Chapter 4
Convolution Singular Integral Operators on Lipschitz Surfaces

As the high-dimensional generalization of the boundedness of singular integrals on Lipschitz curves, the $L^p(\Sigma)$-boundedness of the Cauchy-type integral operators on the Lipschitz surfaces Σ is a meaningful question. The increase of the dimensions means that we need to apply a new method to solve the above question. In 1994, C. Li, A. McIntosh and S. Semmes embedded $\mathbb{R}^{n+1}$ into Clifford algebra $\mathbb{R}_{(n)}$ and considered the class of holomorphic functions on the sectors $S_{w,\pm}$, see [1]. They proved that if the function ϕ belongs to $K(S_{w,\pm})$, then the singular integral operator T_ϕ with the kernel ϕ on Lipschitz surface is bounded on $L^p(\Sigma)$.

In [2], G. Gaudry, R. Long and T. Qian applied Clifford-valued martingales to prove the same result as is proved in [1], that i.e., the L^2-boundedness of the Cauchy integral operators on Lipschitz surfaces [2]. The authors of [2] then indicated how to prove the Clifford $T(b)$ theory. The idea of the proof is similar to that of [3], but there is some difference. We define a suitable sequence of atomic σ-fields on $\mathbb{R}^n$. Because Clifford algebra is non-commutative, it is necessary to associate each atom with a pair of Clifford-valued Haar functions. Hence, the appropriate Haar system is in fact a system of pairs of Clifford-valued functions. We only use the martingale technique to prove the L^2-norm equivalence between the function f and its Littlewood–Paley function $S(f)$.

4.1 Clifford-Valued Martingales

We first state some backgrounds of the martingales and the Littlewood–Paley estimate of Clifford-valued functions. Let X be a set and $\mathcal{B}$ be a σ-field in X. Assume that ν is a non-negative measure on $\mathcal{B}$ and $\{\mathcal{F}_m\}_{m=-\infty}^{\infty}$ is a non-decreasing family of σ-field in X satisfying

T. Qian and P. Li, *Singular Integrals and Fourier Theory on Lipschitz Boundaries*,
https://doi.org/10.1007/978-981-13-6500-3_4

(i) $\bigcup_{m=-\infty}^{\infty} \mathcal{F}_m$ generates $\mathcal{B}$;

(ii) $\bigcap_{m=-\infty}^{\infty} \mathcal{F}_m = \{\emptyset,\ X\}$;

(iii) the measure ν is $\sigma-$finite on $\mathcal{B}$ and on each $\mathcal{F}_m$.

Let $\mathcal{F}$ be a sub-$\sigma-$field of $\mathcal{B}$ such that ν is σ-finite on $\mathcal{F}$. Because $(X,\ \mathcal{F})$ is $\sigma-$finite, X can be written as $X = \bigcup_j U_j$, where $U_j \in \mathcal{F}$ and $\nu(U_j) < +\infty$. If f is a locally integrable scalar-valued function on $(X, \mathcal{B}, \nu)$, i.e., a function whose integral is finite on every set of finite ν-measure, its conditional expectation $\widetilde{E}(f \mid \mathcal{F})$ is well-defined. On each U_j, $\widetilde{E}(f \mid \mathcal{F})$ equals to the conditional expectation of $f\ |_{U_j}$ with respect to $(\mathcal{F}\ |_{U_j},\ \nu\ |_{U_i})$. If A is any set in $\mathcal{F}$ with finite $\nu-$measure, then

$$\int_A \widetilde{E}(f \mid \mathcal{F}) d\nu = \int_A f d\nu. \tag{4.1}$$

If f is integrable, then (4.1) also holds for any $A \in \mathcal{F}$, whether of finite $\nu-$measure or not.

Let $\mathbb{R}_{(n)}$ denote the Clifford algebra generated by $\{e_0, e_1, \ldots, e_n\}$. The definition of the conditional expectation can be extended to locally integrable $\mathbb{R}_{(n)}$-valued functions. In fact, if $f = \sum_S f_S e_S$, then

$$\widetilde{E}(f \mid \mathcal{F}) = \sum_S \widetilde{E}(f_S \mid \mathcal{F}) e_S.$$

The characteristic martingale property (4.1) holds also for $\mathbb{R}_{(n)}$-valued functions f.

We denote by $L^p(\mathcal{F},\ d\nu;\ \mathbb{R}_{(n)})$ or simply $L^p(d\nu;\ \mathbb{R}_{(n)})$, $1 \leqslant p \leqslant \infty$, the Lebesgue spaces of all $\mathbb{R}_{(n)}$-vauled $\mathcal{F}-$ measurable functions on X. The space $L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$ has the obvious interpretation.

Assume that ψ is a fixed L^∞ function on X with values in $\mathbb{R}^{1+n}$.

Definition 4.1.1 Suppose that $\widetilde{E}(\psi \mid \mathcal{F}) \notin 0$ a.e., and let $f \in L^1_{\text{loc}}(d\nu;\ \mathbb{R}_{(n)})$. Then the left and the right conditional expectations E^l and E^r of f respect to $\mathcal{F}$ are given by the following formulas

$$E^l(f) = E^l(f \mid \mathcal{F}) = \widetilde{E}(\psi \mid \mathcal{F})^{-1} \widetilde{E}(\psi f \mid \mathcal{F}) \tag{4.2}$$

and

$$E^r(f) = E^r(f \mid \mathcal{F}) = \widetilde{E}(f\psi \mid \mathcal{F}) \widetilde{E}(\psi \mid \mathcal{F})^{-1}. \tag{4.3}$$

The left conditional expectation of f respect to $\mathcal{F}_m$ is denoted by $E^l(f \mid \mathcal{F}_m)$ or $E^l_m(f)$, and the right conditional expectation of f respect to $\mathcal{F}_m$ is denoted by $E^r(f \mid \mathcal{F}_m)$ or $E^r_m(f)$.

The mapping properties of E^l and E^r are good only under further assumptions on the function ψ.

Proposition 4.1.1 *Let $1 \leqslant p \leqslant \infty$. The operators E^l and E^r are bounded on L^p if there exists a constant $c_0 > 0$ such that for x a.e.,*

$$c_0^{-1} \leqslant |\widetilde{E}(\psi \mid \mathcal{F})(x)| \leqslant c_0. \tag{4.4}$$

Proof This theorem can be proved via modifying the corresponding argument in [4]. □

If a function $\psi \in L^\infty(X; \mathbb{R}^{1+n})$ and satisfies (4.4), we call this function pseudo-accretive with respect to $\mathcal{F}$. Now we assume that for a general $\mathcal{F}$, the condition (4.4) holds, and for all $\mathcal{F}_m$, the constant in (4.4) is independent of n. That being so, it follows that, if $f \in L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$, then $E^l(f)$ and $E^r(f)$ are locally integrable. The main elementary properties of E^l and E^r are as follows:

Proposition 4.1.2 (a) *If $g \in L^\infty(\mathcal{F}, d\nu; \mathbb{R}_{(n)})$, then $E^l(fg) = E^l(f)g$. Similarly, the right conditional expectation E^r commutes with the multiplication on the left by g.*

(b) $E^l(1) = E^r(1) = 1$.

(c) *If $f \in L^1_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$ and A is of finite measure (or $f \in L^1(d\nu; \mathbb{R}_{(n)})$ and A is $\mathcal{F}$-measurable), then*

$$\int_A \psi E^l(f) d\nu = \int_A \psi f d\nu, \tag{4.5}$$

$$\int_A E^r(f)\psi d\nu = \int_A f\psi d\nu. \tag{4.6}$$

(d) *For $m \leqslant \kappa$, we have*

$$E_m(E_\kappa(f)) = E_m(f), \tag{4.7}$$

where E_m denotes the left (or right) conditional expectation with respect to $\mathcal{F}_m$.

(e) *Set $\Delta^l_m = E^l_m - E^l_{m-1}$, $\Delta^r_m = E^r_m - E^r_{m-1}$, and*

$$\langle f,\ g\rangle_\psi = \int f\psi g d\nu.$$

We have for all $m \neq \kappa$ and $f, g \in L^2(d\nu; \mathbb{R}_{(n)})$,

$$\langle \Delta^r_m f,\ \Delta^l_\kappa g\rangle_\psi = 0.$$

Proof (a) and (b) are obvious. To prove (c), assume that $A \in \mathcal{F}$. Because $E^l f$ and A is $\mathcal{F}-$measurable,

$$\int\limits_A \psi E^l f d\nu = \int\limits_X \chi_A \psi E^l f d\nu = \int\limits_X \widetilde{E}(\chi_A \psi E^l f) d\nu = \int\limits_X \chi_A \widetilde{E}(\psi) E^l f d\nu = \int\limits_A \psi f d\nu.$$

For E^r, we can give a similar proof and so is omitted.

The conclusion (d) can be proved as follows. For example, for the left conditional expectation,

$$\begin{aligned} E^l_m(E^l_\kappa(f)) &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi\widetilde{E}_\kappa(\phi)^{-1}\widetilde{E}_\kappa(\psi f)) \\ &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\widetilde{E}_\kappa[\psi\widetilde{E}_\kappa(\psi)^{-1}\widetilde{E}_\kappa(\psi f)]) \\ &= \widetilde{E}_m(\psi)^{-1}\widetilde{E}_m(\psi f) = E^l_m(f). \end{aligned}$$

The proof for the right conditional expectation is similar.

At last, we prove (e). For $n > \kappa$,

$$\begin{aligned} \langle \Delta^r_m,\ \Delta^l_\kappa g\rangle_\psi &= \int \Delta^r_m f\psi\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi\Delta^l_\kappa g)d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi)\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}_{m-1}(\Delta^r_m f\psi)\widetilde{E}_{m-1}(\psi)^{-1}\widetilde{E}_{m-1}(\psi)\Delta^l_\kappa g d\nu \\ &= \int \widetilde{E}^r_{m-1}(\Delta^r_m f)\widetilde{E}_{m-1}(\psi)\Delta^l_\kappa g d\nu = 0, \end{aligned}$$

where in the last step we have used (4.7). The proof for $\kappa > n$ is similar. □

Definition 4.1.2 Let $f \in L^1_{\text{loc}}(d\nu;\mathbb{R}_{(n)})$. The left martingale with respect to $\{\mathcal{F}_m\}_{m=-\infty}^\infty$ generated by f is the sequence $\{f^l_m\}_{m=-\infty}^\infty = \{E^l_m(f)\}_{m=-\infty}^\infty$. If the limit $f^l_{-\infty} = \lim\limits_{m\to-\infty} E^l_m(f)$ exists a.e., the left-Littlewood–Paley square function $S^l(f)$ is defined by

$$S^l(f) = \Big(|f^l_{-\infty}|^2 + \sum_{m=-\infty}^{\infty} |\Delta^l_m f|^2\Big)^{1/2}.$$

The right martingale and the right-Littlewood–Paley square function can be defined similarly. If $f \in \bigcup\limits_{1\leqslant p<\infty} L^p(d\nu;\mathbb{R}_{(n)})$ and $\nu(X) = +\infty$, then $f^l_{-\infty} = 0$.

