Chapter 8
Fourier Multipliers and Singular Integrals on $\mathbb{C}^n$

In this chapter, we introduce a class of singular integral operators on the n-complex unit sphere. This class of singular integral operators corresponds to bounded Fourier multipliers. Similar to the results of Chaps. 6 and 7, we also develop the fractional Fourier multiplier theory on the unit complex sphere.

8.1 A Class of Singular Integral Operators on the n-Complex Unit Sphere

In this section, we study a class of singular integral operators defined on $n-$complex unit sphere. The Cauchy–Szegö kernel and the related theory of singular integrals of several variables have been studied extensively, see [1–4]. The singular integrals studied in this section can be represented as certain Fourier multiplier operators with bounded symbols defined on S_ω. This class of singular integrals constitute an operator algebra, that is, the bounded holomorphic functional calculus of the radial Dirac operator

$$D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

A special example of these singular integrals is the Cauchy integral operator.

We will still use the following sector regions in the complex plane. For $0 \leqslant \omega < \pi/2$, let

$$S_\omega = \Big\{ z \in \mathbb{C} \mid z \neq 0,\ \text{and} \mid \arg z \mid < \omega \Big\},$$

$$S_\omega(\pi) = \Big\{ z \in \mathbb{C} \mid z \neq 0,\ |\text{Re} z| \leqslant \pi,\ \text{and} \mid \arg(\pm z) \mid < \omega \Big\},$$

$$W_\omega(\pi) = \Big\{ z \in \mathbb{C} \mid z \neq 0,\ |\text{Re} z| \leqslant \pi,\ \text{and } \text{Im}(z) > 0 \Big\} \cup S_\omega(\pi),$$

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$$H_\omega = \Big\{ z \in \mathbb{C} \mid z = e^{iw},\ w \in W_\omega(\pi) \Big\}.$$

The sets S_ω, $S_\omega(\pi)$, $W_\omega(\pi)$ and H_ω are cone-shaped, bowknot-shaped region, W-shaped region and heart-shaped region, respectively.

Let

$$\phi_b(z) = \sum_{k=1}^{\infty} b(k) z^k. \tag{8.1}$$

By Lemma 6.1.1, for $b \in H^\infty(S_\omega)$, ϕ_b can be extended to H_ω holomorphically, and

$$\left| \left(z\frac{d}{dz} \right)^l \phi_b(z) \right| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu,\ \mu')|1-z|^{1+l}},\ \ z \in H_\mu, 0 < \mu < \mu' < \omega,\ l = 0, 1, 2, \cdots,$$

where $\delta(\mu,\ \mu') = \min \Big\{ 1/2,\ \tan(\mu' - \mu) \Big\}$. $C_{\mu'}$ is the constant in the definition of $b \in H^\infty(S_\omega)$.

In the sequel, we use z to denote any element in $\mathbb{C}^n$, that is, $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{C}, i = 1, 2, \ldots, n, n \geqslant 2$. Write $\overline{z} = (\overline{z}_1, \cdots, \overline{z}_n)$. z can be seen as a row vector. Denote by B the open ball $\{z \in \mathbb{C}^n : |z| < 1\}$, where $|z| = \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2}$, and ∂B is the boundary, i.e.,

$$\partial B = \Big\{ z \in \mathbb{C}^n : |z| = 1 \Big\}.$$

The open ball centered at z with radius r is denoted by $B(z, r)$. Any element on the unit sphere is usually denoted by ξ or ζ. Below the constant ω_{2n-1} occurring in the Cauchy–Szegö kernel is the surface area of $\partial B = S^{2n-1}$ and equals to $2\pi^n/\Gamma(n)$. For $z, w \in \mathbb{C}^n$, we use the notation $zw' = \sum_{k=1}^{n} z_k w_k$. The object of study in this section is the radial Dirac operator

$$D = \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}.$$

We shall make some modifications on the basis of holomorphic function spaces in B and the corresponding function spaces on ∂B. We apply the form given in [1]. Let k be a non-negative integer. We consider the column vector $z^{[k]}$ with the components

$$\sqrt{\frac{k!}{k_1! \cdots k_n!}} z_1^{k_1} \cdots z_n^{k_n},\ k_1 + k_2 + \cdots k_n = k.$$

The dimension of $z^{[k]}$ is

$$N_k = \frac{1}{k!} n(n+1) \cdots (n+k-1) = C_{n+k-1}^{k}.$$

Set

$$\int_B \overline{z^{[k]'}} \cdot z^{[k]} dz = H_1^k$$

and

$$\int_{\partial B} \overline{\xi^{[k]'}} \xi^{[k]} d\sigma(\xi) = H_2^k,$$

where dz is the Lebesgue volume element in $\mathbb{R}^{2n} = \mathbb{C}^n$, and $d\sigma(\xi)$ is the Lebesgue area element of the unit sphere $S^{2n-1} = \partial B$. It is easy to prove that H_1^k and H_2^k is the positive definite Hermitian matrix of order N_k. Hence there exists a matrix Γ such that

$$\begin{cases} \overline{\Gamma'} \cdot H_1^k \cdot \Gamma = \Lambda, \\ \overline{\Gamma'} \cdot H_2^k \cdot \Gamma = I, \end{cases} \tag{8.2}$$

where $\Lambda = [\beta_1^k, \cdots, \beta_n^k]$ is the diagonal matrix and I is the identity matrix.

We set

$$\begin{cases} z_{[k]} = z^{[k]} \cdot \Gamma, \\ \xi_{[k]} = \xi^{[k]} \cdot \Gamma. \end{cases}$$

and use $\{p_\nu^k(z)\}$ to denote the components of the vector $z_{[k]}$. By (8.2), we have

$$\int_B p_\nu^k(z) \overline{p_\mu^l(z)} dz = \delta_{\nu\mu} \cdot \delta_{kl} \cdot \beta_\nu^k \tag{8.3}$$

and

$$\int_{\partial B} p_\nu^k(\xi) \overline{p_\mu^l(\xi)} d\sigma(\xi) = \delta_{\nu\mu} \cdot \delta_{kl}. \tag{8.4}$$

The following theorem is well-known.

Theorem 8.1.1 ([1]) *The function system*

$$(\beta_\nu^k)^{-1/2} p_\nu^k, \ k = 0, 1, 2, \ldots, \ \nu = 1, 2, \ldots, N_k,$$

is a complete orthogonal system of the holomorphic function space in B. In the space of continuous functions on ∂B, the function system $\{p_\nu^k(\xi)\}$ is orthogonal, but is not complete.

In [1], applying the function system $\{p_\nu^k\}$ and relation

$$H(z, \overline{\xi}) = \sum_{k=0}^{\infty} \sum_{\nu=1}^{N_k} p_\nu^k(z) \overline{p_\nu^k(\xi)}, \ z \in B, \ \xi \in \partial B,$$

L. Hua gave the explicit formula of the Cauchy–Szegö kernel on ∂B:

$$H(z,\ \overline{\xi}) = \frac{1}{\omega_n}\frac{1}{(1 - z\overline{\xi}')^n}. \tag{8.5}$$

In the following, we give a technical result.

Theorem 8.1.2 *Let $b \in H^\infty(S_\omega)$ and*

$$H_b(z,\ \overline{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)},\ z \in B,\ \xi \in \partial B. \tag{8.6}$$

Then for any $z \in B$ and $\xi \in \partial B$ such that $z\overline{\xi}' \in H_\omega$,

$$H_b(z,\ \overline{\xi}) = \frac{1}{(n-1)!\omega_{2n-1}}(r^n\phi_b(r))^{(n-1)}\ |_{r=z\overline{\xi'}} \tag{8.7}$$

are all holomorphic, where ϕ_b is the function defined in (8.1). In addition, for $0 < \mu < \mu' < \omega,\ l = 0, 1, 2, \ldots,$

$$|D_z^l H_b(z,\ \overline{\xi})| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu, \mu')|1 - z\overline{\xi'}|^{n+l}}, z\overline{\xi'} \in H_\mu, \tag{8.8}$$

where $\delta(\mu,\ \mu') = \Big\{1/2,\ \tan(\mu' - \mu)\Big\}$; $C_{\mu'}$ is the constant in the definition of $H^\infty(S_\omega)$.

