Chapter 7
The Fractional Fourier Multipliers on Lipschitz Curves and Surfaces

The main contents of this chapter are based on some new developments on the holomorphic Fourier multipliers which are obtained by the two authors in recent years, see the author's paper joint with Leong [1] and the joint work [2]. In the above chapters, we state the convolution singular integral operators and the related bounded holomorphic Fourier multipliers on the finite and infinite Lipschitz curves and surfaces. Let $S^c_{\mu,\pm}$ and S^c_μ be the regions defined in Sect. 1.1. The multiplier b belongs to the class $H^\infty(S^c_{\mu,\pm})$ defined as

$$H^\infty(S^c_\mu) = \Big\{b:\ S^c_\mu \to \mathbb{C}:\ b_\pm = b\chi_{\{z\in\mathbb{C}:\ \pm\mathrm{Re}z>0\}} \in H^\infty(S^c_{\mu,\pm})\Big\},$$

where $H^\infty(S^c_{\mu,\pm})$ is defined as the set of all holomorphic function b satisfying $|b(z)| \leqslant C_\nu$ in any $S^c_{\nu,\pm}$, $0<\nu<\mu$. A natural question is that whether we can establish the corresponding theory of Fourier multiplier operators if b is dominated by a polynomial?

On the other hand, in new progress of Clifford analysis studies, there exist some examples which can not be included in the theory of singular operator on the Lipschitz graph. We give the following example.

Example 7.0.1 In [3, 4], in order to investigate the so-called Photogenic-Dirac equation which have the singular-valued functional solution, D. Eelbode introduce the Photogenic-Cauchy transform C^α_P on the unit sphere in $\mathbb{R}^n$. To give the definition of this transform, we state some backgrounds on this topic.

Let $\mathbb{R}^{1,n}$ be the real orthogonal space with the orthogonal basis $B_{1,n}(\varepsilon, e_j) = \{\varepsilon, e_1, \ldots, e_n\}$ endowed with the quadratic form

$$Q_{1,n}(T, \underline{X}) = T^2 - \sum_{j=1}^{n} X_j^2 = T^2 - R^2,$$

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where we take

$$R = |\underline{X}| = \left(\sum_{j=1}^{n} X_j^2\right)^{1/2}.$$

The orthogonal space $\mathbb{R}^{1,n}$ is called the m-dimensional space-time, n denotes the spatial dimension. The space-time Clifford algebra $\mathbb{R}_{1,n}$ is generated by the following multiplication rules: for all $1 \leqslant i, j \leqslant n$, $e_i e_j + e_j e_i = -2\delta_{ij}$. For all i and $\varepsilon^2 = 1$, $e_i\varepsilon + \varepsilon e_i = 0$. The vectors in $\mathbb{R}^{1,m}$, i.e., $(m+1)$-tuples $(T, \underline{X})$ or space-time vectors is identified with the 1-vectors in $\mathbb{R}_{1,n}$ under the canonical mapping

$$(T, \underline{X}) = (T, X_1, \ldots, X_n) \longmapsto \varepsilon T + \underline{X} \in \mathbb{R}_{1,n}.$$

The Dirac operator on $\mathbb{R}^{1,n}$ is given by the vector derivative

$$D(T, \underline{X})_{1,n} = \varepsilon \partial_T - \sum_{j=1}^{n} e_j \partial_{X_j},$$

which factorizes the wave operator $\Box_n = \partial_T^2 - \Delta_n$ on $\mathbb{R}^{1,n}$ as

$$\Box_n = \left(\varepsilon \partial_T - \sum_{j=1}^{n} e_j \partial_{X_j}\right)^2.$$

For $\alpha + n \geqslant 0$ and $\underline{\omega} \in \mathbb{S}^{n-1}$, we consider the following Photogenic-Dirac equation

$$(\varepsilon \partial_T - \partial_{\underline{X}})\mathcal{F}_{\alpha,\underline{\omega}}(T, \underline{X}) = T^{\alpha+n-1}\delta(T\underline{\omega} - \underline{X})$$

and take the transformation:

$$\lambda = T \text{ and } \underline{x} = \frac{\underline{X}}{T} = r\underline{\xi} \in B_n(1),$$

where $B_n(1)$ is the unit sphere in $\mathbb{R}^n$ and $|\xi| = 1$. In [3], D. Eelbode proved that

$$\begin{aligned}\mathcal{F}_\alpha(\underline{x}, \underline{\omega}) = {} & (2\alpha + n + 1)c(\alpha, n)(\varepsilon + \underline{x})\frac{(1 - r^2)^{\alpha+(n-1)/2}}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+n}} \\ & + (\alpha + n)c(\alpha, n)(\varepsilon + \underline{\omega})\frac{(1 - r^2)^{\alpha+(n+1)/2}}{(1 - \langle \underline{x}, \underline{\omega} \rangle)^{\alpha+n+1}},\end{aligned}$$

where $c(\alpha, n)$ is the constant depending on α and n. In addition, let $f(\underline{\omega})$ be any function defined on the sphere $\mathbb{S}^{n-1}$. For all $\underline{x} \in B_n(1)$, the Photogenic-Cauchy transform of f $C_P^\alpha[f](\underline{x})$ is defined by

$$C_P^\alpha[f](\underline{x}) = \frac{1}{\Omega_n}\int_{\mathbb{S}^{m-1}} \mathcal{F}_\alpha(\underline{x}, \underline{\omega})\underline{\omega} f(\omega)d\omega,$$

where Ω_n is the surface area of the sphere $\mathbb{S}^{n-1}$.

If we apply this transform C_P^α to the inner and outer spherical monogenic polynomials P_k and Q_k on $\mathbb{R}^n \setminus \{0\}$ and let $r \to 1-$, we can obtain the boundary values $C_P^\alpha[P_k] \uparrow$ and $C_P^\alpha[Q_k] \uparrow$ as follows:

$$C_P^\alpha[P_k] \uparrow (\underline{\xi}) = \frac{\Gamma(n/2-1/2)}{8\pi^{n/1-1/2}} \frac{(\alpha+n+k)\{(\alpha+n+k-1)+(k-\alpha)\underline{\xi}\varepsilon\}P_k(\underline{\xi})}{(\alpha+n/2+1/2)(\alpha+n/2-1/2)},$$

$$C_P^\alpha[Q_k] \uparrow (\underline{\xi}) = \frac{\Gamma(n/2-1/2)}{8\pi^{n/1-1/2}} \frac{(1+\alpha-k)\{(\alpha-k)+(\alpha+n+k-1)\underline{\xi}\varepsilon\}Q_k(\underline{\xi})}{(\alpha+n/2+1/2)(\alpha+n/2-1/2)}.$$

It is obvious that the occurrence of

$$k^2P_k(\underline{\xi}),\ kP_k(\underline{\xi}),\ k^2Q_k(\underline{\xi}),\ kQ_k(\underline{\xi})$$

indicates that for $f \in L^2(\mathbb{S}^{n-1})$, the boundary value $C_P^\alpha[f] \uparrow$ does not belong to $L^2(\mathbb{S}^{n-1})$. Hence, in order to obtain the boundedness of this operator, we need to restrict f into a space smaller that $L^2(\mathbb{S}^{n-1})$. In [3], the author replaced $L^2(\mathbb{S}^{n-1})$ by a special Sobolev space and obtained the boundedness of $C_P^\alpha[f] \uparrow$. Based on the above result, in this chapter, we consider the Fourier multiplier b satisfying $|b(\xi)| \leqslant C|\xi+1|^s$ in some region for $s \neq 0$ and study the boundedness of the integral operators associated with these multipliers.

Remark 7.0.2 Particularly, if we take some special b_k in the definition of the Fourier multiplier (see Definition 7.3.2 and the remark below), we can see that the multiplier operator becomes the boundary value of the Cauchy transform on the hyperbolic sphere which was studied in [3, 4].

Compared with the Photogenic-Cauchy transform in Example 7.0.1, there exist two difficulties for the study of Fourier multipliers:

(1) The kernel $\mathcal{F}_\alpha(\underline{x}, \underline{\omega})$ of the Cauchy transform C_P^α can be derived from the fundamental solution of the wave operator $\Box_n$, while the kernel of the Fourier multiplier does not have an explicit expression.
(2) On the unit sphere in $\mathbb{R}^n$, the Plancherel theorem holds. After obtaining the decomposition of $C_P^\alpha(f)$ with respect to the spherical harmonics, the author of [3] can deduce easily that if f belongs to some Sobolev spaces, the function $C_P^\alpha(f)$ belongs to $L^2(\mathbb{S}^{n-1})$. However, in the case of Lipschitz surfaces, there is no corresponding Plancherel theorem, and the method of [3] is invalid.

To overcome the above difficulties, we use the Fueter theorem to estimate the kernel of the multiplier operator. We prove that the kernel of the Fourier multiplier operator has a decay with the form of a polynomial of degree $-(n+s)$. The proof is similar to that of Chap. 6 but with some modifications. When we deal with the case $s < 0$, the function $|x|^s$ is unbounded in the domain $H_{\omega,+}$. After getting the estimate of the kernel on $H_{\omega,-}$, we can not use the Kelvin inversion to obtain the corresponding estimate on $H_{\omega,+}$, see Theorem 7.2.2 for details.

7.1 The Fractional Fourier Multipliers on Lipschitz Curves

In this section, we generalize the results in Chaps. 1 and 2 to the following cases: $|b_n| \leqslant Cn^s$, $-\infty < s < \infty$. Such result corresponds to the fractional integrations and differentials on the closed Lipschitz curve and has a closed relation with the boundary value problem on Lipschitz domains.

We still use the following sets in the complex plane $\mathbb{C}$. For $\omega \in (0, \pi/2]$, write

$$S_{\omega,\pm} = \left\{z \in \mathbb{C} : \ |\arg(\pm z)| < \omega\right\}$$

as the sets defined in Definition 1.2.1. Define the sets

$$W_{\omega,\pm} = \left\{z \in \mathbb{Z} : \ |\mathrm{Re}(z)| \leqslant \pi \text{ and } \mathrm{Im}(\pm z) > 0\right\} \cup S_{\omega},$$

see the following graph (Figs. 7.1 and 7.2):

The periodization of $W_{\omega,\pm}$ is the following heart shaped regions:

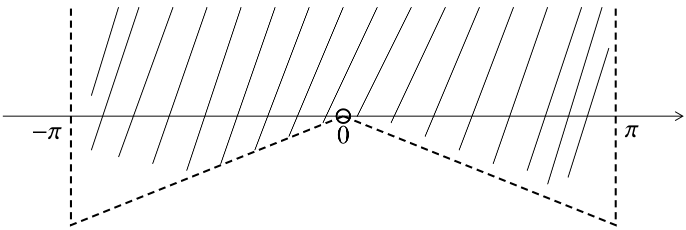


Fig. 7.1 $W_{\omega,+}$

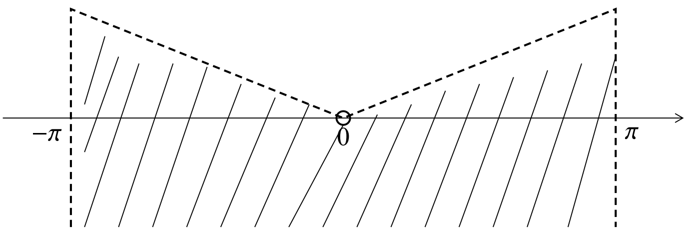


Fig. 7.2 $W_{\omega,-}$

Fig. 7.3 $C_{\omega,+}$

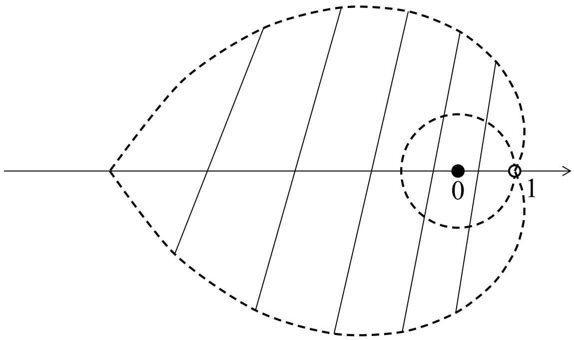
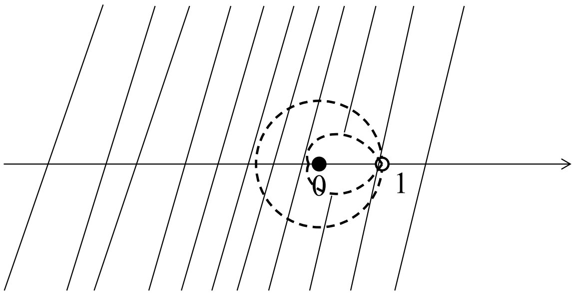


Fig. 7.4 $C_{\omega,-}$



$$C_{\omega,\pm} = \Big\{ z = \exp(i\eta) \in \mathbb{C} : \ \eta \in W_{\omega,\pm} \Big\},$$

which are shown in the following figure (Figs. 7.3 and 7.4):

Define

$$S_\omega = S_{\omega,+} \cup S_{\omega,-},$$

$$W_\omega = W_{\omega,+} \cap W_{\omega,-},$$

and

$$C_\omega = C_{\omega,+} \cap C_{\omega,-}.$$

Let O be a set in the complex plane. If $rz \in O$ for $z \in O$ and all $0 < r \leqslant 1$, we call O the inner starlike region with the pole zero. If $rz \in O$ for $z \in O$ and all $1 \leqslant r < \infty$, we call O the outer starlike region with the pole zero. For $\omega \in (0, \pi/2]$, $C_{\omega,+}$ is heart-shaped and inner starlike with the pole zero, while $C_{\omega,-}$ can be regarded as the complement of a heart shaped region and an outer starlike region with pole zero.

The following function spaces defined on the sectors will be used in the rest of this section. For $-\infty < s < \infty$,

$$H^s(S_{\omega,\pm}) = \Big\{ b : \ S_{\omega,\pm} \to \mathbb{C} \mid b \text{ is holomorphic and satisfies}$$
$$|b(z)| \leqslant C_\mu |z \pm 1|^s \text{ in every } \ S_{\mu,\pm}, 0 < \mu < \omega \Big\}.$$

For $s = -1, -2, \ldots$, we will also use another class of function spaces.

$$H^s_{\ln}(S_{\omega,\pm}) = \Big\{ b : \ S_{\omega,\pm} \to \mathbb{C} \mid b \text{ are holomorphic and satisfies}$$
$$|b(z)| \leqslant C_\mu |z \pm 2|^s \ln |z \pm 2| \text{ in every } \ S_{\mu,\pm}, 0 < \mu < \omega \Big\}.$$

On the double sectors, we can define the corresponding function spaces. For $-\infty < s < \infty$,

$$H^s(S_\omega) = \Big\{ b : S_\omega \to \mathbb{C} \mid b_\pm \in H^s(S_{\omega,\pm}), \ \text{where} \ b_\pm = b\chi_{\{z\in\mathbb{C}, \pm\mathrm{Re}z>0\}} \Big\}$$

and

$$H^s_{\ln}(S_\omega) = \Big\{ b : S_\omega \to \mathbb{C} \mid b_\pm \in H^s_{\ln}(S_{\omega,\pm}), \ \text{where} \ b_\pm = b\chi_{\{z\in\mathbb{C}, \pm\mathrm{Re}z>0\}} \Big\},$$

where χ_E denotes the characteristic function of the set E.

Hence, the function spaces $H^s(S_\omega)$ and $H^s_{\ln}(S_\omega)$ defined above consist of the functions on sectors which are bounded near zero and dominated by $C_\mu |z|^s$ and $C_\mu |z|^s \ln |z|$ at infinity in any smaller sectors than those in which the functions are holomorphically defined.

If a function defined by the Laurent series converges to a holomorphic function in a region, then this function is called holomorphically defined. In this case, by the Abel theorem, the power series part is holomorphically defined in the related inner starlike region with the pole zero. The negative power series part is holomorphically defined in the related outer starlike region with the pole zero.

For $s > -1$, define

$$K^s(C_{\omega,\pm}) = \Big\{ \phi : C_{\omega,\pm} \to \mathbb{C} \mid \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \leqslant \frac{C_\mu}{|1 - z|^{1+s}} \text{ in any } C_{\mu,\pm}, 0 < \mu < \omega \Big\}$$

and

$$K^s(C_\omega) = \Big\{ \phi : C_\omega \to \mathbb{C} \mid \phi \text{ is holomorphic and satisfies}$$
$$|\phi(z)| \leqslant \frac{C_\mu}{|1 - z|^{1+s}} \text{ in any } C_{\mu,\pm}, 0 < \mu < \omega \Big\}.$$

For $-\infty < s \leqslant -1$, we only give the definition of $K^s(C_{\omega,+})$. For $-\infty < s \leqslant -1$, the spaces $K^s(C_{\omega,-})$ and $K^s(S_\omega)$ can be defined similarly. Assume

(i) $\underline{b} = \{b_n\}_{n=0}^{\infty} \in l^\infty$;

(ii) $\phi_{\underline{b}}(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ is holomorphically defined in $C_{\omega,+}$;

(iii) The series $\phi_{\underline{b}}(1) = \sum\limits_{n=0}^{\infty} b_n$ is convergent.

Form the difference

$$\phi_{\underline{b}}(z) - \phi_{\underline{b}}(1) = b_1(z-1) + b_2(z^2-1) + \cdots + b_n(z^n-1) + \cdots + (z-1)\phi_{I(\underline{b})}(z),$$

where

$$I(\underline{b}) = \left(\sum_{k=n}^{\infty} b_k\right)_{n=1}^{\infty} \in l^\infty$$

and

$$\phi_{I(\underline{b})}(z) = \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} b_k\right) z^{n-1}.$$

Then by (ii), $\phi_{I(\underline{b})}$ is holomorphic in $C_{\omega,+}$.