If $f \in L^1_{\text{loc}}(d\nu;\mathbb{R}_{(n)})$, then the BMO-norm of f is defined as

$$\|f\|_{BMO} = \sup_m \|\widetilde{E}_m(|f - \widetilde{E}_{m-1}f|^2)\|^{1/2}_\infty. \tag{4.8}$$

We need the following facts: if $\psi \in L^\infty(d\nu;\mathbb{R}^{1+n})$ then $\psi \in BMO$ and for every m,

$$\widetilde{E}_m\Big(\sum_{k=m}^{\infty} |\widetilde{\Delta}_k(\psi)|^2\Big) \leqslant C\|\psi\|^2_{BMO} \leqslant C\|\psi\|^2_\infty. \tag{4.9}$$

By the John–Nirenberg inequality, the right hand side of (4.8) is equivalent to

$$\sup_m \left\| \widetilde{E}_m\Big(\Big|f - \widetilde{E}_m(f)\Big|\Big)\right\|_\infty,$$

see [5, 6] for the proof.

The following Littlewood–Paley result is one of the essential ingredients of this chapter. We use C to denote a constant which may vary from line to line.

Lemma 4.1.1 *There exists a constant $c > 0$ depending only on c_0 and d such that for all $f \in L^2_{\text{loc}}(d\nu; \mathbb{R}_{(n)})$,*

$$c^{-1}\|S(f)\|_{L^2} \leqslant \|f\|_{L^2} \leqslant c\|S(f)\|_{L^2}, \tag{4.10}$$

where S denotes S^l or S^r.

Proof We only consider the case of left martingales and the case of right martingales can be dealt with similarly. Fix m_0. Consider the sequence $\{\mathcal{F}_m\}_{n\geqslant m_0}$ and the corresponding square function:

$$\Big(\sum_{m\geqslant m_0+1}|\Delta^l_m f|^2\Big)^{1/2}.$$

If $n \geqslant n_0 + 1$, we have

$$\begin{aligned}\Delta^l_m f &= \widetilde{E}(\psi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_m) - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_{m-1})\\ &= \Big[\widetilde{E}(\psi \mid \mathcal{F}_m)^{-1} - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\Big]\widetilde{E}(\psi f \mid \mathcal{F}_m)\\ &\quad+\widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\Big[\widetilde{E}(\psi f \mid \mathcal{F}_m)^{-1} - \widetilde{E}(\psi f \mid \mathcal{F}_{m-1})^{-1}\Big].\end{aligned} \tag{4.11}$$

Hence by (4.4),

$$|\delta^l_m(f)|^2 \leqslant C\Big(|\widetilde{\Delta}_m(\psi)|^2|\widetilde{E}(\psi f \mid \mathcal{F}_m)|^2 + |\widetilde{\Delta}_m(\psi f)|^2\Big). \tag{4.12}$$

Because ν is σ-finite on $\mathcal{F}_{m_0}$, we can write $X = \bigcup_{j=1}^{\infty} U_j$, where $U_1 \subseteq U_2 \subseteq \cdots$, and the set $U_j \subset \mathcal{F}_{m_0}$ that has a finite measure. Fix $M \geqslant 1$. Then by (4.12) and the standard Littlewood–Paley estimate, we get

$$\begin{aligned}&\int_{U_M}\sum_{m\geqslant m_0+1}|\Delta^l_m f|^2 \qquad (4.13)\\ &\leqslant C\Big(\int_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}_m(\psi f \mid \mathcal{F}_m)|^2|\widetilde{\Delta}_n\psi|^2 d\nu + \int_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{\Delta}_m(\psi f)|^2 d\nu\Big)\end{aligned}$$

$$\leqslant C\Big(\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu+\int\limits_X|\psi f|^2d\nu\Big)$$

$$\leqslant C\Big(\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu+\int\limits_X|f|^2d\nu\Big),$$

where

$$\widetilde{E}^*_m(f)=\sup_{m_0+1\leqslant j\leqslant m}\Big|\widetilde{E}(f\mid\mathcal{F}_j)\Big|.$$

For $m\geqslant m_0+1$, let $T_m=\sum_{k=m}^{\infty}|\widetilde{\Delta}_k\psi|^2$ and set $T_{m_0}=0$. If $N>m_0$, we have

$$\begin{aligned}\sum_{m=m_0+1}^{N}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2&=\sum_{m=m_0+1}^{N}|\widetilde{E}^*_m(\psi f)|^2(T_m-T_{m+1})\\&=\sum_{m=m_0}^{N-1}T_{m+1}\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_m(\psi f)|^2\Big]-|\widetilde{E}^*(\psi f)|^2T_{N+1}.\end{aligned}$$

It can be deduced from (4.9) and (4.14) that

$$\begin{aligned}&\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\widetilde{E}^*_m(\psi f)|^2|\widetilde{\Delta}_m(\psi)|^2d\nu \qquad (4.14)\\&\leqslant\int\limits_{U_M}\sum_{m=n_0}^{\infty}\Big(\sum_{k=m+1}^{\infty}|\widetilde{\Delta}_k(\psi)|^2\Big)\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_n(\psi f)|^2\Big]d\nu\\&\leqslant\int\limits_{U_M}\sum_{m=m_0}^{\infty}\widetilde{E}_{m+1}\Big(\sum_{k=m+1}^{\infty}|\widetilde{\Delta}_k(\psi)|^2\Big)\Big[|\widetilde{E}^*_{m+1}(\psi f)|^2-|\widetilde{E}^*_m(\psi f)|^2\Big]d\nu\\&\leqslant\|\psi\|^2_{BMO}\int\limits_{U_M}|\psi f|^{*2}d\nu\\&\leqslant C\|\psi\|^2_\infty\int\limits_{U_M}|f|^2d\nu.\end{aligned}$$

In the last step, we have used the $L^2(U_M)$-boundedness of the maximal function. The constant is independent of M or m_0.

By (4.13) and (4.14), we can obtain

$$\int\limits_{U_M}\sum_{m\geqslant m_0+1}|\Delta^l_m f|^2d\nu\leqslant C\int\limits_{U_M}|f|^2d\nu. \qquad (4.15)$$

In (4.15), letting $M\to\infty$ and then letting $m_0\to-\infty$, we can conclude that the inequality on the left hand side of (4.10).

To prove the inequality on the left hand side of (4.10) we need the following facts. If $g \in L^2(d\nu; \mathbb{R}_{(n)})$, then

(a) $\lim_{m\to+\infty} E^l_m g = g = \lim_{m\to+\infty} E^r_m g$ in the sense of L^2.

(b) $\lim_{m\to-\infty} E^l_m g = 0 = \lim_{m\to-\infty} E^r_m g$ in the sense of L^2-.

(c) $g = \sum_{m=-\infty}^{\infty} \Delta^l_m g = \sum_{m=-\infty}^{\infty} \Delta^r_m g$.

These facts can be proved in the same way as the corresponding scalar-valued results in [5, Chap. 5]. Of course, the condition (4.4) is crucial in the proofs.

Suppose that $f, g \in L^2(d\nu; \mathbb{R}_{(n)})$. By (4.4) and the right hand inequality in (4.10), we can get

$$\begin{aligned}\Big|\int_X f\psi g d\nu\Big| &= \Big|\int_X \Big(\sum_{m=-\infty}^{\infty} \Delta^r_m g\Big)\psi\Big(\sum_{\kappa=-\infty}^{\infty} \Delta^l_\kappa f\Big)d\nu\Big| \\ &= \Big|\int_X \Big(\sum_{m=-\infty}^{\infty} \Delta^r_m g\psi\Delta^l_n f\Big)d\nu\Big| \\ &\leqslant C\|S^r g\|_2\|S^l(f)\|_2.\end{aligned} \tag{4.16}$$

In (4.16), taking supremum over all g satisfying $\|g\|_2 \leqslant 1$ and using again the condition (4.4), we complete the proof. □

We now construct a special example, and the associated Haar functions are appropriate to the analysis of the Cauchy integral. Let $X = \mathbb{R}^n$ and $\mathcal{B}$ be the Borel $\sigma-$field. Assume that $d\nu$ is the Lebesgue measure, also denoted by dx. The Lebesgue measure of a measurable set U is denoted by $|U|$. Let $\mathcal{F}_0$ be the $\sigma-$ field generated by the family $\mathcal{J}_0$ of cubes with side length 1 whose corners lie at the points of the integer lattice.

Let I be any cube in $\mathcal{J}_0$. Divide I equally by the hyperplane that bisects the edges parallel to the x_1-axis, and let $\mathcal{J}_1$ denote the family of dyadic-quasi-cubes so produced. Let $\mathcal{F}_1$ be the $\sigma-$ generated by $\mathcal{J}_1$. Now subdivide each dyadic-quasi-cube by the hyperplane that bisects the edges parallel to the x_2-axis, and let $\mathcal{F}_2$ be the $\sigma-$field generated by the new family of dyadic-quasi-cubes.

Continue in this manner, at each stage bisecting each dyadic-quasi-cube of the previous family by the hyperplane perpendicular to the next coordinate axis. This produces the sequence $\{\mathcal{F}_m\}_{m=0}^{\infty}$. For $m < 0$, the $\sigma-$ field $\mathcal{F}_m$ are produced by the reverse procedure to the one just described-successive doubling in the coordinate directions. Note that each dyadic-quasi-cube in $\mathcal{F}_{kn}$, $k \in \mathbb{Z}$, i.e., atom, is actually a standard dyadic cube of side length 2^{-k}.

At last, let $\mathcal{J} = \bigcup_{m=-\infty}^{\infty} \mathcal{J}_m$. Note that any $I \in \mathcal{J}$ is a dyadic-quasi-cube, say $I \in \mathcal{J}_{m-1}$, and so can be written as $I = I_1 \cup I_2$, where I_1 and I_2 are dyadic-quasi-cubes in $\mathcal{J}_m$.

From now on, we only discuss the left martingale. Hence we simplify the notation by writing E_m, Δ_m, f_m etc. in place of E^l_m, Δ^l_m, f^l_m etc. We still assume that the function $\psi \in L^\infty(X : \mathbb{R}^{1+n}) = L^\infty(\mathbb{R}^n; \mathbb{R}^{1+n})$ satisfies (4.4), but corresponds to the particular sequence $\{\mathcal{F}_m\}_{-\infty}^{\infty}$ in the $\sigma-$field. The following lemma is an essential ingredient of this chapter.