Proof In (8.5), letting $z = r\zeta$ and $|\zeta| = 1$, we obtain

$$H(r\zeta,\ \overline{\xi}) = \frac{1}{\omega_{2n-1}}\frac{1}{(1 - r\zeta\overline{\xi'})^n}. \tag{8.9}$$

Taking $H(r\zeta,\ \overline{\xi})$ as a function of r, we know that in the Taylor expansion of this function, the term with respect to r^k is

$$\begin{aligned} &\frac{1}{k!}\Big(\frac{\partial}{\partial r}\Big)^k\Big(\frac{1}{\omega_{2n-1}}\frac{1}{(1 - r\zeta\overline{\xi'})^n}\Big)\Big|_{r=0} r^k \\ &= \frac{1}{\omega_{2n-1}}\frac{n(n+1)\cdots(n+k-1)}{k!}(r\zeta\overline{\xi'})^k. \end{aligned} \tag{8.10}$$

Let $r\zeta = z$. We get the projection from $H(z,\ \overline{\xi})$ to the k-homogeneous function space of variable z is

$$\sum_{\nu=1}^{N_k} p_\nu^k(z)\overline{p_\nu^k(\xi)} = \frac{1}{\omega_{2n-1}}\frac{n(n+1)\cdots(n+k-1)}{k!}(z\overline{\xi'})^k.$$

By the definition of ϕ_b, a direct computation gives the formula of $H_b(z, \overline{\xi})$. The corresponding estimate can be deduced from Lemma 6.1.1. □

Remark 8.1.1 In the former chapters, the size of ω is very important and is related to the Lipschitz constant of Lipschitz curves or Lipschitz surfaces, see also [5–16]. Now, the Lipschitz constant of the unit sphere is 0, and ω can be chosen as any number in the interval $(0, \pi/2]$. In this section, we always assume that ω is any number in $(0, \pi/2]$ but should be determined via discussion. We also take $\mu = \omega/2$ and $\mu' = 3\omega'/4$ large enough to adapt to our theory.

For $z, w \in B \cup \partial B$, denote by $d(z, w)$ the anisotropic distance between z and w defined as

$$d(z, w) = |1 - z\overline{w'}|^{1/2}.$$

It is easy to prove d is a distance on $B \cup \partial B$. On ∂B, denote by $S(\zeta, \varepsilon)$ the ball centered at ζ with radius ε which is defined via d. The complementary set of $S(\zeta, \varepsilon)$ in ∂B is denoted by $S^c(\zeta, \varepsilon)$.

Let $f \in L^p(\partial B)$, $1 \leqslant p < \infty$. Then the Cauchy integral of f

$$C(f)(z) = \frac{1}{\omega_{2n-1}} \int_{\partial B} \frac{f(\xi)}{(1 - z\overline{\xi'})^n} d\sigma(\xi)$$

is well defined and is holomorphic in B.

It is fairly well known that the operator

$$P(f)(\zeta) = \lim_{r \to 1-0} C(f)(r\zeta)$$

is the projection from $L^p(\partial B)$ to the Hardy space $H^p(\partial B)$ and is bounded from $L^p(\partial B)$ to $H^p(\partial B)$, $1 < p < \infty$. Moreover, $P(f)$ has a singular integral expression [3, 4]

$$P(f)(\zeta) = \frac{1}{\omega_{2n-1}} \lim_{\varepsilon \to 0} \int_{S^c(\zeta, \varepsilon)} \frac{f(\xi)}{(1 - \zeta\overline{\xi'})^n} d\sigma(\xi) + \frac{1}{2} f(\zeta) \text{ a.e.} \zeta \in \partial B.$$

Let

$$\mathscr{A} = \left\{ f : f \text{ is a holomorphic function in } B(0, 1 + \delta) \text{ for some } \delta > 0 \right\}.$$

It is easy to verify that $\mathscr{A}$ is dense in $L^p(\partial B)$, $1 \leqslant p < \infty$. If $f \in \mathscr{A}$, then

$$f(z) = \sum_{k=0}^{\infty} \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z),$$

where $c_{k\nu}$ is the Fourier coefficient of f:

$$c_{k\nu} = \int_{\partial B} \overline{p_\nu^k(\xi)} f(\xi) d\sigma(\xi).$$

Also, for any positive integer l, the series

$$\sum_{k=0}^{\infty} k^l \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(z)$$

uniformly absolutely converges in any ball contained $B(0, 1+\delta)$ on which f is defined.

Let $\mathscr{U}$ be the unitary group consisting of all unitary operators in the sense of complex inner product $\langle z, w\rangle = z\overline{w'}$ on Hilbert spaces in $\mathbb{C}^n$. These operators are linear operators U which keep the inner product invariant:

$$\langle Uz, Uw\rangle = \langle z, w\rangle.$$

Obviously, $\mathscr{U}$ is a compact subset in $O(2n)$. It is easy to prove that $\mathscr{A}$ is invariant under the operation of $U \in \mathscr{U}$. If $f \in \mathscr{A}$, then f is determined by its value on ∂B. Below we shall regard $f\,|_{\partial B}$ as $f \in \mathscr{A}$. For a given function $b \in H^\infty(S_\omega)$, we define an operator $M_b : \mathscr{A} \to \mathscr{A}$ as

$$M_b(f)(\zeta) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=0}^{N_k} c_{k\nu} p_\nu^k(\zeta), \ \zeta \in \partial B,$$

where $c_{k\nu}$ is the Fourier coefficient of the test function $f \in \mathscr{A}$.

The principal value of the Cauchy integral defined via the surface distance

$$d(\eta, \zeta) = |1 - \eta\overline{\zeta'}|^{1/2}$$

can be extended as in the following Theorem 8.1.3:

Theorem 8.1.3 *The operator M_b can be expressed as the form of the singular integral. Precisely, for $f \in \mathscr{A}$,*

$$M_b(f)(\zeta) = \lim_{\varepsilon\to 0}\Big[\int_{S^c(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) f(\xi) d\sigma\xi + f(\zeta)\int_{S(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) d\sigma(\xi)\Big], \tag{8.11}$$

where

$$\int_{S(\zeta,\varepsilon)} H_b(\zeta, \overline{\xi}) d\sigma(\xi)$$

are bounded functions for $\zeta \in \partial B$ and ε.

Proof Let $f \in \mathscr{A}$ and $\rho \in (0, 1)$. On the one hand,

$$M_b(f)(\rho\zeta) = \sum_{k=1}^{\infty} b(k) \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\rho\zeta),$$

where $c_{k\nu}$ is the Fourier coefficient of f. Because $\{b(k)\}_{k=1}^{\infty} \in l^{\infty}$ and the Fourier expansion of $f \in \mathscr{A}$ is convergent, we obtain

$$\lim_{\rho \to 1-0} M_b(f)(\rho\zeta) = M_b(f)(\zeta). \tag{8.12}$$

On the other hand, applying the formula of the Fourier coefficients and the definition of $H_b(z, \overline{\xi})$ given in (8.5), we have

$$M_b(f)(\rho\zeta) = \int_{\partial B} H_b(\rho\zeta, \overline{\xi}) f(\xi) d\sigma(\xi).$$

For any $\varepsilon > 0$, we get

$$\begin{aligned} M_b(f)(\rho\zeta) &= \int_{S^c(\zeta,\ \xi)} H_b(\rho\zeta, \overline{\xi}) f(\xi) d\sigma(\xi) \\ &\quad + \int_{S(\zeta,\ \xi)} H_b(\rho\zeta, \overline{\xi})(f(\xi) - f(\zeta)) d\sigma(\xi) \\ &\quad + f(\zeta) \int_{S(\zeta,\ \xi)} H_b(\rho\zeta, \overline{\xi}) d\sigma(\xi) \\ &= I_1(\rho, \varepsilon) + I_2(\rho, \varepsilon) + f(\zeta) I_3(\rho, \varepsilon). \end{aligned}$$