The sequence $I(b)$ constructed above may or may not satisfy the condition (iii). If this sequence satisfies (iii), then it satisfies (i) automatically. Hence $(I(\underline{b}), \phi_{I(\underline{b})})$ satisfies (i), (ii) and (iii). Then we continue to consider if the sequence $I(I(\underline{b})) = I^2(\underline{b})$ satisfies (iii), and so on. Write

$$I(I^n(\underline{b})) = I^{n+1}(\underline{b}) \text{ and } I^0(\underline{b}) = \underline{b}.$$

If the above procedure can be applied at most k times, then the pairs

$$(I^j(\underline{b}),\ \phi_{I^j(\underline{b})}),\ 0 \leqslant j \leqslant k,$$

all satisfy (i), (ii) and (iii), but $I^{k+1}(\underline{b})$ does not satisfy (iii). In this case, we have

$$\phi_{\underline{b}}(z) = \phi_{\underline{b}}(1) + (z-1)\phi_{I(\underline{b})}(1) + \cdots + (z-1)^k \phi_{I^k(\underline{b})}(z). \tag{7.1}$$

Now we begin to define the function class $K^s(C_{\omega,+})$, $-\infty < s \leqslant -1$:

$$K^s(C_{\omega,+}) = \Big\{\phi_{\underline{b}} :\ C_{\omega,+} \to \mathbb{C} \mid \underline{b} \in l^\infty,\ \text{the above procedure can be applied at most } k_s \text{ times},$$
$$\text{where } k_s = [1-s] \text{ or } [-s] \text{ depending on whether } s \text{ is an integer or not},$$
$$\text{and in any } C_{\mu,+}, 0 < \mu < \omega,\ |(z-1)^{k_s}\phi_{I^{k_s}(\underline{b})}(z)| \leqslant \frac{C_\mu}{|z-1|^{1+s}}\Big\},$$

where for $\alpha > 0$, $[\alpha]$ denotes the largest integer which does not exceed α, that is, $[\alpha] = \max\{n \in \mathbb{Z} \mid n \leqslant \alpha\}$.

For $s = -1, -2, \ldots,$ we consider another class of functions

$$K^s_{\ln}(C_{\omega,+}) = \Big\{\phi_{\underline{b}} : C_{\omega,+} \to \mathbb{C} \mid \underline{b} \in l^\infty, \text{ the above procedure can be applied at most } -s-1 \text{ times, and in any } C_{\mu,+}, 0 < \mu < \omega,\ |(z-1)^{-s-1}\phi_{I^{-s-1}(\underline{b})}(z)| \leqslant C\frac{|\ln|z-1||}{|z-1|^{1+s}}\Big\}.$$

It is easy to see that the above spaces $\{H^s(S_{\omega,\pm})\}$ and $\{K^s(C_{\omega,\pm})\}$ are increasing classes along with $s \to \infty$. Now we state the main results of this section. In the rest of this section, the symbol "±" should be understood as either all + or all −.

Theorem 7.1.1 *Let $-\infty < s < \infty$, $s \neq -1, -2, \ldots$, $b \in H^s(S_{\omega,\pm})$, and $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$. Then $\phi \in K^s(C_{\omega,\pm})$.*

Proof We first consider the case $0 \leqslant s < \infty$. Define

$$\Psi(z) = \frac{1}{2\pi}\int_{\rho_\theta} \exp(iz\zeta)b(\zeta)d\zeta,\ z \in V_{\omega,+},$$

where

$$V_{\omega,+} = \Big\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\Big\} \cup S_\omega$$

and ρ_θ denotes the ray: $r\exp(i\theta)$, $0 < r < \infty$. Here θ satisfies $\rho_\theta \in S_{\omega,+}$ and $\exp(iz\zeta)$ is exponentially decaying as $\zeta \to \infty$ along ρ_θ. It is easy to see that Ψ is well defined and holomorphic in $V_{\omega,+}$. In fact, the definition of Ψ is independent of the choice of θ. For any $\mu \in (0, \omega)$, we can see that

$$|\Psi(z)| \leqslant \frac{C_\mu}{|z|^{1+s}},\ z \in V_{\mu,+}.$$

We further define function

$$\Psi^1(z) = \int_{\delta(z)} \Psi(\zeta)d\zeta,\ z \in S_{\omega,+},$$

where $\delta(z)$ is any path from $-z$ to z in V_ω. It follows from Cauchy's formula that for any $\mu \in (0, \omega)$,

$$|\Psi^1(z)| \leqslant \frac{C_\mu}{|z|^s},\ z \in S_{\mu,+}.$$

By the Poisson summation formula, define

$$\psi(z) = 2\pi\sum_{n=-\infty}^{\infty}\Psi(z + 2n\pi),\ z \in \bigcup_{n=-\infty}^{\infty}(2n\pi + W_{\omega,+}),$$

where $\sum$ denotes the summation in the following sense:

(i) for $s > 0$, the series absolutely and locally uniformly converges to a 2π-periodic holomorphic function ψ, and the function $\phi = \psi \circ \ln /i \in K^s(C_{\omega,+})$;
(ii) for $s = 0$, there exists a subsequence $\{n_k\}_1^\infty$ such that the partial sum

$$s_{n_k}(z) = 2\pi \sum_{|n|\leqslant n_k} \Psi(z + 2n\pi)$$

locally uniformly converges to a 2π-periodic function ψ, and $\phi = \psi \circ \ln /i \in K^s(C_{\omega,+})$.

It can be proved that different functions Ψ defined via different subsequences $\{n_k\}$ differ by bounded constants. By use of the estimate of Ψ, it is easy to prove the case $s > 0$.

Now we consider the case $s = 0$. Consider the decomposition

$$\sum_{k=-n}^{n} \Psi(z + 2k\pi) = \Psi(z) + \sum_1 + \sum_2, \ z \in W_{\mu,+},$$

where

$$\sum_1 =: \sum_{k\neq 0}^{\pm n} \Big(\Psi(z + 2k\pi) - \Psi(2k\pi)\Big)$$

and

$$\sum_2 =: \sum_{k=1}^{n} (\Psi^1)'(2k\pi).$$

We will prove that $\sum_1$ is absolutely convergent and bounded, and $\sum_2$ is bounded and convergent in the sense mentioned above. Hence, as the principal part of the sum, $\Psi(z)$ is dominated by $C|z|^{-1}$ as $z \to 0$ and so is the function ψ. Therefore, the function $\phi = \psi \circ \ln /i$ satisfies the desired estimate. To deal with $\sum_1$, we need the following formula derived by Cauchy's formula:

$$|\Psi'(z)| \leqslant \frac{C_\mu}{|z|^{2+s}}, \ z \in W_{\mu,+}.$$

To deal with $\sum_2$, by the mean value theorem, we obtain

$$\begin{aligned}&\sum_{k=1}^{n}(\Psi^1)'(2k\pi)\\&=\left[\int_{2\pi}^{2(n+1)\pi}(\Psi^1)'(r)dr+\sum_{k=1}^{n}(\Psi^1)'(2k\pi)-\mathrm{Re}((\Psi^1)')(\xi_k)-i\mathrm{Im}((\Psi^1)')(\eta_k)\right]\\&=\Psi^1(2(n+1)\pi)-\Psi^1(2\pi)+\sum_{k=1}\Big[(\Psi^1)'(2k\pi)-\mathrm{Re}((\Psi^1)')(\xi_k)-i\mathrm{Im}((\Psi^1)')(\eta_k)\Big],\end{aligned}$$

where $\xi_k,\ \eta_k \in (2k\pi,\ 2(k+1)\pi)$. Then by the estimate of Ψ', the series part in the above expression is absolutely convergent. Because that part is bounded, by choosing a suitable subsequence $\{n_k\}$, we conclude that the part converges to a constant with the same bounds. This completes the proof of the case $s = 0$.

For the case $s < 0$, we apply induction to the interval $-k-1 \leqslant s < -k$, where $k \geqslant 0$ is an integer. We first consider $-1 < s < 0$. Let $b \in H^s(S_{\omega,+})$ and

$$\phi(z)=\sum_{n=1}^{\infty}b(n)z^n,\quad \phi_0(z)=\sum_{n=1}^{\infty}nb(n)z^n, z\phi'(z)=\phi_0(z).$$

Because $b \in H^s(S_{\omega,+})$, we have $(\cdot)b(\cdot) \in H^{s+1}(S_{\omega,+})$, where $0 < s+1 < 1$. As proved above, we get $\phi_0 \in K^{s+1}(C_{\omega,+})$, and the series ϕ_0 locally uniformly converges. This fact enables us to integrate the series $\phi_0(z)/z$ term by term. Notice that the region $C_{\omega,+}$ is starlike. Denote by $l(0, z)$ the segment from 0 to $1 \approx z = x + iy \in C_{\mu,+}$. By the estimate of the functions in $K^{s+1}(C_{\omega,+})$, we obtain

$$\begin{aligned}|\phi(z)| &\leqslant \int_{l(0,z)}\left|\frac{\phi_0(\zeta)}{\zeta}\right||d\zeta|\\&\leqslant C_\mu\int_{l(0,z)}\frac{|d\zeta|}{|1-\zeta|^{s+2}}\\&\leqslant C_\mu\int_0^1\frac{dt}{(|1-tx|+t|y|)^{s+2}}.\end{aligned}$$

To complete the proof, we divide the rest of the proof into two cases: $x \leqslant 1$ and $x > 1$. For $x \leqslant 1$, the above estimate becomes

$$\begin{aligned}\left|\int_0^1\frac{dt}{(1-t(x-|y|))^{s+2}}\right| &= \frac{1}{s+1}\frac{1}{x-|y|}\left[\frac{1}{(|1-x|+|y|)^{s+1}}-1\right]\\&\leqslant \frac{C_{\mu,s}}{|1-z|^{s+1}},\end{aligned}$$

where we used the condition that $z \approx 1 \Longrightarrow x \approx 1, y \approx 0$.

For $x > 1$, because z belongs to the starlike region $C_{\mu,+}$, we can deduce that

$$x-1=|1-x|\leqslant(\tan(\mu))|y|$$

and

$$|y| \geqslant C_\mu(|1-x|+|y|).$$

This fact together with $x \approx 1$ and $y \approx 0$ implies

$$\begin{aligned}
&\int_0^1 \frac{dt}{(|1-tx|+t|y|)^{s+2}} \\
&= \int_0^{1/x} \frac{dt}{(1-t(x-|y|))^{s+2}} + \int_{1/t}^1 \frac{dt}{(t(x+|y|)-1)^{s+2}} \\
&= \frac{1}{s+1}\left[\frac{2x}{x^2-y^2}\frac{x^{s+1}}{|y|^{s+1}} + \frac{1}{x+|y|}\frac{1}{(|1-x|+|y|)^{s+1}} - \frac{1}{x-|y|}\right] \\
&\leqslant \frac{C_\mu}{|1-z|^{s+1}}.
\end{aligned}$$

For $s=-1$, by using the result of the case $s=0$, we can apply a similar argument to obtain

$$|\phi(z)| \leqslant C_\mu \int_{l(0,z)} \frac{1}{|1-\zeta|}|d\zeta| \leqslant C_\mu |\ln|1-z||,$$

where $z \in C_{\mu,+}$.

This completes the proof for the case $-1 \leqslant s < 0$. Below we use induction to the index s :

Let $-k-1 \leqslant s < -k$, where $k \geqslant 0$ is an integer, and let $b \in H^s_\omega$. We define $\underline{b} = \{b(n)\}_{n=1}^\infty$ and get $\phi_{\underline{b}} \in K^s(C_{\omega,+})$.

Now we consider the case $-k-2 \leqslant s < -k-1$, where $k \geqslant 0$ is an integer and $b \in H^s(S_{\omega,+})$. Set

$$\begin{cases} \phi(z) = \sum\limits_{n=1}^{\infty} b(n)z^n, \\ \phi_0(z) = \sum\limits_{n=1}^{\infty} b_0(n)z^n, \end{cases}$$

where $b_0(z) = \sum\limits_{n=0}^{\infty} b(z+n)$. It is easy to see that $b_0 \in H^{s+1}(S_{\omega,+})$. Because $-k-1 \leqslant s+1 < -k$, by induction, we can obtain that $\phi_0 \in K^{s+1}_\omega$. Hence, if s is an integer, $\phi_{I^{[-s-2]}(\underline{b}_0)}$ can be extended to $C_{\omega,+}$ holomorphically. If s is not an integer, $\phi_{I^{[-s-1]}(\underline{b}_0)}$ can be extended to $C_{\omega,+}$ holomorphically. Here $\underline{b}_0 = \{b_0(n)\}_{n=1}^\infty$. In both cases, for $z \in C_{\mu,+}$, we have

$$|(z-1)^{[-s-2]}\phi_{I^{[-s-2]}(\underline{b}_0)}(z)| \leqslant C_\mu \frac{|\ln|z-1||}{|z-1|^{s+2}}$$

or

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b}_0)}(z)| \leqslant \frac{C_\mu}{|z-1|^{s+2}}.$$

Because $I^k \underline{b}_0 = I^{k+1}\underline{b}$ for any $k \to 0$, we have $\phi_{I^k(\underline{b}_0)} = \phi_{I^{k+1}(\underline{b})}$. When s is an integer,

$$|(z-1)^{[-s-1]}\phi_{I^{[-s-1]}(\underline{b})}(z)| \leqslant C_\mu \frac{|\ln|z-1||}{|z-1|^{s+1}}.$$

If s is not an integer,

$$|(z-1)^{[-s]}\phi_{I^{[-s]}(\underline{b})}(z)| \leqslant \frac{C_\mu}{|z-1|^{s+1}}.$$

This proves $\phi \in K^s_\omega$ for $b \in H^s_\omega$, $-k-2 \leqslant s < -k-1$. □

The cases "+" and "−" in Theorem 7.1.1 are associated with power series and negative power series, respectively. By these results, we obtain the result corresponding to the Laurent series.

Corollary 7.1.1 *Let $-\infty < s < \infty$, $s \neq -1, -2, \ldots$, $b \in H^s(S_\omega)$ and*

$$\phi(z) = \sum_{n=-\infty}^{\infty} b(n) z^n.$$

Then $\phi \in K^s(C_\omega)$.

The inverse of Theorem 7.1.1 is the following.

Theorem 7.1.2 *Let $-\infty < s < \infty$ and $\phi \in K^s(C_{\omega,\pm})$. Then for any $\mu \in (0,\omega)$, there exists a function $b^\mu \in H^s(S_{\mu,\pm})$ such that*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n) z^n.$$

Moreover, for $s < 0$ and $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi}\int_{\lambda_\pm(\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta, \tag{7.2}$$

where

$$\lambda_\pm(\mu) = \Big\{\eta \in H^c_{\omega,\pm} \mid \eta = r\exp(i(\pi \pm \mu)),\ r \text{ is from } \pi\sec\mu \text{ to } 0; \\ \text{and } \eta = r\exp(\mp i\mu),\ r \text{ is from } 0 \text{ to } \pi\sec\mu\Big\}$$

and for $s \geqslant 0$, $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left(\int_{l(\varepsilon,|z|^{-1})\cup c_\pm(|z|^{-1},\mu)\cup\Lambda_\pm(|z^{-1}|,\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta + \phi^{|s|}_{\varepsilon,\pm}(z)\right),$$

where if $r \leqslant \pi$,

$$l(\varepsilon, r) = \Big\{\eta = x + iy \mid y = 0, x \textit{ is from } -r \textit{ to } -\varepsilon, \textit{ and from } \varepsilon \textit{ to } r\Big\},$$

$$c_{\pm}(r, \mu) = \Big\{\eta = r\exp(i\alpha) \mid \alpha \textit{ from } \pi \pm \mu \textit{ to } \pi, \textit{ then from } 0 \textit{ to } \mp\mu\Big\},$$

and

$$\Lambda_{\pm}(r, \mu) = \Big\{\eta \in W_{\omega,\pm} \mid \eta = \rho\exp(i(\pi \pm \mu)), \rho \textit{ is from } \pi\sec\mu \textit{ to } r; \\ \textit{and } \eta = \rho\exp(\mp i\mu), \rho \textit{ from } r \textit{ to } \pi\sec\mu\Big\};$$

If $r > \pi$,

$$l(\varepsilon, r) = l(\varepsilon, \pi), \quad c_{\pm}(r, \mu) = c_{\pm}(\pi, \mu) \\ \Lambda_{\pm}(r, \mu) = \Lambda_{\pm}(\pi, \mu).$$

In any case,

$$\phi^{[s]}_{\varepsilon,\pm}(z) = \int_{L_{\pm}(\varepsilon)} \phi(\exp(i\eta))\left(1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right) d\eta,$$

where $L_{\pm}(\varepsilon)$ *is any contour from* $-\varepsilon$ *to* ε *in* $C_{\omega,\pm}$.

Proof Let $\phi \in K^s(C_{\omega,+})$, $-\infty < s < \infty$. We will apply (7.2) or (7.3) to prove that b^{μ} defined above belongs to $H^s(C_{\mu,+})$, and $\phi(z) = \sum_{n=1}^{\infty} b^{\mu}(n)z^n$.

We first consider the case $-\infty < s < 0$. By the expressions (7.2) and (7.1), using the estimate of the function ϕ and Cauchy's theorem, we can prove

$$\lim_{z\to 0} b^{\mu}(z) = \frac{1}{2\pi}\int_{\lambda(\mu)} \exp(i\eta z)\phi(\exp(i\eta))d\eta, \ z \in S_{\mu,+},$$

where

$$\lambda(\mu) = \Big\{\eta \in W_{\omega,+} \mid \eta = r\exp(i(\pi + \mu)), \ r \text{ is from } \pi\sec(\mu) \text{ to } 0, \\ \text{and } \eta = r\exp(-i\mu), \ r \text{ from } 0 \text{ to } \pi\sec(\mu)\Big\},$$

where $|\arg(z)| < \mu < \omega$. Let $|\arg(z)| < \theta < \mu$. By the estimate of ϕ and the property of the path $\lambda(\mu)$, the function b^{μ} satisfies the following estimate (Fig. 7.5):

$$|b^{\mu}(z)| \leqslant C_{\mu}\left(|z|^s + \int_0^{\infty} \exp(-\sin(\mu - \theta)|z|r)\frac{dr}{r^{1+s}}\right) \leqslant C_{\mu,\theta}|z|^s.$$

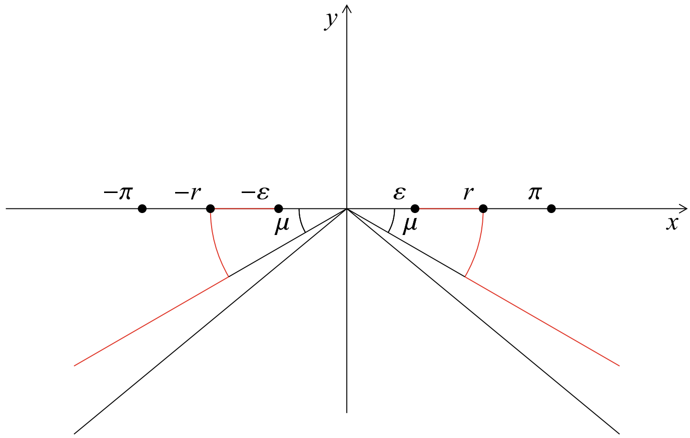


Fig. 7.5 $l_+(\varepsilon, r) \cup c_+(r, \mu) \cup \Lambda_+(r, \mu)$

Now we consider the case $0 \leqslant s < \infty$. By (7.2), without loss of generality, as $z \approx \infty$, assume that $|z|^{-1} \leqslant \pi$. We have

$$\begin{aligned} b^{\mu}(z) &= \frac{1}{2\pi} \lim_{\epsilon \to 0} \left\{ \left(\int_{\epsilon \leqslant |t| \leqslant |z|^{-1}} \exp(-itz)\phi(\exp(it))dt + \phi_{\epsilon}^{[s]}(z) \right) \right. \\ &\quad + \int_{c_+(|z|^{-1},\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta \\ &\quad \left. + \int_{\Lambda_+(|z|^{-1},\mu)} \exp(-i\eta z)\phi(\exp(i\eta))d\eta \right\} \\ &= \frac{1}{2\pi} \lim_{\epsilon \to 0} \left\{ I_1(\epsilon, z) + I_2(z, \mu) + I_3(z, \mu) \right\}, \end{aligned}$$

where $|\arg(z)| < \mu < \omega$,

$$c_+(r, \mu) = \left\{ \eta = r\exp(i\alpha) \mid \alpha \text{ is from } \pi + \mu \text{ to } \pi, \text{ and from } 0 \text{ to } -\mu \right\},$$

and

$$\begin{aligned} \Lambda_+(r, \mu) = \Big\{ &\eta \in W_{\omega,+} \mid \eta = \rho\exp(i(\pi + \mu)),\ \rho \text{ is from } \pi\sec(\mu) \text{ to } r, \\ &\text{and } \eta = \rho\exp(-i\mu),\ \rho \text{ is from } r \text{ to } \pi\sec(\mu) \Big\}. \end{aligned}$$

Now we prove that I_1, I_2, I_3 are uniformly dominated by the bounds indicated in the theorem, and the limit $\lim_{\epsilon \to 0} I_1$ exists.