Lemma 4.1.2 *For any $I \in \mathcal{J}_{m-1}$, where $I = I_1 \cup I_2$ with $I_1, I_2 \in \mathcal{J}_m$, there exist a pair of $\mathbb{R}_{(n)}$-valued functions α_I and β_I on $\mathbb{R}^n$ and a positive constant C such that*

(i)

$$\alpha_I = a_1\chi_{I_1} + a_2\chi_{I_2},\ a_j \in \mathbb{R}_{(n)},$$
$$\beta_I = b_1\chi_{I_1} + b_2\chi_{I_2},\ b_j \in \mathbb{R}_{(n)};$$

(ii) *For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}_{(n)})$,*

$$\Delta_m f(x) = \alpha_I(x)\langle\beta_I,\ f\rangle_\psi,\ x \in I;$$

(iii) $C^{-1}|I|^{-1/2} \leqslant |\alpha_I(x)| \leqslant C|I|^{-1/2}$, *and for all $x \in I$,* $C^{-1}|I|^{-1/2} \leqslant |\beta_I(x)| \leqslant C|I|^{-1/2}$;

(iv)

$$\int \psi\alpha_I dx = \int \beta_I\psi dx = 0.$$

Proof Define α_I and β_I as in (i). We need to choose a_1, a_2, b_1 and b_2 such that (ii)–(iv) hold.

We consider (ii). Because $\mathcal{F}_m$ and $\mathcal{F}_{m-1}$ are atoms, on I, we have

$$\widetilde{E}_{m-1}f = \Big(\frac{1}{|I|}\int_I f(y)dy\Big)\chi_I.$$

For $\widetilde{E}_m(f)$, a similar formula holds. Let

$$u = \int_I \psi(t)dt,\ u_j = \int_{I_j} \psi(t)dt,\ j = 1, 2.$$

Then on I,

$$\begin{aligned}\Delta_m f &= \widetilde{E}(\psi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_m) - \widetilde{E}(\psi \mid \mathcal{F}_{m-1})^{-1}\widetilde{E}(\psi f \mid \mathcal{F}_{m-1})\\ &= u_1^{-1}\Big(\int_{I_1}\psi f dx\Big)\chi_{I_1} + u_2^{-1}\Big(\int_{I_2}\psi f dx\Big)\chi_{I_2}\\ &\quad -u^{-1}\Big(\int_{I_1}\psi f dx + \int_{I_2}\psi f dx\Big)(\chi_{I_1} + \chi_{I_2})\end{aligned}$$

$$= \left((u_1^{-1} - u^{-1}) \int_{I_1} \psi f dx - u^{-1} \int_{I_2} \psi f dx \right) \chi_{I_1}$$
$$+ \left((u_2^{-1} - u^{-1}) \int_{I_2} \psi f dx - u^{-1} \int_{I_1} \psi f dx \right) \chi_{I_2}.$$

On the other hand,

$$\alpha_I \langle \beta_I,\ f \rangle_\psi = \left(a_1 b_1 \int_{I_1} \psi f dx + a_1 b_2 \int_{I_2} \psi f dx \right) \chi_{I_1}$$
$$+ \left(a_2 b_2 \int_{I_2} \psi f dx + a_2 b_1 \int_{I_1} \psi f dx \right) \chi_{I_2}.$$

Comparing the last two expressions, we choose $a_i, b_i, i = 1, 2$, such that

$$a_1 b_1 = u_1^{-1} - u^{-1},\ a_2 b_2 = u_2^{-1} - u^{-1},\ a_1 b_2 = -u^{-1} = a_2 b_1.$$

Letting $u = u_1 + u_2$ and applying the equality

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1} = b^{-1}(b - a)a^{-1}, \tag{4.17}$$

we can see that the above equation has a concise expression:

$$a_1 b_1 = u^{-1} u_2 u_1^{-1},\ a_2 b_2 = u^{-1} u_1 u_2^{-1},\ a_1 b_2 = -u^{-1}, a_2 b_1 = -u^{-1}. \tag{4.18}$$

The solutions of (4.18) can be represented as

$$a_1 = u^{-1} u_2 c,\ a_2 = -u^{-1} u_1 c,\ b_1 = c^{-1} u_1^{-1},\ b_2 = -c^{-1} u_2^{-1}, \tag{4.19}$$

where c is any invertible element in $\mathbb{R}_{(n)}$. We want to choose c such that (iii) holds. In fact, by (i) and (4.19), it is obvious that if c is taken to be $|I|^{-1/2}$, then (iii) holds.

At last, we verify (iv). By (i) and (4.19), we can get

$$\begin{aligned}
\int \psi \alpha_I dx &= \int \psi (a_1 \chi_{I_1} + a_2 \chi_{I_2}) dx \\
&= u_1 a_1 + u_2 a_2 \\
&= (u_1 u^{-1} u_2 - u_2 u^{-1} u_1) c \\
&= u_1 u^{-1} (u - u_1) c - (u - u_1) u^{-1} u_1 c = 0.
\end{aligned}$$

We can deduce from (4.19) that $\int \beta_I \psi dx = 0$. □

4.2 Martingale Type $T(b)$ Theorem

In this section, we prove the boundedness of Cauchy singular integral operators via the Clifford martingale. The main result is as follows. We suppress the fact that the Cauchy singular integral is a principal value by writing our operators in terms of ordinary integrals. The principal values are to be interpreted as the ones obtained by projecting the Euclidean balls in Σ onto $\mathbb{R}^n$ and integrating over their complements.

Theorem 4.2.1 *If Σ is a Lipschitz graph, then the Cauchy singular integral operator is bounded from $L^2(\Sigma; \mathbb{R}_{(n)})$ to $L^2(\Sigma; \mathbb{R}_{(n)})$.*

Let $\phi(v) = A(v)e_0 + v$ $(v \in \mathbb{R}^n)$ be the coordinate system on Σ defined by A. The unit normal of Σ is

$$n(\phi(v)) = (e_0 - \nabla A(v))\sqrt{1 + |\nabla A(v)|^2}.$$

For these coordinates, we have

$$\begin{aligned} T_\Sigma h(\phi(u)) &= \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \mathfrak{n}(\phi(v))h(\phi(v))\sqrt{1 + |\nabla A(v)|^2} dv \\ &= \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \psi(v)h(\phi(v))dv, \end{aligned}$$

where $\psi(v) = e_0 - \nabla A(v)$. Because $|\nabla A(v)| \leqslant C$, we can see that T_Σ is bounded on $L^2(\Sigma; \mathbb{R}_{(n)})$ if and only if the operator

$$T : f \mapsto \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} f(v)dv \tag{4.20}$$

is bounded from $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$ to $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$.

Notice that if I is a dyadic-quasi-cube, then the principal value integral

$$T(\psi\chi_I)(u) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\overline{\phi(v) - \phi(u)}}{|\phi(v) - \phi(u)|^{1+n}} \psi(v)\chi_I(v)dv$$

exists and defines a locally integrable function. The existence and the local integrability of $T(\psi\chi_I)(u)$ on $\mathbb{R}^n \setminus I$ are straightforward. Moreover, in $\mathbb{R}^n \setminus I$, the singularity of $T(\psi\chi_I)(u)$ is $O(\log(\text{dist}(u, \partial I)))$ as $u \to \partial I$. To deal with the case $u \in I$, we only need to consider

$$T_\Sigma F(x) = \text{p.v.} \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)F(y)d\sigma(y),$$

where F vanishes outside $\phi(I)$ and satisfies a uniform Lischitz condition. Write

$$\begin{aligned} T_\Sigma F(x) = \text{p.v.} \int_\Sigma \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)\Big[F(y) - F(x)\Big]d\sigma(y) \\ + \int_\Sigma \frac{\overline{y - x}}{|y - x|^{1+n}} \mathfrak{n}(y)F(x)d\sigma(y). \end{aligned}$$

The Lipschitz condition of F gives an appropriate control on the first integral. By Cauchy's theorem, the monogenicity and cancellation properties of the kernel $\overline{(y-x)}/|y-x|^{1+n}$, we obtain a suitable control on the second integral.

We write the operator in (4.20) as

$$Tf(u) = \int_{\mathbb{R}^n} K(u, v) f(v) dv.$$

In the following lemma, we give some elementary properties of the kernel K.

Lemma 4.2.1 *For all x, x', y such that $x \neq y$ and $|x - x'| < 1/2|x - y|$, the kernel K satisfies*

$$|K(x, y)| \leqslant \frac{C}{|x-y|^n}, \ x \neq y, \tag{4.21}$$

$$|K(x, y) - K(x', y)| \leqslant C \frac{|x - x'|}{|x-y|^{1+n}}, \tag{4.22}$$

and

$$|K(y, x) - K(y, x')| \leqslant C \frac{|x - x'|}{|x-y|^{1+n}}. \tag{4.23}$$

Let $\mathcal{S}$ denote the span over $\mathbb{R}_{(n)}$ of the set of all characteristic functions of dyadic-quasi-cubes. The space $\mathcal{S}\psi$ of pointwise products with the function ψ is a left-linear space over $\mathbb{A}_d$. By use of the idea of [7], we can define $T\psi$ as a Clifford left functional on the subspace $(\mathcal{S}\psi)_0$ of $\mathcal{S}\psi$. The space $(\mathcal{S}\psi)_0$ consists of the functions having integral 0: fix $g\psi \in (\mathcal{S}\psi)_0$ and choose N large enough such that the ball B_N of radius N centered at 0 contains the support of g. Then we define

$$\begin{aligned} T\psi(g\psi) &= T(\psi\chi_{B_N})(g\psi) + \iint g(x)\psi(x)\Big[K(x, y) - K(0, y)\Big]\Big[1 - \chi_{B_N}(y)\Big]\psi(y) dx dy \\ &= I_N^{(1)} + I_N^{(2)}. \end{aligned}$$

By (4.22) and (4.23), this definition is meaningful. An important fact is that

$$\langle \beta_J, \ T\psi \rangle_\psi = T\psi(\beta_J \psi) = 0. \tag{4.24}$$

This can be proved as follows.

(a) When $N \to \infty$, $I_N^{(2)} \to 0$.
(b) By the monogenicity of the Cauchy kernel, using Cauchy's theorem, we can prove that $\lim\limits_{N\to\infty} T(\psi\chi_{B_N})(x)$ exists and is independent of $x \in \text{supp}\beta_J$.

Because the integral of $\beta_J \psi$ is 0, we can conclude that

$$\lim_{N\to\infty} T(\psi\chi_{B_N})(\beta_J\psi) = 0.$$

In establishing (b), one works on the surface Σ.

We note that, if T^t is the operator $f \mapsto \int f(y)K(y,x)dy$, then for all dyadic-quasi-cubes I and J,

$$\langle T^t(\chi_I\psi),\ \chi_J\rangle_\psi = \langle \chi_I,\ T(\psi\chi_J)\rangle_\psi.$$

Similar to T, we have

$$\langle T^t\psi,\ \beta_J\rangle_\psi = T^t\psi(\psi\beta_J) = 0. \tag{4.25}$$

By Lemma 4.1.2, if $f \in L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$, we get

$$f = \sum_{m=-\infty}^{\infty} \Delta_m f = \sum_I \alpha_I\langle \beta_I,\ f\rangle_\psi$$

and

$$\begin{aligned} T(\psi f) &= \sum_{J\in\mathcal{J}} T(\psi\alpha_J)\langle\beta_J,\ f\rangle_\psi \\ &= \sum_{J,I} \alpha_I\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi\langle\beta_J,\ f\rangle_\psi \\ &= \sum_I \alpha_I \sum_J \langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi\langle\beta_J,\ f\rangle_\psi. \end{aligned}$$

Let $u_{IJ} = \langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi$. By Lemmas 4.1.1 and 4.1.2, we only need to prove the linear transform defined by the matrix (u_{IJ}) on $l^2(\mathcal{J};\mathbb{R}_{(n)})$ is bounded. We need the following Schur lemma.

