T(R))$ containing v, then $H \cong G$.*

Proof The statements in (2) hold by [35, Theorems 4.1 and 5.7], and (1) and (3) follow by [35, Corollary 3.1 and Theorem 4.6]. □

In fact, the following characterization shows that if x is idempotent then the structure of $\Gamma_x(R)$ is strikingly similar to that of a zero-divisor graph. Recall that if G and H are graphs then the *direct product* $G \times H$ is the graph whose vertices are the elements of $V(G) \times V(H)$ such that two vertices (a, b) and (x, y) are adjacent if and only if a is adjacent to x in G, and b is adjacent to y in H. Also, denote the complete graph of order n by K_n, and let $\Gamma_0^*(R)$ be the *unrestricted zero-divisor graph* of R with $V(\Gamma_0^*(R)) = V(\Gamma_0(R))$ such that (not necessarily distinct) $a, b \in V(\Gamma_0^*(R))$ are adjacent if and only if $ab = 0$.

Theorem 6 *Let x be an idempotent element of a commutative ring R. If G is a component of $\Gamma_x(R)$ then the following statements hold.*

1. *$y^2 = x$ for some $y \in V(G)$ if and only if $G \cong \Gamma_0(\mathrm{ann}_R(x))$.*
2. *$y^2 \neq x$ for every $y \in V(G)$ if and only if $G \cong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$.*

In particular, $\Gamma_x(R)$ is isomorphic to the zero-divisor graph $\Gamma(S)$ of a commutative ring S if and only if $\Gamma_x(R)$ is connected.

Proof The statements in (1) and (2) are given in [35, Theorem 3.1], and then the "in particular" part follows by setting $y = x$ in (1) since if $\Gamma_x(R)$ is connected then $x \in V(\Gamma_x(R)) = V(G)$ (cf. [35, Corollary 2.6]). □

1.3 General Divisors

What if x is not idempotent? With the exception of the first statement in Theorem 5(3) (see Question 1), all questions regarding the generalizability of Theorems 5 and 6 will be answered in the negative. On the other hand, Sect. 2 is devoted to showing that these results extend rather nicely if R is a von Neumann regular ring (except for the second assertion of Theorem 5(2) and the "in particular" statement of Theorem 6).

First, let $R = \mathbb{Z}_2[X]$, and set $x = X$. Then rR is infinite for every $r \neq 0$, but $\Gamma_x \cong K_2$ (the complete graph on $\{1, x\}$). Thus, Theorem 5(1) does not hold if the "idempotent" condition is dropped.

Next, consider Theorem 5(2). Let $I = X^3\mathbb{Z}_5[X]$, and define $R = (X + I)\mathbb{Z}_5[X]/I$. If $x = X^2 + I$ then xR is the trivial (hence Boolean) ring, but $\Gamma_x(R)$ is the disjoint union of two complete graphs K_5 (on the vertex-sets $\{X + rX^2 \mid r \in \mathbb{Z}_5\}$ and $\{4X + rX^2 \mid r \in \mathbb{Z}_5\}$, respectively), and a complete bipartite graph $K_{5,5}$ (with partite sets $\{2X + rX^2 \mid r \in \mathbb{Z}_5\}$ and $\{3X + rX^2 \mid r \in \mathbb{Z}_5\}$, respectively). Hence, the "idempotent" condition is necessary in the first assertion of Theorem 5(2). Also, the graph $\Gamma_2(\mathbb{Z}_3) \cong K_2$ is connected even though $\mathbb{Z}_3$ is von Neumann regular and $2\mathbb{Z}_3 = \mathbb{Z}_3$ is not Boolean. Therefore, the "idempotent" condition cannot be dropped in the second part of Theorem 5(2).

The situation regarding Theorem 5(3) is nontrivial, and it is addressed below (where it will be shown that its second assertion does not generalize; also, see Question 1 regarding the first assertion). Thus, we consider Theorem 6 next. Note that if $R = \mathbb{Z}_8$ then $2^2 = 4$ while the component of $\Gamma_4(R)$ that contains 2 is a complete graph on $\{2, 6\}$. However, $\Gamma_0(\mathrm{ann}_R(4)) = \Gamma_0(\{0, 2, 4, 6\})$ is a graph on four vertices. Also, if $R = \mathbb{Z}_4$ then $y^2 \neq 2$ for every $y \in R$ while $\Gamma_2(R)$ is a path on the three vertices 1, 2, and 3, but $K_2 \times \Gamma_0^*(\mathrm{ann}_R(2))$ is easily checked to be a cycle on four vertices. Therefore, assertions (1) and (2) of Theorem 6 do not generalize to elements x that are not idempotent. Furthermore, the graph $\Gamma_{(0,2)}(\mathbb{Z}_2 \times \mathbb{Z}_3)$ is a path on the four vertices (1, 1), (0, 2), (0, 1), and (1, 2), which is a connected graph that is not a zero-divisor graph (in a zero-divisor graph, every nonzero vertex is adjacent to the vertex 0; also, cf. [11, Example 2.1(b)]). Hence, the "in particular" statement of Theorem 6 need not hold if x is not idempotent.

The next example provides a commutative ring R such that $T(R)$ is von Neumann regular, and R contains an element x such that $\Gamma_x(R)$ has a component of diameter at least four. By appealing to results from Sect. 2, it will follow that the second statement of Theorem 5(3) does not extend to elements x that are not idempotent. The next lemma handles the technical aspects of the example. Throughout, the leading terms of polynomials will be defined with respect to the lexicographic monomial ordering induced by $X_1 > X_2 > X_3 > X_4 > X_5 > Y$.

Lemma 1 *Let I be the ideal of $D = \mathbb{Z}_2[X_1, X_2, X_3, X_4, X_5, Y]$ (where $X_1, \ldots, X_5$ and Y are algebraically independent indeterminates) that is generated by $\{X_iX_{i+1} + Y \mid i \in \{1, 2, 3, 4\}\}$. The following statements hold.*

1. *If* $r \in X_2 + X_1^{-1}I$ *and* $s \in X_4 + X_5^{-1}I$ *then* $rs \notin Y + I$.
2. *If* $r \in D$ *such that* $rX_1 \in Y + I$ *then* $r \in X_2 + X_1^{-1}I$.
3. *If* $r \in D$ *such that* $rX_5 \in Y + I$ *then* $r \in X_4 + X_5^{-1}I$.
4. *If* $r \in D$ *such that* $rX_1 \in Y + I$ *then* $rX_5 \notin Y + I$.
5. *If* $R = D/I$ *then* $T(R)$ *is von Neumann regular.*

Proof Let $r = X_2 + h$ and $s = X_4 + k$, where $h \in X_1^{-1}I$ and $k \in X_5^{-1}I$. If $rs = Y + i$ for some $i \in I$ then $X_1X_5(X_2X_4 + X_2k + X_4h + hk) = X_1X_5rs = X_1X_5(Y + i)$, and hence $X_1X_2X_4X_5 + X_1X_5Y = X_1X_2X_5k + X_1X_4X_5h + X_1X_5hk + X_1X_5i \in I$. But it is straightforward to check (e.g., using Buchberger's Criterion, and remembering that D has characteristic 2) that

$$\mathscr{G} = \{X_iX_{i+1} + Y \mid i \in \{1, 2, 3, 4\}\} \cup \{X_1Y + X_5Y, X_2Y + X_4Y, X_3Y + X_5Y\}$$

is a Gröbner basis for I, and therefore the containment

$$\begin{aligned} X_1X_2X_4X_5 + X_1X_5Y & \\ &= X_4X_5(X_1X_2 + Y) + Y(X_4X_5 + Y) + X_5(X_1Y + X_5Y) + X_5^2Y + Y^2 \\ &\in X_5^2Y + Y^2 + I \end{aligned}$$

shows (e.g., by [29, Theorem 9.6.23], and noting that none of the terms of $X_5^2Y + Y^2$ are divisible by any of the leading terms of the elements of $\mathscr{G}$) that $X_1X_2X_4X_5 + X_1X_5Y \notin I$. Thus, $rs \notin Y + I$, which verifies (1).

To prove (2), let $r_1, r_2, r_3, r_4 \in D$ such that $rX_1 = Y + \sum_{i=1}^4 r_i(X_iX_{i+1} + Y)$. The polynomials $r_2, r_3, r_4 \in D$ do not have any nonzero constant terms (because there is no $i \in \{2, 3, 4\}$ such that X_iX_{i+1} is a term of rX_1), and it follows that $r_1 = f + 1$ for some $f \in D$ that has no nonzero constant term (since Y is not a term of rX_1). Then X_1X_2 is a term of rX_1, and hence $r = X_2 + g$ for some $g \in D$ such that X_2 is not a term of g. Thus, (2) follows since $X_1g = X_1X_2 + rX_1 = X_1X_2 + Y + \sum_{i=1}^4 r_i(X_iX_{i+1} + Y) \in I$, and (3) holds symmetrically.

Suppose that $r \in D$ such that $rX_1 \in Y + I$. By the proof of (2), $r = X_2 + g$ for some $g \in D$ such that X_2 is not a term of g. Hence, if $rX_5 \in Y + I$ then there exist $r_1, r_2, r_3, r_4 \in D$ such that $X_2X_5 + gX_5 = rX_5 = Y + \sum_{i=1}^4 r_i(X_iX_{i+1} + Y)$. But X_2X_5 is not a term of the latter expression, so the previous equality cannot hold unless X_2 is a term of g. Thus, (4) follows, and it remains to prove (5).

Since every Noetherian ring with a trivial nilradical has a von Neumann regular total quotient ring [37, e.g., Propositions 2.4.1 and 4.6.5], it is sufficient to verify that I is a radical ideal. But the ideal $L = (X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_1Y, X_2Y, X_3Y)$ of D generated by the leading terms of the Gröbner basis $\mathscr{G}$ provided above is radical since its generators are square-free monomials [30, Lemma 2.5.3], and it follows that I is a radical ideal: Indeed, if $r \in D$ then (general) polynomial division by the elements of $\mathscr{G}$ yields $f, g \in D$ with $r = f + g$ such that $f \in I$ and no nonzero monomial term of the "remainder" g is divisible by any of the given generators of L [29, Sect. 9.6]. Thus, if $r^n \in I$ then $g^n = (r - f)^n \in I$ and hence, since $\mathscr{G}$ is a

Gröbner basis, either $g = 0$ or the leading term of g^n belongs to L. But L is a radical ideal, so either $g = 0$ or the leading term of g belongs to L. The latter scenario contradicts the choice of g, so $r = f + g = f \in I$. □

Example 1 Let R be the ring defined in Lemma 1(5). Note that the vertices $X_1 + I$, $X_2 + I$, $X_3 + I$, $X_4 + I$, $X_5 + I$ of the graph $\Gamma_{Y+I}(R)$ belong to a path of length four. Also, Lemma 1(4) implies that $\Gamma_{Y+I}(R)$ contains no vertex that is adjacent to both $X_1 + I$ and $X_5 + I$, eliminating the possibility of any path between $X_1 + I$ and $X_5 + I$ of length two. But if $r + I$ is adjacent to $X_1 + I$, and $s + I$ is adjacent to $X_5 + I$, then $r \in X_2 + X_1^{-1}I$ and $s \in X_4 + X_5^{-1}I$ by Lemma 1(2) and (3), respectively. Therefore $r + I$ is not adjacent to $s + I$ by Lemma 1(1), and this eliminates the possibility of any path between $X_1 + I$ and $X_5 + I$ of length one or three. Hence, every path in $\Gamma_{Y+I}(R)$ between $X_1 + I$ and $X_5 + I$ has length at least four.

Let G be the component of $\Gamma_{Y+I}(R)$ that contains $X_1 + I$. If S is a commutative ring then $\Gamma_0(S)$ has diameter at most two (as every vertex is adjacent to 0) and $K_2 \times \Gamma_0^*(S)$ has diameter at most three (as every vertex is adjacent to one of the two elements of $V(K_2) \times \{0\}$). Therefore, since $T(R)$ is von Neumann regular, Corollary 1 in Sect. 2.2 implies that G is not isomorphic to any component of $\Gamma_{Y+I}(T(R))$. Hence, the second statement of Theorem 5(3) fails in the case $x = Y + I$.

Regarding the generalizability of Theorems 5 and 6, the following question remains unanswered.

Question 1 Let x be an element of a commutative ring R with identity. Does $\Gamma_x(T(R)) \cong \Gamma_x(R)$ hold if $\Gamma_x(T(R))$ is connected?

Furthermore, in contrast to idempotent-divisor graphs, Example 1 provides a graph $\Gamma_x(R)$ containing a component of diameter at least four, and the next question is left open.

Question 2 Let x be an element of a commutative ring R, and suppose that G is a component of $\Gamma_x(R)$. Does there exist a bound on the diameter of G?

1.4 The Shape of Unique Factorization

Let R be an atomic integral domain. If $x \in R \setminus (U(R) \cup \{0\})$ then define $\ell(x) = \sup\{n \mid n \in \mathbb{N} \text{ and } x = p_1 \ldots p_n \text{ for some } p_1, \ldots, p_n \in \operatorname{irr}(R)\} \in \mathbb{N} \cup \{\infty\}$. It is clear that $\ell(xy) \geq \ell(x) + \ell(y)$ for all $x, y \in R \setminus (U(R) \cup \{0\})$, and R is a BFD if and only if $\ell(x) < \infty$ for every $x \in R \setminus (U(R) \cup \{0\})$. (In fact, it is interesting to note that [2, Theorem 2.4] shows that BFDs are completely characterized by the existence of a function $\ell : R \setminus \{0\} \to \mathbb{N} \cup \{0\}$ that is positive at nonunits and obeys the inequality $\ell(xy) \geq \ell(x) + \ell(y)$ for every $x, y \in R \setminus \{0\}$.)

Continuing in the spirit of relating general divisor graphs to zero-divisor graphs, we offer the following observations. To avoid disruption, the proofs of Propositions 4 and 5 are postponed until the end of this section.

Proposition 4 *Suppose that R is an atomic integral domain, and let $x \in R \setminus (\mathrm{irr}(R) \cup U(R))$ be square-free with a unique factorization (up to order and associates) into a product of irreducible elements of R. If B is the finite Boolean ring of order $2^{\ell(x)}$ then following statements hold.*

1. $(\Gamma_E)_x^{d^\times} \cong \Gamma(B)$.
2. *If either $U(R) \neq \{1\}$ or $\ell(x) \geq 3$ then $\widetilde{\Gamma}_x^{d^\times} \cong \Gamma(B)$.*
3. *If $U(R) = \{1\}$ and $\ell(x) = 2$ then $\widetilde{\Gamma}_x^{d^\times} \cong \widetilde{\Gamma(B)} \cong K_1$.*

In particular, if R is a UFD then these statements hold for every square-free $x \in R \setminus (\mathrm{irr}(R) \cup U(R))$.

Naturally, Proposition 4 suggests the question of whether a graph-theoretic characterization of UFDs can be given by the condition "$(\Gamma_E)_x^{d^\times}$ is isomorphic to the zero-divisor graph of a finite Boolean ring for every square-free $x \in R \setminus (\mathrm{irr}(R) \cup U(R))$." However, the next result shows that this is not possible.

Proposition 5 *Let K_1 be a proper subfield of a quadratically closed field K_2 (that is, for every $b \in K_2$ there exists $c \in K_2$ such that $c^2 = b$), and set $R = K_1 + XK_2[X]$. Then R is a non-UFD such that every square-free $F \in R \setminus U(R)$ has a unique factorization (up to order and associates) into a product of irreducible elements of R. In particular, $(\Gamma_E)_x^{d^\times}$ is isomorphic to the zero-divisor graph of a finite Boolean ring for every square-free $x \in R \setminus (\mathrm{irr}(R) \cup U(R))$.*

In search of a characterization of UFDs in terms of the graphs $(\Gamma_E)_x^{d^\times}$(respectively, $\widetilde{\Gamma}_x^{d^\times}$), we have shown that the condition given prior to Proposition 5 admits too many rings. Naively, one might suggest removing the "square-free" assumption, but this would certainly be too restrictive (e.g., the ring of integers $\mathbb{Z}$ is a UFD but $(\Gamma_E)_{12}^{d^\times}(\mathbb{Z}) = \widetilde{\Gamma}_{12}^{d^\times}(\mathbb{Z})$ is a path on the four vertices [4], [3], [2], and [6], which is not even a zero-divisor graph [11, Example 2.1(b)]). What is the appropriate compromise?

A subgraph H of a simple graph G is called a *clique* if it is a complete graph, and it is called *dominant* if every vertex of G is either a vertex of H or adjacent to a vertex of H. For example, if $B \ncong \mathbb{Z}_2$ is a finite Boolean ring (which, for convenience, may be regarded as the power-set of a finite set S; see the proof of Proposition 4) then the subgraph H of $\Gamma(B)$ that is induced by the singleton subsets of S is a dominant clique. The following theorem shows how these concepts yield a condition that gives a purely graph-theoretic classification of UFDs (without any knowledge of irreducible elements).

Let $\mathbb{Z}_{\geq 0}$ denote the (additive) monoid of nonnegative integers. Given a nonempty indexing set I, the monoid $\bigoplus_I \mathbb{Z}_{\geq 0}$ is partially ordered (under the usual product order) with least element equal to its identity $\mathbf{0}$. Recall that if $\alpha \in \bigoplus_I \mathbb{Z}_{\geq 0}$ then $\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$ is the graph whose vertices are the elements of $(\mathbf{0}, \alpha) = \{\beta \in \bigoplus_I \mathbb{Z}_{\geq 0} \mid \mathbf{0} < \beta < \alpha\}$ such that two distinct vertices v and w are adjacent if and only if $v + w \leq \alpha$ (see the discussion prior to Proposition 3). If $(\mathbf{0}, \alpha) \neq \emptyset$ then α will be called a *non-minimal nonzero* element. Thus, $\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$ is not null if and only if $\alpha \in \bigoplus_I \mathbb{Z}_{\geq 0}$ is a non-minimal nonzero element.

Theorem 7 *The following statements are equivalent for an integral domain R.*

1. *R is a UFD.*
2. *If $x \in R \setminus \{0\}$ then $(\Gamma_E)_x^{d^\times}$ is either a null graph, or $(\Gamma_E)_x^{d^\times} \cong \Gamma_{\leq\alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$ for some finite indexing set I and non-minimal nonzero $\alpha \in \bigoplus_I \mathbb{Z}_{\geq 0}$.*
3. *If $x \in R \setminus \{0\}$ then $(\Gamma_E)_x^{d^\times}$ is either a null graph, or a finite graph that contains a dominant clique.*
4. *If $x \in R \setminus \{0\}$ then $\widetilde{\Gamma}_x^{d^\times}$ is either a null graph, or a finite graph that contains a dominant clique.*

The proof of Theorem 7 (which is the main objective of Sect. 3.2) reveals that the vertices of the aforementioned dominant cliques can be taken to be the (equivalence classes of the) vertices of the complete graph in Theorem 1 (although, the dominant clique need not be unique; e.g., the sets {[2]}, {[2], [4]}, and {[2], [8]} induce three distinct dominant cliques in $(\Gamma_E)_{16}^{d^\times}(\mathbb{Z}) = \widetilde{\Gamma}_{16}^{d^\times}(\mathbb{Z})$). On the other hand, in contrast to Theorem 1, $(\Gamma_E)_x^{d^\times}$(respectively, $\widetilde{\Gamma}_x^{d^\times}$) may be connected for every nonzero nonunit $x \in R$ even if R is not a UFD. Furthermore, while Theorem 1 is given without imposing any "finite" conditions on $\overline{G}(x)$, the "finite" assumptions in Theorem 7 cannot be dropped. We close this section with an example to illustrate these claims, along with the proofs of Propositions 4 and 5.