By Cauchy's theorem, we have

$$\begin{aligned} I_1(\epsilon, z) &= \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt \\ &+ \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(1 + \frac{(-itz)}{1!} + \cdots + \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt + \phi^{[s]}_{\epsilon,+}(z) \\ &= \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left(\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right)\phi(\exp(it))dt \\ &+ \phi^{[s]}_{|z|^{-1},+}(z). \end{aligned}$$

Invoking the estimate of ϕ, we obtain

$$\begin{aligned} &\left|\int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} \left[\exp(-itz) - 1 - \frac{(-itz)}{1!} - \cdots - \frac{(-itz)^{[s]}}{[s]!}\right]\phi(\exp(it))dt\right| \\ &\leqslant C_\mu \int_{\epsilon\leqslant|t|\leqslant|z|^{-1}} |t|^{[s]+1}|z|^{[s]+1}\frac{1}{|t|^{1+s}}dt \\ &\leqslant C_\mu|z|^{[s]+1}\int_0^{|z|^{-1}} t^{[s]-s}dt \\ &= C_\mu|z|^s. \end{aligned}$$

The above argument implies that $\lim_{\epsilon\to 0} I_1$ exists.

To estimate $\phi^{[s]}_{|z|^{-1},+}(z)$, we only need to estimate the integral

$$\int_{L_\pm(|z|^{-1})} \frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))d\eta,\ k = 0, 1, \ldots, [s]. \tag{7.3}$$

Taking the contour $L_+(|z|^{-1})$ as the upper half circle centered at 0 with radius $|z|^{-1}$, we get

$$\begin{aligned} \left|\int_{L_+(|z|^{-1})} \frac{(-i\eta z)^k}{k!}\phi(\exp(i\eta))d\eta\right| &\leqslant C_\mu \int_{L_+(|z|^{-1})} |\eta z|^k|\eta|^{-1-s}|d\eta| \\ &\leqslant C_\mu|z|^s. \end{aligned}$$

To estimate I_2, we have

$$|I_2(z, \mu)| \leqslant C_\mu \int_0^\mu \exp\Big(|\eta||z|\sin(\arg(z) + t)\Big)|\eta|\frac{dt}{|\eta|^{1+s}} \leqslant C_\mu|z|^s.$$

Now we consider I_3. Letting $|\arg(z)| < \theta < \mu$, we get

$$\begin{aligned}|I_3(z,\mu)| &\leqslant C_\mu \int_{\Lambda(|z|^{-1},\mu)} \exp(|\eta||z|\sin(\mu-\theta))\frac{|d\eta|}{|\eta|^{1+s}}\\ &\leqslant C_\mu \int_{|z|^{-1}}^{\infty} r^{-1-s}\exp(-r|z|\sin(\mu-\theta))dr\\ &\leqslant C_{\mu,\theta}|z|^s.\end{aligned}$$

For $z \approx 0$, assume that $|z|^{-1} > \pi$. We first prove that the integral on the contour $l(\epsilon,\pi)$ is uniformly bounded and has limit as $\epsilon \to 0$. Except that the contour in (7.3) should be replaced by $L_+(\pi)$, the argument dealing with $I_1(\epsilon, z)$ for $|z|^{-1} \leqslant \pi$ still applies to the integral on $l(\epsilon,\pi)$. Let the contour $L_+(\pi)$ be the upper half circle centered at 0 with radius π. We have

$$\begin{aligned}\left|\int_{L_+(\pi)} \frac{(-\eta z)^k}{k!}\phi(\exp(i\eta))d\eta\right| &\leqslant C_\mu \int_{L_+(\pi)} |\eta z|^k |\eta|^{-1-s}|d\eta|\\ &\leqslant C_\mu |z|^k\\ &\leqslant C_\mu,\end{aligned}$$

where $k = 1, 2, \ldots, [s]$.

To prove the integrals on $c_+(\pi,\mu)$ and $\Lambda_+(\pi,\mu)$ are bounded, we use Cauchy's theorem to change the contour to the following one:

$$\Big\{z = x+iy \mid x=-\pi,\ y \text{ is from } -\pi\tan(\mu) \text{ to } 0,\ \text{ and } x=-\pi,\ y \text{ is from } 0 \text{ to } -\pi\tan(\mu)\Big\}.$$

However, using the fact that $\mathrm{Re}(z) > 0$, we can conclude that the integrals on the above sets are bounded.

Now we are left to prove

$$\phi(z) = \sum_{n=1}^{\infty} b^\mu(n) z^n,\ -\infty < s < \infty,\ 0 < \mu < \omega.$$

This is equivalent to proving $b(n) = b^\mu(n)$, $n = 1, 2, \ldots$ in these cases.

Let $r \in (0,1)$. Since the series $\phi(rz) = \sum\limits_{n=1}^{\infty} b(n) r^n z^n$ is absolutely convergent in $|z| \leqslant 1$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-itn)\phi(r\exp(it))dt = r^n b_n. \tag{7.4}$$

We first deal with the case $s \geqslant 0$. Write $\delta = -\ln(r)$. Then $r \to 1-0$ if and only if $\delta \to 0+$. Taking the limits $\delta \to 0+$ and $r \to 1-0$ on both sides of (7.4), we conclude that the right hand side tends to b_n, while the limit of the left hand side is

$$\lim_{\delta\to 0+}\int_{-\pi}^{\pi} \exp(-itn)\phi(\exp(-\delta+it))dt.$$

For any fixed $\epsilon \in (0, \pi)$, we can get

$$\lim_{\delta\to 0+}\left(\int_{0\leqslant|t|\leqslant\epsilon}+\int_{\epsilon\leqslant|t|\leqslant\pi}\right)\exp(-itn)\phi(\exp(-\delta+it))dt \tag{7.5}$$

$$= \lim_{\delta\to 0+}\left\{\int_{0\leqslant|t|\leqslant\epsilon}\left(\exp(-itn)-1-\frac{(-itn)}{1!}-\frac{(-itn)^2}{2!}-\cdots-\frac{(-itn)^{[s]}}{[s]!}\right)\right.$$

$$\times\,\phi(\exp(-\delta+it))dt$$

$$+\int_{L_+(\epsilon)}\left(1+\frac{(-itn)}{1!}+\frac{(-itn)^2}{2!}+\cdots+\frac{(-itn)^{[s]}}{[s]!}\right)\phi(\exp(-\delta+it))dt$$

$$\left.+\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(-\delta+it))dt\right\}$$

$$= \lim_{\delta\to 0+}\int_{0\leqslant|t|\leqslant\epsilon}\left(\exp(-itn)-1-\frac{(-itn)}{1!}-\frac{(-itn)^2}{2!}-\cdots-\frac{(-itn)^{[s]}}{[s]!}\right)$$

$$\times\phi(\exp(-\delta+it))dt+\phi^{[s]}_{\epsilon,+}(n)+\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(-\delta+it))dt,$$

where we used Cauchy's theorem and the fact that the last two integrals are absolutely integrable as $\delta \to 0+$. Invoking the estimate of ϕ, the last expression of (7.5) is dominated by

$$C_\mu\int_{0\leqslant|t|\leqslant\epsilon}|tn|^{[s]+1}\frac{1}{|t|^{s+1}}dt,$$

which is independent of $\delta > 0$. Taking the limits $\epsilon \to 0$ on (7.5), the integral tends to 0 and (7.5) reduces to

$$b_n = \lim_{\epsilon\to 0}\left(\int_{\epsilon\leqslant|t|\leqslant\pi}\exp(-itn)\phi(\exp(it))dt+\phi^{[s]}_{\epsilon,+}(n)\right),$$

which equals to (7.3). By the periodicity of the integrand function and Cauchy's theorem, this equals $b^\mu(n)$. The proof for the case $s \geqslant 0$ is complete.

For $s < 0$, by the estimate of the function ϕ and the Lebesgue dominated convergence theorem, we take the limit $r \to 1-0$ on both sides of (7.4) and therefore, obtain

$$b(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp(-itn)\phi(\exp(it))dt.$$

Then by the 2π-periodicity of the integral, Cauchy's theorem and (7.2), the above expression equals to $b^\mu(n)$. This completes the proof of the theorem. □

By Theorems 7.1.1 and 7.1.2, we obtain a result for the case $s \in \mathbb{Z}_-$.

Theorem 7.1.3 *Let s be a negative integer.*

(i) *If* $b \in H^s(S_{\omega,\pm})$ *and* $\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b(n)z^n$, *then* $\phi \in K^s_{\ln}(C_{\omega,\pm})$.

(ii) *If* $\phi \in K^s_{\ln}(C_{\omega,\pm})$, *then for any* $\nu \in (0, \omega)$, *there exists a function* b^μ *such that* $b^\mu \in H^s_{\ln}(S_{\mu,\pm})$, *and*

$$\phi(z) = \sum_{n=\pm 1}^{\pm\infty} b^\mu(n) z^n.$$

Moreover, b^μ *is given by (7.2).*

Proof The conclusion (i) was obtained in Theorem 7.1.1. We only need to prove (ii). By (7.2), it is easy to prove that b^μ is bounded near the origin. For large z, invoking (7.1), we obtain that for $|\arg(z)| < \theta < \mu$,

$$\begin{aligned} |b^\mu(z)| &\leqslant C_\mu \left(|z|^s + \int_0^\infty \exp(-r|z|\sin(\mu-\theta)) |\ln r| r^{-s} \frac{dr}{r} \right) \\ &\leqslant C_\mu \left(|z|^s + |z|^s \int_0^\infty \exp(-r\sin(\mu-\theta)) |\ln r - \ln|z|| r^{-s} \frac{dr}{r} \right) \\ &\leqslant C_{\mu,\theta} |z|^s \ln|z|. \end{aligned}$$

This proves $b^\mu \in H^s_{\ln}(S_{\mu,+})$. The verification of $\phi(z) = \sum\limits_{n=1}^{\infty} b^\mu(n) z^n$ is similar to the case $s < 0$ in Theorem 7.1.2. The proof is complete. □

Remark 7.1.1 For $\{b_n\}_{n=1}^\infty \in l^\infty$, the series

$$\phi(z) = \sum_{n=1}^{\infty} b_n z^n$$

is well-defined on the unit disc and holomorphic. Theorem 7.1.1 and (i) of Theorem 7.1.3 indicate that if there exists $b \in H^s(S_{\omega,+})$ such that $b_n = b(n)$, then ϕ can be extended to $C_{\omega,+}$ holomorphically. In any small $C_{\mu,+}$, when s is an integer, this function satisfies the conditions in the definition of $K^s_{\ln}(S_{\omega,+})$. When s is not an integer, this function satisfies the conditions in the definition of $K^s(S_{\omega,+})$. Theorem 7.1.2 and (ii) of Theorem 7.1.3 give the inverse result.

Remark 7.1.2 Under the assumption of Theorem 7.1.2, the mapping $\phi \to b$ satisfying $\phi(z) = \sum b(n) z^n$ is not single-valued. In fact, by Theorem 7.1.2, any b^μ, $0 < \mu < \omega$, gives a solution of b, and if $\mu_1 \neq \mu_2$, then generally, $b^{\mu_1} \neq b^{\mu_2}$, see also the example in Remark 7.1.3.

Remark 7.1.3 In the proof of Theorem 7.1.2, we need the following function space $\tilde{P}^+_\omega$ which consists of all finite linear combinations of the holomorphic functions with the following form

$$g_n(z) = \begin{cases} 1, & \text{if } z = n, \\ \dfrac{[\exp(i\pi(z-n)) - \exp(-i\pi(z-n))]\exp(-\pi(z-n)\tan\omega)}{2i\pi(z-n)}, & \text{if } z \neq n, \end{cases}$$

where n is a non-negative integer. It is easy to prove

$$|g_n(z)| \leqslant C_{\mu,n} \frac{\exp(-\pi(\text{Re}(z)\tan\omega - |\text{Im}(z)|))}{|z+1|}, \ z \in S_{\mu,+}, 0 < \mu < \omega.$$

Hence $g_n \in \bigcup_{s=-\infty}^{\infty} H^s(S_{\omega,+})$. It is remarkable that the functions in $\tilde{P}_\omega^+$ are the inverse Fourier transforms of the finite polynomials of z given by (7.2) in Theorem 7.1.2. Similarly, we can define the space $\tilde{P}^-$ with respect to the negative integer.

Remark 7.1.4 The holomorphic extension given in Theorem 7.1.1 is optimal in the following sense: if ω is the largest angle such that $b \in H^s(S_{\omega,+})$, then ϕ can not be holomorphically extended to any larger heart-shaped region $C_{\omega+\delta,+}$, $\delta > 0$, which satisfies the corresponding estimate. Or else, by Theorem 7.1.2, we can obtain contradiction.

Remark 7.1.5 (i) of Theorem 7.1.3 corresponds to the function $b(z) = z/(1+z^2)$. Take $s = -1$ for example, A. Baernstein studied that how to construct a holomorphic function in the unit disc such that when $z \to 1$,

$$\phi(z) = O(\ln|z-1|) \text{ and } \phi'(z) \neq O(1/|z-1|),$$

see [5]. At the same time he also proved that it is equivalent to considering the matter in the unit disc instead of in the heart-shaped region. The reason is that the estimates for $s = -1$ remain unchanged after applying a suitable conformal mapping. In Theorem 7.1.1, letting $s = 0$, we conclude that $b(z) \neq O(1/|z|)$ at ∞. However, it is still an open problem that the estimates given in (ii) of Theorem 7.1.3 are the best possible in those cases.

7.2 Fractional Fourier Multipliers on Starlike Lipschitz Surfaces

In this section, we consider a class of Fourier multiplier operators whose multipliers are dominated by a polynomial and give the estimates of the kernels of the integral operators associated with the Fourier multipliers. The main tool is still the generalized Fueter theorem obtained in [6] (see Sect. 3.5). The main idea is to establish a relation between the set O in the complex plane $\mathbb{C}$ and the set $\overrightarrow{O}$ in the $(n+1)$-dimensional space $\mathbb{R}_1^n$, and then transfer the estimate for the functions defined on $\overrightarrow{O}$ to the corresponding one defined on O.

As in Chap. 6, we still use the following intrinsic set. We recall

Definition 7.2.1

(i) A set O in the complex plane $\mathbb{C}$ is called an intrinsic set if the set is systemic about the real axis, that is, the set is unchanged under the complex conjugate.

(ii) If a function f^0 is defined on an intrinsic set in $\mathbb{C}$ and $\overline{f^0(z)} = f^0(\overline{z})$ in the domain, then the function f^0 is called an intrinsic function.

The functions of the form $\sum c_k(z-a_k)^k,\ k \in \mathbb{Z},\ a_k, c_k \in \mathbb{R}$, are all intrinsic functions. If $f = u + iv$, where u and v are real-valued, then f^0 is intrinsic if and only if in their domains, $u(x,-y) = u(x,y)$ and $v(x,-y) = -v(x,y)$.

We regard $\mathbb{R}_1^n$ as the $(n+1)$-dimensional Euclidean space and define the intrinsic set in $\mathbb{R}_1^n$ as follows.

Definition 7.2.2 We call a set in $\mathbb{R}_1^n$ an intrinsic set if it is invariant under all rotations in $\mathbb{R}_1^n$ that keep the e_0 axis fixed. If O is a subset in the complex plane, then in $\mathbb{R}_1^n$, we call the intrinsic set

$$\overrightarrow{O} = \left\{x \in \mathbb{R}_1^n : (x_0, |\underline{x}|) \in O\right\}$$

the set induced by O.

Definition 7.2.3 Let $f^0(z) = u(x,y) + iv(x,y)$ be the intrinsic function defined on the intrinsic set $U \subset \mathbb{C}$. Define the function $\overrightarrow{f^0}$ on the induced set $\overrightarrow{U}$ as follows:

$$\overrightarrow{f^0}(x_0 + \underline{x}) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|).$$

We call $\overrightarrow{f^0}$ the function induced by f^0.

We denote by τ the mapping:

$$\tau(f^0) = k_n^{-1} \Delta^{(n-1)/2} \overrightarrow{f^0},$$

where $\Delta = D\overline{D}$ and $\overline{D} = D_0 - \underline{D}$, $k_n = (2i)^{n-1}\Gamma^2(\frac{n+1}{2})$ is the normalized constant such that $\tau((\cdot)^{-1}) = E$. The operator $\Delta^{(n-1)/2}$ is defined via the Fourier multiplier $m(\xi) = (2\pi i|\xi|)^{n-1}$ defined on the tempered distributions $\mathcal{M} : \mathcal{S}' \to \mathcal{S}'$. Precisely,

$$\mathcal{M}f = \mathcal{R}(m\mathcal{F}f),$$

where

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}_1^n} e^{2\pi i \langle x,\xi\rangle} f(x)dx$$

and

$$\mathcal{R}h(x) = \int_{\mathbb{R}_1^n} e^{-2\pi i \langle x,\xi\rangle} h(\xi)d\xi.$$

The monogenic monomials in $\mathbb{R}_1^n$ are defined by

$$P^{(-k)} = \tau((\cdot)^{-k}) \text{ and } P^{(k-1)} = I(P^{(-k)}), k \in \mathbb{Z}^+,$$

where I denotes the Kelvin inversion $I(f)(x) = E(x)f(x^{-1})$.