Lemma 4.2.2 (Schur) *Assume that there exist a family of positive numbers* (ω_I) *and a constant* C *such that*

$$\sum_J |\omega_J u_{IJ}| \leqslant C\omega_I,\ \ I \in \mathcal{J}, \tag{4.26}$$

and

$$\sum_I |\omega_I u_{IJ}| \leqslant C\omega_J,\ \ I \in \mathcal{J}. \tag{4.27}$$

Then the matrix (u_{IJ}) *defines a bounded operator on* $l^2(\mathcal{J};\mathbb{R}_{(n)})$.

Proof This is a natural modification of the proof of the scalar version. □

Now we state some facts associated with the estimate of $|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$. Assume that I and J are atoms in $\mathcal{F}_m$ and $\mathcal{F}_\kappa$, and assume that $m \geqslant \kappa$. If the atom $A \in \mathcal{F}_n$ is not contained in J (or J^c) but a part of its boundary is in common with the boundary of J, then A is said to be contiguous to J (or contiguous to J^c). If the atoms of A are in the same σ−field as I and are contiguous to J, we denote the union of J and such atoms by $I + J$. Specially, $2J$ denotes the union of J with all of atoms in $\mathcal{F}_m$ which are contiguous to J. The bottom-left corner x_J of J is the vertex of J having minimal coordinates.

Lemma 4.2.3 *Let I and J be atoms of $\mathcal{F}_m$ and $\mathcal{F}_\kappa$, respectively, and $m \geqslant \kappa$. There exists a constant C, independent of κ and m, such that if $I \subseteq 2J\backslash J$, then*

$$\int_{I\times J} \frac{dxdy}{|x-y|^n} \leqslant C|I|\Big(\log\frac{|J|}{|I|}+1\Big).$$

Proof We can prove this lemma via a simple calculation and we omit the details. □

Lemma 4.2.4 *Let I and J be atoms in $\bigcup_{j=-\infty}^{\infty} \mathcal{F}_j$. Then*

(i) *for all $x \notin 2J$,*
$$|T(\psi\alpha_J)| \leqslant C|J|^{1/2+1/n}|x-x_J|^{-1-d}; \tag{4.28}$$

(ii) *if $I \subseteq (2J)^c$, then*
$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C|I|^{-1/2}|J|^{1/2+1/n}\int_I |x-x_J|^{-1-n}dx; \tag{4.29}$$

(iii) *for all $x \notin J$,*
$$|T(\psi\alpha_J)(x)| \leqslant C|J|^{-1/2}\int_J |x-y|^{-n}dy;$$

(iv) *if $I \subseteq 2J\backslash J$, then*
$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big).$$

(In the above (i)–(iv), the constant C is independent of I and J).

Proof The assertion (i) can be proved by the canceling property of Haar functions. Hence

$$\begin{aligned} T(\psi\alpha_J) &= \int K(x,y)\psi(y)\alpha_J(y)dy \\ &= \int_J [K(x,y)-K(x,x_J)]\psi(y)\alpha_J(y)dy. \end{aligned}$$

So we can deduce from (4.23) that if $x \notin 2J$, then

$$\begin{aligned}|T(\psi\alpha_J)(x)| &\leqslant C|J|^{-1/2}\int_J \frac{|y-x_J|}{|x-x_J|^{1+n}}dy\\ &\leqslant C|J|^{1/2}|x-x_J|^{-1-n}\sup_{y\in J}|y-x_J|\\ &\leqslant C|J|^{1/2+1/d}\frac{1}{|x-x_J|^{1+n}}.\end{aligned}$$

To prove (ii), we can use (i) and (iii) of Lemma 4.1.2. The assertion (iii) follows from (4.21). The assertion (iv) is clear from (iii) and Lemma 4.2.3. □

We divide the estimate of

$$\sum_I |I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$$

into three parts, each with a number of separate cases based on the relative size and disposition of the atoms I and J.

Case 1. The sum with respect to atoms I larger that J.
Fix $J\in\mathcal{F}_\kappa$ and consider the set $2J$. Let x_J be the bottom-left corner of J. Consider $I\in\mathcal{F}_m$, $m<\kappa$.

(a) If I lies outside $2J$, by (ii) of Lemma 4.2.4 and (iii) of Lemma 4.1.2, we have

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|\leqslant C|I|^{-1/2}|J|^{1/2+1/n}\int_I|x-x_J|^{-1-n}dx.$$

Hence, in this case, if $t<1/2$, the estimate for the Schur sum is

$$\begin{aligned}&\sum_{I\in\bigcup_{m<\kappa}\mathcal{F}_m,I\subseteq(2J)^c}|I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|\\ &\leqslant C\sum_{j=1}^{\infty}(2^j|J|)^{t-1/2}\sum_{I\in\mathcal{F}_{\kappa-j},\ I\subseteq(2J)^c}|J|^{1/2+1/n}\int_I|x-x_J|^{-1-n}dx\\ &\leqslant C\sum_{j=1}^{\infty}2^{j(t-1/2)}|J|^{t+1/d}\int_{(2J)^c}|x-x_J|^{-1-n}dx\\ &\leqslant C\sum_{j=1}^{\infty}2^{j(t-1/2)}|J|^t\\ &\leqslant C|J|^t.\end{aligned}$$

(b) For a fixed $m<\kappa$, the dyadic-quasi-cubes which meet $2J$ are of two kinds: those that lie in $2J\backslash J$, and one that contains J. If I lies in $2J\backslash J$, then because the ration of the measures of I and J is bounded above and away from 0 and independent of I and J, by (iv) Lemma 4.2.4, we know

$$|I|^t|\langle \beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\frac{|I|^{t+1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big) \leqslant C|J|^t.$$

Because the number of such terms is bounded and is independent of I and J, the corresponding part of the Schur sum is $O(|J|^t)$.

If I contains J and is larger than J, the I can be written as $I = I_1 \cup I_2$, where I_1 and I_2 are atoms in $\mathcal{F}_{m+1}$. Assume that $J \subseteq I_1$ and write $\beta_I = \beta_1\chi_{I_1} + \beta_2\chi_{I_2}$. Then similar to (4.24) and (4.25), we can get

$$\Big\langle \beta_1\chi_{I_1},\ T(\psi\alpha_J)\Big\rangle_\psi = -\Big\langle \beta_1\chi_{I_1^c},\ T(\psi\alpha_J)\Big\rangle_\psi.$$

Now I_1^c contains part of the region $2J\backslash J$. We can use (i) of Lemma 4.2.4 on this region. In particular,

$$\begin{aligned} |\langle \beta_1\chi_{I_1},\ T(\psi\alpha_J)\rangle_\psi| &= \Big|\beta_1\int_{I_1^c}\psi(x)T(\psi\alpha_J)(x)dx\Big| \qquad (4.30)\\ &\leqslant C|\beta_1|\Big(\int_{2J\backslash J}|T(\psi\alpha_J)(x)|dx + \int_{(2J)^c}|T(\psi\alpha_J)(x)|dx\Big)\\ &\leqslant C|I|^{-1/2}|J|^{-1/2}\int_{2J\backslash J}dx\int_J|x-y|^{-n}dy\\ &+ C|I|^{-1/2}|J|^{1/2+1/n}\int_{(2J)^c}|x-x_J|^{-1-n}dx\\ &\leqslant C\Big\{|I|^{-1/2}|J|^{-1/2} + |I|^{-1/2}|J|^{1/2}\Big\} \leqslant C\frac{|J|^{1/2}}{|I|^{1/2}}, \end{aligned}$$

where in the second-last step we have used Lemma 4.2.3. As for $\langle \beta_2\chi_{I_2},\ T(\psi\alpha_J)\rangle_\psi$, we have I_2 is disjoint with J, so we can obtain an estimate similar to that of (4.30).

The estimate for the Schur sum of the dyadic-quasi-cubes satisfying $I \supseteq J$ is

$$\sum_{I\in\bigcup_{m<\kappa}\mathcal{F}_m,\ I\supseteq J}|I|^t|\langle \beta_I,\ T(\psi\alpha_J)\rangle_\psi| \leqslant C\sum_{k=1}^{\infty}(2^k|J|)^{t-1/2}|J|^{1/2} \leqslant C|J|^t,$$

where $t < 1/2$.

Case 2. The sum with respect to atoms I smaller than J.
For this case, we deal with the atoms $J \in \mathcal{F}_\kappa$ and $I \in \mathcal{F}_m$ with $m > \kappa$.

(a) If I lies outside $2J$, then J lies outside $2I$. Hence we apply (i) of Lemma 4.2.4 to T^t and get

$$|T^t(\beta_I\psi)(x)| \leqslant \frac{C|I|^{1/2+1/n}}{|x-x_I|^{1+n}},$$

which implies that

$$\begin{aligned}|\langle T^t(\beta_I\psi),\ \alpha_J\rangle_\psi| &\leqslant C|I|^{1/2+1/n}|J|^{-1/2}\int_J \frac{dx}{|x-x_I|^{1+n}}\\ &\leqslant C|I|^{1/2+1/n}|J|^{1/2}\frac{1}{|x-x_J|^{1+n}}\\ &\leqslant C|I|^{1/n-1/2}|J|^{1/2}\int_I \frac{dx}{|x-x_J|^{1+n}},\end{aligned}$$

where in the middle step we have used the fact that $I \subseteq (2J)^c$. The estimate of the corresponding Schur sum is

$$\begin{aligned}&\sum_{I\in\bigcup_{m>\kappa}\mathcal{F}_m,\ I\cap 2J=\emptyset} |I|^{t+1/n-1/2}|J|^{1/2}\int_I \frac{dx}{|x-x_J|^{1+n}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{1/2}\int_{(2J)^c}\frac{dx}{|x-x_J|^{1+n}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{-j})^{t+1/n-1/2}|J|^t \leqslant C|J|^t,\end{aligned}$$

where $t > 1/2 - 1/n$.
(b) If $I \cap J = \emptyset$ and $I \subseteq 2J \setminus (I + J)$, then $J \subseteq (2I)^c$. So for T^t, we can use (ii) of Lemma 4.2.4 to obtain

$$\begin{aligned}|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| &= C\left|\left\langle T^t(\beta_I\psi),\ \alpha_J\right\rangle_\psi\right|\\ &\leqslant C|J|^{-1/2}|I|^{1/2+1/n}\int_J \frac{dx}{|x-x_I|^{1+n}}.\end{aligned} \tag{4.31}$$

Let $d(x,\ J)$ denote the distance of the point x from J. The atom I may have unequal side length. Let $l(I)$ be the smallest side length. We can deduce from (4.31) that

$$\begin{aligned}|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi| &\leqslant C|J|^{-1/2}|I|^{1/2+1/n}\frac{1}{d(x_I,\ J)}\\ &\leqslant C|J|^{-1/2}|I|^{1/2+1/n}|I|^{-1}\int_I \frac{dx}{d(x,\ J)+l(I)}.\end{aligned} \tag{4.32}$$