Example 2 Let $\mathbb{C}[X; \mathbb{Q}_{\geq 0}] = \{\sum_{i \in \mathbb{Q}_{\geq 0}} a_i X^i \mid a_i \in \mathbb{C}$ with $a_i = 0$ for all but finitely many $i\}$ be the *monoid domain* induced by $X^i X^j = X^{i+j}$, where $\mathbb{Q}_{\geq 0}$ is the additive monoid of nonnegative rational numbers (see [31] for more on monoid domain constructions). Define $R = \mathbb{C}[X; \mathbb{Q}_{\geq 0}]_{\mathfrak{m}}$, where $\mathfrak{m} = \sum\{X^q \mathbb{C}[X; \mathbb{Q}_{\geq 0}] \mid q \in \mathbb{Q}_{\geq 0} \setminus \{0\}\}$. Then R is an integral domain that is not a UFD (in fact, $\operatorname{irr}(R) = \emptyset$; cf. [4, Theorem 1] or [26, Proposition 2.3]).

If $x \in R$ is a nonzero nonunit then $x = X^q f$ for some $q \in \mathbb{Q}_{\geq 0} \setminus \{0\}$ and $f \in U(R)$, and if $y, z \in V(\Gamma_x^{d^\times})$ then $y = X^i g$ and $z = X^j h$ for some $g, h \in U(R)$ and $i, j \in \mathbb{Q}_{\geq 0}$ with $i, j < q$. Hence, if $n \in \mathbb{Q}_{\geq 0} \setminus \{i, j\}$ with $0 < n < \min\{q - i, q - j\}$ then $X^n \in N(y) \cap N(z)$, which shows that $\Gamma_x^{d^\times}$ (and hence $(\Gamma_E)_x^{d^\times}$ and $\widetilde{\Gamma}_x^{d^\times}$) is connected. In fact, similar reasoning shows that the set $\{X^n \mid n \in \mathbb{Q}_{\geq 0} \setminus \{0\}$ and $n \leq q/2\}$ induces a dominant clique of $\Gamma_x^{d^\times}$, and hence $(\Gamma_E)_x^{d^\times}$ and $\widetilde{\Gamma}_x^{d^\times}$ also contain dominant cliques.

Proof of Proposition 4. Suppose that $x = p_1 \ldots p_n$ $(n \geq 2)$, where $p_1, \ldots, p_n \in \operatorname{irr}(R)$ are mutually nonassociate (by the "square-free" hypothesis), and let B be the power-set of $P = \{p_1, \ldots, p_n\}$. Thus, $\ell(x) = n$, and B is a Boolean ring of order 2^n (with addition given by symmetric-difference, and multiplication given by $\cap$). Moreover, it is straightforward to check that if $n = 2$ then $(\Gamma_E)_x^{d^\times} \cong K_2$ (on the vertices $\overline{p_1}$ and $\overline{p_2}$), while $\widetilde{\Gamma}_x^{d^\times} \cong K_2$ (on the vertices $[p_1]$ and $[p_2]$) if $U(R) \neq \{1\}$, and $\widetilde{\Gamma}_x^{d^\times} \cong K_1$ (on the vertex $[p_1] = [p_2]$) if $U(R) = \{1\}$. The result follows immediately for the case when $\ell(x) = 2$, so it remains to prove (1) and (2) when $\ell(x) \geq 3$.

Henceforth, suppose that $n \geq 3$. Note that $\widetilde{\Gamma}_x^{d^\times} = (\Gamma_E)_x^{d^\times}$ by Proposition 6 in Sect. 3.1. Therefore, it is sufficient to prove that $(\Gamma_E)_x^{d^\times} \cong \Gamma(B)$.

It is clear that $V((\Gamma_E)_x^{d^\times}) = \{\overline{\prod S} \mid S \in B \setminus \{\emptyset, P\}\}$ (indeed, if $r \in d(x)^\times$ then "unique factorization" guarantees that r and $\prod S$ are associates for some $S \in B \setminus \{\emptyset, P\}$). Also, since x is square-free, two vertices $\overline{\prod S_1}$ and $\overline{\prod S_2}$ are adjacent in $(\Gamma_E)_x^{d^\times}$ if and only if $\prod(S_1 \sqcup S_2) = (\prod S_1)(\prod S_2) \in d(x)$, if and only if $S_1 \sqcup S_2 \subseteq P$, if and only if $S_1 \cap S_2 = \emptyset$. Therefore, the mapping $V(\Gamma(B)) \to V((\Gamma_E)_x^{d^\times})$ by $S \mapsto \overline{\prod S}$ for every $S \in B \setminus \{\emptyset, P\}$ preserves and reflects adjacency. Thus, it is an graph-isomorphism because the "unique factorization" of x guarantees that it is injective, and it is surjective since clearly $V(\Gamma(B)) = B \setminus \{\emptyset, P\}$.

The "in particular" statement follows trivially. □

Proof of Proposition 5. Note that R is an HFD (hence, it is atomic) by [3, Theorem 5.3]. If $a \in K_2 \setminus K_1$ then aX and X are nonassociate irreducible elements of R, and $(aX)(a^{-1}X) = X^2$. Therefore, R is not a UFD.

Next, observe that $\mathrm{irr}(R) \subseteq \mathrm{irr}(K_2[X])$. To see this, it is enough to show that if $f, g \in K_2[X] \setminus K_2$ with $f \notin R$ and $fg \in R$ then $fg = rs$ for some $r, s \in R \setminus K_1$. But if $f \notin R$ and $fg \in R$ then $f \in a + XK_2[X]$ for some $a \in K_2 \setminus K_1$, and either $g \in a^{-1}K_1 + XK_2[X]$ or $g \in XK_2[X]$. In any case, it follows that $a^{-1}f, ag \in R \setminus K_1$ with $\big(a^{-1}f\big)\big(ag\big) = fg$, and the claim is verified.

Let $F \in R \setminus (U(R) \cup \{0\})$. It will be shown that either F factors uniquely (up to order and associates) into a product of irreducible elements of R, or F is not square-free. Note that either $F \in u + XK_2[X]$ for some $u \in K_1 \setminus \{0\}$, or $F \in XK_2[X]$, so first assume that $F \in u + XK_2[X]$.

Every irreducible factor of F in R belongs to $v + XK_2[X]$ for some $v \in K_1 \setminus \{0\}$. But $\mathrm{irr}(R) \subseteq \mathrm{irr}(K_2[X])$, so the existence of two "distinct" factorizations of F into products of irreducibles in R would imply (since $K_2[X]$ is a UFD) that F has two nonassociate (in R) irreducible factors f and g that are associates in $K_2[X]$. That is, there exist $v_1, v_2 \in K_1 \setminus \{0\}$ and distinct $f \in v_1 + XK_2[X]$ and $g \in v_2 + XK_2[X]$ such that $f = ag$ for some $a \in K_2 \setminus K_1$. This contradicts that $v_1 \in K_1$, so F factors uniquely (up to order and associates) in R.

Suppose that $F \in XK_2[X]$; say $F = aX + h$ for some $a \in K_2$ and $h \in X^2K_2[X]$. If $a \neq 0$ then $F = aX(1 + h/(aX))$ is a factorization of F in R. Thus, since $K_2[X]$ is a UFD, every factorization of F into irreducible elements of R includes a $K_2[X]$-associate of aX. But if $b \in K_2$ such that $F = bX(ab^{-1} + h/(bX))$ is a factorization in R then $ab^{-1} \in K_1$, i.e., $a \in bK_1$, so aX and bX are associates in R. Moreover, $1 + h/(aX)$ and $ab^{-1} + h/(bX)$ are associates in R and, since the above argument implies that $1 + h/(aX) \in 1 + XK_2[X]$ factors "uniquely" in R, it follows again that any factorization of F in R is unique (up to order and associates).

Finally, suppose that $a = 0$. Then there exists $n \geq 2$ such that $F = bX^n + h'$ for some $b \in K_2 \setminus \{0\}$ and $h' \in X^{n+1}K_2[X]$. Thus, if $c \in K_2$ with $c^2 = b$ then $F = (cX)^2(X^{n-2} + h'/(bX^2))$ is a factorization in R. Therefore, if $a = 0$ then F is not square-free.

It has been verified that if $F \in R \setminus U(R)$ is square-free then F factors uniquely (up to order and associates) into a product of irreducible elements of R. The "in particular" statement follows by Proposition 4. □

2 von Neumann Regular Rings

In this section, it is shown that the "idempotent" hypotheses in Theorems 5 and 6 can be dropped if R is a von Neumann regular ring, with the exception of the second assertion of Theorem 5(2) (consider the von Neumann regular ring $\mathbb{Z}_3$ and the graph $\Gamma_2(\mathbb{Z}_3)$ that was mentioned in the discussion prior to Lemma 1) and the "in particular" statement of Theorem 6 (consider the von Neumann regular ring $\mathbb{Z}_2 \times \mathbb{Z}_3$ and the graph $\Gamma_{(0,2)}(\mathbb{Z}_2 \times \mathbb{Z}_3)$ that was mentioned in the discussion prior to Lemma 1). If R is von Neumann regular then $T(R) = R$, so Theorem 5(3) holds trivially (even if $\Gamma(T(R))$ is not connected). Hence, the arguments of this section focus on Theorem 5(1), the first assertion of Theorem 5(2), and statements (1) and (2) of Theorem 6.

Recall that a commutative ring R is von Neumann regular if for every $r \in R$ there exists $t_r \in R$ such that $r = r^2 t_r$ (note that t_r is not necessarily unique, but this is a harmless abuse of notation whose convenience will be enjoyed throughout). Also, the element e_r defined by $e_r = rt_r$ is idempotent, and $re_r = r$ (note that e_r is unique since if $r = r^2 s = r^2 t$ then $rt = r^2 st = (rs)(rt)$ and, symmetrically, $rs = (rs)(rt)$). These facts, along with the observations in the following lemma, will be used repeatedly.

Lemma 2 *Let r and s be elements of a commutative von Neumann regular ring R. If s divides r then $e_s e_r = e_r$. In particular, $e_s r = r$.*

Proof Let $w \in R$ such that $sw = r$. Then $e_r = rt_r = swt_r = (s^2 t_s)wt_r = (st_s) swt_r = e_s rt_r = e_s e_r$, and then the "in particular" statement holds by multiplying by r. □

2.1 *Generalization of Theorem 5*

To begin, consider the assertions of Theorem 5. First, it is proved that the statement in (1) generalizes if R is von Neumann regular. Note that both conditions in Theorem 8 below are necessary to conclude that $\Gamma_x(R)$ is finite. For example, consider the von Neumann regular rings $\mathbb{Q}$ and $\prod_{i=1}^{\infty} \mathbb{Z}_2$. If $x = 1$ then $\Gamma_x(\mathbb{Q})$ is infinite even though $(x-1)\mathbb{Q} = \{0\}$ is finite, and $\Gamma_{(1,0,0,\ldots)}(\prod_{i=1}^{\infty} \mathbb{Z}_2)$ is infinite even though $U(\prod_{i=1}^{\infty} \mathbb{Z}_2) = \{\mathbf{1}\}$ is finite.

Theorem 8 *Let x be an element of a commutative von Neumann regular ring R with identity. Then $\Gamma_x(R)$ is finite if and only if $(1-x)R$ and $U(R)$ are finite.*

Proof Suppose that $(1-x)R$ and $U(R)$ are finite. If $r \in R$ then $(r + 1 - e_r)(e_r t_r + 1 - e_r) = 1$, so the set $A = \{r + 1 - e_r \mid r \in V(\Gamma_x(R))\} \subseteq U(R)$ is finite. Note that if $r \in V(\Gamma_x(R))$ then $e_r e_x = e_x$ by Lemma 2, so multiplying by x yields $e_r x = x$. Hence, the set $\{rx \mid r \in V(\Gamma_x(R))\} = \{(r + 1 - e_r)x \mid r \in V(\Gamma_x(R)\} = \{ax \mid a \in A\}$ is finite. But $\{r(1-x) \mid r \in V(\Gamma_x(R))\} \subseteq (1-x)R$ is also finite, and therefore

$V(\Gamma_x(R)) = \{r \mid r \in V(\Gamma_x(R))\} = \{rx + r(1-x) \mid r \in V(\Gamma_x(R))\}$ is finite. This verifies the "if" portion of the result.

Conversely, if $u \in U(R)$ then $u(u^{-1}x) = x$, so $u \in V(\Gamma_x(R))$. Therefore, if $\Gamma_x(R)$ is finite then $U(R)$ is finite. By way of contraposition, we next assume that $(1-x)R$ is infinite, and prove that $\Gamma_x(R)$ is infinite. Note that the equality $a = ae_x + a(1-e_x)$ implies that either $\{ae_x \mid a \in (1-x)R\}$ is infinite or $\{a(1-e_x) \mid a \in (1-x)R\}$ is infinite.

If $\{a(1-e_x) \mid a \in (1-x)R\}$ is infinite then $B = \{e_x + a(1-e_x) \mid a \in (1-x)R\}$ is infinite, and $B \subseteq V(\Gamma_x(R))$ since $bx = x$ for every $b \in B$. Thus, if $\{a(1-e_x) \mid a \in (1-x)R\}$ is infinite then $\Gamma_x(R)$ is infinite. Therefore, assume that $\{ae_x \mid a \in (1-x)R\}$ is infinite.

If $\{e_{ae_x} \mid a \in (1-x)R\}$ is finite then $C = \{ae_x + e_x - e_{ae_x} \mid a \in (1-x)R\}$ is infinite, and the equality $(ae_x + e_x - e_{ae_x})(x + xt_{ae_x}e_{ae_x} - xe_{ae_x}) = x$ shows that $C \subseteq V(\Gamma_x(R))$. Hence, if $\{e_{ae_x} \mid a \in (1-x)R\}$ is finite then $\Gamma_x(R)$ is infinite. Henceforth, suppose that $\{e_{ae_x} \mid a \in (1-x)R\}$ is infinite.

The set $D = \{e_x - xe_{ae_x} \mid a \in (1-x)R\}$ is infinite; indeed, if $xe_{ae_x} = xe_{be_x}$ for some $a, b \in (1-x)R$ then multiplying by t_x yields $e_xe_{ae_x} = e_xe_{be_x}$, and hence $e_{ae_x} = e_{be_x}$ by Lemma 2 since $x(t_xa) = ae_x$ and $x(t_xb) = be_x$. Moreover, if $a \in (1-x)R$ then $(e_x - xe_{ae_x})(x + xe_{ae_x}t_{1-x} - xe_{ae_x}) = x + xe_{ae_x}t_{1-x} - xe_{ae_x} - x^2e_{ae_x} - x^2e_{ae_x}t_{1-x} + x^2e_{ae_x} = x - xe_{ae_x} + x(1-x)e_{ae_x}t_{1-x} = x - xe_{ae_x} + xe_{1-x}e_{ae_x} = x - xe_{ae_x} + xe_{ae_x} = x$, where the fourth equality holds by Lemma 2 since $a \in (1-x)R$. Therefore, $D \subseteq V(\Gamma_x(R))$, and it follow that $\Gamma_x(R)$ is infinite. This exhausts all cases, so if $\Gamma_x(R)$ is finite then $(1-x)R$ and $U(R)$ are finite. □

The following result proves that if R is von Neumann regular then the first statement in Theorem 5(2) holds (albeit vacuously) without the "idempotent" condition.

Theorem 9 *Let x be an element of a commutative von Neumann regular ring R. If xR is Boolean then x is idempotent. In particular, if xR is a Boolean ring then $\Gamma_x(R)$ is connected.*

Proof The "in particular" statement holds if x is idempotent by Theorem 5(2), so it suffices to verify that if xR is Boolean then x is idempotent. Since xR is Boolean, the result will follow if $x \in xR$ (which, of course, is obvious if R has identity). But $x = x^2t_x \in x^2R \subseteq xR$ (actually, $x^2R = xR$ since $x = x^2t_x \in x^2R$). □

2.2 Generalization of Theorem 6

Now, statements (1) and (2) of Theorem 6 will be generalized to include non-idempotent elements x of a von Neumann regular ring R (Corollary 1). The objective in Theorem 10 is to establish an isomorphism $\Gamma_x \cong \Gamma_{e_x}$ whenever there exists $y \in R$ such that $y^2 = x$, and then Theorem 11 provides an isomorphism $G \cong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$ for any component G of Γ_x in the case where $y^2 \neq x$ for every $y \in R$. The next lemma will be useful when proving that the desired isomorphisms are bijective.

Lemma 3 *Let x be an element of a commutative von Neumann regular ring R. If $y \in R$ such that $e_x = e_y$ then the following statements hold.*

1. *If $r \in V(\Gamma_{e_x})$ then there exists $s \in R$ such that $(yr + r - e_x r)s = x$.*
2. *If $r, s \in R$ such that $yr + r - e_x r = ys + s - e_x s$ then $r = s$.*
3. *If $s \in V(\Gamma_x)$ then there exists $r \in V(\Gamma_{e_x})$ such that $s = yr + r - e_x r$.*

Proof Set $e = e_x = e_y$. If $r \in V(\Gamma_e)$ then there exists $z \in R$ with $rz = e$, and then (1) is easily verified by setting $s = xzt_y$. Also, multiplying the equality $yr + r - er = ys + s - es$ by e yields $yr = ys$, which becomes $er = es$ upon multiplying by t_y. Hence, $yr - er = ys - es$, so if $yr + r - er = ys + s - es$ then $r = s$, and this proves (2).

To show that (3) holds, let $z \in R$ such that $sz = x$. Define $r = et_y s + s - es$. Then $r \in V(\Gamma_e)$ since $r(yt_s) = e(yt_y)(st_s) + y(st_s) - (ey)(st_s) = eee_s + ye_s - ye_s = e$, where the last equality follows by Lemma 2. Similarly, straightforward computations show that $yr = es$ and $er = et_y s$, and hence $yr + r - er = es + r - et_y s = s$, where the last equality holds by the definition of r. □

The following theorem shows that if x is a perfect square in a von Neumann regular ring then Γ_x is identical to an idempotent-divisor graph.

Theorem 10 *Let x be an element of a commutative von Neumann regular ring R. If there exists $y \in R$ such that $y^2 = x$ then $\Gamma_x \cong \Gamma_{e_x}$.*

Proof The equality $e_x = e_y e_x$ holds by Lemma 2. Moreover, $e_y = e_y^2 = (yt_y)^2 = xt_y^2 = xe_x t_y^2 = y^2 e_x t_y^2 = (yt_y)^2 e_x = e_y^2 e_x = e_y e_x$. Henceforth, set $e = e_x = e_y$. It will be proved that the mapping $\varphi : V(\Gamma_e) \to V(\Gamma_x)$ defined by

$$\varphi(r) = yr + r - er$$

is a graph-isomorphism.