We also need the following set in the complex plane. For $\omega \in (0, \frac{\pi}{2})$, let

$$\begin{aligned}
S^c_{\omega,\pm} &= \left\{z \in \mathbb{C} : |arg(\pm z)| < \omega\right\}, \text{ the angle } \arg(z) \in (-\pi, \pi],\\
S^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant \pi,\ z \in S^c_{\omega,\pm}\right\},\\
S^c_{\omega} &= S^c_{\omega,+} \cup S^c_{\omega,-} \text{ and } S^c_{\omega}(\pi) = S^c_{\omega,+}(\pi) \cup S^c_{\omega,-}(\pi),\\
W^c_{\omega,\pm}(\pi) &= \left\{z \in \mathbb{C} : |\mathrm{Re}\, z| \leqslant \pi \text{ and } \pm\ \mathrm{Im} z > 0\right\} \cup S^c_{\omega}(\pi),\\
H^c_{\omega,\pm} &= \left\{z = \exp(i\eta) \in \mathbb{C}, \eta \in W^c_{\omega,\pm}(\pi)\right\}\\
H^c_{\omega} &= H^c_{\omega,+} \cap H^c_{\omega,-}.
\end{aligned}$$

We define the Fourier multipliers in the following function space

$$\begin{aligned}
K^s(H^c_{\omega,\pm}) = \Big\{\phi^0 :\ & H^c_{\omega,\pm} \to \mathbb{C}, \phi^0 \text{ is holomorphic and}\\
&\text{in any } H^c_{\mu,\pm},\ 0 < \mu < \omega, |\phi^0(z)| \leqslant \frac{C_\mu}{|1-z|^{1+s}}\Big\},
\end{aligned}$$

and

$$K^s(H^c_{\omega}) = \left\{\phi^0 :\ H^c_{\omega} \to \mathbb{C},\ \phi^0 = \phi^{0,+} + \phi^{0,-}, \phi^{0,\pm} \in K^s(H^c_{\omega,\pm})\right\}.$$

The corresponding multiplier spaces are

$$\begin{aligned}
H^s(S^c_{\omega,\pm}) = \Big\{b :\ & S^c_{\omega,\pm} \to \mathbb{C},\ b \text{ is holomorphic and in any } S^c_{\mu,\pm},\\
&0 < \mu < \omega, |b(z)| \leqslant C_\mu |z \pm 1|^s\Big\}.
\end{aligned}$$

and

$$H^s(S^c_{\omega}) = \left\{b :\ S^c_{\omega} \to \mathbb{C},\ b_\pm = b\chi_{\{z\in\mathbb{C}:\pm Rez>0\}} \in H^s(S^c_{\omega,\pm})\right\}.$$

Let

$$H_{\omega,\pm} = \left\{x \in \mathbb{R}^n_1 : \frac{(\pm \ln|x|)}{\arg(e_0, x)} < \tan\omega\right\} = \overrightarrow{H^c_{\omega,\pm}},$$

and

$$H_\omega = H_{\omega,+} \cap H_{\omega,-} = \left\{x \in \mathbb{R}^n_1 :\ \frac{|\ln|x||}{\arg(e_0, x)} < \tan\omega\right\} = \overrightarrow{H^c_\omega}.$$

Hence, the corresponding function spaces in $\mathbb{R}^n_1$ are

$$\begin{aligned}
K^s(H_{\omega,\pm}) = \Big\{\phi :\ & H_{\omega,\pm} \to \mathbb{C}_{(n)}, \phi \text{ is monogenic and}\\
&|\phi(x)| \leqslant \frac{C_\mu}{|1-x|^{n+s}},\ x \in H_{\mu,\pm},\ 0 < \mu < \omega\Big\}
\end{aligned}$$

and

$$K^s(H_\omega) = \left\{\phi : H_\omega \to \mathbb{C}_{(n)},\ \phi = \phi^+ + \phi^-,\ \phi^\pm \in K^s(H_{\omega,\pm})\right\}.$$

Now we consider the multipliers $b \in H^s(S^c_{\omega,\pm})$. At first, in the following lemma, we estimate the jth derivative of the intrinsic function ϕ^0.

Lemma 7.2.1 *Assume that $b \in H^s(S^c_{\omega,-})$. For the multiplier defined by $\phi^0(z) = \sum_{k=1}^{\infty} b(-k)z^{-k}$, its jth derivative satisfies*

$$|(\phi^0)^{(j)}(z)| \leqslant \frac{C}{|1-z|^{s+j+1}},$$

where $z \in H^c_{\mu,-}$, $0 < \mu < \omega$ and j is a positive integer.

Proof Without loss of generality, for $b \in H^s(S^c_{\omega,-})$, we assume that $|b(-k)| \leqslant |k|^s$. By Theorem 7.1.1, for $\phi^0(z) = \sum\limits_{k=1}^{\infty} b(-k)z^{-k}$,

$$|\phi^0(z)| \leqslant \frac{C}{|1-z|^{s+1}}.$$

Take a circle $C(z, r)$ centered at z with radius r. By Cauchy's formula, we obtain

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{C_j}{2\pi}\int_{C(z,r)} \frac{|\phi^0(\xi)|}{|z-\xi|^{j+1}}|d\xi|.$$

Let $r = \frac{1}{2}|1-z|$. Then $\xi \in C(z, r)$ implies that

$$|1-\xi| \geqslant |1-z| - |z-\xi| = |1-z| - \frac{1}{2}|1-z| = \frac{1}{2}|1-z|.$$

Therefore we obtain

$$\left|(\phi^0)^{(j)}(z)\right| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)}\frac{1}{|1-z|^{j+s+2}}|1-z| \leqslant C_{\mu,j}\frac{1}{|1-z|^{j+s+1}}.$$

This proves Lemma 7.2.1. □

Lemma 7.2.2 enables us to estimate the kernels of the Fourier multipliers generated by the functions in $H^s(S^c_\omega)$ and the spherical monogenic functions.

Theorem 7.2.1 *For $s > 0$, if $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum\limits_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(x)$, then $\phi \in K^s(H_{\omega,\pm})$.*

Proof Similar to Theorem 6.1.1, we divide the proof into two cases according to the parity of n.

Case 1. n is odd: We assume that $n = 2m + 1$ and restrict the proof to $x \approx 1$. By Lemma 3.5.1, we only need to estimate the corresponding u_l and v_l. There are two subcases to be considered.

Subcase (1.1). $|\underline{x}| > (\delta(\mu)/2^{m+1/2})|1 - x|$. For this case, we write $z = x_0 + i|\underline{x}|$. $x \approx 1$ implies that $z \approx 1$. We can write $z = s + it$, where $s = x_0$ and $t = |\underline{x}|$. We have $t = |\underline{x}| = |1 - z|$.

For $l = 0$, $u_l = u_0 = u$ and $v_l = v_0 = v$. By the estimate of ϕ_0, we have

$$|u_0|,\ |v_0| \leqslant |\phi_0| \leqslant \frac{C}{\delta^0(\mu)} \frac{1}{|1-z|^{s+1}}.$$

For $l = 1$ and $t \approx |1 - z|$, we get

$$|u_1| = \left| 2l \frac{1}{t} \frac{\partial u_0}{\partial t} \right| \leqslant \frac{1}{|1-z|} \frac{1}{|1-z|^{s+2}} = \frac{1}{|1-z|^{s+3}};$$

and

$$\begin{aligned} |v_1| &= \left| \frac{1}{t} \frac{\partial v_0}{\partial t} - \frac{v_0}{t^2} \right| \\ &\leqslant \frac{1}{|1-z|} \frac{1}{|1-z|^{s+2}} + \frac{1}{|1-z|^2} \frac{1}{|1-z|^{s+1}} \\ &= \frac{1}{|1-z|^{s+3}}. \end{aligned}$$

Because $\Delta^1 \phi^0(x) = u_1(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v_1(x_0, |\underline{x}|)$, we have

$$\left| \Delta^1 \phi^0(x) \right| \leqslant C \left| u_1(x_0, |\underline{x}|) \right| + \left| \frac{\underline{x}}{|\underline{x}|} v_1(x_0, |\underline{x}|) \right| \leqslant C \frac{1}{|1-z|^{s+3}}.$$

Repeating the above procedure m times, for u_m and v_m, we obtain

$$|u_m(x)|,\ |v_m(x)| \leqslant \frac{C}{|1-z|^{s+2m+1}} = \frac{1}{|1-z|^{n+s}}.$$

Subcase (1.2). $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1 - x|$. The points x in $H_{\omega,-}$ satisfying $x \approx 1,\ x_0 \leqslant 1$ belong to *Subcase (1.1)*. Hence we assume that $x_0 > 1$. Now we prove the following conclusion: if $z = s + it \approx 1, s > 1, z \in H^c_{\mu,-}$ and $|t| \leqslant (\delta(\mu)/2^{m+1/2}|1 - z|)$, then

(1) the function u_l is an even function with respect to the second variable t.

(2) the jth derivation satisfies

$$\left| \frac{\partial^j}{\partial t^j} u_l(s,t) \right| \leqslant \frac{C_\mu C_l 2^{lj} C_j}{\delta^{2l+j}} \frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is

$$C_j = \begin{cases} (j+4l)!, & j \text{ is even}, \quad (7.7) \\ (j+5l)!, & j \text{ is odd}. \quad (7.7') \end{cases}$$

We apply the mathematical induction to l in order to to prove (1) and (2). Clearly, for $l = 0$, by Lemma 7.2.1, we have

$$\left|\frac{\partial^j}{\partial t^j}u_0(s,t)\right|, \ \left|\frac{\partial^j}{\partial t^j}v_0(s,t)\right| \leqslant \left|\frac{\partial^j}{\partial t^j}\phi^0(s,t)\right| \leqslant \frac{j!}{(\delta(\mu))^j}\frac{1}{|1-z|^{j+s+1}}.$$

Now we assume that (1) and (2) for $0 \leqslant l \leqslant m-1$. Because

$$u_{l+1} = 2(l+1)(1/t)(\partial u_l/\partial t)(s,t)$$

and u_l is even, u_{l+1} is also an even function. This proves (1).

For (2), we first consider the case that j is even. By the definition and (1), $\partial u_l/\partial t$ is an odd function with respect to the second variable t. We can obtain

$$\frac{\partial u_l}{\partial t}(s,0) = \frac{\partial^{2k+1}u_l}{\partial^{2k+1}t}(s,0) = 0.$$

By Taylor's expansion, we have

$$\begin{aligned} u_{l+1}(s,t) &= \frac{2(l+1)}{t}\left(\sum_{k=0}^{\infty}\frac{1}{(2k)!}\frac{\partial^{2k+1}u_l}{\partial t^{2k+1}}(s,0)t^{2k} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k+1}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{(2k)!}\frac{\partial^{2k+1}u_l}{\partial t^{2k+1}}(s,0)t^{2k}. \end{aligned}$$

Letting $k = j/2 + k'$ and noticing that $\left(\frac{t}{\delta|1-z|}\right)^{2k'} \leqslant \left(\frac{1}{2^{m+1/2}}\right)^{2k'}$, we conclude that

$$\begin{aligned} &\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right| \\ &= \left|2(l+1)\sum_{k=j/2}^{\infty}\frac{(2k)(2k-1)\cdots(2k-j+1)}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k-j}\right| \\ &\leqslant 2(l+1)\sum_{k'=0}^{\infty}\frac{(2k'+j)(2k'+j-1)\cdots(2k'+1)}{(2k'+j+1)!}\frac{C_\mu C_l 2^{l(2k'+j+2)(2k'+j+2+4l)}}{\delta^{2l+2k'+j+2}} \\ &\times \frac{t^{2k'}}{|1-z|^{2l+2k'+j+2+s+1}} \end{aligned}$$

$$\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k}.$$

The rest of the proof is similar to that of Theorem 6.1.1. By use of (6.7), we obtain that the series in the last inequality converges and satisfies

$$\sum_{k=0}^{\infty}\frac{(j+2k+2+4l)\cdots(2k+2)}{2^k}\leqslant 2^{j+4l-1}(j+4l+4)!.$$

Finally, we have

$$\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|\leqslant 2(l+1)\frac{C_\mu C_l 2^{l(j+2)}}{\delta^{2(l+1)+j}|1-z|^{2(l+1)+j+1+s}}2^{j+4l-1}(j+4l+4)!.$$

Now we verify that $\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|$ satisfies the estimate for odd j. Similar to the proof for j even, by Taylor's expansion, we have

$$\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)=2(l+1)t\sum_{k=\frac{j+1}{2}}^{\infty}\frac{2k(2k-1)\cdots(2k+1-j)}{(2k+1)!}\frac{\partial^{2k+2}u_l}{\partial t^{2k+2}}(s,0)t^{2k-1-j}.$$

Let $2k-1-j=2k'$. We can obtain

$$\begin{aligned}
&\left|\frac{\partial^j}{\partial t^j}u_{l+1}(s,t)\right|\\
&\leqslant 2(l+1)t\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}\frac{C_\mu C_l 2^{l(2k+3+j)}}{\delta^{2l(2k+3+j)}}\frac{(2k+3+j+5l)!}{|1-z|^{2l+2k+3+j+s+1}}t^{2k}\\
&\leqslant 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}\\
&\quad\sum_{k=0}^{\infty}\frac{(2k+j+1)(2k+j)\cdots(2k+2)}{(2k+j+2)!}2^{kl}\left(\frac{1}{2^{m+1/2}}\right)^{2k}(2k+3+j+5l)!\\
&\leqslant 2(l+1)\left(\frac{t}{\delta|1-z|}\right)\frac{1}{\delta^{2(l+1)+j}}\frac{C_\mu C_l 2^{l(j+3)}}{|1-z|^{2(l+1)+j+s+1}}2^{j+5l+4}((j+5l+3)/2)!
\end{aligned}$$

Letting $j=0$ and $l=m$, we have

$$|u_m(s,t)|\leqslant\frac{C_\mu C_0(4m)!}{\delta^{2m}}\frac{1}{|1-z|^{2m+s+1}}\leqslant\frac{C}{|1-z|^{n+s}}.$$

Now we estimate v_m. As before, we divide the discussion into two cases.
Subcase (1.3). $|\underline{x}|>(\delta(\mu)/2^{m+1/2})$. When $l=0$, noticing that $|t|\approx|1-z|$, we have

$$|v_0(s,t)| = |v(s,t)| \leqslant C\frac{2C_\mu}{|1-z|^{1+s}}.$$

For $l = 1$, because

$$|(\phi^0)^j(z)| \leqslant \frac{2j!C_\mu}{\delta^j(\mu)}\frac{1}{|1-z|^{1+j+s}},$$

we have

$$\begin{aligned}|v_1(s,t)| &\leqslant \frac{2C_\mu}{\delta(\mu)}\left(\frac{1}{|1-z|^{2+s}}\frac{1}{|1-z|} + \frac{1}{|1-z|^2}\frac{1}{|1-z|^{1+s}}\right)\\ &\leqslant \frac{C_\mu}{|1-z|^{s+3}}.\end{aligned}$$

Repeating this procedure m times, we know

$$|v_m(s,t)| \leqslant \frac{C_\mu}{|1-z|^{2m+1+s}} = \frac{C_\mu}{|1-z|^{n+s}}.$$

Subcase (1.4). $|\underline{x}| \leqslant (\delta(\mu)/2^{m+1/2})|1-x|$. For this case, we assume that $x_0 > 1$. For $0 \leqslant l \leqslant m$, we have the following conclusion:

Conclusion (1). $v_l(s,t)$ is odd with respect to the second variable t. In fact, for $l = 0$, $v_0(s,t) = \mathrm{Im}\phi^0(s,t)$. Because $\phi^0(z) = \sum\limits_{k=1}^{\infty} b(-k)z^{-k}$, we have

$$\phi^0(\overline{z}) = \sum_{k=1}^{\infty} b(-k)\overline{z}^{-k} = \overline{\sum_{k=0}^{\infty} b(-k)z^{-k}} = \overline{\phi^0(z)}.$$

Let $\phi^0(z) = u(x,y) + iv(x,y)$, where u and v are real-valued functions. Then

$$u(x,-y) + iv(x,-y) = \overline{u(x,y)} - i\overline{v(x,y)} = u(x,y) - iv(x,y).$$

Hence $v(x,-y) = -v(x,y)$, that is, v_0 is an odd function for the second variable.

For $l = 1$, because (v_0/t) is an even function, $v_1 = 2\frac{\partial}{\partial t}(\frac{v_0}{t})$ is an odd function. We assume that for $0 \leqslant l \leqslant m-1$, v_l is odd. Hence

$$v_m = 2m\left(\frac{1}{t}\frac{\partial v_{m-1}}{\partial t} - \frac{v_{m-1}}{t^2}\right)$$

is also odd. This proves *Conclusion (1)*.

Conclusion (2). For $0 \leqslant l \leqslant m$,

$$\left|\frac{\partial^j}{\partial t^j}v_l(s,t)\right| \leqslant \frac{C_\mu C_l C_j j!}{\delta^j}\frac{1}{|1-z|^{2l+j+s+1}},$$

where the constant C_j is defined by

$$C_j = \begin{cases} (j+5l)!, & \text{if } j \text{ is even,} \\ (j+4l)!, & \text{if } j \text{ is odd.} \end{cases}$$

For simplicity, we only consider the case j is odd. When $l = 0$, it follows from the estimate of $|(\phi^0)^{(j)}|$ that

$$\left|\frac{\partial^j}{t^j} v_0(s,t)\right| \leqslant \frac{C_\mu C_j j!}{(\delta^j)} \frac{1}{|1-z|^{j+s+1}}.$$

Because $v_l(s,t)$ is odd with respect to the second variable, $(\partial^{2k} v_l / \partial t^{2k})(s,0) = 0$. By Taylor's expansion, we have

$$v_{l+1}(s,t) = 2(l+1)\frac{1}{t^2}\sum_{k=0}^{\infty}\left(\frac{1}{(2k)!} - \frac{1}{(2k+1)!}\right) t^{2k+1}\frac{\partial^{2k+1} v_l}{\partial t^{2k+1}}(s,0).$$

Let $k = k' + 1$ and write $k = k'$. We get

$$\frac{\partial^j v_{l+1}}{\partial t^j}(s,t) = 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}\frac{\partial^{2k+3} v_l}{\partial t^{2k+3}}(s,0)(2k+1)\cdots(2k+2-j)t^{2k+1-j}.$$

We assume that *Conclusion (2)* holds for $1 \leqslant l \leqslant m-1$. Letting $2k - j = 2k'$, by $t/(\delta|1-z|) \leqslant 2^{-(m+1/2)}$, we have

$$\begin{aligned} &\left|\frac{\partial^j v_{l+1}}{\partial t^j}(s,t)\right| \\ &\leqslant 2(l+1)\sum_{k=0}^{\infty}\frac{2k+2}{(2k+3)!}(2k+1)\cdots(2k+2-j)\left|\frac{\partial^{2k+3} v_l}{\partial t^{2k+3}}(s,0)\right| t^{2k+1-j} \\ &\leqslant 2(l+1)\frac{1}{2^{m+1/2}}\frac{2^{l(j+3)}}{\delta^{2(l+1)+j}}\frac{1}{|1-z|^{2(l+1)+j+s+1}} \\ &\sum_{k=0}^{\infty}\frac{(2k+j+3+5l)\cdots(2k+j+4)(2k+j+2)\cdots(2k+2)}{2^k}. \end{aligned}$$

This proves Conclusion (2).