Denote by L the maximal side length of J and by l the minimal side length of J, respectively. Then $L \leqslant 2l$ and $l^n \leqslant |J| \leqslant 2^n l^n$. The smallest side length of the dyadic-quasi-cubes $I \in \mathcal{F}_{\kappa+j}$ is $l(I) \geqslant l/2^{k/n+1}$. It follows from (4.32) that the estimate of the relevant part of the Schur sum is: if $t > 1/2 - 1/n$, then

$$\sum_{I\in\bigcup_{m>\kappa},\ I\subseteq 2J\backslash(I+J)} |I|^t|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|$$

$$\leqslant \sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}\int_{2J\backslash(I+J)}\frac{dx}{d(x,J)+2^{-j/n-1}l}$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}\int_0^{3L}dx_1\cdots\int_0^{3L}dx_{d-1}\int_{2^{-j/n-1}l}^{2l}\frac{du}{u+2^{-j/n-1}l}$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/n-1/2}|J|^{-1/2}|J|^{(n-1)/n}\log\Big(\frac{2l+2^{-j/d-1}l}{2^{-j/n}l}\Big)$$

$$\leqslant C\sum_{j=1}^{\infty}(2^{-j})^{t+1/n-1/2}\frac{j}{n}|J|^t\leqslant C|J|^t.$$

(c) If $I\subseteq(I+J)\backslash J$, we have $I\subseteq 2J\backslash J$. By (iv) of Lemma 4.2.4,

$$|\langle\beta_I,\ T(\psi\alpha_J)\rangle_\psi|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}\Big(\log\frac{|J|}{|I|}+1\Big). \tag{4.33}$$

In the region $(I+J)\backslash J$, there exist $O(L^{d-1}/(2^{-j/n-1}l)^{n-1})$ atoms which belong to $\mathcal{F}_m$. In other words, there exist $O(2^{j(1-1/n)})$ atoms. By (4.33), if $t>1/2-1/n$, the corresponding estimate of the Suchr sum is

$$C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2}|J|^{-1/2}j2^{j(1-1/n)}=C|J|^t\sum_{j=1}^{\infty}j(2^{-j})^{t-1/2+1/n}\leqslant C|J|^t.$$

(d) If $I\subseteq J$ and L is contiguous to J^c, we write $J=J_1+J_2$, where J_1 and J_2 are atoms in $\mathcal{F}_{m+1}$. Let $\alpha_J=\alpha_1\chi_{J_1}+\alpha_2\chi_{J_2}$, and assume that $I\subseteq J_1$.

We first consider the atoms $I\subseteq J_1$ which are contiguous to J_1^c. We have

$$\begin{aligned}\Big|\Big\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\Big\rangle_\psi\Big| &= \Big|\Big\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1^c})\Big\rangle_\psi\Big|\\ &= \Big|\Big\langle T(\beta_1\psi),\ \alpha_I\chi_{J_1^c}\Big\rangle_\psi\Big|\\ &\leqslant \Big|\int_{J_1^c\cap 2I}T^t(\beta_I\psi)(x)\alpha_1dx\Big|+\Big|\int_{J_1^c\backslash 2I}T^t(\beta_I\psi)(x)\alpha_1dx\Big|.\end{aligned}$$

Hence by Lemma 4.2.3, applying (i) of Lemma 4.2.4 to $T^t(\beta_I\psi)$, we get

$$\begin{aligned}\left|\left\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\right\rangle_\psi\right| &\leqslant C|I|^{-1/2}|J|^{-1/2}\int_{2I\setminus I}dx\int_I\frac{dy}{|x-y|^n}\\ &+C|I|^{1/2+1/n}\int_{(2I)^c}\frac{dx}{|x-x_I|^{1+n}}\\ &\leqslant C|I|^{-1/2}|J|^{-1/2}|I|\log\left(\frac{|I|}{|I|}+1\right)+C|I|^{1/2}|J|^{-1/2}\\ &\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}.\end{aligned}\tag{4.34}$$

Because $J_2\subseteq J_1^c$, we can use an estimate similar to (4.34) to obtain

$$\left|\left\langle\beta_I,\ T(\psi\alpha_2\chi_{J_2})\right\rangle_\psi\right|\leqslant C\frac{|I|^{1/2}}{|J|^{1/2}}.\tag{4.35}$$

In $\mathcal{F}_{\kappa+j}$, there exist $O(2^{j(1-1/n)})$ atoms that are contiguous to J_1^c. It follows from (4.34) and (4.35) that for the atoms which are contiguous to J_1^c, the corresponding estimate of the Schur sum is

$$C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2}|J|^{-1/2}|J|^{-1/2}2^{j(1-1/n)}=C|J|^t\sum_{j=1}^{\infty}(2^{-j})^{t-1/2+1/n}\leqslant C|J|^t,$$

where $t>1/2-1/n$.

(e) If $I\subseteq J$ and I is disjoint with J_1^c, similar to (i) of Lemma 4.2.4, we have

$$\begin{aligned}\left|\left\langle\beta_I,\ T(\psi\alpha_1\chi_{J_1})\right\rangle_\psi\right| &=\left|\int T^t(\beta_I\psi)(x)\psi(x)\alpha_1\chi_{J_1^c}(x)dx\right|\\ &\leqslant C|J|^{-1/2}\int_{J_1^c}|T^t(\beta_I\psi)(x)|dx\\ &\leqslant C|I|^{1/2+1/n}|J|^{-1/2}\int_{J_1^c}\frac{dx}{|x-x_I|^{1+n}}\\ &\leqslant C|I|^{1/2+1/n}|J|^{-1/2}\frac{1}{d(x,\ J_1^c)}.\end{aligned}$$

For $|\langle\beta_I,\ T(\phi\alpha_2\chi_{J_2})\rangle_\psi|$, a similar estimate holds. So the corresponding estimate of the Schur sum is

$$\begin{aligned}&\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2+1/n}|J|^{-1/2}\sum_{j=1}^{2^{j(1-1/n)}}\frac{1}{j2^{-j/n}}\\ &\leqslant C\sum_{j=1}^{\infty}(2^{-j}|J|)^{t+1/2+1/n}|J|^{-1/2-1/n}2^j\log(2^{j(1-1/n)})\end{aligned}$$

$$\leqslant C\sum_{j=1}^{\infty} k(2^{-j})^{t-1/2+1/n}|J|^t$$
$$\leqslant C|J|^t,$$

where $t > 1/2 - 1/n$.

Case 3. Atoms of the same size.
Here we only need to estimate the term $\langle \beta_I, T(\psi\alpha_I)\rangle_\psi$ since the arguments for Case 1 can be used to estimate the other parts of the Schur sum.

By Lemma 4.1.2, it suffices to prove that for all dyadic-quasi-cubes I,

$$|\langle \chi_I, T(\psi\chi_I)\rangle_\psi| \leqslant C|I|.$$

For this, we need to use the monogenicity of the Cauchy kernel. So we pass from T back to T_Σ. The coordinate mapping is $\phi(v) = A(v)e_0 + v$. For small $\epsilon > 0$ and $x = \phi(u)(u \in I)$, consider

$$\int_{|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y). \tag{4.36}$$

Let P_z be the tangent hyperplane Σ to at x. Set $a(u) = \text{dist}(u, \partial\phi I)$ and $b = b(x) = \text{dist}(x, \partial\phi(I))$. Write (4.36) as $I_1 + I_2$, where

$$I_1 := \int_{b>|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y)$$

and

$$I_2 := \int_{|x-y|>b} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y).$$

Then

$$|I| \leqslant C\log\Big(\frac{C|I|^{1/n}}{a(u)}\Big).$$

By Cauchy's theorem, we write

$$I_1 = \int_{S_b} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y) + \int_{S_\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y)$$
$$+ \int_{x,\ y\in P_x,\ b>|x-y|>\epsilon} \frac{\overline{y-x}}{|y-x|^{1+n}} \mathfrak{n}(y)\chi_{\phi(I)}(y)d\sigma(y), \tag{4.37}$$

where S_b and S_ϵ are the portions of the sphere of radii b and ϵ, respectively, that lie between Σ and P_x. Because the kernel is anti-systemic and the integrals on S_b and S_ϵ are dominated by a constant, independent of x, ϵ and b, then the third integral in (4.37) is 0. Hence

$$|\langle \chi_I,\ T(\psi\chi_I)\rangle_\psi| \leqslant C|I| + C\int_I \log\Big(\frac{|I|^{1/n}}{a(u)}\Big)du \leqslant C|I|.$$

Assume that b_1 and b_2 are two pseudoaccretive functions. The space $b_1L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$ is defined as the set of all products of the form $b_1 f,\ f \in L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$. Similarly, we can define $L^2(\mathbb{R}^n;\mathbb{R}_{(n)})b_2$. These spaces are isomorphic to $L^2(\mathbb{R}^n;\mathbb{R}_{(n)})$. Let $\mathcal{S}$ denote the space of finite linear combinations over $\mathbb{R}_{(n)}$ of characteristic functions of dyadic-quasi-cubes. Then $b_1\mathcal{S}$ is dense in $b_1L^2(\mathbb{R}_{(n)})$. Denote by $(\mathcal{S}b_2)^*$ the space of all Clifford left linear functionals on $\mathcal{S}b_2$ with values in $\mathbb{R}_{(n)}$. Similarly, $(b_1\mathcal{S})^*$ denotes the space of all Clifford right linear functionals on $b_1\mathcal{S}$.

Let T be a Clifford right linear mapping from $b_1\mathcal{S}$ to $(\mathcal{S}b_2)^*$ and let $\Delta = \{(x, y) : x = y\}$. We call T a standard Calderón-Zygmund operator, if there exists a C^∞ function K in $\mathbb{R}^n \times \mathbb{R}^n\backslash\Delta$ with values in $\mathbb{R}_{(n)}$ satisfying:

(i) for $x \neq y$,

$$|K(x,\ y)| \leqslant C\frac{1}{|x-y|^n}; \tag{4.38}$$

(ii) there exist a constant δ such that for $0 < \delta \leqslant 1$ and $|y - y_0| < |y - x|/2$,

$$|K(x, y) - K(x, y_0)| + |K(y, x) - K(y_0, x)| \leqslant C\frac{|y-y_0|^\delta}{|x-y|^{n+\delta}}; \tag{4.39}$$

(iii) for all $f, g \in \mathcal{S}$ having disjoint supports,

$$T(b_1 f)(gb_2) = \iint g(x)b_2(x)K(x, y)b_1(y)f(y)dxdy. \tag{4.40}$$

In conformity with (4.40), we write

$$T(b_1 f)(gb_2) = \langle g,\ T(b_1 f)\rangle_{b_2}.$$

If T^t is a left linear mapping from $\mathcal{S}b_2$ to $(b_1\mathcal{S})^*$ such that for all $f, g \in \mathcal{S}$,

$$\langle g,\ T(b_1 f)\rangle_{b_2} = \langle T^t(gb_2),\ f\rangle_{b_1}$$

and T is associated with the kernel K, then T^t is associated with the kernel $K(y, x)$ in the sense that

$$T^t(gb_2)(b_1 f) = \int\Big(\int g(x)b_2(x)K(x, y)dx\Big)b_1(y)f(y)dy.$$

If there exists a constant C such that for all dyadic-quasi-cubes Q,

$$|T(b_1\chi_Q)(\chi_Q b_2)| \leqslant C|Q|,$$

We say that T is weakly bounded with respect to b_1 and b_2. This definition is formally different from the usual one in [7, 8], in which the test functions are taken to be smooth. However, the two definitions are equivalent.