For $\rho \to 1 - 0$, we have

$$I_1(\rho, \varepsilon) \to \int_{S^c(\zeta,\ \varepsilon)} H_b(\zeta, \overline{\xi}) f(\xi) d\sigma(\xi).$$

Now we consider $I_2(\rho, \varepsilon)$. Because the metric d, the Euclidean metric $|\cdot|$ and the function class $\mathscr{A}$ are all $\mathscr{U}-$invariant, without loss of generality, we can assume that $\zeta = (1, 0, \ldots, 0)$. For the variable $\xi \in \partial B$, we adopt the parameter system

$$\xi_1 = re^{1\theta},\ \xi_2 = v_2, \ldots, \xi_n = v_n.$$

Write $v = (v_2, \ldots, v_n)$. The integral region $S(\zeta, \varepsilon)$ is defined by the following condition:

$$v\overline{v'} = 1 - r^2, \quad \cos\theta \geqslant \frac{1 + r^2 - \varepsilon^4}{2r}. \tag{8.13}$$

Now, because $\frac{1+r^2-\varepsilon^4}{2r} \leqslant \cos\theta \leqslant 1$, we have $(1-r)^2 \leqslant \epsilon^4$. Then $1 - r \leqslant \varepsilon^2$, or $1 - \varepsilon^2 \leqslant r$. This implies that

$$v\overline{v'} = 1 - r^2 \leqslant 1 - (1-\varepsilon^2)^2 = 2\varepsilon^2 - \varepsilon^4.$$

Write

$$a = a(r,\ \varepsilon) = \arccos\left(\frac{1 + r^2 - \varepsilon^4}{2r}\right).$$

Because $(1-r)^2 \leqslant \varepsilon^4$ and $1 - y = O(\arccos^2(y))$, we obtain $a = O(\varepsilon^2)$.

It is not difficult to verify that

$$\begin{aligned} |\zeta - \xi|^2 &= |1 - re^{i\theta}|^2 + (|v_2|^2 + \cdots + |v_n|^2) \\ &= (1 + r^2 - 2r\cos\theta) + (1 - r^2) \\ &= 2 - 2r\cos\theta \end{aligned} \tag{8.14}$$

and

$$\begin{aligned} d^4(\zeta,\ \xi) &= |1 - \zeta\overline{\xi'}|^2 = 1 + r^2 - 2r\cos\theta \\ &= (2 - 2r\cos\theta) - (1 - r^2) \\ &= |\zeta - \xi|^2 - (1+r)(1-r). \end{aligned} \tag{8.15}$$

Now, it follows from (8.14) that $1 - r^2 \leqslant d^2(\zeta,\ \xi)$. This fact together with (8.15) implies that

$$d^4(\zeta,\ \xi) + (1+r)d^2(\zeta,\ \xi) \geqslant |\zeta - \xi|^2.$$

Because $d^2(\zeta,\ \xi) < 2$, the last inequality indicates that

$$|\zeta - \xi| \leqslant 2d(\zeta,\ \xi). \tag{8.16}$$

Noticing that for $f \in \mathscr{A}$,

$$|f(\zeta) - f(\xi)| \leqslant C|\zeta - \xi|.$$

Hence

$$|f(\zeta) - f(\xi)| \leqslant Cd(\zeta,\ \xi).$$

For any $\rho \in (0, 1)$, because (8.13), we have

$$\begin{aligned} |I_2(\rho,\ \varepsilon)| &\leqslant \int_{S(\zeta,\ \varepsilon)} |H_b(\rho\zeta,\ \overline{\xi})||f(\zeta) - f(\xi)|d\sigma(\zeta) \\ &\leqslant C\int_{S(\zeta,\ \varepsilon)} \frac{1}{d^{2n-1}(\zeta,\ \xi)} d\sigma(\xi) \\ &\leqslant C\int_{v\overline{v'} \leqslant 2\varepsilon^2 - \varepsilon^4} \int_{-a}^{a} \frac{1}{|1 - re^{i\theta}|^{n-1/2}} d\theta dv. \end{aligned}$$

Now we estimate the inner integral. For $n = 2$, Hölder's inequality gives

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{2-1/2}}d\theta &\leqslant \Big(\frac{1}{2a}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\Big)^{3/4}\\ &\leqslant \Big(\frac{1}{2a}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\Big)^{3/4}\\ &\leqslant \Big(\frac{1}{2a}\Big)^{3/4}\frac{1}{(1-r^2)^{3/4}}.\end{aligned}$$

In this case, when $\varepsilon \to 0$,

$$\begin{aligned}|I_2(\rho,\ \varepsilon)| &\leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{(1-r^2)^{3/4}}dv\\ &\leqslant C\varepsilon^{1/2}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\overline{v'})^{3/4}}dv\\ &\leqslant C\varepsilon^{1/2}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{1}{t^{3/2}}dt\\ &\leqslant C\varepsilon \to 0.\end{aligned}$$

For $n > 2$, because r approaches 1, we have

$$\begin{aligned}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{n-(1/2)}}d\theta &\leqslant \frac{C}{(1-r^2)^{n-5/2}}\int_{-\pi}^{\pi}\frac{1}{|1-re^{i\theta}|^{2}}d\theta\\ &\leqslant \frac{C}{(1-r^2)^{n-3/2}}.\end{aligned}$$

Hence as $\varepsilon \to 0$,

$$|I_2(\rho,\ \varepsilon)| \leqslant C\int^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}dt \leqslant C\varepsilon \to 0.$$

Now we prove that if $\rho \to 1-0$, then $I_3(\rho,\varepsilon)$ has a uniform bound for ε near 0. Similar to the above integral, we have

$$\begin{aligned}I_3(\rho,\ \varepsilon) &= \int_{S(\zeta,\ \varepsilon)} H_b(\rho\zeta,\ \overline{\xi})d\sigma(\xi)\\ &= \int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}(t^{n-1}\phi_b(t))^{(n-1)}\Big|_{t=\rho re^{i\theta}}d\theta dv\\ &= \frac{1}{i}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{(t^{n-1}\phi_b(t))^{(n-1)}}{t}dtdv.\end{aligned}$$

Using integration by parts, the inner product for the variable t reduces to

$$\Big[\sum_{k=1}^{n-1}(k-1)!\frac{(t^{n-1}\phi_b(t))^{(n-1-k)}}{t^k}\Big]_{\rho re^{-ia}}^{\rho re^{ia}} + (n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\phi_b(t)}{t}dt$$
$$= \sum_{k=1}^{n-1}\Big[J_k(t)\Big]_{\rho re^{-ia}}^{\rho re^{ia}} + L(r,\ a).$$

We first estimate the integral of J_k. We have

$$\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} J_k(\rho re^{\pm ia})dv \leqslant C\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{|1-\rho re^{\pm ia}|^{n-k}}dv.$$

It can be directly verified that

$$|1-\rho re^{\pm ia}| \geqslant |1-re^{\pm ia}| = \varepsilon^2.$$

So the above integral is dominated by

$$\frac{1}{\varepsilon^{2n-2k}}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4}dv \leqslant \frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}dt$$
$$\leqslant C\varepsilon^{2k-2},$$

where the terms are bounded when $k=1$, tends to zero when $k\geqslant 2$. When $\rho\to 1-0$, the existence of the limit can be deduced from the Lebesgue dominated convergence theorem.

Now,

$$(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\phi_b(t)}{t}dt = (n-1)!i\int_{-a}^{a}\phi_b(t)\Big|_{t=\rho re^{i\theta}}d\theta.$$

By Cauchy's theorem and the estimate of ϕ_b, we can prove that for any $\rho\to 1-0$, the above is a bounded function. This implies that

$$\lim_{\varepsilon\to 0}\int_{v\overline{v'}\leqslant 2\varepsilon^2-\varepsilon^4} L(\rho r,\ a)dv = 0.$$

At last we obtain $\lim\limits_{\rho\to 1-0} I_3(\rho,\ \varepsilon)$ exists and is bounded for small $\varepsilon>0$. This proves Theorem 8.1.3. □

Remark 8.1.2 A corollary of (8.14) is

$$d(\zeta,\ \xi)\leqslant|\zeta-\xi|^{1/2},$$

which is not used in the proof.