Similarly, we can prove the case that n are even. For $j = 0$ and $l = m$, we obtain

$$\left|v_m(s,t)\right| \leqslant \frac{C_\mu C_m (4m)!}{\delta^{2m}} \frac{1}{|1-z|^{2m+1+s}} \leqslant \frac{C_{\mu,\delta}}{|1-z|^{n+s}}.$$

Now we deal with the multipliers defined on the region $S^c_{\omega,+}$. By the Kelvin inversion, for $b \in H^{s,r}(S^c_{\omega,+})$, we estimate the function $\phi(x) = \sum_{i=1}^{\infty} b(i)P^{(i)}(x)$. We have

$$I(\phi)(x) = \sum_{i=-1}^{-\infty} \widetilde{b}(i)P^{(i-1)}(x),$$

where $\widetilde{b}(z) = b(-z) \in H^{s,r}(S^c_{\omega,-})$. Because $I(\phi) = \tau(\phi^0)$, where

$$\phi^0(z) = \sum_{i=-1}^{-\infty} \widetilde{b}(i)z^{i-1} = \frac{1}{z}\sum_{i=-1}^{-\infty} \widetilde{b}(i)z^i \in H^{s,c}_{\omega,-},$$

we have $\phi(x) = I^2(\phi) = E(x)I(\phi)(x^{-1})$ and

$$|\phi(x)| = \left|E(x)I(\phi)(x^{-1})\right| \leqslant \frac{1}{|x|^n}\frac{C_\mu}{|1-x^{-1}|^{n+s}} = \frac{C_\mu|x|^s}{|1-x|^{n+s}}.$$

Because $x \in H_{\nu,+} = \overrightarrow{H^c_{\nu,+}}$, we can see that $(x_0, |\underline{x}|) \in H^c_{\nu,+}$ and

$$|x| = (x_0^2 + |\underline{x}|^2)^{1/2} \leqslant 1 + e^{\tan\nu}.$$

Finally we obtain that $|\phi(x)| \leqslant C_\nu/|1-x|^{n+s}$. This completes the proof of Case 1.

Case 2. n is even. As above, we only need to estimate the kernel ϕ defined on $H_{\omega,-}$. Let $b \in H^{s,r}(S^c_{\omega,-})$. Consider $\phi(x) = \sum_{k=1}^{\infty} b(-k)P_n^{(-k)}(x)$. Because $n+1$ is odd, we have

$$\begin{aligned}
c_{n+1}\phi(x) &= \sum_{k=1}^{\infty} b(-k)\int_{-\infty}^{\infty} P_{n+1}^{(-k)}(x + x_{n+1}e_{n+1})dx_{n+1}\\
&\leqslant c_\mu \int_{-\infty}^{\infty} \frac{1}{|1-(x+x_{n+1}e_{n+1})|^{n+1+s}}dx_{n+1}\\
&= \frac{1}{|1-x|^{n+s}}\int_0^{\infty} \frac{|1-x|}{\left[1+(x_{n+1}/|1-x|)^2\right]^{(n+1+s)/2}}d\left(\frac{x_{n+1}}{|1-x|}\right)\\
&\leqslant \frac{C}{|1-x|^{n+s}}.
\end{aligned}$$

This completes the proof of Theorem 7.2.1. □

The following corollary can be deduced from Theorem 7.2.1.

Corollary 7.2.1 *Let $s > 0$, $b \in H^s(S^c_\omega)$ and*

$$\phi(x) = \left(\sum_{i=1}^{\infty} + \sum_{i=-1}^{-\infty}\right) b(i)P^{(i)}(x).$$

Then $\phi \in K^s(H_\omega)$.

For the case $s < 0$, the proof of the conclusion for the function ϕ is similar to that given in the above theorem. In the following theorem, we prove the conclusion of Theorem 7.2.1 holds for the spaces whose dimension n are odd.

Theorem 7.2.2 *For $s < 0$, $b \in H^s(S^c_{\omega,\pm})$ and $\phi(x) = \sum_{k=\pm 1}^{\pm\infty} b(k)P^{(k)}(x)$, if the spatial dimension n is odd, we have $\phi \in K^s(H_{\omega,\pm})$.*

Proof Because the index s is negative, we can not use the method of Theorem 7.2.1 directly. Precisely, for $s < 0$, $|z|^s$ is unbounded as z approaches the origin. Hence, after getting the estimate of the function ϕ^0 on the region $S^c_{\omega,-}$, we will not use the Kelvin inversion to obtain the estimate on the region $S^c_{\omega,+}$.

To deal with this case, we estimate the function ϕ on the regions $H_{\omega,+}$ and $H_{\omega,-}$. On the region $H_{\omega,-}$, the estimate for the function ϕ is the same as that of Theorem 7.2.1. We omit the details.

For the region $H_{\omega,+}$, because the Kelvin inversion is invalid, we need to estimate the intrinsic function ϕ^0 in the region $H^c_{\omega,+}$. For this purpose, we use Theorem 3.5.1 to obtain that for the odd n, $P^{(k-1)} = \tau((\cdot)^{n+k-2})$, where the mapping τ denotes the operator $\tau(f^0) = k_n^{-1}\Delta^{(n-1)/2}\overrightarrow{f^0}$ and

$$\overrightarrow{f^0}(x) = u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0, |\underline{x}|).$$

Now we complete the estimate for the kernel ϕ. We assume that $b \in H^{s,r}(S^c_{\omega,+})$ and consider $\phi(x) = \sum_{k=1}^{\infty} b(k)P^{(k)}(x)$. By Fueter's theorem, we have

$$\phi(x) = \Delta^m \phi^0(x_0, |\underline{x}|), \text{ where } \phi^0(z) = \sum_{k=1}^{\infty} b(k)z^{n+k-1}.$$

For simplicity, $\phi^0(z) = z^{n-1}\phi^0_1(z)$, where $\phi^0_1(z) = \sum_{k=1}^{\infty} b(k)z^k$. By Theorem 7.1.1, for $b \in H^s(S^c_{\omega,+})$,

$$|\phi^0_1(z)| \leqslant \frac{C}{|1-z|^{1+s}},$$

where $z \in H^c_{\omega,+}$. Then we have

$$|\phi^0(z)| \leqslant \frac{|z|^{n-1}}{|1-z|^{1+s}} \leqslant \frac{C_\omega}{|1-z|^{1+s}},$$

where in the last inequality we have used the fact that the function $|z|^{n-1}$ is bounded on $H^c_{\omega,+}$. Then repeating the procedure used in Theorem 7.2.1, by the estimate of the intrinsic function ϕ^0, we can deduce the estimate of the induced function ϕ. This completes the proof. □

As a direct corollary of Theorem 7.2.2, we have

Corollary 7.2.2 *For the case that the spatial dimension n is odd, Corollary 7.2.1 holds for* $s < 0$.

On $\mathbb{R}^n$, The Fourier theory indicates that there exists a one-one correspondence between the kernels of singular integrals and the symbols of Fourier multipliers. By Theorem 7.2.1, for $b \in H^s(S^c_\omega)$, there exists a function $\phi \in K^s(H_\omega)$. Now we consider the converse of Theorem 7.2.1. For $\phi \in K^s(H_{\omega,\pm})$, we prove that there exists a function $b^\nu(z) \in H^s(S^c_{\nu,\pm})$ such that $b_k = b^\nu(k)$, $0 < \nu < \omega$.

Let $n = 3$. For the case $s = 0$, such function b^ν was obtained by T. Qian in [7]. The main tool is the following polynomial $P^{(k)}$. For any $z \in S^c_\omega$, let

$$\begin{cases} P^{(z)}_- = \tau^0((\cdot)^z), \ z \in S^c_{\omega,-}, \\ P^{(z)}_+ = \tau^0((\cdot)^{z+2}), \ z \in S^c_{\omega,+}, \end{cases}$$

where $(\cdot)^z = \exp(z \ln(\cdot))$. In the first case, the function ln is defined by cutting the positive half x-axis; while in the section case, the function is defined by cutting the negative half x-axis.

By the new functions $P^{(z)}_-$ and $P^{(z)}_+$, we can obtain the following result. For the sake of simplicity, we assume that $n = 3$.

Theorem 7.2.3 *Let* $n = 3$ *and* $-\infty < s < -2$. *If* $\phi(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$, *then for any* $\nu \in (0, \omega)$, *there exists a function* $b^\nu \in H^{s+2}(S^c_{\nu,\pm})$ *such that* $b_i = b^\nu(i)$, $i = \pm 1, \pm 2, \ldots$. *In addition,*

$$b^\nu(z) = \lim_{r \to 1-} \frac{1}{2\pi^2} \int_{L^\pm(\nu)} P^{(z)}(y^{-1}) E(y) n(y) \phi(r^{\pm 1} y) d\sigma(y),$$

where $L^\pm(\nu) = \overrightarrow{\exp(il^\pm(\nu))}$ *and the path* $l^\pm(\nu))$ *is defined as*

$$l^\pm(\nu) = \Big\{ z \in \mathbb{C} : z = r\exp(i(\pi \pm \nu)),\ r \text{ is from } \pi \sec(\nu) \text{ to } 0; \\ \text{and } z = r\exp(-(\pm i\nu)),\ r \text{ is from } 0 \text{ to } \pi \sec(\nu) \Big\}.$$

Proof Recall that $\tau^0 : f^0 \longrightarrow \frac{1}{4}\Delta\overrightarrow{f^0}$. Write $f^0 = \eta^z$, where $\eta = x + iy$. For $x = (x_0, |\underline{x}|) \in L^\pm(\nu)$, there exists $\eta \in \exp(il^\pm(\nu))$ such that $\eta = (x_0, |\underline{x}|)$. Write $\underline{e} = \underline{x}/|\underline{x}|$. We have

$$\Delta\overrightarrow{f^0} = \Delta\overrightarrow{((\cdot)^z)} = \frac{2}{|\underline{x}|}\frac{\partial u}{\partial y}(x_0, |\underline{x}|) + 2\mathbf{e}\left(\frac{1}{|\underline{x}|}\frac{\partial v}{\partial y}(x_0, |\underline{x}|) - \frac{1}{|\underline{x}|^2}v(x_0, |\underline{x}|)\right).$$

Now $f^0 = e^{i\eta z}$, where $\eta \in l^{\pm}(\nu)$. Then $f = u + iv$, where u and v are the real part and the imaginary part of f, respectively. We have $\frac{\partial}{\partial\eta}(e^{i\eta z}) = ize^{i\eta z}$. Let $\eta = re^{-i\mu}$ and $z = |z|e^{i\theta}$. We can get

$$e^{-i\eta z} = \exp(-ir|z|e^{i(\theta-\mu)}) = \exp(r|z|\sin(\theta-\mu))\exp(-ir|z|\cos(\theta-\mu)).$$

Because $\phi \in K^s(S_\omega)$, we have

$$|\phi(x)| \leqslant \frac{C}{|1-x|^{s+3}}, \text{ where } x = x_0 + \underline{x} \in L^{\pm}(\nu).$$

For such a x, there exists a $z = x + iy \in \exp(il^{\pm}(\nu))$ such that $z = e^{i\eta} = \exp(r\sin\mu + ir\cos\mu)$ and $|\underline{x}| = e^{r\sin\mu}\sin(r\cos\mu)$. Then we get

$$|b^\mu(z)| \leqslant C\int_0^{\pi\sec\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{|1-e^{i\eta}|^{s+3}}\frac{1}{|x|}\frac{1}{|\underline{x}|}r^2dr.$$

For the factor $1/|1-e^{i\eta}|^{s+3}$, we have

$$|1-e^{i\eta}|^2 = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu).$$

Let $f(r) = r^2$ and $g(r) = 1 + e^{2r\sin\mu} - 2e^{r\sin\mu}\cos(r\cos\mu)$. We obtain $\lim_{r\to 0}\frac{f(r)}{g(r)} = 1$. Hence we can find a constant C such that

$$\frac{r}{|1-e^{r\sin\mu}e^{ir\cos\mu}|} \leqslant C, \ r \in (0, \pi\sec\mu),$$

that is, $1/|1-e^{r\sin\mu}e^{ir\cos\mu}|^{s+3} \sim r^{s+3}$. Finally we have

$$\begin{aligned}|b^\mu(z)| &\leqslant C\int_0^{\pi\sin\mu} |z|e^{-r|z|\sin(\mu-\theta)}\frac{1}{r^{s+3}}\frac{1}{e^{3r\sin\mu}}\frac{r^2}{e^{r\sin\mu}\sin(r\cos\mu)}dr \\ &\leqslant C|z|\int_0^{\pi\sin\mu} e^{-r|z|\sin(\mu-\theta)}\frac{r^2}{r^{s+4}}e^{-4r\sin\mu}dr \\ &\leqslant C|z|^{s+2},\end{aligned}$$

where in the last inequality we used $s < -2$. □

Theorem 7.2.3 indicates that using the method in [7], for $s \neq 0$, we only get $b \in H^{s+2}(S^c_{\omega,\pm})$ rather than $b \in H^s(S^c_{\omega,\pm})$. To obtain a more precise result, we need apply a new method. It will be based on the following things. First, the desired function b is defined on $S^c_{\omega,\pm} \subset \mathbb{C}$. Secondly, by Proposition 6.1.1, we know that if the dimension n is odd, the polynomials $P^{(-k)}$ and $P^{(k-1)}$, $k \in \mathbb{Z}_+$, satisfy the following relation:

$$P^{(-k)} = \tau((\cdot)^{-k}), \quad P^{(k-1)} = \tau((\cdot)^{k+n-2}).$$

Our idea is to construct a function $\phi^0 \in K^s(H^c_{\omega,\pm})$ by use of $\phi \in K^s(H_{\omega,\pm})$. Then we can express the function b via ϕ^0 by using techniques in complex analysis. At first we give a lemma to show the relation between $H^c_{\omega,\pm}$ and $H_{\omega,\pm}$.

For any element $\mathbf{e}$ in the vector space Q, the linear span of 1 and $\mathbf{e}$ in $\mathbb{R}$ is called the complex plane induced by $\mathbf{e}$ in $\mathbb{R}^n_1$ denoted by $\mathbb{C}^{\mathbf{e}}$. Denote by $H^{\mathbf{e}}_{\omega,\pm}$ and $H^{\mathbf{e}}_{\omega}$ the images on $\mathbb{C}^{\mathbf{e}} \subset \mathbb{R}^{(n)}_1$ of the sets $H^c_{\omega,\pm}$ and H^c_{ω} in $\mathbb{C}$ under the mapping $i_{\mathbf{e}} : a + bi \longrightarrow a + b\mathbf{e}$, respectively. By the same method as that of [7, Lemma 4], we can prove the following lemma.

Lemma 7.2.2

$$H_{\omega,\pm} = \bigcup_{\mathbf{e}\in\mathbf{J}} H^{\mathbf{e}}_{\omega,\pm} \text{ and } H_{\omega,\pm} = \bigcup_{\mathbf{e}\in\mathbf{J}} H^{\mathbf{e}}_{\omega,\pm},$$

where the index set is the set of all unit elements.

Lemma 7.2.2 establishes the relation between the class of monogenic functions and the corresponding holomorphic Fourier multipliers.

Theorem 7.2.4 *Let n be odd and $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$. If the series $\sum_{k\in\mathbb{Z}\setminus\{0\}} b_k z^k$ converges in $H^c_{\omega,\pm}$, then for any $\nu \in (0, \omega)$, there exists a function $b^\nu \in H^s(S^c_{\nu,\pm})$ such that $b_k = b^\nu(k)$, $k \in \mathbb{Z} \setminus \{0\}$.*

Proof We already know that if n is odd, for $k \in \mathbb{Z}_+$,

$$P^{(-k)} = \tau^0((\cdot)^{-k}) \text{ and } P^{(k-1)} = \tau^0((\cdot)^{n+k-1}).$$

For $\phi(x) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x)$ on $H_{\omega,\pm}$, we define the following function ϕ^0 on $H^c_{\omega,\pm}$ as $\phi^0(z) = \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k z^k$, where $z \in H^c_{\omega,\pm}$. For simplicity, we only estimate ϕ^0 in $H^c_{\omega,+}$. Let $\mathbf{e} = \frac{\mathbf{x}}{|\mathbf{x}|}$. For any $z = u + iv \in H^c_{\omega,+}$, by Lemma 7.2.2, we get $x = u + v\mathbf{e} = (x_0, \underline{x}) \in H^{\mathbf{e}}_{\omega,+} \subset H_{\omega,+}$. We have proved that for $z \in H^c_{\omega,+}$, there exists a constant $\delta(\nu) = \min\{1/2, \tan(\omega - \nu)\}$ such that the ball $S_r(z)$ is contained in $H^c_{\omega,\pm}$, where z is the center and the radius is $\delta(\nu)|1 - z|$. We denote by $B(x, r)$ the ball $\Big\{y \in \mathbb{R}^n_1, |x - y| < \delta(\nu)|1 - x|\Big\}$ and have $B(x, r) \subset H^{\mathbf{e}}_{\omega,+} \subset H_{\omega,+}$.