If $h \in L^\infty(\mathbb{R}^n; \mathbb{R}^{1+n})$, then Th can be defined as a linear functional on the subspace $(\mathcal{S}b_2)_0$ of $\mathcal{S}b_2$ consisting of functions having integral 0. In the next theorem, we say $T(b_1) \in BMO$ if there exists a locally integrable BMO function ϕ such that for all $g \in (\mathcal{S}b_2)_0$, $\langle g,\ T(b_1)\rangle_{b_2} = \langle g,\ \phi\rangle_{b_2}$. A similar interpretation applies to $T^t(b_2)$. For the sequence of $\sigma-$fields, the space BMO is the one defined in (4.8).

Theorem 4.2.2 ($T(b)$ theorem) *Let T and T^t ba as above and T be associated with the standard Calderón-Zygmund kernel K. Then T is extendible to a bounded linear operator from $b_1L^2(\mathbb{R}^n; \mathbb{R}_{(n)})$ to $L^2(\mathbb{R}^n; \mathbb{R}_{(n)})b_2$ if and only if*

(i) *$T(b_1)$, $T^t(b_2) \in BMO$;*

(ii) *T is weakly bounded for b_1 and b_2.*

Proof The necessity of the conditions (i) and (ii) was proved in the classical case by [9–11]. Their proof adapted to the more general Clifford algebra setting.

To prove the sufficiency, we first deal with the case $T(b_1) = T^t(b_2) = 0$. For every pair of pesduoaccretive functions b_1 and b_2, we associate a Haar basis and denote the respective pair-base by $\{(\alpha_I^{(1)},\ \beta_I^{(1)})\}_{I\in\mathcal{J}}$ and $\{(\alpha_I^{(2)},\ \beta_I^{(2)})\}_{I\in\mathcal{J}}$. Formally, we have the following expansion

$$T(b_1 f) = \sum_{I,J} \alpha_I^{(2)} \left\langle \beta_I^{(2)},\ Tb_1\alpha_J^{(1)} \right\rangle_{b_2} \left\langle \beta_J^{(1)},\ f \right\rangle_{b_1}.$$

Let

$$u_{IJ} = \left\langle \beta_I^{(2)},\ Tb_1\alpha_J^{(1)} \right\rangle_{b_2}.$$

It suffices to prove for a suitable number t, when ω_I is taken to be $|I|^t$, the conditions of Lemma 4.2.2 are satisfied.

Because $T(b_1) = T^t(b_2) = 0$ and the kernel with respect to T satisfies (4.38) and (4.39), for the present more general operator T, the statement and the proof of Lemma 4.2.4 still hold. Because of the assumption that $T(b_1) = T^t(b_2) = 0$, we find that the estimates for Case 1 and Case 2 go through unchanged. The estimate of the part of the Schur sum corresponding to Case 3 holds by virtue of the weak boundedness assumption.

The general case: $T(b_1), T^t(b_2) \in BMO$. Let $T(b_1) = \phi_1$ and $T^t(b_2) = \phi_2$. We define

$$U_i f = \sum_{k=-\infty}^{\infty} \Delta_k^{(j)}(\phi_i) E_{k-1}^{(i)}(b_i^{-1} f),\ i, j = 1, 2, i \neq j, \tag{4.41}$$

where $E_k^{(i)}$ and $\Delta_k^{(i)}$ are the left conditional expectation operator and the left martingale difference with respect to the pseudoaccretive function b_i. It is obvious that $U_i b_i = \phi_i$, $i = 1, 2$. The kernel K_i of the operator U_i is given by the expression

$$K_i(x,\ y)=\sum_{k=-\infty}^{\infty}\sum_{I\in\mathcal{J}_{k-1}}\chi_I(x)\alpha_I^{(j)}(x)\Big\langle\beta_I^{(j)},\ \phi_i\Big\rangle_{b_j}\left(\int_I b_i\right)^{-1}\chi_I(y). \tag{4.42}$$

By (4.42), it is easy to verify

$$\Delta_m^{(i)}U_i f=\Delta_m^{(j)}(\phi_i)E_{m-1}^{(i)}(b_i^{-1}f).$$

We claim

$$\|S^{(i)}(U_i f)\|_2\leqslant C\|f\|_2, \tag{4.43}$$

where $S^{(i)}$ denotes the Littlewood–Paley square function with respect to b_i. Hence U_i is bounded on L^2. To prove (4.43), note that

$$\begin{aligned}
&\|S^{(i)}(U_i f)\|_2^2 \qquad (4.44)\\
&=\int\sum_k|\Delta_k^{(j)}(\phi_i)E_{k-1}^{(i)}(b_i^{-1}f)|^2dx\\
&\leqslant C\int\sum_k|\Delta_k^{(j)}(\phi_i)|^2\Big(E_{k-1}^{(i)*}(b_i^{-1}f)\Big)^2dx\\
&\leqslant C\int\sum_{k=-\infty}^{\infty}\widetilde{E}_{k-1}\Big(\sum_{m=k}^{\infty}|\Delta_m^{(j)}(\phi_i)|^2\Big)\Big[\Big(E_{k-1}^{(i)*}(b_i^{-1}f)\Big)^2-\Big(E_{k-2}^{(i)*}(b_i^{-1}f)\Big)^2\Big]dx,
\end{aligned}$$

where $E_k^{(i)*}g=\sup\limits_{m\leqslant k}|E_m^{(i)}g|$. Now, for every k,

$$\widetilde{E}_{k-1}\Big(\sum_{m=k}^{\infty}|\Delta_m^{(j)}(\phi_i)|^2\Big)\leqslant C\|\phi_i\|_{BMO}^2. \tag{4.45}$$

This is because, if $I\in\mathcal{J}_{k-1}$, then we can restrict $\sigma-$field $\{\mathcal{F}_m\}_{m=k-1}^{\infty}$ to I and deduce that on I,

$$\begin{aligned}
\widetilde{E}_{k-1}\Big(\sum_{m=k}^{\infty}|\Delta_m^{(j)}(\phi_i)|^2\Big)&=\frac{1}{|I|}\int_I\sum_{m=k}^{\infty}|\Delta_m^{(j)}(\phi_i)|^2dx\\
&=\frac{1}{|I|}\int_I\sum_{m=k}^{\infty}|\Delta_m^{(j)}(\phi_i-E_{k-1}^{(j)}(\phi_i)|^2dx\\
&=\frac{C}{|I|^{(j)}}\int_I\Big|\phi_i-\frac{1}{|I|^{(j)}}\int_I b_j\phi_i\Big|^2dx\\
&=\frac{C}{|I|}\int_I\Big|\phi_i-\frac{1}{|I|}\int_I\phi_i dy+\frac{1}{|I|^{(j)}}\int_I b_j\Big(\phi_i-\frac{1}{|I|}\int_I\phi_i dz\Big)dx\Big|^2\\
&\leqslant C\|\phi_i\|_{BMO}^2,
\end{aligned}$$

where we have used the fact that $|I^{(j)}| = \int_I b_j dx$. This gives (4.45). Returning to (4.43), we have

$$\|S^{(i)}(U_i f)\|_2^2 \leqslant C\|\phi_i\|_{BMO}^2 \int (Mf(x))^2 dx \leqslant C\|f\|_2^2,$$

where Mf denotes the usual Hardy–Littlewood maximal function. This proves (4.43).

By Lemma 4.1.1, U_i is bounded on L^2. The operator U_i^t is still bounded on L^2. If $i \neq j$, because $\int b_j \alpha_I^{(j)} dx = 0$,

$$\begin{aligned}\langle U_i^t(b_j),\ f\rangle_{b_i} &= \langle b_j,\ U_i(b_i f)\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{I \in \mathcal{J}_{k-1}} \Big(\int b_j \alpha_I^{(j)}\Big)\langle \beta_I^{(j)},\ \phi_i\rangle_{b_j} \Big(\int_I b_i\Big)^{-1}\Big(\int_I b_i f\Big) \\ &= 0.\end{aligned}$$

Hence if $i \neq j$, $U_i^t(b_j) = 0$. Letting $R = t - U_1 - U_2^t$, we have

$$R(b_1) = R^t(b_2) = 0. \tag{4.46}$$

The operator R is also weakly bounded. Applying the method of Theorem 4.2.1, we wish to show that R and T are bounded on L^2. This effectively reduces to checking that the operator R and R^t satisfy the same kind of conditions as those given in Lemma 4.2.4. The proofs of (iii) and (iv) of Lemma 4.2.4 use only the property (4.21) of the kernel K. Consider the kernels associated with the operators U_1 and U_2^t. For $i = 1, 2$, they are given by (4.42). Now for fixed $x \neq y$, and k, there exists at most one $I \in \mathcal{J}_{k-1}$, denoted by I_{k-1}, such that the summand in (4.42) is nonzero. For such a term,

$$|x - y| \leqslant C2^{-k}, \tag{4.47}$$

where C is independent of x, y and k. Let k_0 be the largest integer such that (4.47) holds. By (4.47), the sum in (4.42) is then, in norm, at most

$$\begin{aligned}& C\sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1/2}\frac{1}{|I_{k-1}|}\int_{I_{k-1}} |\beta_{I_{k-1}}^{(j)}(y)b_j(y)||\phi_i - (\phi_i)_{I_{k-1}}|dy \\ &\leqslant C\|\phi_i\|_{BMO}\sum_{k=-\infty}^{k_0} |I_{k-1}|^{-1} \\ &\leqslant C\|\phi_i\|_{BMO}\sum_{k=-\infty}^{k_0} 2^{nk} \\ &\leqslant C\|\phi_i\|_{BMO}2^{nk_0} \\ &\leqslant C\|\phi_i\|_{BMO}|x - y|^{-n}.\end{aligned}$$

As to (i) and (ii) of Lemma 4.2.4, we note that, if J is a dyadic-quasi-cube with $x \notin 2J$, then

$$U_1(b_1\alpha_J^{(1)})(x) = \sum_{-\infty}^{\infty}\sum_{I\in\mathcal{J}_{k-1}} \alpha_I^{(2)}(x)\chi_I(x)\langle\beta_I^{(2)},\ \phi_1\rangle_{b_2}\left(\int_I b_1\right)^{-1}\left(\int_I b_1\alpha_J^{(1)}\right) = 0.$$

In fact, the last factor in a term of the double summation is nonzero only when $I \subseteq J$. But because $x \notin 2J$, then $\chi_I(x) = 0$. So this term is 0. A similar argument applies to U_2^t. Hence, (i)–(iv) of Lemma 4.2.4 hold for the operator R. The operator R^t can be dealt with similarly. Assume that $R(b_1) = R^t(b_2) = 0$. With some appropriate modifications, the proof of Theorem 4.2.1 applies to the operator R. □

4.3 Clifford Martingale Φ−Equivalence Between $S(f)$ and f^*

In Sect. 4.2, the L^2-norm equivalence between a Clifford martingale and its square function plays an important role in the proof of the main results. The L^2-boundedness of the maximal function f^* indicates the L^2 equivalence between f^* and its square function. The later mentioned result is associated with $\Phi(t) = t^2$. In this section, we will generalize this result to more general functions Φ.