Theorem 8.1.4 *The operator M_b can be extended a bounded operator from $L^p(\partial B)$ to $L^p(\partial B)$, $1 < p < \infty$, and from $L^1(\partial B)$ to weak $L^1(\partial B)$.*

Proof The boundedness of $M_b = M_b P$ from $L^2(\partial B)$ to $H^2(\partial B)$ is a direct corollary of the orthogonality of the function system $\{p_\nu^k(\xi)\}$. We only prove the operator is bounded from $L^1(\partial B)$ to weak $-L^1(\partial B)$, that is, the operator is weak (1,1) type. For $1 < p < 2$, the $L^p(\partial B)-$boundedness can be deduced from Marcinkiewicz's interpolation. For $2 < p < \infty$, the L^p-boundedness can be obtained by the property of the kernel

$$\overline{H_b(\zeta,\ \overline{\xi})} = H_b(\xi,\ \overline{\zeta})$$

and the bilinear pair

$$\langle f,\ g\rangle = \int_{\partial B} f(\zeta)\overline{g(\zeta)}d\sigma(\zeta),$$

in the standard duality method.

The weak (1, 1) type boundedness of M_b is based on a Hömander type inequality. The proof given below is different from that of the Cauchy integral in [3]. We will use the non-tangential approach regions

$$D_\alpha(\zeta) = \left\{z \in \mathbb{C}^n :\ |1 - z\overline{\zeta'}| < \frac{a}{2}(1 - |z|^2)\right\},\ \zeta \in \partial B,\ a > 1.$$

□

We shall prove

Lemma 8.1.1 *Assume that $\xi,\ \zeta,\ \eta \in \partial B$, $d(\xi,\ \zeta) < \delta$, $d(\xi,\ \eta) > 2\delta$, and $z \in D_\alpha(\eta)$. Then*

$$\left|H_b(z,\ \overline{\xi}) - H_b(z,\ \overline{\zeta})\right| \leqslant \delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-1/2}.$$

Proof By the estimate

$$\left|\left(r^{n-1}\phi_b(r)\right)^{(n)}\right| \leqslant \frac{C_\omega}{|1 - r|^{n+1}},$$

and the mean value theorem, for some $t \in (0, 1)$, the real part

$$\begin{aligned}
&\left|\mathrm{Re}(r^{n-1}\phi_b(r))^{(n-1)}\ |_{r=z\overline{\xi'}} - \mathrm{Re}(r^{n-1}\phi_b(r))^{(n-1)}\ |_{r=z\overline{\zeta'}}\right| \\
&\leqslant \left|\mathrm{Re}(r^{n-1}\phi_b(r))^{(n)}\ |_{r=z\overline{w'}}\right| \cdot |z\overline{\xi'} - z\overline{\zeta'}| \\
&\leqslant \frac{C_\omega |z\overline{\xi'} - z\overline{\zeta'}|}{|1 - z\overline{w'}_t|^{n+1}},
\end{aligned} \tag{8.17}$$

where $w_t = t\overline{\xi'} + (1 - t)\overline{\zeta'} \in B$.

The imaginary part satisfies a similar inequality.

Denote by ξ_t the projection onto ∂B of ω_t. We can easily prove

(i) as $\delta \to 0$, $|\xi_t - w_t| = 1 - |z_t| = A(t) \to 0$;
(ii) $\xi_t \in S(\xi, \delta) \cap S(\zeta, \delta)$.

It follows from (i) that $\xi_t = \frac{1}{1-A(t)} w_t$. Because $D_\alpha(\eta)$ is an open set, for small $\delta > 0$, i.e., $0 < \delta \leqslant \delta_0$, we have $z_t = (1 - A(t))z \in D_\alpha(\eta)$. We write

$$|1 - z\overline{w'}_t| = |1 - z_t\overline{\xi'}_t|. \tag{8.18}$$

On the other hand, by (4) on page 92 of [3], we have

$$\begin{aligned} |z\overline{\xi'} - z\overline{\zeta'}| &= \frac{1}{1-A(t)}|z_t\overline{\xi'} - z_t\overline{\zeta'}| \\ &\leqslant \frac{1}{1-A(t)}\Big(|z_t\overline{\xi'} - z_t\overline{\xi'}_t| + |z_t\overline{\zeta'} - z_t\overline{\xi'}_t|\Big) \\ &\leqslant \frac{6}{1-A(t)}\delta\alpha^{1/2}|1 - z_t\overline{\xi'}_t|^{1/2} \\ &\leqslant \delta C_\alpha |1 - z_t\overline{\xi'}_t|^{1/2}. \end{aligned} \tag{8.19}$$

By (3) on page 92 of [3], we have

$$|1 - z_t\overline{\xi'}_t|^{-1} \leqslant 16\alpha|1 - \xi\overline{\eta'}|^{-1}. \tag{8.20}$$

The relations (8.18)–(8.20) imply that for $\delta \leqslant \delta_0$, the last part of the inequality (8.17) is dominated by $\delta C_\alpha |1 - \xi\overline{\eta'}|^{-n-1/2}$.

For $\delta \geqslant \delta_0$, on the right hand side of the desired inequality,

$$\delta|1 - \xi\overline{\eta'}|^{-n-1/2}$$

has a positive lower bound which depends on δ_0. Hence it is easy to choose $C = C_{\alpha,\,\delta_0}$ such that the inequality holds. This proves Lemma 8.1.1. □

The weak (1, 1) type boundedness is a special case of Theorem 8.1.5.

Theorem 8.1.5 *For any $\alpha > 1$, there exists a constant $C_\alpha < \infty$ such that for any $f \in \mathscr{A}$ and $t > 0$,*

$$\sigma\Big(\{M_\alpha M_b(f) > t\Big\} \leqslant C_\alpha t^{-1}\|f\|_{L^1(\partial B)},$$

where

$$M_\alpha M_b(f)(\zeta) = \sup\Big\{|M_b(f)(z)| : z \in D_\alpha(\zeta)\Big\}$$

is defined as the non-tangential maximal function of $M_b(f)$ in the region $D_\alpha(\zeta)$.

The proof of Theorem 8.1.5 is based on Lemma 8.1.1 and a covering lemma [3]. To adapt to this case, we can make some modifications on the proof for the corresponding result of the Cauchy integral operator in [3].

It should be pointed out that the class of bounded operators M_b generates an operator algebra. In fact, this operator class is equivalent to the Cauchy–Dunford bounded holomorphic functional calculus of DP, where D is the radial Dirac operator and P is the projection operator from L^p to H^p.

The operator M_b has the following properties, and hence the operator class $\{M_b, b \in H^\infty(S_\omega)\}$ is called the bounded holomorphic functional calculus.

Let $b, b_1, b_2 \in H^\infty(S_\omega)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$, $1 < p < \infty$, $0 < \mu < \omega$. Then

$$\|M_b\|_{L^p(\partial B)\to L^p(\partial B)} \leqslant C_{p,\ \mu}\|b\|_{L^\infty(S_\mu)},$$

$$M_{b_1 b_2} = M_{b_1} \circ M_{b_2},$$

$$M_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 M_{b_1} + \alpha_2 M_{b_2}.$$

The first property follows from Theorem 8.1.4. The second and the third properties can be obtained by the Taylor series expansion of test functions.