Assume that f and g are the real part and the imaginary part of $\phi^0(z)$, respectively. The induced function is defined by

$$\overrightarrow{\phi^0}(x) = f(x_0, |\underline{x}|) + \mathbf{e}g(x_0, |\underline{x}|)$$

and satisfies $\Delta^{(n-1)/2}\overrightarrow{\phi^0}(x) = \phi(x)$, where $x = (x_0, \underline{x}) = u + v\mathbf{e}$. We can see that

$$|\overrightarrow{\phi^0}(x)| \leqslant \int_{B(x,r)} \frac{c}{|x-y|^2} \frac{C_\nu}{|1-y|^{n+s}} dy.$$

For any $y \in B(x, \delta(\nu)|1-x|)$,

$$|1-y| \geqslant |1-x| - |x-y| > (1-\delta(\nu))|1-x|.$$

We get

$$\begin{aligned} |\overrightarrow{\phi^0}(x)| &\leqslant \frac{C_\nu}{|1-x|^{n+s}} \int_0^{\delta(\nu)|1-x|} \frac{1}{|x-y|^2} |x-y|^{n-1} d(|x-y|) \\ &\leqslant \frac{C_\nu}{|1-x|^{1+s}}. \end{aligned}$$

By the definition of $|\overrightarrow{\phi^0}|$, we have

$$|\phi^0(z)| = |\overrightarrow{\phi^0}(x)| \leqslant \frac{C_\nu}{|1-x|^{1+s}} = \frac{C_\nu}{|1-z|^{1+s}}.$$

By the above estimate, we can construct the function $b \in H^s(S^{\omega}_{\omega,\pm})$ as follows.
For $s < 0$ and $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi} \int_{\lambda_\pm(\mu)} \exp(-i\eta z)\phi^0(\exp(i\eta))d\eta,$$

where

$$\begin{aligned} \lambda_\pm(\mu) = \Big\{ &\eta \in H^c_{\omega,\pm} \mid \eta = r\exp(i(\pi \pm \mu)),\, r \text{ is from } \pi \sec\mu \text{ to } 0 \\ &\text{and } \eta = r\exp(\mp i\mu),\, r \text{ is from } 0 \text{ to } \pi \sec\mu \Big\} \end{aligned}$$

and for $s \geqslant 0$, $z \in S^c_{\mu,\pm}$,

$$b^\mu(z) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \left(\int_{l(\varepsilon,|z|^{-1}) \cup c_\pm(|z|^{-1},\mu) \cup \Lambda_\pm(|z^{-1}|,\mu)} \exp(-i\eta z)\phi^0(\exp(i\eta))d\eta + \phi^{|s|}_{\varepsilon,\pm}(z) \right),$$

where if $r \leqslant \pi$,

$$l(\varepsilon, r) = \Big\{ \eta = x + iy \mid y = 0, x \text{ is from } -r \text{ to } -\varepsilon, \text{ then from } \varepsilon \text{ to } r \Big\},$$

$$c_\pm(r,\mu) = \Big\{ \eta = r\exp(i\alpha) \mid \alpha \text{ is from } \pi \pm \mu \text{ to } \pi, \text{ then from } 0 \text{ to } \mp\mu \Big\},$$

and

$$\Lambda_{\pm}(r,\mu) = \Big\{\eta \in W_{\omega,\pm} \mid \eta = \rho \exp(i(\pi \pm \mu)),\ \rho \text{ is from } \pi \sec\mu \text{ to } r;$$
$$\text{then } \eta = \rho\exp(\mp i\mu),\ \rho \text{ is from } r \text{ to } \pi \sec \mu\Big\},$$

and if $r > \pi$,

$$l(\varepsilon, r) = l(\varepsilon,\pi), \quad c_{\pm}(r,\mu) = c_{\pm}(\pi,\mu), \quad \Lambda_{\pm}(r,\mu) = \Lambda_{\pm}(\pi,\mu).$$

In any case,

$$\phi^{[s]}_{\varepsilon,\pm}(z) = \int_{L_{\pm}(\varepsilon)} \phi^0(\exp(i\eta))\left[1 + (-i\eta z) + \cdots + \frac{(-i\eta z)^{[s]}}{[s]!}\right] d\eta,$$

where $L_{\pm}(\varepsilon)$ is any contour from $-\varepsilon$ to ε in $C_{\omega,\pm}$.

By Cauchy's theorem and the Taylor series expansion, we can use the estimate for ϕ^0 to show $b^\nu \in H^s(S^c_\omega)$ and $b_i = b^\nu(i)$, $i = \pm 1, \pm 2, \ldots$, see Sect. 7.1 for details. □

7.3 Integral Representation of Sobolev–Fourier Multipliers

In this section, we consider a class of Fourier multipliers defined on Sobolev spaces on starlike Lipschitz surfaces. If a Lipschitz surface Σ is n-dimensional and starlike about the origin and there exists a constant $M < \infty$ such that $x_1, x_2 \in \Sigma$,

$$\frac{\left|\ln |x_1^{-1} x_2|\right|}{\arg(x_1, x_2)} \leqslant M, \tag{7.8}$$

we call Σ a starlike Lipschitz surface. We denote by $N = Lip(\Sigma)$ the minimum of M such that (7.8) holds.

Let $s \in \mathbb{R}^n_1$. For $x \in \mathbb{R}^n_1$, we define the mapping $r_s : x \to sxs^{-1}$. By (i) and (iv) of Lemma 6.2.1, we can prove that if x' and x belong to a starlike Lipschitz surface with the Lipschitz constant N, then

$$\left(\left|\ln |x^{-1}x'|\right| / \arg(x, x')\right) = \left|\ln ||x|^{-1}\tilde{x}|\right| / \arg(1, |x|^{-1}\tilde{x}) \leqslant N,$$

that is, $|x|^{-1}\widetilde{x} \in H_\omega$. This gives the relation between the set H_ω and the starlike Lipschitz surface.

We use $\mathcal{M}_k$ for the finite dimensional right module of k homogeneous monogenic functions in $\mathbb{R}^n_1$ and use $\mathcal{M}_{-(k+n)}$ for the right dimensional right module of $-(k+n)$-homogeneous monogenic functions in $\mathbb{R}^n_1 \setminus \{0\}$. The spaces $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$ are eigenspaces of the left Dirac operator Γ_ξ. We define

$$P_k : f \to P_k(f) \text{ and } Q_k : f \to Q_k(f)$$

as the projections on $\mathcal{M}_k$ and $\mathcal{M}_{-(k+n)}$, respectively.

The Fourier multipliers are defined on the following test function space:

$$\mathcal{A} = \Big\{ f : \text{ for some } s > 0, f(x) \text{ is left monogenic in } \rho - s < |x| < l + s \Big\}.$$

For $f \in \mathcal{A}$, in the annuals where f is defined, we have the Laurant series expansion

$$f(x) = \sum_{k=0}^{\infty} P_k(f)(x) + \sum_{k=0}^{\infty} Q_k(f)(x).$$

Here we have used the projection operators P_k and Q_k defined as follows:

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^k C^{+}_{n+1,k}(\xi, \eta) E(y) \mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} |y^{-1}x|^{-n-k} C^{-}_{n+1,k}(\xi, \eta) E(y) \mathfrak{n}(y) f(y) d\sigma(y),$$

where $x = |x|\xi, y = |y|\eta$ and $\mathfrak{n}(y)$ is the outer unit normal of Σ at y. Here $C^{+}_{n+1,k}(\xi, \eta)$ and $C^{-}_{n+1,k}(\xi, \eta)$ are the functions defined as

$$\begin{aligned} C^{+}_{n+1,k}(\xi, \eta) = \frac{1}{1-n} \Big[& -(n+k-1) C_k^{(n-1)/2}(\langle \xi, \eta \rangle) \\ & + (1-n) C_{k-1}^{(n+1)/2}(\langle \xi, \eta \rangle)(\langle \xi, \eta \rangle - \overline{\xi}\eta) \Big] \end{aligned}$$

and

$$\begin{aligned} C^{-}_{n+1,k}(\xi, \eta) = \frac{1}{n-1} \Big[& (k+1) C_{k+1}^{(n-1)/2}(\langle \xi, \eta \rangle) \\ & + (1-n) C_k^{(n+1)/2}(\langle \eta, \xi \rangle)(\langle \eta, \xi \rangle - \overline{\eta}\xi) \Big], \end{aligned}$$

where C_k^{ν} is the Gegenbaur polynomial of degree k associated with ν (see [8]).

Now, on the starlike Lipschitz surface Σ, we give the Fourier multiplier induced by the sequence $\{b_k\}$, where $b_k = b(k)$ for any function $b \in H^s(S^c_\omega)$. We can see from Theorem 7.2.1 that the corresponding kernel ϕ satisfies

$$|\phi(x)| \leqslant C_\mu / |1-x|^{n+s} \text{ for } s > 0.$$

The regularity index s indicates that we can not define the Fourier multipliers for $f \in L^2(\Sigma)$ as the bounded Fourier multipliers in Sect. 6.2. To compensate the role of s, we need to restrict these multipliers on some subspace of $L^2(\Sigma)$. Hence we use the following Sobolev spaces on the starlike Lipschitz surface Σ.

Definition 7.3.1 Let $s \in \mathbb{Z}^+ \cup \{0\}$ and Σ be a starlike Lipschitz surface. For $1 \leqslant p < \infty$, define the norm of Sobolev space $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$ as

$$\|f\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)} = \|f\|_{L^p(\Sigma)} + \sum_{j=0}^{s} \|\Gamma_\xi^j f\|_{L^p(\Sigma)}.$$

The Sobolev space associated with the spherical Dirac operator Γ_ξ is defines as the closure of the class $\mathcal{A}$ under the norm $\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}$, that is, $\overline{\mathcal{A}}^{\|\cdot\|_{W^{p,s}_{\Gamma_\xi}(\Sigma)}}$.

Now we give the definition of the Fourier multipliers. By Definition 7.3.1, $\mathcal{A}$ is dense in $W^{p,s}_{\Gamma_\xi}$. Hence when we define the Fourier multipliers, we assume that $f \in \mathcal{A}$.

Definition 7.3.2 For the sequence $\{b_k\}_{k\in\mathbb{Z}}$ satisfying $|b_k| \leqslant k^s$, we define the Fourier multiplier $M_{(b_k)}$ as follows:

$$M_{(b_k)}f(x) = \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x).$$

Remark 7.3.1 When Σ is the unit sphere, if we take two sequences $\{b_k^{(1)}\}$and $\{b_k^{(2)}\}$, where $b_k^{(1)} = k^2$ and $b_k^{(2)} = k$, the Fourier multipliers in Definition 7.3.2 reduce to the boundary values of the Photogenic-Cauchy integrals on the hyperbolic unit sphere, see Example 7.0.1.

Now for $k \geqslant 0$, we define

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^+_{n+1,k}(\xi, \eta)$$

and

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^-_{n+1,k}(\xi, \eta).$$

The projections P_k and Q_k can be expressed by

$$P_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(k)}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y)$$

and

$$Q_k(f)(x) = \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^{(-k-1)}(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y).$$

If we use

$$\widetilde{\phi}(y^{-1}x) = \sum_{-\infty}^{\infty} b_k \widetilde{P}^{(-k)}(y^{-1}x)$$

to denote the kernel of the Fourier multiplier $M_{(b_k)}$ in Definition 7.3.2, we get the following estimate.

Theorem 7.3.1 *Let* $\omega \in (\arctan(N), \pi/2)$ *and* $b \in H^s(S^c_\omega)$. *The kernel* $\widetilde{\phi}(y^{-1}x)$ $E(y)$ *associated with* $\{b_k\}$ *in the manner given above is monogeneically defined in a neighborhood of* $\Sigma \times \Sigma \setminus \{(x, y) : x = y\}$. *In addition, in this neighborhood,*

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^{n+s}}.$$

Proof The proof of this theorem is similar to Proposition 6.2.3. We omit the details. □

For $f \in \mathcal{A}$, the multiplier $M_{(b_k)}$ introduced above is well-defined. For $b \in H^s(S^c_\omega)$, we consider the following multiplier $M^r_{(b_k)}$:

$$M^r_{(b_k)}(f)(x) = \sum_{k=0}^{\infty} b_k P_k(f)(rx) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x), \ \rho - s < |x| < l + s,$$

where $x \in \Sigma, \ r \approx 1$ and $r < 1$.

We use M_1 and M_2 to denote the two sums in the expression of $M^r_{(b_k)}$. Because $b \in H^s(S^c_\omega)$, b is bounded near the origin and $|b(z)| \leqslant |z|^s$ when $|z| > 1$. We deduce that for $|z| > 1$, $|b(z)| \leqslant |z|^s < |z|^{s_1}$. Hence for $s_1 = [s] + 1$, $b \in H^{s_1}(S^c_\omega)$. Write $b_1(z) = z^{-s_1} b(z)$. We see that $|b_1(z)| \leqslant |b(z)/z^{s_1}| \leqslant C$ implies $b_1(z) \in H^\infty(S^c_\omega)$, where

$$H^\infty(S^c_{\mu,\pm}) = \Big\{ b : \ S^c_{\mu,\pm} \to \mathbb{C} : \ b \text{ is holomorphic, and satisfies}$$
$$|b(z)| \leqslant C_\nu \text{ in any } S^c_{\nu,\pm}, \ 0 < \nu < \mu \Big\}$$

and

$$H^\infty(S^c_\mu) = \Big\{ b : \ S^c_\mu \to \mathbb{C} : \ b_\pm = b\chi_{\{z \in \mathbb{C}: \ \pm \mathrm{Re} z > 0\}} \in H^\infty(S^c_{\mu,\pm}) \Big\},$$

where $S^c_{\mu,\pm}$ and S^c_μ are sectors.

For M_1, $|b_k| = |b(k)| \leqslant k^{s_1}$, we take $b_1(z) = z^{-s_1}b(z)$. It is easy to see that b_1 is also holomorphic in S^c_ω. Then we have

$$M_1 = \sum_{k=0}^{\infty} b_k P_k(f)(rx) = \sum_{k=0}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx),$$

where $b_{1,k} = b_1(k) = \frac{b_k}{k^{s_1}}$. Because M_k is an eigenspace of the spherical Dirac operator Γ_ξ, we have

$$\Gamma_\xi P_k(f)(rx) = kP_k(f)(rx)$$

and

$$M_1 = \sum_{k=0}^{\infty} b_{1,k} \Gamma_\xi^{s_1} P_k(f)(rx) = \Gamma_\xi^{s_1} \left(\sum_{k=0}^{\infty} b_{1,k} P_k(f)(rx) \right).$$

By a result of [8], we obtain another expression of $P_k(f)$.

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n} \int_\Sigma \widetilde{P}^k(y^{-1}rx) E(y) \mathfrak{n}(y) f(y) d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(rx) W_{\underline{\alpha}}(y) \mathfrak{n}(y) f(y) d\sigma(y), \end{aligned}$$

where we have used the Cauchy–Kovalevska expansion

$$\widetilde{P}^{(k)}(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}}(x) \in \mathcal{M}_k$ and $W_{\underline{\alpha}}(y) \in \mathcal{M}_{-n-k}$ (see [8, Chap. 2, (1.15)]). By the above relation, we have

$$\begin{aligned} \Gamma_\xi P_k(f)(x) &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} (\Gamma_\xi V_{\underline{\alpha}})(x) W_{\underline{\alpha}}(y) \mathfrak{n}(y) f(y) d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} k V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y) \mathfrak{n}(y) f(y) d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} \frac{k}{n+k-2} V_{\underline{\alpha}}(x)(n+k-2) W_{\underline{\alpha}}(y) \mathfrak{n}(y) f(y) d\sigma(y) \\ &= \frac{k}{(n+k-2)\Omega_n} \int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta W_{\underline{\alpha}})(y) \mathfrak{n}(y) f(y) d\sigma(y). \end{aligned}$$

Because the Fourier expansion of the functions in $\mathcal{A}$ is rapidly decaying, via integration by parts, we have

$$\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} b_{1,k} k^{s_1} P_k(f)(rx) \\
&= \sum_{k=1}^{\infty} b_{1,k} \left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)(\Gamma_\eta^{s_1} W_{\underline{\alpha}})(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=1}^{\infty} b_{1,k} \left(\frac{k}{n+k-2}\right)^{s_1} \frac{r^k}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x) W_{\underline{\alpha}}(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1} f)(y) d\sigma(y).
\end{aligned}$$

Since $|b_{1,k}(\frac{k}{n+k-2})^{s_1}| \leqslant C$, if we denote $b_{1,k}(\frac{k}{n+k-2})^{s_1}$ by $b_{1,k}$, we can obtain the following singular integral expression of M_1:

$$\begin{aligned}
M_1 &= \sum_{k=1}^{\infty} b_{1,k} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^k(y^{-1}rx) E(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \left(\sum_{k=1}^{\infty} b_{1,k} \widetilde{P}^k(y^{-1}rx)\right) E(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_1(y^{-1}rx) E(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1} f(y)) d\sigma(y).
\end{aligned}$$

Similarly, for M_2, applying the Cauchy–Kovalevska expansion again ([8, Chap. II, (1.16)]), we have

$$\begin{aligned}
M_2 &= \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(r^{-1}x) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x) k^{s_1} \overline{V}_{\underline{\alpha}}(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x)(\Gamma_\eta^{s_1} \overline{V}_{\underline{\alpha}})(y)\mathfrak{n}(y) f(y) d\sigma(y) \\
&= \sum_{k=0}^{\infty} \frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1} \frac{1}{\Omega_n} \int_{\Sigma} \sum_{|\underline{\alpha}|=k} W_{\underline{\alpha}}(r^{-1}x) \overline{V}_{\underline{\alpha}}(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1} f)(y) d\sigma(y).
\end{aligned}$$

As above, we still denote $\frac{b_{-k-1}}{(k+1)^{s_1}} \left(\frac{k+1}{k}\right)^{s_1}$ by b_{-1-k}, and obtain the singular integral expression of M_2 as

$$
\begin{aligned}
M_2 &= \sum_{k=0}^{\infty} b_{-k-1} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{P}^{-k-1}(y^{-1}r^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y)d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \left(\sum_{k=0}^{\infty} b_{-k-1}\widetilde{P}^{-k-1}(y^{-1}r^{-1}x) \right) E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f)(y)d\sigma(y) \\
&= \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_2(y^{-1}r^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_{\eta}^{s_1} f(y))d\sigma(y).
\end{aligned}
$$

Finally we rewrite the multiplier $M^r_{(b_k)}(f)(x)$ as

$$
M^r_{(b_k)}(f)(x) = \lim_{r\to 1-} \frac{1}{\Omega_n} \int_{\Sigma} (\widetilde{\phi}_1(y^{-1}rx) + \widetilde{\phi}_2(y^{-1}r^{-1}x))E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y),
$$

where we have used the fact that for $f \in \mathcal{A}$, the series which defines $M^r_{b_k}(f)$ is uniformly convergent as $r \to 1-$.

For $M_{(b_k)}(f)(x)$, we have the following boundary value result.