Note that φ is well defined by Lemma 3(1), and it is bijective by Lemma 3(2) and (3). Thus, it remains to show that φ preserves and reflects adjacency relations, and it is sufficient to verify that if $r, s \in V(\Gamma_e)$ then $rs = e$ if and only if $\varphi(r)\varphi(s) = x$.

If $rs = e$ then the equalities $\varphi(r)\varphi(s) = (yr + r - er)(ys + s - es) = y^2 e = xe = x$ are easily verified. Thus, suppose that $\varphi(r)\varphi(s) = x$. That is, $x = (yr + r - er)(ys + s - es) = xrs + rs - ers$. Multiplying by e yields the equality $x = xrs$, from which $e = ers$ is deduced by multiplying by t_x. Therefore, $rs = x - xrs + ers = x - x + e = e$. □

Let e be an idempotent element of a commutative ring R, and let G_e be the component of Γ_e that contains the vertex e. Define G_e^* to be the *unrestricted component* of Γ_e containing e, where $V(G_e^*) = V(G_e)$ and (not necessarily distinct) $r, s \in V(G_e^*)$

are adjacent if and only if $rs = e$ (in particular, $G_0^* = \Gamma_0^*(R)$ is the unrestricted zero-divisor graph of R, as defined in Sect. 1.2).

It is proved in [35, Proposition 2.5(1)] that if r is a vertex of G_e then $re = e$. This fact will be used freely in the proof of the next theorem, which addresses the structure of Γ_x when x is not a perfect square.

Theorem 11 *Let x be an element of a commutative von Neumann regular ring R. If $y^2 \neq x$ for every $y \in R$ then $G \cong K_2 \times \Gamma_0^*(\text{ann}_R(x))$ for every component G of Γ_x.*

Proof Let G be a component of Γ_x, and set $e = e_x$. Since $y^2 \neq x$ for every $y \in R$, it follows that $|V(G)| \geq 2$. Hence, there exist $v_1, v_2 \in V(G)$ that are adjacent in G.

Let $y_i = v_i e$ ($i \in \{1, 2\}$). Then $y_1 y_2 = v_1 v_2 e^2 = xe = x$, and therefore $y_1 \neq y_2$ since $y_i^2 \neq x$. Furthermore, since $y_1 y_2 = x$ and $v_i x t_x = v_i e = y_i$, Lemma 2 implies $e_{y_i} e = e$ and $e_{y_i} e = e_{y_i}$, respectively. Thus, $e_{y_1} = e = e_{y_2}$.

Suppose that G_e is the component of Γ_e containing the vertex e. Clearly $\text{ann}_R(x) = \text{ann}_R(e)$ (indeed, $e = xt_x$ and $xe = x$). Also, [35, Proposition 2.5(2)] shows that $V(G_e)$ and $\text{ann}_R(e)$ are isomorphic (multiplicative) semigroups under a mapping that satisfies $e \mapsto 0$, so $G_e^* \cong \Gamma_0^*(\text{ann}_R(x))$. Therefore, it is enough to show that $G \cong K_2 \times G_e^*$. Hence, let $V(K_2) = \{1, 2\}$, and define $\varphi : V(K_2 \times G_e^*) = \{1, 2\} \times V(G_e) \to V(G)$ by

$$\varphi(i, r) = \begin{cases} y_1 + r - e, & i = 1 \\ y_2 + r - e, & i = 2 \end{cases}.$$

Pick $i \in \{1, 2\}$. If $r \in V(G_e)$ then $re = e$ (recall the above comments), so if $w \in R$ with $v_i w = x$ then the equalities $(y_i + r - e)(we) = v_i ewe + we - we = xe = x$ are easily verified. Hence, $(y_i + r - e)(we) = x = xe = v_i(we)$, which shows that $y_i + r - e$ and v_i belong to the same component, i.e., $y_i + r - e \in V(G)$. Therefore, φ is well defined.

Let $i, j \in \{1, 2\}$, and suppose that $r, s \in V(G_e)$ such that $\varphi(i, r) = \varphi(j, s)$. Then $y_i = y_i e = (y_i + r - e)e = \varphi(i, r)e = \varphi(j, s)e = (y_j + s - e)e = y_j e = y_j$ (in particular, $i = j$, as it has already been observed that $y_1 \neq y_2$), and therefore the equalities $r = \varphi(i, r) - y_i + e = \varphi(j, s) - y_j + e = s$ hold. Thus, φ is injective.

To prove that φ is surjective, let $s \in V(G)$. By induction (since paths between vertices of a connected graph are necessarily finite), generality is not lost by assuming that s is adjacent to $\varphi(i, r)$ for some $i \in \{1, 2\}$ and $r \in V(G_e)$. Furthermore, by symmetry, it can be assumed that $i = 1$, so suppose that s is adjacent to $\varphi(1, r)$ (in particular, $\varphi(1, r)s = x$).

By Lemma 3(3), there exists $\rho \in V(\Gamma_e)$ such that $s = y_2\rho + \rho - e\rho$. Thus, $x = \varphi(1, r)s = (y_1 + r - e)(y_2\rho + \rho - e\rho) = x\rho + y_2 r\rho + r\rho - e\rho - y_2\rho$. Multiplying by et_x yields $e = ee = et_x x = et_x(x\rho + y_2 r\rho + r\rho - e\rho - y_2\rho) = e(t_x x)\rho + (er)t_x y_2\rho + (er)t_x\rho - (ee)t_x\rho - et_x y_2\rho = ee\rho + et_x y_2\rho + et_x\rho - et_x \rho - et_x y_2\rho = e\rho$, and hence $\rho \in V(G_e)$. Therefore, $\varphi(2, \rho) = y_2 + \rho - e = y_2 e + \rho - e\rho = y_2(e\rho) + \rho - e\rho = (y_2 e)\rho + \rho - e\rho = y_2\rho + \rho - e\rho = s$.

To show that φ preserves adjacency, suppose that $(i, r), (j, s) \in V(K_2 \times G_e^*)$ are adjacent. Then $i \neq j$ and $rs = e$. Note that $\varphi(i, r) \neq \varphi(j, s)$ since φ is injective, so it is sufficient to verify that $\varphi(i, r)\varphi(j, s) = x$.

The equalities $er = e = es$ hold since $r, s \in V(G_e)$. Hence, $y_i s = (v_i e)s = v_i(es) = v_i e = y_i$. Similarly, $y_j r = y_j$, and it follows that $\varphi(i, r)\varphi(j, s) = (y_i + r - e)(y_j + s - e) = y_i y_j = x$.

To complete the proof, it only remains to show that φ reflects adjacency. Suppose that $(i, r), (j, s) \in V(K_2 \times G_e^*)$ such that $\varphi(i, r)$ and $\varphi(j, s)$ are adjacent in G. As noted above, the equalities $y_i s = y_i$ and $y_j r = y_j$ hold. Hence, if $i = j$ then $x = ex = e\varphi(i, r)\varphi(i, s) = e(y_i + r - e)(y_i + s - e) = e(y_i^2 + rs - e) = y_i^2$, which contradicts the hypothesis. Therefore, $i \neq j$, and it is sufficient to verify that $rs = e$. But this follows immediately from $x = \varphi(i, r)\varphi(j, s) = (y_i + r - e)(y_j + s - e) = x + rs - e$, and the proof is complete. □

Theorems 10 and 11 are summarized in the following corollary, which completes the task of determining the extent to which statements (1) and (2) of Theorem 6 generalize to non-idempotent elements of von Neumann regular rings.

Corollary 1 *Let x be an element of a commutative von Neumann regular ring R. If G is a component of Γ_x then the following statements hold.*

1. *$y^2 = x$ for some $y \in V(G)$ if and only if $G \cong \Gamma_0(\mathrm{ann}_R(x))$.*
2. *$y^2 \neq x$ for every $y \in V(G)$ if and only if $G \cong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$.*

Proof It is not difficult to check that $\Gamma_0(\mathrm{ann}_R(x)) \ncong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$ (details are given in the first paragraph of the proof of [35, Theorem 3.1]), so it is sufficient to prove the "only if" portions of (1) and (2). Thus, suppose that $y^2 = x$ for some $y \in V(G)$. The first paragraph of the proof of Theorem 10 shows that $e_y = e_x$, and then it was proved that $\varphi : V(\Gamma_{e_x}) \to V(\Gamma_x)$ defined by $\varphi(r) = yr + r - e_x r$ is a graph-isomorphism. In particular, since $\varphi(e_x) = y$, the restricted map $\varphi : V(G_{e_x}) \to V(G)$ is a well-defined graph-isomorphism. Hence, $G \cong G_{e_x} \cong \Gamma_0(\mathrm{ann}_R(e_x)) \cong \Gamma_0(\mathrm{ann}_R(x))$, where the second and third isomorphisms follow from Theorem 6(1) and the equality $\mathrm{ann}_R(e_x) = \mathrm{ann}_R(x)$, respectively. Therefore, the "only if" statement in (1) holds.

Suppose that $y^2 \neq x$ for every $y \in V(G)$. If $z^2 \neq x$ for every $z \in R$ then the result follows by Theorem 11. Thus, suppose that $z \in R$ such that $z^2 = x$. As before, $e_z = e_x$, and the mapping $\varphi : V(\Gamma_{e_x}) \to V(\Gamma_x)$ defined by $\varphi(r) = zr + r - e_x r$ is a graph-isomorphism.

To the contrary, suppose that $G \ncong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$. Since G is isomorphic to the component of Γ_{e_x} that is induced by $\varphi^{-1}(V(G))$, it follows by Theorem 6(1) that $r^2 = e_x$ for some $r \in \varphi^{-1}(V(G))$. But then $\varphi(r)^2 = (zr + r - e_x r)^2 = z^2 = x$, contradicting that $y^2 \neq x$ for every $y \in V(G)$. Hence, if $y^2 \neq x$ for every $y \in V(G)$ then $G \cong K_2 \times \Gamma_0^*(\mathrm{ann}_R(x))$. □

3 Integral Domains

The main goal of this section is to provide a proof of Theorem 7, along with a graph-theoretic characterization of FFDs. It is shown that an integral domain R is an FFD if and only if $(\Gamma_E)_x^{d^\times}(R)$ (respectively, $\widetilde{\Gamma}_x^{d^\times}(R)$) is a finite graph for every $x \in R \setminus \{0\}$

(Theorem 13; cf. [25, Proposition 3.1]). To proceed, it is necessary to consider the connection between associate elements of an integral domain R and equivalence classes of vertices in $\Gamma_x^{d^\times}(R)$. This is done in Sect. 3.1, where it is proved that if $|[r]| \geq 3$ then the relation $r \equiv s$ holds if and only if r and s are associates in R.

To ease notation, set $R' = R \setminus (\mathrm{irr}(R) \cup U(R) \cup \{0\})$. Thus, if x is a nonzero element of an integral domain R then $\Gamma_x^{d^\times}(R)$ (and hence $(\Gamma_E)_x^{d^\times}(R)$ and $\widetilde{\Gamma}_x^{d^\times}(R)$) is not null if and only if $x \in R'$. Recall that two distinct vertices r and s of $\Gamma_x^{d^\times}(R)$ (respectively, distinct vertices $\overline{r}$ and $\overline{s}$ of $(\Gamma_E)_x^{d^\times}(R)$, and distinct vertices $[r]$ and $[s]$ of $\widetilde{\Gamma}_x^{d^\times}(R)$) are adjacent if and only if $rs \in d(x)$, if and only if there exists $q \in R$ such that $rsq = x$. A graph is called *totally disconnected* if it has at least one vertex and it has no edges. Also, the *clique number* of a graph G is 0 if G is null, and otherwise it is $\sup\{n \in \mathbb{N} \mid G \text{ contains a clique of order } n\} \in \mathbb{N} \cup \{\infty\}$.

3.1 Equivalent Vertices and Associates

Suppose that R is an integral domain, and let $x \in R'$. The following lemma records the basic observations that every equivalence class of $\equiv$ in $\Gamma_x^{d^\times}$ induces a subgraph of $\Gamma_x^{d^\times}$ that is either complete or totally disconnected, and if two vertices of $\Gamma_x^{d^\times}$ are associates in R then they are equivalent in $\Gamma_x^{d^\times}$.

Lemma 4 *Let R be an integral domain. If $x \in R'$ then the following statements hold.*

1. *If $r \in V(\Gamma_x^{d^\times})$ then the subgraph of $\Gamma_x^{d^\times}$ induced by $[r]$ is either complete or totally disconnected.*
2. *If $r \in V(\Gamma_x^{d^\times})$ and $u \in U(R)$ then $[r] = [ur]$.*

Proof Suppose that the subgraph of $\Gamma_x^{d^\times}$ induced by $[r]$ is neither complete nor totally disconnected. Then there exist distinct $q, s, t \in [r]$ such that q and s are adjacent in $\Gamma_x^{d^\times}$, but q and t are not. Of course, this is absurd since $s \equiv t$, so the subgraph of $\Gamma_x^{d^\times}$ induced by $[r]$ is either complete or totally disconnected.

To prove (2), note that if $t \in N(r) \setminus \{ur\}$ then $t \notin \{r, ur\}$, and there exists $q \in R$ such that $rtq = x$. Hence, $(ur)tq' = x$ where $q' = qu^{-1}$, and therefore $t \in N(ur) \setminus \{r\}$. This shows that $N(r) \setminus \{ur\} \subseteq N(ur) \setminus \{r\}$, and the reverse inclusion follows similarly by noting that if $(ur)tq = x$ then $rtq' = x$ where $q' = uq$. Thus, $r \equiv ur$, i.e., $[r] = [ur]$. □

While it was straightforward to verify that associates are equivalent in $\Gamma_x^{d^\times}$, the core of this subsection is devoted to the converse. The following example shows that it can fail, regardless of whether the subgraph induced by $[r]$ is complete or totally disconnected.

Example 3 1. Let $R = \mathbb{Z}_2[X^2, X^3]$. The equivalence class $[X^2] = [X^4] = \{X^2, X^4\}$ induces a complete subgraph of $\Gamma_{X^6}^{d^\times}$. Also, X^2 and X^4 are not associates in R.

2. Let $R = \mathbb{Z}_2[Y]$. The equivalence class $[Y^2] = [Y^3] = \{Y^2, Y^3\}$ induces a totally disconnected subgraph of $\Gamma_{Y^4}^{d^\times}$. Also, Y^2 and Y^3 are not associates in R.

As a special case, however, it is not difficult to show that equivalent vertices are necessarily associates whenever x is a square-free element of a UFD such that $\ell(x) \geq 3$. This is recorded in the next proposition (which was used in the proof of Proposition 4). Note that Example 3(2) shows that the result does not extend to general elements of UFDs.

Proposition 6 *Suppose that R is an atomic integral domain, and let $x \in R$ be square-free with a unique factorization (up to order and associates) into a product of irreducible elements of R. If $\ell(x) \geq 3$ and $r, s \in V(\Gamma_x^{d^\times})$ then $[r] = [s]$ if and only if r and s are associates.*

Proof Let $r, s \in V(\Gamma_x^{d^\times})$, and set $x = p_1 \dots p_n$ for some mutually distinct $p_1, \dots, p_n \in \operatorname{irr}(R)$ $(n \geq 3)$. Since R is atomic, unique factorization of x implies r and s factor "uniquely" into products of associates of some of the irreducibles p_i. If r and s are not associates in R then it can be assumed, without loss of generality, that p_1 divides r but does not divide s. Thus, if $r \neq p_1$ then p_1 is an element of $N(s) \setminus \{r\}$ that is not adjacent to r in $\Gamma_x^{d^\times}$. But if $r = p_1$ then it can be assumed, without loss of generality, that p_2 divides s, and it follows that either p_2 or $p_2 p_3$ is an element of $N(r) \setminus \{s\}$ that is not adjacent to s in $\Gamma_x^{d^\times}$. This shows that $N(r) \setminus \{s\} = N(s) \setminus \{r\}$ only if r and s are associates, and the converse holds by Lemma 4(2). $\square$

The illustrations given in Example 3 are perhaps not very surprising since $U(R) = \{1\}$ (i.e., no two distinct elements are associates) in both cases. However, Theorem 12 and Corollary 2 show that if $[r]$ contains two elements that are not associates then it is necessarily true that $|[r]| = 2$ and $U(R) = \{1\}$ (in particular, R has characteristic equal to 2). Furthermore, in the case where the subgraph induced by $[r] = \{r, s\}$ is totally disconnected, if r and s are not associates then the following lemma shows that either $r^2 = x$ or $s^2 = x$ (and they cannot both equal x since if $r^2 = x = s^2$ then the containment $r \in \{-s, s\}$ implies r and s are associates). In this sense, Example 3 turns out to illustrate the unfavorable situation in a rather general way.

Lemma 5 *Suppose that R is an integral domain, and let $x \in R'$. Assume that $r, s \in V(\Gamma_x^{d^\times})$ such that $[r] = [s]$ induces a totally disconnected subgraph of $\Gamma_x^{d^\times}$. If r and s are not associates, then $x \in \{r^2, s^2\}$.*

Proof Suppose that $r^2 \neq x$. Since $r \in d(x)^\times$, there exists $t \in d(x)^\times$ such that $rt = x$. Note that $t \neq r$ since $r^2 \neq x$, so r and t are adjacent in $\Gamma_x^{d^\times}$. But $t \neq s$ since $rs \notin d(x)$ (as r and s are vertices of a totally disconnected subgraph), so the equality $[r] = [s]$ implies that s and t are adjacent in $\Gamma_x^{d^\times}$. Hence, there exists $q \in R$ such that $stq = x$. Thus, to prove that $s^2 = x$, it remains to show that $tq = s$.

To the contrary, suppose that $tq \neq s$. In particular, the equality $stq = x$ implies that s and tq are adjacent in $\Gamma_x^{d^\times}$. Also, $tq \neq r$ since r and s are vertices of a totally disconnected subgraph. Since $[r] = [s]$, it follows that r and tq are adjacent in $\Gamma_x^{d^\times}$. Hence, there exists $p \in R$ such that $r(tq)p = x$, i.e., $rtqp = rt$. Thus, $qp = 1$, so

$q \in U(R)$. But the equality $rt = x = stq$ implies that $r = sq$, and this contradicts the hypothesis that r and s are not associates. Therefore, $tq = s$. $\square$

The next result gives the analogue of Lemma 5 for the case where $[r]$ induces a complete subgraph of $\Gamma_x^{d^\times}$.

Lemma 6 *Suppose that R is an integral domain, and let $x \in R'$. If $[r]$ induces a clique of $\Gamma_x^{d^\times}$ and $|[r]| \geq 3$ then $s^2 \in d(x)$ for every $s \in [r]$.*

Proof Let $s \in [r]$, and pick $q, t \in [r]$ such that q, s, and t are mutually distinct. Since $[r]$ induces a clique, s is adjacent to both q and t. Hence, there exist $j, k \in R$ such that $sqj = x = stk$. Since $q \neq t$, we have $sq \neq st$, so either $j \neq 1$ or $k \neq 1$. Without loss of generality, assume that $j \neq 1$. In particular, $s \neq sj$.