Denote by

$$R(\lambda,\ DP) = (\lambda I - DP)^{-1}$$

the resolvent operator of DP at $\lambda \in \mathbb{C}$. For $\lambda \notin [0,\ \infty)$, we prove

$$R(\lambda,\ DP) = M_{\frac{1}{\lambda-(\cdot)}}.$$

In fact, by the relation

$$DP(f)(\zeta) = \sum_{k=1}^{\infty} k \sum_{\nu=1}^{N_k} c_{k\nu} p_\nu^k(\zeta),\ \ f \in \mathscr{A},$$

where $c_{k\nu}$ are the Fourier coefficients of f, the Fourier multiplier $(\lambda - k)$ is associated with the operator $\lambda I - DP$. Hence the Fourier multiplier $(\lambda - k)^{-1}$ is associated with $R(\lambda,\ DP)$. The properties of the functional calculus in relation to the boundedness indicate that for $1 < p < \infty$,

$$\|R(\lambda,\ DP)\|_{L^p(\partial B)\to L^p(\partial B)} \leqslant \frac{C_\mu}{|\lambda|},\ \ \lambda \notin S_\mu.$$

By this estimate, for a function $b \in H^\infty(S_\omega)$ with good decay properties at both the origin and the infinity, the Cauchy–Dunford integral

$$b(DP)f = \frac{1}{2\pi i}\int_{II} b(\lambda)R(\lambda,\ DP)d\lambda f$$

is well defined and is a bounded operator, where II denotes the path containing two rays in

$$S_\omega = \left\{s\exp(i\theta) : \ s \text{ is from } \infty \text{ to } 0\right\} \bigcup \left\{s\exp(-i\theta) : \ s \text{ is from } 0 \text{ to } \infty\right\}, \ 0 < \theta < \omega.$$

Such functions b generate a dense subclass of $H^\infty(S_\omega)$ in the sense of the covering lemma of [17]. By this lemma, we can generalize the definition given by the Cauchy–Dunford integral and define a functional calculus for $b \in H^\infty(S_\omega)$.

Now we prove $b(DP) = M_b$. Assume that b has good decay properties at both the origin and at the infinity, and $f \in \mathscr{A}$. In the following deductions, the order of the integral and the summation can be exchanged. Then we have

$$\begin{aligned} b(DP)(f)(\zeta) &= \frac{1}{2\pi i}\int_{II} b(\lambda)R(\lambda,\ DP)d\lambda f(\zeta) \\ &= \frac{1}{2\pi i}\int_{II} b(\lambda)\sum_{k=1}^{\infty}(\lambda-k)^{-1}\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta)d\lambda \\ &= \sum_{k=1}^{\infty}\left(\frac{1}{2\pi i}\int_{II} b(\lambda)(\lambda-k)^{-1}d\lambda\right)\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta) \\ &= \sum_{k=1}^{\infty} b(k)\sum_{\nu=1}^{N_p} c_{k\nu}p_\nu^k(\zeta) \\ &= M_b(f)(\zeta). \end{aligned}$$

It follows from the estimate of the norm of the resolvent operator $R(\lambda,\ DP)$ that DP is a type ω operator (see [17]). For the bilinear pair and the dual pair $(L^2(\partial B),\ L^2(\partial B))$ used in the proof of Theorem 8.1.4, the operator DP equals to the dual operator on $L^2(\partial B)$, that is,

$$\Big\langle DP(f),\ g\Big\rangle = \Big\langle f,\ DP(g)\Big\rangle,\ f, g \in \mathscr{A},$$

which can be deduced from the Parseval identity

$$\sum_{k=0}^{\infty}\sum_{\nu=1}^{N_k} c_{k\nu}\overline{c'_{k\nu}} = \int_{\partial B} f(\zeta)\overline{g(\zeta)}d\sigma(\zeta).$$

The Parseval identity follows from the orthogonality of $\{p_\nu^k\}$, where $c_{k\nu}$ and $c'_{k\nu}$ are the Fourier coefficients of f and g, respectively.

Under the same bilinear pair, a counterpart result holds for the Banach space dual pair $(L^p(\partial B),\ L^{p'}(\partial B))$, $1 < p < \infty$, $1/p + 1/p' = 1$. In [17, 18], the authors studied the properties on Hilbert spaces and Banach spaces for the generalized type ω operator. It can be verified, without difficulty, that the results of [17, 18] hold for the operator DP.

8.2 Fractional Multipliers on the Unit Complex Sphere

The contents of this section is an extension of the results in Sect. 8.1. We state some new developments of the study on unbounded Fourier multipliers on the unit complex ball, see Li–Qian–Lv [19]. Let

$$
\begin{aligned}
S_\omega &= \Big\{z \in \mathbb{C} \mid z \neq 0 \ \text{ and } \ |\arg z| < \omega\Big\},\\
S_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leqslant \pi \ \text{ and } \ |\arg(\pm z)| < \omega\Big\},\\
W_\omega(\pi) &= \Big\{z \in \mathbb{C} \mid z \neq 0, |\mathrm{Re}(z)| \leqslant \pi \ \text{ and } \ \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega(\pi),\\
H_\omega &= \Big\{z \in \mathbb{C} \mid z = e^{i\omega}, \omega \in W_\omega(\pi)\Big\}.
\end{aligned}
$$

We also need the following function space:

Definition 8.2.1 Let $-1 < s < \infty$. $H^s(S_\omega)$ is defined as the set of all functions in S_ω which satisfy the following conditions:

(1) for $|z| < 1$, b is bounded;
(2) $|b(z)| \leqslant C_\mu |z|^s$, $z \in S_\mu$, $0 < \mu < \omega$.

Remark 8.2.1 The spaces $H^s(S_\omega)$ are extensions of $H^\infty(S_\omega)$ introduced by A. McIntosh et al. For further information on $H^\infty(S_\omega)$, see [10, 17, 20, 21] and the reference therein.

Letting

$$\varphi_b(z) = \sum_{k=1}^{\infty} b(k) z^k.$$

we have the following result.

Lemma 8.2.1 *Let $b \in H^s(S_\omega)$, $-1 < s < \infty$. Then φ_b can be extended holomorphically to H_ω. In addition, for $0 < \mu < \mu' < \omega$ and $l = 0, 1, 2, \ldots,$*

$$\left|\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^l \varphi_b(z)\right| \leqslant \frac{C_{\mu'} l!}{\delta^l(\mu, \mu')\,|1 - z|^{l+1+s}}, \ z \in H_\mu,$$

where $\delta(\mu, \mu') = \min\{1/2, \tan(\mu, \mu'\}$ and $C_{\mu'}$ is the constant in Definition 8.2.1.

Proof Let

$$V_\omega = \Big\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\Big\} \bigcup S_\omega \bigcup (-S_\omega),$$

$$W_\omega = V_\omega \cap \Big\{z \in \mathbb{C} : \ -\pi \leqslant \mathrm{Re} z \leqslant \pi\Big\}$$

and ρ_θ is the ray $r\exp(i\theta)$, $0 < r < \infty$, where θ is chosen such that $\rho_\theta \subsetneq S_\omega$. Define

$$\Psi_b(z) = \frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z)b(\xi)\mathrm{d}\xi,\ z \in V_\omega,$$

where as $\xi \to \infty$, $\exp(iz\xi)$ is decreasing exponentially along ρ_θ. Then we obtain

$$\begin{aligned} \left||z|^{1+s}\,\Psi_b(z)\right| &= \left|\frac{1}{2\pi}\int_{\rho(\theta)} \exp(i\xi z)\,|z|^{1+s}\,b(\xi)\mathrm{d}z\right| \\ &\leqslant \frac{C_{\mu'}}{2\pi}\int_0^\infty \exp(-r|z|\sin(\theta+\arg z))(r|z|)^s\mathrm{d}(r|z|)^s \\ &\leqslant C_{\mu'}. \end{aligned} \tag{8.21}$$

Hence we get $|\Psi_b(z)| \leqslant 1/|z|^{1+s}$. Define

$$\psi_b(z) = 2\pi\sum_{n=-\infty}^{\infty}\Psi_b(z+2n\pi), \quad z \in \bigcup_{n=-\infty}^{\infty}(2n\pi + W_\omega).$$