Theorem 7.3.2 *Let $s > 0$. If $b \in H^s(S^c_\omega)$, then for $f \in \mathcal{A}$ and $x \in \Sigma$, we have*

$$
\begin{aligned}
M_{(b_k)}(f)(x) &= \lim_{r\to 1-} \frac{1}{\Omega_n} \int_{\Sigma} (\widetilde{\phi}_1(y^{-1}rx) + \widetilde{\phi}_2(y^{-1}r^{-1}x))E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\Omega_n} \left\{ \int_{|y-x|>\varepsilon, y\in\Sigma} [\widetilde{\phi}_1(y^{-1}x) + \widetilde{\phi}_2(y^{-1}x)]E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y) \right. \\
&\quad \left. +(\widetilde{\phi}_1(\varepsilon, x) + \widetilde{\phi}_2(\varepsilon, x))f(x) \right\}.
\end{aligned}
$$

Here

$$
\widetilde{\phi}_1(\varepsilon, x) = \int_{S(\varepsilon,x,+)} \widetilde{\phi}_1(y^{-1}x)E(y)\mathfrak{n}(y)d\sigma(y)
$$

and

$$
\widetilde{\phi}_2(\varepsilon, x) = \int_{S(\varepsilon,x,-)} \widetilde{\phi}_2(y^{-1}x)E(y)\mathfrak{n}(y)d\sigma(y),
$$

where $S(\varepsilon, x, \pm)$ is the part of the sphere $|y - x| = \varepsilon$ inside or outside Σ depending on the index of $\widetilde{\phi}_i$ taking $i = 1$ or $i = 2$.

Proof The proof of this theorem is similar to the classical Plemelj formula of the Cauchy integral. For simplicity, we only consider

$$
\lim_{r\to 1-} I = \lim_{r\to 1-} \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_{\xi}^{s_1} f)(y)d\sigma(y).
$$

The other integral can be dealt with similarly. For a fixed $\varepsilon > 0$, the above integral can be divided into three parts:

$$\begin{aligned} I &= \frac{1}{\Omega_n}\int_{\Sigma}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &= \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)(\Gamma_\xi^{s_1}f)(y)d\sigma(y)\\ &+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)[(\Gamma_\xi^{s_1}f)(y)-(\Gamma_\xi^{s_1}f)(x)]d\sigma(y)\\ &+ \frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)d\sigma(y)(\Gamma_\xi^{s_1}f)(x)\\ &=: I_1+I_2+I_3, \end{aligned}$$

where the symbol $\Gamma_\xi f(y)$ denotes the spherical Dirac operator Γ_ξ acting on the variable η of f, where $y=|y|\eta$.

Let $r\to 1-$. The integral I_1 tends to

$$\frac{1}{\Omega_n}\int_{y\in\Sigma,|y-x|>\varepsilon}\widetilde{\phi}_1(y^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_\eta^{s_1}f)(y)d\sigma(y).$$

For I_2, because $f\in\mathcal{A}$ implies $\Gamma_\xi^{s_1}f$ is a Lipschitz function, we have

$$\begin{aligned} \lim_{\varepsilon\to0}\lim_{r\to1-} I_2 &= \lim_{r\to1-}\lim_{\varepsilon\to0}\int_{y\in\Sigma,\ |y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)\\ &\quad\times\Big[(\Gamma_\xi^{s_1}f)(y)-(\Gamma_\xi^{s_1}f)(x)\Big]d\sigma(y)=0. \end{aligned}$$

Finally we estimate I_3. By Cauchy's theorem, for any fixed $\varepsilon>0$, we have

$$\begin{aligned} \lim_{r\to1-} I_3 &= \lim_{r\to1-}\int_{y\in\Sigma,\ |y-x|\leqslant\varepsilon}\widetilde{\phi}_1(y^{-1}rx)E(y)\mathfrak{n}(y)d\sigma(y)(\Gamma_\xi^{s_1}f)(x)\\ &= \widetilde{\phi}_1(\varepsilon,x)(\Gamma_\xi^{s_1}f)(x). \end{aligned}$$

This completes the proof of the theorem. □

As a useful tool in the study of boundary value problems on the non-smooth domains, the theory of Hardy spaces on Lipschitz curves and surfaces has attracted attention of many mathematicians. In 1980s, Jerison and Kenig [9, 10] considered the complex variable case. In [11], Mitrea introduced the theory of Clifford-valued Hardy spaces on high-dimensional Lipschitz graphs.

Let Δ and Δ^c be the bounded and unbounded connected components of $\mathbb{R}_1^n\setminus\Sigma$, respectively. For $\alpha>0$, define the non-tangential approach regions $\Lambda_\alpha(x)$ and $\Lambda_\alpha^c(x)$ to a point $x\in\Sigma$ as

$$\Lambda_\alpha(x)=\Big\{x\in\Delta,\ |y-x|<(1+\alpha)dist(y,\Sigma)\Big\}$$

and

$$\Lambda^c_\alpha(x) = \Big\{ y \in \Delta^c,\ |y-x| < (1+\alpha)dist(y,\Sigma)\Big\}.$$

Let f be defined in Δ (Δ^c). The interior non-tangential maximal function $N_\alpha(f)$ is defined as

$$N_\alpha(f)(x) = \sup\Big\{ |f(y)| \ :\ y \in \Lambda_\alpha(x)(y \in \Lambda^c_\alpha(x))\Big\}.$$

For $0 < p < \infty$, Hardy spaces $\mathcal{H}^p(\Delta)$ and $\mathcal{H}^p(\Delta^c)$ are defined as

$$\mathcal{H}^p(\Delta) = \Big\{f : f \text{ is left monogenic in } \Delta \text{ and } N_\alpha(f) \in L^p(\Sigma)\Big\},$$
$$\mathcal{H}^p(\Delta^c) = \Big\{f : f \text{ is left monogenic in } \Delta^c \text{ and } N_\alpha(f) \in L^p(\Sigma)\Big\}.$$

The theory of monogenic Hardy spaces in [11] indicates that for $p > 1$, the $\mathcal{H}^p(\Delta)$-norm of a function is equivalent to the L^p-norm of its non-tangential maximal function on the boundary. For the spaces $\mathcal{H}^p(\Delta^c)$, similar conclusions hold. Precisely, if $f \in \mathcal{H}^p(\Delta)$ for $p > 1$, we have

$$C_1\|f\|_{\mathcal{H}^p(\Delta)} \leqslant \|f\|_{L^p(\Sigma)} \leqslant C_2\|f\|_{\mathcal{H}^p(\Delta)}.$$

If $f \in \mathcal{M}_k$ and $k \neq -1, -2, \ldots, -n+1$, because $\mathcal{M}_k$ is the subspace consisting of all k-homogeneous left monogenic functions, we have $\Gamma_\xi f(\xi) = kf(\xi)$. For $f \in \mathcal{A}$, we define $\Gamma(f\,|_\Gamma)$ as the restriction of the monogenic extension of $\Gamma_\xi(f\,|_{S_{\mathbb{R}^n_1}})$ to Γ. Then the definition of Γ_ξ can be extended to $\Gamma_\xi : \mathcal{A} \to \mathcal{A}$.

In [3], Eelbode studied the boundary value of the Photogenic-Cauchy transform C^α_P on the unit hyperbolic sphere. In Example 7.0.1, The occurrence of the factors $k^2P_k(f)$ and $k^2Q_k(f)$ implies that the boundary value $C^\alpha_P[f]\uparrow$ of C^α_P is not a bounded operator from $L^2(\mathbb{S}^{n-1})$ to itself. If we restrict this operator to some smaller subspaces of $L^2(\mathbb{S}^{n-1})$, we can obtain the corresponding boundedness.

Now we give the main result of this section.

Theorem 7.3.3 *Let* $\omega \in (\arctan(N), \pi/2)$. *If* $b \in H^s(S^c_\omega)$, $s > 0$, *then with the assumption* $b(0) = 0$, *the multipliers introduced in Definition 7.3.2 can be extended to a bounded operator from* $W^{2,s_1}_{\Gamma_\xi}(\Sigma)$ *to* $L^2(\Sigma)$, *where* $s_1 = \lceil s\rceil$. *In addition,*

$$\|M_{(b(k))}\|_{op} \leqslant C_\nu \left\| \frac{b}{|z+1|^s} \right\|_{L^\infty(S^c_\nu)},\ \arctan N < \nu < \omega.$$

Proof For $f \in W^{2,s_1}_{\Gamma_\xi}(\Sigma) \subset L^2(\Sigma)$, by Proposition 6.2.7, we have $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$ such that

$$\|f^\pm\|_{L^2(\Sigma)} \leqslant C_N\|f\|_{W^{2,s_1}(\Sigma)}.$$

By the linearity and Theorem 7.3.2, we have $M_b(f) = M_{b^+}f^+ + M_{b^-}f^-$, where

$$M_{b^{\pm}}f^{\pm}(x) = \lim_{r\to -}\int_{\Sigma}\widetilde{\phi}_{\pm}(r^{\pm 1}y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y), x\in\Sigma.$$

Hence, we only need to prove

$$\|M_{b^{\pm}}f^{\pm}\|_{\mathcal{H}^2}\leqslant C_N\|\Gamma_{\xi}^{s_1}f^{\pm}\|_{\mathcal{H}^2}.$$

We only prove the above inequality for f^+. For the sake of simplicity, we omit the symbol "+". The f^- part can be similarly dealt.w

By Theorem 7.3.1, for $b\in H^s(S_\omega^c)$, we have

$$|\widetilde{\phi}(y^{-1}x)|\leqslant\frac{C}{|1-y^{-1}x|^{n+s}}.$$

Hence by Hölder's inequality, we obtain

$$\begin{aligned}
&|\Gamma_{\xi}^{1+s_1}M_bf(x)|\\
&\leqslant\left(\int_{\Sigma_{\sqrt{t}}}|\phi(y^{-1}x)|\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}|\phi(y^{-1}x)||\Gamma_{\xi}^{s_1+1}f(y)|^2\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\\
&\leqslant C\left(\int_{\Sigma_{\sqrt{t}}}\frac{1}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}\left(\int_{\Sigma_{\sqrt{t}}}\frac{|\Gamma_{\xi}^{s_1+1}f(y)|^2}{|1-y^{-1}x|^{n+s}}\frac{d\sigma(y)}{|y|^n}\right)^{1/2}.
\end{aligned}$$

Through change of variable, we have

$$\begin{aligned}
|\Gamma_{\xi}^{1+s_1}M_bf(x)|\leqslant C&\left(\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\sigma(y)\right)^{1/2}\\
&\times\left(\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}|\Gamma_{\xi}^{1+s_1}f(y)|^2d\sigma(y)\right)^{1/2},
\end{aligned}$$

where the integral in the last inequality satisfies

$$\begin{aligned}
\int_{\Sigma}\frac{1}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\sigma(y)&\leqslant\int_0^{\pi}\frac{\sin^{n-1}\theta_0}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}}d\theta_0\\
&\leqslant C\frac{1}{(1-\sqrt{t})^s}.
\end{aligned}$$

Hence by the equivalent characterization given in Proposition 6.2.6, we have

$$
\begin{aligned}
&\|M_b f\|^2_{H^2(\Delta)} \\
&\leqslant \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} M_b f(tx)|^2 (1-t)^{2s_1+1} d\sigma(x) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma \frac{(1-\sqrt{t})^{2s_1+1}}{(1-\sqrt{t})^s} \left(\int_\Sigma \frac{|\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(y) \right) d\sigma(x) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma |\Gamma_\xi^{1+s_1} f(\sqrt{t}y)|^2 \left(\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x) \right) (1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\leqslant C \int_0^1 \int_\Sigma \left| \Gamma_\xi (\Gamma_\xi^{s_1} f)(\sqrt{t}y) \right|^2 (1-\sqrt{t}) d\sigma(y) \frac{dt}{t} \\
&\leqslant C \|\Gamma_\xi^{s_1} f\|_{\mathcal{H}^2(\Delta)},
\end{aligned}
$$

where in the forth inequality we used the fact that for $t \in (0, 1)$,

$$
(1-\sqrt{t})^{2s_1+1-s} = (1-\sqrt{t})^{1+s+2s_1-s} \leqslant (1-\sqrt{t})^{1+s}
$$

and

$$
\int_\Sigma \frac{(1-\sqrt{t})^s}{[(1-\sqrt{t})^2+\theta_0^2]^{\frac{n+s}{2}}} d\sigma(x) \leqslant C(1-\sqrt{t})^s \frac{1}{(1-\sqrt{t})^s} \leqslant C.
$$

In the last inequality, we used Proposition 6.2.6. This completes the proof of Theorem 7.3.3. □

For the classical convolution singular integral operator T_ϕ on $\mathbb{R}^n$, one of the basic facts is the endpoint estimate, that is, the weak-(1, 1) boundedness. If for all $\lambda > 0$,

$$
|\{x \in \Sigma : |T(f)(x)| > \lambda\}| \leqslant \frac{C}{\lambda} \|f\|_1,
$$

we call an operator T is weak-(1, 1) bounded on Σ. In other words, we say that this operator is bounded from L^1 to the weak type space WL^1, see [12–14] and the reference therein. By this weak boundedness, we can use the interpolation theory and the duality of operators to deduce the L^p-boundedness of T_ϕ, $1 < p < \infty$. In the rest of this section, we study the endpoint estimate of the Fourier multipliers.

Theorem 7.3.4 *Let $\omega \in (\arg(N), \frac{\pi}{2})$. If $b \in H^s(S^c_\omega)$, $s > 0$ and $b(0) = 0$. Then the multiplier $M_{(b_k)}$:*

$$
M_{(b_k)}(f)(x) = \sum_{k=0}^\infty b_k P_k(f)(x) + \sum_{k=0}^\infty b_{-k-1} Q_k(f)(x)
$$

is bounded from $W^{1,s_1}_{\Gamma_\xi}(\Sigma)$ to $WL^1(\Sigma)$, where $s_1 = \lceil s \rceil$.

Proof For $b \in H^s(S^c_\omega)$ and $z \in S^c_\omega$, $|b(z)| \leqslant C|z|^s$, $s > 0$. Hence it is natural to get $|b(z)/z^s| \leqslant C$, where C is a constant. On the other hand, $b \in H^s(S^c_\omega)$ implies that b is holomorphic in S^c_ω. Then $z^{-s}b(z)$ is also holomorphic in S^c_ω. Now for the Fourier multiplier $M_{(b_k)}$, we have

$$\begin{aligned} M_{(b_k)}f(x) &= \sum_{k=0}^{\infty} b_k P_k(f)(x) + \sum_{k=0}^{\infty} b_{-k-1} Q_k(f)(x) \\ &= I + II. \end{aligned}$$

For simplicity, we only deal with the term I. As above, I can be represented as

$$I = \frac{1}{\Omega_n} \int_\Sigma \widetilde{\phi}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y).$$

If we write $b(z) = z^{s_1} b_1(z)$ and $b_1(z) \in H^\infty(S^c_\omega)$, then the corresponding sequence is $\{b_{1,k}\}$ whose the elements is $b_k = k^{s_1} b_{1,k}$. Therefore we can rewrite I as the following form

$$I = \sum_{k=0}^{\infty} b_{1,k} k^{s_1} P_k(f)(x).$$

The kernel associated to $M_{b_{1,k}}$ is denoted by $\widetilde{\phi_1}(y^{-1}x)E(y)$ that satisfies

$$\Gamma_\xi(\widetilde{\phi_1}(y^{-1}x))E(y) = \sum_{k=1}^{\infty} k b_1(k) \widetilde{P}^{(k)}(y^{-1}x)E(y).$$

By integration by parts, we get

$$\begin{aligned} I &= \frac{1}{\Omega_n} \int_\Sigma \Gamma_\xi^{s_1}(\widetilde{\phi_1}(y^{-1}x)) E(y) \mathfrak{n}(y) f(y) d\sigma(y) \\ &= \frac{1}{\Omega_n} \int_\Sigma \widetilde{\phi_1}(y^{-1}x) E(y) \mathfrak{n}(y) \Gamma_\eta^{s_1}(f)(y) d\sigma(y). \end{aligned}$$

Similarly, if we take $s = 0$ in Theorem 7.3.1, $\widetilde{\phi_1}(y^{-1}x)$ satisfies

$$|\widetilde{\phi_1}(y^{-1}x)| \leqslant \frac{C}{|1 - y^{-1}x|^n}.$$

Hence the multiplier $M_{b_{1,k}}$ reduces to a H^∞-Fourier multiplier on starlike Lipschitz graph and is weak-(1, 1) bounded. Then we have

$$\begin{aligned} \left|\left\{x \in \Sigma : |M_{b_k} f(x)| > \lambda\right\}\right| &= \left|\left\{x \in \Sigma : |M_{b_{1,k}}(\Gamma_\xi^{s_1} f)(x)| > \lambda\right\}\right| \\ &\leqslant \frac{C}{\lambda} \left\| \Gamma_\xi^{s_1} f \right\|_{L^1}. \end{aligned}$$

This completes the proof of this theorem. □

At last, we consider the boundedness of the Fourier multipliers for $s < 0$. Let $-n < s < 0$ and $\{b_k\}$ be a sequence which satisfies $|b_k| \leqslant k^s$. We define the Fourier multiplier $M_{(b_k)}$ as follows.

$$M_{(b_k)}(f)(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1} Q_k(f)(x).$$

Similar to the case $s > 0$, we can express the multiplier as

$$M_{(b_k)}(f)(x) = \frac{1}{\Omega_n} \int_{\Sigma} \widetilde{\phi}(y^{-1}x) E(y) \mathfrak{n}(y) f(y) d\sigma(y).$$

Here $x \in \Sigma$ and

$$\widetilde{\phi}(y^{-1}x) = \left(\sum_{k=1}^{\infty} + \sum_{-\infty}^{-1} \right) b_k \widetilde{P}^{(k)}(y^{-1}x),$$

where $\widetilde{P}^{(k)}$ is the polynomial defined as

$$\widetilde{P}^{(k)}(y^{-1}x) = |y^{-1}x|^k C^{+}_{n+1,k}(\xi, \eta)$$

or

$$\widetilde{P}^{(-k-1)}(y^{-1}x) = |y^{-1}x|^{-k-n} C^{-}_{n+1,k}(\xi, \eta).$$

To obtain the boundedness of the multiplier, we need to estimate the function $\widetilde{\phi}(x)$. By the method of Theorem 1.3.2, we can prove that the kernel $\phi(x) = \sum\limits_{k=-\infty}^{\infty} b_k P^k(x)$ satisfies

$$|\phi(x)| \leqslant \frac{C|x|^s}{|1-x|^{n+s}}, \quad \text{where } x \in H_\omega.$$

For the kernel $\widetilde{\phi}(y^{-1}x)$ defined above, we can use the method of Proposition 6.2.3 to obtain

$$|\widetilde{\phi}(y^{-1}x)| \leqslant \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}.$$

For any two points x_1, x_2 on the starlike Lipschitz surface, we have $x_2^{-1}x_1 \in H_\omega$, that is, there exist two constants $C_1,\ C_2$ such that $C_1 \leqslant |x_2^{-1}x_1| \leqslant C_2$. Hence for any points $x_1,\ x_2 \in \Sigma$, the equality

$$|x_1| = |x_2 x_2^{-1} x_1| = |x_2||x_2^{-1}x_1|$$

implies that $C_1|x_1| \leqslant |x_2| \leqslant C_2|x_1|$. In other words, the norms of the two points on the starlike Lipschitz surface are approximately a constant associated with Σ, denoted by C_Σ. Hence we can obtain the estimate

$$\begin{aligned}|\widetilde{\phi}(y^{-1}x)E(y)\mathfrak{n}(y)| &\leqslant \frac{C|y^{-1}x|^s}{|1-y^{-1}x|^{n+s}}\frac{1}{|y|^n}\\ &\leqslant \frac{C|x|^s}{|y-x|^{n+s}}\\ &\leqslant \frac{C_\Sigma}{|y-x|^{n+s}}.\end{aligned}$$

Because the Lipschitz surface is a homogeneous space, our Fourier multiplier $M_{(b_k)}f(x)$ can be regarded as the fractional integral operator on Σ. By the classical theory of the fractional integral operator on homogeneous spaces, we can obtain the $L^p - L^q$ boundedness of the Fourier multiplier as follows.