Let $(\Omega, \mathcal{F}, \nu)$ be a nonnegative σ−finite space and let ϕ be a bounded Clifford-valued measurable function. Consider the Clifford-valued measure $d\mu = \phi\nu$. The martingales are with respect to $d\mu$ and a family of $\{\mathcal{F}_m\}_{-\infty}^{\infty}$ of sub-σ-field satisfying

$$\{\mathcal{F}_m\}_{-\infty}^{\infty} \text{ nondecreasing}, \mathcal{F} = \cup\mathcal{F}_m,\ \cap\mathcal{F}_m = \emptyset, \tag{4.48}$$

and

$$(\Omega, \mathcal{F}_m, \nu) \text{ complete}, \sigma - \text{finite } \ \forall\, m. \tag{4.49}$$

Let $e_1, \cdots, e_n$ be the basic vectors of $\mathbb{R}^n$ satisfying

$$e^2 = -1,\ e_ie_j = -e_je_i,\ i \neq j,\ i, j = 1, 2, \ldots, n, \tag{4.50}$$

and $\mathbb{R}_{(n)}$ be the Clifford algebra on 2^n-dimensional real number field generated by the increasingly ordered subset e_A, $\{1, \cdots, n\}$, where $e_A = e_{j_1}\cdots e_{j_l}$, $A = \{j_1, \ldots, j_l\}$, $1 \leqslant l \leqslant n$, $e_\emptyset = e_0 = 1$. We will use the following norm in $\mathbb{R}_{(n)}$:

$$|\lambda| = \left(\sum_A \lambda_A^2\right)^{1/2},\ \lambda = \sum_A \lambda_A e_A. \tag{4.51}$$

For this norm, we have the following relation:

$$|\lambda\mu| \leqslant k|\lambda||\mu| \; \forall \, \lambda, \mu \in \mathbb{R}_{(n)}, \tag{4.52}$$

where k is a constant which depends only on the dimension m. When at least one of λ and μ, say λ, is of the form $\lambda = \sum_{i=0}^{d} \lambda_i e_i$, i.e., a vector in $\mathbb{R}^{n+1} \subset \mathbb{R}_{(n)}$, we have

$$k^{-1}|\lambda||\mu| \leqslant |\lambda|. \tag{4.53}$$

For a martingale $f = (f_m)_{-\infty}^{\infty}$, the maximal square function is defined as

$$f_m^* = \sup_{k \leqslant m} |f_k|, \;\; f^* = f_\infty^*. \tag{4.54}$$

For $1 \leqslant p \leqslant \infty$, $f = \{f_m\}_{-\infty}^{\infty}$ is called bounded on L^p if

$$\|f\|_p = \sup_m \|f_m\|_p < \infty. \tag{4.55}$$

In the next proposition, we prove the boundedness of the maximal operator f^*.

Proposition 4.3.1 *Let $1 < p \leqslant \infty$. The maximal operator "$*$" is (p, p) type and weak $(1, 1)$ type. For $1 < p \leqslant \infty$, every L^p-bounded martingale $f = \{f_m\}_{-\infty}^{\infty}$ is generated by some function $f \in L^p(\nu)$ which satisfies $\|f\|_p \approx \sup_m \|f_m\|_p$.*

Proof Let $f = \{f_m\}_{-\infty}^{\infty}$ be a martingale. On the one hand,

$$f_m = E(f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m).$$

On the other hand,

$$\begin{aligned} f_m &= E(f_{n+2} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{m+2} \mid \mathcal{F}_m) \\ &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\widetilde{E}(\phi f_{m+2} \mid \mathcal{F}_{m+1}) \mid \mathcal{F}_m). \end{aligned}$$

The above estimates give

$$\widetilde{E}(\phi f_{m+1}) = \widetilde{E}(\widetilde{E}(\phi f_{n+2} \mid \mathcal{F}_m) \mid \mathcal{F}_m).$$

Hence $\{\widetilde{E}(\phi f_{m+1})\}_{-\infty}^{\infty}$ is the martingale with respect to $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$. We can deduce from the expression of f_m that the following relation holds:

$$\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi \mid \mathcal{F}_m) f_m.$$

Then it is L^p-bounded. Moreover, we have

$$\sup_n \|f_n\|_p \approx \sup_m \|\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\|_p,$$
$$f^* \approx \sup_m |\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)|.$$

Because of the result in the classical case, $*$ is (p, p) type and weak $(1, 1)$ type. For $1 < p \leqslant \infty$ and any integer $M > 0$, we decompose $\Omega = \cup_k \Omega_k$, where $\Omega_k \in \mathcal{F}_{-M}$ and $|\Omega_k| < \infty$. Because for any k, $\{\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{n \geqslant -M}$ is a classical martingale, we can obtain some $\phi f \in L^p(\Omega_k, \nu)$ such that on Ω_k,

$$\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \widetilde{E}(\phi f \mid \mathcal{F}_m), \ n \geqslant -M.$$

Therefore, for $n \geqslant -M$,

$$\begin{aligned} f_m &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f_{n+1} \mid \mathcal{F}_m) \\ &= \widetilde{E}(\phi \mid \mathcal{F}_m)^{-1}\widetilde{E}(\phi f \mid \mathcal{F}_m) \\ &= E(f \mid \mathcal{F}_m). \end{aligned}$$

Letting $M \to \infty$, we can see that $f_m = E(f \mid \mathcal{F}_m)\ \forall\, n$. Moreover, we have

$$\|f\chi_{\Omega_k}\|_p \leqslant C \sup_n \|f_m \chi_{\Omega_k}\|_p$$

and

$$\|f\|_p \leqslant C \sup_m \|f_m\|_p.$$

In addition, $\sup_m \|f_m\|_p \leqslant C\|f\|_p$ and $\|f\|_p \approx \sup_m \|f_m\|_p$. □

By Proposition 4.3.1, we can identify a L^p-bounded martingale with the function that generalizes the martingale as follows

$$f = \{f_m\}_{-\infty}^{\infty} = \{E(f \mid \mathcal{F}_m)\ \forall\, m\}_{-\infty}^{\infty}.$$

Proposition 4.3.2 *Let $1 \leqslant p \leqslant \infty$ and $f = \{f_m\}_{-\infty}^{\infty}$ be a L^p-bounded martingale. Then*

$$\lim_{m\to\infty} f_m = f, 1 < p \leqslant \infty, \tag{4.56}$$

where f is the L^p-function which generates $\{f_m\}_{-\infty}^{\infty}$ in Proposition 4.3.1, and for $p = 1$, the following limits exists:

$$\lim_{m\to\infty} f_m \text{ exists }, p = 1 \tag{4.57}$$

and

$$\lim_{m\to-\infty} f_m = 0, 1 \leqslant p < \infty. \tag{4.58}$$

Proof Let $\Omega = \cup\Omega_k$, where $\Omega_k \in \mathcal{F}_0$ with $|\Omega_k| < \infty\ \forall\, k$. Then $\{\widetilde{E}(\phi \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{m>0}$ and $\{\widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m)\chi_{\Omega_k}\}_{m>0}$ are L^p-bounded martingales with respect to $(\Omega_k, \mathcal{F} \cap \Omega_k, \{\mathcal{F}_m \cap \Omega_k\}_{m\geqslant 0})$, and have their respective limits:

$$\begin{cases} \text{on every } \Omega_k, \ \lim\limits_{m\to\infty} \widetilde{E}(\phi \mid \mathcal{F}_m) = \phi \text{ a.e.;} \\ \text{on every } \Omega_k, \ \text{for some } g, \lim_{m\to\infty} \widetilde{E}(\phi f_{m+1} \mid \mathcal{F}_m) = \phi g \text{ a.e.;} \\ \text{for } 1 < p \leqslant \infty, g = f. \end{cases}$$

The last two limits imply that (4.56) and (4.57) hold. Now we prove (4.58). Write $\theta(\omega) = \overline{\lim}_{m\to-\infty}|f_m|$. Then $\theta(\omega) \leqslant f^*(\omega)$ and $\theta(\omega)$ are $\cap\mathcal{F}_m$ measurable. This means that $\theta(\omega) = a \geqslant 0$ a.e. Because $*$ is weak (p, p) type, for $1 \leqslant p < \infty$, we have

$$|\{\theta(\omega) > \lambda\}|_\nu \leqslant |\{f^* > \lambda\}|_\nu \leqslant \left(\frac{C}{\lambda}\|f\|_p\right)^p \ \forall\, \lambda > 0.$$

Hence $a = 0$. This gives (4.58). □

Let Φ be a nondecreasing and continuous function from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying $\Phi(0) = 0$ with the moderate growth condition

$$\Phi(2u) \leqslant C_1\phi(u), u > 0. \tag{4.59}$$

We begin to establish the $\Phi-$equivalence between $S(f)$ and f^*, where f is the martingale such that for any m,

$$|\Delta_m f| \leqslant D_{m-1}, \tag{4.60}$$

where $D = \{D_m\}$ is a nonnegative nondecreasing and adapted process to $\{\mathcal{F}_m\}$. We only consider the case $\{\mathcal{F}_m\}_{m\geqslant 0}$.

Theorem 4.3.1 *Let $f = \{f_m\}_{m\geqslant 0}$ be a $l-$martingale or a $r-$martingale satisfying (4.60). Then*

$$\int_\Omega \Phi(S(f))d\nu \leqslant C \int_\Omega \Phi(f^* + D_\infty)d\nu \tag{4.61}$$

and

$$\int_\Omega \Phi(f^*)d\nu \leqslant C \int_\Omega \Phi(S(f) + D_\infty)d\nu, \tag{4.62}$$

where the involved constants depend only on C_0 and C_1.