It is easy to see that ψ_b is holomorphic, 2π-periodic and satisfies $|\psi_b(z)| \leqslant C/|z|^{1+s}$. Let

$$\varphi_b(z) = \psi_b\left(\frac{\log z}{i}\right).$$

For $z \in \exp(iS_\omega)$, we write $z = e^{iu}$, where $u \in S_\omega$. Then $\sin(|u|/2) \leqslant c|u|/2$. This implies that $2 - 2\cos|u| \leqslant c|u|^2$ and $|1 - e^{i|u|}| \leqslant c|u|$. Therefore, (8.21) yields

$$\begin{aligned} |\varphi_b(z)| &\leqslant \frac{C_{\mu'}}{|\log z|^{1+s}} \leqslant \frac{C_{\mu'}}{|\log|z||^{1+s}} \\ &\leqslant \frac{C_{\mu'}}{|1-z|^{1+s}}. \end{aligned}$$

Take the ball

$$B(z,r) = \Big\{\xi : |z-\xi| < \delta(\mu,\mu')|1-z|\Big\}.$$

By Cauchy's formula, we have

$$\varphi_b^{(l)}(z) = \frac{l!}{2\pi i}\int_{\partial B(z,r)} \frac{\varphi(\eta)}{(\eta-z)^{1+l}}d\eta.$$

For any $\eta \in \partial B(z,r)$, we have $|\eta - z| \geqslant (1-\delta(\mu,\mu'))|1-z|$. Then we obtain

$$\begin{aligned}\left|\varphi_b^{(l)}(z)\right| &\leqslant \frac{Cl!\|b\|_{H^s(S^c_\omega)}}{\delta^l(\mu,\mu')|1-z|^l}\left|\int_{\partial B(z,r)}\frac{1}{|1-\eta|^{1+s}}d\eta\right| \\ &\leqslant \frac{Cl!}{\delta^l(\mu,\mu')|1-z|^{l+1+s}}.\end{aligned}$$

□

Theorem 8.2.1 *Let $b \in H^s(S_\omega)$ and*

$$H_b(z,\bar{\xi}) = \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)}, \quad z \in \mathbb{B}_n,\ \xi \in \partial\mathbb{B}_n.$$

Then for $z \in \mathbb{B}_n$, $\xi \in \partial\mathbb{B}_n$ such that $z\bar{\xi}' \in H_\omega$,

$$H_b(z,\bar{\xi}) = \frac{1}{(n-1)!\omega_{2n-1}}(r^{n-1}\varphi_b(r))^{(n-1)}\Big|_{r=z\bar{\xi}'}$$

is holomorphic, where φ_b is the function defined in Lemma 8.2.1. In addition, for $0<\mu<\mu'<\omega$ and $l=0,1,2,\ldots,$

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_{\mu'}l!}{\delta^l(\mu,\mu')\left|1-z\bar{\xi}'\right|^{n+l+s}}, \quad z\bar{\xi}' \in H_\mu,$$

where $\delta(\mu,\mu') = \min\{1/2, \tan(\mu'-\mu)\}$ and $C_{\mu'}$ is the constant in the definition of the function space $H^s(S_\omega)$.

Proof We know that

$$\begin{cases}\varphi_b(z) = \sum_{k=1}^{\infty} b(k)z^k, \\ r^{n-1}\varphi_b(r) = \sum_{k=1}^{\infty} b(k)r^{n+k-1}.\end{cases}$$

Then we have

$$\begin{aligned}\frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)} &= \frac{1}{(n-1)!}\sum_{k=1}^{\infty} b(k)(n+k-1)(n+k-2)\ldots(k+1)r^k \\ &= \sum_{k=1}^{\infty} b(k)r^k\frac{(n+k-1)!}{(n-1)!k!} \\ &= \sum_{k=1}^{\infty}\frac{(n+k-1)(n+k-2)(n+1)n}{k!}b(k)r^k.\end{aligned}$$

Therefore,

$$\frac{1}{(n-1)!}\left(r^{n-1}\varphi_b(r)\right)^{(n-1)}\Big|_{r=z\bar{\xi}'} = \sum_{k=1}^{\infty} b(k)\frac{(n+k-1)(n+k-2)(n+1)n}{k!}(z\bar{\xi}')^k$$
$$= \omega_{2n-1}\sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)\overline{p_v^k(\xi)}$$
$$= \omega_{2n-1}H_b(z,\bar{\xi}).$$

□

By [12, Theorem 3], we can get the following result.

Theorem 8.2.2 *Let s be a negative integer. If* $b \in H^s(S_{\omega,\pm})$,

$$H_b(z,\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} p_v^k(z)p_\mu^l(\xi),\ z\in\mathbb{B},\ \xi\in\partial\mathbb{B}_n,$$

then

$$\left|D_z^l H_b(z,\bar{\xi})\right| \lesssim \frac{C_\mu l!\left[|\ln|1-z\bar{\xi}'||+1\right]}{\delta^l(\mu,\mu')|1-z\bar{\xi}'|^{n+l+s}}.$$

Proof The proof is similar to that of Theorem 8.2.1. We omit the details. □

Given $b \in H^s(S_\omega)$. We define the Fourier multiplier operator $M_b : \mathcal{A} \to \mathcal{A}$ as

$$M_b(f)(\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=0}^{N_k} c_{kv}p_v^k(\xi),\ \xi\in\partial\mathbb{B}_n,$$

where $\{c_{kv}\}$ is the Fourier coefficient of the test function $f \in \mathcal{A}$.

For the above operator M_b, there holds a Plemelj type formula.

Theorem 8.2.3 *Let* $b \in H^s(S_\omega)$, $s > 0$. *Take* $b_1(z) = z^{-s_1}b(z)$, *where* $s_1 = [s]+1$. *The operator* M_b *has a singular integral expression. Precisely, for* $f \in \mathcal{A}$,

$$M_b(f)(\xi) = \lim_{\varepsilon\to 0}\left[\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})D_\eta^{s_1}f(\eta)d\sigma(\eta) + (D_z^{s_1}f)(\xi)\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})d\sigma(\eta)\right],$$

where $\int_{S(\xi,\varepsilon)} H_{b_1}(\xi,\overline{\eta})d\sigma(\eta)$ *is a bounded function of* $\xi\in\partial\mathbb{B}_n$ *and* ε.

Proof Let

$$M_b(f)(\rho\xi) = \sum_{k=1}^{\infty} b(k)\sum_{v=1}^{N_k} c_{kv}p_v^k(\rho\xi),\quad \xi\in\partial\mathbb{B}_n,$$

where

$$c_{kv} = \int_{\partial B}\overline{p_v^k(\eta)}f(\eta)\mathrm{d}\sigma(\eta).$$

We can see that

$$\begin{aligned}
D_z z^{[l]} &= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}} \sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} \left(z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \right) \\
&= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}} \sum_{k=1}^{n} z_k l_k z_1^{l_1} z_2^{l_2} \cdots z_{k-1}^{l_{k-1}} z_k^{l_k-1} z_{k+1}^{l_{k+1}} \cdots z_n^{l_n} \\
&= \sqrt{\frac{l!}{l_1!l_2!\cdots l_n!}} \Big(\sum_{k=1}^{n} l_k \Big) z_1^{l_1} z_2^{l_2} \cdots z_n^{l_n} \\
&= l z^{[l]},
\end{aligned}$$

which yields $D_z p_v^k = k p_v^k$. Then we have

$$\begin{aligned}
M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k) \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta) \\
&= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) k^{s_1} \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta) \\
&= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) D_\eta^{s_1} \overline{p_v^k(\eta)} f(\eta) \mathrm{d}\sigma(\eta).
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
M_b(f)(\rho\xi) &= \sum_{k=1}^{\infty} b(k) \frac{1}{k^{s_1}} \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} (D_\eta^{s_1} f)(\eta) \mathrm{d}\sigma(\eta) \\
&= \sum_{k=1}^{\infty} b_1(k) \sum_{v=1}^{N_k} \int_{\partial B} p_v^k(\rho\xi) \overline{p_v^k(\eta)} (D_\eta^{s_1} f)(\eta) \mathrm{d}\sigma(\eta).
\end{aligned}$$