Theorem 7.3.5 *Let* $-n < s < 0$, $1 \leqslant p < q < \infty$ *and* $\frac{1}{q} = \frac{1}{p} + \frac{s}{n}$. *If* $b \in H^s(S^c_\omega)$, *the Fourier multipliers on starlike Lipschitz surface:*

$$M_{(b_k)}f(x) = \sum_{k=1}^{\infty} b_k P_k(f)(x) + \sum_{k=1}^{\infty} b_{-k-1}Q_k(f)(x)$$

with $b_k = b(k)$ *is bounded from* $L^p(\Sigma)$ *to* $L^q(\Sigma)$.

Proof For a starlike Lipschitz surface Σ, if $x_1, x_2 \in \Sigma$, then $x_2^{-1}x_1 \in H_\omega$, i.e., there exist two constants c_1, c_2 depending on ω and Σ such that $C_1 \leqslant |x_2^{-1}x_1| \leqslant C_2$. For any points $x_1, x_2 \in \Sigma$, the equality

$$|x_1| = |x_2x_2^{-1}x_1| = |x_2||x_2^{-1}x_1|$$

indicates that $C_1|x_1| \leqslant |x_2| \leqslant C_2|x_1|$. In other words, the norm of any point on Σ is about a constant C_Σ which is related to Σ. Then the kernel $\phi(y^{-1}x)E(y)$ satisfies

$$\begin{aligned}|\phi(y^{-1}x)E(y)| &= |\phi(y^{-1}x)||E(y)|\\ &\leqslant \frac{C}{|1-y^{-1}x|^{n+s}}\frac{1}{|y|^n}\\ &\leqslant \frac{C|y|^s}{|y-x|^{n+s}}\\ &\leqslant \frac{C_\Sigma}{|y-x|^{n+s}}.\end{aligned}$$

In addition, for any ball $B(x,r) = \left\{y \in \Sigma,\ |x-y| < r\right\}$, we have

$$\sigma(B(x,r)) = \int_{B(x,r)} d\sigma(y) \leqslant Cr^n,$$

that is, the surface measure of $B(x, r)$ is dominated by the area of a sphere in $\mathbb{R}^n$. Hence, we can use the classical method to prove the boundedness. Below we give the details. At first, we define the auxiliary function $\Omega(x)$ by

$$\Omega(x) = \sup_{r>0} \frac{\sigma(B(x, r))}{r^n}.$$

For the integral representation of M_b, we divide the integral into two parts.

$$|M_b(f)(x)| \leqslant \left(\int_{B(x,r)} + \int_{\Sigma\setminus B(x,r)}\right) |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y) =: I_1 + I_2.$$

For I_1, we have

$$\begin{aligned} I_1 &\leqslant \int_{B(x,r)} |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y) \\ &= \sum_{k=0}^{\infty} \int_{B(x,2^{-k}r)\setminus B(x,2^{-k-1}r)} |f(y)| \frac{1}{|y-x|^{n+s}} d\sigma(y). \end{aligned}$$

Because $|y-x| \leqslant 2^{-k}r$ for $y \in B(x, 2^{-k}r) \setminus B(x, 2^{-k-1}r)$, we can obtain

$$\begin{aligned} I_1 &\leqslant \sum_{k=0}^{\infty} (2^{-k-1}r)^{-n-s} \sigma(B(x, 2^{-k}r)) \frac{1}{\sigma(B(x, r))} \int_{B(x,2^{-k}r)} |f(y)| d\sigma(y) \\ &\leqslant \sum_{k=0}^{\infty} (2^{-k-1}r)^{-n-s} \sigma(B(x, 2^{-k}r)) M(f)(x). \end{aligned}$$

By the definition of $\Omega(x)$, we have

$$\sigma(B(x, 2^{-k}r)) = \frac{\sigma(B(x, 2^{-k}r))}{(2^{-k}r)^n} \leqslant \Omega(x)(2^{-k}r)^n.$$

Then by $-s > 0$, we get

$$I_1 \leqslant r^{-s}\Omega(x)M(f)(x) \sum_{k=0}^{\infty} (2^{-k-1})^{-s} \leqslant r^{-s}\Omega(x)M(f)(x).$$

For I_2, we have

$$
\begin{aligned}
I_2 &\leqslant \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(y)|}{|x-y|^{n+s}} d\sigma(y) \\
&\leqslant \sum_{k=0}^{\infty} (2^k r)^{-s-n} (\sigma(B(x,2^{k+1}r)))^{1-\lambda/p} (\sigma(B(x,2^{k+1}r)))^{\lambda/p-1} \int_{B(x,2^{k+1}r)} |f(y)| d\sigma(y) \\
&\leqslant \sum_{k=0}^{\infty} (2^k r)^{-s-n} (2^{k+1} r)^{n(1-\lambda/p)} (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x) \\
&= r^{-s-n\lambda/p} \left(\sum_{k=0}^{\infty} 2^{k(-n-s)} 2^{nk(1-\lambda/p)} \right) (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x).
\end{aligned}
$$

Because $s - n\lambda/p < 0$ for $1 \leqslant p < n\lambda/s$, then

$$
|M_b(f)(x)| \leqslant r^{-s}\Omega(x) M(f)(x) + r^{-s-n\lambda/p} (\Omega(x))^{1-\lambda/p} M_{\lambda/p}(f)(x).
$$

Letting

$$
r = \left(\frac{M_{\lambda/p}(f)(x)}{M(f)(x)} \right)^{p/n\lambda} \frac{1}{\Omega^{1/n}(x)},
$$

we obtain

$$
\begin{aligned}
|M_b(f)(x)| \leqslant\ & \Big(M_{\lambda/p}(f)(x)\Big)^{-sp/n\lambda} \Big(\Omega(x)\Big)^{1+s/n} \Big(M(f)(x)\Big)^{1+sp/n\lambda} \\
&+ \Big(M_{\lambda/p}(f)(x)\Big)^{-ps/n\lambda-1+1} \Big(M(f)(x)\Big)^{-sp/n\lambda+1} \Big(\Omega(x)\Big)^{1+s/n} \\
&\leqslant \Big(\Omega(x)\Big)^{s/n+1} \Big(M_{\lambda/p}(f)(x)\Big)^{-sp/n\lambda} \Big(M(f)(x)\Big)^{1+sp/n\lambda}.
\end{aligned}
$$

Now we get

$$
\Big\| (\Omega(x))^{-s/n-1} M_b(f)(x) \Big\|_{L^q}^q \leqslant \int_{\Sigma} \Big(M_{\lambda/p}(f)(x)\Big)^{-spq/n\lambda} \Big(M(f)(x)\Big)^{(1+sp/n\lambda)q} d\sigma(x).
$$

Let $\lambda = 1$. Because $\sigma(B(x,r)) \leqslant cr^n$, then $\Omega^{-s/n-1}(x) \geqslant C^{-s/n-1}$ for $-n < s < 0$. By the fact that $M_{1/p} f(x) \leqslant C\|f\|_p$, we see that

$$
\begin{aligned}
\big\| (\Omega(x))^{-s/n-1} M_b(f)(x) \big\|_{L^q}^q &= \int_{\Sigma} |M_{1/p}(f)(x)|^{q-p} |M(f)(x)|^p d\sigma(x) \\
&\leqslant \|M_{1/p} f\|_{\infty}^{q-p} \|M(f)\|_p^p \\
&\leqslant C\|f\|_p^{q-p} \|f\|_p^p \\
&\leqslant C\|f\|_p^q.
\end{aligned}
$$

This completes the proof of Theorem 7.4.1. □

7.4 The Equivalence of Hardy–Sobolev Spaces

In this section, we give an application of Fourier multipliers on the starlike Lipschitz surface Σ. In the proof of Theorem 7.3.1, we used the Hardy decomposition of $L^2(\Sigma)$: for $f \in L^2(\Sigma)$, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^2(\Delta)$ and $f^- \in \mathcal{H}^2(\Delta^c)$. If $f \in W^{2,s}_{\Gamma_\xi}(\Sigma)$, f^+ and f^- belong to the so-called Hardy–Sobolev spaces. For these spaces, there exist two methods to given the definitions.

Method I. For $f \in L^2(\Sigma)$, $f = f^+ + f^-$, where $f^+ \in \mathcal{H}^{2,+}$ and $f^- \in \mathcal{H}^{2,-}$. That is f^+ belongs to the Hardy space, while f^- belongs to the conjugate Hardy space. We define the Hardy–Sobolev space on Σ as

$$\mathcal{H}^{2,s}_{+,1}(\Sigma) = \Big\{ f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } \\ f = g^+ \in L^2(\Sigma) \text{ and } \Gamma^j_\xi(g^+) \in L^2(\Sigma),\ j = 1, 2, \ldots, s \Big\}$$

and

$$\mathcal{H}^{2,s}_{-,1}(\Sigma) = \Big\{ f :\in L^2(\Sigma) \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } \\ f = g^- \in L^2(\Sigma) \text{ and } \Gamma^j_\xi(g^-) \in L^2(\Sigma),\ j = 1, 2, \ldots, s \Big\}.$$

Method II. At first for any $f \in W^{2,s}_{\Gamma_\xi}$, $\Gamma^j_\xi f \in L^2(\Sigma)$, $j = 1, 2, \ldots, s$. We obtain the decomposition $\Gamma^j_\xi f = (\Gamma^j_\xi f)^+ + (\Gamma^j_\xi f)^-$, where $(\Gamma^j_\xi f)^+ \in \mathcal{H}^{2,+}$ and $(\Gamma^j_\xi f)^- \in \mathcal{H}^{2,-}$. The Hardy–Sobolev spaces are defined as follows.

$$\mathcal{H}^{2,s}_{+,2}(\Sigma) = \Big\{ f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } \\ f = g^+ \in L^2(\Sigma) \text{ and } (\Gamma^j_\xi g)^+ \in L^2(\Sigma),\ j = 1, 2, \ldots, s \Big\}$$

and

$$\mathcal{H}^{2,s}_{-,2}(\Sigma) = \Big\{ f : \text{ there exists a function } g \in L^2(\Sigma) \text{ such that } \\ f = g^- \in L^2(\Sigma) \text{ and } (\Gamma^j_\xi g)^- \in L^2(\Sigma),\ j = 1, 2, \ldots, s \Big\}.$$

On the unit sphere, because we can exchange the order of the Riesz transform and the Dirac operator, the above two Hardy–Sobolev spaces are the same one obviously. On a general starlike Lipschitz surface, we will use the theory of Fourier multipliers to show that the two kinds of Hardy–Sobolev spaces are equivalent on Σ.

Theorem 7.4.1 *For the starlike Lipschitz surface Σ, let s be a positive integer, the Hardy–Sobolev spaces $\mathcal{H}^{2,s}_{\pm,1}(\Sigma)$ and $\mathcal{H}^{2,s}_{\pm,1}(\Sigma)$ are equivalent.*

Proof Because $\mathscr{A}$ is dense in $L^2(\Sigma)$, without loss of generality, we assume that $f \in \mathscr{A}$. By the spherical harmonic expansion, we have

$$f = \sum_{k=1}^{\infty} P_k(f)(x) + \sum_{k=1}^{\infty} Q_k(f)(x).$$

Then letting $f^+ = \sum_{k=1}^{\infty} P_k(f)(x)$ and $f^- = \sum_{k=1}^{\infty} Q_k(f)(x)$, we get

$$\Gamma_\xi(f^+) = \Gamma_\xi\left(\sum_{k=1}^{\infty} P_k(f)(x)\right).$$

Because $P_k(f)(x)$ belongs to the k-homogeneous eigenspace $\mathcal{M}_k$, we can deduce that

$$\Gamma_\xi(f^+)(x) = \sum_{k=1}^{\infty} kP_k(f)(x) \text{ for } f \in \mathscr{A}.$$

On the other hand,

$$\begin{aligned} P_k(f)(x) &= \frac{1}{\Omega_n}\int_\Sigma \widetilde{P}^k(y^{-1}x)E(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y), \end{aligned}$$

where we use the Cauchy–Kovalevska expansion

$$\widetilde{P}^k(y^{-1}x)E(y) = \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)W_{\underline{\alpha}}(y),$$

where $V_{\underline{\alpha}}(x) \in \mathcal{M}_k$ and $W_{\underline{\alpha}}(y) \in \mathcal{M}_{-3-k}$. Hence we can get

$$\begin{aligned} \Gamma_\xi(f^+)(x) &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\frac{k}{k+1}(k+1)W_{\underline{\alpha}}(y)\mathfrak{n}(y)f(y)d\sigma(y) \\ &= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}\int_\Sigma \sum_{|\underline{\alpha}|=k} V_{\underline{\alpha}}(x)\Gamma_\eta W_{\underline{\alpha}}(x)\mathfrak{n}(y)f(y)d\sigma(y). \end{aligned}$$

Because f decays fast for $f \in \mathscr{A}$, we can use integration by parts to obtain that

$$\Gamma_\xi(f^+)(x) = \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}\int_\Sigma \widetilde{P}^{(k)}(y^{-1}x)E(y)\mathfrak{n}(y)(\Gamma_\eta f)(y)d\sigma(y)$$
$$= \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k}{k+1}P_k(\Gamma_\xi f)(x).$$

Let $b_k = \frac{k}{k+1}$. We have $\Gamma_\xi(f^+)(x) = M_{(b_k)}((\Gamma_\xi f)^+)$. Since $|b_k| \leqslant C$, it follows from the theory of Fourier multipliers on Σ that $M_{(b_k)}$ is bounded on $L^2(\Sigma)$, that is, there exists a constant C_1 such that

$$\|(\Gamma_\xi f^+)\|_{L^2(\Sigma)} \leqslant C_1\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)}.$$

Conversely, let $b'_k = \frac{k+1}{k}$. Similarly, we can get

$$(\Gamma_\xi f)^+(x) = \frac{1}{\Omega_n}\sum_{k=1}^{\infty}\frac{k+1}{k}(\Gamma_\xi P_k(f))(x) = M_{(b'_k)}(\Gamma_\xi(f^+))(x),$$

and there exists a constant C_2 such that

$$\|(\Gamma_\xi f)^+\|_{L^2(\Sigma)} \leqslant C_1\|\Gamma_\xi(f^+)\|_{L^2(\Sigma)}.$$

This proves Theorem 7.4.1. □

7.5 Remarks

Remark 7.5.1 The definitions of $H^s_{\ln}$ & $K^s_{\ln}$ and Theorem 7.1.3 only concern the case of the first power of the log function. In fact, if k is a positive integer, by the same proof, we can extend (ii) of Theorem 7.1.3 to the kth power of the log function.

Remark 7.5.2 By the following method, we can obtain variations of Theorems 7.1.1–7.1.3. Denote by $\exp(-i\theta\cdot)$ the function $z \to \exp(i\theta z)$. Define the spaces

$$H^{s,\theta}(S_{\omega,\pm}) = \exp(i\theta\cdot)H^s(S_{\omega,\pm}), \quad H^{s,\theta}(S_\omega) = \exp(i\theta\cdot)H^s(S_\omega),$$

$$K^{s,\theta}(C_{\omega,\pm}) = \left\{\phi \mid \phi\circ\exp(-i\theta) \in K^s(C_{\omega,\pm})\right\}$$

and

$$K^{s,\theta}(S_\omega) = \left\{\phi \mid \phi\circ\exp(-i\theta) \in K^s(S_\omega)\right\}.$$

If we change the statements of the theorems by using these spaces with the parameter θ, then the singular point $z = 1$ of the functions ϕ_+ and ϕ will be shifted to the point $z = \exp(i\theta)$ on the unit circle.

Remark 7.5.3 For the case $s = 0$, the main results of Sect. 7.1 are corollaries of the Fourier theory of holomorphic functions on the sectors established in [15]. In [16], the authors proved that if the Lipschitz constant of the curve is smaller than $\tan(\omega)$, as the kernel, any element in $K^0(C_{\omega,\pm})$ and $K^0(S_\omega)$ induces a L^2-bounded convolution singular integral operator on this starlike Lipschitz curve. In fact, these operators can be represented as the H^∞-functional calculus of the Dirac operator $z(d/dz)$ on the closed curve. By the conformal mapping, we can deduce a corresponding singular integral operator on any simply-connected Lipschitz curve. The cases of $s \neq 0$ correspond to the fractional integrations and differentials on these curves. All those mentioned are closely related to boundary value problems associated with Lipschitz domains. We refer to [17–19] for further information.

Remark 7.5.4 In [20], D. Khavinson proved the following result. Let $f(z) = \sum_{n=1}^{\infty} b_n z^n$, where $b_n = g(n)$, g is a bounded holomorphic function in the sector $S_\phi = \left\{z : |\arg z| \leqslant \phi\right\}$, $0 < \phi \leqslant \frac{\pi}{2}$. Then f can be extended to a holomorphic function on the heart-shaped region $G_\phi = \left\{z = re^{i\theta},\ \ 2\pi - \cot\phi \cdot \log r > \theta > \cot\phi \cdot \log r\right\}$. Hence, in Sect. 7.1, the result of the fractional integrals on the closed Lipschitz curves can be deduced from the result of the unit circle.

Remark 7.5.5 If $b \in H^s(S^c_\omega)$, $s > 0$, there exists a holomorphic function b_1 such that $|b_1(z)| \leqslant C_\mu$ and $\phi(x) = \Gamma_\xi^{s_1}\phi_1(x)$, where $s_1 = [s] + 1$. Here ϕ_1 is the kernel associated with b_1 in Theorem 7.2.1. However, in this way, we only obtain the following estimate: $|\phi(x)| \leqslant C/|1 - x|^{n+s_1}$, which is not precise compared with the result of Theorem 7.2.1.

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