If $q = sj$ then $s^2 j^2 = sqj = x$, and hence $s^2 \in d(x)$. If $q \neq sj$ then q and sj are adjacent in $\Gamma_x^{d^\times}$, and it follows that s and sj are adjacent since $s \neq sj$ and $[s] = [q]$. Thus, $s^2 j = s(sj) \in d(x)$, and therefore $s^2 \in d(x)$. $\square$

The preparation to begin establishing the connection between $\equiv$ and the "associate relation" is now in place. The next lemma handles the case when $[r]$ induces a complete subgraph, and then the full result is provided in the following theorem.

Lemma 7 *Suppose that R is an integral domain, and let $x \in R'$. Assume that $r \in V(\Gamma_x^{d^\times})$ such that $|[r]| \geq 3$. If $[r]$ induces a clique of $\Gamma_x^{d^\times}$ then r and s are associates for every $s \in [r]$.*

Proof Let $s \in [r]$. By Lemma 6, there exist $j, k \in R$ such that $r^2 j = x = s^2 k$. If $r^2 = s^2$ then $r \in \{-s, s\}$, and the result follows. Also, if $s = rj$ then $xjk = (r^2 j)jk = (rj)^2 k = s^2 k = x$, and thus $jk = 1$. In particular, $j \in U(R)$, so r and s are associates. Similarly, the result holds if $r = sk$. Henceforth, assume that $r^2 \neq s^2$, $s \neq rj$, and $r \neq sk$.

Since $r^2 \neq s^2$, either $j \neq 1$ or $k \neq 1$. Without loss of generality, assume that $j \neq 1$. Thus, $r \neq rj$, and hence r and rj are adjacent in $\Gamma_x^{d^\times}$. But $s \neq rj$, so the equality $[r] = [s]$ implies that s and rj are adjacent. It follows that there exists $q \in R$ such that $rsjq = x$. That is, $rsjq = r^2 j$, and therefore $r = sq \in (s)$.

By similar reasoning as above, if $k \neq 1$ then $s \in (r)$. But if $k = 1$, i.e., if $s^2 = x$, then $rsjq = x = s^2$ implies $s = rjq \in (r)$. In either case, the containments $r \in (s)$ and $s \in (r)$ hold. Therefore, r and s are associates. $\square$

Theorem 12 *Suppose that R is an integral domain, and let $x \in R'$. If $r \in V(\Gamma_x^{d^\times})$ and $|[r]| \geq 3$ then r and s are associates for every $s \in [r]$.*

Proof Let $s \in [r]$. The desired outcome is trivial if $r = s$, so assume $r \neq s$. By Lemmas 4(1) and 7, it only remains to verify the result for the case where $[r]$ induces a totally disconnected subgraph of $\Gamma_x^{d^\times}$. In particular, by Lemma 5, if $x \notin \{r^2, s^2\}$ then there is nothing left to prove. Therefore, it is enough to verify that the condition "$x \in \{r^2, s^2\}$" leads to a contradiction.

Without loss of generality, assume that $r^2 = x$. Let $t \in [r] \setminus \{r, s\} = [s] \setminus \{r, s\}$. If s and t are not associates then, by Lemma 5, there exists $q \in \{s, t\}$ such that $q^2 = x$.

Then $q^2 = r^2$, so $q = -r$ (since $q \neq r$). But this means $rq(-1) = r^2 = x$, which implies that r and q are adjacent in $\Gamma_x^{d^\times}$. This contradicts that $[r]$ induces a totally disconnected subgraph of $\Gamma_x^{d^\times}$, so s and t are associates.

Let $u \in U(R)$ such that $s = ut$. Then $u \neq 1$ since $s \neq t$, so $r \neq ur$. But $r(ur)u^{-1} = r^2 = x$, which implies r and ur are adjacent in $\Gamma_x^{d^\times}$. By Lemma 4(2), this contradicts that $[r]$ induces a totally disconnected subgraph, and the result follows. □

This subsection is closed with a corollary to further demonstrate that the occurrence of nonassociate elements in $[r]$ is a rather special situation.

Corollary 2 *Suppose that R is an integral domain such that $U(R) \neq \{1\}$, and let $x \in R'$. If $r, s \in V(\Gamma_x^{d^\times})$ then $[r] = [s]$ if and only if r and s are associates. In particular, this is the case if the characteristic of R is not equal to* 2.

Proof If r and s are associates then $[r] = [s]$ by Lemma 4(2). Conversely, suppose that $[r] = [s]$. If $s \in rU(R)$ then there is nothing left to show, so assume $s \notin rU(R)$. If $u \in U(R) \setminus \{1\}$ then $ur \neq r$, so ur, r, and s are distinct elements of $[r]$. Thus, $|[r]| \geq 3$, and it follows that r and s are associates by Theorem 12.

The "in particular" statement holds since if the characteristic of R is not equal to 2 then $-1 \in U(R) \setminus \{1\}$. □

3.2 FFDs and UFDs

Let R be an integral domain. Now, the results of Sect. 3.1 are applied to obtain purely graph-theoretic characterizations of FFDs and UFDs. Statements that can be applied to both $(\Gamma_E)_x^{d^\times}$ and $\widetilde{\Gamma}_x^{d^\times}$ will often be made simultaneously. For this, it will be notationally convenient to regard $(\Gamma_E)^{d^\times}$ and $\widetilde{\Gamma}^{d^\times}$ as functions on the set $R \setminus \{0\}$, where $(\Gamma_E)^{d^\times}: R \setminus \{0\} \to \{(\Gamma_E)_x^{d^\times} \mid x \in R \setminus \{0\}\}$ is defined by the rule $x \mapsto (\Gamma_E)_x^{d^\times}$, and $\widetilde{\Gamma}^{d^\times}: R \setminus \{0\} \to \{\widetilde{\Gamma}_x^{d^\times} \mid x \in R \setminus \{0\}\}$ is defined by the rule $x \mapsto \widetilde{\Gamma}_x^{d^\times}$. Thus, if $G \in \{(\Gamma_E)^{d^\times}, \widetilde{\Gamma}^{d^\times}\}$ and $x \in R \setminus \{0\}$ then we shall write $G_x = (\Gamma_E)_x^{d^\times}$ whenever $G = (\Gamma_E)^{d^\times}$, and otherwise $G_x = \widetilde{\Gamma}_x^{d^\times}$.

It is shown in [25, Proposition 3.1] that an atomic integral domain R is an FFD if and only if the irreducible divisor graph of x is finite for every $x \in R \setminus (U(R) \cup \{0\})$. By considering the graphs $(\Gamma_E)_x^{d^\times}$ and $\widetilde{\Gamma}_x^{d^\times}$, we can drop the "atomic" hypothesis, and the irreducible elements of R do not have to be acknowledged. It will be useful to note that Lemma 4(2) implies that if $r, s \in d(x)^\times$ such that $[r]$ and $[s]$ are distinct then $\overline{r}$ and $\overline{s}$ are distinct. In particular, the inequality $|V(\widetilde{\Gamma}_x^{d^\times})| \leq |V((\Gamma_E)_x^{d^\times})|$ holds, and if $[r]$ and $[s]$ are adjacent in $\widetilde{\Gamma}_x^{d^\times}$ then $\overline{r}$ and $\overline{s}$ are adjacent in $(\Gamma_E)_x^{d^\times}$.

Theorem 13 *Suppose that R is an integral domain, and let $G \in \{(\Gamma_E)^{d^\times}, \widetilde{\Gamma}^{d^\times}\}$. The following statements are equivalent.*

1. *R is an FFD.*
2. *G_x is finite for every $x \in R \setminus \{0\}$.*

3. *G_x has finite clique number for every $x \in R \setminus \{0\}$.*
4. *G_x has no infinite clique for every $x \in R \setminus \{0\}$.*

Proof It is clear that (1) implies (2) since G_x is null (hence finite) if $x \in \text{irr}(R) \cup U(R)$, and if $x \in R'$ then $(\Gamma_E)_x^{d^\times}$ (and hence $\widetilde{\Gamma}_x^{d^\times}$) is finite since x has only finitely many nonassociate irreducible divisors. Also, the implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial. It remains to prove that (4) implies (1).

By Lemma 4(2) and Theorem 12, it is straightforward to check that $(\Gamma_E)_x^{d^\times}$ has an infinite clique if and only if $\widetilde{\Gamma}_x^{d^\times}$ has an infinite clique, so it is sufficient to verify the result for $G = (\Gamma_E)^{d^\times}$. Hence, suppose that $(\Gamma_E)_x^{d^\times}$ has no infinite clique for every $x \in R \setminus \{0\}$. To the contrary, let S be an infinite set of nonassociate irreducible divisors of an element $x \in R \setminus \{0\}$. Then the set $\{\overline{s} \mid s \in S\}$ is infinite, and if $s_1, s_2 \in S$ with $\overline{s_1} \neq \overline{s_2}$ then $\overline{s_1}$ is adjacent to $\overline{s_2}$ in the graph $(\Gamma_E)_{x^2}^{d^\times}$. Thus, $\{\overline{s} \mid s \in S\}$ induces an infinite clique of $(\Gamma_E)_{x^2}^{d^\times}$. This is a contradiction, so the result follows. □

The remainder of this section is devoted to the graph-theoretic properties of UFDs. The following result shows that if irreducible elements can be distinguished then a characterization is readily obtained. Note that if R is a UFD then R is an FFD, so G_x is finite for every $x \in R'$ by Theorem 13. Also, the assumption "G_x is finite for every $x \in R'$" implies that R is an FFD by Theorem 13 (in particular, R is atomic), so Proposition 7 easily follows by the Coykendall–Maney result (Theorem 1). An independent proof is provided below.

Proposition 7 *Suppose that R is an integral domain, and let $G \in \{(\Gamma_E)^{d^\times}, \widetilde{\Gamma}^{d^\times}\}$. Then R is a UFD if and only if, for every $x \in R'$, the graph G_x is finite, and every set of nonassociate irreducible divisors of x induces a clique of $\Gamma_x^{d^\times}$.*

Proof By Theorem 13, the conditions of the "if" statement imply that R is atomic, so it is sufficient to verify that irreducible elements of R are prime. For this, suppose that $p \in R$ is irreducible, and $p \in d(ab)$ for some $a, b \in R \setminus (U(R) \cup \{0\})$. Since R is atomic, there exist factorizations $a = q_1 \ldots q_m$ and $b = q_{m+1} \ldots q_n$ ($m \geq 1$) where every q_i is irreducible. Since nonassociate irreducible divisors of ab are assumed adjacent in $\Gamma_{ab}^{d^\times}$, if p and q_1 are not associates then $pq_1 \in d(ab)$. Thus, $p \in d(q_2 \ldots q_n)$. If $n = 2$ then clearly p and q_2 are associates. If $n > 2$ then either p and q_2 are associates, or the hypothesis implies that p and q_2 are adjacent in $\Gamma_{q_2 \ldots q_n}^{d^\times}$. This process will terminate, so p and q_i are associates for some i; that is, $p \in d(a)$ or $p \in d(b)$, and it follows that R is a UFD.

To verify the "only if" statement, suppose that R is a UFD and $x \in R'$. Then R is an FFD, so G_x is finite by Theorem 13. Also, by "unique factorization," if $A \subseteq d(x)$ is a set of nonassociate irreducible elements then $\prod A \in d(x)$, and therefore A induces a clique of $\Gamma_x^{d^\times}$. □

The next lemma takes steps toward abstracting the ring-theoretic information in the hypothesis of Proposition 7. The remaining arguments are rather technical, but they serve to finalize the claims of Sect. 1.4.

Lemma 8 *Let R be an atomic integral domain. If $x \in R'$ then the following statements hold.*

1. *If $\widetilde{\Gamma}_x^{d^\times}$ contains a dominant clique then there exists $Q \subseteq \operatorname{irr}(R)$ and a dominant clique K of $\widetilde{\Gamma}_x^{d^\times}$ such that $V(K) = \{[q] \mid q \in Q\}$.*
2. *Suppose that $q_1 \in \operatorname{irr}(R) \cap d(x)$ such that $\{[q_1]\}$ induces a dominant clique of $\widetilde{\Gamma}_x^{d^\times}$. If $p \in \operatorname{irr}(R) \cap d(x)$ such that $p \neq q_1$ and $pq_1 \notin d(x)$ then there exists $q_2 \in \operatorname{irr}(R) \cap d(x)$ with $[q_1] \neq [q_2]$ such that p and q_2 are not associates and $\{[q_1], [q_2]\}$ induces a dominant clique of $\widetilde{\Gamma}_x^{d^\times}$.*
3. *Suppose that R is a BFD. If $Q \subseteq \operatorname{irr}(R)$ such that $\{[q] \mid q \in Q\}$ induces a dominant clique K of $\widetilde{\Gamma}_x^{d^\times}$ with $|V(K)| \geq 2$ then for every nonempty multiset S of elements of Q with $\prod S \in d(x)$ there exist $u \in U(R)$ and a (possibly empty) multiset T of elements of Q such that $x = u \prod(S \sqcup T)$.*

Proof (1) Let C be a dominant clique of $\widetilde{\Gamma}_x^{d^\times}$; say $V(C) = \{C_i \mid i \in I\}$ (I an indexing set). For every $i \in I$, pick $s \in C_i$ and let $q_i \in \operatorname{irr}(R) \cap d(s)$ (such q_i exists since R is atomic and $s \in d(x)^\times$). Set $Q = \{q_i \mid i \in I\}$, and let K be the subgraph of $\widetilde{\Gamma}_x^{d^\times}$ induced by $\{[q] \mid q \in Q\}$.

To show that K is dominant, let $r \in d(x)^\times$ such that $[r] \notin V(K)$. If there does not exist $i \in I$ such that $[r]$ is adjacent to C_i then, evidently (since C is a dominant clique), $V(C) = \{[r]\}$. Hence, $[r]$ is adjacent to every element of $V(\widetilde{\Gamma}_x^{d^\times}) \setminus \{[r]\}$ and, in particular, it is adjacent to every element of $V(K)$.

Suppose that $[r]$ is adjacent to C_i for some $i \in I$. Then $rs \in d(x)$ for every $s \in C_i$. But $q_i \in d(s)$ for some $s \in C_i$, so $rq_i \in d(rs) \subseteq d(x)$. Hence, $[r]$ is adjacent to $[q_i]$ (recall that $[r] \neq [q_i]$ since $[r] \notin V(K)$ by assumption). Therefore, K is dominant.

To show that K is a clique, let $[q_i], [q_j] \in V(K)$ be distinct (in particular, $i, j \in I$ are distinct). By the definition of q_i and q_j, there exist $r \in C_i$ and $s \in C_j$ such that $q_i \in d(r)$ and $q_j \in d(s)$. Note that $[r] = C_i \neq C_j = [s]$ since $i \neq j$, so $rs \in d(x)$ (since C is a clique). Hence, $q_i q_j \in d(rs) \subseteq d(x)$, which shows that $[q_i]$ and $[q_j]$ are adjacent in $\widetilde{\Gamma}_x^{d^\times}$. Therefore, (1) holds.

(2) Let $p \in \operatorname{irr}(R) \cap d(x)$ such that $p \neq q_1$ and $pq_1 \notin d(x)$. Note that $[p]$ and $[q_1]$ are not adjacent since $pq_1 \notin d(x)$, so the equality $[p] = [q_1]$ must hold by the dominance of $\{[q_1]\}$. Moreover, since $\{[q_1]\}$ already induces a dominant clique, $\{[q_1], [q_2]\}$ will also induce a dominant clique for every $q_2 \in \operatorname{irr}(R) \cap d(x)$. Furthermore, if $[q_1] \neq [q_2]$ (i.e., if $[p] \neq [q_2]$) then p and q_2 are not associates by Lemma 4(2). Thus, it is sufficient to show that there exists $q_2 \in \operatorname{irr}(R) \cap d(x)$ with $[q_1] \neq [q_2]$.

If no such q_2 exists then $\operatorname{irr}(R) \cap d(x) \subseteq [q_1]$. But $x \in R'$ (in particular, x is not irreducible), so if the elements of $\operatorname{irr}(R) \cap d(x)$ were associates then $pq_1 \in d(x)$. This is a contradiction, so it follows by Theorem 12 and Corollary 2 that $[p] = [q_1] = \{p, q_1\}$ and $U(R) = \{1\}$. Therefore, $\operatorname{irr}(R) \cap d(x) = \{p, q_1\}$ and $U(R) = \{1\}$, so the assumption $pq_1 \notin d(x)$ implies that $x = p^m = q_1^n$ for some $m, n \geq 2$.

If $m = n = 2$ then $p \in \{-q_1, q_1\}$ (which implies $p = q_1$ since $U(R) = \{1\}$), contradicting that $p \neq q_1$. Hence, without loss of generality, assume that $m \geq 3$. Then p and p^2 are adjacent in $\Gamma_x^{d^\times}$, so the equality $[p] = [q_1]$ implies that q_1 and p^2 are adjacent in $\Gamma_x^{d^\times}$. This contradicts that $pq_1 \notin d(x)$, and the result follows.

(3) Let $S \neq \emptyset$ be a multiset of elements of Q such that $\prod S \in d(x)$. Consider the (possibly empty) multisets T consisting only of elements from Q such that $\prod(S \sqcup T) \in d(x)$. Since R is a BFD, such a T can be chosen "maximally" so that $q \prod(S \sqcup T) \notin d(x)$ for every $q \in Q$. But if $r \in R \setminus U(R)$ with $x = r \prod(S \sqcup T)$ then $[\prod(S \sqcup T)] \in V(\widetilde{\Gamma}_x^{d\times})$, from which it follows by the "dominant clique" hypothesis that $[\prod(S \sqcup T)]$ is adjacent to $[q]$ for some $q \in Q$ (and this is also where the assumption "$|V(K)| \geq 2$" is needed). This contradicts the maximality of T, so the containment $\prod(S \sqcup T) \in d(x)$ implies there exists $u \in U(R)$ such that $x = u \prod (S \sqcup T)$. □

Finally, we present a proof of Theorem 7.

Proof of Theorem 7. Suppose that R is a UFD, and let $\{p_i R \mid i \in J\}$ (J an indexing set) be the set of principal prime ideals of R (with $p_i R \neq p_j R$ if $i \neq j$). As R is a UFD, recall that the mapping $\varphi : (R \setminus \{0\})/U(R) \to \bigoplus_J \mathbb{Z}_{\geq 0}$ by $(\prod_J p_i^{a_i})U(R) \mapsto (a_i \mid i \in J)$ (with $a_i = 0$ for all but finitely many $i \in J$) is an isomorphism of partially ordered monoids. If $x \in R \setminus \{0\}$ such that $(\Gamma_E)_x^{d\times}$ is not null (i.e., $x \in R'$), say $u \in U(R)$ with $x = u \prod_J p_i^{a_i}$, then it is clear that $\alpha' = (a_i \mid i \in J)$ is a non-minimal nonzero element of $\bigoplus_J \mathbb{Z}_{\geq 0}$, and the restriction of φ to $\{rU(R) \mid r \in d(x)^\times\}$ induces a graph-isomorphism between $(\Gamma_E)_x^{d\times}$ and $\Gamma_{\leq \alpha'}(\bigoplus_J \mathbb{Z}_{\geq 0})$. Furthermore, let $I = \{i \in J \mid a_i \neq 0\}$ and $\alpha = (a_i \mid i \in I)$. Then I is finite, α is a non-minimal nonzero element of $\bigoplus_I \mathbb{Z}_{\geq 0}$, and the natural projection $V(\Gamma_{\leq \alpha'}(\bigoplus_J \mathbb{Z}_{\geq 0})) \to V(\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0}))$ defined by $(r_i \mid i \in J) \mapsto (r_i \mid i \in I)$ is trivially an isomorphism between $\Gamma_{\leq \alpha'}(\bigoplus_J \mathbb{Z}_{\geq 0})$ and $\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$. Hence, (1) implies (2).