For any $\varepsilon > 0$, we have

$$\begin{aligned}
M_b(f)(\rho\xi) \;&= \int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi, \bar{\eta}) D_\eta^{s_1} f(\eta) \mathrm{d}\sigma(\eta) \\
&\quad + \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi, \bar{\eta}) (-D_\xi^{s_1} f(\xi) + D_\eta^{s_1} f(\eta)) \mathrm{d}\sigma(\eta) \\
&\quad + D_\xi^{s_1} f(\xi) \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi, \bar{\eta}) \mathrm{d}\sigma(\eta) \\
&=: I_1(\rho, \varepsilon) + I_2(\rho, \varepsilon) + D_\xi^{s_1} f(\xi) I_3(\rho, \varepsilon),
\end{aligned}$$

where

$$I_1(\rho,\varepsilon)=\int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})D_\eta^{s_1} f(\eta)\mathrm{d}\sigma(\eta),$$
$$I_2(\rho,\varepsilon)=\int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})(-D_\xi^{s_1} f(\xi)+D_\eta^{s_1} f(\eta))\mathrm{d}\sigma(\eta),$$
$$I_3(\rho,\varepsilon)=\int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta).$$

For $\rho\to 1-0$, we have

$$\lim_{\rho\to 1-0} I_1(\rho,\varepsilon)=\lim_{\rho\to 1-0}\int_{S^c(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})D_\eta^{s_1} f(\eta)\mathrm{d}\sigma(\eta)$$
$$=\int_{S^c(\xi,\varepsilon)} H_{b_1}(\xi,\bar{\eta})D_\eta^{s_1} f(\eta)\mathrm{d}\sigma(\eta).$$

Now we consider $I_2(\rho,\varepsilon)$. Let $\xi=(1,0,\ldots,0)$. For $\eta\in\partial\mathbb{B}_n$, write

$$\begin{cases}\eta_1=re^{i\theta},\eta_2=v_2,\eta_3=v_3,\ldots,\eta_n=v_n,\\ v=[v_2,v_3,\ldots,v_n].\end{cases}$$

For such $\eta\in\partial\mathbb{B}_n$, $v\bar{v}'=1-r^2$. Without loss of generality, assume that $\xi=1$. We get

$$\left|1-\xi\bar{\eta}'\right|^{1/2}=\left|1-re^{i\theta}\right|^{1/2}=[(1-r\cos\theta)^2+(r\sin\theta)^2]^{1/4}\leqslant\varepsilon.$$

This implies

$$\cos\theta\geqslant\frac{1+r^2-\varepsilon^4}{2r}.$$

The above estimate indicates

$$S(\xi,\varepsilon)=\left\{\eta\mid v\bar{v}'=1-r^2,\cos\theta\geqslant\frac{1+r^2-\varepsilon^4}{2r}\right\}.$$

Because

$$\frac{1+r^2-\varepsilon^4}{2r}\leqslant\cos\theta\leqslant 1,$$

we obtain $1-r\leqslant\varepsilon^2$ and

$$v\bar{v}'=1-r^2\leqslant 1-(1-\varepsilon^2)^2=2\varepsilon^2-\varepsilon^4.$$

Set

$$a=a(r,\varepsilon)=\arccos\left(\frac{1+r^2-\varepsilon^4}{2r}\right).$$

Because $(1-r)^2 \leqslant \varepsilon^4$ and $1-y=O(\arccos^2 y)$, we get $a=O(\varepsilon^2)$. It is easy to see

$$
\begin{aligned}
|\xi-\eta|^2 &= |1-re^{i\theta}|^2+\sum_{k=2}^{n}|v_k|^2 \\
&= (1+r^2-2r\cos\theta)+(1-r^2) \\
&= 2-2r\cos\theta
\end{aligned}
$$

and

$$
\begin{aligned}
d^4(\xi,\eta) &= 1+r^2-2r\cos\theta \\
&= (2-2r\cos\theta)-(1-r^2) \\
&= |\xi-\eta|^2-(1+r)(1-r),
\end{aligned}
$$

that is, $d^2(\xi,\eta) \leqslant |\xi-\eta|$. Since

$$
d^2(\xi,\eta)=[1+r^2-2r\cos\theta]^{1/2} \geqslant 1-r,
$$

we have $1-r \leqslant d^2(\xi,\eta)$, and thus

$$
|\xi-\eta|^2 \leqslant d^4(\xi,\eta)+(1+r)d^2(\xi,\eta).
$$

The fact that $d^2(\xi,\eta) \leqslant 2$ implies

$$
|\xi-\eta|^2 \leqslant 2d^2(\xi,\eta)+2d^2(\xi,\eta)=4d^2(\xi,\eta),
$$

that is, $|\xi-\eta| \leqslant 2d(\xi,\eta)$. Since $f \in \mathcal{A}$, we have

$$
|f(\xi)-f(\eta)| \leqslant C|\xi-\eta| \leqslant Cd(\xi,\eta).
$$

For $\rho \in (0,1)$

$$
\begin{aligned}
\Big|I_2(\rho,\varepsilon)\Big| &\leqslant C\int_{S(\xi,\varepsilon)}\Big|H_{b_1}(\rho\xi,\bar{\eta})\Big|\Big|f(\xi)-f(\eta)\Big|\mathrm{d}\sigma(\eta) \\
&\leqslant C\int_{S(\xi,\varepsilon)}\frac{d(\xi,\eta)}{|1-\xi\bar{\eta}'|^n}\mathrm{d}\sigma(\eta) \\
&\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}\frac{1}{|1-re^{i\theta}|^{n-1/2}}\mathrm{d}\theta\mathrm{d}v.
\end{aligned}
$$

For $n=2$,

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2-1/2}}\mathrm{d}\theta &\leqslant \left(\frac{1}{2a}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4}\\ &\leqslant \left(\frac{1}{2a}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\right)^{3/4}\\ &\leqslant \left(\frac{1}{2a}\right)^{3/4}\frac{1}{(1-r^2)^{3/4}}.\end{aligned}$$

Then we obtain

$$\begin{aligned}|I_2(\rho,\varepsilon)| &\lesssim \int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} a^{1/4}\frac{1}{\left(1-r^2\right)^{3/4}}\mathrm{d}v\\ &\lesssim \varepsilon^{1/2}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{(v\bar{v}')^{3/4}}\mathrm{d}v\\ &= \varepsilon^{1/2}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}}\frac{t}{t^{3/2}}\mathrm{d}t\\ &\lesssim \varepsilon\to 0.\end{aligned}$$

For $n>2$, we have

$$\begin{aligned}\int_{-a}^{a}\frac{1}{\left|1-re^{i\theta}\right|^{n-1/2}}\mathrm{d}\theta &\leqslant C\int_{-a}^{a}\frac{\left|1-r^2\right|^{n-1/2-2}}{\left|1-re^{i\theta}\right|^{n-1/2}}\frac{1}{\left|1-r^2\right|^{n-1/2-2}}\mathrm{d}\theta\\ &\leqslant C\frac{1}{\left|1-r^2\right|^{n-1/2-1}}\int_{-\pi}^{\pi}\frac{1}{\left|1-re^{i\theta}\right|^{2}}\mathrm{d}\theta\\ &\leqslant C\frac{1}{\left|1-r^2\right|^{n-1/2-1}}.\end{aligned}$$

Then we obtain

$$|I_2(\rho,\varepsilon)|\lesssim\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\frac{1}{t^{2n-3}}\mathrm{d}t\lesssim\sqrt{2\varepsilon^2}\to 0.$$

Now we prove that if $\rho\to 1-0$, $I_3(\rho,\varepsilon)$ has a uniformly bounded limit for ε near 0. Integrating as above, we can deduce that

$$\begin{aligned}I_3(\rho,\varepsilon) &= \int_{S(\xi,\varepsilon)} H_{b_1}(\rho\xi,\bar{\eta})\mathrm{d}\sigma(\eta)\\ &= \int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{-a}^{a}\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\mathrm{d}v.\end{aligned}$$

Let $s = \rho r e^{i\theta}$. Then $\mathrm{d}s = is\mathrm{d}\theta$. We can obtain

$$I_3(\rho,\varepsilon) = -i\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\int_{\rho re^{-ia}}^{\rho re^{ia}}\left(s^{n-1}\varphi_{b_1}(s)\right)^{(n-1)}\mathrm{d}s\mathrm{d}v.$$