Proof We shall use the stopping time argument and the good $\lambda-$ inequality. Let α be any real number larger than 1, $\beta > 0$ to be determined and λ be any level. Notice that

$$|f_m| \leqslant |f_{m-1}| + |\Delta_m f| \leqslant f^*_{m-1} + D_{m-1} = \rho_{m-1}.$$

Define the stopping time $\tau = \inf\{m : \rho_m > \beta\lambda\}$ and the associated stopping martingale

$$f^{(\tau)} = \{f_m^{(\tau)}\}_{m\geqslant 0} = \{f_{\min\{m,\tau\}}\}_{m\geqslant 0}.$$

Then we have

$$\{\tau < \infty\} = \{\rho_\infty > \beta\lambda\},\ f^{(\tau)*} = \sup_m |f_{\min\{m,\tau\}}| \leqslant f^*_\tau \leqslant \rho_{\tau-1} \leqslant \beta\lambda.$$

Now consider the adapted process $\{S_m(f^{(\tau)}) > \lambda\}$ and define the stopping time

$$T = \inf\{m : S_m(f^{(\tau)}) > \lambda\}.$$

Then we have

$$\{T < \infty\} = \{S(f^{(\tau)}) > \lambda\},\ S_{T-1}(f^{(\tau)}) \leqslant \lambda.$$

Hence

$$\begin{aligned}\{S(f) > \alpha\lambda\} &\subset \{\tau < \infty\} \cup \{\tau = \infty,\ S_\tau(f)^2 > \alpha^2\lambda^2\}\\ &\subset \{\tau < \infty\} \cup \{S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 > (\alpha^2 - 1)\lambda^2\}\end{aligned}$$

and

$$\begin{aligned}&\widetilde{E}(\chi_{S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 > (\alpha^2-1)\lambda^2} \mid \mathcal{F}_T)\\ &\leqslant \frac{1}{(\alpha^2-1)\lambda^2}\widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^\tau)^2 \mid \mathcal{F}_T).\end{aligned}$$

Now we consider a new underlying space $(\Omega, \mathcal{F}, \nu, \{\mathcal{J}_m\}_{m\geqslant 0})$ with $J_m = \mathcal{F}_{T+m}$, and the martingale

$$g = \{g_m\}_{m\geqslant 0} \text{ such that } g_m = f^{(\tau)}_{T+m} - f^{(\tau)}_{T-1}.$$

Then we have

$$\Delta_m g = f^{(\tau)}_{T+m} - f^{(\tau)}_{T-1} - (f^{(\tau)}_{T+m-1} - f^{(\tau)}_{T-1}) = \Delta_{T+m} f^{(\tau)}$$

and

$$\begin{aligned}S(g)^2 &= \sum_{m=0}^{\infty} |\Delta_m g|^2 = \sum_{m=0}^{\infty} |\Delta_{T+m} f^{(\tau)}|^2\\ &= \sum_{k=T}^{\infty} |\Delta_k f^{(\tau)}|^2 = S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2.\end{aligned}$$

By Lemma 4.1.1, we get

$$\begin{aligned}\widetilde{E}(S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 \mid \mathcal{F}_T) &= \widetilde{E}(S(g)^2 \mid \mathcal{J}_\prime) \\ &\leqslant C\widetilde{E}(|g|^2 \mid \mathcal{J}_0) \\ &= C\widetilde{E}(|f^{(\tau)} - f^{(\tau)}_{T-1}| \mid \mathcal{F}_T) \\ &\leqslant C\beta^2\lambda^2.\end{aligned}$$

Now, because $\{S(f^\tau) > \alpha\lambda\} \subset \{T \leqslant \infty\}$, we obtain

$$\begin{aligned}|\{S(f^{(\tau)}) > \alpha\lambda\}|_\nu &\leqslant \int_{\{T<\infty\}} \chi_{\{S(f^{(\tau)})>\alpha\lambda\}} d\nu \\ &= \int_{\{T<\infty\}} \widetilde{E}(\chi_{\{S(f^{(\tau)})>\alpha\lambda\}} \mid \mathcal{F}_T) d\nu \\ &\leqslant \int_{\{T<\infty\}} \widetilde{E}(\chi_{\{S(f^{(\tau)})^2 - S_{T-1}(f^{(\tau)})^2 > (\alpha^2-1)\lambda^2\}} \mid \mathcal{F}_T) d\nu \\ &\leqslant \frac{C\beta^2}{\alpha^2-1}|\{S(f^{(\tau)}) > \lambda\}|_\nu \leqslant \frac{C\beta^2}{\alpha^2-1}|\{S(f) > \lambda\}|_\nu,\end{aligned}$$

and hence

$$|\{S(f) > \alpha\lambda\}|_\nu \leqslant |\{\rho_\infty > \beta\lambda\}|_\nu + \frac{C\beta^2}{\alpha^2-1}|\{S(f) > \lambda\}|_\nu,$$

which is the desired good λ inequality for the couple $(S(f),\ f^* + D_\infty)$. The one for the couple $(f^*,\ S(f) + D_\infty)$ is similar. From them, we obtain (4.61) and (4.62). □

We can get rid of D_∞ in the following cases:

(i) Φ is convex;
(ii) $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$ is regular in some sense.

For simplicity, we only consider the simplest regularity, i.e., the dyadic type one: each $\mathcal{F}_m$ is atomic, whose atom $I^{(m)} = I_1^{(m+1)} + I_2^{(m+1)}$ satisfies $||I_1^{(m+1)}|_\mu| = ||I_2^{(m+1)}|_\mu|$. A little more general regularity is applicable to our case. We have

Theorem 4.3.2 *Under the additional condition (i) on* Φ *or (ii) on* $(\Omega, \mathcal{F}, \nu, \{\mathcal{F}_m\}_{-\infty}^{\infty})$*, we have*

$$\int_\Omega \Phi(S(f))d\nu \approx \int_\Omega \Phi(f^*)d\nu,$$

where in the above equivalence, all the constants only depend on C_0 *and* C_1.

Proof We first consider $\{\mathcal{F}_m\}_{m\geqslant 0}$. Davis' decomposition holds in such case: every Clifford martingale $f = \{f_m\}_{m\geqslant 0}$ can be decomposed into a sum of two martingales: $g = \{g_m\}_{m\geqslant 0}$ and $h = \{h_m\}_{m\geqslant 0}$ satisfying

$$|\Delta_m g| \leqslant 4d^*_{m-1},\ d^* = \sup_{k\leqslant m} |d_k|,\ d_k = \Delta_k f, \tag{4.63}$$

and

$$\int_\Omega \Phi(\sum_{m=0}^{\infty} |\Delta_m h|)d\nu \leqslant C \int_\Omega \Phi(d^*)d\nu\ \forall\ \text{ convex } \Phi. \tag{4.64}$$

Now for $f = \{f_m\}_{m\geqslant 0}$, we have

$$\begin{aligned}\int_\Omega \Phi(S(f))d\nu &\leqslant C \int_\Omega \Phi(S(g))d\nu + C \int_\Omega \Phi(S(h))d\nu \\ &\leqslant C \int_\Omega \Phi(g^*) + C \int_\Omega \Phi(d^*) + C \int_\Omega \Phi(\sum_{m=0}^{\infty} |\Delta_m h|)d\nu \\ &\leqslant C \int_\Omega \Phi(f^*)d\nu.\end{aligned}$$

The proof for the reverse inequality is similar. Next we consider the dyadic type case. We claim that in such case, (4.60) holds for any martingale $f = \{f_m\}_{-\infty}^{\infty}$ and suitably defined $D = \{D_m\}$. In fact,

$$D_{m-1}\ |_{I^{m-1}} = \sup_{k\leqslant m} \max(|\Delta_k f|\ |_{I_1^{(k)}},\ |\Delta_k f|\ |_{I_2^{(k)}})$$

is a nonnegative, nondecreasing and adapted process such that

$$|\Delta_m f| \leqslant D_{m-1}$$

and

$$D_\infty \leqslant C \min(f^*,\ S(f)).$$

Only the last assertion needs to be verified. In fact,

$$\int_{I^{(k-1)}} \Delta_k f d\mu = 0$$

implies

$$\int_{I_1^{(k-1)}} \Delta_k f d\mu = -\int_{I_2^{(k-1)}} \Delta_k f d\mu.$$

This gives

$$\Delta_k f\ |_{I_1^{(k)}}\ |I_1^{(k)}|_\mu = -\Delta_k f\ |_{I_2^{(k)}}\ |I_2^{(k)}|_\mu$$

or

$$\frac{|\Delta_k f\ |_{I_1^{(k)}}\ |}{|\Delta_k f\ |_{I_2^{(k)}}\ |} = \frac{||I_2^{(k)}|_\mu|}{||I_1^{(k)}|_\mu|}.$$

Therefore, on $I^{(k-1)}$,

$$\max(|\Delta_k f|\ |_{I_1^{(k)}},\ |\Delta_k f|\ |_{I_2^{(k)}}) \leqslant C|\Delta_k f|$$

and

$$D_\infty \leqslant C \sup_k |\Delta_k f| \leqslant C \min\{S(f),\ f^*\}.$$

□

4.4 Remarks

Remark 4.4.1 Another method to prove the boundedness of Calderón-Zygmund operators lays on the multi-resolution technique developed by R. Coifman, Y. Meyer etc. That method is usually called the fast algorithm of Calderón-Zygmund. The basic idea is to decompose the kernel of the Calderón-Zygmund operator T under consideration by wavelet basis and then represent T as a linear combination of quasi-annular operators. Then applying the smoothness and the canceling condition of the regular wavelets, we estimate the coefficients of the kernel and obtain that the L^2−norms of the quasi-annular operators have a good rate of decay. This implies the L^2-boundedness of Calderón-Zygmund operators. In 1994, similar to the result on $\mathbb{R}^n$, using Clifford-valued regular wavelets, M. Mitrea obtained the L^2-boundedness of singular integral operators on Lipschitz surface, see [12].

References

1. Li C, McIntosh A, Semmes S. Convolution singular integrals on Lipschitz surfaces. J Am Math Soc. 1992;5:455–81.
2. Gaudry G, Long R, Qian T. A martingale proof of L^2-boundedness of Clifford-valued singular integrals. Ann Math Pura Appl. 1993;165:369–94.
3. Coifman R, Jones P, Semmes S. Two elementary proofs of the L^2 boundedness of Cauchy integrals on Lipschitz curves. J Am Math Soc. 1989;2:553–64.
4. Cowling M, Gaudry G, Qian T. A note on martingales with respect to complex measures. In: Miniconference on operators in analysis, Macquare University, September 1989, Proceedings of the center for mathematical analysis, Australian National University, vol. 24; 1989. p. 10–27.
5. Edwards R, Gaudry G. Littlewood-Paley and multiplier theory. Berlin: Springer; 1977.
6. Garsla A. Martingale inequalities. New York: W. A. Benjamin Inc; 1973.
7. David G, Journé J-L. A boundedness criterion for generalized Calderón-Zygmund operators. Ann Math. 1984;120:371–97.

8. David G, Journé J-L, Semmes S. Opérateurs de Caldéron-Zygmund sur les espaces de nature homogène. Preprint.
9. Peetre J. On convolution operators leaving $L^{p,\lambda}$ invariant. Ann Mat Pura Appl. 1966;72:295–304.
10. Spanne S. Sur l'interpolation entre les espaces $\mathfrak{L}_k^{p,\phi}$. Ann Sc Norm Sup Pisa. 1964;20:625–48.
11. Stein E-M. Singular integrals, harmonic functions, and differentiability properties of functions of several variables. In: Proceedings of symposium in pure mathematics, vol. 10; 1967. p. 316–35.
12. Mitrea M. Clifford wavelets, singular integrals, and hardy spaces. Lecture notes in mathematics, vol. 1575. Berlin: Springer; 1994.