Assume (2), and that $x \in R \setminus \{0\}$ such that $(\Gamma_E)_x^{d\times}$ is not null. If $\alpha \in \bigoplus_I \mathbb{Z}_{\geq 0}$ such that $(\Gamma_E)_x^{d\times} \cong \Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$ then it is clear that the minimal elements of $(\mathbf{0}, \alpha)$ (that is, the elements less than α with a 1 in exactly one coordinate and 0 elsewhere) induce a dominant clique of $\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$. Also, $(\mathbf{0}, \alpha)$ is finite (i.e., $\Gamma_{\leq \alpha}(\bigoplus_I \mathbb{Z}_{\geq 0})$ is a finite graph) since I is finite, so (2) implies (3).

If $r, s \in d(x)^\times$ such that $\overline{r}$ and $\overline{s}$ are adjacent in $(\Gamma_E)_x^{d\times}$ then either $[r] = [s]$, or $[r]$ and $[s]$ are adjacent in $\widetilde{\Gamma}_x^{d\times}$. Thus, it is straightforward to check that (3) implies (4). Note, however, that the converse does not follow as trivially (e.g., if $R = \mathbb{Z}_2[Y]$ then $\{[Y^2]\}$ induces a dominant clique of $\widetilde{\Gamma}_{Y^4}^{d\times}$, but $\{\overline{Y^2}\}$ does not induce a dominant clique of $(\Gamma_E)_{Y^4}^{d\times}$). Hence, we will complete the proof by showing that (4) implies (1).

Suppose that $\widetilde{\Gamma}_x^{d\times}$ is either a null graph or a finite graph that contains a dominant clique for every $x \in R \setminus \{0\}$. To the contrary, assume that R is not a UFD. Proposition 7 implies that there exist $x \in R'$ and a set of nonassociate irreducible divisors of x that does not induce a clique of $\Gamma_x^{d\times}$. Since $x \in R'$ (so that $\widetilde{\Gamma}_x^{d\times}$ is not null), the hypothesis implies $\widetilde{\Gamma}_x^{d\times}$ contains a dominant clique.

By Lemma 8(1), there exists a set $Q \subseteq \text{irr}(R)$ such that $\{[q] \mid q \in Q\}$ induces a dominant clique K of $\widetilde{\Gamma}_x^{d\times}$. Since K is a clique, it can be assumed (by omitting redundant elements of Q when necessary) that any two distinct elements of Q are adjacent in $\Gamma_x^{d\times}$. Also, since K is finite, Q can be assumed finite; say $Q = \{q_1, \ldots, q_n\}$.

Since the subgraph of $\Gamma_x^{d\times}$ induced by Q is a clique, the choice of x implies that there exists an irreducible $p \in d(x)$ such that p and q are not associates for every

$q \in Q$. If p and q are adjacent in $\Gamma_x^{d^\times}$ for every $q \in Q$ then redefine Q (and hence K, possibly) by letting $Q = \{p, q_1, \ldots, q_n\}$, and note that the above conditions on Q and K are still satisfied. By relabeling, set $Q = \{q_1, \ldots, q_n\}$.

Note that R is an FFD by Theorem 13. Since x has only finitely many nonassociate irreducible divisors, the choice of x guarantees that iterating the above process will eventually yield an irreducible $p \in d(x)$ that is not adjacent to q in $\Gamma_x^{d^\times}$ for some $q \in Q$. Therefore, it can be assumed that $q_1, \ldots, q_n \in \operatorname{irr}(R)$ such that $\{[q_1], \ldots, [q_n]\}$ induces a dominant clique K in $\widetilde{\Gamma}_x^{d^\times}$, the elements p and q_i are nonassociate for every $i \in \{1, \ldots, n\}$, and $pq_j \notin d(x)$ for some $j \in \{1, \ldots, n\}$. Moreover, if $|V(K)| = 1$ then Lemma 8(2) supplies an extension of K to a dominant clique of order 2 on which all of the above conditions are preserved. Thus, it can be assumed that $|V(K)| \geq 2$.

Consider the nonempty multisets S consisting only of elements from $\{q_1, \ldots, q_n\}$ such that $p \prod S \in d(x)$ (and note that such a nonempty S exists since K is dominant with $|V(K)| \geq 2$). Since R is an FFD (and hence a BFD), such an S can be chosen "maximally" so that $pq_i \prod S \notin d(x)$ for every $i \in \{1, \ldots, n\}$. If $p \prod S$ is not an associate of x then $[p \prod S] \in V(\widetilde{\Gamma}_x^{d^\times})$. But this contradicts that K is a dominant clique since $pq_i \prod S \notin d(x)$ for every $i \in \{1, \ldots, n\}$. Thus, $x = pv \prod S$ for some $v \in U(R)$.

By Lemma 8(3), $x = u \prod (S \sqcup T)$ for some $u \in U(R)$ and multiset T of elements of Q, and it follows that $p = w \prod T$ where $w = v^{-1}u$. This implies that if $T = \emptyset$ then $p = w \in U(R)$, if $|T| = 1$ then p is associate to an element of Q, and if $|T| \geq 2$ then $p \notin \operatorname{irr}(R)$. This is a contradiction, and therefore R is a UFD. □

References

1. D.D. Anderson, M. Naseer, Beck's coloring of a commutative ring. J. Algebra **159**, 500–514 (1993)
2. D.D. Anderson, D.F. Anderson, M. Zafrullah, Factorization in integral domains. J. Pure Appl. Algebra **69**, 1–19 (1990)
3. D.D. Anderson, D.F. Anderson, M. Zafrullah, Rings between $D[X]$ and $K[X]$. Houst. J. Math. **17**, 109–129 (1991)
4. D.D. Anderson, J. Coykendall, L. Hill, M. Zafrullah, Monoid domain constructions of antimatter domains. Commun. Algebra **35**, 3236–3241 (2007)
5. D.F. Anderson, A. Badawi, The total graph of a commutative ring. J. Algebra **320**, 2706–2719 (2008)
6. D.F. Anderson, A. Badawi, The zero-divisor graph of a commutative semigroup: a survey, in *Groups, Modules, and Model Theory-Surveys and Recent Developments*, ed. by M. Droste, L. Fuchs, B. Goldsmith, L. Strüngmann (Springer, Berlin, 2017), pp. 23–39
7. D.F. Anderson, J.D. LaGrange, Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. J. Pure Appl. Algebra **216**, 1626–1636 (2012)
8. D.F. Anderson, J.D. LaGrange, Abian's poset and the ordered monoid of annihilator classes in a reduced commutative ring. J. Algebra Appl. **13**, 1450070(18 pp.) (2014)
9. D.F. Anderson, J.D. LaGrange, Some remarks on the compressed zero-divisor graph. J. Algebra **447**, 297–321 (2016)
10. D.F. Anderson, E.F. Lewis, A general theory of zero-divisor graphs over a commutative ring. Int. Electron. J. Algebra **20**, 111–135 (2016)

11. D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring. J. Algebra **217**, 434–447 (1999)
12. D.F. Anderson, D. Weber, The zero-divisor graph of a commutative ring without identity. Int. Electron. J. Algebra **23**, 176–202 (2018)
13. D.F. Anderson, R. Levy, J. Shapiro, Zero-divisor graphs, von Neumann regular rings, and Boolean algebras. J. Pure Appl. Algebra **180**, 221–241 (2003)
14. D.F. Anderson, M.C. Axtell, J.A. Stickles Jr., Zero-divisor graphs in commutative rings, in *Commutative Algebra, Noetherian and Non-Noetherian Perspectives*, ed. by M. Fontana, S.-E. Kabbaj, B. Olberding, I. Swanson (Springer, New York, 2011), pp. 23–45
15. M. Axtell, J. Stickles, Irreducible divisor graphs in commutative rings with zero-divisors. Commun. Algebra **36**, 1883–1893 (2008)
16. M. Axtell, M. Baeth, J. Stickles, Irreducible divisor graphs and factorization properties of domains. Commun. Algebra **39**, 4148–4162 (2011)
17. M. Axtell, M. Baeth, J. Stickles, Survey article-graphical representations of factorizations in commutative rings. Rocky Mt. J. Math. **43**, 1–36 (2013)
18. M. Axtell, M. Baeth, J. Stickles, Cut structures in zero-divisor graphs of commutative rings. J. Commut. Algebra **8**, 143–171 (2016)
19. A. Badawi, On the annihilator graph of a commutative ring. Commun. Algebra **42**, 108–121 (2014)
20. A. Badawi, On the dot product graph of a commutative ring. Commun. Algebra **43**, 43–50 (2015)
21. I. Beck, Coloring of commutative rings. J. Algebra **116**, 208–226 (1988)
22. M. Behboodi, Z. Rakeei, The annihilating-ideal graph of commutative rings I. J. Algebra Appl. **10**, 727–739 (2011)
23. B. Bollobás, *Modern Graph Theory* (Springer, New York, 1998)
24. P.M. Cohn, Bézout rings and their subrings. Math. Proc. Camb. Philos. Soc. **64**, 251–264 (1968)
25. J. Coykendall, J. Maney, Irreducible divisor graphs. Commun. Algebra **35**, 885–895 (2007)
26. J. Coykendall, D.E. Dobbs, B. Mullins, On integral domains with no atoms. Commun. Algebra **27**, 5813–5831 (1999)
27. J. Coykendall, S. Sather-Wagstaff, L. Sheppardson, S. Spiroff, On zero divisor graphs, in *Progress in Commutative Algebra II: Closures, Finiteness and Factorization*, ed. by C. Francisco, L.C. Klinger, S.M. Sather-Wagstaff, J.C. Vassilev (de Gruyter, Berlin, 2012), pp. 241–299
28. F.R. DeMeyer, T. McKenzie, K. Schneider, The zero-divisor graph of a commutative semigroup. Semigroup Forum **65**, 206–214 (2002)
29. D.S. Dummit, R.M. Foote, *Abstract Algebra*, 3rd edn. (Wiley, New York, 2004)
30. J. Fröberg, *An Introduction to Gröbner Bases* (Wiley, New York, 1997)
31. R. Gilmer, *Commutative Semigroup Rings* (The University of Chicago Press, Chicago, 1984)
32. R. Halaš, M. Jukl, On Beck's coloring of partially ordered sets. Discret. Math. **309**, 4584–4589 (2009)
33. E. Hashemi, M. Abdi, A. Alhevaz, On the diameter of the compressed zero-divisor graph. Commun. Algebra **45**, 4855–4864 (2017)
34. V. Joshi, S. Sarode, Beck's conjecture and multiplicative lattices. Discret. Math. **338**, 93–98 (2015)
35. C.F. Kimball, J.D. LaGrange, The idempotent-divisor graphs of a commutative ring. Commun. Algebra **46**, 3899–3912 (2018)
36. J.D. LaGrange, The x-divisor pseudographs of a commutative groupoid. Int. Electron. J. Algebra **22**, 62–77 (2017)
37. J. Lambek, *Lectures on Rings and Modules* (Blaisdell Publishing Company, Waltham, 1966)
38. D. Lu, T. Wu, The zero-divisor graphs of partially ordered sets and an application to semigroups. Graph Comb. **26**, 793–804 (2010)
39. X. Ma, D. Wang, J. Zhou, Automorphisms of the zero-divisor graph over 2×2 matrices. J. Korean Math. Soc. **53**, 519–532 (2016)
40. J. Maney, Irreducible divisor graphs II. Commun. Algebra **36**, 3496–3513 (2008)
41. C.P. Mooney, Generalized irreducible divisor graphs. Commun. Algebra **42**, 4366–4375 (2014)

42. S.B. Mulay, Cycles and symmetries of zero-divisors. Commun. Algebra **30**, 3533–3558 (2002)
43. S.P. Redmond, An ideal based zero-divisor graph of a commutative ring. Commun. Algebra **31**, 4425–4443 (2003)
44. S. Spiroff, C. Wickham, A zero divisor graph determined by equivalence classes of zero divisors. Commun. Algebra **39**, 2338–2348 (2011)
45. A. Zaks, Half-factorial domains. Bull. Am. Math. Soc. **82**, 721–724 (1976)

Isomorphisms and Planarity of Zero-Divisor Graphs

Jesse Gerald Smith Jr.

Abstract Let R be a commutative ring with nonzero identity and I a proper ideal of R. The *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the graph on vertices $R^* = R \setminus \{0\}$ where distinct vertices x and y are adjacent if and only if $xy = 0$. The *ideal-based zero-divisor graph* of R with respect to the ideal I, denoted by $\Gamma_I(R)$, is the graph on vertices $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we cover two main topics: isomorphisms and planarity of zero-divisor graphs. For each topic, we begin with a brief overview on past research on zero-divisor graphs. Whereafter, we provide extensions of that material to ideal-based zero-divisor graphs.

1 Preliminaries

Let R be a commutative ring with nonzero identity, I a proper ideal of R, and $Z(R)$ the set of zero-divisors of R. Throughout this paper, a *graph* will always be a simple graph, i.e. an undirected graph without multiple edges or loops. In 1988, I. Beck used zero-divisors to produce a graph given a ring R [8]; he was interested in colorings of these graphs. In 1999, D.F. Anderson and P.S. Livingston modified Beck's definition to the following [4, 13]; the *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the graph on the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, where two distinct vertices x and y are adjacent if and only if $xy = 0$. In 2001, S.P. Redmond gave the following definition [16, 17] as a generalization of the zero-divisor graph; the graph on vertex set $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This is called the *ideal-based zero-divisor graph* of R with respect to the ideal I and is denoted by $\Gamma_I(R)$. Note that $\Gamma_I(R)$ and $\Gamma(R/I)$ are non-empty if and only if I is not a prime ideal of R.

Throughout this paper, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $nil(R)$ its ideal of nilpotent elements, and total quotient ring $T(R) = R_S$, where $S = R \setminus \{0\}$. Given an ideal I of R, we define $\sqrt{I} = \{r \in R \mid$

J. G. Smith Jr. (✉)
Maryville College, 502 E. Lamar Alexendar Pkwy, Maryville, TN 37804, USA
e-mail: jesse.smith@maryvillecollege.edu

A. Badawi and J. Coykendall (eds.), *Advances in Commutative Algebra*,
Trends in Mathematics, https://doi.org/10.1007/978-981-13-7028-1_12

$r^k \in I$ for some $k \in \mathbb{N}\}$. A ring R is reduced if $nil(R) = \sqrt{\{0\}} = \{0\}$. Notice that R/I is reduced if and only if $\sqrt{I} = I$. An ideal I is a radical ideal if $\sqrt{I} = I$. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the integers and the integers modulo n, respectively. A graph G is a pair of sets V and E, of vertices and edges respectively, where E consists of sets $\{a, b\}$ and $a, b \in V$. Graphs are often visualized by drawing the vertices as dots and the edges by lines connecting the dots. Let the graph G be defined by vertex set V and edge set E. Let H be the graph defined by vertex set V' and edge set E'. We say that a graph is complete on n vectices, denoted by K^n, if it is a graph on n vectices in which each vertex is connected to all other vertices. Given a graph G, we will let $V(G)$ be its vertex set and $E(G)$ be its edge set. A *complete bipartite graph G* is a graph for which there exists disjoint non-empty subsets A, B of vertices such that two vertices of G are adjacent if and only if one vertex is in A and the other vertex is B; we denote such a graph by $K^{m,n}$, where $m = |A|$ and $n = |B|$. The *girth* of a graph G, denoted $gr(G)$, is defined to be the length of a shortest cycle in G provided a cycle exists and ∞ otherwise.

2 Isomorphisms

An important concept in abstract algebra is that of isomorphisms. In this section, we study the relationship between ring isomorphisms and graph isomorphisms on two types of zero-divisor graphs ($\Gamma(R)$ and $\Gamma_I(R)$). Recall that a *graph isomorphism* from G to H is a bijection $\phi : V \to V'$ such that $\{\phi(a), \phi(b)\} \in E'$ if and only if $\{a, b\} \in E$. In other words, a graph isomorphism is a bijection between the vertex sets which preserves edges.

2.1 A Brief Survey on Isomorphisms of Zero-Divisor Graphs

We begin by reviewing the research done on graph isomorphisms of zero-divisor graphs by David F. Anderson and his colleagues. It is clear that $R \cong S$ implies that $\Gamma(R) \cong \Gamma(S)$. But does $\Gamma(R) \cong \Gamma(S)$ imply $R \cong S$? It was in Anderson's second paper on zero-divisor graphs that we first saw results on isomorphisms of zero-divisor graphs. From arguably the simplest zero-divisor graph (a graph with a single vertex) it is quickly noted that zero-divisor graphs being isomorphic does not imply that the corresponding rings are isomorphic. The authors (D.F. Anderson, A Frazier, A. Lauve, and P.S. Livingston) go on to find a case when the answer to our question is yes. The key to the following result involves being able to express the rings as a nontrivial product of finite fields.

Theorem 2.1 ([2]) *Let R and S be commutative reduced rings that are not fields. Then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.*

In the same paper, the authors go on to find the finite hypothesis in the preceding result is required. Their example being the rings $R = \mathbb{Z}_2 \times \mathbb{Z}$ and $S = \mathbb{Z}_2 \times \mathbb{Q}$. Here both rings have zero-divisor graph $K^{1,\aleph_0}$ but $R \ncong S$.

In 2003, D.F. Anderson, R. Levy, and J. Shapiro, proved a rather interesting result. Recall that given a ring R we denote its total quotient ring by $T(R)$. The result is as follows. It has been reported that the proof of this theorem is one of D.F. Anderson's favorites (being referred to as rather beautiful and elegant). The proof builds a graph isomorphism beginning with an equivalence relation which give rise to equivalence classes and eventually the isomorphism.

Theorem 2.2 ([3]) *Let R be a commutative ring with total quotient ring $T(R)$. Then $\Gamma(T(R)) \cong \Gamma(R)$.*

Research by J.D. LaGrange found another case in which the answer to our original question is yes. In one paper, LaGrange considers the concepts of complements, ends, and isomorphisms (among others). The proof for the following results utilizes a finding from [3].

Theorem 2.3 ([12]) *Let R and S be Boolean rings, then $\Gamma(R) \cong \Gamma(S) \Leftrightarrow R \cong S$.*

We know turn our attention to the ideal-based zero-divisor graph. Recall that the ideal-based zero-divisor graph of a commutative ring R with ideal I is the graph on vertex set $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$.