Using integration by parts, we can see that the inner integral for the variable t reduces to

$$\begin{aligned}
&\int_{-a}^{a}\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-1)}\Big|_{t=\rho re^{i\theta}}\mathrm{d}\theta\\
&=\left[\sum_{k=1}^{n-1}(k-1)!\frac{\left(t^{n-1}\varphi_{b_1}(t)\right)^{(n-k-1)}}{t^k}\right]\Bigg|_{\rho re^{-ia}}^{\rho re^{ia}}+(n-1)!\int_{\rho re^{-ia}}^{\rho re^{ia}}\frac{\varphi_{b_1}(t)}{t}\mathrm{d}t\\
&=\sum_{k=1}^{n-1}[J_k(t)]_{\rho re^{-ia}}^{\rho re^{ia}}+L(r,a).
\end{aligned}$$

We first estimate J_k as

$$\begin{aligned}
&\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}J_k\left(\rho re^{\pm ia}\right)\mathrm{d}v\\
&\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}(k-1)!\frac{\left(\rho re^{\pm ia}\right)^k}{\left(\rho re^{\pm ia}\right)^k}\frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v\\
&\leqslant C\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\frac{1}{\left|1-\rho re^{\pm ia}\right|^{n-k}}\mathrm{d}v.
\end{aligned}$$

Since $\left|1-\rho re^{\pm ia}\right|^2 = 1+\rho^2r^2-2\rho r\cos a$, we have

$$\begin{aligned}
\left|1-\rho re^{\pm ia}\right|^2-\left|1-re^{\pm ia}\right|^2 &= \rho^2r^2-2\rho r\cos a-(r^2-2r\cos a)\\
&= r^2(\rho^2-1)+2r\cos a(1-\rho).
\end{aligned}$$

It follows from the relation $\cos a = (1+r^2-\varepsilon^4)/2r$ that we have

$$\begin{aligned}
\left|1-\rho re^{\pm ia}\right|^2-\left|1-re^{\pm ia}\right|^2 &= r^2(\rho^2-1)+(1+r^2-\varepsilon^4)(1-\rho)\\
&= (1-\rho)[1+r^2-\varepsilon^4-(1+\rho)r^2]\\
&= (1-\rho)(1-\rho r^2-\varepsilon^4) > 0.
\end{aligned}$$

Therefore,

$$\left|1-\rho re^{\pm ia}\right| \geqslant \left|1-re^{\pm ia}\right| = \varepsilon^2.$$

For any fixed k, as $\varepsilon\to 0$, we obtain

$$\begin{aligned}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} J_k\left(\rho r e^{\pm ia}\right)\mathrm{d}v &\leqslant C\frac{1}{\varepsilon^{2n-2k}}\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4}\mathrm{d}v\\ &\leqslant C\frac{1}{\varepsilon^{2n-2k}}\int_0^{\sqrt{2\varepsilon^2-\varepsilon^4}} t^{2n-3}\mathrm{d}t\\ &\leqslant C\frac{\varepsilon^{2n-2}}{\varepsilon^{2n-2k}}\lesssim 1.\end{aligned}$$

On the other hand, as $\rho \to 0$,

$$\begin{aligned}(n-1)!\int_{\rho r e^{-ia}}^{\rho r e^{ia}}\frac{\varphi_{b_1}(t)}{t}\mathrm{d}t &= i(n-1)!\int_{-a}^{a}\varphi_{b_1}(t)\big|_{t=\rho r e^{i\theta}}\,\mathrm{d}\theta\\ &\leqslant C,\end{aligned}$$

which implies

$$\int_{v\bar{v}'\leqslant 2\varepsilon^2-\varepsilon^4} L(\rho r, a)\mathrm{d}v.$$

□

8.3 Fourier Multipliers and Sobolev Spaces on Unit Complex Sphere

We define Sobolev spaces on the n-complex unit sphere $\partial\mathbb{B}_n$ through defining as follows. We define the fractional integrals $\mathcal{I}^s$ on $\partial\mathbb{B}_n$. Let

$$f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k} c_{kv}\,p_v^k(z).$$

For $-\infty < s < \infty$, the operator $\mathcal{I}^s$ is defined as

$$\mathcal{I}^s f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k} k^s c_{kv}\,p_v^k(z).$$

For $s \in \mathbb{Z}_+$, we see that the operator $\mathcal{I}^s$ reduces to the high-order ordinary differential operator.

Theorem 8.3.1 *Let $s \in \mathbb{Z}_+$. $D_z^s = \mathcal{I}^s$ on $L^2(\partial\mathbb{B}_n)$.*

Proof Without loss of generalization, we assume that $f \in \mathcal{A}$. Then

$$f(z)=\sum_{k=0}^{\infty}\sum_{v=0}^{N_k} c_{kv}\,p_v^k(z),$$

where c_{kv} is the Fourier coefficient of f:

$$c_{kv} = \int_{\partial\mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi).$$

So

$$\begin{aligned} D_z^s f(z) &= \sum_{k=0}^{\infty} \sum_{v=0}^{N_k} \int_{\partial\mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi) D_z^s(p_v^k)(z) \\ &= \sum_{k=0}^{\infty} k^s \sum_{v=0}^{N_k} \int_{\partial\mathbb{B}_n} \overline{p_v^k(\xi)} f(\xi) d\sigma(\xi) p_v^k(z). \end{aligned}$$

□

Definition 8.3.1 Let $s \in [0, +\infty)$. The Sobolev norm $\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}$ on $\partial\mathbb{B}_n$ is defined as

$$\|f\|_{W^{2,s}(\partial\mathbb{B}_n)} =: \|\mathcal{I}^s f\|_2 < \infty.$$

The Sobolev space on $\partial\mathbb{B}_n$ is defined as the closure of $\mathcal{A}$ under the norm $\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}$, that is,

$$W^{2,s}(\partial\mathbb{B}_n) = \overline{\mathcal{A}}^{\|\cdot\|_{W^{2,s}(\partial\mathbb{B}_n)}}.$$

Remark 8.3.1 According to Plancherel's theorem, $f \in W^{2,s}(\partial\mathbb{B}_n)$ if and only if

$$\left(\sum_{k=1}^{\infty} k^{2s} \sum_{v=0}^{N_k} |c_{kv}|^2 \right)^{1/2} < \infty.$$

Now we study the boundedness properties of M_b on Sobolev spaces.

Theorem 8.3.2 *Given $r, s \in [0, +\infty)$ and $b \in H^s(S_\omega)$. The Fourier multiplier operator M_b is bounded from $W^{2,r+s}(\partial\mathbb{B}_n)$ to $W^{2,r}(\partial\mathbb{B}_n)$.*

Proof Set

$$\mathcal{I}^s f(z) = \sum_{k=0}^{\infty} \sum_{v=0}^{N_k} c_{kv}^s p_v^k(z).$$

By the orthogonality of $\{p_v^k\}$, we see that $c_{kv}^s = k^s c_{kv}$. Let $b(z) = z^{-s} b(z)$. Because $b \in H^s(S_\omega)$, we have $b_1 \in H^\infty(S_\omega)$. This implies that

$$\begin{aligned} \mathcal{I}^r(M_b(f))(\xi) &= \sum_{k=1}^{\infty} b(k)k^r \sum_{v=0}^{N_k} c_{kv} p_v^k(\xi) \\ &= \sum_{k=1}^{\infty} b_1(k)k^{r+s} \sum_{v=0}^{N_k} c_{kv} p_v^k(\xi) \\ &= M_{b_1}(\mathcal{I}^{r+s} f)(\xi). \end{aligned}$$

Finally, by Theorem 8.1.4, we get

$$\begin{aligned} \|M_b(f)\|_{W^{2,r}} &= \|\mathcal{I}^r(M_b(f))\|_2 \\ &= \|M_{b_1}(\mathcal{I}^{r+s} f)\|_2 \\ &\leqslant C\|\mathcal{I}^{r+s} f\|_2. \end{aligned}$$

This completes the proof of Theorem 8.3.2.

□

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