2.2 Isomorphisms of $\Gamma_I(R)$

In this section, we consider the nature of isomorphisms on the ideal-based zero-divisor graph. It is evident that $R \cong S$ does not imply $\Gamma_I(R) \cong \Gamma_J(S)$ (as the ideals I and J could vary in size or structure). Properties of the ideal-based zero-divisor graph have been studied by various authors. In both [16, 17], Redmond notes a strong connection between $\Gamma_I(R)$ and $\Gamma(R/I)$. In his early papers, Redmond describes a three-step construction method for $\Gamma_I(R)$ based on $\Gamma(R/I)$. Notice that the key factors in the construction method are $\Gamma(R/I)$, $|I|$, and the concept of connected columns.

The motivation for our research in this topic begins with a Theorem published in 2006 [14, Theorem 2.2]. It was stated as follows:

Let I be a finite ideal of R and J be a finite ideal of S such that $I = \sqrt{I}$ and $J = \sqrt{J}$. Then the following hold:

(a) If $|I| = |J|$ and $\Gamma(R/I) \cong \Gamma(S/J)$, then $\Gamma_I(R) \cong \Gamma_J(S)$.
(b) If $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$.

Remark 2.1 We first note that (b) of the preceding result does not hold. Consider the following example. Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $I = 0$. Let $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and

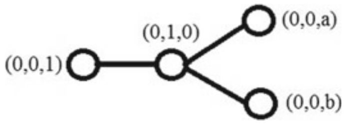


Fig. 1 $\Gamma(R/I)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = \mathbb{Z}_2 \times 0 \times 0$

$J = \mathbb{Z}_2 \times 0 \times 0$. Then $\Gamma_I(R)$ and $\Gamma_J(S)$ are both 4-cycles, and hence isomorphic. In both cases, I and J are finite radical ideals of their respective rings. However, $\Gamma(R/I)$ is a 4-cycle and $\Gamma(S/J)$ is a line graph on 2 vertices; thus $\Gamma(R/I) \not\cong \Gamma(S/J)$. This example also provides a counterexample to [6, Theorem 5.3] as both I and J are also non-maximal ideals.

The proof of [14, Theorem 2.2(b)] seems to have two shortcomings. The first is in the line: "Now if $a, b \in V(K')$, then $a + J \neq b + J$; otherwise, $a^2 \in J = \sqrt{J}$, and hence $a \in J$, which is a contradiction." It is a common argument in proofs regarding ideal-based zero-divisor graphs that if a, b are adjacent vertices of $\Gamma_J(S)$ and $J = \sqrt{J}$, then $a + J \neq b + J$. However, we do not have here that the two vertices in the argument are necessarily adjacent. By considering the example in the above remark, one can see that two different coset representatives in $\Gamma(R/I)$ may map to equivalent coset representatives in $\Gamma(S/J)$.

The second shortcoming in [14, Theorem 2.2(b)] can be seen by considering Example 2.1. The example shows that the restriction of a graph isomorphism between $\Gamma_I(R)$ and $\Gamma_J(S)$ to a set of coset representatives for $V(\Gamma(R/I))$ may not map to a set of distinct coset representatives of $V(\Gamma(S/J))$.

Example 2.1 Let $R = S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = J = \mathbb{Z}_2 \times 0 \times 0$, where $\mathbb{F}_4 = \{0, 1, a, b\}$ is the field with four elements. For a graph of $\Gamma(R/I)$, see Fig. 1. For a graph of $\Gamma_I(R)$, see Fig. 2. We may choose a complete set of coset representatives for $\Gamma(R/I)$ to be $K = \{(0, 0, 1), (0, 1, 0), (0, 0, a), (0, 0, b)\}$. Then consider the graph isomorphism given by Table 1.

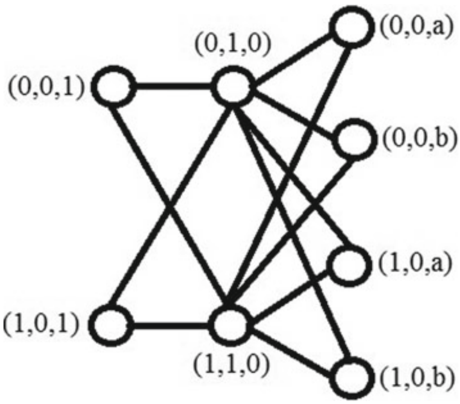


Fig. 2 $\Gamma_I(R)$, where $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ and $I = \mathbb{Z}_2 \times 0 \times 0$

Table 1 A graph isomorphism

$x \in \Gamma_I(R)$	$\phi(x)$
(0, 0, 1)	(1, 0, 1)
(0, 1, 0)	(0, 1, 0)
(0, 0, a)	(0, 0, a)
(0, 0, b)	(1, 0, a)
(1, 0, 1)	(0, 0, 1)
(1, 1, 0)	(1, 1, 0)
(1, 0, a)	(1, 0, b)
(1, 0, b)	(0, 0, b)

We then have that $K' = \phi(K) = \{(1, 0, 1), (0, 1, 0), (0, 0, a), (1, 0, a)\}$. But K' is not a set of **distinct** coset representatives for $\Gamma(S/J)$ as $(0, 0, a) + J = (1, 0, a) + J$ since $(1, 0, 0) \in J$.

We believe that the proposition in question will hold if we assume beforehand that $|I| = |J|$. That is, changing [14, Theorem 2.2(b)] to be "If $|I| = |J|$ and $\Gamma_I(R) \cong \Gamma_J(S)$, then $\Gamma(R/I) \cong \Gamma(S/J)$."

Because of the shortcomings in the proof of [14, Theorem 2.2(b)] and reservations of this author regarding the statement "Part (a) is an easy consequence of Theorem 2.1" [14], we seek to first give a direct proof of part (a) of [14, Theorem 2.2]. We should note that the following results should be fairly intuitive from Redmond's three step construction method for $\Gamma_I(R)$. In this proof, notice the subtle use of the radical ideal hypothesis. It would not be hard for one to construct an incorrect proof overlooking the requirement that the ideals must be radical.

Theorem 1 *Let R and S be commutative rings with nonzero identity and I and J radical ideals of R and S, respectively. If $\Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$, then $\Gamma_I(R) \cong \Gamma_J(S)$.*

Proof Since $\Gamma(R/I) \cong \Gamma(S/J)$, there exists a graph isomorphism $\phi : \Gamma(R/I) \to \Gamma(S/J)$. Let $K = \{a_\lambda\}_{\lambda\in\Lambda}$ be a complete set of distinct coset representatives of $V(\Gamma(R/I))$. Consider $\phi(K) = \{\phi(a_\lambda)\}_{\lambda\in\Lambda}$; this will be a complete set of distinct coset representatives of $V(\Gamma(S/J))$ as $\phi : V(\Gamma(R/I)) \to V(\Gamma(S/J))$ is a bijection. For ease of notation, set $\phi(a_\lambda) = b_\lambda$ and $\phi(K) = \{b_\lambda\}_{\lambda\in\Lambda}$. Since $|I| = |J|$, there exists a bijection $f : I \to J$. Consider the correspondence $\psi : V(\Gamma_I(R)) \to V(\Gamma_J(S))$ given by $\psi(a_\lambda + i) = \phi(a_\lambda) + f(i) = b_\lambda + f(i)$. This correspondence is a well-defined function by [17, Corollary 2.7]; the fact that ψ is onto follows from [17, Corollary 2.7] and that both ϕ and f are onto. Assume that $b_{\lambda_1} + f(i_1) = \phi(a_{\lambda_1} + i_1) = \phi(a_{\lambda_2} + i_2) = b_{\lambda_2} + f(i_2)$. Then $b_{\lambda_1} - b_{\lambda_2} \in J$, and hence $b_{\lambda_1} + J = b_{\lambda_2} + J$. But $\phi(K)$ is a set of distinct coset representatives of $V(\Gamma(S/J))$, and therefore $\lambda_1 = \lambda_2$. It is then evident that $i_1 = i_2$ as f is injective. Therefore ψ is also injective.

We now show that ψ preserves edges. Let r and s be adjacent in $V(\Gamma_I(R))$. Then since I is a radical ideal, $r + I \neq s + I$ and $r + I$ is adjacent to $s + I$ [17, Theorem 2.5]. Since $r + I \neq s + I$, there exist distinct $\lambda_1, \lambda_2 \in \Lambda$ and $i, j \in I$ such that $r = a_{\lambda_1} + i$ and $s = a_{\lambda_1} + j$. Since ϕ is a graph isomorphism, $\phi(r + I) = \phi(a_{\lambda_1} + I) = b_{\lambda_1} + J$ is adjacent to $\phi(s + I) = \phi(a_{\lambda_2} + I) = b_{\lambda_2} + J$. In other words, $b_{\lambda_1} b_{\lambda_2} \in J$. Therefore, $\psi(r) = b_{\lambda_1} + f(i)$ is adjacent to $\psi(s) = b_{\lambda_2} + f(j)$ in $\Gamma_J(S)$. The proof of the reverse direction for edge preservation is similar. Thus, $\psi : \Gamma_I(R) \to \Gamma_J(S)$ is a graph isomorphism. □

If we leave out the radical ideal hypothesis, we know the result does not hold. For the non-radical case, the proof fails when we try to prove that edges are preserved. In particular, if we had an edge created by a connected column, we would not be guaranteed that a corresponding edge exists in the second graph.

We return our focus to finding a converse of this result; we begin by proving a weaker result. Instead of assuming that R/I and S/J are reduced, let us assume that they are Boolean rings. We then quickly get that the desired implication holds. The following Lemma follows from the fact that $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))|$.

Lemma 2.1 *Let R and S be finite commutative rings with nonzero identity and I and J ideals of R and S respectively. If $|I| = |J|$ and $\Gamma_I(R) \cong \Gamma_J(S)$, then $|V(\Gamma(R/I))| = |V(\Gamma(S/J))|$.*

Lemma 2.2 *Let R and S be finite Boolean rings. Then $R \cong S$ if and only if $|Z(R)| = |Z(S)|$.*

Proof The forward direction is evident. So let us prove the converse. It is well known that for finite Boolean rings R and S, we have $R \cong \prod_{i=1}^{m} \mathbb{Z}_2$ and $S \cong \prod_{i=1}^{n} \mathbb{Z}_2$, for $m, n \in \mathbb{Z}^+$. It suffices to show that $m = n$. Assume to the contrary, that is $m \neq n$. Without loss of generality, $m < n$. Then R can be viewed as a subring of S in the natural way, namely $R \cong R' = \prod_{i=1}^{m} \mathbb{Z}_2 \times \prod_{j=m+1}^{n} 0 \subseteq S$. Let $x = (1, 1, 1, \ldots, 1, 0)$. Then $x \in Z(S) \setminus Z(R')$ since $m < n$. Hence $|Z(R)| = |Z(R')| < |Z(S)|$. But this is a contradiction of the hypothesis, and therefore we must have that $m = n$, and hence $R \cong S$. □

Alternative Proof

Proof Since R and S are finite Boolean rings, they are isomorphic to a product of $\mathbb{Z}_2$'s. Thus $|R| = 2^m$ and $|S| = 2^n$. Notice then that $R \cong S$ if and only if $m = n$. Thus $R \cong S$ if and only if $|R| = |S|$. Notice that if $1 \neq x \in R$, then $1 - x \neq 0$ and $x(1 - x) = 0$. Thus $x \in Z(R)$. Since $1 \neq x \in R$ was arbitrary, it follows that $R = Z(R) \cup \{1\}$. Hence $|R| = |S|$ if and only if $|Z(R)| = |Z(S)|$. □

Notice that the argument in the *Alternative Proof* shows that for a Boolean ring R, $R = Z(R) \cup \{1\}$.

Theorem 2 *Let R and S be finite commutative rings with nonzero identities and ideals I and J, respectively. Moreover, assume that R/I and S/J are Boolean and $|I| = |J|$. Then $\Gamma_I(R) \cong \Gamma_J(S)$ implies that $\Gamma(R/I) \cong \Gamma(S/J)$.*

Proof By the Lemma 2.1, $|V(\Gamma(R/I))| = |V(\Gamma(S/J))|$.

Hence $|Z(R/I)| = |V(\Gamma(R/I))| + 1 = |V(\Gamma(S/J))| + 1 = |Z(S/J)|$. Thus $R/I \cong S/J$ by Lemma 2.2, and hence $\Gamma(R/I) \cong \Gamma(S/J)$. □

The preceding arguments gave rise to the following conjecture and its proof. Here we find that if we assume that R and S are finite Boolean rings, then $\Gamma_I(R) \cong \Gamma_J(S) \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$ and $|I| = |J|$.

Theorem 3 *Let R and S be finite Boolean rings with I and J proper non-prime ideals of R and S, respectively. Then $\Gamma_I(R) \cong \Gamma_J(S)$ if and only if $R \cong S$ and $|I| = |J|$.*

Proof Assume that $\Gamma_I(R) \cong \Gamma_J(S)$. Then $|I||V(\Gamma(R/I))| = |J||V(\Gamma(S/J))|$. Since R and S are isomorphic to a direct product of $\mathbb{Z}_2$'s, we have $|R| = 2^m$ and $|S| = 2^n$ for some positive integers m, n.

If either $m = 1$ or $n = 1$, then the corresponding ring(s) will be isomorphic to $\mathbb{Z}_2$, and hence will not contain a proper non-prime ideal; the only proper ideal of $\mathbb{Z}_2$ is $\{0\}$, which is maximal, and hence prime. Thus $m \geq 2$ and $n \geq 2$, and therefore $m - 2 \geq 0$ and $n - 2 \geq 0$.

Moreover, we know that I will be isomorphic to a product of $\mathbb{Z}_2$'s and $\{0\}$'s, whence $|I| = 2^i$ for some $0 \leq i < m$. Similarly, $|J| = 2^j$ for some $0 \leq j < n$. Note that if $i = m - 1$, then R/I will be an integral domain, and hence I a prime ideal. But this is contrary to the hypothesis. Hence $0 \leq i \leq m - 2$; similarly, $0 \leq j \leq m - 2$. Since the number of zero-divisors of a Boolean ring is one less than the cardinality of the ring, we have $|Z(R/I)| = |R/I| - 1 = 2^{m-i} - 1$ and $|Z(S/J)| = 2^{n-j} - 1$. Hence $|V(\Gamma(R/I))| = 2^{m-i} - 2$ and $|V(\Gamma(S/J))| = 2^{n-j} - 2$ with $0 \leq i \leq m - 2$ and $0 \leq j \leq n - 2$ (using that $|Z^*(R)| = |Z(R)| - 1$). Therefore, $|I||V(\Gamma(R/I))| = |J||V(\Gamma(S/J))|$ implies that $2^m - 2^{i+1} = 2^n - 2^{j+1}$.

We claim that $m = n$ and $i = j$. It is evident that $m = n \Leftrightarrow i = j$; hence it suffices to show that $m \neq n$ and $i \neq j$ (along with $0 \leq i \leq m - 2$ and $0 \leq j \leq n - 2$) implies that $2^m - 2^{i+1} \neq 2^n - 2^{j+1}$.

Assume that $m \neq n$, $i \neq j$, $0 \leq i \leq m - 2$ and $0 \leq j \leq n - 2$. Without loss of generality, we may assume that $m < n$. First notice that for all $0 \leq j \leq n - 2$, we have that $2^n - 2^{j+1} \geq 2^n - 2^{n-1}$ (since $f(x) = 2^n - 2^x$ is decreasing). Now $2^n - 2^{n-1} = 2^n(1 - 2^{-1}) = 2^n(2^{-1}) = 2^{n-1}$; thus $2^n - 2^{j+1} \geq 2^{n-1}$ for all $0 \leq j \leq n - 2$. Since $m < n$, we have $2^m < 2^n$, and hence $2^m \leq 2^{n-1}$. Thus $m < n$ implies that $2^m - 2 < 2^m \leq 2^{n-1}$. Hence

$$2^n - 2^{j+1} \geq 2^{n-1} > 2^m - 2 \text{ for all } 0 \leq j \leq m - 2.$$

But $2^m - 2 \geq 2^m - 2^{i+1}$ for all $0 \leq i \leq n - 2$. Thus

$$2^n - 2^{j+1} > 2^m - 2 \geq 2^m - 2^{i+1} \text{ for } 0 \leq i \leq m - 2 \text{ and } 0 \leq j \leq n - 2.$$

Hence $2^n - 2^{j+1} \neq 2^m - 2^{i+1}$ for all $0 \leq i \leq m - 2$ and $0 \leq j \leq n - 2$, as desired.

Thus we must have $m = n$ and $i = j$, whence $|R| = |S|$ and $|I| = |J|$. Since R and S are finite Boolean rings, $|R| = |S| \Rightarrow R \cong S$. Therefore, $R \cong S$ and $|I| = |J|$, as desired.

For the converse, note that Boolean rings are reduced, and thus I and J are radical ideals. The result then follows by Theorem 1. □

Remark 2.2 If we simply assume that at least one of the ideals in the preceding proposition is non-prime, it follows that the ideal-based zero-divisor graph relative to the non-prime ideal will be non-empty. Thus in the forward implication, the other graph is non-empty; therefore we also have the remaining ideal is non-prime. In the reverse implication, $Spec(R) = Max(R)$ and $Spec(S) = Max(S)$. Thus the prime ideals are those that are maximal. By viewing R and S as a product of $\mathbb{Z}_2$'s, it is evident that an ideal I is prime if and only if maximal, if and only if $|I| = |R|/2$. Thus the conditions $|I| = |J|$ and $R \cong S$ ensure that if at least one of the two ideals is prime, then so is the other.

Therefore, although the theorem is true if we only assume that one of the ideals is non-prime, we are not losing generality by assuming both are non-prime.

If both of the ideals are prime, then the theorem does not hold. Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $I = 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S = \mathbb{Z}_2 \times \mathbb{Z}_2$, $J = 0 \times \mathbb{Z}_2$. Then $\Gamma_I(R)$ and $\Gamma_J(S)$ are empty, hence isomorphic; however, $R \ncong S$ and $|I| \neq |J|$.

Remark 2.3 The converse of Theorem 3 does not hold for infinite Boolean rings. Consider $R = S = \prod_{i=1}^{\infty} \mathbb{Z}_2$, where $I = 0 \times 0 \times \prod_{i=3}^{\infty} \mathbb{Z}_2$ and $J = 0 \times 0 \times 0 \times \prod_{i=4}^{\infty} \mathbb{Z}_2$. Then $S/J \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, whence $gr(\Gamma(S/J)) = 3 = gr(\Gamma_J(S))$. However $\Gamma_I(R) \cong K^{\aleph_0,\aleph_0}$; to see this, consider the vertex sets $V = \{(a_i)_{i\in\mathbb{N}} \in R \mid a_1 = 0, a_2 = 1\}$ and $W = \{(a_i)_{i\in\mathbb{N}} \in R \mid a_1 = 1, a_2 = 0\}$.

Here we have that $gr(\Gamma_I(R)) = 4$ and $gr(\Gamma_J(S)) = 3$. Therefore $\Gamma_I(R) \ncong \Gamma_J(S)$; however $R \cong S$ and $|I| = |J|$.

Recall that one of our goals is to determine when the following implication holds: $\Gamma_I(R) \cong \Gamma_J(S) \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$.

We have seen that even in the finite radical case that the above implication does not hold. We then assumed that we need the ideals to have the same cardinality. That is, we hoped to prove the following implication (at least in the reduced case): $\Gamma_I(R) \cong \Gamma_J(S)$ and $|I| = |J| \Rightarrow \Gamma(R/I) \cong \Gamma(S/J)$.

However, the following example dashes the hopes of this holding in the case the ideals are infinite.

Example 2.2 Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, $S = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}$, $I = 0 \times 0 \times \mathbb{Z}$, and $J = 0 \times 0 \times \mathbb{Z}$. Then $|I| = |J|$. Notice that $\Gamma(R/I) \cong K^2$ and $\Gamma(S/J)$ is a 4-cycle; hence $\Gamma(R/I) \ncong \Gamma(S/J)$. However, $\Gamma_I(R) \cong \Gamma_J(S)$ since both graphs are a $K^{\aleph_0,\aleph_0}$.

To see the that $\Gamma_J(S) = K^{\aleph_0,\aleph_0}$, consider the sets $V = \{(0, 1, k) \mid k \in \mathbb{Z}\} \cup \{(0, 2, k) \mid k \in \mathbb{Z}\}$ and $W = \{(1, 0, k) \mid k \in \mathbb{Z}\} \cup \{(2, 0, k) \mid k \in \mathbb{Z}\}$. Notice that no vertex of V is adjacent to any other vertex of V. The same is true of W. However, every vertex of V is adjacent to every vertex of W (and vice-versa). Thus $\Gamma_J(S) = K^{\aleph_0,\aleph_0}$.

Similarly, $\Gamma_I(R) = K^{\aleph_0,\aleph_0}$.

In this example, we have commutative rings R and S with radical ideals I and J, respectively, such that $\Gamma_I(R) \cong \Gamma_J(S)$ and $|I| = |J|$, but $\Gamma(R/I) \ncong \Gamma(S/J)$.

As of yet, there has not been a counterexample for the implication when both ideals are finite. Based of the underlying similarities to Theorem 4.1 from [2], we believe that the implication will hold in this case. As such we close here leaving open the following conjecture.

Conjecture 2.1 *Let R and S be commutative rings with* **finite** *radical ideals I of R and J of S and $|I| = |J|$. Then $\Gamma_I(R) \cong \Gamma_J(S) \Leftrightarrow \Gamma(R/I) \cong \Gamma(S/J)$.*

3 On Planarity of Zero-Divisor Graphs

Recall that a graph G is *planar* if it can be drawn in a plane so that no two edges cross. The following is a well-known result that we use throughout this section.

Lemma 1 *Let R be a commutative local ring with nonzero identity and $|R| = 4$. Then*

$$R \cong \mathbb{Z}_4,\ \mathbb{Z}_2[X]/(X^2),\ \text{or } \mathbb{F}_4.$$

In this paper, one of our goals is to classify when an ideal-based zero-divisor graph is planar. We consider $\Gamma_I(R)$ to be *nontrivial* if I is a nonzero, proper, non-prime ideal of R. The latter requirements forces $\Gamma_I(R)$ to be distinct from $\Gamma(R)$ and to be nonempty. In order to achieve this goal, we will use the celebrated Kurtowski's Theorem from Graph Theory [10, Theorem 6.13]. To state the theorem, we need to define a graph subdivision.

Definition: Let G and H be graphs. Then H is a *subdivision* of G if H can be derived from G by applying the following operations:

1. Adding a vertex on an edge, that is, replacing $v - w$ (vertices v, w are adjacent) by $v - a - w$, where a is a new vertex.
2. Replacing a vertex adjacent to only two vertices by only an edge (undoing item 1).

Theorem 4 (Kuratowski's Theorem) *A graph G is planar if and only if it does not contain a subgraph which is a subdivision of K^5 or $K^{3,3}$.*

A particular case of the above theorem is that a graph containing a K^5 or $K^{3,3}$ is nonplanar. So throughout the following investigation we will be looking for subgraphs (or subdivisions) that are either K^5 or $K^{3,3}$.

3.1 A Brief Survey on the Planarity of Zero-Divisor Graphs

Research on classifying all finite commutative rings with nonzero identity having nonempty planar zero-divisor graph has been done in [1, 2, 9, 19, 20, 22]. Work has also been done regarding when infinite commutative rings have planar zero-divisor graphs [21].

The earliest work on planar zero-divisor graphs can be found in the Anderson, Lauve, Frazier, and Livingston paper from 1999. In this paper, the authors classified when $\mathbb{Z}_n$, $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ (for $r > 2$), and $Z_n[X]/(X^m)$ have planar zero-divisor graphs [2]. A student of David F. Anderson continued this work in the mid 2000s. Neal O. Smith classified all finite commutative rings (up to isomorphism) which have a nontrivial planar zero-divisor graph. The result is as follows:

Theorem 5 ([21]) *Let R be a finite commutative ring that is not a field and K a finite field. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following* 44 *types of rings:*

$$\begin{aligned}
&\mathbb{Z}_2 \times K, \mathbb{Z}_3 \times K, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X)^2,\\
&\mathbb{Z}_3 \times \mathbb{Z}_4, \mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\\
&\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2),\\
&\mathbb{Z}_2 \times \mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2),\\
&\mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2),\\
&\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2), \mathbb{Z}_9, \mathbb{Z}_3[X]/(X)^2,\\
&\mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2 - 2), \mathbb{Z}_{16}, \mathbb{Z}_2[X]/(X^4),\\
&\mathbb{Z}_4[X]/(2X, X^3 - 2), \mathbb{Z}_4[X]/(X^2 - 2), \mathbb{Z}_4[X]/(X^2 + 2X + 2), \mathbb{F}_4[X]/(X^2),\\
&\mathbb{Z}_4[X]/(X^2 + X + 1), \mathbb{Z}_2[X, Y]/(X, Y)^2, \mathbb{Z}_4[x]/(2, X)^2,\\
&\mathbb{Z}_{27}, \mathbb{Z}_3[X]/(X^3), \mathbb{Z}_9[X]/(X^2 - 3, 3X), \mathbb{Z}_9[X]/(X^2 - 6, 3X),\\
&\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY), \mathbb{Z}_2[X, Y]/(X^2, Y^2), \mathbb{Z}_8[X]/(2X - 4, X^2),\\
&\mathbb{Z}_4[X]/(X^2), \mathbb{Z}_4[X]/(X^2 - 2X),\\
&\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y), \mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y),\\
&\mathbb{Z}_{25}, \textit{ or } \mathbb{Z}_5[X]/(X^2).
\end{aligned}$$

Research has also been done on when ideal-based zero-divisor graphs are planar. This work began with [17]. It is from this paper's work that we begin. In 1999, Anderson and Livingston proved that there are arbitrarily large planar zero-divisor graphs. In this 2003 paper, Redmond extends this result to ideal-based zero-divisor graphs. By considering the foundation laid by Redmond, we seek to find all planar ideal-based zero-divisor graphs.

3.2 Restraints on $|I|$ and $gr(\Gamma(R/I))$

We begin by investigating what restraints planarity forces on the graphs of $\Gamma(R/I)$ and $\Gamma_I(R)$. Much like our last topic of research being motivated by another author's claim. Our work on planarity was spurred on by considering a result by Redmond [17, Theorem 7.2]. This will be discussed shortly.

Proposition 1 *Let R be a commutative ring with nonzero identity and I an ideal of R. If $\Gamma_I(R)$ is planar, then $|I| \le 2$ or $|V(\Gamma(R/I))| \le 1$.*

Proof (By Contrapositive) Assume $|I| \geq 3$ and $|V(\Gamma(R/I))| \geq 2$. Then there are distinct adjacent vertices $x + I$, $y + I$ in $\Gamma(R/I)$. Since $|I| \geq 3$, there are distinct elements $0, i, j$ of I.

Note that the subgraph of $\Gamma_I(R)$ generated by $\{x, y, x + i, y + i, x + j, y + j\} = \{x, x + i, x + j\} \cup \{y, y + i, y + j\}$ contains a subgraph isomorphic to $K^{3,3}$. Thus $\Gamma_I(R)$ is nonplanar by Kuratowski's Theorem. □

Proposition 2 *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|V(\Gamma(R/I))| = 1$, then $\Gamma_I(R)$ is planar if and only if $1 \leq |I| \leq 4$.*

Proof $|\Gamma(R/I)| = 1 \Leftrightarrow |Z(R/I)^*| = 1 \Leftrightarrow R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. In both cases, I is not a radical ideal. Thus by Redmond's construction method of $\Gamma_I(R)$, $\Gamma_I(R) = K^{|I|}$. The result then follows since $K^{|I|}$ is planar if and only if $1 \leq |I| \leq 4$ (by Kuratowski's Theorem). □

It now suffices to consider the case when $|I| = 2$ and $\Gamma(R/I)$ has at least two distinct vertices. We will approach the problem by considering the different possibilities for $gr(\Gamma(R/I))$. Recall that $gr(\Gamma(R/I)) \in \{3, 4, \infty\}$ [7, 11, 15, 23].

Proposition 3 *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$ and $gr(\Gamma(R/I)) = 4$, then $\Gamma_I(R)$ is nonplanar. Moreover, if I is nonzero and $gr(\Gamma(R/I)) = 4$, then $\Gamma_I(R)$ is nonplanar.*

Proof Since $gr(\Gamma(R/I)) = 4$, there exists vertices $a + I, b + I, c + I, d + I$ of $\Gamma(R/I)$ that form a 4-cycle. Moreover, since $|I| = 2$, there exists $0 \neq i \in I$. Thus by Redmond's construction of $\Gamma_I(R)$ based on $\Gamma(R/I)$, we see that $\Gamma_I(R)$ will have a subgraph as in Fig. 3.

Notice that the vertex sets $A = \{a, a + i, c\}$ and $B = \{b, b + i, d + i\}$ induce a subgraph isomorphic to $K^{3,3}$. Whence by Kuratowski's Theorem, $\Gamma_I(R)$ is nonplanar. The "moreover statement" follows by combining Proposition 1 and this result. □

The following is [17, Theorem 7.2]:

Let I be a proper, nonzero ideal of a ring R that is not a prime ideal. Then $\Gamma_I(R)$ is planar if and only if $\omega(\Gamma(R/I)) \leq 2$ (i.e., $\Gamma(R/I)$ has no cycles) and either (a) $|I| = 2$ or (b) $\Gamma(R/I)$ consists of a single vertex and $|I| \leq 4$.

Here $\omega(G)$ is the clique number of a graph G. A *clique* of a graph G is a subgraph of G that is isomorphic to K^n for some $n \in \mathbb{N}$. If a graph has no cliques, we set the *clique number* of G to be zero; otherwise we set the clique number to the $\sup\{n \mid K^n$ is isomorphic to a subgraph of $G\}$. Notice that the clique number of a graph can be infinity.

If $\Gamma(R/I)$ consists solely of a four-cycle (as the subgraph in our previous proof), then $\omega(\Gamma(R/I)) = 2$. So Redmond's [17, Theorem 7.2] would imply that the induced subgraph from the preceding proof would be planar. However, we exhibited that this was not the case. For a concrete counterexample, consider $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$ and $I = 0 \times 0 \times \mathbb{Z}_2$ (See Fig. 3 for an isomorphic copy of $\Gamma_I(R)$). We note that in Redmond's statement of the theorem, Redmond wrote "$\omega(\Gamma(R/I)) \leq 2$ (i.e., $\Gamma(R/I)$

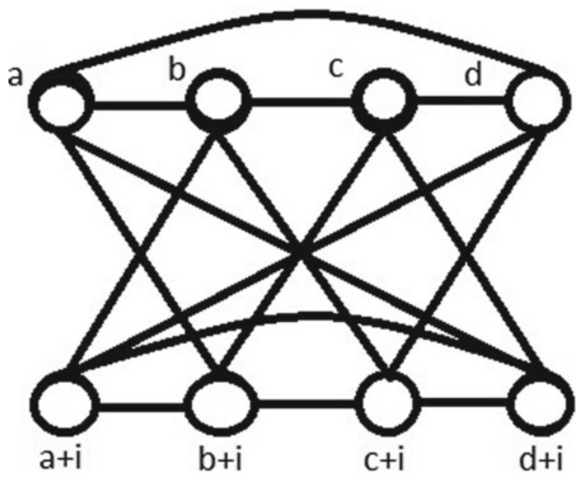


Fig. 3 Subgraph when $gr(\Gamma(R/I)) = 4$

has no cycles)". Although this statement is invalid, the theorem holds if we replace the clique number hypothesis with "$\Gamma(R/I)$ has no cycles" (i.e., $gr(\Gamma(R/I) = \infty$).

We continue our investigation of the problem by now considering the girth 3 case. The proof of the following can be derived by using Redmond's three-step construction and inspection.

Proposition 4 *Let R be a commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$, $I = \sqrt{I}$, and $\Gamma(R/I) = K^3$, then $\Gamma_I(R)$ is planar.*

It turns out the preceding result is rendered mute. In Example 2.1 of [2, pp. 2–3], it was shown that $\Gamma(R) \cong K^3$ if and only R is isomorphic to one of the following four rings: $\mathbb{F}_4[X]/(X^2)$, $\mathbb{Z}_4[X]/(X^2 + X + 1)$, $\mathbb{Z}_4[X]/(2, X)^2$, or $\mathbb{Z}_2[X, Y]/(X, Y)^2$. Thus in the preceding proposition, $\Gamma(R/I) \cong K^3$ if and only if R/I is isomorphic to one of the four previously mentioned rings. Since these rings are non-reduced, it follows that R/I is non-reduced. Since R/I is non-reduced if and only if I is not a radical ideal of R, it follows that the hypothesis of the preceding proposition is vacuous.

This observation lends light to Redmond's argument in [17, Theorem 7.2] in the following manner. He argues in his proof that if $\Gamma(R/I) \cong K^3$, then one can verify that $\Gamma_I(R)$ is nonplanar by exhibiting a subgraph of $\Gamma_I(R)$ isomorphic to $K^{3,3}$. This is the case if one takes into consideration that I must be a non-radical ideal of R, and hence $\Gamma_I(R)$ has a connected column.

In light of this observation, we come to the following proposition.

Proposition 5 *Let R be a commutative ring with nonzero identity and I an ideal of R. If $gr(\Gamma(R/I)) = 3$ and $|I| = 2$, then $\Gamma_I(R)$ is nonplanar. Moreover, if $gr(\Gamma(R/I)) = 3$ and I is nonzero, then $\Gamma_I(R)$ is nonplanar.*

Proof First assume that $\Gamma(R/I) \cong K^3$. By the preceding observations, it follows that I is not a radical ideal. Since $\Gamma(R/I) \cong K^3$, we have that $\Gamma(R/I)$ consists

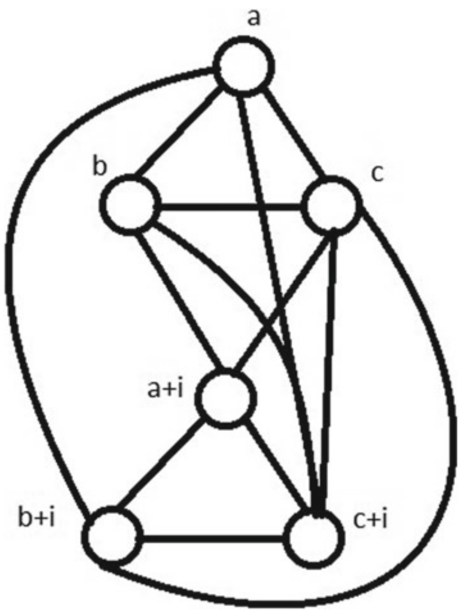


Fig. 4 Subgraph of $\Gamma_I(R)$ when $gr(R/I) = 3$ and $|I| \geq 2$

solely of a three-cycle, say $a+I - b+I - c+I - a+I$. Since $I \neq \sqrt{I}$, at least one of the elements a, b, c is in $\sqrt{I}$. Without loss of generality, assume that $c^2 \in I$.

Then using Redmond's Construction Method, $\Gamma_I(R)$ will have a subgraph as in Fig. 4.

The vertex sets $\{a, c, a+i\}$ and $\{b, b+i, c+i\}$ induce a subgraph of the preceding graph isomorphic to $K^{3,3}$. Thus $\Gamma_I(R)$ is nonplanar by Kuratowski's Theorem.

If $\Gamma(R/I) \ncong K^3$, then since $gr(\Gamma(R/I)) = 3$, we have that $\Gamma(R/I)$ does not consist solely of a three-cycle. Thus it follows that $\Gamma(R/I)$ would have a subgraph as in Fig. 5a. Therefore, again using Redmond's construction method, $\Gamma_I(R)$ would have the subgraph as in Fig. 5b.

Taking a subdivision of this graph by replacing $c - d - c + i$ with $c - c + i$, we get a subdivision of $\Gamma_I(R)$ which contains a subgraph isomorphic to the graph in Fig. 4. Thus $\Gamma_I(R)$ is nonplanar (since we have already shown that the graph in Fig. 4 was nonplanar). The "moreover statement" follows from Proposition 1 and this result. □

It now remains only to investigate the case when $gr(\Gamma(R/I)) = \infty$ (i.e., $\Gamma(R/I)$ has no cycles) and $|I| = 2$. A natural question is whether or not I being a radical ideal of R will affect the planarity of $\Gamma_I(R)$; in this case, it turns out that it does not. We will need a special type of graph defined in [5] denoted $\overline{K}^{1,3}$. The graph $\overline{K}^{1,3}$ is the graph formed by adding an additional vertex to $K^{1,3}$ adjacent only to one leaf (non-center vertex) of $K^{1,3}$.

Proposition 6 *Let R be a* **finite** *commutative ring with nonzero identity and I an ideal of R. If $|I| = 2$ and $gr(\Gamma(R/I)) = \infty$, then $\Gamma_I(R)$ is planar.*

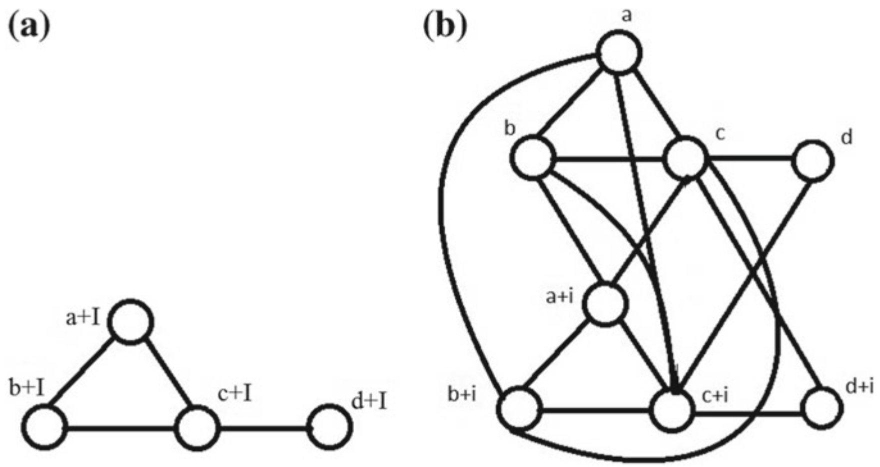


Fig. 5 Subgraphs when $gr(\Gamma(R/I)) = 3$, $|I| = 2$, and $\Gamma_I(R) \ncong K^3$

Proof If I is a prime ideal of R or $I = R$, then both $\Gamma(R/I)$ and $\Gamma_I(R)$ are empty, and hence planar. Assume that I is a proper, non-prime ideal of R.

Now $\Gamma(R/I)$ is nonempty since I is a proper, non-prime ideal of R. We handled the case when $V(\Gamma(R/I))$ is a singleton in Proposition 2; so we may assume that $|V(\Gamma(R/I))| \geq 2$. It then follows from [5, Theorems 2.4 and 2.5] that $\Gamma(R/I)$ is isomorphic to either $\overline{K}^{1,3}$ or $K^{1,n}$ for some $n \geq 1$.

We begin with the case that $\Gamma(R/I)$ is a star graph ($\Gamma(R/I) \cong K^{1,n}$), say with center c, ends a_k, and $I = \{0, i\}$. Using Redmond's construction method and the fact that $|I| = 2$, we can draw $\Gamma_I(R)$ as in Fig. 6a. The dotted or hash-mark lines indicate lines that occur if and only if the vertex is in a connected column (recall connected columns exist if and only if I is a non-radical ideal). As drawn, we see that $\Gamma_I(R)$ is planar. It is important to note that we are using the finite hypothesis here. In order for the drawing of Fig. 6a to be make *sense*, there needs to be some constraints on the cardinality of vertices of $\Gamma(R/I)$.

If $\Gamma(R/I) \cong \overline{K}^{1,3}$, then one can see (regardless of whether or not $I = \sqrt{I}$) that $\Gamma_I(R)$ is planar. Using dotted or hash-mark lines as before, we can draw $\Gamma_I(R)$ as in Fig. 6b.

Thus in all cases, $\Gamma_I(R)$ is planar as desired. □

Combining all these results, we get a theorem which turns out to be only a slight modification of Redmond's Theorem 7.2. As previously mentioned, Redmond's statement of the theorem seems incorrect due to using the hypothesis $\omega(\Gamma(R/I)) \leq 2$ instead of $gr(\Gamma(R/I)) = \infty$. Moreover, it seems that in Redmond's proof a key observation (that appears to go unmentioned) was that $\Gamma(R/I) \cong K^3$ implies that R/I is non-reduced. Combining the previous propositions (and noting that Proposition 4 can not happen) yields the following theorem.

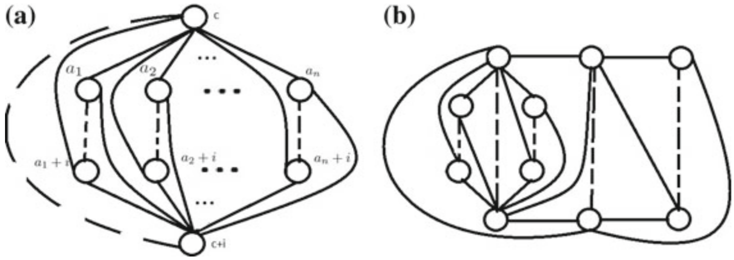


Fig. 6 Graphs for Proposition 6

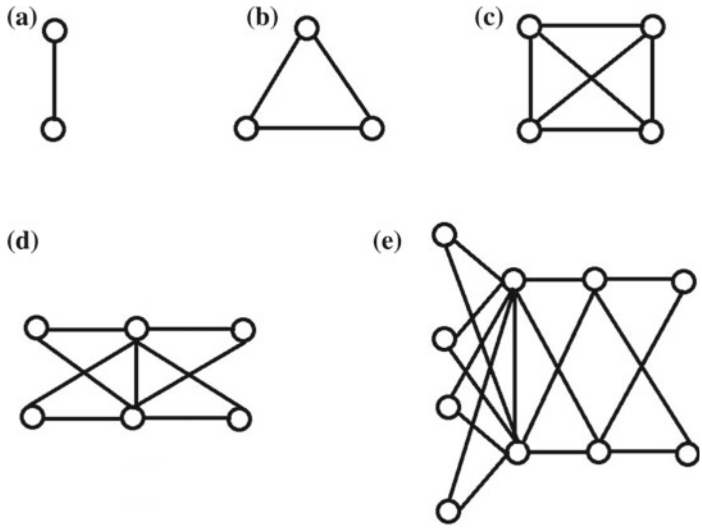


Fig. 7 The five finite planar graphs with non-radical nontrivial ideals

Theorem 6 *Let I be a nonzero, proper, non-prime ideal of a* **finite** *commutative ring R with nonzero identity. Then $\Gamma_I(R)$ is planar if and only if $gr(\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$.*

Notice that the only place we required the finite hypothesis in the preceding was when $|I| = 2$ and $gr(R/I) = \infty$. We can now draw all finite planar graphs corresponding to non-empty $\Gamma_I(R)$ with I a non-radical, nonzero ideal of a ring R. By connecting some results from [3, 5], we can make short work of determining these graphs.

Theorem 7 *Let R be a commutative ring with nonzero identity and I a non-radical, nonzero ideal of R. Then $\Gamma_I(R)$ is planar if and only if $\Gamma_I(R)$ is isomorphic to one of the 5 graphs in Fig. 7.*

Proof The converse is evident. For the forward implication, by Theorem 6 we have that $gr(\Gamma(R/I)) = \infty$ and either (a) $|I| = 2$ or (b) $|V(\Gamma(R/I))| = 1$ and $|I| \in \{2, 3, 4\}$. In case (b), $\Gamma_I(R)$ is isomorphic to K^2, K^3, or K^4.

Assume case (a) holds. Recall that I is not a radical ideal of R if and only if R/I is not reduced. So we begin our search by considering which non-reduced rings have corresponding zero-divisor graph with infinite girth. By [5, Theorem 2.5], we have that a non-reduced ring A has $gr(\Gamma(A)) = \infty$ if and only if $A \cong B$ or $A \cong B \times \mathbb{Z}_2$, where $B \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, or $\Gamma(A)$ is a star graph. In the proof of the proceeding result, the authors show that $\Gamma(A)$ is complemented when $nil(R) \neq 0$ and $gr(\Gamma(R)) = \infty$. They then split the situation into two cases: when the graph is uniquely complemented or not. The uniquely complemented case is when $\Gamma(A)$ is a star graph. But using [3, Theorem 3.9], we have that $\Gamma(A)$ uniquely complemented with $nil(R)$ nonzero implies that either $\Gamma(A)$ is a star graph on at most two edges or an infinite star graph. However, since we are considering finite rings, $\Gamma(A)$ must be a star graph on at most two edges. Now using Redmond's construction of $\Gamma_I(R)$ from $\Gamma(R/I)$, we can deduce the possible graphs for $\Gamma_I(R)$.

If $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, then $|V(\Gamma(R/I))| = 1$. Hence $|I| = 2$ gives that $\Gamma_I(R) \cong K^2$.

If $R/I \cong \mathbb{Z}_2 \times B$ (where B is as before), then R/I is $\overline{K}^{1,3}$. Notice that the vertex of degree 3 is the only element whose square is zero, thus $|I| = 2$ implies that $\Gamma_I(R)$ is isomorphic to (E) in Fig. 7.

If $\Gamma(R/I) \cong K^{1,1}$, then $R/I \cong \mathbb{Z}_9$ or $\mathbb{Z}_3[X]/(X^2)$ [3, pp. 2–3], and whence each vertex of the graph has the property that its square is zero. So with $|I| = 2$, we get $\Gamma_I(R) \cong K^4$.

If $\Gamma(R/I) \cong K^{1,2}$ and $I \neq \sqrt{I}$, then $R/I \cong \mathbb{Z}_8$, $\mathbb{Z}_2[X]/(X^3)$, or $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ [3, pp. 2–3]. In each of the latter cases, the only vertex whose square is zero is the center vertex. Thus with $|I| = 2$, we have that $\Gamma_I(R)$ will be isomorphic to (D) of Fig. 7. □

We now consider when $\Gamma_I(R)$ is a finite, planar graph and I is a radical ideal. In this case, by Theorem 6, it follows that $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$. To conclude our work here, we prove the following Proposition.

Proposition 7 *Let I be a nonzero, proper, non-prime,* **radical** *ideal of a* **finite** *commutative ring R with nonzero identity. Then $gr(\Gamma(R/I)) = \infty$ and $|I| = 2$ if and only if R is isomorphic to a ring with corresponding ideal from Table 2, where K is a finite field.*

Table 2 Rings for Proposition 7

Ring	Ideal
$\mathbb{Z}_4 \times K$	$(2) \times 0$
$\mathbb{Z}_2[X]/(X^2) \times K$	$(x) \times 0$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times K$	$\mathbb{Z}_2 \times 0 \times 0$, $0 \times \mathbb{Z}_2 \times 0$, or $0 \times 0 \times K$ (when $K = \mathbb{Z}_2$)

Proof In the following argument, we use, without direct reference, that an ideal of $\prod_{i=1}^{n} R_i$ is of the form $\prod_{i=1}^{n} I_i$, where I_i is an ideal of R_i. In the case of an ideal with only two elements, it must be of the form $\prod_{i=1}^{n} I_i$, where $I_i = 0$ for all i except a fixed $k \in \{1, \ldots, n\}$ and $|I_k| = 2$.

Since I is a radical ideal, we have that R/I is reduced. By [5, Theorem 2.4], we have that $T(R/I) \cong \mathbb{Z}_2 \times K$, where K is a field. Since R is finite, and hence also R/I, we have $R/I \cong T(R/I)$; whence $R/I \cong \mathbb{Z}_2 \times K$. Since R is a finite commutative ring, we have that $R \cong \prod_{i=1}^{n} R_i$, where each R_i is a finite local ring. If $n \geq 4$, then R/I will be isomorphic to a product of at least 3 nonzero local rings. But this is a contradiction as R/I is a product of 2 local rings. Thus we must have that $n \leq 3$.

If $n = 1$, then R is local. Thus R/I is also local; so R/I can be expressed as a product of only one local ring. Thus $n \neq 1$ as R/I is a product of two local rings.

If $n = 2$, then $R \cong R_1 \times R_2$, where R_1, R_2 are local. Hence either $I = I_1 \times 0$ or $I = 0 \times I_2$, where $|I_1| = |I_2| = 2$. Thus $R/I \cong R_1/I_1 \times R_2$ or $R/I \cong R_1 \times R_2/I_2$. In either case, we have that R_i is a local ring with ideal I_i such that $|I_i| = 2$ and R_i/I_i is a field. Thus I_i is a maximal ideal of R_i. Notice that $Z(R_i)$ is nonzero, since otherwise R_i would be a finite integral domain, and hence a field (which contradicts the existence of a proper ideal with 2 elements). Since $0 \subsetneq Z(R_i) \subseteq I_i$ and $|I_i| = 2$, it follows that $Z(R_i) = I_i$. Therefore $|Z(R_i)| = 2$. Hence either $|Z(R_1)| = 2$ or $|Z(R_2)| = 2$. Since one of the rings has exactly 2 zero-divisors either R_1 or R_2 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. The only constraint on the remaining factor (the one which is neither $\mathbb{Z}_4$ nor $\mathbb{Z}_2[X]/(X^2)$) is that it must a field or $\mathbb{Z}_2$. Since $\mathbb{Z}_2$ is a field, the preceding requirement reduces to simply being a field.

Whence $R_1/I_1 \cong \mathbb{Z}_2$ or $R_2/I_2 \cong K$ where K is field. Thus the ideals I_1, I_2 are maximal ideals of their respective local factors. If $\{0\} \subsetneq I_1 \subsetneq R_1$ where I_1 is maximal, then R_1 is not a field and hence $\{0\} \neq Z(R_1)$ (since R_1 must be finite and a finite integral domain is a field). Thus we would have $Z(R_1) = I_1$ and therefore $|Z(R_1)| = 2$. Similarly if $0 \subsetneq I_2 \subsetneq R_2$, then $|Z(R_2)| = 2$.

If $n = 3$, then since R/I is a product of two local rings, I is of the form one of the three rings times 2 zero ideals. Thus we must have that $R \cong \mathbb{Z}_2 \times K \times \mathbb{Z}_2$, where $I = \mathbb{Z}_2 \times 0 \times 0$, $0 \times 0 \times \mathbb{Z}_2$, or $0 \times K \times 0$ (in the case that $K = \mathbb{Z}_2$).

Thus in conclusion, R is isomorphic to one of the rings in Table 2 with corresponding ideal I. The converse is evident. □

Therefore by Proposition 7, in the case that $\Gamma_I(R)$ is a finite, planar graph and I is a radical ideal we must have that $\Gamma(R/I)$ will be a star graph and $|I| = 2$. It then follows that $\Gamma_I(R)$ will be isomorphic to the graph in Fig. 8. Thus [17, Remark 7.3] actually captures all planar non-radical ideal-based zero-divisor *graphs*.

In conclusion, we have the following theorem.

Theorem 8 *Let R be a commutative ring, I a proper, non-prime ideal of R. Then $\Gamma_I(R)$ is planar if and only if it is isomorphic to one of the graphs in Fig. 7 or Fig. 8.*

Thus, we have classified all the planar nontrivial ideal-based zero-divisor graphs. In our dissertation, we also classified all commutative rings (and corresponding ideals) which produce planar ideal-based zero-divisor graphs. The process begins with

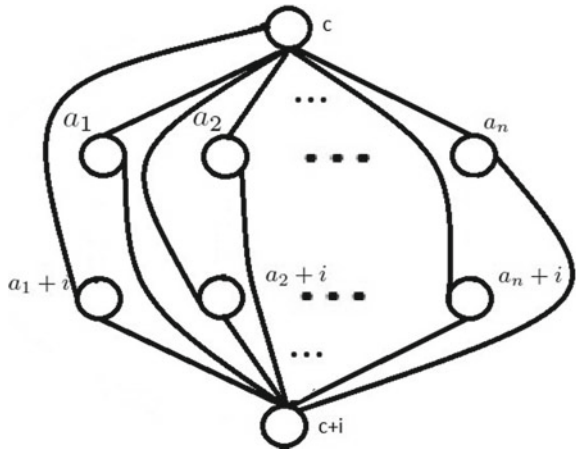


Fig. 8 Finite planar graphs with I radical and nonzero

classifying all complete ideal-based zero-divisor graphs on up to 5 vertices. Using factorization techniques and exhaustive methods, we are able to classify all finite commutative rings with nontrivial, finite, planar ideal-based zero-divisor graphs. In a future paper, we hope to present that work in a more organized and concise manner. We will close with an interesting fact, Neal O. Smith proved that there were 44 types of finite commutative rings (non-fields) which give rise to planar zero-divisor graphs. It turns out that there are also 44 types of rings (some of these different from the 44 in N.O. Smith's result) with nontrivial planar ideal-based zero-divisor graphs (3 infinite and 41 finite).

Acknowledgements I would like to thank my advisor, David F. Anderson, for his contribution and comments in my graduate research. This material is derived from dissertation research performed at the University of Tennessee, Knoxville [18]. Some of my fondest memories are discussing the rings, graphs, and isomorphisms found in this paper with this amazing man. I affectionately remember being called a *trouble maker* when meeting with him about the error that gave birth to my work on graph isomorphisms. We spent many hours talking about many things (sometimes things other than mathematics). David, I offer you my sincerest thanks. I would also like to thank my colleagues and mentors at Maryville College for encouraging me in the transition to a faculty member. Additional thanks to Chase Worley for helping me with edits to this document.

References

1. S. Akbari, H.R. Maimani, S. Yassemi, When a zero-divisor graph is planar or a complete r-partite graph. J. Algebra **270**(1), 169–180 (2003)
2. D.F. Anderson, A. Frazier, A. Lauve, P.S. Livingston, The zero-divisor graph of a commutative ring. II, in *Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999)*, Lecture Notes in Pure and Applied Mathematics, vol. 220 (Dekker, New York, 2001), pp. 61–72
3. D.F. Anderson, R. Levy, J. Shapiro, Zero-divisor graphs, von Neumann regular rings, and Boolean algebras. J. Pure Appl. Algebra **180**(3), 221–241 (2003)

4. D.F. Anderson, P.S. Livingston, The zero-divisor graph of a commutative ring. J. Algebra **217**(2), 434–447 (1999)
5. D.F. Anderson, S.B. Mulay, On the diameter and girth of a zero-divisor graph. J. Pure Appl. Algebra **210**(2), 543–550 (2007)
6. D.F. Anderson, S. Shirinkam, Some remarks on $\Gamma_I(R)$. Commun. Algebra, To Appear
7. M. Axtell, J. Coykendall, J. Stickles, Zero-divisor graphs of polynomials and power series over commutative rings. Commun. Algebra **33**(6), 2043–2050 (2005)
8. I. Beck, Coloring of commutative rings. J. Algebra **116**(1), 208–226 (1988)
9. R. Belshoff, J. Chapman, Planar zero-divisor graphs. J. Algebra **316**(1), 471–480 (2007)
10. G. Chartrand, L. Lesniak, *Graphs and Digraphs*. Chapman and Hall, CRC (1996)
11. F. DeMeyer, K. Schneider, Automorphisms and zero divisor graphs of commutative rings. Int. J. Commut. Rings 93–106 (2002)
12. J.D. LaGrange, Complemented zero-divisor graphs and Boolean rings. J. Algebra **315**(2), 600–611 (2007)
13. P.S. Livingston, *Structure in Zero-divisor Graphs of Commutative Rings*, (Thesis, University of Tennessee Knoxville, 1997)
14. H.R. Maimani, M.R. Pournaki, S. Yassemi, Zero-divisor graph with respect to an ideal. Commun. Algebra **34**(3), 923–929 (2006)
15. S.B. Mulay, Cycles and symmetries of zero-divisors. Commun. Algebra **30**(7), 3533–3558 (2002)
16. S.P. Redmond, *Generalizations of the Zero-divisor Graph of a Ring*. (ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–The University of Tennessee, 2001)
17. S.P. Redmond, An ideal-based zero-divisor graph of a commutative ring. Commun. Algebra **31**(9), 4425–4443 (2003)
18. J.G. Smith, Jr., *Properties of Ideal-Based Zero-Divisor Graphs of Commutative Rings*. (ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–The University of Tennessee, 2014)
19. N.O. Smith, *Properties of the Zero-Divisor Graph of a Ring*. (Dissertation, University of Tennessee Knoxville, 2004)
20. N.O. Smith, Planar zero-divisor graphs. in *Focus on Commutative Rings Research*, (Nova Science Publishers, New York, 2006), pp. 177–186
21. N.O. Smith, Infinite planar zero-divisor graphs. Commun. Algebra **35**(1), 171–180 (2007)
22. H.-J. Wang, Zero-divisor graphs of genus one. J. Algebra **304**(2), 666–678 (2006)
23. S.E. Wright, Lengths of paths and cycles in zero-divisor graphs and digraphs of semigroups. Commun. Algebra **35**(6), 1987–1991 (2007)